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Assessing the number of mean square derivatives of a Gaussian process

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Abstract

We consider a real Gaussian process X with unknown smoothness $r_0 \in \mathbb{N}_0$ where the mean square derivative $X^{(r_0)}$ is supposed to be Hölder continuous in quadratic mean. First, from selected sampled observations, we study the reconstruction of $X(t)$, $t \in [0, 1]$, with $\widetilde{X}_r(t)$ a piecewise polynomial interpolation of degree $r \geq 1$. We show that the mean square error of the interpolation is a decreasing function of r but becomes stable as soon as $r \geq r_0$. Next, from an interpolation-based empirical criterion and n sampled observations of X, we derive an estimator $\widehat{r}_n$ of r_0 and prove its strong consistency by giving an exponential inequality for $\mathbb{P}(\widehat{r}_n \neq r_0)$. Finally, we establish the strong consistency of $\widetilde{X}_{\max(\widehat{r}_n,1)}(t)$ with an almost optimal rate.

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1. Introduction

Let $X = \{X(t), t \in [0, 1]\}$ be a real Gaussian process whose r_0-th ($r_0 \in \mathbb{N}_0$) derivative satisfies a Hölder condition in quadratic mean with exponent $\beta_0 \in [0, 1[$. In several topics of approximation, integration, prediction and estimation, processes of interest are supposed to

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belong to some regularity class depending on r_0. More precisely concerning prediction, one may refer to works of [6,13,4], while for statistical inference, related information could be required in e.g. the works of [11,2,24,3,7]. In this paper, we suppose that r_0 and β_0 are both unknown and that X is observed over [0, 1] from a regular sequence of sampling times denoted by $u_{0,n}, \ldots, u_{n,n}$. Basing considerations on properties of interpolation, we propose and study an estimator of the regularity r_0.

Numerous methods had been proposed and studied for reconstruction of a sample path from discrete observations. For processes satisfying the so-called Sacks and Ylvisaker (SY) conditions, recent works include: ([14], orthogonal projection, optimal designs), ([15], linear interpolation, optimal designs), ([16], linear interpolation, adaptive designs). Under Hölder type conditions, one may cite e.g. works ([21], linear interpolation), ([22], Hermite interpolation splines, optimal designs), ([23], best approximation order). Note that a more detailed survey may be found in the book [19]. As in [17,18], we consider piecewise Lagrange polynomial interpolation for general classes of processes, but observed on some regular sequence of times. Namely if $\widetilde{X}_r(t)$ denotes a piecewise polynomial interpolation with degree r ($r \geq 1$), it may be noticed that the quadratic mean error of interpolation is a decreasing function of r that stabilizes as soon as r exceeds r_0. This key point allows us to estimate r_0 with the help of an empirical criterion based on interpolation. For this purpose, sampling times $u_{i,n}$ are divided into knots used for computing the interpolation while the remaining ones evaluate the quality of approximations obtained.

Main assumptions on X are given and discussed in the Section 2: in particular, SY conditions of order r_0 are included in the proposed examples. Choices of knots and their basic properties are also presented in this section. In Section 3, we derive an estimator for r_0, denoted by $\widehat{r_n}$. We show that $\widehat{r_n}$ is strongly consistent and give an exponential bound for $\mathbb{P}(\widehat{r_n} \neq r_0)$. Finally, we establish the strong consistency of $\widetilde{X}_{\max(\widehat{r_n},1)}(t)$ with an almost optimal rate. All the proofs are postponed to the final section.

2. The general framework

2.1. Assumptions

Let $X = \{X(t), t \in [0, 1]\}$ be a real measurable Gaussian process, defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We will say that $X \in \mathcal{H}(r_0, \beta_0)$ if it fulfills the following assumptions.

Assumption 2.1. (A2.1)

(i) X has continuous derivatives in quadratic mean up to order r_0, $r_0 \in \mathbb{N}_0$, denoted by $X^{(0)}, \ldots, X^{(r_0)}$.

(ii) $\mathrm{E}\left(X^{(r_0)}(s) - X^{(r_0)}(t)\right)^2 \leq A_1^2\,|s-t|^{2\beta_0}$, $(s,t) \in [0,1]^2$, with $\beta_0 \in [0, 1[$, $r_0 + \beta_0 > 0$ and $A_1 > 0$.

(iii) On $[0,1]^2 \setminus \{s = t\}$, $K^{(r_0+1,r_0+1)}(s,t)$ exists and satisfies for some $A_2 > 0$, $\left|K^{(r_0+1,r_0+1)}(s,t)\right| \leq A_2\,|s-t|^{-(2-2\beta_0)}$.

(iv) $\mathrm{E}\left(X(t) - \overline{X}_{a,b}(t)\right)^2 \geq A_3^2 \left(\frac{(b-t)(t-a)}{b-a}\right)^{2(r_0+\beta_0)}$, $t \in]a, b[$, $0 \leq a < b \leq 1$, where $\overline{X}_{a,b}(t) = \mathrm{E}\,(X(t)/X(s), s \in [0,a] \cup [b,1])$ and $A_3 > 0$ does not depend on the values a and b.

Let us give some precision concerning Assumption 2.1. Under A2.1(i), (ii), the process is said to satisfy a mean square Hölder condition of order (r_0, β_0); this allows us to give upper bounds for approximation. Recall that A2.1(i) implies in particular that the covariance kernel $K(s,t) = \operatorname{Cov}(X(s), X(t))$ is also continuously differentiable with $K^{(r,r)}(s,t) = \operatorname{Cov}(X^{(r)}(s), X^{(r)}(t))$, $r = 0, \ldots, r_0$. Using A2.1(i), (ii), one obtains also that the mean function $\mu(t) := \mathrm{E}\,X(t)$ is r_0-times continuously differentiable with $\mathrm{E}\,X^{(r)}(t) = \mu^{(r)}(t)$ and that $\mu^{(r_0)}(t)$ is Hölder continuous with order β_0 and constant A_1. A2.1(iii) is due to [9] (with $r_0 = 0$); let us notice that existence of $K^{(r_0+1,r_0+1)}(s,s)$ is not required, and in fact it is not wanted since then the process would have (r_0+1) derivatives in quadratic mean. A2.1(iv) was introduced by Plaskota et al. [17] with a large variety of examples that we present and develop below. This technical condition is involved in lower bounds of approximation.

Finally, our aim is to estimate the maximal r (denoted by r_0) such that one has $X \in \mathcal{H}(r, \beta_0)$. That's why we have excluded, in condition A2.1(ii), the case $\beta_0 = 1$ to avoid any possible problem of identifiability.

Example 1 (r_0-*Fold-Integrated Fractional Brownian Motion*). Let us define X by

$$X(t) = \int_0^t \int_0^{s_{r_0}} \int_0^{s_{r_0-1}} \cdots \int_0^{s_2} W_{\beta_0}(s_1)\,\mathrm{d}s_1\,\mathrm{d}s_2 \cdots \mathrm{d}s_{r_0},$$

so that $X^{(r_0)} = W_{\beta_0}$, $0 < \beta_0 < 1$, where $(W_{\beta_0}(t), t \in \mathbb{R}^+)$ is a fractional standard Brownian motion. X is a zero-mean Gaussian process with covariance function $K^{(r_0,r_0)}(s,t) = \frac{1}{2}(s^{2\beta_0} + t^{2\beta_0} - |s-t|^{2\beta_0})$. It is shown in [17] that X fulfills conditions A2.1(ii), (iv). Moreover, since one has $\mathrm{E}\,(X^{(r_0)}(t+\tau) - X^{(r_0)}(t))^2 = |\tau|^{2\beta_0}$, condition A2.1(iii) is also satisfied (see [9]). Finally note that $\beta_0 = 1/2$ yields the r_0-fold-integrated standard Brownian motion and, in this case, one gets [17] that $A_3^2 = 1/((2r_0+1)(r_0!)^2)$.

Example 2 (*Sacks–Ylvisaker (SY) Conditions*). We take SY conditions of order r_0 as stated in [19, p. 68] in the case of a zero-mean process (excluding in particular stationary processes for $r_0 \geq 1$). Assumptions A2.1 are then satisfied with $\beta_0 = 1/2$ and $A_3^2 = (2r_0+1)^{-1}((1+c)r_0!)^{-2}$ where c is a positive constant depending only on the covariance kernel K. We refer the reader to [17] and results of [20] for details.

