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Lagging and leading coupled continuous time random walks, renewal times and their joint limits

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Abstract

Subordinating a random walk to a renewal process yields a continuous time random walk (CTRW), which models diffusion and anomalous diffusion. Transition densities of scaling limits of power law CTRWs have been shown to solve fractional Fokker–Planck equations. We consider limits of CTRWs which arise when both waiting times and jumps are taken from an infinitesimal triangular array. Two different limit processes are identified when waiting times precede jumps or follow jumps, respectively, together with two limit processes corresponding to the renewal times. We calculate the joint law of all four limit processes evaluated at a fixed time t.

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1. Introduction

An i.i.d. sequence of jumps J_i in $\mathbb{R}^d$ separated by an i.i.d. sequence of positive waiting times W_i yields a jump process known as a continuous time random walk (CTRW) [23]. Continuous time random walks, and evolution equations for their limiting distributions, obtained as the step size tends to zero and the number of steps tends to infinity, have been widely studied over the

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past few decades as physical models of diffusion. CTRWs with power law waiting time densities and/or infinite variance jumps, and evolution equations for their limiting distributions, formulated in terms of fractional order partial differential equations, have been of particular interest, as physical models for anomalous diffusion [22,12]. The limiting distributions of CTRWs have also been investigated using a mathematical approach based on renewal theory and limit theorems for sums of random jumps [18,2]. More general results have been obtained using "triangular array limits" [4,21,29]. A statement of the problem in this context is: for every $n \in \mathbb{N}$, consider a CTRW X^n arising from an i.i.d. sequence $(J_i^n, W_i^n)_{i\in\mathbb{N}}$ with common law Π^n on $\mathbb{R}^d \times \mathbb{R}^+$. Then assume that Π^n converges weakly to the Dirac measure concentrated at (0, 0), and consider possible stochastic process limits of X^n.

In this paper, we consider coupled CTRWs where the J_i are not independent of W_i. These are of particular interest in finance [28,20,16], as there is empirical evidence for stock markets with correlated waiting times and log-returns [24]; also see [2] for a comprehensive list of coupled CTRWs having appeared in the literature. We show that in general there are two different limit processes X and Y that arise when each W_i precedes or succeeds J_i, respectively. In the transition to the limit, we also keep track of the processes G^n and D^n given by the last renewal times G_t^n before t and first renewal times D_t^n after t, and show that they jointly converge with X and Y. As X and Y turn out to be constant on every interval $[G_t, D_t)$, this enables us to model the time intervals in which the diffusing particle is trapped. After defining the age process A and the remaining lifetime process R via $A_t = t - G_t$ and $R_t = D_t - t$, respectively, we calculate the joint laws of (X_t, A_t, Y_t, R_t) for each fixed time $t \geq 0$.

The remainder of this paper is organized as follows: in Section 2 we introduce our notation and establish general properties for functions that are right-continuous with left-hand limits (*rcll*) and left-continuous with right-hand limits (*lcrl*) and define two continuous mappings on a measurable subset of Skorohod space. In Section 3 we consider triangular array limits for CTRWs and we define the stochastic processes "lagging CTRW" X^n, "leading CTRW" Y^n, "last time of renewal" G^n and "next time of renewal" D^n. Using the continuous mapping theorem, we prove limit theorems for the weak convergence of these processes (Theorem 3.6). In Section 4 we consider the stochastic processes A and R corresponding to the age and remaining lifetime respectively and we obtain an integral equations for the joint law of (X_t, A_t, Y_t, R_t) (Theorem 4.9). Concluding remarks are in Section 5.

2. Two continuous mappings on Skorohod space

For a separable complete metric space E, let $\mathbb{D}(E)$ be the set of all functions defined on $\mathbb{R}^+ := [0, \infty)$ with values in E which are right-continuous and have limits from the left (in short, the set of all *rcll paths*). We assume that $\mathbb{D}(E)$ is endowed with the (metrizable) Skorohod topology J [14]. Equipped with the corresponding Borel-σ-algebra $\mathcal{D}(E)$, $\mathbb{D}(E)$ is a measurable space.

Skorohod subspaces. For an element $\alpha \in \mathbb{D}(\mathbb{R}^d \times \mathbb{R}^+)$, we write $\alpha = (\beta, \sigma)$, where $\beta \in \mathbb{D}(\mathbb{R}^d)$ and $\sigma \in \mathbb{D}(\mathbb{R}^+)$. We write D_u, $D_\uparrow$ and $D_{\uparrow\uparrow}$ for the sets of all such α which have unbounded, non-decreasing and increasing σ, respectively. Slight variations of the proofs of [30, lem.13.2.3, lem.13.6.1] show that $D_\uparrow$ is closed in $\mathbb{D}(\mathbb{R}^d \times \mathbb{R}^+)$, D_u is a G_δ-subset of $\mathbb{D}(\mathbb{R}^d \times \mathbb{R}^+)$ (i.e. a countable intersection of open subsets) and $D_{\uparrow\uparrow}$ is a G_δ-subset of $D_\uparrow$. Hence we have

Lemma 2.1. *The sets* $D_{\uparrow,u} := D_\uparrow \cap D_u$ *and* $D_{\uparrow\uparrow,u} := D_{\uparrow\uparrow} \cap D_u$ *are Borel measurable.*

For an rcll path ξ we write ξ^- for its *lcrl* version (the corresponding left-continuous path having right-hand limits) given by

$$t \mapsto \lim_{\varepsilon \downarrow 0} \xi(t-\varepsilon), \quad t > 0, \qquad \xi^-(0) = \xi(0),$$

and similarly, if ξ is lcrl, we write ξ^+ for its rcll version

$$t \mapsto \lim_{\varepsilon \downarrow 0} \xi(t+\varepsilon), \quad t \geq 0.$$

It will be convenient to use both notations $\xi^-(t) = \xi(t-)$ and $\xi^+(t) = \xi(t+)$. For an unbounded $\xi \in \mathbb{D}(\mathbb{R}^+)$ we define its generalized inverse via

$$\xi^{-1}(t) := \inf\{s \geq 0 : \xi(s) > t\}$$

and note that ξ^{-1} is a non-decreasing element of $\mathbb{D}(\mathbb{R}^+)$.

