SPA: 3487 Model 1 pp. 1–40 (col. fig: NIL)

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Stochastic Processes and their Applications xxx (xxxx) xxx

stochastic processes and their applications

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Estimates of Dirichlet heat kernel for symmetric Markov processes

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Received 8 June 2017; received in revised form 4 September 2018; accepted 21 March 2019
Available online xxxx

Abstract

We consider a large class of symmetric pure jump Markov processes dominated by isotropic unimodal Lévy processes with weak scaling conditions. First, we establish sharp two-sided heat kernel estimates for these processes in $C^{1,1}$ open sets. As corollaries of our main results, we obtain sharp two-sided Green function estimates and a scale invariant boundary Harnack inequality with explicit decay rates in $C^{1,1}$ open sets.
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MSC: primary 31B25; 60J50; secondary 60J75; 60J35

Keywords: First exit time; Dirichlet heat kernel; Heat kernel; Markov process; Transition density; Green function; Boundary Harnack inequality

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[1] The first author was partially supported by National Science Centre (Poland): grant 2014/14/M/ST1/00600 and the Alexander von Humboldt Foundation.

[2] The second author was partially supported by DAAD PaJaKo Programme and the German Research Foundation (DFG) through the International Research Training Group (IRTG) 2235.

[3] This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government(MSIP) (No. NRF-2015R1A4A1041675).

https://doi.org/10.1016/j.spa.2019.03.017
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Please cite this article as: T. Grzywny, K.-Y. Kim and P. Kim, Estimates of Dirichlet heat kernel for symmetric Markov processes, Stochastic Processes and their Applications (2019), https://doi.org/10.1016/j.spa.2019.03.017.

SPA: 3487
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1. Introduction

The study of the heat kernel of a semigroup is a field of interactions between probability, analysis and geometry. The transition density function provides direct access to the path properties of a Markov process. In addition, it is the fundamental solution(or heat kernel) of the heat equation with an infinitesimal generator of the corresponding process. The Dirichlet heat kernel describes an operator with zero exterior conditions. For instance, the Green function and the solutions to Cauchy and Poisson problems with Dirichlet conditions are expressed by the heat kernel. In this paper, we consider a large class of symmetric pure jump Markov processes dominated by isotropic unimodal Lévy processes with weak scaling conditions. We estimate the transition density $p_D(t,x,y)$ of such Markov processes killed upon leaving an open set $D \subset \mathbb{R}^d$ with $C^{1,1}$ smoothness of the boundary. In other words, we establish sharp two-sided estimates of the Dirichlet heat kernel of the integro-differential operators with the maximum principle. Such operators are commonly used to model nonlocal phenomena [8,22,28,30,32,37]

The Dirichlet heat kernel estimates of the Laplacian (the Brownian motion) on the bounded $C^{1,1}$ domains were obtained in [23,29] (the upper bound) and [45] (the lower bound). See [20] for the Dirichlet heat kernel estimates for more general diffusions, and see [43] for bounds of the Dirichlet heat kernel of the Laplacian on the bounded Lipschitz domain.

For the fractional Laplacian, in 2010, Chen et al. [14] gave sharp (two-sided) explicit estimates for the Dirichlet heat kernel $p_D(t,x,y)$ of the fractional Laplacian in any $C^{1,1}$ open set D and over any finite time interval (see [3] for an extension to non-smooth open sets). When D is bounded, large-time Dirichlet heat kernel estimates can be deduced easily from short-time estimates using a spectral analysis.

The approach developed in [14] provides a road map for establishing sharp two-sided Dirichlet heat kernel estimates of other discontinuous processes, and the result has been generalized to more general stochastic processes: purely discontinuous symmetric Lévy processes [5,18], symmetric Lévy processes with Gaussian component [13], symmetric non-Lévy processes [15,33] and non-symmetric stable processes with gradient perturbation [16].

Let $\mathbb{P}_y(\tau_D > t)$ be the survival probability of the corresponding process and $p(t,x,y) = p_{\mathbb{R}^d}(t,x,y)$ be the (free) heat kernel for $D = \mathbb{R}^d$. Another form of two-sided heat kernel estimates is the following factorization;

$$c_1\mathbb{P}_x(\tau_D > t)\mathbb{P}_y(\tau_D > t)p(t,x,y) \leqslant p_D(t,x,y) \leqslant c_2\mathbb{P}_x(\tau_D > t)\mathbb{P}_y(\tau_D > t)p(t,x,y). \tag{1.1}$$

In fact, (1.1) holds for more general sets such as the Lipschitz open set. See [3,18]. See [6] for a direct approach to obtain sharp estimates of the survival probabilities of unimodal Lévy processes.

Extensions of the result in [14] were obtained for a quite large class of symmetric Lévy processes, including general unimodal Lévy processes with Lévy densities satisfying weak scaling conditions, in [5,18]. However, the extension to symmetric Markov processes with jumping kernels satisfying similar weak scaling conditions is unknown. In this paper, we extend the results of [5] and [33] to more general processes that are non-isotropic and non-Lévy. Our results cover not only a large class of symmetric Markov processes with jumping kernels satisfying weak scaling conditions, but also a large class of symmetric Markov processes with jumping kernels that decay exponentially with the damping exponent $\beta \in (0,\infty)$, and symmetric finite range Markov processes.

Please cite this article as: T. Grzywny, K.-Y. Kim and P. Kim, Estimates of Dirichlet heat kernel for symmetric Markov processes, Stochastic Processes and their Applications (2019), https://doi.org/10.1016/j.spa.2019.03.017.

For two non-negative functions f and g, the notation $f \asymp g$ means that there are positive constants c_1 and c_2 such that $c_1 g(x) \leqslant f(x) \leqslant c_2 g(x)$ in the common domain of the definition of f and g. We use the symbol ":=", which is read as "is defined to be". For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. We use dx to denote the Lebesgue measure in $\mathbb{R}^d$. For a Borel set $A \subset \mathbb{R}^d$, we use $|A|$ to denote its Lebesgue measure.

For $0 < \underline{\alpha} \leqslant \overline{\alpha} < 2$, let ϕ be an increasing function on $[0, \infty)$ satisfying that there exist positive constants $\underline{c} \leqslant 1$ and $1 \leqslant \overline{C}$ such that

(WS)

$$\underline{c}\left(\frac{R}{r}\right)^{\underline{\alpha}} \leqslant \frac{\phi(R)}{\phi(r)} \leqslant \overline{C}\left(\frac{R}{r}\right)^{\overline{\alpha}} \qquad \text{for } \ 0 < r \leqslant R.$$

Using this ϕ, we define

$$\nu(r) := \frac{1}{\phi(r) r^d} \qquad \text{for } r > 0. \tag{1.2}$$

Note that according to **(WS)** and (1.2), there exists $c = c(\overline{\alpha}, \overline{C}, d)$ such that

$$\nu(r) \leqslant c\, \nu(2r) \qquad \text{for any } \ r > 0. \tag{1.3}$$

A measure on $\mathbb{R}^d$ is called *isotropic unimodal*, if it is absolutely continuous on $\mathbb{R}^d \setminus \{0\}$ with a radial non-increasing density function and a Lévy process is called *isotropic unimodal*, if the one-dimensional distributions are unimodal. Since **(WS)** implies

$$\int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu(|x|)dx \leqslant c\left(\int_0^1 s^{-\overline{\alpha}+1} ds + \int_1^\infty s^{-\underline{\alpha}-1}\right) ds < \infty,$$

$\nu(dx) := \nu(|x|)dx$ is unimodal Lévy measure, and by Proposition in [44] there exists a pure jump isotropic unimodal Lévy process Z corresponding to ν.

Throughout this paper, we assume that $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)$ is a symmetric measurable function and there exists $L_0 > 1$ such that

$$L_0^{-1} \leqslant \kappa(x, y) \leqslant L_0, \quad x, y \in \mathbb{R}^d. \tag{1.4}$$

Let $J : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)$ be a symmetric measurable function, which is the jumping kernel of our process. We consider two sets of conditions on J. The first set is as follows:

(J1) **(J1.1)** $J(x, y) = \kappa(x, y)\nu(|x - y|)$ on $|x - y| \leqslant 1$,
(J1.2) $\sup_{x\in\mathbb{R}^d} \int_{|x-y|>1} J(x, y)dy < \infty$,
(J1.3) For any $M > 0$, there exists $C_M > 1$ such that $C_M^{-1}\nu(|x - y|) \leqslant J(x, y) \leqslant C_M \nu(|x - y|)$ for $|x - y| < M$.

The constant 1 in the condition **(J1.1)** plays no special role, and it can be changed to any small positive real number. Since **(J1.3)** implies **(J1.1)**, J satisfies conditions **(J1.2)** and **(J1.3)** if and only if J satisfies condition **(J1)**.

For the second set of conditions on J, let χ be a non-decreasing function on $(0, \infty)$ with $\chi(r) \equiv \chi(0)$, $r \in (0, 1]$, and there exists γ_1, γ_2, L_1 $L_2 > 0$ and $\beta \in [0, \infty]$ such that

$$L_1 e^{\gamma_1 r^\beta} \leqslant \chi(r) \leqslant L_2 e^{\gamma_2 r^\beta}, \quad r > 1. \tag{1.5}$$

Then, the second condition on J is as follows:

(J2) $J(x, y) = \kappa(x, y)\nu(|x - y|)\chi(|x - y|)^{-1}, \quad x, y \in \mathbb{R}^d$,

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which is equal to

$$\begin{cases} \kappa(x,y)\left(\phi(|x-y|)|x-y|^d \cdot \chi(|x-y|)\right)^{-1} & \text{if } \beta \in [0,\infty), \\ \kappa(x,y)\left(\phi(|x-y|)|x-y|^d\right)^{-1} \mathbf{1}_{\{|x-y|\leqslant 1\}} & \text{if } \beta = \infty. \end{cases}$$

Clearly **(J2)** implies **(J1.1)** and **(J1.2)**. Moreover, if **(J2)** holds and $\beta \neq \infty$, then **(J1)** holds.

We consider the Dirichlet form $(\mathcal{E}, \mathcal{F})$ associated with the jumping kernel J:

$$\mathcal{E}(u,v) := \frac{1}{2}\int\int (u(x)-u(y))(v(x)-v(y))J(x,y)dxdy,$$

and $\mathcal{F} := \{u \in L^2(\mathbb{R}^d) : \mathcal{E}(u,u) < \infty\}$. Under conditions **(J1.1)** and **(J1.2)**, according to [40, Theorem 2.1] and [41, Theorem 2.4], $(\mathcal{E}, \mathcal{F})$ is a regular (symmetric) Dirichlet form on $L^2(\mathbb{R}^d, dx)$. Moreover, the corresponding Hunt process Y is conservative and Y has the Hölder continuous transition density $p(t,x,y)$ on $MN(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ (see [11]).

Now, we state the estimates of the transition density $p(t,x,y)$ of Y with the jumping intensity kernel J satisfying either the conditions **(J1.2)** and **(J1.3)**, or the condition **(J2)**. The proof of the upper bound of Theorem 1.1 is almost the same as that of [18, (2.6)] using the condition **(J1.3)** instead of that of [18, (1.5)]. Thus, we skip the proof of the upper bound. The proof of the lower bound is given in Section 6.

