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Slow recurrent regimes for a class of one-dimensional stochastic growth models

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Abstract

We classify the possible behaviors of a class of one-dimensional stochastic recurrent growth models. In our main result, we obtain nearly optimal bounds for the tail of hitting times of some compact sets. If the process is an aperiodic irreducible Markov chain, we determine whether it is null recurrent or positive recurrent and in the latter case, we obtain a subgeometric convergence of its transition kernel to its invariant measure. We apply our results in particular to state-dependent Galton–Watson processes and we give precise estimates of the tail of the extinction time.

MSC: 60J10; 60J80

Keywords: Markov chains; Hitting-times; Recurrence classification; Lyapunov function; Stochastic difference equation

1. Introduction and main result

1.1. Introduction

We consider a stochastic growth model $(X_n)_{n\in\mathbb{N}}$, taking values in $\mathcal{X}$, an unbounded subset of $\mathbb{R}_+$, and satisfying a stochastic difference equation of the form

$$X_{n+1} = X_n + g(X_n) + \xi_n, \tag{1}$$

where g is a given function and $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of random variables such that almost surely,

$$\mathbb{E}\left(\xi_n \middle| \mathcal{F}_n\right) = 0,$$

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$$\mathbb{E}\left(\xi_n^2 \middle| \mathcal{F}_n\right) = \sigma^2(X_n) < \infty,$$

for some positive function $\sigma^2(x)$. The filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is such that $(X_n)_{n\in\mathbb{N}}$ is $\mathcal{F}_n$ measurable for all $n \in \mathbb{N}$.

Provided that the following limit exists

$$\theta = \lim_{x\to\infty} \frac{2xg(x)}{\sigma^2(x)},$$

and belongs to $(-\infty, 1)$, Kersting proved in [10] that $\mathbb{P}(\{X_n \underset{n\to\infty}{\longrightarrow} \infty\}) = 0$ and said that $(X_n)_{n\in\mathbb{N}}$ is recurrent, adopting the terminology from Markov chain theory, whereas if $\theta \in (1, \infty)$ then $\mathbb{P}(\{X_n \underset{n\to\infty}{\longrightarrow} \infty\}) > 0$. A similar criterion for the multidimensional case was recently given in [1].

The aim of this article is to determine how quickly the process $(X_n)_{n\in\mathbb{N}}$, started from $x > A$, goes into the interval $[0, A]$, where $A > 0$ is arbitrary. If $(X_n)_{n\in\mathbb{N}}$ is an aperiodic irreducible Markov chain, we deduce therefrom a criterion of positive recurrence and how fast $(X_n)_{n\in\mathbb{N}}$ converges to its invariant measure. Moreover, if we have in mind population models, where a natural assumption is the dichotomy property, *i.e.*,

$$\mathbb{P}\left(\left\{X_n \underset{n\to\infty}{\longrightarrow} \infty\right\}\right) + \mathbb{P}\left(\{\exists n \text{ such that } X_n = 0\}\right) = 1,$$

we obtain precise estimates of the tail of the extinction time.

The first key ingredient of this article is to consider power functions as Lyapunov functions for growth models. Kersting [10] proved recurrence and transience of growth models by using the logarithm as a Lyapunov function. However, we cannot get more information on the behavior of $(X_n)_{n\in\mathbb{N}}$ with this function. Considering power functions yields an inequality of the form

$$\mathbb{E}\left(X_{n+1}^\alpha \middle| \mathcal{F}_n\right) - X_n^\alpha \le -C X_n^{\alpha-1} g(X_n) + b\mathbb{1}_{\{X_n \le A\}},$$

for all $n \in \mathbb{N}$, where $\alpha \in (0, 1)$, A, C and b some positive constants. From this equation, we deduce that

$$\mathbb{E}\left(f(Y_{n+1}) \middle| \mathcal{F}_n\right) - f(Y_n) \le -Cf'(Y_n) + b\mathbb{1}_{\{Y_n \le A\}}, \tag{2}$$

where Y_n is a transform of X_n, f is an increasing differentiable function, and A, C and b are some positive constants. Inequality (2) enables us to give all possible behaviors of our class of recurrent growth models. In a series of papers [3–5], Aspandiiarov et al. proved upper and lower bounds for the tail of hitting-time into compact sets, for processes verifying some conditions, improving previous results of Lamperti [13]. The second key ingredient, is to apply these results on a transform $Y_n = G(X_n)$ of our process to get an upper bound of hitting-time into compact sets. If $(X_n)_{n\in\mathbb{N}}$ is an aperiodic irreducible Markov chain, we give a criterion for null recurrence or positive recurrence. Moreover, if $(X_n)_{n\in\mathbb{N}}$ is positive recurrent, we obtain from [4] in the countable state space, from [7] in a general state space, subgeometric rate of convergence to its invariant probability measure. Thus, we give a complete classification of behaviors of stochastic recurrent growth processes of the form (1). By applying our results, we deduce nearly optimal upper and lower bounds of the tail of the extinction time of state-dependent Galton–Watson processes that seem to have never been studied before, to the best of our knowledge. We also recover a weaker version of results of Zubkov [16] on the return time to zero of critical Galton–Watson process with immigration, but without using probability generating functions.

The article is organized as follows. In the next subsection, our main results Theorems 1.1 and 1.2 are stated. Then, in Section 2 we state and prove a series of lemmas needed for the proof

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of Theorems 1.1 and 1.2. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we consider that $(X_n)_{n\in\mathbb{N}}$ is an aperiodic irreducible Markov chain and we prove Theorem 1.2. In Section 5, we give various examples, in particular extinction time of state-dependent Galton–Watson processes. In the last section, we prove a key lemma for the lower bound of Theorem 1.1 and we recall some results from [4] that we use throughout this article.

1.2. Main results

We list the assumptions we need to formulate our main results.

Assumptions.
(A1) The function g is positive, regular varying with index $1-\lambda$ with $\lambda > 0$ and ultimately monotone.
(A2) There exists $\theta \in (0, 1)$ such that

$$\lim_{x\to\infty} \frac{2xg(x)}{\sigma^2(x)} = \theta.$$

(A3)There exists $\delta > 0$ such that for all $n \in \mathbb{N}$, $\mathbb{E}\left(|\xi_n|^{2+\delta}\middle|\mathcal{F}_n\right) \le C\sigma^{2+\delta}(X_n)$.

Let us comment on these assumptions.

In assumption (A1), since $\lambda > 0$ we have that $g(x) = o(x)$ and this precludes X_n from growing geometrically. Assumption (A3) is usual : in [10], Kersting needs the existence of $2+\delta$-moments, to prove the recurrence of X_n.

Before stating the theorem, we introduce two transforms. Let

$$G(x) = \int_1^x \frac{\mathrm{d}y}{g(y)},$$

and for $\alpha > 0$, let

$$\ell_\alpha = \left(G^{-1}(x)\right)^\alpha,$$

where G^{-1} is the inverse of G *i.e.* $G(G^{-1}(x)) = x$.

Since $\frac{1}{g}$ is regular varying with index $\lambda - 1$, by Proposition 1.5.8 in [6] function G is regular varying with index λ and ℓ_α with index $\frac{\alpha}{\lambda}$.

Theorem 1.1. *Assume that (A1), (A2) and (A3) hold. Then there exists $A > 0$ such that for all $x_0 \in \mathcal{X} \cap (A,\infty)$, for all α and β such that $0 < \alpha < 1-\theta < \beta$, there exist two constants C_α and C_β such that for all $n \in \mathbb{N}$,*

$$\frac{C_\beta}{\ell_\beta(n)} \le \mathbb{P}_{x_0}(\tau_A > n) \le \frac{C_\alpha}{\ell_\alpha(n)}, \tag{3}$$

with $\tau_A = \inf\{n \in \mathbb{N} : X_n \le A\}$.

Remark 1.1. We prove the upper bound in (3) by showing that $\mathbb{E}_{x_0}(\ell_\alpha(\tau_A)) < \infty$ for all $0 < \alpha < 1-\theta$ and $x_0 \in \mathcal{X}$. An easy consequence of this lower bound is that $\mathbb{E}_{x_0}(\ell_\beta(\tau_A)) = \infty$ for all $\beta > 1-\theta$ and $x_0 \in \mathcal{X} \cap (A,\infty)$. We cannot determine if $\mathbb{E}_{x_0}(\ell_{1-\theta}(\tau_A))$ is finite or not.

Remark 1.2. In the proof of the theorem, we get explicit constants C_α and C_β and in particular, the dependence of these constants on x_0.