Definition 2.2. For $\alpha = (\beta, \sigma) \in D_{\uparrow,u}$ write $\ell_\alpha := \sigma^{-1}$. Then put

$$\begin{array}{ll} \Phi : D_{\uparrow,u} \to \mathbb{D}(\mathbb{R}^d \times \mathbb{R}^+) \quad \text{and} & \Psi : D_{\uparrow,u} \to \mathbb{D}(\mathbb{R}^d \times \mathbb{R}^+) \\ \alpha \mapsto (\alpha^- \circ \ell_\alpha^-)^+ & \alpha \mapsto \alpha \circ \ell_\alpha. \end{array}$$

Since ℓ_α is non-decreasing and rcll, $\alpha \circ \ell_\alpha$ is rcll, and so Ψ is well defined. Moreover, it is not hard to see that $\alpha^- \circ \ell_\alpha^-$ is lcrl and hence that Φ is well defined.

Note that the topology J on $\mathbb{D}(\mathbb{R}^d \times \mathbb{R}^+)$ induces the *relative topology* or *subspace topology* on the subset $D_{\uparrow,u}$. The following is the key ingredient of the continuous mapping theorem in Section 3.

Proposition 2.3. *The mappings Φ and Ψ are continuous at $D_{\uparrow\uparrow,u}$.*

The proof is based on the characterization of convergence in $\mathbb{D}(E)$ taken from [11, th 3.6.5]; for convenience, it is restated in Proposition 2.4. We use the abbreviation $x_n \to A$ if $A \subset \mathbb{D}(E)$ contains all limit points of the sequence $\{x_n\}$.

Proposition 2.4. *Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{D}(E)$ and $x \in \mathbb{D}(E)$. Then $x_n \to x$ in $(\mathbb{D}(E), J)$ if and only if whenever $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$, $t \geq 0$, and $\lim t_n = t$, the following conditions hold:*

(I) $x_n(t_n) \to \{x(t), x^-(t)\}$.
(II) *If $x_n(t_n) \to x(t)$ and $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ is such that $s_n \geq t_n$, $s_n \to t$, then $x_n(s_n) \to x(t)$.*
(III) *If $x_n(t_n) \to x^-(t)$ and $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ is such that $0 \leq s_n \leq t_n$, $s_n \to t$, then $x_n(s_n) \to x^-(t)$.*

We will also need the corresponding version for lcrl paths:

Lemma 2.5. *Proposition 2.4 holds if conditions* (I)–(III) *are replaced by*

(I^-) $x_n^-(t_n) \to \{x(t), x^-(t)\}$
(II^-) *If $x_n^-(t_n) \to x(t)$ and $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ is such that $s_n \geq t_n$, $s_n \to t$, then $x_n^-(s_n) \to x(t)$.*
(III^-) *If $x_n^-(t_n) \to x^-(t)$ and $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ is such that $0 \leq s_n \leq t_n$, $s_n \to t$, then $x_n^-(s_n) \to x^-(t)$.*

Proof. Without loss of generality we assume $t_n > 0$ for all n. We write d for the metric in E. Then there is a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}},\ t_n > \varepsilon_n > 0,\ \lim \varepsilon_n = 0$, such that

$$d(x_n^-(t_n), x_n(t_n - \varepsilon_n)) \to 0, \qquad d(x_n(t_n), x_n^-(t_n + \varepsilon_n)) \to 0.$$

Since $\lim t_n - \varepsilon_n = \lim t_n + \varepsilon_n = t$, the equivalence (I) $\Leftrightarrow$(I$^-$) follows. Suppose now that (II) holds, and that $s_n \geq t_n$ is such that $\lim s_n = \lim t_n = t,\ x_n^-(t_n) \to x(t)$. Then there are sequences ε_n and ε_n' tending to 0 and satisfying

$$d(x_n^-(t_n), x(t_n - \varepsilon_n)) \to 0, \qquad d(x_n^-(s_n), x(s_n - \varepsilon_n')) \to 0$$

which we can choose in such a way that $0 < \varepsilon_n' < \varepsilon_n$ for all n. Then $s_n - \varepsilon_n' \geq t_n - \varepsilon_n$ and $s_n - \varepsilon_n' \to t$, hence

$$x(t) = \lim x_n^-(t_n) = \lim x_n(t_n - \varepsilon_n) = \lim x_n(s_n - \varepsilon_n') = \lim x_n^-(s_n),$$

where the third equality follows from (II). This shows the implication (II) $\Rightarrow$(II$^-$). The remaining parts (II$^-$) $\Rightarrow$ (II) and (III) $\Leftrightarrow$(III$^-$) are shown similarly. □

Proof of Proposition 2.3. Throughout this proof, let $\{\alpha_n\}_{n\in\mathbb{N}} \subset D_{\uparrow,u},\ \alpha \in D_{\uparrow\uparrow,u},\ \alpha_n \to \alpha$ in $\mathbb{D}(\mathbb{R}^d \times \mathbb{R}^+)$ with respect to the J-topology, $t \geq 0,\ \{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+,\ \{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+,\ t_n \to t,\ s_n \to t$. We put $\gamma_n := \Phi(\alpha_n),\ \gamma := \Phi(\alpha),\ \delta_n := \Psi(\alpha_n),\ \delta := \Psi(\alpha)$ and show that (I)–(III) are satisfied with x_n and x replaced by δ_n and δ, and that (I$^-$)–(III$^-$) are satisfied with x_n and x replaced by γ_n and γ.