Theorem 1.1. *Suppose that Y is a symmetric pure jump Hunt process whose jumping intensity kernel J satisfies the conditions* **(J1.2)** *and* **(J1.3)**. *Then, for each $M > 0$ and $T > 0$, there is a positive constant $C_{1.1} \geqslant 1$ that depends on ϕ, L_0, M and T such that for every $(t,x,y) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ with $|x-y| < M$, the function $p(t,x,y)$ has the following estimates:*

$$C_{1.1}^{-1}\left([\phi^{-1}(t)]^{-d} \wedge t\nu(|x-y|)\right) \leqslant p(t,x,y) \leqslant C_{1.1}\left([\phi^{-1}(t)]^{-d} \wedge t\nu(|x-y|)\right)$$

where $\phi^{-1}(t)$ is the inverse function of $\phi(t)$.

For notational convenience, for each $a, \gamma, T > 0$, we define a function $F_{a,\gamma,T}(t,r)$ on $(0,T] \times [0,\infty)$ as

$$F_{a,\gamma,T}(t,r) := \begin{cases} [\phi^{-1}(t)]^{-d} \wedge t\nu(r)e^{-\gamma r^\beta} & \text{if } \beta \in [0,1], \\ [\phi^{-1}(t)]^{-d} \wedge t\nu(r) & \text{if } \beta \in (1,\infty] \text{ with } r < 1, \\ t\exp\left\{-a\left(r\left(\log\frac{Tr}{t}\right)^{\frac{\beta-1}{\beta}} \wedge r^\beta\right)\right\} & \text{if } \beta \in (1,\infty) \text{ with } r \geqslant 1, \\ (t/(Tr))^{ar} = \exp\left\{-ar\left(\log\frac{Tr}{t}\right)\right\} & \text{if } \beta = \infty \text{ with } r \geqslant 1. \end{cases} \tag{1.6}$$

Theorem 1.2. *Suppose that Y is a symmetric pure jump Hunt process whose jumping intensity kernel J satisfies the condition* **(J2)**. *Then, the process Y has a continuous transition density function $p(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$. For each $T > 0$, there are positive constants $C_{1.2} \geqslant 1$, c_1 and $c_2 \geqslant 1$ that depend on ϕ, L_0, β, χ and T such that for every $t \in (0,T]$ the function $p(t,x,y)$ has the following estimates:*

$$c_2^{-1}F_{c_1,\gamma_2,T}(t,|x-y|) \leqslant p(t,x,y) \leqslant c_2 F_{C_{1.2},\gamma_1,T}(t,|x-y|).$$

Theorem 1.2 for the case $\beta \in (1,\infty]$ is basically derived from [12, Theorems 1.2 and 1.4]. Despite using 1 instead of T, the proof is the same. When $\beta \in [0,1]$, the upper bound in Theorem 1.2 is derived from [31, Theorem 2, Proposition 1]. The lower bound in Theorem 1.2

is proved as a special case of the preliminary lower bound on the heat kernel of the killed process in Section 6.

For any open set $D \subset \mathbb{R}^d$, the *first exit time of D* by process Y is defined by the formula $\tau_D := \inf\{t > 0 : Y_t \notin D\}$ and we use Y^D to denote the process obtained by killing process Y upon exiting D. The strong Markov property is used to easily verify that $p_D(t,x,y) := p(t,x,y) - \mathbb{E}_x[p(t-\tau_D, Y_{\tau_D}, y); t > \tau_D]$ is the transition density of Y^D. Using the continuity and estimate of p, it is routine to show that $p_D(t,x,y)$ is symmetric and continuous (e.g., see the proof of Theorem 2.4 in [21]).

Let $D \subset \mathbb{R}^d$ (when $d \geqslant 2$) be a $C^{1,1}$ open set, that is, there exists a localization radius $R_0 > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$ there exists a $C^{1,1}$-function $\varphi = \varphi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi(0) = 0$, $\nabla\varphi(0) = (0, \dots, 0)$, $\|\nabla\varphi\|_\infty \leqslant \Lambda$, $|\nabla\varphi(x) - \nabla\varphi(w)| \leqslant \Lambda|x-w|$ and an orthonormal coordinate system CS_z of $z = (z_1, \dots, z_{d-1}, z_d) := (\widetilde{z}, z_d)$ with an origin at z such that $D \cap B(z, R_0) = \{y = (\tilde{y}, y_d) \in B(0, R_0) \text{ in } CS_z : y_d > \varphi(\widetilde{y})\}$. The pair (R_0, Λ) is called the $C^{1,1}$ characteristics of the open set D. Note that a $C^{1,1}$ open set D with characteristics (R_0, Λ) can be unbounded and disconnected, and the distance between two distinct components of D is at least R_0. By a $C^{1,1}$ open set in $\mathbb{R}$ with a characteristic $R_0 > 0$, we mean an open set that can be written as the union of disjoint intervals so that the infimum of the lengths of all these intervals is at least R_0 and the infimum of the distances between these intervals is at least R_0.

It is well known that if D is $C^{1,1}$ open set with the characteristics (R_0, Λ), then D satisfies the interior and exterior ball conditions with the characteristic $R_1 \leqslant R_0$. That is, there exist balls $B_1, B_2 \subset \mathbb{R}^d$ with radius R_1 such that $B_1 \subset D \subset B_2^c$ satisfying $\delta_{B_1}(x) = \delta_D(x) = \delta_{B_2}(x)$ for any $x \in B_1$. Throughout this paper, without loss of generality, we always assume that $R_0 = R_1$.

To obtain the sharp estimates of the exit distributions for Y (see Theorem 4.2), we need additional conditions for the regularity of κ and ϕ.

(K$_\eta$) There are $L_3 > 0$ and $\eta > \overline{\alpha}/2$ such that $|\kappa(x, x+h) - \kappa(x,x)| \leqslant L_3|h|^\eta$ for every $x, h \in \mathbb{R}^d$, $|h| \leqslant 1$.

Note that the condition **(K$_\eta$)** implies that

$$|\kappa(x+h_1, x+h_2) - \kappa(x,x)| \leqslant 2L_3(|h_1|^\eta + |h_2|^\eta), \qquad \text{for } |h_1|, |h_2| < 1.$$

(SD) $\phi \in C^1(0,\infty)$ and $r \to -\nu'(r)/r$ is decreasing.

(See Remark 1.4.)

Now, we state the following theorem, which is one of the main results of this paper. Let $\delta_D(x)$ be the distance between x and D^c, and let

$$\Psi(t,x) := \left(1 \wedge \sqrt{\frac{\phi(\delta_D(x))}{t}}\right). \tag{1.7}$$

Theorem 1.3. *Suppose that Y is a symmetric pure jump Hunt process whose jumping intensity kernel J satisfies the conditions* **(J1)**, **(SD)**, *and* **(K$_\eta$)**. *Suppose that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ). Then, for each $T > 0$, there exists $c_1 = c_1(\phi, L_0, L_3, \eta, R_0, \Lambda, T, d)$ and $c_2 = c_2(\phi, L_0, L_3, \eta, R_0, \Lambda, T, d, \mathrm{diam}(D)) > 0$ such that the transition density $p_D(t,x,y)$ of Y^D has the following estimates.*

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(1) *For any* $(t,x,y) \in (0,T] \times D \times D$, *we have*

$$c_1^{-1}\,\Psi(t,x)\,\Psi(t,y)\,p(t,x,y) \leqslant p_D(t,x,y) \leqslant c_1\,\Psi(t,x)\,\Psi(t,y)\,p(t,x,y)$$

(2) *For any* $(t,x,y) \in [T,\infty) \times D \times D$, *we have*

$$c_2^{-1}e^{-t\lambda^D}\sqrt{\phi(\delta_D(x))}\sqrt{\phi(\delta_D(y))} \leqslant p_D(t,x,y) \leqslant c_2 e^{-t\lambda^D}\sqrt{\phi(\delta_D(x))}\sqrt{\phi(\delta_D(y))},$$

where $-\lambda^D < 0$ *is the largest eigenvalue of the generator of* Y^D.

Remark 1.4. Conditions **(SD)** and **(WS)** hold for a large class of pure jump isotropic unimodal Lévy processes including all subordinate Brownian motions with weak scaling conditions (see (1.8)): let $W=(W_t,\mathbb{P}_x)$ be a Brownian motion in $\mathbb{R}^d$ and $S=(S_t)$ be an independent driftless subordinator with Laplace exponent φ_1. The Laplace exponent φ_1 is a Bernstein function with $\varphi_1(0+)=0$. Since φ_1 has no drift part, φ_1 can be written in the form

$$\varphi_1(\lambda) = \int_0^\infty (1-e^{-\lambda t})\,\mu(dt)\,.$$

Here μ is a σ-finite measure on $(0,\infty)$ satisfying $\int_0^\infty (t\wedge 1)\,\mu(dt) < \infty$. μ is called the Lévy measure of the subordinator S.

The subordinate Brownian motion $Z=(Z_t,\mathbb{P}_x)$ is defined by $Z_t = W_{S_t}$. The density of the Lévy measure of Z with respect to the Lebesgue measure is given by $x \to \nu_d(|x|)$ with

$$\nu_d(r) = \int_0^\infty (4\pi t)^{-d/2}\exp\left(-\frac{r^2}{4t}\right)\mu(t)dt, \qquad r \neq 0.$$

Thus, $r \to \nu_d(r)$ is smooth for $r>0$, and

$$-\frac{\nu_d'(r)}{r} = 2\pi\,\nu_{d+2}(r), \quad r>0$$

which is decreasing. Suppose that

$$\underline{c}\left(\frac{R}{r}\right)^{\underline{\alpha}/2} \leqslant \frac{\varphi_1(R)}{\varphi_1(r)} \leqslant \overline{C}\left(\frac{R}{r}\right)^{\overline{\alpha}/2} \qquad \text{for } 0<r\leqslant R. \tag{1.8}$$

Then, by [4, Theorem 26]

$$\nu_d(r) \asymp r^{-d}\varphi_1(r^{-2}).$$

Let $\widehat{\phi}(r) := r^{-d}\nu_d(r)^{-1}$ (so that $\nu_d(r) = \widehat{\phi}(r)^{-1}r^{-d}$). Then $\widehat{\phi}$ is smooth, and since $\widehat{\phi}(r) \asymp \varphi_1(r^{-2})^{-1}$, it satisfies **(WS)**.

When either D is unbounded or $\beta=\infty$, we need precise information on J, which is encoded in **(J2)**, for large $|x-y|$. Moreover, when $\beta \in (1,\infty]$, we need to impose an addition assumption for D in order to obtain the sharp lower bound of $p_D(t,x,y)$; We say that *the path distance in an open set* U *is comparable to the Euclidean distance with characteristic* λ_1 if for every x and y in U there is a rectifiable curve l in U that connects x to y such that the length of l is less than or equal to $\lambda_1|x-y|$. Clearly, such a property holds for all bounded $C^{1,1}$ connected open sets, $C^{1,1}$ connected open sets with compact complements, and connected open sets above graphs of $C^{1,1}$ functions.