Example 3 (*Stationary Processes with Spectral Density* φ). Suppose that φ satisfies both for u large enough, $\varphi(u) \leq c_1\,|u|^{-2\gamma}$ with $c_1 > 0$, $\gamma > 1/2$, and for every real u, $\varphi(u) \geq c_0(1+u^2)^{-\gamma}$ with $c_0 > 0$; then results of [17] imply that for $\gamma - \frac{1}{2} \notin \mathbb{N}$ the conditions A2.1(ii), (iv) are fulfilled with $r_0 = \left[\gamma - \frac{1}{2}\right]$ and $\beta_0 = \gamma - \frac{1}{2} - r_0$. Next, we strengthen the first condition via $\left|\varphi(u) - c_1\,|u|^{-2\gamma}\right| \leq c_2\,|u|^{-2(\gamma+1)}$ with $c_2 > 0$, $\gamma > 1/2$ and u large enough. In this case, from $K^{(r_0,r_0)}(s,t) = \int_{-\infty}^{\infty} u^{2r_0} e^{\iota(s-t)u}\varphi(u)\,\mathrm{d}u$ and by adapting the proof of [9] to the case $r_0 \geq 1$, we obtain the required condition A2.1(iii). For instance, one has $K(s,t) = (2\theta)^{-1}\exp(-\theta\,|s-t|)$ and $\varphi(u) = (2\pi)^{-1}(\theta^2+u^2)^{-1}$ for an Ornstein–Uhlenbeck (OU) process, which implies in turn that $\gamma = 1$, $r_0 = 0$ and $\beta_0 = 1/2$.

Example 4 (r_0-*Fold-Integrated Stationary Processes*). Let $Y = \{Y_t, t \in [0,1]\}$ be a zero-mean stationary Gaussian process with covariance $\rho_0(|t-s|)$. Lasinger [12] establishes that stationarity should be preserved under r_0-fold integration of Y, if ρ_0 is either linear: $\rho_0(t) = 1 - \lambda\,|t|\,(0 < \lambda < 2)$ or exponential: $\rho_0(t) = (2\theta)^{-1}\exp(-\theta\,|t|)$ $(\theta > 0)$. Then, the same methodology as in Example 2 (using Lemma IV.4 of [19, p. 73]) yields that X (with $X^{(r_0)} = Y$) satisfies condition A2.1(iv) with $\beta_0 = 1/2$ in both cases. Note that conditions A2.1(ii), (iii) are stated on $K^{(r_0,r_0)}(s,t) = \rho_0(|s-t|)$ and consequently are easily checked. Finally, the linear case occurs for example when $X^{(r_0)}(t) = W(t+1) - W(t)$ whereas the exponential one corresponds to the OU process (see Example 3).

Example 5 (*Non-Centered Case*). Suppose that the zero-mean process X satisfies Assumptions A2.1; this is also clearly the case for the non-centered process $Z(t) = X(t) + \mu(t)$ as soon as $\mu^{(r_0)}$ is well defined and Hölder continuous with order β_0.

2.2. *The sampling scheme*

We consider a regular sequence design, which means that we suppose that X is observed at instants $0 = u_{0,n} < u_{1,n} < \cdots < u_{n,n} = 1$ satisfying

$$\int_0^{u_{i,n}} \psi(t)\,\mathrm{d}t = \frac{i}{n}$$

for a positive and continuous density function ψ on $[0, 1]$. Clearly, one gets the equidistant scheme with the choice $\psi \equiv 1$ but also points might be irregularly located. From a practical point of view, this flexibility may allow us to recognize inhomogeneities in the process (e.g. the presence of peaks in environmental pollution monitoring; see [8] and references therein) or else to describe situations where data are collected at equidistant times but become irregularly spaced after some screening (see for example the Wolfcamp aquifer data in [5]).

For $j > i$, one easily gets that

$$m_1 \frac{j-i}{n} \le u_{j,n} - u_{i,n} \le m_2 \frac{j-i}{n} \tag{2.1}$$

where $m_1 = (\sup_{t\in[0,1]} \psi(t))^{-1}$ and $m_2 = (\inf_{t\in[0,1]} \psi(t))^{-1}$ are constants independent of i, j and n.

Half of the knots (denoted by $t_{j,n}$) will be used for the interpolation problem while the remaining ones will be reserved for the estimation problem (namely to evaluate the quality of the approximations performed). More precisely, we set $p_n = \left[\log_a(n)\right] := \left[\underbrace{\ln\ln\cdots\ln}_{a \text{ times}} n\right]$ (for some $a \ge 2$ such that $p_n \ge 1$) and for each $r = 1, \ldots, p_n$, we consider

$$\widetilde{n}_r = \left[\frac{n}{2r}\right] \tag{2.2}$$

piecewise polynomials of degree (at most) r, where $[x]$ denotes the integer part of x. Next, we set

$$t_{j,n} = u_{2j,n}, \quad j = 0, \ldots, r\widetilde{n}_r \tag{2.3}$$

and

$$\bar{t}_{j,n} = u_{2j+1,n}, \quad j = 0, \ldots, r\widetilde{n}_r - 1. \tag{2.4}$$

Remark that (2.1)–(2.4) induce the straightforward properties

$$\frac{2m_1(j-i)}{n} \le t_{j,n} - t_{i,n} \le \frac{2m_2(j-i)}{n} \tag{2.5}$$

for $j > i$, and $\bar{t}_{j,n} \in]t_{j,n}, t_{j+1,n}[\subset \left[t_{[\frac{j}{r}]r,n}, t_{([\frac{j}{r}]+1)r,n} \right]$ with

$$\max(t_{j+1,n} - \bar{t}_{j,n}, \bar{t}_{j,n} - t_{j,n}) \le \tfrac{m_2}{n} \tag{2.6}$$

$$\min(t_{j+1,n} - \bar{t}_{j,n}, \bar{t}_{j,n} - t_{j,n}) \ge \tfrac{m_1}{n}. \tag{2.7}$$

3. Main results

For all $s \in [0,1]$ and each $r \in \{1, \ldots, p_n\}$, there exist $k = 0, \ldots, \widetilde{n}_r - 1$ such that $s \in \mathcal{I}_k := [t_{kr,n}, t_{(k+1)r,n}]$. One may approximate $X(s)$ by $\widetilde{X}_r(s)$, the unique polynomial of degree (at most) r which interpolates $X(s)$ at the $(r+1)$ knots: $t_{kr+i,n}$, $i = 0, \ldots, r$, and defined by

$$\widetilde{X}_r(s) = \sum_{i=0}^{r} L_{i,k,r}(s) X(t_{kr+i,n}),$$

where $L_{i,k,r}(s)$ is the Lagrange interpolator polynomial given by

$$L_{i,k,r}(s) = \prod_{\substack{j=0 \\ j \neq i}}^{r} \frac{(s - t_{kr+j,n})}{(t_{kr+i,n} - t_{kr+j,n})}.$$

These polynomials present several advantages: they are easy to build and to implement, they give sharp upper bounds for approximation (see Proposition 3.1 and the following remarks). Moreover conversely to Hermite polynomials, they do not require observation of derivatives of the process X.

3.1. Upper and lower bounds for the error of interpolation

Using properties of reproducing kernel Hilbert spaces, Plaskota et al. [17] give error estimates for piecewise Lagrange interpolation of order $r \geq r_0$ for equidistant knots. Concerning the regular case, we obtain the following result.

Proposition 3.1. *Under conditions* A2.1(i), (ii)*, we obtain*

$$\sup_{s \in [0,1]} \mathrm{E}\left(X(s) - \widetilde{X}_r(s)\right)^2 \le C_1^2(r, \beta_0)\, n^{-2(r^*+\beta^*)}, \tag{3.1}$$

with $r^* = \min(r, r_0)$, $\beta^* = \begin{cases} 1 & \text{if } r = 1, \ldots, r_0 - 1 \\ \beta_0 & \text{if } r = r_0, r_0 + 1, \end{cases}$ *and* $C_1(r, \beta_0)$ *is a positive constant.*

Remark 3.1. (1) Since our upper bounds only involve the covariance kernel of the process X, they hold also for non-Gaussian processes satisfying conditions A2.1(i), (ii). As a by-product, one has also

$$\sup_{s \in [0,1]} \left| \mathrm{E}\left(X(s) - \widetilde{X}_r(s)\right) \right| \le C_1(r, \beta_0)\, n^{-(r^*+\beta^*)}. \tag{3.2}$$

(2) For $r \geq r_0$, the rate of L^2-approximation is of order $n^{-(r_0+\beta_0)}$. This rate appears to be optimal in some sense for the Hölder class $\mathcal{H}(r_0, \beta_0)$; see [23].

Now, let us turn to a lower bound of approximation at points $\bar{t}_{k,n}$; see (2.4).

Proposition 3.2. *Let us assume condition* A2.1(iv). *Then for* $r = 1, \ldots, r_0 + 1$ *and* $k = 0, \ldots, r\widetilde{n}_r - 1$ *we have*

$$\mathrm{E}\left(X(\bar{t}_{k,n}) - \widetilde{X}_r(\bar{t}_{k,n})\right)^2 \geq C_2^2(r_0, \beta_0)\, n^{-2(r_0+\beta_0)},$$

where $C_2(r_0, \beta_0)$ *is a positive constant.*

Note that a similar result for $\int_0^1 \mathrm{E}\,(X(s) - \mathcal{A}X(s))^2\,\mathrm{d}s$ is obtained in [17] for any algorithm $\mathcal{A}$ using n knots.