First, note that by [30, cor. 13.6.4],

$$\ell_{\alpha_n} \to \ell_\alpha \quad \text{in } \mathbb{D}(\mathbb{R}^+). \tag{2.6}$$

We define $\tau_n := \ell_{\alpha_n}(t_n),\ \tau := \ell_\alpha(t),\ \xi_n := \ell_{\alpha_n}(s_n)$. Since $\alpha \in D_{\uparrow\uparrow,u},\ \ell_\alpha$ is continuous, and hence (I) applied to ℓ_{α_n} yields $\lim \tau_n = \lim \xi_n = \tau$.

If $t = 0$, then $\tau = 0$ and $\delta_n(t_n) = \alpha_n(\tau_n) \to \alpha(0) = \delta(0)$ by (I) applied to α_n, and one checks that (I)–(III) are satisfied for δ_n.

Assume now that $t > 0$, and that ℓ_α is left-increasing at t, i.e. $s < t \Rightarrow \ell_\alpha(s) < \ell_\alpha(t)$. Then $\delta^-(t) = \alpha^-(\tau)$, and

$$\delta_n(t_n) = \alpha_n(\tau_n) \to \{\alpha^-(\tau), \alpha(\tau)\} = \{\delta(t), \delta^-(t)\},$$

showing (I) for δ_n. In order to show (II) (resp. (III)) for δ_n, suppose $\delta_n(t_n) \to \delta(t)$ (resp. $\delta_n(t_n) \to \delta^-(t)$). Then

$$\alpha_n(\tau_n) = \delta_n(t_n) \to \delta(t) = \alpha(\tau) \quad (\text{resp. } \alpha_n(\tau_n) = \delta_n(t_n) \to \delta^-(t) = \alpha^-(\tau)).$$

Since ℓ_{α_n} is non-decreasing, $s_n \geq t_n$ (resp. $s_n \leq t_n$) implies $\xi_n \geq \tau_n$ (resp. $\xi_n \leq \tau_n$). Then (II) (resp. (III)) applied to α_n yields

$$\lim \delta_n(s_n) = \lim \alpha_n(\xi_n) = \alpha(\tau) = \delta(t),$$
$$(\text{resp. } \lim \delta_n(s_n) = \lim \alpha_n(\xi_n) = \alpha^-(\tau) = \delta^-(t),)$$

showing (II) (resp. (III)) for δ_n.

If ℓ_α is left-constant at t, i.e. not left-increasing, then $\alpha(\ell_\alpha(t-)) = \alpha(\ell_\alpha(t))$ and hence δ is continuous at t. Hence (I)–(III) for δ_n reduces to showing $\delta_n(t_n) \to \delta(t)$. (I) applied to α_n yields $\delta_n(t_n) = \alpha_n(\tau_n) \to \{\alpha(\tau), \alpha^-(\tau)\}$. Since ℓ_α is left-constant at t, we have $\sigma^-(\tau) < t \leq \sigma(\tau)$. As σ is the last coordinate of α, we see that $\alpha(\tau) \neq \alpha^-(\tau)$. Suppose now that $\alpha_n(\tau_n) \to \alpha^-(\tau)$.

It follows that $\sigma_n(\tau_n) \to \sigma^-(\tau) < t$, which contradicts $\sigma_n(\tau_n) \geq t_n \to t$. Thus $\alpha_n(\tau_n) \to \alpha(\tau)$ and $\delta_n(t_n) \to \delta(t)$. We have proven the continuity statement about Ψ.

We turn to γ_n and the conditions (I$^-$)–(III$^-$), and define $\tau_n^- := \ell_{\alpha_n}^-(t_n)$ and $\xi_n^- := \ell_{\alpha_n}^-(s_n)$. As before, (2.6), the continuity of ℓ_α and (I) applied to ℓ_{α_n} yield $\lim \tau_n^- = \lim \xi_n^- = \tau$. Note that $\gamma^- = \alpha^- \circ \ell_\alpha$. Assume first that ℓ_α is right-increasing at t. Then $\gamma(t) = \alpha(\tau)$, and

$$\gamma_n^-(t_n) = \alpha_n^-(\tau_n^-) \to \{\alpha(\tau), \alpha^-(\tau)\} = \{\gamma(t), \gamma^-(t)\},$$

by (I$^-$) applied to α_n^-, showing (I$^-$) for γ_n^-. In order to show (II$^-$) (resp. (III$^-$)) for γ_n^-, suppose $\gamma_n^-(t_n) \to \gamma(t)$ (resp. $\gamma_n^-(t_n) \to \gamma^-(t)$). Then

$$\alpha_n^-(\tau_n^-) = \gamma_n^-(t_n) \to \gamma(t) = \alpha(\tau) \quad (\text{resp. } \alpha_n^-(\tau_n^-) = \gamma_n^-(t_n) \to \gamma^-(t) = \alpha^-(\tau)).$$

Since ℓ_{α_n} and $\ell_{\alpha_n}^-$ are non-decreasing, $s_n \geq t_n$ (resp. $s_n \leq t_n$) implies $\xi_n^- \geq \tau_n^-$ (resp. $\xi_n^- \leq \tau_n^-$). Then (II$^-$) (resp. (III$^-$)) applied to α_n yields

$$\lim \gamma_n^-(s_n) = \lim \alpha_n(\xi_n^-) = \alpha(\tau) = \gamma(t),$$
$$(\text{resp. } \lim \gamma_n^-(s_n) = \lim \alpha_n(\xi_n^-) = \alpha^-(\tau) = \gamma^-(t),)$$

showing (II$^-$) (resp. (III$^-$)) for γ_n^-.