Theorem 1.5. *Suppose that* Y *is a symmetric pure jump Hunt process whose jumping intensity kernel* J *satisfies the conditions* **(J2)**, **(SD)** *and* $(\mathbf{K}_\eta)$. *Suppose that* D *is a* $C^{1,1}$ *open set in* $\mathbb{R}^d$ *with characteristics* (R_0,Λ). *Then, for each* $T>0$, *the transition density* $p_D(t,x,y)$ *of* Y^D *has the following estimates.*

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(1) *There is a positive constant* $c_1 = c_1(\beta, \chi, \phi, L_0, L_3, \eta, R_0, \Lambda, T, d)$ *such that for all* $(t, x, y) \in (0, T] \times D \times D$ *we have*

$$p_D(t,x,y) \leqslant c_1\, \Psi(t,x)\Psi(t,y) \begin{cases} F_{C_{1.2}\wedge\gamma_1,\gamma_1,T}(t,|x-y|/6) & \text{if } \beta \in [0,\infty), \\ F_{C_{1.2},\gamma_1,T}(t,|x-y|/4) & \text{if } \beta = \infty, \end{cases}$$

where $C_{1.2}$ *is the constant in Theorem 1.2.*

(2) *There is a positive constant* $c_2 = c_2(\beta, \chi, \phi, L_0, L_3, \eta, R_0, \Lambda, T, d)$ *such that for all* $t \in (0, T]$ *we have*

$$\begin{aligned} p_D(t,x,y) &\geqslant c_2\, \Psi(t,x)\Psi(t,y) \\ &\times \begin{cases} [\phi^{-1}(t)]^{-d} \wedge t e^{-\gamma_2|x-y|^\beta}\nu(|x-y|) & \text{if } \beta \in [0,1], \\ [\phi^{-1}(t)]^{-d} \wedge t\nu(|x-y|) & \text{if } \beta \in (1,\infty) \text{ and } |x-y|<1, \\ & \text{or } \beta = \infty \text{ and } |x-y| \leqslant 4/5. \end{cases} \end{aligned}$$

(3) *Suppose, in addition, that the path distance in* D *is comparable to the Euclidean distance with characteristic* λ_1. *Then, there are positive constants* $c_i = c_i(\beta, \chi, \phi, L_0, L_3, \eta, R_0, \Lambda, T, d, \lambda_1)$, $i = 3, 4$, *such that if* $x, y \in D$ *and* $t \in (0, T]$, *we have*

$$p_D(t,x,y) \geqslant c_3\, \Psi(t,x)\Psi(t,y) \begin{cases} F_{c_4,\gamma_2,T}(t,|x-y|) & \text{if } \beta \in (1,\infty) \text{ and } |x-y| \geqslant 1, \\ F_{c_4,\gamma_2,T}(t,5|x-y|/4) & \text{if } \beta = \infty \text{ and } |x-y| \geqslant 4/5. \end{cases}$$

(4) *If* $\beta \in (1, \infty)$, *there is a positive constant* $c_5 = c_5(\beta, \chi, \phi, L_0, L_3, \eta, R_0, \Lambda, T, d)$ *such that for every* x, y *in the different components of* D *with* $|x - y| \geqslant 1$ *and* $t \in (0, T]$ *we have*

$$p_D(t,x,y) \geqslant c_5\, \Psi(t,x)\Psi(t,y) t e^{-\gamma_2(5|x-y|/4)^\beta}\nu(|x-y|).$$

(5) *Suppose in addition that* $\beta = \infty$ *and* D *is bounded and connected. Then the claim of Theorem 1.3 (2) holds.*

Recall that the Green function $G_D(x, y)$ of Y on D is defined as $G_D(x, y) = \int_0^\infty p_D(t, x, y)\, dt$. As an application of Theorems 1.3 and 1.5, we derive the sharp two sided estimate on the Green function $G_D(x, y)$ of Y on bounded $C^{1,1}$ open sets. For notational convenience, let

$$a(x,y) := \sqrt{\phi(\delta_D(x))}\sqrt{\phi(\delta_D(y))} \tag{1.9}$$

and

$$g(x,y) := \begin{cases} \dfrac{\phi(|x-y|)}{|x-y|^d}\left(1 \wedge \dfrac{\phi(\delta_D(x))}{\phi(|x-y|)}\right)^{1/2}\left(1 \wedge \dfrac{\phi(\delta_D(y))}{\phi(|x-y|)}\right)^{1/2} & \text{when } d \geqslant 2, \\ \dfrac{a(x,y)}{|x-y|} \wedge \left(\dfrac{a(x,y)}{\phi^{-1}(a(x,y))} + \left(\displaystyle\int_{|x-y|}^{\phi^{-1}(a(x,y))} \frac{\phi(s)}{s^2}\,ds\right)^+\right) & \text{when } d = 1 \end{cases}$$

where $x^+ := x \vee 0$.

Theorem 1.6. *Suppose that* D *is a bounded* $C^{1,1}$ *open set in* $\mathbb{R}^d$ *with characteristics* (R_0, Λ). *Let* Y *be a symmetric pure jump Hunt process whose jumping intensity kernel* J *satisfies* $(\mathbf{K}_\eta)$ *and* **(SD)**. *Suppose either (1) the jumping intensity kernel* J *satisfies the condition* **(J1)**, *or (2)* D *is connected and the jumping intensity kernel* J *satisfies the condition* **(J2)** *with* $\beta = \infty$. *Then for every* $(x, y) \in D \times D$, *we have* $G_D(x, y) \asymp g(x, y)$.

Please cite this article as: T. Grzywny, K.-Y. Kim and P. Kim, Estimates of Dirichlet heat kernel for symmetric Markov processes, Stochastic Processes and their Applications (2019), https://doi.org/10.1016/j.spa.2019.03.017.

Remark 1.7. When $d = 1$, if either $\underline{\alpha} < 1$ or $\overline{\alpha} > 1$, one can write the Green function estimates in simpler forms. (see, [18, Corollary 7.4 and Remark 7.5])

In addition, we obtain the *uniform and scale-invariant* boundary Harnack inequality with *explicit decay rates* in $C^{1,1}$ open sets as an application of Theorems 1.5 and 1.6(1). A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be *harmonic* in the open set D with respect to Y if for every open set $U \subset D$ whose closure is a compact subset of D, $\mathbb{E}_x[|f|(Y_{\tau_U})] < \infty$ for every $x \in U$ and

$$f(x) = \mathbb{E}_x[f(Y_{\tau_U})] \qquad \text{for every } x \in U. \tag{1.10}$$

It is said that f is regular harmonic in D with respect to Y if f is harmonic in D with respect to Y and (1.10) holds for $U = D$.

The next condition guarantees that $C_c^2(\mathbb{R}^d)$ is in the domain of the Feller generator.

(L) Y is Feller and there exists a function $q(r)$ such that $J(x, y) \leqslant q(|x - y|)$ and

$$\lim_{R\to\infty} \int_{|h|>R} q(|h|)dh = 0. \tag{1.11}$$

Note that if **(J2)** holds, then clearly (1.11) holds, and Y is Feller based on Theorem 1.2. Thus, condition **(L)** is weaker than condition **(J2)**.

The next condition on J is necessary for the boundary Harnack inequality to hold (see [7, Assumption C and Example 5.14]).

(C) For any $0 < r < R \leqslant 2$ there exists $C^* = C^*(\phi, d, r/R)$ such that for any $x_0 \in \mathbb{R}^d$, $x \in B(x_0, r)$ and $y \in B(x_0, R)^c$, $(C^*)^{-1} J(x_0, y) \leqslant J(x, y) \leqslant C^* J(x_0, y)$.

Note that when $\beta \in [0, 1]$, condition **(J2)** implies **(C)**. In contrast, when $\beta \in (1, \infty]$, the boundary Harnack inequality does not hold under condition **(J2)**.

Theorem 1.8. *Suppose D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ). Let Y be a symmetric pure jump Hunt process whose jumping intensity kernel J satisfies conditions* **(J1)**, **(L)**, **(C)**, **(K$_\eta$)**, *and* **(SD)**. *Then, there exists $c = c(\phi, L_0, L_3, \eta, \Lambda, d)$ such that for any $0 < r < R_0 \wedge 1$, $z \in \partial D$ and any non-negative function f in $\mathbb{R}^d$ that is regular harmonic in $D \cap B(z, r)$ with respect to Y, and vanishes in $D^c \cap B(z, r)$, we have*

$$\frac{f(x)}{f(y)} \leqslant c\sqrt{\frac{\phi(\delta_D(x))}{\phi(\delta_D(y))}} \qquad \text{for any } x, y \in D \cap B(z, r/2).$$

The rest of this paper is organized as follows. In Section 2, we solve a martingale-type problem for Y which yields a Dynkin-type formula. Section 3 deals with the isotropic Lévy process Z with Lévy measure $\nu(|x|)dx$. We compute some key upper bounds of the generator of Z on our testing function for $C^{1,1}$ open sets. In Section 4, we present the key estimates of exit distributions (Theorem 4.2). Section 5 contains the proof of the upper bound of $p_D(t, x, y)$. We use Meyer's construction when $|x - y| < c$. Then, by using Lemma 5.1 twice, we prove the upper bound of $p_D(t, x, y)$ without using the lower bound of $p(t, x, y)$. In Section 6 and Section 7, we prove the lower bound estimates for $p_D(t, x, y)$. First, we consider the case $\delta_D(x) \wedge \delta_D(y) \geqslant t^{1/\alpha}$; that is, x and y are kept away from the boundary of D. These results are presented in Section 6 and the key estimates of the exit distributions obtained in Section 4 are used in Section 7 to prove the lower bound for all $x, y \in D$. Finally, in Section 8, as an application of Theorem 4.2, we derive the Green function estimates and the uniform scale-invariant Boundary Harnack inequality with explicit decay rates in $C^{1,1}$ open sets.

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Throughout the rest of this paper, positive constants $L_0, L_1, L_2, L_3, \gamma_1, \gamma_2$ can be regarded as fixed. In the statements and the proofs of results, constants $c_i = c_i(a, b, c, \ldots)$, $i = 1, 2, 3, \ldots$, denote generic constants that depend on $a, b, c, \ldots$, the exact values of which are unimportant. These are given anew in each statement and each proof. The dependence of the constants on the dimension $d \geqslant 1$ is not be mentioned explicitly.

For a function space $\mathbb{H}(U)$ on an open set U in $\mathbb{R}^d$, we let $\mathbb{H}_c(U) := \{f \in \mathbb{H}(U) : f \text{ has compact support}\}$, $\mathbb{H}_0(U) := \{f \in \mathbb{H}(U) : f \text{ vanishes at infinity}\}$ and $\mathbb{H}_b(U) := \{f \in \mathbb{H}(U) : f \text{ is bounded}\}$.

2. Generator of *Y*

In this section, we assume that Y is the symmetric pure jump Hunt process with the jumping intensity kernel J satisfying the conditions **(J1.1)**, **(J1.2)** and **(K$_\eta$)**. Recall that these conditions imply that Y is strong Feller (see [11, Theorem 3.1]).