If $(X_n)_{n\in\mathbb{N}}$ is an aperiodic irreducible Markov chain, we determine when it is positive recurrent and the rate of convergence to the invariant probability measure. We denote by $P(.,.)$ the transition kernel of the Markov chain $(X_n)_{n\in\mathbb{N}}$. We deal with both countable state space and general state space.

Assumptions.
(A4) $(X_n)_{n\in\mathbb{N}}$ is an aperiodic irreducible Markov chain taking values in a countable set $\mathcal{X} \subset \mathbb{R}_+$, such that for all $A > 0$, $[0, A] \cap \mathcal{X}$ is finite.
(A4') $(X_n)_{n\in\mathbb{N}}$ is an aperiodic ψ-irreducible Markov chain taking values in a general state space $\mathcal{X} \subset \mathbb{R}_+$ and sets $[0, A] \cap \mathcal{X}$ are small sets for all $A > 0$.

We recall the definition of ψ-irreducibility (see [15, p. 84]):

We say that a Markov chain $(X_n)_{n\in\mathbb{N}}$ is ψ-irreducible if there exists a non trivial measure ψ such that for all set $K \subset \mathcal{X}$,

$$\psi(K) > 0 \Rightarrow \mathbb{P}_x(\exists n \text{ such that } X_n \in K) > 0, \tag{4}$$

and for all measures φ satisfying (4), φ is absolutely continuous with respect to ψ.

Theorem 1.2. *Assume that (A1), (A2), (A3) and (A4) or (A4') hold.*

Then $(X_n)_{n\in\mathbb{N}}$ is Harris-recurrent. Moreover
(i) If $\lambda > 1-\theta$, then $(X_n)_{n\in\mathbb{N}}$ is null recurrent.
(ii) If $\lambda < 1-\theta$, then $(X_n)_{n\in\mathbb{N}}$ is positive recurrent. Denote by π its invariant probability measure. Then for all $\alpha \in (\lambda, 1-\theta)$ and if (A4) holds then for all probability measures ν on $\mathcal{X}$ such that

$$\mathbb{E}_\nu(\ell'_\alpha(\tau_A)) < \infty,$$

we have

$$\lim_{n\to\infty} \ell'_\alpha(n) \left\| \nu P^n - \pi \right\|_{\mathrm{TV}} = 0, \tag{5}$$

and if (A4') holds then for all $x \in \mathcal{X}$

$$\lim_{n\to\infty} \ell'_\alpha(n) \left\| P^n(x,.) - \pi(.) \right\|_{\mathrm{TV}} = 0 \tag{6}$$

where ℓ'_α is the derivative of ℓ_α.

Remark 1.3. If $\lambda = 1-\theta$ then (X_n) could be either null recurrent or positive recurrent. We refer to [14] for examples when $g(x) = \frac{c}{x}$.

Example 1.1. We consider a stochastic growth model defined by the stochastic difference equation (1)

$$X_{n+1} = X_n + cX_n^\gamma + \xi_n$$

with $\gamma \in (-1, 1)$, $c > 0$ and $\sigma^2(X_n) = \mathbb{E}(\xi_n^2 \big| \mathcal{F}_n) = dX_n^{1+\gamma}$ with $d > 0$. Then

- $\theta = \frac{2c}{d}$
- $\lambda = 1-\gamma$
- $G(x) \propto x^{1-\gamma}$
- $\ell_\alpha(x) \propto x^{\frac{\alpha}{1-\gamma}}$.

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By Theorem 1.1, for all $\beta < 1-\theta < \alpha$, there exists $A > 0$ such that for all $x_0 > A$, there exist $C_\beta > 0$ and $C_\alpha > 0$ such that

$$\frac{C_\alpha}{n^{\frac{\alpha}{1-\gamma}}} \le \mathbb{P}_{x_0}(\tau_A > n) \le \frac{C_\beta}{n^{\frac{\beta}{1-\gamma}}}.$$

If $\gamma > \theta$ and (X_n) is a Markov chain satisfying the assumptions of Theorem 1.2, then (X_n) is positive recurrent and for all $\alpha < 1-\theta$, for all $x \in \mathcal{X} \subset \mathbb{R}_+$,

$$\lim_{n\to\infty} n^{\frac{\alpha}{1-\gamma}-1} \| P^n(x,.) - \pi(.)\|_{\mathrm{TV}} = 0,$$

where π is the invariant probability measure of $(X_n)_{n\in\mathbb{N}}$.

If c and d are fixed, then by increasing γ, we make $(X_n)_{n\in\mathbb{N}}$ positive recurrent. Actually, the parameter γ is related to both the drift $g(x)$ and the variance $\sigma^2(x)$, by increasing γ we increase both of them but we can see that its effect on the variance is more important.

2. Preliminary results

We state and prove here some important lemmas which will be useful for the proofs of Theorems 1.1 and 1.2. In the first lemma, we prove that $\left(X^\alpha_{n\wedge\tau_A}\right)_{n\in\mathbb{N}}$ is a supermartingale if $\alpha \in (0, 1-\theta)$, and a submartingale if $\alpha \in (1-\theta, 1)$.

Lemma 2.1. *Assume that (A1), (A2) and (A3) hold.*
(i) If $\alpha \in (0, 1-\theta)$, then there exist $A > 0$, $C > 0$ and $b > 0$ such that for all $n \in \mathbb{N}$,

$$\mathbb{E}\left(X^\alpha_{n+1} \middle| \mathcal{F}_n\right) \le X_n^\alpha - Cg(X_n)X_n^{\alpha-1} + b\mathbb{1}_{\{X_n\le A\}} \text{ a.s.} \tag{7}$$

(ii) If $\alpha \in (1-\theta, 1)$, then there exist $B > 0$ and $b_1 > 0$ such that for all $n \in \mathbb{N}$

$$\mathbb{E}\left(X^\alpha_{n+1} \middle| \mathcal{F}_n\right) \ge X_n^\alpha - b_1\mathbb{1}_{\{X_n\le B\}} \text{ a.s.} \tag{8}$$

Proof. For $D > 0$ large enough, we have by Taylor's expansion

$$(1+u)^\alpha \le 1 + \alpha u + \frac{\alpha(\alpha-1)}{2}u^2 + D|u|^{2+\delta}, \tag{9}$$

for all $u \in (-1, +\infty)$, with δ as in (A3). We obtain, for all $n \in \mathbb{N}$, if $X_n > 0$,

$$\begin{aligned}
\mathbb{E}\left(X^\alpha_{n+1} \middle| \mathcal{F}_n\right) &\le \mathbb{E}\left(X_n^\alpha\left(1+\frac{g(X_n)+\xi_n}{X_n}\right)^\alpha \middle| \mathcal{F}_n\right)\\
&\le \mathbb{E}\left(X_n^\alpha\left(1+\alpha\left(\frac{g(X_n)+\xi_n}{X_n}\right)+\frac{\alpha(\alpha-1)}{2}\left(\frac{g(X_n)+\xi_n}{X_n}\right)^2\right)\middle|\mathcal{F}_n\right)\\
&\quad + \mathbb{E}\left(X_n^\alpha\left(D\left|\frac{g(X_n)+\xi_n}{X_n}\right|^{2+\delta}\right)\middle|\mathcal{F}_n\right)\\
&\le X_n^\alpha + \alpha\left(g(X_n)X_n^{\alpha-1} - \frac{1-\alpha}{2}\sigma^2(X_n)X_n^{\alpha-2}\right) + R_n,
\end{aligned}$$

with

$$R_n = \frac{\alpha(\alpha-1)}{2}g(X_n)^2X_n^{\alpha-2} + D\mathbb{E}\left(|g(X_n)+\xi_n|^{2+\delta}X_n^{\alpha-3}\middle|\mathcal{F}_n\right).$$

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By Hölder's inequality and (A3),

$$R_n \le \frac{\alpha(\alpha-1)}{2} g(X_n)^2 X_n^{\alpha-2} + D'|g(X_n)|^{2+\delta} X_n^{\alpha-2-\delta} + D''\sigma^{2+\delta}(X_n) X_n^{\alpha-2-\delta}.$$

By (A1) and (A2), $\sigma(x) = o(x)$ when x tends to infinity and then $\sigma^{2+\delta}(x)x^{\alpha-2-\delta} = o\left(g(x)x^{\alpha-1}\right)$ when x tends to infinity, and then there exist $C, B, b > 0$ such that

$$\mathbb{E}\left(X_{n+1}^{\alpha} \middle| \mathcal{F}_n\right) \le X_n^{\alpha} - Cg\left(X_n\right) X_n^{\alpha-1} + b\mathbb{1}_{\{X_n \le B\}}.$$

Since by Taylor's expansion there exists a positive constant D such that

$$(1+u)^{\alpha} \ge 1 + \alpha u + \frac{\alpha(\alpha-1)}{2} u^2 - D|u|^{2+\delta},$$

for all $u \in (-1, +\infty)$, the proof of (8) is similar. $\square$

The first two statements of the next lemma, on the top of the previous one, give us a better understanding of the criterion of Theorem 1.2, *i.e.*, the comparison between λ and $1-\theta$. Some points of this lemma are stated and proved in [11] with different assumptions. We recall that a function f is ultimately concave or ultimately convex if there exists $x_0 > 0$ such that the restriction of f to $[x_0, \infty)$ is concave or convex respectively.