3.2. Estimation of the parameter r_0

Proposition 3.1 underlines that the error of interpolation decreases as the degree r of Lagrange polynomials increases, but stabilizes as soon as r exceeds r_0. Taking into account this property, we define

$$\widehat{r}_n = \min\left\{ r \in \{1, \ldots, p_n\} : \frac{1}{r\widetilde{n}_r} \sum_{k=0}^{r\widetilde{n}_r - 1} \left(X(\bar{t}_{k,n}) - \widetilde{X}_r(\bar{t}_{k,n})\right)^2 \geq n^{-2r} h_n \right\} - 1$$

where $\widetilde{n}_r$ is given by (2.2). If the above set is empty, we fix $\widehat{r}_n = l_0$ for an arbitrary value $l_0 \notin \mathbb{N}_0$. Here, the threshold $h_n \to \infty$ is supposed to satisfy both conditions $n^{2\beta_0 - 2} h_n \to 0$ and $n^{2\beta_0} h_n \to \infty$, for all $\beta_0 \in [0, 1[$. For example, an omnibus choice is given by $h_n = \ln n$. Furthermore, note that if an upper bound B is known for r_0, one can choose $p_n = B + 1$ in the definition of $\widehat{r}_n$.

We now present the main result of our paper, namely an exponential upper bound for the probability of the event $\{\widehat{r}_n \neq r_0\}$.

Theorem 3.1. *Let us assume Hypotheses A2.1. Then*

$$\mathbb{P}(\widehat{r}_n \neq r_0) = \mathcal{O}\left(\exp\left(-C_3(r_0, \beta_0)\, \varphi_n(\beta_0)\right)\right)$$

for some positive constant $C_3(r_0, \beta_0)$ *and where*

$$\varphi_n(\beta_0) = n\left(\mathbb{I}_{[0,1/2[}(\beta_0) + \frac{1}{\ln n}\mathbb{I}_{\{1/2\}}(\beta_0) + n^{1-2\beta_0}\mathbb{I}_{]1/2,1[}(\beta_0)\right). \tag{3.3}$$

Remark 3.2. (1) Note that a more explicit bound is established during the proof; see relations (4.10) and (4.19). Furthermore, under the more restrictive Baxter's condition [1]: *on* $[0, 1]^2 \setminus \{s = t\}$, $K^{(r_0+1, r_0+1)}(s, t)$ *exists and it is bounded* (e.g. processes satisfying SY conditions of order r_0), the special case $\beta_0 = 1/2$ disappears: $\varphi_n(\beta_0) = n\,\mathbb{I}_{[0,1/2]}(\beta_0) + n^{2-2\beta_0}\,\mathbb{I}_{]1/2,1[}(\beta_0)$.

(2) The rate of convergence is exponential but it is a decreasing function of β_0: as expected, the case $\beta_0 = 1$ turns out to be degenerate.

Finally, Theorem 3.1 allows us to study the pointwise almost sure convergence of $\widetilde{X}_{\max(\widehat{r}_n, 1)}(s)$ toward $X(s)$.

Theorem 3.2. *Under Assumptions* A2.1, *we obtain the following results:*

(i) $\widehat{r}_n = r_0$ *almost surely for n large enough;*

(ii) *for all* $s \in [0, 1]$, *one gets that almost surely,*

$$\limsup_{n\to\infty} \frac{n^{(r_0+\beta_0)}}{\sqrt{\ln n}} \left| X(s) - \widetilde{X}_{\max(\widehat{r}_n,1)}(s) \right| < +\infty.$$

Regarding this last result, the rate of convergence is, up to a logarithmic term, the same as the one obtained in the case 'r_0 known'; see Proposition 3.1. Finally note that the asymptotic constant is also of the same order since an examination of the proof gives the constant $\sqrt{2}\, C_1(\max(1, r_0), \beta_0)$, with $C_1(r, \beta_0)$ introduced in Proposition 3.1.

4. Auxiliary results and proofs

4.1. Auxiliary results

For further results, we express the function

$$\mathbb{C}_r(s_1, s_2) := \mathrm{Cov}\left(X(s_1) - \widetilde{X}_r(s_1), X(s_2) - \widetilde{X}_r(s_2)\right) \tag{4.1}$$

with $r \geq 1$, in terms of the covariance $K(s_1, s_2)$ and its partial derivatives for $(s_1, s_2) \in \mathcal{I}_k \times \mathcal{I}_\ell$ for some $(k, \ell) \in \{0, \ldots, \widetilde{n}_r - 1\}^2$. Throughout the section, we abbreviate the notation $t_{j,n}$ to t_j for the sake of simplicity.

Lemma 4.1. *Suppose that condition* A2.1(i) *holds and let* $r^* = \min(r_0, r)$ *be such that* $r^* \geq 1$. *Then for* $(s_1, s_2) \in \mathcal{I}_k \times \mathcal{I}_\ell$ *one has*

$$\mathbb{C}_r(s_1, s_2) = \sum_{i,j=0}^{r} \frac{L_{i,k,r}(s_1) L_{j,\ell,r}(s_2)}{((r^*-1)!)^2} (t_{kr+i} - t_{kr})^{r^*} (t_{\ell r+j} - t_{\ell r})^{r^*} \times \iint_{[0,1]^2} ((1-v)(1-w))^{r^*-1}\, \mathcal{R}(v, w)\, \mathrm{d}v\, \mathrm{d}w$$

where

$$\begin{aligned} \mathcal{R}(v, w) = {} & K^{(r^*,r^*)}(t_{kr} + (s_1 - t_{kr})v, t_{\ell r} + (s_2 - t_{\ell r})w) \\ & - K^{(r^*,r^*)}(t_{kr} + (s_1 - t_{kr})v, t_{\ell r} + (t_{\ell r+j} - t_{\ell r})w) \\ & - K^{(r^*,r^*)}(t_{kr} + (t_{kr+i} - t_{kr})v, t_{\ell r} + (s_2 - t_{\ell r})w) \\ & + K^{(r^*,r^*)}(t_{kr} + (t_{kr+i} - t_{kr})v, t_{\ell r} + (t_{\ell r+j} - t_{\ell r})w). \end{aligned} \tag{4.2}$$

In addition,

$$\mathrm{E}\left(X(s_1) - \widetilde{X}_r(s_1)\right) = \sum_{i=0}^{r} \frac{L_{i,k,r}(s_1)}{(r^*-1)!} (t_{kr+i} - t_{kr})^{r^*} \times \int_{[0,1]} (1-v)^{r^*-1} \left(\mu^{(r^*)}(t_{kr} + (s_1 - t_{kr})v) - \mu^{(r^*)}(t_{kr} + (t_{kr+i} - t_{kr})v)\right) \mathrm{d}v.$$

Remark 4.1. In the case $r_0 = 0$ and $r \geq 1$, one easily obtains

$$\mathbb{C}_r(s_1, s_2) = \sum_{i,j=0}^{r} L_{i,k,r}(s_1) L_{j,\ell,r}(s_2) \times \left\{K(s_1, s_2) - K(s_1, t_{\ell r+j})\right.$$

$$-K(t_{kr+i}, s_2) + K(t_{kr+i}, t_{\ell r+j})\big\}, \tag{4.3}$$

$$\mathrm{E}\left(X(s_1) - \widetilde{X}_r(s_1)\right) = \sum_{i=0}^{r} L_{i,k,r}(s_1)\,(\mu(s_1) - \mu(t_{kr+i})) \tag{4.4}$$

using uniqueness of Lagrange polynomials which implies $\sum_{i=0}^{r} L_{i,k,r}(s_1) = \sum_{j=0}^{r} L_{j,\ell,r}(s_2) = 1$.