We define an operator $\mathcal{L}$ by

$$\mathcal{L}g(x) := P.V. \int (g(x+h) - g(x)) J(x, x+h) dh := \lim_{\varepsilon \downarrow 0} \mathcal{L}^\varepsilon g(x), \tag{2.1}$$

where

$$\mathcal{L}^\varepsilon g(x) := \int_{|h|>\varepsilon} (g(x+h) - g(x)) J(x, x+h) dh,$$

whenever these exist pointwise. Let $g \in C_c^2(\mathbb{R}^d)$ and $\varepsilon < r < 1$, then by **(J1.1)**, we have

$$\begin{aligned}
\mathcal{L}^\varepsilon g(x) &= \kappa(x,x) \int_{\varepsilon<|h|<r} (g(x+h) - g(x) - h \cdot \nabla g(x)) \nu(|h|) dh \\
&+ \int_{\varepsilon<|h|<r} (g(x+h) - g(x))(\kappa(x, x+h) - \kappa(x,x)) \nu(|h|) dh \\
&+ \int_{r \leqslant |h| \leqslant 1} (g(x+h) - g(x)) \kappa(x, x+h) \nu(|h|) dh \\
&+ \int_{1<|h|} (g(x+h) - g(x)) J(x, x+h) dh.
\end{aligned}$$

Since **(K$_\eta$)** holds, we have that

$$|(g(x+h) - g(x))(\kappa(x, x+h) - \kappa(x,x))| \leqslant \|\nabla g\|_\infty (L_3 + 2L_0) |h|^{\eta+1}, \quad x, h \in \mathbb{R}^d.$$

Since **(WS)** and the inequality $\eta > \overline{\alpha}/2 > \overline{\alpha} - 1$ imply $\int_{|h|<r} |h|^2 \nu(|h|) dh \leqslant \int_{|h|<r} |h|^{\eta+1} \nu(|h|) dh < \infty$, $\mathcal{L}g$ is well defined and $\mathcal{L}^\varepsilon g$ converges to $\mathcal{L}g$ locally uniformly on $\mathbb{R}^d$. Furthermore, for every $0 < r < 1$,

$$\begin{aligned}
\mathcal{L}g(x) &= \kappa(x,x) \int_{|h|<r} (g(x+h) - g(x) - h \cdot \nabla g(x)) \nu(|h|) dh \\
&+ \int_{|h|<r} (g(x+h) - g(x))(\kappa(x, x+h) - \kappa(x,x)) \nu(|h|) dh \\
&+ \int_{r \leqslant |h| \leqslant 1} (g(x+h) - g(x)) \kappa(x, x+h) \nu(|h|) dh \\
&+ \int_{1<|h|} (g(x+h) - g(x)) J(x, x+h) dh.
\end{aligned} \tag{2.2}$$

Now we are ready to prove the following lemma.

Please cite this article as: T. Grzywny, K.-Y. Kim and P. Kim, Estimates of Dirichlet heat kernel for symmetric Markov processes, Stochastic Processes and their Applications (2019), https://doi.org/10.1016/j.spa.2019.03.017.

Lemma 2.1. *There is $C_{2.1} = C_{2.1}(\phi, \eta, L_0, L_3) > 0$ such that for any function $g \in C_c^2(\mathbb{R}^d)$ and $0 < r < 1$,*

$$\|\mathcal{L}g\|_\infty \leqslant \frac{C_{2.1}}{\phi(r)} \left(r^2\|\partial^2 g\|_\infty + r^{\eta+1}\|\nabla g\|_\infty + \|g\|_\infty\right). \tag{2.3}$$

Proof. By (2.2), (1.4), $(\mathbf{K}_\eta)$ and **(J1.2)**, we obtain that

$$\begin{aligned}|\mathcal{L}g(x)| \leqslant c_0\Big(L_0\|\partial^2 g\|_\infty \int_0^r \frac{s}{\phi(s)}ds + L_3\|\nabla g\|_\infty \int_0^r \frac{s^\eta}{\phi(s)}ds \\ + 2L_0\|g\|_\infty \int_r^1 \frac{ds}{s\phi(s)}\Big) + c_1\|g\|_\infty.\end{aligned} \tag{2.4}$$

For $s \leqslant r$, since $\phi(r)/\phi(s) \leqslant \overline{C}(r/s)^{\overline{\alpha}}$ by **(WS)** and $\eta > \overline{\alpha}/2 > \overline{\alpha} - 1$, we have

$$\int_0^r \frac{s^\eta}{\phi(s)}ds \leqslant \frac{\overline{C}}{\phi(r)} \frac{1}{\eta + 1 - \overline{\alpha}} r^{\eta+1}. \tag{2.5}$$

For $r < s$, since $\underline{c}(s/r)^{\underline{\alpha}} \leqslant \phi(s)/\phi(r)$ by **(WS)**, we have

$$\int_r^\infty \frac{ds}{s\phi(s)} \leqslant \frac{\underline{c}^{-1}}{\phi(r)} r^{\underline{\alpha}} \int_r^\infty s^{-1-\overline{\alpha}}ds < (\underline{c}\,\underline{\alpha})^{-1}\frac{1}{\phi(r)}. \tag{2.6}$$

Applying (2.5) and (2.6) to (2.4), we conclude that (2.3) hold. □

Lemma 2.2. *For any $u \in C_c^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, there exists a $\mathbb{P}_x$-martingale M_t^u with respect to the filtration of Y such that*

$$M_t^u = u(Y_t) - u(Y_0) - \int_0^t \mathcal{L}u(Y_s)ds$$

$\mathbb{P}_x$-a.s. In particular, for any stopping time S with $\mathbb{E}_x S < \infty$ we have

$$\mathbb{E}_x u(Y_S) - u(x) = \mathbb{E}_x \int_0^S \mathcal{L}u(Y_s)ds. \tag{2.7}$$

Proof. Let $(A, D(A))$ be the L^2-generator of the semigroup T_t with respect to Y. Due to [40, Proposition 2.5], we have $C_c^2(\mathbb{R}^d) \subset D(A)$ and $A|_{C_c^2(\mathbb{R}^d)} = \mathcal{L}|_{C_c^2(\mathbb{R}^d)}$. Since T_t is strongly continuous (see [27, Theorem 1.3.1 and Lemma 1.3.2]) we have that for any $f \in D(A)$ and $t \geqslant 0$,

$$\left\|(T_t f - f) - \int_0^t T_s A f\, ds\right\|_{L^2} = 0$$

(see e.g. [24, Proposition 1.5]). Hence for $u \in C_c^2(\mathbb{R}^d)$,

$$T_t u(x) - u(x) = \int_0^t T_s \mathcal{L}u(x)ds, \quad a.e.\ x \in \mathbb{R}^d, \tag{2.8}$$

and $\mathcal{L}u$ is bounded by Lemma 2.1.

Let us denote $g_t(x) = \int_0^t T_s\mathcal{L}u(x)ds$. First, we show that $g_t \in C_b(\mathbb{R}^d)$, $t > 0$. Note that $g_t(x) \leqslant t\|\mathcal{L}u\|_\infty$. Hence, since T_ε is strong Feller for any $\varepsilon > 0$, we have $T_\varepsilon g_{t-\varepsilon} \in C_b(\mathbb{R}^d)$ for all $\varepsilon \in (0, t)$. Moreover,

$$|g_t(x) - T_\varepsilon g_{t-\varepsilon}(x)| = |g_\varepsilon(x)| \leqslant \varepsilon\|\mathcal{L}u\|_\infty.$$

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Hence, g_t is continuous and (2.8) holds for any $x \in \mathbb{R}^d$. This and Markov property imply that

$$M_t^u = u(Y_t) - u(Y_0) - \int_0^t \mathcal{L}u(Y_s)ds$$

is $\mathbb{P}_x$-martingale for any $x \in \mathbb{R}^d$. Since $|M_t^u| \leqslant 2\|u\|_\infty + t\|\mathcal{L}u\|_\infty$, by the optional stopping theorem (2.7) follows. □

Lemma 2.3. *There exists a constant $C_{2.3} = C_{2.3}(\phi, \eta, L_0, L_3) > 0$ such that, for any $r \in (0, 1]$, $x_0 \in \mathbb{R}^d$, and any stopping time S (with respect to the filtration of Y), we have*

$$\mathbb{P}_x(|Y_S - x_0| \geqslant r) \leqslant C_{2.3}\frac{\mathbb{E}_x[S]}{\phi(r)}, \qquad x \in B(x_0, r/2).$$

Proof. Fix $x_0 \in \mathbb{R}^d$. Since this lemma is clear for $\mathbb{E}_x[S] = \infty$, we consider the case that $\mathbb{E}_x[S] < \infty$ for $x \in B(x_0, r/2)$. Define a radial function $g \in C_c^\infty(\mathbb{R}^d)$ such that $-1 \leqslant g \leqslant 0$, with

$$g(y) := \begin{cases} -1, & \text{if } |y| < 1/2 \\ 0, & \text{if } |y| \geqslant 1. \end{cases}$$

Then,

$$\sum_{i=1}^d \left\| \frac{\partial}{\partial y_i} g \right\|_\infty + \sum_{i,j=1}^d \left\| \frac{\partial^2}{\partial y_i \partial y_j} g \right\|_\infty = c_1 < \infty.$$

For any $r \in (0, 1]$, define $g_r(y) = g(\frac{x_0 - y}{r})$ so that $-1 \leqslant g_r \leqslant 0$,

$$g_r(y) = \begin{cases} -1, & \text{if } |x_0 - y| < r/2 \\ 0, & \text{if } |x_0 - y| \geqslant r, \end{cases} \tag{2.9}$$

and

$$\sum_{i=1}^d \left\| \frac{\partial}{\partial y_i} g_r \right\|_\infty < c_1 r^{-1} \quad \text{and} \quad \sum_{i,j=1}^d \left\| \frac{\partial^2}{\partial y_i \partial y_j} g_r \right\|_\infty < c_1 r^{-2}. \tag{2.10}$$

By Lemma 2.1, there exists $c_2 = c_2(\underline{\alpha}, \underline{c}, \overline{\alpha}, \overline{C}, \eta, L_0, L_3) > 0$ such that for $0 < r < 1$,

$$\|\mathcal{L}g_r\|_\infty \leqslant \frac{c_2}{\phi(r)}. \tag{2.11}$$

Combining Lemma 2.2, (2.9) and (2.11), we find that for any $x \in B(x_0, r/2)$ with $\mathbb{E}_x S < \infty$, we have

$$\begin{aligned}
&\mathbb{P}_x(|Y_S - x_0| \geqslant r) = \mathbb{E}_x[1 + g_r(Y_S); |Y_S - x_0| \geqslant r] \\
&\leqslant \mathbb{E}_x[1 + g_r(Y_S)] = -g_r(x) + \mathbb{E}_x[g_r(Y_S)] = \mathbb{E}_x\left[\int_0^S \mathcal{L}g_r(Y_t)dt\right] \leqslant \|\mathcal{L}g_r\|_\infty \mathbb{E}_x[S] \\
&\leqslant c_2 \frac{\mathbb{E}_x[S]}{\phi(r)}. \quad \square
\end{aligned}$$

Recall that for any open set $D \subset \mathbb{R}^d$, $\tau_D = \inf\{t > 0 : Y_t \notin D\}$ denote the first exit time of D by the process Y_t.