Lemma 2.2. *Assume that (A1) holds.*

1. *For $\alpha \in (0, \lambda)$, the function ℓ_α is ultimately concave.*
2. *For $\alpha \in (\lambda, +\infty)$, the function ℓ_α is ultimately convex.*
3. *We have*

$$\lim_{x\to\infty} \frac{x}{G(x)\, g(x)} = \lambda. \tag{10}$$

4. *For all $\mu < \lambda$,*

$$g(x) = \mathcal{O}(x^{1-\mu}), \tag{11}$$

when x tends to infinity.

5. *Let $\alpha > 0$, for all $r \in (0, +\infty)$, there exists a positive constant A_r such that for all $x \in \mathbb{R}_+$,*

$$A_r \ell_\alpha(x) \ge \ell_\alpha(rx). \tag{12}$$

Proof. We first prove statements 1 and 2. Since g is ultimately monotone, G is either ultimately concave or convex and so is ℓ_α. The regular variation of ℓ_α with index $\frac{\alpha}{\lambda}$ implies the two first statements.

The third statement is a direct consequence of Proposition 1.5.8 of [6, p. 26].

The Potter's bound (see Theorem 1.5.6 [6, p. 25]) states that

$$\forall M > 0, \forall v > M, g(v) \le \left(\frac{v}{M}\right)^{1-\lambda+\varepsilon} g(M), \tag{13}$$

thus $g(x) = \mathcal{O}(x^{1-\mu})$ for all $\mu < \lambda$.

Finally, we prove the last statement. We know that ℓ_α is a regular varying function with index $\frac{\alpha}{\lambda}$, then for all $r \in (0, +\infty)$

$$\lim_{x\to+\infty} \frac{\ell_\alpha(rx)}{\ell_\alpha(x)} = r^{1-\lambda}.$$

Since $\ell_\alpha \ge 1$, we get the fifth statement. $\square$

Remark 2.1. Since we further consider ℓ_α, with $\alpha \neq \lambda$, we know by Theorem 1.8.3 in [6, p. 45], that there exists a $\mathcal{C}^\infty$-function $\tilde{\ell_\alpha}$, all of whose derivatives are monotone, with $\ell_\alpha \sim \tilde{\ell_\alpha}$. Since we always consider asymptotic properties of ℓ_α, we henceforth consider that ℓ_α is a $\mathcal{C}^\infty$-function with all of whose derivatives are monotone.

3. Polynomial asymptotics of the tail of hitting times

The aim of this section is to prove Theorem 1.1.

We first prove the upper bound of the inequality (3) by using Theorem 2 and Theorem 3 in [4], which we recall them in the last section. Let $\mathcal{A}$ be the set of positive functions f such that there exists a positive constant A_f such that

$$\limsup_{x\to\infty} \frac{f(2x)}{f(x)} \le A_f.$$

For all real valued functions h, let $\mathcal{B}_h$ be the set of positive functions $f \in \mathcal{C}^2(0,\infty)$ ultimately concave, such that $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} f'(x) = 0$, and such that the integral

$$\int_1^\infty \frac{f'(x)\mathrm{d}x}{h\circ r(x)} \quad \text{converges,} \tag{14}$$

with $r(x) = \sup\{y \ge A,\ f'(x) = h'(y)\}$.

Proposition 3.1. *Assume that (A1), (A2) and (A3) hold. Then there exists $A > 0$ such that for all $x_0 > A$, γ and η such that $\gamma < \eta < 1-\theta$, there exists a constant $K(\gamma,\eta)$ such that for all $n \in \mathbb{N}$,*

$$\mathbb{P}_{x_0}(\tau_A > n) \le \frac{K(\gamma,\eta)x_0^\eta}{\ell_\gamma(n)}.$$

Proof. If $\gamma > \lambda$, then ℓ_γ is ultimately convex. We know by (12) that $\ell_\gamma \in \mathcal{A}$ and then we apply Theorem B.1 and get the upper bound by Chebyshev's inequality.

If $\eta < \lambda$, then ℓ_η is ultimately concave. To apply Theorem B.2 with $f = \ell_\gamma$ and $h = \ell_\eta$, we need also to check that the integral (14) converges. Let $r(x) = \sup\{y \ge A,\ \ell'_\gamma(x) = \ell'_\eta(y)\}$. We first prove that for x large enough, we have $x \le r(x)$.

We recall that $\ell'_\gamma(x) = \gamma g\left(G^{-1}(x)\right)\left(G^{-1}(x)\right)^{\gamma-1}$. Thus,

$$\frac{\ell'_\gamma(x)}{\ell'_\eta(x)} = \frac{\gamma}{\eta}\left(G^{-1}(x)\right)^{\gamma-\eta} \underset{x\to\infty}{\longrightarrow} 0.$$

Since $G^{-1}(x)$ increases to infinity, there exists $A_1 > 0$ such that for all $x > A_1$, $\ell'_\gamma(x) \le \ell'_\eta(x)$ and then, for all $x > A_1$, $r(x) \ge x$.

Since ℓ_η is an increasing function, we obtain by substitution

$$\begin{aligned}\int^\infty \frac{\ell'_\gamma(x)\mathrm{d}x}{\ell_\eta\circ r(x)} &\le \int^\infty \frac{\ell'_\gamma(x)\mathrm{d}x}{\ell_\eta(x)}\\ &\le \int^\infty \frac{\ell'_\gamma(x)\mathrm{d}x}{(\ell_\gamma(x))^{\eta/\gamma}}\\ &\le C\int^\infty \frac{\mathrm{d}u}{u^{\eta/\gamma}} < \infty.\end{aligned}$$

Finally, we obtain the upper bound by Chebyshev's inequality. □

Before proving the lower bound of Theorem 1.1, we recall an important lemma from [5]:

Lemma 3.1 *([5], Lemma 2).* *Let Y_n be a $\mathcal{F}_n$-adapted stochastic process taking values in an unbounded subset of $\mathbb{R}_+$ and for $A > 0$, let $\tilde{\tau}_A = \inf\{n \geq 0, Y_n \leq A\}$. Suppose there exist positive constants A, C and D such that for all $n \in \mathbb{N}$, on $\{\tilde{\tau}_A > n\}$,*

$$\mathbb{E}(Y_{n+1} - Y_n \big| \mathcal{F}_n) \geq -C$$

and, for some $r > 1$,

$$\mathbb{E}\left(Y_{n+1}^r - Y_n^r \big| \mathcal{F}_n\right) \leq D Y_n^{r-1}.$$

Then, for any $\nu \in (0, 1)$, there exist positive ε and d that do not depend on A such that for any $n \in \mathbb{N}$, on $\{Y_{n\wedge\tilde{\tau}_A} > A(1+d)\}$,

$$\mathbb{P}\left(\tilde{\tau}_A > n + \varepsilon Y_{n\wedge\tilde{\tau}_A} \big| \mathcal{F}_n\right) \geq 1 - \nu.$$

The next lemma is crucial. We defer its proof, which is rather technical, to Appendix A:

Lemma 3.2. *For all $n \in \mathbb{N}$, let $Y_n = G(X_n)$. Assume that (A1), (A2), (A3) hold. Then $(Y_n)_{n\in\mathbb{N}}$ satisfies the assumptions of Lemma 3.1 with $r = 2$.*

Proposition 3.2. *Assume that (A1), (A2) and (A3) hold. Let $\beta > 1 - \theta$. There exists $A > 0$ such that for all $x_0 > A$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $n \in \mathbb{N}$,*

$$\mathbb{P}_{x_0}(\tau_A > n) \geq C \frac{x_0^\beta - A^\beta}{\ell_\beta(n/\varepsilon_0)}.$$

Proof. The proof of the lower bound is as follows: we know by Lemma 3.2 that Y_n verifies the assumptions of Lemma 3.1 and then we follow the proof of Theorem 1 in [3]. We relax the assumption of bounded jumps of this theorem by using Hölder's inequality.

Let $\beta > 1 - \theta$.