Proof of Lemma 4.1. Since the second result follows directly using similar arguments, we only prove the expansion of the covariance function. Recall that condition A2.1(i) holds iff $K \in C^{r_0,r_0}([0,1]^2)$; in other words, K has continuous partial derivatives $K^{(p,m)}$ on $[0,1]^2$ for all integers $p, m \le r_0$. From (4.3), we apply a Taylor series expansion (with integral remainder) of order $r^* = \min(r, r_0)$:

$$\begin{aligned} K(s_1, s_2) &= \sum_{p=0}^{r^*-1} \frac{(s_1 - t_{kr})^p}{p!} K^{(p,0)}(t_{kr}, s_2) \\ &\quad + \int_0^1 \frac{(s_1 - t_{kr})^{r^*}}{(r^*-1)!}(1-v)^{r^*-1} K^{(r^*,0)}(t_{kr} + (s_1 - t_{kr})v, s_2)\,\mathrm{d}v, \end{aligned}$$

but one has also

$$\begin{aligned} K^{(p,0)}(t_{kr}, s_2) &= \sum_{m=0}^{r^*-1} \frac{(s_2 - t_{\ell r})^m}{m!} K^{(p,m)}(t_{kr}, t_{\ell r}) \\ &\quad + \int_0^1 \frac{(s_2 - t_{\ell r})^{r^*}}{(r^*-1)!}(1-w)^{r^*-1} K^{(p,r^*)}(t_{kr}, t_{\ell r} + (s_2 - t_{\ell r})w)\,\mathrm{d}w \end{aligned}$$

and

$$\begin{aligned} K^{(r^*,0)}(t_{kr} + (s_1 - t_{kr})v, s_2) &= \sum_{m=0}^{r^*-1} \frac{(s_2 - t_{\ell r})^m}{m!} K^{(r^*,m)}(t_{kr} + (s_1 - t_{kr})v, t_{\ell r}) \\ &+ \int_0^1 \frac{(s_2 - t_{\ell r})^{r^*}}{(r^*-1)!}(1-w)^{r^*-1} K^{(r^*,r^*)}(t_{kr} + (s_1 - t_{kr})v, t_{\ell r} + (s_2 - t_{\ell r})w)\,\mathrm{d}w. \end{aligned}$$

Now similar expansions hold for $K(s_1, t_{\ell r+j})$, $K(t_{kr+i}, s_2)$ and $K(t_{kr+i}, t_{\ell r+j})$. This yields: $\mathbb{C}_r(s_1, s_2) = T_1 + T_2 + T_3 + T_4$ with the T_i respectively defined by

$$\begin{aligned} T_1 &= \sum_{p,m=0}^{r^*-1} \frac{K^{(p,m)}(t_{kr}, t_{\ell r})}{p!m!} \sum_{i,j=0}^{r} L_{i,k,r}(s_1) L_{j,\ell,r}(s_2) \big\{(s_1 - t_{kr})^p (s_2 - t_{\ell r})^m \\ &\quad - (s_1 - t_{kr})^p (t_{\ell r+j} - t_{\ell r})^m - (t_{kr+i} - t_{kr})^p (s_2 - t_{\ell r})^m \\ &\quad + (t_{kr+i} - t_{kr})^p (t_{\ell r+j} - t_{\ell r})^m\big\}, \\ T_2 &= \sum_{p=0}^{r^*-1} \sum_{i,j=0}^{r} \frac{L_{i,k,r}(s_1) L_{j,\ell,r}(s_2)}{(r^*-1)!p!} \big\{(s_1 - t_{kr})^p - (t_{kr+i} - t_{kr})^p\big\} \\ &\quad \times \int_0^1 (1-w)^{r^*-1} \Big\{(s_2 - t_{\ell r})^{r^*} K^{(p,r^*)}(t_{kr}, t_{\ell r} + (s_2 - t_{\ell r})w) \\ &\quad - (t_{\ell r+j} - t_{\ell r})^{r^*} K^{(p,r^*)}(t_{kr}, t_{\ell r} + (t_{\ell r+j} - t_{\ell r})w)\Big\}\,\mathrm{d}w, \end{aligned}$$

$$
\begin{aligned}
T_3 &= \sum_{m=0}^{r^*-1}\sum_{i,j=0}^{r} \frac{L_{i,k,r}(s_1)L_{j,\ell,r}(s_2)}{(r^*-1)!m!}\left\{(s_2-t_{\ell r})^m-(t_{\ell r+j}-t_{\ell r})^m\right\}\\
&\quad\times\int_0^1(1-v)^{r^*-1}\Big\{(s_1-t_{kr})^{r^*}K^{(r^*,m)}(t_{kr}+(s_1-t_{kr})v,t_{\ell r})\\
&\quad-(t_{kr+i}-t_{kr})^{r^*}K^{(r^*,m)}(t_{kr}+(t_{kr+i}-t_{kr})v,t_{\ell r})\Big\}\,\mathrm{d}v,\\
T_4 &= \sum_{i,j=0}^{r}\frac{L_{i,k,r}(s_1)L_{j,\ell,r}(s_2)}{(r^*-1)!^2}\times\iint_{[0,1]^2}(1-v)^{r^*-1}(1-w)^{r^*-1}\\
&\quad\times\Big\{(s_1-t_{kr})^{r^*}(s_2-t_{\ell r})^{r^*}K^{(r^*,r^*)}(t_{kr}+(s_1-t_{kr})v,t_{\ell r}+(s_2-t_{\ell r})w)\\
&\quad-(s_1-t_{kr})^{r^*}(t_{\ell r+j}-t_{\ell r})^{r^*}K^{(r^*,r^*)}(t_{kr}+(s_1-t_{kr})v,t_{\ell r}+(t_{\ell r+j}-t_{\ell r})w)\\
&\quad-(t_{kr+i}-t_{kr})^{r^*}(s_2-t_{\ell r})^{r^*}K^{(r^*,r^*)}(t_{kr}+(t_{kr+i}-t_{kr})v,t_{\ell r}+(s_2-t_{\ell r})w)\\
&\quad+(t_{kr+i}-t_{kr})^{r^*}(t_{\ell r+j}-t_{\ell r})^{r^*}K^{(r^*,r^*)}(t_{kr}+(t_{kr+i}-t_{kr})v,t_{\ell r}\\
&\quad+(t_{\ell r+j}-t_{\ell r})w)\Big\}\,\mathrm{d}v\,\mathrm{d}w.
\end{aligned}
$$

Now, one may easily obtain $T_1 = T_2 = T_3 = 0$. Let us consider for example the term $\sum_{i=0}^{r} L_{i,k,r}(s_1)(t_{kr+i}-t_{kr})^p$ in T_1 (with $p = 0,\dots,r^*-1 \le r-1$); it corresponds to the unique polynomial of degree at most r which interpolates a function taking values $(t_{kr+i}-t_{kr})^p$ at points t_{kr+i} for $i = 0,\dots,r$. A candidate is the polynomial $(x-t_{kr})^p$, so by uniqueness, $\sum_{i=0}^{r} L_{i,k,r}(s_1)(t_{kr+i}-t_{kr})^p = (s_1-t_{kr})^p$. In the same way, $\sum_{j=0}^{r} L_{j,\ell,r}(s_2)(t_{\ell r+j}-t_{\ell r})^m = (s_2-t_{\ell r})^m$.

Still using the Lagrange polynomial properties, one may write T_4 as

$$
\begin{aligned}
T_4 &= \sum_{i,j=0}^{r}\frac{L_{i,k,r}(s_1)L_{j,\ell,r}(s_2)(t_{kr+i}-t_{kr})^{r^*}(t_{\ell r+j}-t_{\ell r})^{r^*}}{r^{2r^*}(r^*-1)!^2}\\
&\quad\times\iint_{[0,1]^2}((1-v)(1-w))^{r^*-1}\,\mathcal{R}(v,w)\,\mathrm{d}v\,\mathrm{d}w
\end{aligned}
$$

with $\mathcal{R}$ given by (4.2). $\square$

To establish Theorem 3.1, we need also the following proposition concerning $\mathbb{C}(\cdot,\cdot)$ defined in (4.1).

Proposition 4.1. *Let us assume conditions* A2.1(i)–(iii)*; for some positive constant* $C_4(r,\beta_0)$ *and any* $r \in \{1,\dots,r_0+1\}$*, one obtains that*

$$
\begin{aligned}
&\max_{0\le k\le r\tilde{n}_r-1}\sum_{\ell=0}^{r\tilde{n}_r-1}\left|\mathbb{C}_r(\bar{t}_{k,n},\bar{t}_{\ell,n})\right|\\
&\quad\le C_4(r,\beta_0)\begin{cases} n^{-(2r+1)} & \text{if } r=1,\dots,r_0-1,\\ n^{-2(r_0+\beta_0)} & \text{if } r=r_0,r_0+1 \text{ and } 0\le\beta_0<1/2,\\ n^{-(2r_0+1)}\ln(n) & \text{if } r=r_0,r_0+1 \text{ and } \beta_0=1/2,\\ n^{-(2r_0+1)} & \text{if } r=r_0,r_0+1 \text{ and } 1/2<\beta_0<1.\end{cases}
\end{aligned}
\tag{4.5}
$$

Remark 4.2. 1. Let us notice that, if $r_0 = 0$ (resp. $r_0 = 1$) only the three last cases subsist for $r = 1$ (resp. $r = 1, 2$).

2. Under Baxter's condition, see Remark 3.2(1), one obtains a similar bound but without the logarithmic term in the case $\beta_0 = 1/2$.

Proof of Proposition 4.1. (a) Case where $r = r_0$ ($r_0 \geq 1$) or $r = r_0 + 1$.