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Corollary 2.4. *There exists a constant $C_{2.4} = C_{2.4}(\phi, \eta, L_0, L_3) > 0$ such that, for any $r \in (0, 1]$, $x_0 \in \mathbb{R}^d$, and any open sets U and D with $D \cap B(x_0, r) \subset U \subset D$, we have*

$$\mathbb{P}_x\left(Y_{\tau_U} \in D\right) \leqslant C_{2.4} \frac{\mathbb{E}_x[\tau_U]}{\phi(r)}, \qquad x \in D \cap B(x_0, r/2). \tag{2.12}$$

Proof. Since $D \setminus U \subset B(x_0, r)^c$, by Lemma 2.3 we have that for $x \in D \cap B(x_0, r/2)$

$$\mathbb{P}_x\left(Y_{\tau_U} \in D\right) \leqslant \mathbb{P}_x\left(|Y_{\tau_U} - x_0| \geqslant r\right) \leqslant C_{2.3} \frac{\mathbb{E}_x[\tau_U]}{\phi(r)}. \quad \square$$

3. Analysis on Z

Recall that Z is a pure jump isotropic unimodal Lévy process with Lévy measure $\nu(|x|)dx$. Moreover, we assume **(SD)** holds in this section. The Lévy–Khintchine (characteristic) exponent of Z has the form

$$\psi(|\xi|) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot x))\, \nu(|x|)dx, \quad \xi \in \mathbb{R}^d. \tag{3.1}$$

Let Z^d be the last coordinate of Z and $M_t = \sup_{s \leqslant t} Z^d_s$ and let L_t be the local time at 0 for $M_t - Z^d_t$, the last coordinate of Z reflected at the supremum. We consider its right-continuous inverse, L^{-1}_s which is called the ascending ladder time process for X^1_t. Define the ascending ladder-height process as $H_s = Z^d_{L^{-1}_s} = M_{L^{-1}_s}$. The Laplace exponent of H_s is

$$\kappa(\xi) = \exp\left\{\frac{1}{\pi}\int_0^\infty \frac{\log \psi(\theta\xi)}{1+\theta^2}\, d\theta\right\}, \quad \xi \geqslant 0.$$

(See [25, Corollary 9.7].) The renewal function V of the ascending ladder-height process H is defined as

$$V(x) = \int_0^\infty \mathbb{P}(H_s \leqslant x)ds, \quad x \in \mathbb{R}.$$

then $V(x) = 0$ if $x < 0$ and V is non-decreasing. Also V is subadditive (see [2, p. 74]), that is,

$$V(x+y) \leqslant V(x) + V(y), \quad x, y \in \mathbb{R}, \tag{3.2}$$

and $V(\infty) = \infty$. Since the distribution of Z^d_t is absolutely continuous for every $t > 0$ the resolvent measures of Z^d_t as well, (see [26, Theorem 6]), it follows by [42, Theorem 2] that $V(x)$ is absolutely continuous and harmonic on $(0, \infty)$ for the process Z^d_t. Also, V' is a positive harmonic function for Z^d_t on $(0, \infty)$, hence V is actually (strictly) increasing.

For $r > 0$, define Pruitt's function $h(r) = \int_{\mathbb{R}^d} 1 \wedge |z|^2 r^{-2} \nu(dz)$ (e.g., see [38]). By [4, Corollary 3] and [6, Proposition 2.4] (see also [2, p. 74]), we first note that

$$h(r) \asymp [V(r)]^{-2} \asymp \kappa(r^{-1}) \asymp \psi(r^{-1}) \qquad \text{for any} \quad r > 0.$$

Clearly, **(WS)** implies that $s \to \phi(s^{-1})^{-1}$ also satisfies **(WS)**, that is, using the notation in [4], $s \to \phi(s^{-1})^{-1} \in \mathrm{WLSC}(\underline{\alpha}, \underline{c}, 0) \cap \mathrm{WUSC}(\overline{\alpha}, \overline{C}, 0)$. So by (1.2) and [4, Proposition 28], we have that

$$\psi(r) \asymp \phi(r^{-1})^{-1} \qquad \text{for any} \quad r > 0.$$

Combining these observations, we conclude that

$$V(r) \asymp [\phi(r)]^{1/2} \quad \text{and} \quad \nu(r) \asymp [V(r)]^{-2} r^{-d} \qquad \text{for any} \quad r > 0. \tag{3.3}$$

So by **(WS)**, there exists $C_V := (\underline{c}, \overline{C}, d) > 1$ such that

$$C_V^{-1}\left(\frac{R}{r}\right)^{\underline{\alpha}/2} \leqslant \frac{V(R)}{V(r)} \leqslant C_V\left(\frac{R}{r}\right)^{\overline{\alpha}/2} \qquad \text{for any } 0<r\leqslant R. \tag{3.4}$$

Define $w(x) := V((x_d)^+)$ and $\mathbb{H} := \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$. Since the renewal function V is harmonic on $(0,\infty)$ for Z^d, by the strong Markov Property w is harmonic in $\mathbb{H}$ with respect to Z.

Proposition 3.1. *$x \to V(x)$ is twice-differentiable for any $x > 0$, and there exists $C_{3.1} > 0$ such that*

$$|V''(x)| \leqslant C_{3.1}\frac{V'(x)}{x\wedge 1} \quad \text{and} \quad V'(x) \leqslant C_{3.1}\frac{V(x)}{x\wedge 1}, \quad x>0.$$

Proof. Let $f((\widetilde{y}, x_d)) = V'((x_d)^+)$ for $\widetilde{y} \in \mathbb{R}^{d-1}$. Then f is harmonic in $\mathbb{H}$. The assumption [36, (A)] is satisfied by **(SD)**. Hence, by Theorem 1.1 therein, we get for any $x > 0$

$$\left|\frac{\partial}{\partial x_d} f((\widetilde{0}, x))\right| \leqslant C_{3.1}\frac{f((\widetilde{0}, x))}{x\wedge 1} \quad \text{and} \quad \frac{\partial}{\partial x_d} w(\widetilde{0}, x) \leqslant C_{3.1}\frac{w(\widetilde{0}, x)}{x\wedge 1}.$$

These imply the claim of proposition, because $V(x) = w(\widetilde{0}, x)$ and $V'(x) = f(\widetilde{0}, x)$, $x > 0$. □

Proposition 3.2. *For $\lambda > 0$, there exists $C_{3.2} = C_{3.2}(d, \lambda) > 0$ such that for any $r > 0$, we have*

$$\sup_{\{x\in\mathbb{R}^d : 0<x_d\leqslant\lambda r\}} \int_{B(x,r)^c} w(y)\nu(|x-y|)dy < \frac{C_{3.2}}{V(r)}. \tag{3.5}$$

Proof. Since $w(x + z) = V(x_d + z_d) \leqslant V(x_d) + V(|z|)$ for $x_d > 0$, it follows that

$$\begin{aligned}\int_{B(x,r)^c} w(y)\nu(|x-y|)dy &= \int_{B(0,r)^c} w(x+z)\nu(|z|)dz \\ &\leqslant V(x_d)\int_{B(0,r)^c}\nu(|z|)dz + \int_{B(0,r)^c} V(|z|)\nu(|z|)dz.\end{aligned}$$

By [6, (2.23) and Lemma 3.5], we have that

$$\sup_{\{x\in\mathbb{R}^d : 0<x_d\leqslant\lambda r\}} \int_{B(x,r)^c} w(y)\nu(|x-y|)dy \leqslant c_1\left(\frac{V(\lambda r)}{[V(r)]^2} + \frac{1}{V(r)}\right).$$

Since V is subadditive, $V(\lambda r) \leqslant (\lambda + 1)V(r)$ for any $\lambda > 0$, and therefore we conclude the result. □

For any function $f : \mathbb{R}^d \to \mathbb{R}$ and $x \in \mathbb{R}^d$, we define an operator as follows:

$$\mathcal{L}_Z f(x) := P.V.\int_{\mathbb{R}^d}(f(y)-f(x))\nu(|x-y|)dy := \lim_{\varepsilon\downarrow 0}\mathcal{L}_Z^{\varepsilon} f(x),$$

$$\mathcal{D}(\mathcal{L}_Z) := \left\{f \in C^2(\mathbb{R}^d) : P.V.\int_{\mathbb{R}^d}(f(y)-f(x))\nu(|x-y|)dy \text{ exists and is finite.}\right\},$$

where

$$\mathcal{L}_Z^{\varepsilon} f(x) := \int_{B(x,\varepsilon)^c}(f(y)-f(x))\nu(|x-y|)dy. \tag{3.6}$$

Recall that $C_0^2(\mathbb{R}^d)$ be the collection of C^2 functions in $\mathbb{R}^d$ vanishing at infinity. It is well known that $C_0^2(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{L}_Z)$ and that, by the rotational symmetry of Z, $A_Z|_{C_0^2(\mathbb{R}^d)} = \mathcal{L}_Z|_{C_0^2(\mathbb{R}^d)}$ where A_Z is the infinitesimal generator of Z(e.g. [39, Theorem 31.5]). Hence, we see that Dynkin formula holds for $\mathcal{L}_Z$: for each $g \in C_0^2(\mathbb{R}^d)$ and any bounded open subset U of $\mathbb{R}^d$ we have

$$\mathbb{E}_x \int_0^{\tau_U} \mathcal{L}_Z g(Z_t)dt = \mathbb{E}_x[g(Z_{\tau_U})] - g(x). \tag{3.7}$$

Theorem 3.3. *For any $x \in \mathbb{H}$, $\mathcal{L}_Z w(x)$ is well-defined and $\mathcal{L}_Z w(x) = 0$.*

Proof. By subadditivity of V, $|w(y) - w(x)| \leqslant V(|y_d - x_d|) \leqslant V(|x - y|)$ for $x \in \mathbb{H}$. By [6, Lemma 3.5], it follows that for any $\varepsilon \in (0, 1/2)$

$$\left| \int_{B(x,\varepsilon)^c} (w(y) - w(x))\nu(|x-y|)dy \right| \leqslant \int_{B(0,\varepsilon)^c} V(|z|)\nu(|z|)dz < \frac{c_1}{V(\varepsilon)} < \infty. \tag{3.8}$$

Thus $\mathcal{L}_Z^\varepsilon w(x)$ is well defined in $\mathbb{H}$ and

$$\mathcal{L}_Z^\varepsilon w(x) = \int_{B(x,\varepsilon)^c} (w(y) - w(x) - \mathbf{1}_{\{|x-y|<1\}}(x-y) \cdot \nabla w(x))\nu(|x-y|)dy.$$

Since Proposition 3.1 implies $V''(s)$ exists and so w is twice differentiable in $\mathbb{H}$, we have that

$$x \mapsto \int_{B(x,\varepsilon)} (w(y) - w(x) - (x-y) \cdot \nabla w(x))\nu(|x-y|)dy$$

converges to 0 locally uniformly in $\mathbb{H}$ as $\varepsilon \downarrow 0$. From (3.8), we see that $\mathcal{L}_Z^\varepsilon w(x)$ converges to

$$\mathcal{L}_Z w(x) = \int_{\mathbb{R}^d} (w(y) - w(x) - \mathbf{1}_{\{|x-y|<1\}}(x-y) \cdot \nabla w(x))\nu(|x-y|)dy$$

locally uniformly in $\mathbb{H}$ as $\varepsilon \downarrow 0$.