If ℓ_α is right-constant at t, i.e. not right-increasing, then $\alpha^-(\ell_\alpha(t+)) = \alpha^-(\ell_\alpha(t))$ and hence γ^- is continuous at t. Hence (I$^-$)–(III$^-$) for γ_n^- reduces to showing $\gamma_n^-(t_n) \to \gamma(t)$. (I$^-$) applied to α_n yields $\gamma_n^-(t_n) = \alpha_n^-(\tau_n^-) \to \{\alpha(\tau), \alpha^-(\tau)\}$. Since ℓ_α is right-constant at t, we have $\sigma^-(\tau) \leq t < \sigma(\tau)$, and we see that $\alpha(\tau) \neq \alpha^-(\tau)$. Suppose now that $\alpha_n^-(\tau_n^-) \to \alpha(\tau)$. It follows that $\sigma_n^-(\tau_n^-) \to \sigma(\tau) > t$, which contradicts $\sigma_n^-(\tau_n^-) \leq t_n \to t$. Thus $\alpha_n^-(\tau_n^-) \to \alpha^-(\tau)$ meaning that $\gamma_n^-(t_n) \to \gamma^-(t) = \gamma(t)$. We have proven the statement about Φ and thus our lemma. □

3. CTRW limit theorems

Triangular arrays. Following [21], we define the triangular array

$$\Delta := \{(J_i^n, W_i^n); n, i \in \mathbb{N}\} \tag{3.1}$$

of $\mathbb{R}^d$-valued jumps J_i^n and $\mathbb{R}^+$-valued waiting times W_i^n, which are random variables on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. For every $n \in \mathbb{N}$, the sequence $\{(J_i^n, W_i^n)\}_{i\in\mathbb{N}}$ is assumed i.i.d.; note however that we do not assume that J_i^n and W_i^n are independent. We define the sequence of row sum processes $(B^n, S^n) := \{(B_t^n, S_t^n)\}_{t\geq 0}$ via

$$(B_t^n, S_t^n) := \sum_{k=1}^{\lfloor nt \rfloor} (J_k^n, W_k^n) \tag{3.2}$$

and note that all these processes start at $(0, 0) \in \mathbb{R}^d \times \mathbb{R}^+$. A standard argument involving finite collections of coordinate projections shows that (B^n, S^n), seen as a map from $\tilde{\Omega}$ to $\mathbb{D}(\mathbb{R}^d \times \mathbb{R}^+)$, is measurable, i.e. an rcll path-valued random variable. We write $\mathbb{P}_n$ for its distribution, which is a probability measure on $\mathbb{D}(\mathbb{R}^d \times \mathbb{R}^+)$, and note that $\mathbb{P}_n(D_{\uparrow,u}) = 1$. Similarly to [21], we assume that the laws $\mathbb{P}^n$ converge weakly on $(\mathbb{D}(\mathbb{R}^d \times \mathbb{R}^+), J)$ to a limit law $\mathbb{P}$, which is the law of a Lévy process. Setting $(\Omega, \mathcal{F}^0) := (\mathbb{D}(\mathbb{R}^d \times \mathbb{R}^+), \mathcal{D}(\mathbb{R}^d \times \mathbb{R}^+))$ and letting

$$(B_t, S_t) : \mathbb{D}(\mathbb{R}^d \times \mathbb{R}^+) \to \mathbb{R}^d \times \mathbb{R}^+,$$
$$\omega \mapsto \omega(t),$$

for all $t \geq 0$, we produce the probability space $(\Omega, \mathcal{F}^0, \mathbb{P})$ on which $(B, S) = \{(B_t, S_t)\}_{t\geq 0}$ is a Lévy process. We let $\mathcal{F}_t^0$ be the σ-field generated by all (B_u, S_u) where $0 \leq u \leq t$, we denote the $\mathbb{P}$-completion of $\mathcal{F}^0$ and $\mathcal{F}_t^0$ by $\mathcal{F}$ and $\mathcal{F}_t$, and work on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ from now on. Observe that S is a subordinator. Finally, we make the following assumption for the rest of this paper:

$$\text{Neither } B \text{ nor } S \text{ is a compound Poisson process.} \tag{3.3}$$

As will become clear in the course of the paper, this condition guarantees that the limit processes X and Y below are *not* ordinary CTRWs, i.e. are not pure step processes with finitely many jumps on finite time intervals.

Continuous time random walks.

Definition 3.4. For each $n \in \mathbb{N}$, the process $N^n := \{N_t^n\}_{t\geq 0}$ given by

$$N_t^n = \max\{m \in \mathbb{N} \cup \{0\} : W_1^n + \cdots + W_m^n \leq t\}$$

is called a *renewal process*. Moreover, the processes X, Y, G and D given by

$$(X_t^n, G_t^n) = \sum_{k=1}^{N_t^n} (J_k^n, W_k^n), \qquad (Y_t^n, D_t^n) = \sum_{k=1}^{N_t^n+1} (J_k^n, W_k^n),$$

are called a *lagging CTRW*, a *leading CTRW*, the *last time of renewal* and the *next time of renewal*, respectively.

Lemma 3.5. *Let Φ and Ψ be as in Definition 2.2. Then*

$$\Phi \circ (B^n, S^n) = (X^n, G^n), \qquad \Psi \circ (B^n, S^n) = (Y^n, D^n), \quad n \in \mathbb{N}.$$

Proof. Let $L^n = (S^n)^{-1}$, and note that by definition of N_t^n, S_t^n and the generalized inverse, the paths of L^n are rcll step functions starting at $1/n$ with jumps of size $1/n$ occurring at the end of each renewal epoch; this yields $nL_t^n = N_t^n + 1$, and the second equality follows.

Turning to the first statement, note that the left-hand limits of (B^n, S^n) and L^n satisfy

$$(B_{t-}^n, S_{t-}^n) = \sum_{k=1}^{\lceil nt-1 \rceil} (J_k^n, W_k^n), \qquad nL_{t-}^n = N_{t-}^n + 1.$$

From the formula $(\Phi(\beta, \sigma))^- = \alpha^- \circ (\sigma^{-1})^-$, valid for all $\alpha = (\beta, \sigma) \in D_{\uparrow,u}$, it follows that

$$(\Phi \circ (B^n, S^n))(t-) = \sum_{k=1}^{N_{t-}^n} (J_k^n, W_k^n) = (X_{t-}^n, G_{t-}^n).$$

Taking right-hand limits again yields the first equality. □

We have prepared everything for the proof of the first main result.