Setting $s_1 = \bar{t}_{k,n}$ and $s_2 = \bar{t}_{\ell,n}$ in Eq. (4.1), so that $(s_1, s_2) \in \mathcal{I}_{p_1} \times \mathcal{I}_{p_2}$ with $p_1 = [k/r]$ and $p_2 = [\ell/r]$, we write

$$\max_{0\leq k\leq r\widetilde{n}_r-1} \sum_{\ell=0}^{r\widetilde{n}_r-1} \left| \mathbb{C}_r(\bar{t}_{k,n}, \bar{t}_{\ell,n}) \right| = \max_{0\leq k\leq r\widetilde{n}_r-1} \left\{ \sum_{\substack{\ell=0 \\ |p_2-p_1|\geq 2}}^{r\widetilde{n}_r-1} + \sum_{\substack{\ell=0 \\ |p_2-p_1|\leq 1}}^{r\widetilde{n}_r-1} \right\} \left| \mathbb{C}_r(\bar{t}_{k,n}, \bar{t}_{\ell,n}) \right|,$$

and in fact such decomposition is compulsory as condition A2.1(iii) cannot be used on the diagonal of $[0, 1]^2$. Let us first consider the case $|p_2 - p_1| \geq 2$: using Lemma 4.1, one arrives at

$$\mathbb{C}_r(\bar{t}_{k,n}, \bar{t}_{\ell,n}) = \sum_{i=0}^{r}\sum_{j=0}^{r} \frac{L_{i,p_1,r}(\bar{t}_{k,n}) L_{j,p_2,r}(\bar{t}_{\ell,n})}{((r_0-1)!)^2} (t_{p_1r+i} - t_{p_1r})^{r_0} (t_{p_2r+j} - t_{p_2r})^{r_0}$$
$$\times \int_0^1 \int_0^1 (1-v)^{r_0-1}(1-w)^{r_0-1}\mathcal{R}(v, w)\,\mathrm{d}v\,\mathrm{d}w \tag{4.6}$$

with

$$\begin{aligned}\mathcal{R}(v, w) = {} & K^{(r_0,r_0)}\left(t_{p_1r} + (\bar{t}_{k,n} - t_{p_1r})v, t_{p_2r} + (\bar{t}_{\ell,n} - t_{p_2r})w\right) \\ & - K^{(r_0,r_0)}\left(t_{p_1r} + (\bar{t}_{k,n} - t_{p_1r})v, t_{p_2r} + (t_{p_2r+j} - t_{p_2r})w\right) \\ & - K^{(r_0,r_0)}\left(t_{p_1r} + (t_{p_1r+i} - t_{p_1r})v, t_{p_2r} + (\bar{t}_{\ell,n} - t_{p_2r})w\right) \\ & + K^{(r_0,r_0)}\left(t_{p_1r} + (t_{p_1r+i} - t_{p_1r})v, t_{p_2r} + (t_{p_2r+j} - t_{p_2r})w\right)\end{aligned} \tag{4.7}$$

which can be rewritten as

$$\mathcal{R}(v, w) = \int_{t_{p_2r}+(t_{p_2r+j}-t_{p_2r})w}^{t_{p_2r}+(\bar{t}_{\ell,n}-t_{p_2r})w} \left(\int_{t_{p_1r}+(t_{p_1r+i}-t_{p_1r})v}^{t_{p_1r}+(\bar{t}_{k,n}-t_{p_1r})v} K^{(r_0+1,r_0+1)}(u_1, u_2)\mathrm{d}u_1 \right) \mathrm{d}u_2.$$

Next for all $(v, w) \in [0, 1]^2$ and $(i, j) \in \{0, \ldots, r\}^2$, one gets the following bound:

$$|\mathcal{R}(v, w)| \leq A_2 \int_{\mathcal{I}_{p_2}} \int_{\mathcal{I}_{p_1}} |u_1 - u_2|^{-2(1-\beta_0)}\,\mathrm{d}u_1\mathrm{d}u_2$$

as soon as condition A2.1(iii) is satisfied since $|p_2 - p_1| \geq 2$ implies that $\mathcal{I}_{p_1} \times \mathcal{I}_{p_2}$ does not contain the diagonal. From the lower bound of (2.5), one has

$$|u_1 - u_2| \geq 2m_1 \left(|p_2 - p_1| - 1\right) n^{-1}$$

so that

$$|\mathcal{R}(v, w)| \leq c\ (|p_2 - p_1| - 1)^{-2(1-\beta_0)}\, n^{-2\beta_0}$$

where, here and throughout the following, c denotes a generic positive constant (independent of n, p_1, p_2) whose value may vary from line to line. Now, one may bound $L_{i,p_1,r}(s_1)$ independently from i, p_1, s_1 and n:

$$\left|L_{i,p_1,r}(s_1)\right| = \prod_{\substack{j=0 \\ j\neq i}}^{r} \left| \frac{(s_1 - t_{p_1r+j})}{(t_{p_1r+i} - t_{p_1r+j})} \right| \leq \left(\frac{t_{(p_1+1)r} - t_{p_1r}}{\min\limits_{i=0,\ldots,r-1} (t_{p_1r+i+1} - t_{p_1r+i})} \right)^r \leq c\, r^r \tag{4.8}$$

where c only depends on m_1, m_2 by relation (2.5). This yields

$$\max_{0\le k\le r\widetilde{n}_r-1}\sum_{\substack{\ell=0\\|p_2-p_1|\ge 2}}^{r\widetilde{n}_r-1}\left|\mathbb{C}_r(\bar{t}_{k,n},\bar{t}_{\ell,n})\right|$$
$$\le c\,n^{-2(r_0+\beta_0)}\max_{p_1=0,\ldots,\widetilde{n}_r-1}\sum_{\substack{p_2=0\\|p_2-p_1|\ge 2}}^{\widetilde{n}_r-1}(|p_2-p_1|-1)^{-2(1-\beta_0)}$$
$$\le c\,n^{-2(r_0+\beta_0)}\sum_{i=1}^{\widetilde{n}_r} i^{-2(1-\beta_0)}$$

which implies in turn the successive bounds $n^{-2(r_0+\beta_0)}$ if $0\le\beta_0<1/2$, $n^{-(2r_0+1)}\ln(n)$ if $\beta_0=1/2$ and $n^{-(2r_0+1)}$ if $1/2<\beta_0<1$.

Consider now the case $p_2\in\{p_1-1,p_1,p_1+1\}$. For each of the four terms of (4.7), we add and remove additional terms of the form $\mu^{(r_0)}(x_i)$, $K^{(r_0,r_0)}(x_i,x_i)$ (with adequate x_i) so that we can use condition A2.1(ii) rewritten as

$$\left|\mathbb{L}^{(r_0,r_0)}(x_1,x_1)+\mathbb{L}^{(r_0,r_0)}(x_2,x_2)-2\,\mathbb{L}^{(r_0,r_0)}(x_1,x_2)\right|\le A_1^2\,|x_2-x_1|^{2\beta_0}, \tag{4.9}$$

where $\mathbb{L}^{(r_0,r_0)}(x_1,x_2):=K^{(r_0,r_0)}(x_1,x_2)+\mu^{(r_0)}(x_1)\mu^{(r_0)}(x_2)$, together with

$$\left|\mu^{(r_0)}(x_2)-\mu^{(r_0)}(x_1)\right|=\left|\mathrm{E}\,X^{(r_0)}(x_2)-\mathrm{E}\,X^{(r_0)}(x_1)\right|\le A_1\,|x_2-x_1|^{\beta_0}.$$

Next, $L_{i,p_1,r}(\cdot)$ is bounded by $c\,r^r$, see Eq. (4.8), so one gets

$$\max_{0\le k\le r\widetilde{n}_r-1}\sum_{\substack{\ell=0\\|p_2-p_1|\le 1}}^{r\widetilde{n}_r-1}\left|\mathbb{C}_r(\bar{t}_{k,n},\bar{t}_{\ell,n})\right|\le c\,n^{-2r_0}\Big(2A_1^2\max\Big(\left|t_{(p_1+1)r}-t_{p_2r}\right|^{2\beta_0},$$
$$\left|t_{(p_2+1)r}-t_{p_1r}\right|^{2\beta_0}\Big)+A_1^2\left|t_{(p_2+1)r}-t_{p_2r}\right|^{\beta_0}\left|t_{(p_1+1)r}-t_{p_1r}\right|^{\beta_0}\Big).$$

From (2.5), it is easy to see that this last term is at most of the same order, i.e. $c\,n^{-2(r_0+\beta_0)}$, as the previous one.

(b) Case $r=1,\ldots,r_0-1$ ($r_0\ge 2$).

Using the relations (4.6) and (4.7) with r_0 replaced by r, one may proceed as in the case $|p_2-p_1|\ge 2$ since under condition A2.1(i), $K^{(r+1,r+1)}$ exists on $[0,1]^2$ and is a continuous and bounded function, so one may work with the special value $\beta_0=1$. In this way, we obtain easily the bound $c\,n^{-(2r+1)}$.