By Lemma 3.2, we know that Lemma 3.1 applies to $Y_n = G(X_n)$. By Lemmas 3.1 and 3.2, there exist $\varepsilon_0 > 0$ and $d > 0$ such that for any n :

$$\mathbb{P}\left(\tau_A > n + \varepsilon_0 Y_{n\wedge\tau_A} \big| \mathcal{F}_n\right) \geq 1 - \upsilon \qquad \text{on } \left\{Y_{n\wedge\tau_A} > G(A)(1+d)\right\}.$$

This implies that for any stopping time μ we have

$$\mathbb{P}\left(\tau_A > \mu + \varepsilon_0 Y_{\mu\wedge\tau_A} \big| \mathcal{F}_\mu\right) \geq 1 - \upsilon \qquad \text{on } \left\{Y_{\mu\wedge\tau_A} > G(A)(1+d)\right\} \cap \{\mu < \infty\}.$$

For each $S > 0$, let

$$\tilde{\tau}_S = \inf\{n \geq 0, Y_n \geq S\}.$$

Let us fix B such that $B > G(A)(1+d)$.

Then,

$$\begin{aligned}\mathbb{P}(\tau_A \geq \varepsilon_0 B) &\geq \mathbb{P}\left(\tau_A > \tilde{\tau}_B + \varepsilon_0 Y_{\tilde{\tau}_B\wedge\tau_A}, \tilde{\tau}_B < \tau_A\right) \\ &= \mathbb{E}\left(\mathbb{1}_{\{\tilde{\tau}_B<\tau_A\}}\mathbb{P}\left(\tau_A > \tilde{\tau}_B + \varepsilon_0 Y_{\tilde{\tau}_B\wedge\tau_A} \big| \mathcal{F}_{\tilde{\tau}_B}\right)\right) \\ &\geq (1-\upsilon)\mathbb{P}(\tilde{\tau}_B < \tau_A). \end{aligned} \tag{15}$$

Since $(\tau_A \wedge \tilde{\tau}_B) < \infty$ and $\ell_\beta\left(Y_{n\wedge\tau_A\wedge\tilde{\tau}_B}\right)$ is a submartingale by Lemma 2.1, we have

$$x_0^\beta = \ell_\beta(Y_0) \leq \mathbb{E}\left(\ell_\beta(Y_{\tau_A\wedge\tilde{\tau}_B})\right).$$

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Since $g(x) = o(x)$, there exists $K > 0$ such that $\mathbb{E}\left(\ell_1\left(Y_{\tilde{\tau}_B-1}\right) + g\left(X_{\tilde{\tau}_B-1}\right)\right) \le K^{1/\beta}\mathbb{E}\left(X_{\tilde{\tau}_B-1}\right)$ and then

$$\begin{aligned}\mathbb{E}\left(\ell_\beta\left(Y_{\tilde{\tau}_B}\right)\mathbb{1}_{\{\tilde{\tau}_B<\tau_A\}}\right) &\le \mathbb{E}\big(\ell_1\left(Y_{\tilde{\tau}_B}\right)\big)^\beta\mathbb{P}\left(\tilde{\tau}_B < \tau_A\right)\\ &\le \mathbb{E}\big(\mathbb{E}\left(\ell_1\left(Y_{\tilde{\tau}_B}\right)\middle|\mathcal{F}_{\tilde{\tau}_B-1}\right)\big)^\beta\mathbb{P}\left(\tilde{\tau}_B < \tau_A\right)\\ &\le \mathbb{E}\big(\ell_1\left(Y_{\tilde{\tau}_B-1}\right) + g\left(X_{\tilde{\tau}_B-1}\right)\big)^\beta\mathbb{P}\left(\tilde{\tau}_B < \tau_A\right)\\ &\le K\ell_\beta\left(B\right)\mathbb{P}\left(\tilde{\tau}_B < \tau_A\right).\end{aligned} \tag{16}$$

Hence,

$$\begin{aligned}x_0^\beta &\le \mathbb{E}\left(\ell_\beta\left(Y_{\tau_A}\right)\mathbb{1}_{\{\tilde{\tau}_B>\tau_A\}}\right) + \mathbb{E}\left(\ell_\beta\left(Y_{\tilde{\tau}_B}\right)\mathbb{1}_{\{\tilde{\tau}_B<\tau_A\}}\right)\\ &\le \ell_\beta\left(G(A)\right) + K\ell_\beta\left(B\right)\mathbb{P}\left(\tilde{\tau}_B < \tau_A\right),\end{aligned}$$

by (16) and

$$\mathbb{P}\left(\tilde{\tau}_B < \tau_A\right) \ge \frac{x_0^\beta - \ell_\beta\left(G(A)\right)}{Kl_\beta\left(B\right)}.$$

Then, by (15), for $n > \varepsilon_0 G(A)\left(1+\delta\right)$,

$$\mathbb{P}\left(\tau_A > n\right) \ge \left(1-\upsilon\right)\frac{x_0^\beta - A^\beta}{K\ell_\beta\left(n/\varepsilon_0\right)}. \qquad \square \tag{17}$$

Proof of Theorem 1.1. The upper bound is a direct consequence of Proposition 3.1. The lower bound comes from Proposition 3.2 and (12). $\square$

4. The Markov case : subgeometric rate of convergence

In this section, we prove Theorem 1.2, firstly the countable state space case and secondly the general state space case. We apply some results from [4] that we recall in the last section.

Let $\mathcal{G}$ be the set of positive functions f such that there exist a positive function h such that $h(x) \to 0$ as $x \to \infty$ and a positive constant c such that for any positive $m \ge 1$, $x_1 \ge 1, \ldots, x_m \ge 1$,

$$f\left(\sum_{k=1}^m x_k\right) \le ce^{mh(m)}\sum_{k=1}^m f(x_k).$$

Let $\mathcal{G}'$ be the set of non decreasing in a neighborhood of infinity functions f such that $\ln(f(x))/x$ is non increasing in a neighborhood of infinity and tends to zero when x tends to infinity.

Proof of Theorem 1.2 for a countable state space. Let A be defined as in Theorem 1.1. We know by (A4) that $F = [0, A] \cap \mathcal{X}$ is finite. First note that for all $z \in F$, by Markov property we have

$$\mathbb{E}_z(\tau_F) = \mathbb{P}_z(X_1 \in F) + \sum_{s\in\mathcal{X}\setminus F}\mathbb{P}_z(X_1 = s)\mathbb{E}_s(\tau_F). \tag{18}$$

(i) Let us assume that $\lambda > 1-\theta$. We prove that for all $s \in \mathcal{X}\setminus F$, $\mathbb{E}_s(\tau_F) = \infty$. Let $\beta \in (1-\theta, \lambda)$. By Theorem 1.1 we know that if $\sum 1/\ell_\beta(n)$ diverges, then $\mathbb{E}_s(\tau_F) = \infty$, for all $s \in \mathcal{X}\setminus F$. The sum $\sum 1/\ell_\beta(n)$, is of the same nature as the integral $\int \mathrm{d}x/\ell_\beta(x)$. By the substitution $u = G^{-1}(x)$, we obtain

$$\int_{\cdot}^{\infty} \frac{\mathrm{d}x}{\ell_\beta(x)} = \int_{\cdot}^{\infty} \frac{\mathrm{d}u}{u^\beta g(u)} = \infty,$$

since $g(u) \le K u^{1-\lambda+(\lambda-\beta)/2}$.

Since $(X_n)_{n\in\mathbb{N}}$ is irreducible, there exists $(z_0, s_o) \in F \times \mathcal{X} \setminus F$ such that $\mathbb{P}_{z_0}(X_1 = s_0) > 0$. Thus $\mathbb{E}_{z_0}(\tau_F) = \infty$ and by Proposition B.1, $\mathbb{E}_{z_0}(\tau) = \infty$, then $(X_n)_{n\in\mathbb{N}}$ is null recurrent.
(ii) Let us assume that $\lambda < 1 - \theta$. Let $\eta \in (\lambda, 1 - \theta)$. We first prove that there exists a positive constant K such that for all $s \in \mathcal{X} \setminus F$, $\mathbb{E}_s(\tau_F) \le K s^\eta$. Let $\gamma \in (\lambda, \eta)$. By Proposition 3.1, we know that

$$\mathbb{P}_s(\tau_F > n) \le \frac{K(\gamma, \eta)s^\eta}{\ell_\gamma(n)}.$$

We check that $\sum_{n=1}^{\infty} 1/\ell_\gamma(n) < \infty$. Since ℓ_η is convex, there exists a constant C such that

$$\sum_{n=1}^{\infty} \frac{1}{\ell_\gamma(n)} \le C \sum_{n=1}^{\infty} \frac{\ell'_\eta(n)}{\ell_\gamma(n)}.$$

This series is of the same nature as the integral

$$\int_{\cdot}^{\infty} \frac{\ell'_\eta(x)\mathrm{d}x}{\ell_\gamma(x)} = \int_{\cdot}^{\infty} \frac{\ell'_\eta(x)\mathrm{d}x}{\ell_\eta(x)^{\gamma/\eta}} = K \int_{\cdot}^{\infty} \frac{\mathrm{d}u}{u^{\gamma/\eta}} < \infty.$$

Thus

$$\mathbb{E}_s(\tau_F) \le K s^\eta. \tag{19}$$

By (19) and (18), we obtain

$$\mathbb{E}_z(\tau_F) \le 1 + K\mathbb{E}_z(X_1^\eta) < \infty,$$

thus by Proposition B.1, for any $z \in F$, $\mathbb{E}_z(\tau) < \infty$ so $(X_n)_{n\in\mathbb{N}}$ is positive recurrent.