Theorem 3.6. *Define $(X, G) = \Phi(B, S)$ and $(Y, D) = \Psi(B, S)$. Then*

$$(X^n, G^n) \Rightarrow (X, G), \qquad (Y^n, D^n) \Rightarrow (Y, D),$$

where "$\Rightarrow$" denotes weak convergence in $(\mathbb{D}(\mathbb{R}^d \times \mathbb{R}^+), J)$ as $n \to \infty$.

Proof. By Lemma 2.1, $D_{\uparrow,u}$ is measurable, and we find that $\mathbb{P}(D_{\uparrow,u}) = \mathbb{P}^n(D_{\uparrow,u}) = 1$ for all $n \in \mathbb{N}$. Hence by [11, ch. 3, cor. 3.2] the restrictions of $\mathbb{P}^n$ to $D_{\uparrow,u}$ (endowed with the relative topology and Borel σ-field), converge weakly to the same restriction of $\mathbb{P}$. By Proposition 2.3, $D_{\uparrow\uparrow,u}$ is contained in the set of continuity points of Φ and Ψ, and $\mathbb{P}(D_{\uparrow\uparrow,u}) = 1$ by assumption (3.3). Then Lemma 3.5 and the continuous mapping theorem (see e.g. [6, p. 30]) yield the statement. □

Remark 3.7. Silvestrov and Teugels have shown the weak J-convergence of (Y^n, D^n) to (Y, D) for $d = 1$ in [29, th 3.4], using a compactness approach.

Remark 3.8. In our notation, Becker-Kern et al. [2, th 3.1] have shown the weak convergence

$$X^n \Rightarrow Y \quad \text{in } \mathbb{D}(\mathbb{R}^d)$$

with respect to the weaker M_1-topology under the assumption

$$\mathrm{disc}(B) \cap \mathrm{disc}(S) = \emptyset \quad \text{a.s.},$$

where $\mathrm{disc}(B)$ and $\mathrm{disc}(S)$ denote the (random) sets of discontinuities of B and S, respectively. This assumption is typically only satisfied by *un*coupled CTRW limits. The following observation explains why Theorem 3.6 does not contradict their limit theorem (see also [15, Remark 4.4]).

Lemma 3.9. *Let* $(\beta, \sigma) \in D_{\uparrow\uparrow,u}$ *and write* $\Phi(\beta, \sigma) = (\xi, \iota)$, $\Psi(\beta, \sigma) = (\eta, \upsilon)$. *If* $\mathrm{disc}(\beta) \cap \mathrm{disc}(\sigma) = \emptyset$, *then* $\xi = \eta$.

Proof. Let $\ell = \sigma^{-1}$, and see that ℓ is continuous since σ is increasing. Then by definition of Φ and Ψ we have $\xi = (\beta^- \circ \ell)^+$ and $\eta = \beta \circ \ell$. Both paths start at the same point $\beta(0) \in \mathbb{R}^d$, hence it suffices to show that their left limits coincide at every $t > 0$. We find

$$\eta(t-) = \begin{cases} \beta^-(\ell(t)) & \text{if } \ell \text{ is left-increasing at } t \\ \beta(\ell(t)) & \text{otherwise} \end{cases}$$

where "left-increasing at t" means $s < t \Rightarrow \ell(s) < \ell(t)$. In the second case, $\ell(t) \in \mathrm{disc}(\sigma)$, hence $\ell(t) \notin \mathrm{disc}(\beta)$ by assumption, so that $\beta(\ell(t)) = \beta^-(\ell(t))$. Thus we have shown $\eta^- = \beta^- \circ \ell = ((\beta^- \circ \ell)^+)^- = \xi^-$. □

Remark 3.10. Under the assumption that B and S are independent, the process $B \circ L^-$ appears as a CTRW limit in [3]. The interesting generalization there is that S is *not* assumed to be strictly increasing. Convergence has only been shown for all finite dimensional distributions, and it is an interesting open question whether or not the rcll versions converge weakly on $\mathbb{D}(\mathbb{R}^d)$. The continuous mapping theorem does not give a simple answer in this case, as the mapping Φ is not continuous on $D_{u,\uparrow}$, even if the uniform convergence topology is chosen on the domain and the M_1-topology is chosen on the codomain. To see this, let (β_n, σ_n) and (β, σ) be given by

$$\beta_n(t) = \beta(t), \qquad \beta(t) = \begin{cases} 1, & 1 \le t < 2 \\ 0, & \text{else}, \end{cases}$$

$$\sigma_n(t) = \begin{cases} 2 - 1/n, & 0 \le t < 1 \\ 2 + 1/n, & 1 \le t < 2 + 1/n \\ t, & 2 + 1/n \le t \end{cases}, \qquad \sigma(t) = \begin{cases} 2, & 0 \le t < 2 \\ t, & 2 \le t. \end{cases}$$

Then $(\beta_n, \sigma_n) \to (\beta, \sigma)$ with the (strongest) uniform topology, and $\beta \circ \sigma^{-1} \equiv 0$, but $\beta_n \circ \sigma_n^{-1}(2) = 1$ for all n.

4. CTRW limit laws

Definition 4.1. The $\mathbb{R}^+$-valued processes A and R given by

$$A_t = t - G_t, \qquad R_t = D_t - t$$

are called the *age* and the *remaining lifetime*, respectively. The random set

$$\mathbf{M} = \{(t, \omega) \in \mathbb{R}^+ \times \Omega : t = S_u(\omega) \text{ for some } u \geq 0\}.$$

is called the *regenerative set*. Its ω-slices and t-slices are the sets

$$\mathbf{M}^\omega = \{t \in \mathbb{R}^+ : (t, \omega) \in \mathbf{M}\}, \qquad \mathbf{M}_t = \{\omega \in \Omega : (t, \omega) \in \mathbf{M}\}.$$

Note that for $\mathbb{P}$-almost every ω,

$$G_t(\omega) = \sup([0, t] \cap \mathbf{M}^\omega), \qquad D_t(\omega) = \inf([t, \infty) \cap \mathbf{M}^\omega).$$

The term "regenerative" refers to the property that the part of $\mathbf{M}$ to the right of any $D_t(\omega) \in \mathbf{M}^\omega$ has the same distribution as $\mathbf{M}$. This can be easily inferred from the strong Markov property of S; for details see e.g. [5].