4.2. Proof of Proposition 3.1

One may write

$$\sup_{s\in[0,1]}\mathrm{E}\left(X(s)-\widetilde{X}_r(s)\right)^2=\max_{k=0,\ldots,\widetilde{n}_r-1}\ \sup_{s\in[t_{kr},t_{(k+1)r}]}\Big(\mathbb{C}_r(s,s)+(\mathrm{E}\,X(s)-\mathrm{E}\,\widetilde{X}_r(s))^2\Big).$$

(i) For $r_0=0$ and $r\ge 1$, the relations (2.5), (4.3), (4.4), (4.8) and (4.9) ($s=x_1=x_2$, $k=\ell$) yield the required result (3.1), namely with $r^*=0$ and $\beta^*=\beta_0$.

(ii) For $r^* = r_0$ ($r \geq r_0 \geq 1$), we apply results of Lemma 4.1 with the choices $s_1 = s_2 = s$ and $k = \ell$:

$$\begin{aligned}
\mathrm{E}\left(X(s) - \widetilde{X}_r(s)\right)^2 &= \sum_{i,j=0}^{r} \frac{L_{i,k,r}(s)L_{j,k,r}(s)}{((r_0-1)!)^2}(t_{kr+i} - t_{kr})^{r_0}(t_{kr+j} - t_{kr})^{r_0} \\
&\quad \times \iint_{[0,1]^2} ((1-v)(1-w))^{r_0-1} \Big\{\mathbb{L}^{(r_0,r_0)}(t_{kr} + (s-t_{kr})v, t_{kr} + (s-t_{kr})w) \\
&\quad - \mathbb{L}^{(r_0,r_0)}(t_{kr} + (s-t_{kr})v, t_{kr} + (t_{kr+j} - t_{kr})w) \\
&\quad - \mathbb{L}^{(r_0,r_0)}(t_{kr} + (t_{kr+i} - t_{kr})v, t_{kr} + (s-t_{kr})w) \\
&\quad + \mathbb{L}^{(r_0,r_0)}(t_{kr} + (t_{kr+i} - t_{kr})v, t_{kr} + (t_{kr+j} - t_{kr})w)\Big\}\, \mathrm{d}v\, \mathrm{d}w.
\end{aligned}$$

Next, the result follows obviously from (4.9), by adding and removing additional terms of the form $\mathbb{L}^{(r_0,r_0)}(x_i, x_i)$.

(iii) Finally if $r^* = r$ ($1 \leq r \leq r_0 - 1$ and $r_0 \geq 2$), one may write

$$\begin{aligned}
\mathrm{E}\left(X(s) - \widetilde{X}_r(s)\right)^2 &\leq \sum_{i,j=0}^{r} \frac{L_{i,k,r}(s)L_{j,k,r}(s)}{((r-1)!)^2}(t_{kr+i} - t_{kr})^{r}(t_{kr+j} - t_{kr})^{r} \\
&\quad \times \iint_{\mathcal{I}_k \times \mathcal{I}_k} \left|\mathbb{L}^{(r+1,r+1)}(u_1, u_2)\right| \mathrm{d}u_1 \mathrm{d}u_2
\end{aligned}$$

where $\mathbb{L}^{(r+1,r+1)}(\cdot,\cdot)$ is a bounded function. Details are left to the reader.

4.3. *Proof of Proposition 3.2*

Using condition A2.1(iv) with $a = t_k$ and $b = t_{k+1}$, we obtain

$$\begin{aligned}
\mathrm{E}\left(X(\bar{t}_{k,n}) - \widetilde{X}_r(\bar{t}_{k,n})\right)^2 &\geq \mathrm{E}\left(X(\bar{t}_{k,n}) - \overline{X}_{t_k,t_{k+1}}(\bar{t}_{k,n})\right)^2 \\
&\geq C_2^2(r_0, \beta_0) n^{-2(r_0+\beta_0)}
\end{aligned}$$

with the help of (2.5)–(2.7).

4.4. *Proof of Theorem 3.1*

We study $\mathbb{P}(\widehat{r}_n \neq r_0)$ for n sufficiently large to ensure that $r_0 \in \{0, \ldots, p_n\}$.
Let us define

$$B_n(r) = \left\{ \frac{n^{2r}}{\widetilde{n}r} \sum_{k=0}^{\widetilde{n}r-1} \left(X(\bar{t}_{k,n}) - \widetilde{X}_r(\bar{t}_{k,n})\right)^2 \geq h_n \right\}.$$

One gets

(i) if $r_0 = 0$, $\{\widehat{r}_n = 0\} = B_n(1)$,
(ii) if $r_0 \geq 1$, $\{\widehat{r}_n = r_0\} = \bigcap_{r=1}^{r_0} (B_n(r)^c) \cap B_n(r_0+1)$

so that

$$\{\widehat{r}_n \neq r_0\} = \begin{cases} B_n(1)^c & \text{if } r_0 = 0, \\ B_n(1) \cup \cdots \cup B_n(r_0) \cup B_n(r_0+1)^c & \text{if } r_0 \geq 1. \end{cases}$$

Next for all $r_0 \geq 0$,

$$\mathbb{P}(\widehat{r_n} \neq r_0) \leq T_{1,n}(r_0, \beta_0) + T_{2,n}(r_0, \beta_0), \tag{4.10}$$

where $T_{1,n}(r_0, \beta_0)$, $T_{2,n}(r_0, \beta_0)$ are defined by $T_{2,n}(0, \beta_0) := 0$ and

$$T_{1,n}(r_0, \beta_0) := \mathbb{P}\left(B_n(r_0+1)^c\right) \tag{4.11}$$

$$T_{2,n}(r_0, \beta_0) := \sum_{r=1}^{r_0} \mathbb{P}\left(B_n(r)\right), \quad r_0 \geq 1. \tag{4.12}$$

Finally the result follows immediately from the following lemma (whose proof is given in the following Section 4.5).

Lemma 4.2. *Assume A2.1. In addition if $n^{2\beta_0-2}h_n \to 0$ and $n^{2\beta_0}h_n \to +\infty$, for all $\beta_0 \in [0, 1[$, then for n large enough*

$$T_{1,n}(r_0, \beta_0) + T_{2,n}(r_0, \beta_0) \leq C_5(r_0) \exp\left(-C_6(r_0, \beta_0)\varphi_n(\beta_0)\right)$$

with $\varphi_n(\beta_0)$ defined in Eq. (3.3) and positive constants $C_5(r_0)$, $C_6(r_0, \beta_0)$.

4.5. *Proof of Lemma* 4.2

Let us introduce the quantity

$$Z_{k,r} := \frac{n^r}{\sqrt{r\widetilde{n}_r}}\left(X(\bar{t}_{k,n}) - \widetilde{X}_r(\bar{t}_{k,n})\right). \tag{4.13}$$

(i) *Study of the term $T_{1,n}(r_0, \beta_0)$.*

From (4.11) and setting $\kappa = r_0 + 1$, one can write

$$T_{1,n}(r_0, \beta_0) = \mathbb{P}\left(\sum_{j=0}^{\kappa\widetilde{n}_\kappa - 1} Z_{j,\kappa}^2 < h_n\right) = \mathbb{P}\left(\mathrm{E}\left(\sum_{j=0}^{\kappa\widetilde{n}_\kappa - 1} Z_{j,\kappa}^2\right) - \sum_{j=0}^{\kappa\widetilde{n}_\kappa - 1} Z_{j,\kappa}^2 > \eta_{1,\kappa}(n)\right),$$

where $\eta_{1,\kappa}(n) = \mathrm{E}\,(\sum_{j=0}^{\kappa\widetilde{n}_\kappa - 1} Z_{j,\kappa}^2) - h_n$. Using Propositions 3.1 and 3.2, we can deduce that

$$\eta_{1,\kappa}(n) \in \left[C_2^2(r_0, \beta_0)n^{2(1-\beta_0)} - h_n,\, C_1^2(r_0+1, \beta_0)n^{2(1-\beta_0)} - h_n\right].$$