Let $\alpha \in (\lambda, 1-\theta)$ and $\beta \in (\alpha, 1-\theta)$. To apply Theorem B.3 with $f = \ell_\alpha$ and $\phi = \ell_\beta$, we need to check that $\ell_\alpha \in \mathcal{G}$ and $\ell'_\alpha \in \mathcal{G}'$. Since ℓ_α is convex, we have for all $m \ge 1, \boldsymbol{x}_1 \ge 1, \dots, x_m \ge 1$

$$\ell_\alpha\left(\sum_{k=1}^m x_k\right) \le \frac{1}{m}\sum_{k=1}^m \ell_\alpha(m x_k),$$

and by 5 of Lemma 2.2,

$$\ell_\alpha\left(\sum_{k=1}^m x_k\right) \le \frac{(4m)^{2\alpha/\lambda}}{m}\sum_{k=1}^m \ell_\alpha(x_k) \le 4^{2\alpha/\lambda} e^{(2\alpha/\lambda-1)\ln(m)}\sum_{k=1}^m \ell_\alpha(x_k),$$

thus $\ell_\alpha \in \mathcal{G}$.

We recall that $\ell'_\alpha(x) = \alpha g(G^{-1}(x))(G^{-1}(x))^{\alpha-1}$. Since ℓ_α is regular varying with index $\frac{\alpha}{\lambda}$ we only need to prove that $\ln(\ell'_\alpha(x))/x$ is non increasing in a neighborhood of infinity.

The derivative of $\ln(\ell'_\alpha(x))/x$ is $\frac{x\ell''_\alpha(x)/\ell'_\alpha(x)-\ln(\ell'_\alpha(x))}{x^2}$ since we can consider that ℓ'_α is regular varying with variation $\alpha/\lambda - 1 > 0$, $x\ell''_\alpha(x)/\ell'_\alpha(x)$ tends to $\alpha/\lambda - 1$ but $\ln(\ell'_\alpha(x))$ tends to infinity, so $\ln(\ell'_\alpha(x))/x$ is non-increasing in a neighborhood of infinity and so $\ell'_\alpha \in \mathcal{G}'$. □

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In the general state space case, we use a drift condition which comes from [7]:

Definition 4.1. We say that the condition $\boldsymbol{D}\,(\phi, V, \Gamma)$ is verified if there exist a function V, a concave monotone non-decreasing differentiable function $\phi : [1, \infty] \mapsto (0, \infty]$, a measurable set Γ and a finite constant b such that for all $x \in \mathbb{R}_+$

$$\mathbb{E}_x\left(V(X_1)\right) + \phi \circ V(x) \le V(x) + b\mathbb{1}_{\{x \in \Gamma\}}.$$

Proposition 4.1 ([7], *Proposition 2.5*)**.** *Let P be a ψ-irreducible and aperiodic kernel. Assume that $\boldsymbol{D}(\phi, V, \Gamma)$ holds for a function ϕ such that $\lim_{t\to\infty} \phi'(t) = 0$, a petite set Γ and a function V such that $\{V < \infty\} \neq \emptyset$. Then, there exists an invariant probability measure π, and for all x in the full and absorbing set $\{V < \infty\}$, i.e. $\pi(\{V < \infty\}) = 1$,*

$$\lim_{n\to\infty} r_\phi(n) \left\| P^n(x, .) - \pi(.)\right\|_{\mathrm{TV}} = 0,$$

with $r_\phi\,(x) = \phi \circ \Phi^{-1}\,(x)$ and $\Phi\,(x) = \int_1^x \frac{\mathrm{d}u}{\phi(u)}$.

The proof of Theorem 1.2 in the general state space case consists essentially in checking that the condition $\boldsymbol{D}\,(\phi, V, \Gamma)$ holds.

We also recall that a set C is regular if for all set B such that $\psi(B) > 0$,

$$\sup_{x\in C} \mathbb{E}_x(\tau_B) < \infty,$$

where τ_B is the first hitting-time of the set B. A Markov chain is called regular if there exists a countable cover of $\mathcal{X}$ by regular sets.

Proof of Theorem 1.2 for a general state space. Since $[0, A] \cap \mathcal{X}$ is petite and since for all $x \in \mathcal{X}$, $\mathbb{P}_x(\tau_A < \infty) = 1$, we know from [15, Proposition 9.1.7 p.205] that $(X_n)_{n\in\mathbb{N}}$ is Harris-recurrent.
(i) We assume that $\lambda > 1-\theta$. By Theorem 1.1, $\forall x \in (A, \infty)\cap\mathcal{X}$, $\mathbb{E}_x(\tau_A) = \infty$. We assume that $(X_n)_{n\in\mathbb{N}}$ is positive recurrent to get a contradiction. By [15, Theorem 11.1.4 p.260], we know that there exists a decomposition $\mathcal{X} = \mathcal{S} \cup \mathcal{N}$ with $\mathcal{S}$ full and absorbing and $(X_n)_{n\in\mathbb{N}}$ restricted to $\mathcal{S}$ is regular. Since $\mathcal{S}$ is absorbing, we know that $[0, A] \cap \mathcal{S} \neq \emptyset$ and $(A, \infty) \cap \mathcal{S} \neq \emptyset$. Let $C \subset \mathcal{S}$ be a regular set of the countable cover of $\mathcal{S}$ such that $C \cap (A, \infty) \neq \emptyset$. Then there exists $x \in C \cap (A, \infty)$, and we know that $\mathbb{E}_x(\tau_A) = \infty$ which contradicts the regularity of C. Then $(X_n)_{n\in\mathbb{N}}$ is not positive recurrent but null recurrent.
(ii) We assume that $\lambda < 1-\theta$. Let $\alpha \in (\lambda, 1-\theta)$ and $\phi(x) = g(x^{\frac{1}{\alpha}})\,x^{\frac{\alpha-1}{\alpha}}$. Using Lemma 2.1,

$$\mathbb{E}_x\left(X_1^\alpha\right) \le x^\alpha - C\phi\left(x^\alpha\right) + b\mathbb{1}_{\{x\le A\}}.$$

We know that ϕ is a regular varying function with index $(\alpha-\lambda)/\alpha \in (0, 1)$ then ϕ is an ultimately concave non-decreasing function. Thus, the condition $\boldsymbol{D}\,(\phi, V, \Gamma)$ holds. By a short computation, we see that $r_\phi(x) = \ell'_\alpha(x)$. Since $[0, A]\cap\mathcal{X}$ is a petite set by assumption, we apply Proposition 4.1 and there exists an invariant probability measure π such that for all x

$$\lim_{n\to\infty} \ell'_\alpha(n) \left\| P^n(x, .) - \pi(.)\right\|_{\mathrm{TV}} = 0. \qquad \square$$

5. Examples and applications

We now illustrate our results by applying Theorems 1.1 and 1.2 to several models.

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5.1. Bessel-like walks

A Bessel-like walk is a random walk on $\mathbb{N}$, reflecting at 0, with steps ± 1 and transition probabilities of the form

$$\mathbb{P}(X_{n+1} = x + 1 \big| X_n = x) = p_x = \frac{1}{2}\left(1 - \frac{\delta}{2x} + o\left(\frac{1}{x}\right)\right)$$

and

$$\mathbb{P}(X_{n+1} = x - 1 \big| X_n = x) = 1 - p_x$$

where $x \geq 1$, $\delta \in \mathbb{R}$ and the $o(1/x)$ holds for x tending to infinity. A Bessel-like walk is recurrent if $\delta > -1$, positive recurrent if $\delta > 1$ and transient if $\delta < -1$.

We assume here that $\delta \in (-1, 0)$. There exists $A > 0$ such that we obtain an estimation of the tail of the hitting-time of the compact set $[0, A]$.