The set $\mathbf{M}^\omega$ can be interpreted as the range of the path $S_\cdot(\omega)$. Almost every such path is increasing, rcll and has countably many jumps, hence the complement $(\mathbf{M}^\omega)^\complement$ can be written as a countable union of intervals of the form $[g_i, d_i)$; such intervals are commonly termed *contiguous to* $\mathbf{M}$. The collection of all such g_i (resp. d_i) defines the set $\mathbf{G}^\omega$ (resp. $\mathbf{D}^\omega$), which in turn defines a random set $\mathbf{G}$ (resp. $\mathbf{D}$). In what follows, if we apply a topological operation (e.g. "complement", "union" or "closure") to a random set $\subset \mathbb{R}^+ \times \Omega$, then we mean the application of this operation to all ω-slices $\subset \mathbb{R}^+$; for instance $\overline{\mathbf{M}} = \mathbf{M} \cup \mathbf{G}$.

Lemma 4.2. *Fix* $t \geq 0$.

i. *On* $\mathbf{M}$ *we have* $Y = X$ *and* $R = A \equiv 0$.
ii. $\mathbb{P}(\mathbf{G}_t) = \mathbb{P}(\mathbf{D}_t) = 0$.
iii. $\mathbb{P}(\{\omega : L_t(\omega) \in \operatorname{disc}(B, S)^\omega \setminus \operatorname{disc}(S)^\omega\}) = 0$.
iv. $\mathbb{P}((X, A)_t^- = (X, A)_t) = 1$, *i.e.* (X, A) *admits no fixed discontinuities.*
v. $(Y_t, D_t) - (X_t, G_t) = \Delta(B, S)_{L_t}$ *almost surely.*

Proof. i. For almost all $\omega \in \Omega$, every point in $\mathbf{M}^\omega$ is a right-limit point. Hence $L_\cdot(\omega)$ is right-increasing at $t \in \mathbf{M}^\omega$, i.e. $t < u \Rightarrow L_t < L_u$, and thus $(X_t, G_t) = (B, S)^-(L_{t+}) = (B, S)(L_t) = (Y_t, D_t)$. As, by definition, $G_t \leq t \leq D_t$, we have $G_t = t = D_t$, i.e. $A_t = R_t = 0$.
ii. $\mathbb{P}(\mathbf{G}_t) = 0$ has been shown in [5, proof of prop.1.9 (ii)]. Using a similar argument based on the compensation formula, we find that

$$\mathbb{P}(\mathbf{D}_t) = \mathbb{P}(\exists u \geq 0 : S_u^- < t, \Delta S_u = t - S_u^-) = \int_{[0,t)} \Pi(\{t - y\}) U(\mathrm{d}y),$$

where U and Π denote the 0-potential (or "renewal measure") and the Lévy-measure of the subordinator S, respectively. Since the integrand equals zero except at, at most countably many points and since U has no atoms [5, p. 10], the integral vanishes.
iii. We abbreviate $\mathbf{K} := \operatorname{disc}(B, S) \setminus \operatorname{disc}(S)$. For each $u \geq 0$, $\mathbf{K}_u$ is measurable with respect to the σ-field $\sigma(Z_v : v \geq 0)$ where the process Z is defined via

$$Z_t(\omega) := \int_{\{\|x\|<1, z=0\}} x \tilde{N}((0, t], d(x, z)) + \int_{\{\|x\| \geq 1, z=0\}} x N((0, t], d(x, z)).$$

Here N denotes the Poisson point process on $\mathbb{R}^+ \times (\mathbb{R}^d \times \mathbb{R}^+)$ of jumps of (B, S) and $\tilde{N}$ its compensated version. By the Lévy-Itō decomposition of (B, S) and by the Poisson process nature of the jumps, the processes Z and $(B, S) - (Z, 0)$ are independent. In particular, $\mathbf{K}$ is independent of S and hence of L. This yields

$$\begin{aligned}
\mathbb{P}(\{\omega : L_t(\omega) \in \mathbf{K}^\omega\}) &= \mathbb{E}[\mathbb{P}(\{\omega : L_t(\omega) \in \mathbf{K}^\omega\} | L_t)] \\
&= \int_0^\infty \mathbb{P}(L_t \in \mathrm{d}u)\mathbb{P}(\{\omega : u \in \mathbf{K}^\omega\} | L_t = u) \\
&= \int_0^\infty \mathbb{P}(L_t \in \mathrm{d}u)\mathbb{P}(\mathbf{K}_u).
\end{aligned}$$

Since the Lévy process Z has no fixed discontinuities, $\mathbb{P}(\mathbf{K}_u) = 0$ for all $u \geq 0$, and the integral vanishes.

iv. Observe that if $\omega \in \Omega$ is fixed then L is constant on every open interval which is contiguous to $\overline{\mathbf{M}}$. Hence the same is true for (X, G), and thus $\mathrm{disc}(X, A) \cap \overline{\mathbf{M}}^{\complement} = \emptyset$. Thus we established

$$\mathrm{disc}(X, A) \subset \overline{\mathbf{M}} = \mathbf{G} \cup (\mathbf{M} \setminus \mathbf{D}) \cup \mathbf{D},$$

and according to part ii, it now suffices to show $\mathbb{P}((\mathrm{disc}(X, A) \cap (\mathbf{M} \setminus \mathbf{D}))_t) = 0$. Suppose $(t, \omega) \in \mathbf{M} \setminus \mathbf{D}$. Then t is both left- and right-limit point, and one consequence is that G and thus A are continuous at t. Another consequence is that $L.(\omega)$ is both left- and right-increasing at t, i.e. $s < t < u \Rightarrow L_s < L_t < L_u$. Since moreover L is continuous, and since $G^- = S^- \circ L$ is continuous at t, S must be continuous at L_t. Item iii then implies that B is continuous at L_t a.s., and hence $X^- = B^- \circ L$ and X are both continuous at $t \in \mathbf{M} \setminus \mathbf{D}$ a.s.