For every $x \in \mathbb{H}$, $z \in B(x, (\varepsilon \wedge x_d)/2)$ and $y \in B(z, \varepsilon)^c$, it holds that $|y-z|/2 \leqslant |x-y| \leqslant 3|y-z|/2$. Since $r \to \nu(r)$ is decreasing, using Proposition 3.1

$$\begin{aligned} &\mathbf{1}_{\{|y-z|>\varepsilon\}} \left| w(y) - w(z) - \mathbf{1}_{\{|y-z|<1\}}(y-z) \cdot \nabla w(z) \right| \nu(|y-z|) \\ &\leqslant c_2 \left(\sup_{\varepsilon/2<s<x_d+2} V''(s) \right) |x-y|^2 \mathbf{1}_{\{\varepsilon/2<|x-y|<2\}} \nu(|x-y|/2) \\ &\quad + (w(y) + V(x_d+1)) \mathbf{1}_{\{|x-y|>1/2\}} \nu(|x-y|/2). \end{aligned}$$

So applying the dominated convergence theorem with Proposition 3.2 and the fact that ν is a Lévy density, we obtain that $x \to \mathcal{L}_Z^\varepsilon w(x)$ is continuous for each ε. Therefore, the function $\mathcal{L}_Z w(x)$ is continuous in $\mathbb{H}$.

Let U_1 and U_2 be relatively compact open subsets of $\mathbb{H}$ satisfying $\overline{U_1} \subset U_2 \subset \overline{U_2} \subset \mathbb{H}$ and $0 < r_0 := \mathrm{dist}(U_1, U_2^c) < 1$. By Proposition 3.2,

$$\begin{aligned} \int_{U_1} \int_{U_2^c} w(y)\nu(|x-y|)dydx &\leqslant |U_1| \sup_{x \in U_1} \int_{U_2^c} w(y)\nu(|x-y|)dy \\ &\leqslant |U_1| \sup_{x \in U_1} \int_{B(x,r_0)^c} w(y)\nu(|x-y|)dy < \infty. \end{aligned} \tag{3.9}$$

Since w is harmonic, $w(Z_{\tau_{U_1}}) \in L^1(\mathbb{P}_x)$ and

$$\sup_{x\in U_1} \mathbb{E}_x[\mathbf{1}_{U_2^c}(Z_{\tau_{U_1}})w(Z_{\tau_{U_1}})] \leqslant \sup_{x\in U_1} \mathbb{E}_x[w(Z_{\tau_{U_1}})] = \sup_{x\in U_1} w(x) < \infty. \tag{3.10}$$

From (3.9) and (3.10), the conditions [9, (2.4), (2.6)] hold and by [9, Lemma 2.3, Theorem 2.11(ii)], we have that for any $f \in C_c^2(\mathbb{H})$,

$$0 = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (w(y) - w(x))(f(y) - f(x))\nu(|x-y|)dxdy. \tag{3.11}$$

For $f \in C_c^2(\mathbb{H})$ with $\mathrm{supp}(f) \subset \overline{U_1} \subset U_2 \subset \overline{U_2} \subset \mathbb{H}$,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |w(y) - w(x)||f(y) - f(x)|\nu(|x-y|)dxdy \\
&= \int_{U_1}\int_{U_2} |w(y) - w(x)||f(y) - f(x)|\nu(|x-y|)dxdy \\
&\qquad + 2\int_{U_1}\int_{U_2^c} |w(y) - w(x)||f(y)|\nu(|x-y|)dxdy \\
&\leqslant c_2 \int_{U_1\times U_2} |x-y|^2\nu(|x-y|)dxdy + 2\|f\|_\infty \int_{U_1}\int_{U_2^c} w(y)\nu(|x-y|)dxdy \\
&\qquad + 2\|f\|_\infty |U_1| \left(\sup_{y\in U_1} w(y)\right) \int_{U_1}\int_{U_2^c} \nu(|x-y|)dxdy,
\end{aligned} \tag{3.12}$$

and it is finite from (3.9) and the fact that ν is a Lévy density. Applying the dominated convergence theorem with (3.11) and (3.12), for any $f \in C_c^2(\mathbb{H})$, we have

$$\begin{aligned}
0 &= \lim_{\varepsilon\downarrow 0} \int_{\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d : |x-y|>\varepsilon\}} (w(y) - w(x))(f(y) - f(x))\nu(|x-y|)dxdy \\
&= -2\lim_{\varepsilon\downarrow 0} \int_{\mathbb{H}} f(x) \left(\int_{\{y\in\mathbb{R}^d : |x-y|>\varepsilon\}} (w(y) - w(x))\nu(|x-y|)dy \right) dx \\
&= -2\int_{\mathbb{H}} f(x)\mathcal{L}_Z w(x)dx.
\end{aligned}$$

We have used Fubini's theorem and the fact that $\mathcal{L}_Z^\varepsilon w \to \mathcal{L}_Z w$ converges uniformly on the support f. Hence, by the continuity of $\mathcal{L}_Z w$, we have $\mathcal{L}_Z w(x) = 0$ in $\mathbb{H}$. $\square$

Proposition 3.4. *Suppose that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ). For any $z \in \partial D$ and $r \leqslant 1 \wedge R_0$, we define*

$$h_r(y) = h_{r,z}(y) := V(\delta_D(y))\mathbf{1}_{D\cap B(z,r)}(y).$$

There exists $C_{3.4} = C_{3.4}(\phi, \Lambda, d) > 0$ independent of z such that $\mathcal{L}_Z h$ is well-defined in $D \cap B(z, r/4)$ and

$$|\mathcal{L}_Z h_r(x)| \leqslant \frac{C_{3.4}}{V(r)} \qquad \text{for all } x \in D \cap B(z, r/4). \tag{3.13}$$

Proof. Since the case of $d = 1$ is easier, we give the proof only for $d \geqslant 2$. Without loss of generality, we assume that $R_0 < 1$ and $\Lambda > 1$. For $x \in D \cap B(z, r/4)$, let $z_x \in \partial D$ be the point satisfying $\delta_D(x) = |x - z_x|$. Let φ be a $C^{1,1}$ function and $CS = CS_{z_x}$ be an orthonormal coordinate system with z_x chosen so that $\varphi(\widetilde{0}) = 0$, $\nabla\varphi(\widetilde{0}) = (0, \dots, 0)$,

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$\|\nabla\varphi\|_\infty \leqslant \Lambda$, $|\nabla\varphi(\widetilde{y}) - \nabla\varphi(\widetilde{z})| \leqslant \Lambda|\widetilde{y} - \widetilde{z}|$, and $x = (\widetilde{0}, x_d)$, $D \cap B(z_x, R_0) = \{y = (\widetilde{y}, y_d) \in B(0, R_0)$ in $CS : y_d > \varphi(\widetilde{y})\}$. We fix the function φ and the coordinate system CS, and we define a function $g_x(y) = V(\delta_{\mathbb{H}}(y)) = V(y_d)$, where $\mathbb{H} = \{y = (\widetilde{y}, y_d)$ in $CS : y_d > 0\}$ is the half space in CS.

Note that $h_r(x) = g_x(x)$, and that $\mathcal{L}_Z(h_r - g_x) = \mathcal{L}_Z h_r$ by Theorem 3.3. So, it suffices to show that $\mathcal{L}_Z(h_r - g_x)$ is well defined and that there exists a constant $c_1 = c_1(\underline{\alpha}, \underline{c}, \overline{\alpha}, \overline{C}, \Lambda, d) > 0$ independent of $x \in D \cap B(z, r/4)$ and $z \in \partial D$ such that

$$\int_{D\cup\mathbb{H}} |h_r(y) - g_x(y)|\nu(|x - y|)dy \leqslant c_1 V(r)^{-1}. \tag{3.14}$$

We define $\widehat{\varphi} : B(\widetilde{0}, r) \to \mathbb{R}$ by $\widehat{\varphi}(\widetilde{y}) := 2\Lambda|\widetilde{y}|^2$. Since $\nabla\varphi(\widetilde{0}) = 0$, by the mean value theorem we have $-\widehat{\varphi}(\widetilde{y}) \leqslant \varphi(\widetilde{y}) \leqslant \widehat{\varphi}(\widetilde{y})$ for any $y \in D \cap B(x, r/2)$ and so that

$$\begin{aligned}\{z = (\widetilde{z}, z_d) \in B(x, r/2) : z_d \geqslant \widehat{\varphi}(\widetilde{z})\} &\subset D \cap B(x, r/2)\\ &\subset \{z = (\widetilde{z}, z_d) \in B(x, r/2) : z_d \geqslant -\widehat{\varphi}(\widetilde{z})\}.\end{aligned}$$

Let $A := \{y \in (D \cup \mathbb{H}) \cap B(x, r/4) : -\widehat{\varphi}(\widetilde{y}) \leqslant y_d \leqslant \widehat{\varphi}(\widetilde{y})\}$ and $E := \{y \in B(x, r/4) : y_d > \widehat{\varphi}(\widetilde{y})\} \subset D$. We will prove (3.14) by showing that $\mathrm{I} + \mathrm{II} + \mathrm{III} \leqslant c_1 V(r)^{-1}$, where

$$\begin{aligned}\mathrm{I} &:= \int_{B(x,r/4)^c} (h_r(y) + g_x(y))\nu(|x - y|)dy,\\ \mathrm{II} &:= \int_A (h_r(y) + g_x(y))\nu(|x - y|)dy, \quad \text{and} \quad \mathrm{III} := \int_E |h_r(y) - g_x(y)|\nu(|x - y|)dy.\end{aligned}$$

First, since $h_r \leqslant V(r)$, by [6, (2.23)] and Proposition 3.2, we have

$$\begin{aligned}\mathrm{I} &\leqslant V(r)\int_{B(x,r/4)^c} \nu(|x - y|)dy + \sup_{\{z\in\mathbb{R}^d:0<z_d<r\}} \int_{B(z,r/4)^c\cap\mathbb{H}} g_x(y)\nu(|z - y|)dy\\ &\leqslant c_2 V(r)^{-1} + \sup_{\{z\in\mathbb{R}^d:0<z_d<r\}} \int_{B(z,r/4)^c\cap\mathbb{H}} w(y)\nu(|z - y|)dy \leqslant (c_2 + C_{3.2})V(r)^{-1}.\end{aligned}$$

For $y \in A$, since $\widehat{\varphi}(\widetilde{y}) \leqslant 2\Lambda|\widetilde{y}|$ and V is increasing and subadditive, we observe that $h_r(y) + g_x(y) \leqslant 2V(2\widehat{\varphi}(\widetilde{y})) \leqslant c_3 V(|\widetilde{y}|)$. For $s \leqslant r/4$, note that $m_{d-1}(\{y : |\widetilde{y}| = s, -\widehat{\varphi}(\widetilde{y}) \leqslant y_d \leqslant \widehat{\varphi}(\widetilde{y})\}) \leqslant c_4 s^d$ where $m_{d-1}(dy)$ is the surface measure on $\mathbb{R}^{d-1}$. Thus $\int_{|\widetilde{y}|=s} \mathbf{1}_A(y)V(|\widetilde{y}|)\nu(|\widetilde{y}|)\, m_{d-1}(dy) \leqslant c_4 V(s)\nu(s)s^d$ for $0 < s < r/4$. From (3.4), we note that $V(s)^{-1} \leqslant C_V V(r)^{-1}(r/s)^{\overline{\alpha}/2}$ for $s \leqslant r$. Hence, by (3.3) and (1.2),

$$\begin{aligned}\mathrm{II} &\leqslant c_3 c_4 \int_0^r V(s)\nu(s)s^d ds\\ &\leqslant c_5 \int_0^r V(s)^{-1} s ds \leqslant c_4 \cdot C_V V(r)^{-1} r^{\overline{\alpha}/2} \int_0^r s^{-\overline{\alpha}/2} ds\\ &= c_5 \cdot C_V V(r)^{-1} \frac{1}{1 - \overline{\alpha}/2} r^1 \leqslant c_6 V(r)^{-1}\end{aligned}$$

for some positive constant $c_6 = c_6(\overline{\alpha}, \underline{c}, \overline{C}, \Lambda, d)$.