Proposition 5.1. *For all α, β such that $\alpha < 1 + \delta < \beta$, there exists $A > 0$ such that for all $x_0 > A$, there exist two positive constants C_α and C_β such that*

$$\frac{C_\beta}{n^{\beta/2}} \leq \mathbb{P}_{x_0}(\tau_A > n) \leq \frac{C_\alpha}{n^{\alpha/2}}.$$

In this example, $g(x) = \frac{-\delta}{2x}$, $\theta = -\delta$, $\lambda = 2$, $G(x) = \frac{x^2-1}{-\delta}$, $\ell_\alpha(x) = \mathcal{O}(x^{\frac{\alpha}{2}})$. For more precise results on Bessel-like walks and in particular asymptotic behaviors of $\mathbb{P}_x(\tau_0 > n)$ and $\mathbb{P}_x(\tau_0 = n)$, we refer to [2].

5.2. Critical Galton–Watson process with immigration

We consider a critical Galton–Watson process with immigration $(X_n)_{n\in\mathbb{N}}$ defined by

$$X_{n+1} = \sum_{k=1}^{X_n} \xi_{k,n} + I_n,$$

where $\left(\xi_{k,n}\right)_{k,n\in\mathbb{N}}$ are i.i.d. integer-valued random variables such that $\mathbb{E}(\xi_{1,1}) = 1$, $\mathrm{Var}(\xi_{1,1}) = d > 0$ and $\mathbb{E}(\xi_{1,1}^{2+\delta}) < \infty$ for some $\delta > 0$ and i.i.d. integer-valued random variables $(I_n)_{n\in\mathbb{N}}$ such that $\mathbb{E}(I_1) = c > 0$, $\mathbb{E}(I_1^{2+\delta}) < \infty$ and the variables $(\xi_{k,n})_{k,n\in\mathbb{N}}$ and $(I_n)_{n\in\mathbb{N}}$ are independent.

Zubkov proved in [16] that the Markov chain $(X_n)_{n\in\mathbb{N}}$ is recurrent if $\theta = \frac{2c}{d} < 1$ and gave the asymptotic behavior of the tail of the return-time to zero $T_0 = \inf\{n \geq 1 \text{ such that } X_n = 0\}$:

$$\mathbb{P}_0(T_0 > n) \sim L(n) n^{\theta-1},$$

with L a slowly varying function. He also needed weaker moments assumptions.

We get here a weaker version of his result but without using neither the branching property nor probability generating functions.

Proposition 5.2. *There exists $A > 0$, such that for all $x_0 > A$, α, β such that $\alpha < 1 - \theta < \beta$, there exist some positive constants C_α and C_β such that for all $n \in \mathbb{N}$*

$$\frac{C_\beta}{n^\beta} \leq \mathbb{P}_{x_0}(\tau_A > n) \leq \frac{C_\alpha}{n^\alpha}.$$

In this example, $X_{n+1} = \sum_{k=1}^{X_n} \xi_{k,n} + I_n = X_n + c + (\sum_{k=1}^{X_n}(\xi_{k,n} - 1) + I_n - c$, so $\xi_n = (\sum_{k=1}^{X_n}(\xi_{k,n} - 1) + I_n - c$, $g(x) = c$, $\theta = \frac{2c}{d}$, $\lambda = 1$, $G = \mathcal{O}(x)$, $\ell_\alpha = \mathcal{O}(x^\alpha)$.

5.3. Extinction time of state-dependent Galton–Watson process

State-dependent Galton–Watson processes were introduced by Klebaner in [12] and Höpfner in [8]. They both gave condition for extinction and gamma-type limiting distribution for the process. However, to the best of our knowledge, extinction times of state-dependent Galton–Watson processes were never investigated.

Let $(X_n)_{n\in\mathbb{N}}$ be state-dependent Galton–Watson process defined as follows :

$$X_{n+1} = \sum_{k=1}^{X_n} A_{k,n}(X_n),$$

where $\mathbb{E}(A_{k,n}(X_n)\big|X_n = x) = 1 + \frac{c}{x}$ and $\mathbb{V}ar(A_{k,n}(X_n)\big|X_n = x) = \sigma^2 + o(1)$ with $c > 0$ and $\sigma^2 > 0$. We assume that 0 is an absorbing state and that for all $A > 0$ and all $n \in \mathbb{N}$, there exist $\varepsilon > 0$ and $k_A \in \mathbb{N}^*$,

$$\mathbb{P}(X_{n+k_A} = 0\big|X_n \le A) \ge \varepsilon. \tag{20}$$

This assumption implies the dichotomy property (see Theorem 3.1 in [9]), that is to say,

$$\mathbb{P}\left(\{\exists n \text{ such that } X_n = 0\}\right) + \mathbb{P}\left(\left\{X_n \underset{n\to\infty}{\longrightarrow} \infty\right\}\right) = 1.$$

We denote the extinction time by $\tau_0 = \inf\{n \in \mathbb{N} \text{ such that } X_n = 0\}$.

Theorem 5.1. *Let $\theta = \frac{2c}{\sigma^2}$ and assume that $\theta \in (0, 1)$. Then, for all $\alpha < 1 - \theta < \beta$, for all $x \in \mathbb{N}^*$, there exist two constants D_α and D_β such that*

$$\frac{D_\beta}{n^\beta} \le \mathbb{P}_x\left(\tau_0 > n\right) \le \frac{D_\alpha}{n^\alpha}.$$

Proof. Let α and β such that $\alpha < 1 - \theta < \beta$. We apply Theorem 1.1 and then there exists $A > 0$, such that for all $x > A$, there exist $C_\alpha > 0$ and $C_\beta > 0$ such that

$$\frac{C_\beta}{n^\beta} \le \mathbb{P}_x\left(\tau_A > n\right) \le \frac{C_\alpha}{n^\alpha}.$$

Since $\{0\} \subset [0, A]$, we obtain $\mathbb{P}_x(\tau_A > n) \le \mathbb{P}_x(\tau_0 > n)$ and then

$$\frac{C_\beta}{n^\beta} \le \mathbb{P}_x\left(\tau_A > n\right) \le \mathbb{P}_x(\tau_0 > n).$$

Let $(T_\ell)_{\ell\ge 0}$ be a sequence of stopping times defined as below

$$T_\ell = \inf\{n \ge k_A + T_{\ell-1} \text{ such that } X_n \in [0, A]\},$$

with $T_0 = 1$ and k_A is the integer associated to A such that (20) holds. By (20), we get

$$\mathbb{P}_x(\tau_0 > T_\ell) \le (1 - \varepsilon)^l.$$

For $\alpha \in (0, 1 - \theta)$, we get

$$\mathbb{E}_x\left(\tau_0^\alpha\right) = \sum_{\ell=0}^{\infty} \mathbb{E}_x\left(\mathbb{1}_{\{T_\ell < \tau_0 \le T_{\ell+1}\}}\tau_0^\alpha\right) \le \sum_{\ell=0}^{\infty} \mathbb{E}_x\left(\mathbb{1}_{\{T_\ell < \tau_0\}}T_{\ell+1}^\alpha\right)$$

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$$\leq \sum_{\ell=0}^{\infty} \mathbb{E}_x \left(\mathbb{1}_{\{T_\ell < \tau_0\}} (T_\ell + k_A + (T_{\ell+1} - k_A - T_\ell))^\alpha \right)$$

$$\leq \sum_{\ell=0}^{\infty} \mathbb{E}_x \left(\mathbb{1}_{\{T_\ell < \tau_0\}} \left(T_\ell^\alpha + k_A^\alpha + (T_{\ell+1} - k_A - T_\ell)^\alpha \right) \right).$$

Let $\tau_{A,k_A} = \inf\{n \geq k_A \text{ such that } X_n \in [0, A]\}$.