v. By definition, $(Y_t, D_t) - (X_t^-, G_t^-) = \Delta(B, S)_{L_t}$, so the statement follows from item iv and $A_t = t - G_t$. $\square$

Lévy-characteristics. For a bounded Borel measure ν on $\mathbb{R}^d \times \mathbb{R}^+$ we define its *Fourier–Laplace transform* (FLT) as the map

$$\mathbb{R}^d \times \mathbb{R}^+ \ni (k, s) \mapsto \int_{\mathbb{R}^d \times \mathbb{R}^+} \exp(\mathrm{i}\langle x, k\rangle - st)\nu(\mathrm{d}x, \mathrm{d}t) \in \mathbb{C},$$

with $\langle \cdot, \cdot \rangle$ denoting the inner product in $\mathbb{R}^d$. Lemma 2.1 in [2], which is derived from a general theorem on integral transforms on semigroups in [25], states that for $t \geq 0$ the FLT of the law of (B_t, S_t) is $\exp(-t\psi(k, s))$, where

$$\psi(k, s) = \mathrm{i}\langle b, k\rangle + \gamma s + \sigma^2(k) + \int \left(1 - \exp(\mathrm{i}\langle x, k\rangle - st) + \frac{\mathrm{i}\langle k, x\rangle}{1 + \|x\|^2}\right) \Pi(\mathrm{d}x, \mathrm{d}t)$$

for some $(b, \gamma) \in \mathbb{R}^d \times \mathbb{R}^+$, a quadratic form σ^2 on $\mathbb{R}^d$ and a Borel measure Π on $\mathbb{R}^d \times \mathbb{R}^+ \setminus \{(0, 0)\}$ which assigns finite mass to sets bounded away from $(0, 0)$ and satisfies

$$\int_{0 < t + \|x\|^2 < 1} (\|x\|^2 + t)\Pi(\mathrm{d}x, \mathrm{d}t) < \infty.$$

The distribution of (B_1, S_1) uniquely determines b, γ, σ^2 and Π, which are called the *drift of* B, the *drift of* S, the *diffusivity of* B and the *jump measure* of (B, S), respectively. We write $\overline{\Pi}(a) := \Pi(\mathbb{R}^d \times (a, \infty))$, $a \geq 0$, for the *tail function* of Π. Note that assumption (3.3) is then equivalent to "$\overline{\Pi}(0) = \infty$ or $\gamma > 0$". The *potential measure* U of (B, S) is the unique measure on $\mathbb{R}^d \times \mathbb{R}^+$ which satisfies

$$\int f(b,s)U(\mathrm{d}b, \mathrm{d}s) = \mathbb{E}\left[\int_0^\infty f((B,S)_t)\mathrm{d}t\right], \quad f \in p\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^+)$$

where $p\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^+)$ is the set of all real-valued $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^+)$-measurable functions which only attain non-negative values.

For fixed $t \geq 0$, we want to find the joint law of (X_t, A_t, Y_t, R_t). As we have seen that (X_t, A_t) does not admit any fixed discontinuities, this law follows easily from the joint law of $(X_t^-, A_t^-, Y_t, R_t) = (B_{L_t}^-, t - S_{L_t}^-, B_{L_t}, S_{L_t} - t)$. Observing that the Lévy process (B, S) can be viewed as a special case of a Markov additive process [9], we can find this law in [10] (or in [19], where an easier proof is given); however, only for t outside of some Lebesgue nullset. With a few additional assumptions on the regularity of U, we show that the formula in the latter reference holds true for *all* $t \geq 0$.

Proposition 4.3. *Suppose that $\gamma > 0$, that the laws of (B_t, S_t) are Lebesgue absolutely continuous for $t > 0$, and that the Lebesgue density $u(\cdot,\cdot)$ of U is continuous. Then*

$$\mathbb{P}(Y_t \in C, R_t = 0) = \gamma \int_C u(b,t)\mathrm{d}b, \quad C \in \mathcal{B}(\mathbb{R}^d).$$

Proof. Note that absolute continuity of the laws (B_t, S_t) $(t > 0)$ implies absolute continuity of U [27, 41.13]. The marginal $U(\mathbb{R}^d, \cdot)$ of U is the potential of the subordinator S and is usually called a "renewal measure". Its Laplace transform satisfies

$$\int_0^\infty \mathrm{e}^{-sx} U(\mathbb{R}^d, \mathrm{d}x) = \frac{1}{\psi(0,s)} \sim \frac{1}{\gamma s}, \quad (s \to \infty).$$

A Tauberian theorem then implies that

$$U(\mathbb{R}^d \times [0,\varepsilon]) \sim \varepsilon/\gamma, \quad (\varepsilon \to 0+). \tag{4.4}$$

For any $\delta > 0$ define $B_\delta = \{x \in \mathbb{R}^d : \|x\| < \delta\}$. Then using $L_\varepsilon \leq \varepsilon/\gamma$ and the stochastic continuity of B at 0 we have

$$U(B_\delta^\complement \times [0,\varepsilon]) = \mathbb{E}\left[\int_0^{L_\varepsilon} \mathbf{1}\{B_t \geq \delta\}\right] \leq \int_0^{\varepsilon/\gamma} \mathbb{P}(B_t \geq \delta)\mathrm{d}t = o(\varepsilon).$$