When $y \in E$, we have that $|y_d - \delta_D(y)| \leqslant \Lambda|\widetilde{y}|^2$. Indeed, if $0 < y_d = \delta_{\mathbb{H}}(y) \leqslant \delta_D(y)$ and $y \in E$, $\delta_D(y) \leqslant y_d + |\varphi(\widetilde{y})| \leqslant y_d + \Lambda|\widetilde{y}|^2$. Since we assume that $R_0 < 1$ and $\Lambda > 1$, if

$y_d = \delta_{\mathbb{H}}(y) \geqslant \delta_D(y)$ and $y \in E$, the interior ball condition implies that

$$y_d - \delta_D(y) \leqslant y_d - R_0 + \sqrt{|\widetilde{y}|^2 + (R_0 - y_d)^2} = \frac{|\widetilde{y}|^2}{\sqrt{|\widetilde{y}|^2 + (R_0 - y_d)^2} + (R_0 - y_d)}$$
$$\leqslant \frac{|\widetilde{y}|^2}{2(R_0 - y_d)} \leqslant \frac{|\widetilde{y}|^2}{R_0} \leqslant \Lambda |\widetilde{y}|^2.$$

Furthermore, $|y_d - \delta_D(y)| \leqslant \Lambda|\widetilde{y}|^2$ yields that

$$\Lambda|\widetilde{y}|^2 \leqslant y_d - \Lambda|\widetilde{y}|^2 \leqslant y_d \wedge \delta_D(y) \quad \text{and} \quad y_d \vee \delta_D(y) - (y_d - \Lambda|\widetilde{y}|^2) \leqslant 2\Lambda|\widetilde{y}|^2.$$

Hence, by the mean value Theorem and the scale invariant Harnack inequality for Z^d ([11, Theorem 1.4]) applying to V', we have that

$$\begin{aligned} |h_r(y) - g_x(y)| =& |V(\delta_D(y)) - V(y_d)| \leqslant \sup_{u \in [y_d \wedge \delta_D(y),\ y_d \vee \delta_D(y)]} V'(u)|\delta_D(y) - y_d| \\ &\leqslant \sup_{u \in [y_d - \Lambda|\widetilde{y}|^2,\ y_d \vee \delta_D(y)]} V'(u)|\delta_D(y) - y_d| \\ &\leqslant c_7 \inf_{u \in [y_d - \Lambda|\widetilde{y}|^2,\ y_d \vee \delta_D(y)]} V'(u)|\widetilde{y}|^2 \leqslant 2\Lambda c_7 V'\left(y_d - \tfrac{1}{2}\widehat{\varphi}(\widetilde{y})\right)|\widetilde{y}|^2. \end{aligned} \tag{3.15}$$

Since $E \subset \{(\widetilde{y}, y_d) : |\widetilde{y}| < r/4,\ \widehat{\varphi}(\widetilde{y}) < y_d < \frac{1}{2}\widehat{\varphi}(\widetilde{y}) + r/2\}$, using with (3.15) and the polar coordinates for $|\widetilde{y}| = v$, we first see that

$$\begin{aligned} \mathrm{III} &\leqslant 2\Lambda c_7 \int_E V'\left(y_d - \tfrac{1}{2}\widehat{\varphi}(\widetilde{y})\right)|\widetilde{y}|^2 \nu(|x-y|)dy \\ &\leqslant c_8 \int_0^{r/4} \int_{\widehat{\varphi}(v)}^{\frac{1}{2}\widehat{\varphi}(v)+r/2} V'(y_d - \tfrac{1}{2}\widehat{\varphi}(v))\nu((v^2 + |y_d - x_d|^2)^{1/2})v^d dy_d dv \end{aligned}$$

Let $s := y_d - \frac{1}{2}\widehat{\varphi}(v)$. Since $(v^2 + |y_d - x_d|^2)^{1/2} \geqslant (v + |y_d - x_d|)/2$ and ν is decreasing, by (1.2) and (3.3), we have that

$$\begin{aligned} \nu((v^2 + |y_d - x_d|^2)^{1/2})v^d &\leqslant \nu((v + |s + \tfrac{1}{2}\widehat{\varphi}(r) - x_d|)/2)(v + (|s + \tfrac{1}{2}\widehat{\varphi}(v) - x_d|))^d \\ &\leqslant c_9 V(v + |s + \tfrac{1}{2}\widehat{\varphi}(v) - x_d|)^{-2} \end{aligned}$$

For $g(s) := \sup_{u \geqslant s}(V(u)/u)$, $V'(s) \leqslant C_{3.1}(V(s)/s) \leqslant C_{3.1}g(s)$ by Proposition 3.1. Therefore we have that

$$\begin{aligned} \mathrm{III} &\leqslant c_8 \cdot c_9 \int_0^{r/2} \int_0^{r/2} V'(s)V(v + |s + \tfrac{1}{2}\widehat{\varphi}(v) - x_d|)^{-2} ds dv \\ &\leqslant c_{10} \int_0^{r/2} \int_0^{r/2} g(s)V(v + |s + \tfrac{1}{2}\widehat{\varphi}(v) - x_d|)^{-2} ds dv. \end{aligned}$$

Applying [34, Lemma 4.4] with non-increasing functions $g(s)$ and $f(s) := V(s)^{-2}$ and $x(r) = s + \frac{1}{2}\widehat{\varphi}(r)$, we have that

$$\mathrm{III} \leqslant c_{11} \int_0^{3r/4} G(u)V(u)^{-2} du.$$

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where $G(u)=\int_0^u g(s)ds$. By subadditivity of V and (3.4), $G(u)\leqslant 2\int_0^u (V(s)/s)\,ds\leqslant c_{12}V(u)$. Using (3.4) again, we conclude that

$$\mathrm{III}\leqslant c_{11}\cdot c_{12}\int_0^r V(u)^{-1}du\leqslant c_{13}V(r)^{-1}r^{\overline{\alpha}/2}\int_0^r u^{-\overline{\alpha}/2}du\leqslant c_{14}V(r)^{-1}$$

for some positive constant $c_{14}:=c_{14}(\underline{\alpha},\overline{\alpha},\underline{c},\overline{C},d)$. □

4. Estimates on exit distributions for Y

In this section we give some key estimates on exit distributions for Y. Throughout this section, we assume that Y is the symmetric pure jump Hunt process with the jumping intensity kernel J satisfying the conditions **(J1.1)**, **(J1.2)**, **(SD)** and **($\mathbf{K}_\eta$)**.

For any $x\in\mathbb{R}^d$, stopping time S (with respect to the filtration of Y), and non-negative measurable function f on $\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d$ with $f(s,y,y)=0$ for all $y\in\mathbb{R}^d$ and $s\geqslant 0$, we have the following Lévy system:

$$\mathbb{E}_x\left[\sum_{s\leqslant S} f(s,Y_{s-},Y_s)\right]=\mathbb{E}_x\left[\int_0^S\left(\int_{\mathbb{R}^d} f(s,Y_s,y)J(Y_s,y)dy\right)ds\right] \tag{4.1}$$

(e.g., see [19, Appendix A]).

Throughout this section, we assume that D is a $C^{1,1}$ open set with $C^{1,1}$ characteristics (R_0,Λ), and without loss of generality, we will assume that $R_0<1$ and $\Lambda>1$. Recall that the function $h_r(y)=h_{r,z}(y)$ is defined in Proposition 3.4.

Lemma 4.1. *Let $r\leqslant R_0/2$. For any $z\in\partial D$ and $k\in\mathbb{N}$, let $B_k:=\{y\in D\cap B(z,r/4): \delta_{D\cap B(z,r/4)}(y)\geqslant 2^{-k}\}$. Then, for every $u\in\mathbb{R}^d$ and $k\in\mathbb{N}$ with $|u|<2^{-k}<2^{-8}r$,*

$$\mathcal{L}^u h_{r,z}(w):=\lim_{\varepsilon\downarrow 0}\int_{|(w-u)-y|>\varepsilon}(h_{r,z}(y)-h_{r,z}(w-u))J(w,u+y)dy$$

is well defined in B_k and there exists $C_{4.1}=C_{4.1}(\phi,L_0,L_3,\Lambda,\eta,d)>0$ independent of $z\in\partial D$, $k\in\mathbb{N}$ with $2^{-k+8}<r\leqslant R_0$ such that

$$|\mathcal{L}^u h_{r,z}(w)|\leqslant\frac{C_{4.1}}{V(r)}\quad\text{for all}\quad w\in B_k,\ |u|<2^{-k}.$$

Proof. We fix $z\in\partial D$ and use the short notation $h_r(y)=h_{r,z}(y)$. For any $w\in B_k$ and $|u|<2^{-k}<2^{-8}r$, let $x:=w-u\in B(z,r/4)$. Define $\kappa_u(x,y):=\kappa(u+x,u+y)$, and for $\varepsilon<2^{-k-1}$ denote $A_\varepsilon(x)$ and $L(x)$ by

$$A_\varepsilon(x):=\int_{\varepsilon<|x-y|\leqslant 1}(h_r(y)-h_r(x))(\kappa_u(x,y)-\kappa_u(x,x))\nu(|x-y|)dy,$$
$$L(x):=\int_{1<|x-y|}(h_r(y)-h_r(x))(J(x,y)-\kappa_u(x,x)\nu(|x-y|))dy,$$

so that

$$\int_{\varepsilon<|x-y|\leqslant 1}(h_r(y)-h_r(x))\kappa_u(x,y)\nu(|x-y|)dy+\int_{1<|x-y|}(h_r(y)-h_r(x))J(x,y)dy$$
$$=A_\varepsilon(x)+\kappa_u(x,x)\cdot\mathcal{L}_Z^\varepsilon h_r(x)+L(x)$$

where $\mathcal{L}_Z^\varepsilon$ is defined in (3.6).