Since $(T_{\ell+1} - k_A - T_\ell)^\alpha \leq \mathbb{E}_{X_{T_\ell + k_A}} \left(\tau_A^\alpha \right)$, then by induction we obtain

$$\begin{aligned}
\mathbb{E}_x \left(\tau_0^\alpha \right) &\leq \sum_{\ell=0}^{\infty} \mathbb{E}_x \left(\mathbb{1}_{\{T_\ell < \tau_0\}} \left(T_0^\alpha + \ell k_A^\alpha + \sum_{i=0}^{\ell} \mathbb{E}_{X_{T_{i+k_A}}} \left(\tau_A^\alpha \right) \right) \right) \\
&\leq \mathbb{E}_x \left(T_0^\alpha \right) + \sum_{\ell=0}^{\infty} (1-\varepsilon)^\ell \ell k^\alpha + \mathbb{E}_x \left(\sum_{\ell=0}^{\infty} \mathbb{1}_{\{T_\ell < \tau_0\}} \sum_{i=0}^{\ell} \mathbb{E}_{X_{T_{i+k_A}}} \left(\tau_A^\alpha \right) \right) \\
&\leq \mathbb{E}_x \left(T_0^\alpha \right) + \sum_{\ell=0}^{\infty} (1-\varepsilon)^\ell \ell k_A^\alpha + \sum_{\ell=0}^{\infty} (1-\varepsilon)^\ell \sup_{y \in [0,A]} \mathbb{E}_y \left(\tau_{A,k_A}^\alpha \right) \\
&< \infty.
\end{aligned}$$

We obtain the expected upper bound for $\mathbb{P}_x (\tau_0 > n)$ by Chebyshev's inequality. □

In this example, $g(x) = c$, $\theta = \frac{2c}{\sigma^2}$, $\lambda = 1$, $G = \mathcal{O}(x)$, $\ell_\alpha = \mathcal{O}(x^\alpha)$.

5.4. A non-markovian example

Let $(X_n)_{n\in\mathbb{N}}$ be a process defined by

$$X_{n+1} = X_n + 1 + K \varepsilon_n \sqrt{R_n}$$

where $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables such that for all $n \in \mathbb{N}$, $\mathbb{P}(\varepsilon_n = -1) = \mathbb{P}(\varepsilon_n = 1) = \frac{1}{2}$, $K > 2$ and R_n defined as follows :

- Let $(N_n)_{n\in\mathbb{N}}$ be a sequence of independent integer-valued random variables such that $\forall i \in \{0, \dots, n\}$, $\mathbb{P}(N_n = i) = \frac{1}{n+1}$.
- Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. integer-valued random variables such that $\mathbb{P}(U_n = 0) = \mathbb{P}(U_n = 1) = \frac{1}{2}$.

We also assume that the random sequences $(N_n)_{n\in\mathbb{N}}$, $(U_n)_{n\in\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ are independent.

Let

$$R_n = U_n \frac{X_n^2}{X_n + X_{N_n}} + (1 - U_n) \frac{X_n X_{N_n}}{X_n + X_{N_n}}.$$

If there exists $n \in \mathbb{N}$ such that $X_n \leq 0$, then for all $k \in \mathbb{N}$, $X_{n+k} = 0$.

By construction, $(X_n)_{n\in\mathbb{N}}$ is not a Markov chain of any order. Let us check that $(X_n)_{n\in\mathbb{N}}$ satisfies the stochastic difference equation $X_{n+1} = X_n + g(X_n) + \xi_n$ with $\mathbb{E}\left(\xi_n \middle| \mathcal{F}_n\right) = 0$ and $\mathbb{E}\left(\xi_n^2 \middle| \mathcal{F}_n\right) = \sigma^2 (X_n)$. Let $\xi_n = \varepsilon_n K \sqrt{R_n}$. By independence, one has immediately $\mathbb{E}\left(\xi_n \middle| \mathcal{F}_n\right) = 0$.

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A short computation gives

$$\mathbb{E}\left(\xi_n^2\middle|\mathcal{F}_n\right) = \frac{K^2}{2(n+1)}\sum_{k=0}^{n}\frac{X_n^2+X_nX_k}{X_n+X_k} = \frac{K^2}{2}X_n.$$

Thus, $\theta = \frac{4}{K^2}$. If $K > 2$, then we know that $\mathbb{P}(\{X_n \underset{n\to\infty}{\longrightarrow} \infty\}) = 0$ and we can apply Theorem 1.1 and get lower and upper bounds of tail of the hitting-time of X_n in a compact set $[0, A]$.

Proposition 5.3. *Assume that $K > 2$. For all α and β such that $\alpha < 1 - 4/K^2 < \beta$, there exists $A > 0$ such that for all $x > A$ there exist $C_\alpha > 0$ and $C_\beta > 0$ such that*

$$\frac{C_\alpha}{n^\alpha} \le \mathbb{P}_x\left(\tau_A > n\right) \le \frac{C_\beta}{n^\beta}.$$

In this example, $g(x) = 1$, $\theta = \frac{4}{K^2}$, $\lambda = 1$, $G(x) = \mathcal{O}(x)$, $\ell_\alpha(x) = \mathcal{O}(x)$.

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Appendix A. Proof of Lemma 3.2

In this section, we turn to the proof of our key result, Lemma 3.2.

Proof of Lemma 3.2. We first verify that $(Y_n)_{n\in\mathbb{N}}$ satisfies the first inequality of Lemma 3.1. Since $\{Y_n \ge A\} = \{X_n \ge \ell_1(A)\}$ and $\lim_{x\to+\infty} \ell_1(x) = +\infty$, then Y_n large implies X_n large too. Let $n \in \mathbb{N}$, then on $\tilde{\tau_A} > n$

$$\begin{aligned}\mathbb{E}\left(Y_{n+1} - Y_n\middle|\mathcal{F}_n\right) &= \mathbb{E}\left((Y_{n+1} - Y_n)\left(\mathbb{1}_{\{\xi_n \le -g(X_n)-\varepsilon X_n\}} + \mathbb{1}_{\{\xi_n > -g(X_n)-\varepsilon X_n\}}\right)\middle|\mathcal{F}_n\right)\\ &\ge -Y_n\mathbb{P}\left(\xi_n \le -g\left(X_n\right) - \varepsilon X_n\middle|\mathcal{F}_n\right)\\ &\quad + \mathbb{E}\left((Y_{n+1} - Y_n)\,\mathbb{1}_{\{\xi_n > -g(X_n)-\varepsilon X_n\}}\middle|\mathcal{F}_n\right).\end{aligned}$$

We first give a lower bound of the first term. We know that $\mathbb{P}\left(\xi_n \le -g\left(X_n\right) - \varepsilon X_n\middle|\mathcal{F}_n\right) \le \mathbb{P}\left(\xi_n^2 \ge \left(-g\left(X_n\right) - \varepsilon X_n\right)^2\middle|\mathcal{F}_n\right) \le \frac{\mathbb{E}(\xi_n^2|\mathcal{F}_n)}{(g(X_n)+\varepsilon X_n)^2} \le \frac{C_1\sigma^2(X_n)}{X_n^2}$ by Chebyshev’s inequality. By (10) and $(A2)$, we obtain that

$$-Y_n\mathbb{P}\left(\xi_n \le -g\left(X_n\right) - \varepsilon X_n\middle|\mathcal{F}_n\right) \ge -C_2.$$

We now give a bound of the second term. If $\xi_n + g(X_n) > -\varepsilon X_n$, then $X_{n+1} > (1-\varepsilon)X_n$. Moreover, only the case $X_{n+1} \le X_n$ must be considered otherwise $Y_{n+1} - Y_n \ge 0$ since G is an increasing function. For $Y_n > A$ and large enough A, X_n is large too and then by regular variation and Potter’s bound, there exists $K > 0$ such that

$$g(x) \le K^{-1}g(X_n)$$

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for $x \in [X_{n+1}, X_n]$ and thus

$$\begin{aligned}
&\mathbb{E}((Y_{n+1} - Y_n)\mathbb{1}_{0>\xi_n+g(X_n)>-\varepsilon X_n}\big|\mathcal{F}_n)\\
&\qquad = \mathbb{E}\left(\int_{X_n}^{X_{n+1}} \frac{\mathrm{d}x}{g(x)}\mathbb{1}_{0>\xi_n+g(X_n)>-\varepsilon X_n}\Big|\mathcal{F}_n\right)\\
&\qquad \geq \frac{K}{g(X_n)}\mathbb{E}((X_{n+1} - X_n)\mathbb{1}_{0>\xi_n+g(X_n)>-\varepsilon X_n}\big|\mathcal{F}_n)\\
&\qquad \geq \frac{K}{g(X_n)}\mathbb{E}(\xi_n\mathbb{1}_{0>\xi_n+g(X_n)>-\varepsilon X_n}\big|\mathcal{F}_n)\\
&\qquad \geq -C\frac{\mathbb{E}(\xi_n^2\big|\mathcal{F}_n)}{X_n g(X_n)} = -C\frac{\sigma^2(X_n)}{X_n g(X_n)} \geq -C_3 \text{ by (A2).}
\end{aligned}$$

Thus, $(Y_n)_{n\in\mathbb{N}}$ verifies the first inequality of Lemma 3.1.