Now for h_n such that $h_n n^{-2(1-\beta_0)} \to 0$ for all $\beta_0 \in [0, 1[$, one gets that $\eta_{1,\kappa}(n)$ is positive for n large enough and $\eta_{1,\kappa}(n) \to \infty$ with same order as $n^{2(1-\beta_0)}$. Now, one may write

$$\left|\sum_{j=0}^{\kappa\widetilde{n}_\kappa - 1}\left(Z_{j,\kappa}^2 - \mathrm{E}\,Z_{j,\kappa}^2\right)\right| = \left|\sum_{j=0}^{\kappa\widetilde{n}_\kappa - 1}\left\{(Z_{j,\kappa} - \mathrm{E}\,Z_{j,\kappa})^2 - \mathrm{E}\,(Z_{j,\kappa} - \mathrm{E}\,Z_{j,\kappa})^2\right\}\right.$$
$$\left. + \left\{2\left(\mathrm{E}\,Z_{j,\kappa}\right)(Z_{j,\kappa} - \mathrm{E}\,Z_{j,\kappa})\right\}\right|,$$

so that

$$\mathbb{P}\left(\left|\sum_{j=0}^{\kappa\widetilde{n}_\kappa - 1}\left(Z_{j,\kappa}^2 - \mathrm{E}\,Z_{j,\kappa}^2\right)\right| > \eta_{1,\kappa}(n)\right) \leq S_1 + S_2 \tag{4.14}$$

with

$$S_1 := \mathbb{P}\left(\left|\sum_{j=0}^{\kappa\widetilde{n}_\kappa-1}(Z_{j,\kappa}-\mathrm{E}\,Z_{j,\kappa})^2-\mathrm{E}\,(Z_{j,\kappa}-\mathrm{E}\,Z_{j,\kappa})^2\right|>\frac{\eta_{1,\kappa}(n)}{2}\right) \tag{4.15}$$

$$S_2 := \mathbb{P}\left(\left|\sum_{j=0}^{\kappa\widetilde{n}_\kappa-1}\left(\mathrm{E}\,Z_{j,\kappa}\right)(Z_{j,\kappa}-\mathrm{E}\,Z_{j,\kappa})\right|\geq\frac{\eta_{1,\kappa}(n)}{4}\right). \tag{4.16}$$

(a) Study of the term $\mathbf{S_1}$

We consider an ortho-Gaussian basis $\{Y_i\}$, Y_i i.i.d. with distribution $\mathcal{N}(0,1)$, of the subspace of L^2 spanned by $\{Z_{j,\kappa}-\mathrm{E}\,Z_{j,\kappa}\}_{j=0,\dots,\kappa\widetilde{n}_\kappa-1}$. Let denote the size of the basis by d_n. Then we can write $Z_{j,\kappa}-\mathrm{E}\,Z_{j,\kappa}=\sum_{i=1}^{d_n}\mathrm{Cov}\left(Z_{j,\kappa},Y_i\right)Y_i := \sum_{i=1}^{d_n} b_{j,i}Y_i$.

Next, if $Y=(Y_1,\dots,Y_{d_n})^\top$, we obtain

$$\sum_{j=0}^{\kappa\widetilde{n}_\kappa-1}(Z_{j,\kappa}-\mathrm{E}\,Z_{j,\kappa})^2=\sum_{k,\ell=1}^{d_n}c_{k,\ell}Y_kY_\ell=Y^\top CY$$

with $c_{k,\ell}=\sum_{j=0}^{\kappa\widetilde{n}_\kappa-1}\mathrm{Cov}\,(Z_{j,\kappa},Y_k)\mathrm{Cov}\,(Z_{j,\kappa},Y_\ell)$ so that $C=B^\top B$. Let us define $C=\left(c_{k,\ell}\right)_{k,\ell=1,\dots,d_n}$ and $B=\left(b_{k,\ell}\right)_{k=0,\dots,\kappa\widetilde{n}_\kappa-1,\ell=1,\dots,d_n}$. The matrix C is real, symmetric and positive semidefinite, so there exists an orthogonal matrix P such that $\mathrm{diag}(\lambda_1,\dots,\lambda_{d_n})=P^\top CP$, where the quantities λ_i are the eigenvalues of the matrix C. Then we can transform the quadratic form

$$\sum_{j=0}^{\kappa\widetilde{n}_\kappa-1}(Z_{j,\kappa}-\mathrm{E}\,Z_{j,\kappa})^2=\sum_{j=1}^{d_n}\lambda_j(P^\top Y)_j^2$$

where $(P^\top Y)_j$ denotes the j-th component of the $(d_n\times 1)$ vector $P^\top Y$. As $E(P^\top Y)_j^2=1$, we arrive at

$$S_1=\mathbb{P}\left(\left|\sum_{j=1}^{d_n}\lambda_j\left((P^\top Y)_j^2-1\right)\right|\geq\frac{\eta_{1,\kappa}(n)}{2}\right).$$

Now, using the exponential bound of [10], one gets

$$S_1\leq 2\exp\left(-\min\left(\frac{c_1\,\eta_{1,\kappa}(n)}{2\max\limits_{i=1,\dots,d_n}\lambda_i},\frac{c_2\,\eta_{1,\kappa}^2(n)}{4\sum\lambda_i^2}\right)\right)\leq 2\exp\left(-\frac{c\,\eta_{1,\kappa}(n)}{\max\limits_{i=1,\dots,d_n}\lambda_i}\right)$$

for n large enough and with easy calculation. Next, since $B^\top B$ and $BB^\top$ have the same non-zero eigenvalues, we can write

$$\begin{aligned}\max_{i=1,\dots,d_n}\lambda_i &\leq \max_{0\leq j\leq\kappa\widetilde{n}_\kappa-1}\sum_{\ell=0}^{\kappa\widetilde{n}_\kappa-1}\left|\mathrm{E}\,\left(Z_{j,\kappa}-\mathrm{E}\,(Z_{j,\kappa})\right)\left(Z_{\ell,\kappa}-\mathrm{E}\,(Z_{\ell,\kappa})\right)\right|\\ &\leq \frac{n^{2\kappa}}{\kappa\widetilde{n}_\kappa}\max_{0\leq j\leq\kappa\widetilde{n}_\kappa-1}\sum_{\ell=0}^{\kappa\widetilde{n}_\kappa-1}\left|\mathbb{C}_\kappa(\bar{t}_{j,\kappa},\bar{t}_{\ell,\kappa})\right|\end{aligned}$$

with $\mathbb{C}_\kappa(\cdot,\cdot)$ defined in (4.1). By Proposition 4.1, one obtains

$$\max_{i=1,\ldots,d_n} \lambda_i \le c \begin{cases} n^{1-2\beta_0} & \text{if } 0 \le \beta_0 < 1/2, \\ \ln n & \text{if } \beta_0 = 1/2, \\ 1 & \text{if } 1/2 < \beta_0 < 1 \end{cases}$$

so that

$$S_1 \le 2\exp\left(-\frac{c\,\eta_{1,\kappa}(n)}{n^{1-2\beta_0}\mathbb{I}_{\{0\le\beta_0<1/2\}} + (\ln n)\mathbb{I}_{\{\beta_0=1/2\}} + \mathbb{I}_{\{1/2<\beta_0<1\}}}\right).$$

Therefore, recalling $\eta_{1,\kappa}(n) \propto n^{2(1-\beta_0)}$, one gets for all n large enough that

$$S_1 \le 2\exp\left(-c\varphi_n(\beta_0)\right) \tag{4.17}$$

with $\varphi_n(\beta_0)$ given in (3.3). Note that by Remark 4.2, the logarithmic term disappears if Baxter's condition is fulfilled.

(b) Study of the term $\mathbf{S_2}$

Let us recall the following well-known lemma:

Lemma 4.3. *If $Y \sim \mathcal{N}(0,\sigma^2)$, $\sigma > 0$, then for all $\varepsilon > 0$,*

$$\mathbb{P}\left(|Y| \ge \varepsilon\right) \le 2\exp\left(-\frac{\varepsilon^2}{2\sigma^2}\right).$$

Since X is a Gaussian process, $\sum_{j=0}^{\kappa\widetilde{n}_\kappa-1}\left(\mathrm{E}\,Z_{j,\kappa}\right)\left(Z_{j,\kappa} - \mathrm{E}\,Z_{j,\kappa}\right)$ with $Z_{j,\kappa}$ defined in (4.13) is a zero-mean Gaussian random variable. Therefore, one may apply Lemma 4.3 to the term S_2, defined in (4.16), and get

$$S_2 \le 2\exp\left(-\frac{\eta_{1,\kappa}^2(n)}{32\sigma_n^2}\right),$$

where $\sigma_n^2 := \mathrm{Var}\left(\sum_{j=0}^{\kappa\widetilde{n}_\kappa-1}\left(\mathrm{E}\,Z_{j,\kappa}\right)\left(Z_{j,\kappa} - \mathrm{E}\,Z_{j,\kappa}\right)\right)$. Moreover, using results (3.2) and (4.5), one may obtain successively

$$\begin{aligned}
\sigma_n^2 &= \sum_{j,\ell=0}^{\kappa\widetilde{n}_\kappa-1}\left(\mathrm{E}\,Z_{j,\kappa}\right)\left(\mathrm{E}\,Z_{\ell,\kappa}\right)\mathrm{Cov}\left(Z_{j,\kappa}, Z_{\ell,\kappa}\right) \\
&\le (\kappa\widetilde{n}_\kappa)\left(\max_{j=0,\ldots,\kappa\widetilde{n}_\kappa-1}\left|\mathrm{E}\,Z_{j,\kappa}\right|\right)^2 \max_{j=0,\ldots,\kappa\widetilde{n}_\kappa-1}\sum_{\ell=0}^{\kappa\widetilde{n}_\kappa-1}\left|\mathrm{Cov}\left(Z_{j,\kappa}, Z_{\ell,\kappa}\right)\right| \\
&\le cn^{-2(1-\beta_0)} \times \left\{n^{1-2\beta_0}\mathbb{I}_{\{0\le\beta_0<1/2\}} + (\ln n)\mathbb{I}_{\{\beta_0=1/2\}} + \mathbb{I}_{\{1/2<\beta_0<1\}}\right\}.
\end{aligned}$$

Finally, for n large enough and using the order of $\eta_{1,\kappa}(n)$, we get

$$S_2 \le 2\exp\left(-c\,\varphi_n(\beta_0)\right) \tag{4.18}$$

with $\varphi_n(\beta_0)$ given by (3.3). In conclusion, by collecting results from (4.14)–(4.18), for n large enough, we arrive at

$$T_{1,n}(r_0,\beta_0) \le 4\exp\left(-C_7(r_0,\beta_0)\varphi_n(\beta_0)\right)$$

for all h_n such that $n^{2\beta_0-2}h_n \to 0$ and for some positive constant $C_7(r_0,\beta_0)$.