Without loss of generality, let $C \subset \mathbb{R}^d$ be open; then (4.4) can be refined to

$$\lim_{\varepsilon \downarrow 0} \frac{U(C \times [0,\varepsilon])}{\varepsilon} = \frac{\mathbf{1}\{0 \in C\}}{\gamma} \tag{4.5}$$

if $0 \notin \partial C$. The continuity of u and an application of the Markov property at the time L_t yield

$$\begin{aligned}
\int_C u(b,t)\mathrm{d}b &= \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} U(C \times (t, t+\varepsilon]) \\
&= \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \mathbb{E}\left[\int_{L_t}^\infty \mathbf{1}\{(B,S)_u \in C \times (t, t+\varepsilon]\}\mathrm{d}u\right] \\
&= \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \int_{\mathbb{R}^d \times [t,t+\varepsilon]} \mathbb{P}((B,S)_{L_t} \in (\mathrm{d}b, \mathrm{d}s)) U((C-b) \times [0, t+\varepsilon-s)).
\end{aligned}$$

The last displayed integral, if instead taken over the set $\mathbb{R}^d \times (t, t+\varepsilon]$, is bounded by

$$\mathbb{P}(S_{L_t} \in (t, t+\varepsilon])U(\mathbb{R}^d \times [0, \varepsilon)),$$

which is of order $o(\varepsilon)$ by (4.4). Hence, by dominated convergence and (4.5),

$$\int_C u(b,t)\mathrm{d}b = \int_{\mathbb{R}^d \times \{t\}} \mathbb{P}((B,S)_{L_t} \in (\mathrm{d}b, \mathrm{d}s))\gamma^{-1}\mathbf{1}\{b \in C\} = \gamma^{-1}\mathbb{P}(Y_t \in C, D_t = t),$$

where we have assumed without loss of generality that the boundary of C has Lebesgue measure 0. □

Proposition 4.6. *Let $f \in p\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^+)$. Then*

$$\mathbb{E}[f(X_t, A_t, Y_t, R_t)\mathbf{1}\{R_t > 0\}] = \int_{\mathbb{R}^d \times [0,t]} U(\mathrm{d}y, \mathrm{d}s) \int_{\mathbb{R}^d \times [t-s, \infty)} \Pi(\mathrm{d}\xi, \mathrm{d}\eta) f(y, t-s, y+\xi, s+\eta-t). \tag{4.7}$$

Proof. An application of the compensation formula [26, prop. XII.1.10] yields

$$\mathbb{E}\left[\sum_{s>0} F((B,S)_s^-, \Delta(B,S)_s)\right] = \mathbb{E}\left[\int_0^\infty \mathrm{d}s \int F((B,S)_s^-, (\xi,\eta))\Pi(\mathrm{d}\xi, \mathrm{d}\eta)\right], \tag{4.8}$$

valid for all $F \in p\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^+)$ such that $F(\cdot, \cdot, 0, 0) = 0$. Letting

$$F((x,z),(\xi,\eta)) = f(x, t-z, x+\xi, z+\eta-t)\mathbf{1}\{z \le t, z+\eta > t\},$$

we find that $s = L_t$ yields the only possibly non-zero summand. After an application of Lemma 4.2(iv) we can match the left sides of (4.7) and (4.8). The Lévy process (B, S) has no fixed discontinuities, and a short calculation shows that the right sides are equal as well. □

We are now ready to give our second main result.

Theorem 4.9. *Suppose that for $t > 0$ the laws of (B_t, S_t) are Lebesgue absolutely continuous. Then we have for $t \ge 0$ and $f \in p\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^+)$, we have*

$$\mathbb{E}[f(X_t, A_t, Y_t, R_t)] = \gamma \int f(y, 0, y, 0)u(\mathrm{d}y, t) + \int_{\mathbb{R}^d \times [0,t]} U(\mathrm{d}y, \mathrm{d}s) \times \int_{\mathbb{R}^d \times [t-s,\infty)} \Pi(\mathrm{d}\xi, \mathrm{d}\eta) f(y, t-s, y+\xi, s+\eta-t). \tag{4.10}$$

Proof. Observe that $\{(t,\omega) : R_t(\omega) = 0\} = \mathbf{M}$. Combining Lemma 4.2(i), Proposition 4.3 and Proposition 4.6 then yields the above formulae. □

Note that the joint law of (X_t, A_t) admits the slightly simpler formula

$$\mathbb{E}[f(X_t, A_t)] = \gamma \int f(y, 0)u(\mathrm{d}y, t) + \int_{\mathbb{R}^d \times [0,t]} U(\mathrm{d}y, \mathrm{d}s) f(y, t-s)\bar{\Pi}(t-s).$$

Moreover, note that in the uncoupled case the jump measure Π is supported by $\mathbb{R}^d \times \{0\} \cup \{0\} \times \mathbb{R}^+$, whence the integration variable ξ in (4.10) vanishes and X_t and Y_t have the same law, which agrees with Lemma 3.9.

Remark 4.11. The law of X_t has appeared in [2, th. 4.1] and [21, th. 3.6]; however, it has been overlooked that in general X_t is not equal to $B \circ L(t) = Y_t$. Moreover, we do not need to impose any growth condition on $\overline{\Pi}(t)$, and we have relaxed assumptions from "$\overline{\Pi}(0) = \infty$ and $\gamma = 0$" to "$\overline{\Pi}(0) = \infty$ or $\gamma > 0$". We believe that Theorem 4.9 holds true also in the case "$\overline{\Pi}(0) < \infty$ and $\gamma = 0$". But then X and Y are essentially CTRWs and not CTRW limit processes, and we do not investigate this any further.

5. Concluding remarks

The main result of this paper established that coupled leading CTRWs (jumps precede waiting times) and coupled lagging CTRWs (waiting times precede jumps) have different limit processes. In a very recent related study Jurlewicz et al. [15] determined the governing equations for the limits of leading CTRWs with infinite mean waiting times and showed that these differ from the governing equations for limits of lagging CTRWs.

Most of the recent mathematical literature on CTRW limits is concerned with jumps and waiting times which are homogeneous in both time and space [3,2,4,15,21]. For recent progress with space-dependent resp. space- and time-dependent jumps we refer the reader to [17] resp. [13].

Finally, we point out that CTRWs with infinite mean waiting times are models for physical processes which exhibit very slow relaxation and "ageing"; see [7,1], and also [8] for a mathematical account. We have yet to connect the age process A with the ageing phenomenon in our future work.

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