We now check that there exists $D > 0$ such that for all $n \in \mathbb{N}$

$$\mathbb{E}\left(Y_{n+1}^2 - Y_n^2\big|\mathcal{F}_n\right) \leq DY_n.$$

First, note that we can consider $X_{n+1} \geq (1-\varepsilon)X_n$:

$$\mathbb{E}\left(Y_{n+1}^2 - Y_n^2\big|\mathcal{F}_n\right) \leq \mathbb{E}\left(\left(Y_{n+1}^2 - Y_n^2\right)\mathbb{1}_{\{\xi_n+g(X_n)>-\varepsilon X_n\}}\big|\mathcal{F}_n\right).$$

By Potter's bound, if $X_{n+1} < X_n$ then there exists $K_1 > 0$ such that $\forall t \in [X_{n+1}, X_n]$, $\frac{K_1}{g(X_n)} \leq \frac{1}{g(t)}$; if $X_{n+1} \geq X_n$ then there exists $K_2 > 0$ such that $\forall t \in [X_n, X_{n+1}]$, $\frac{1}{g(t)} \leq \frac{K_2}{g(X_n)}$. Thus, we obtain that

$$Y_{n+1} - Y_n \leq \frac{K_3(X_{n+1} - X_n)}{g(X_n)}.$$

Since $Y_n > A$, X_n is large enough and so is X_{n+1}, then, by (10), there exists $C_4 > 0$ such that $Y_{n+1} = G(X_{n+1}) \leq \frac{X_{n+1}}{g(X_{n+1})} \leq C_5\frac{X_{n+1}}{g(X_n)}$.

Then

$$\begin{aligned}
\mathbb{E}\left(Y_{n+1}^2 - Y_n^2\big|\mathcal{F}_n\right) &\leq \frac{K_3}{g(X_n)}\mathbb{E}\left((X_{n+1} - X_n)(Y_{n+1} + Y_n)\mathbb{1}_{\{\xi_n+g(X_n)>-\varepsilon X_n\}}\big|\mathcal{F}_n\right)\\
&\leq \frac{C_6}{g(X_n)^2}\mathbb{E}\left((X_{n+1} - X_n)(X_{n+1} + X_n)\mathbb{1}_{\{\xi_n+g(X_n)>-\varepsilon X_n\}}\big|\mathcal{F}_n\right)\\
&\leq \frac{C_6}{g(X_n)^2}\mathbb{E}\left((g(X_n) + \xi_n)(2X_n + g(X_n) + \xi_n)\mathbb{1}_{\{\xi_n+g(X_n)>-\varepsilon X_n\}}\big|\mathcal{F}_n\right).
\end{aligned}$$

By (10), there exists $C_7 > 0$ such that $\frac{X_n}{g(X_n)} \leq C_7 G(X_n)$ and by $(A2)$, there exists C_8 such that $\frac{\sigma^2(X_n)}{g^2(X_n)} \leq C_8\frac{X_n}{g(X_n)} \leq C_7C_8G(X_n)$. We now check that the other terms are negligible.

$$\begin{aligned}
\mathbb{E}(\xi_n\mathbb{1}_{g(X_n)+\xi_n>-\varepsilon X_n}\big|\mathcal{F}_n) &= -\mathbb{E}(\xi_n\mathbb{1}_{g(X_n)+\xi_n\leq-\varepsilon X_n}\big|\mathcal{F}_n)\\
&\leq \frac{\sigma^2(X_n)}{(\varepsilon X_n + g(X_n))} \leq \frac{\sigma^2(X_n)}{\varepsilon X_n}.
\end{aligned}$$

Thus by (A2) and (10),

$$\frac{1}{g^2(X_n)}\mathbb{E}(X_n\xi_n\mathbb{1}_{g(X_n)+\xi_n>-\varepsilon X_n}\big|\mathcal{F}_n) \leq C_9G(X_n)$$

and there exists D such that $\mathbb{E}\left(Y_{n+1}^2 - Y_n^2\big|\mathcal{F}_n\right) \leq DY_n$, then $(Y_n)_{n\in\mathbb{N}}$ satisfies the assumptions of Lemma 3.1. □

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Appendix B. Auxiliary results

In this last section, we recall some results from [4] that we applied above.

We recall that $\mathcal{A}$ is the set of positive function f such that there exists a positive constant A_f such that

$$\limsup_{x\to\infty} \frac{f(2x)}{f(x)} \le A_f.$$

Theorem B.1 (*[4], Theorem 2*). *Let $(X_n)_{n\in\mathbb{N}}$ be an $\mathcal{F}_n$-adapted stochastic process taking values in an unbounded subset of $\mathbb{R}^+$. Let $f \in \mathcal{A}$ be an ultimately convex function. Suppose there exist positive constants A_0, ε such that $(f(X_{n\wedge\tau_{A_0}}))_{n\in\mathbb{N}}$ is a supermartingale and for any $n \in \mathbb{N}$, on the event $\{\tau_{A_0} > n\}$,*

$$\mathbb{E}(f(X_{n+1}) - f(X_n)\big|\mathcal{F}_n) \le -\varepsilon f'(X_n).$$

Then, there exists a positive constant c such that for all $x \ge A_0$,

$$\mathbb{E}_x(f(\tau_{A_0})) \le cf(x).$$

For all real valued functions h, let $\mathcal{B}_h$ be the set of positive functions $f \in \mathcal{C}^2(0,\infty)$ ultimately concave, such that $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} f'(x) = 0$, and such that the integral

$$\int_1^\infty \frac{f'(x)\mathrm{d}x}{h\circ r(x)} \quad \text{converges,}$$

with $r(x) = \sup\{y \ge A,\ f'(x) = h'(y)\}$.

Theorem B.2 (*[4], Theorem 3*). *Let $(X_n)_{n\in\mathbb{N}}$ be an $\mathcal{F}_n$-adapted stochastic process taking values in an unbounded subset of $\mathbb{R}^+$. Let $h \in \mathcal{C}^1([0,\infty))$ be a real-valued function such that h' decreases in a neighborhood of ∞ and $h'(x) \to 0$ as $x \to \infty$. Suppose there exist positive constants A_0, ε such that h increases on $[A_0,\infty)$ and for any $n \in \mathbb{N}$, on the event $\{\tau_{A_0} > n\}$,*

$$\mathbb{E}(h(X_{n+1}) - h(X_n)) \le -\varepsilon h'(X_n).$$

Then, for any $f \in \mathcal{B}_h$, there exist positive constants c, $A \ge A_0$ such that for all $x \ge A_0$,

$$\mathbb{E}_x(f(\tau_A)) \le ch(x).$$

We now recall a proposition from [4] which gives a link between integrability of hitting times of a finite set and of first return times to the initial state.

Proposition B.1 (*Proposition 1, [4]*). *Let F be a finite subset of $\mathcal{X}$, $\tau_F = \inf\{n > 0, X_n \in F\}$ the hitting time of F and $\tau = \inf\{n > 0, X_n = X_0\}$ be the first return time.*
(i) If for any $z \in F$,

$$\mathbb{E}_z(\tau_F) < \infty,$$

then for any $z \in F$, $\mathbb{E}_z(\tau) < \infty$.
(ii) If for some $z_0 \in F$, we have $\mathbb{E}_{z_0}(\tau_F) = \infty$, then $\mathbb{E}_{z_0}(\tau) = \infty$.

The following theorem gives the speed of convergence to the invariant measure of probability of $(X_n)_{n\in\mathbb{N}}$ in the recurrent positive case. We first introduce two sets of positive functions.

Let $\mathcal{G}$ be the set of positive functions f such that there exist a positive function h such that $h(x) \to 0$ as $x \to \infty$ and a positive constant c such that for any positive $m \ge 1$,

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$x_1 \geq 1, \ldots, x_m \geq 1$,

$$f\left(\sum_{k=1}^{m} x_k\right) \leq c e^{mh(m)} \sum_{k=1}^{m} f(x_k).$$

Let $\mathcal{G}'$ be the set of non decreasing in a neighborhood of infinity functions f such that $\ln(f(x))/x$ is non increasing in a neighborhood of infinity and tends to zero when x tends to infinity.

Theorem B.3 (*Theorem 1, [4]*). *Let $f \in \mathcal{G}$ such that $f' \in \mathcal{G}'$. Suppose there exists a positive function ϕ defined on $\mathcal{X}$ such that for all $s \in \mathcal{X} \setminus F$,*

$$\mathbb{E}_s(f(\tau_F)) \leq \phi(s),$$

and also that for all $z \in F$, $\mathbb{E}_z(\phi(X_1)) < \infty$. Then, for any initial distribution ν on $\mathcal{X}$ such that

$$\mathbb{E}_\nu(f'(\tau_F)) < \infty,$$

we have

$$\lim_{n\to\infty} f'(n) \sum_{i\in\mathcal{X}} \sum_{j\in\mathcal{X}} \nu(i)|P^n(i,j) - \pi(j)| = 0.$$

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