(ii) *Study of the term* $T_{2,n}(r_0, \beta_0)$.

Recall that this term, defined in Eq. (4.12), occurs only when $r_0 \geq 1$. For $Z_{j,r}$ defined in Eq. (4.13), one has for $\eta_{2,r}(n) = h_n - \sum_{j=0}^{r\widetilde{n}_r - 1} \mathrm{E}\, Z_{j,r}^2$,

$$T_{2,n}(r_0, \beta_0) = \sum_{r=1}^{r_0} \mathbb{P}\left(\sum_{j=0}^{r\widetilde{n}_r - 1} \left(Z_{j,r}^2 - \mathrm{E}\, Z_{j,r}^2 \right) \geq \eta_{2,r}(n) \right).$$

- Suppose that $r = r_0$. Propositions 3.1 and 3.2 lead to $\eta_{2,r_0}(n) \in [h_n - C_1^2(r_0, \beta_0) n^{-2\beta_0}, h_n - C_2^2(r_0, \beta_0) n^{-2\beta_0}]$. Now for h_n such that $h_n n^{2\beta_0} \to +\infty$ for all $\beta_0 \in [0, 1[$, one gets that $\eta_{2,r_0}(n)$ is positive for n large enough and $\eta_{2,r_0}(n) \to +\infty$ with the same order as h_n. Now using the same bounding method as in part (i), with $\eta_{1,\kappa}(n)$ replaced by $\eta_{2,r_0}(n)$, we obtain that
$$\mathbb{P}\left(\sum_{j=0}^{r_0\widetilde{n}_{r_0} - 1} \left(Z_{j,r_0}^2 - \mathrm{E}\, Z_{j,r_0}^2 \right) \geq \eta_{2,r_0}(n) \right) \leq 2 \exp\left(-c\, h_n n^{2\beta_0} \varphi_n(\beta_0) \right) + 2 \exp\left(-c\, h_n^2 n^{4\beta_0} \varphi_n(\beta_0) \right)$$
for n large enough and $\varphi_n(\beta_0)$ given in Eq. (3.3).
- Suppose that $r = 1, \ldots, r_0 - 1$ (and hence $r_0 \geq 2$). Propositions 3.1 and 3.2 imply that $\eta_{2,r}(n) \in \left[h_n - C_1^2(r, \beta_0) n^{-2}, h_n - C_2^2(r_0, \beta_0) n^{-2(r_0 - r + \beta_0)} \right]$ and we get for n large enough
$$\mathbb{P}\left(\sum_{j=0}^{r\widetilde{n}_r - 1} \left(Z_{j,r}^2 - \mathrm{E}\, Z_{j,r}^2 \right) \geq \eta_{2,r}(n) \right) \leq 2 \exp\left(-c\, h_n n^2 \right) + 2 \exp\left(-c\, h_n^2 n^4 \right).$$
- Collecting these results, one obtains
$$T_{2,n}(r_0, \beta_0) \leq 4 r_0 \exp\left(-C_8(r_0, \beta_0) h_n n^{2\beta_0} \varphi_n(\beta_0) \right)$$
for all h_n such that $n^{2\beta_0} h_n \to +\infty$ and for some positive constant $C_8(r_0, \beta_0)$.

Finally, for n large enough, we conclude that

$$T_{1,n}(r_0, \beta_0) + T_{2,n}(r_0, \beta_0) \leq 4 \exp\left(-C_7(r_0, \beta_0) \varphi_n(\beta_0) \right) + 4 r_0 \exp\left(-C_8(r_0, \beta_0) h_n n^{2\beta_0} \varphi_n(\beta_0) \right). \tag{4.19}$$

4.6. *Proof of Theorem* 3.2

(i) Clear from Theorem 3.1 and the Borel–Cantelli lemma.

(ii) If $\widetilde{r}_n = \max(\widehat{r}_n, 1)$, one may write

$$\begin{aligned} &\mathbb{P}\left(\left| X(s) - \widetilde{X}_{\widetilde{r}_n}(s) \right| \geq \varepsilon_n \right) \\ &\quad = \mathbb{P}\left(\left| X(s) - \widetilde{X}_{\widetilde{r}_n}(s) \right| \geq \varepsilon_n, \widehat{r}_n = r_0 \right) + \mathbb{P}\left(\left| X(s) - \widetilde{X}_{\widetilde{r}_n}(s) \right| \geq \varepsilon_n, \widehat{r}_n \neq r_0 \right) \\ &\quad \leq \mathbb{P}\left(\left| X(s) - \widetilde{X}_{\widetilde{r}_0}(s) \right| \geq \varepsilon_n \right) + \mathbb{P}\left(\widehat{r}_n \neq r_0 \right) \end{aligned} \tag{4.20}$$

where we have set $\widetilde{r}_0 = \max(r_0, 1)$. Now since

$$\left| X(s) - \widetilde{X}_{\widetilde{r}_0}(s) \right| \leq \left| X(s) - \widetilde{X}_{\widetilde{r}_0}(s) - \mathrm{E}\left(X(s) - \widetilde{X}_{\widetilde{r}_0}(s) \right) \right| + \left| \mathrm{E}\left(X(s) - \widetilde{X}_{\widetilde{r}_0}(s) \right) \right|,$$

we get

$$\mathbb{P}\left(\left|X(s)-\widetilde{X}_{\widetilde{r_0}}(s)\right|\geq\varepsilon_n\right)\leq\mathbb{P}\left(\left|X(s)-\widetilde{X}_{\widetilde{r_0}}(s)-\mathrm{E}\left(X(s)-\widetilde{X}_{\widetilde{r_0}}(s)\right)\right|\right.$$
$$\left.\geq\varepsilon_n-\left|\mathrm{E}\left(X(s)-\widetilde{X}_{\widetilde{r_0}}(s)\right)\right|\right).$$

The choice $\varepsilon_n=\eta\sqrt{\ln n}\,n^{-(r_0+\beta_0)}$ and (3.2) yield

$$\varepsilon_n-\left|\mathrm{E}\left(X(s)-\widetilde{X}_{\widetilde{r_0}}(s)\right)\right|\geq\eta\sqrt{\ln n}\,n^{-(r_0+\beta_0)}\alpha_n$$

with $\alpha_n\to_{n\to\infty}1^-$. Then, since $\mathrm{Var}\left(X(s)-\widetilde{X}_{\widetilde{r_0}}(s)\right)\leq C_1^2(\widetilde{r_0},\beta_0)n^{-2(r_0+\beta_0)}$, one gets

$$\mathbb{P}\left(\left|X(s)-\widetilde{X}_{\widetilde{r_0}}(s)\right|\geq\varepsilon_n\right)\leq 2\exp\left(-\frac{\eta^2\alpha_n^2\ln n}{2C_1^2(\widetilde{r_0},\beta_0)}\right)$$

as a consequence of Lemma 4.3. Next, from (4.20) and Theorem 3.1,

$$\sum_n\mathbb{P}\left(\left|X(s)-\widetilde{X}_{\widetilde{r_n}}(s)\right|\geq\eta\sqrt{\ln n}\,n^{-(r_0+\beta_0)}\right)<\infty$$

for all $\eta>\sqrt{2}\,C_1(\widetilde{r_0},\beta_0)$. The Borel–Cantelli lemma implies that, almost surely,

$$\limsup_{n\to\infty}\frac{n^{(r_0+\beta_0)}}{\sqrt{\ln n}}\left|X(s)-\widetilde{X}_{\max(1,\widehat{r_n})}(s)\right|\leq\sqrt{2}\,C_1(\widetilde{r_0},\beta_0).$$

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