By the definition of h_r, (1.4), (1.2) and **(J1.2)**, for $r \leqslant R_0 < 1$, we first obtain that

$$|L(x)| \leqslant c_0\left(\left|\int_{1<|x-y|} J(x,y)dy\right| + \left|\int_{1<|x-y|} \nu(|x-y|)dy\right|\right) \leqslant c_1 V(r)^{-1}. \tag{4.2}$$

On the other hand,

$$|A_\varepsilon(x)| \leqslant \left(\int_{|x-y|<r/2} + \int_{r/2\leqslant|x-y|}\right) |h_r(y)-h_r(x)||\kappa_u(x,y)-\kappa_u(x,x)|\nu(|x-y|)dy$$
$$=: \mathrm{I}(x) + \mathrm{II}(x).$$

For $|x-y| < r/2$, $|h_r(y)-h_r(x)| \leqslant V(|x-y|)$ by subadditivity of V, and $V(r)/V(|x-y|) \leqslant C_V(r/|x-y|)^{\overline{\alpha}/2}$ by (3.4). Also $|\kappa_u(x,y)-\kappa_u(x,x)| \leqslant L_3|x-y|^\eta$ by the assumption $(\mathbf{K}_\eta)$. Hence (1.2) and (3.3) imply that

$$|\mathrm{I}(x)| \leqslant c_2 \int_{|x-y|<r} V(|x-y|)^{-1}|x-y|^{\eta-d}dy$$
$$\leqslant c_2 C_V V(r)^{-1} r^{\overline{\alpha}/2} \int_{|x-y|<r} |x-y|^{-\overline{\alpha}/2+\eta-d}dy \leqslant c_3 V(r)^{-1} \tag{4.3}$$

for some positive constant $c_3 := c_3(\overline{\alpha}, \underline{c}, \overline{C}, L_3, \eta, d)$. The last inequality holds since $\eta > \overline{\alpha}/2$. To obtain the upper bound of $\mathrm{II}(x)$, note that $|h_r(y)-h_r(x)| \leqslant 2V(|x-y|)$ for $r/2 \leqslant |x-y|$. Indeed, if $y \in (D\cap B(z,r))^c$, then $|h_r(y)-h_r(x)| = |h_r(x)| \leqslant V(r) \leqslant 2V(|x-y|)$ by subadditivity of V. If $y \in D\cap B(z,r)$, $|h_r(y)-h_r(x)| \leqslant V(|\delta_D(y)-\delta_D(x)|) \leqslant V(|x-y|)$ by subadditivity of V. Hence by (1.4) and [6, Lemma 3.5], we obtain that

$$|\mathrm{II}(x)| \leqslant 4L_0 \int_{r/2\leqslant|x-y|} V(|x-y|)\nu(|x-y|)dy \leqslant c_5 V(r)^{-1}. \tag{4.4}$$

From (4.3) and (4.4),

$$\lim_{\varepsilon\downarrow 0} A_\varepsilon(x) \text{ exists} \quad \text{and} \quad |\lim_{\varepsilon\downarrow 0} A_\varepsilon(x)| \leqslant (c_3+c_5)V(r)^{-1}. \tag{4.5}$$

Finally from Proposition 3.4, $\lim_{\varepsilon\downarrow 0} \mathcal{L}^\varepsilon_Z h_r(x)$ exists and

$$\left|\lim_{\varepsilon\downarrow 0} \mathcal{L}^\varepsilon_Z h_r(x)\right| \leqslant C_{3.4}\, V(r)^{-1}. \tag{4.6}$$

Hence combining (4.2), (4.5) and (4.6), we have the conclusion. □

Theorem 4.2. *For any $x \in D$, let $z_x \in \partial D$ be a point satisfying $\delta_D(x) = |x - z_x|$.*

(1) *There are constants $A_{4.2} = A_{4.2}(\phi, L_0, L_3, \Lambda, \eta) \in (0,1)$ and $C_{4.2.1} = C_{4.2.1}(\phi, L_0, L_3, \Lambda, \eta) > 0$, such that for any $s \leqslant A_{4.2}R_0/2$ and $x \in D$ with $\delta_D(x) < s$,*

$$\mathbb{E}_x\left[\tau_{D\cap B(z_x,s)}\right] \leqslant C_{4.2.1}\, V(s)V(\delta_D(x)). \tag{4.7}$$

(2) *There is a constant $C_{4.2.2} = C_{4.2.2}(\phi, L_0, L_3, \Lambda, \eta) > 0$, such that for any $s \leqslant R_0/2$, $\lambda \geqslant 4$ and $x \in D$ with $\delta_D(x) < \lambda^{-1}s/2$,*

$$\mathbb{P}_x\left(Y_{\tau_{D\cap B(z_x,\lambda^{-1}s)}} \in \{2\Lambda|\widetilde{y}| < y_d, \lambda^{-1}s < |y| < s \text{ in } CS_{z_x}\}\right) \geqslant C_{4.2.2}\frac{V(\delta_D(x))}{V(s)}. \tag{4.8}$$

Please cite this article as: T. Grzywny, K.-Y. Kim and P. Kim, Estimates of Dirichlet heat kernel for symmetric Markov processes, Stochastic Processes and their Applications (2019), https://doi.org/10.1016/j.spa.2019.03.017.

Proof. Without loss of generality, we assume that $z_x = 0$. For $R \leqslant R_0/2$, let $h_R(y) = V(\delta_D(y))\mathbf{1}_{D\cap B(0,R)}(y)$. Let $f \geqslant 0$ be a smooth radial function such that $f(y) = 0$ for $|y| > 1$ and $\int_{\mathbb{R}^d} f(y)dy = 1$. For $k \geqslant 1$, define $f_k(y) := 2^{kd} f(2^k y)$ and $h_R^{(k)} := f_k * h_R \in C_c^2(\mathbb{R}^d)$, and let $B_k^\lambda := \{y \in D \cap B(0, \lambda^{-1}R/4) : \delta_{D\cap B(0,\lambda^{-1}R)}(y) \geqslant 2^{-k}\}$ for $\lambda \geqslant 4$.

Since $h_R^{(k)}$ is a C_c^2 function, $\mathcal{L}h_R^{(k)}$ is well defined everywhere. By Lemma 4.1, for $w \in B_k^\lambda$ and $u \in B(0, 2^{-k})$ the following limit

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\int_{|w-y|>\varepsilon} (h_R(y-u) - h_R(w-u))J(w, y)dy\\ &= \lim_{\varepsilon\downarrow 0}\int_{|(w-u)-y'|>\varepsilon} (h_R(y') - h_R(w-u))J(w, u+y')dy' = \mathcal{L}^u h_R(w)\end{aligned}$$

exists and $-C_{4.1}V(R)^{-1} \leqslant \mathcal{L}^u h_R(w) \leqslant C_{4.1}V(R)^{-1}$. We note that

$$\begin{aligned}&\int_{|w-y|>\varepsilon} (h_R^{(k)}(y) - h_R^{(k)}(w))J(w, y)dy\\ &= \int_{|w-y|>\varepsilon}\int_{\mathbb{R}^d} f_k(u)\,(h_R(y-u) - h_R(w-u))\,du\,J(w, y)dy\\ &= \int_{|u|<2^{-k}} f_k(u)\int_{|w-y|>\varepsilon} (h_R(y-u) - h_R(w-u))\,J(w, y)dydu.\end{aligned}$$

By letting $\varepsilon \downarrow 0$ and using the dominated convergence theorem, it follows that for $w \in B_k^\lambda$ and $2^{-k+8} < \lambda^{-1}R$,

$$\begin{aligned}|\mathcal{L}h_R^{(k)}(w)| &= \left|\int_{|u|<2^{-k}} f_k(u)\mathcal{L}^u h_R(w)\,du\right| \leqslant C_{4.1}V(R)^{-1}\int_{|u|<2^{-k}} f_k(u)\,du\\ &= C_{4.1}V(R)^{-1}.\end{aligned} \tag{4.9}$$

Applying Lemma 2.2 to B_k^λ and $h_R^{(k)}$, and using (4.9), for any $x \in B_k^\lambda$, we have

$$\begin{aligned}&\mathbb{E}_x\left[h_R^{(k)}(Y_{\tau_{B_k^\lambda}})\right] - C_{4.1}V(R)^{-1}\mathbb{E}_x\left[\tau_{B_k^\lambda}\right] \leqslant h_R^{(k)}(x)\\ &\leqslant \mathbb{E}_x\left[h_R^{(k)}(Y_{\tau_{B_k^\lambda}})\right] + C_{4.1}V(R)^{-1}\mathbb{E}_x\left[\tau_{B_k^\lambda}\right].\end{aligned}$$

By letting $k \to \infty$, for any $x \in D \cap B(0, \lambda^{-1}R)$, we obtain

$$V(\delta_D(x)) = h_R(x) \geqslant \mathbb{E}_x\left[h_R\left(Y_{\tau_{D\cap B(0,\lambda^{-1}R)}}\right)\right] - C_{4.1}V(R)^{-1}\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}R)}\right] \tag{4.10}$$

$$\text{and}\quad V(\delta_D(x)) = h_R(x) \leqslant \mathbb{E}_x\left[h_R\left(Y_{\tau_{D\cap B(0,\lambda^{-1}R)}}\right)\right] + C_{4.1}V(R)^{-1}\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}R)}\right]. \tag{4.11}$$

For any $z \in D \cap B(0, \lambda^{-1}R)$ and $y \in D \cap (B(0, R) \setminus B(0, \lambda^{-1}R))$, by the fact that ν is decreasing and (1.3), $\nu(|y-z|) \geqslant \nu(2|y|) \geqslant c_1\nu(|y|)$. So by (1.4), **(J1.1)** and (4.1), we obtain

$$\begin{aligned}\mathbb{E}_x\left[h_R\left(Y_{\tau_{D\cap B(0,\lambda^{-1}R)}}\right)\right] &\geqslant L_0^{-1}\mathbb{E}_x\int_{D\cap(B(0,R)\setminus B(0,\lambda^{-1}R))}\int_0^{\tau_{D\cap B(0,\lambda^{-1}R)}} \nu(|Y_t - y|)dt\,h_R(y)dy\\ &\geqslant L_0^{-1}c_1\,\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}R)}\right]\int_{D\cap(B(0,R)\setminus B(0,\lambda^{-1}R))} \nu(|y|)h_R(y)dy.\end{aligned} \tag{4.12}$$

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Let $A := \{(\widetilde{y}, y_d) : 2\Lambda|\widetilde{y}| < y_d\}$. For any $y \in A \cap B(0, R)$, since $y_d > 2\Lambda|\widetilde{y}| > 2\Lambda|\widetilde{y}|^2 > \varphi(\widetilde{y})$, we have $A \cap B(0, R) \subset D \cap B(0, R)$ and

$$\begin{aligned}\delta_D(y) &\geqslant (1+\Lambda)^{-1}\left(y_d - \varphi(\widetilde{y})\right) \geqslant (2\Lambda)^{-1}(y_d - \Lambda|\widetilde{y}|) > (4\Lambda)^{-1} y_d \\ &\geqslant (4\Lambda((2\Lambda)^{-2}+1)^{1/2})^{-1}|y|.\end{aligned}$$

Combining this and (3.4), $V(\delta_D(y)) \geqslant c_2 V(|y|)$. By changing to polar coordinates with $|y| = t$, (3.3) and Proposition 3.1, we obtain that

$$\begin{aligned}\int_{D\cap(B(0,R)\setminus B(0,\lambda^{-1}R))} \nu(|y|)h_R(y)dy &\geqslant c_2 \int_{A\cap(B(0,R)\setminus B(0,\lambda^{-1}R))} \nu(|y|)V(|y|)dy \\ &\geqslant c_3 \int_{\lambda^{-1}R}^{R} \nu(t)V(t)t^{d-1}dt \geqslant c_4 \int_{\lambda^{-1}R}^{R} V(t)^{-1}t^{-1}dt \\ &\geqslant c_4 \cdot C_{3.1} \int_{\lambda^{-1}R}^{R} \frac{V'(t)}{V(t)^2}dt = c_4 \cdot C_{3.1}\left(V(\lambda^{-1}R)^{-1} - V(R)^{-1}\right).\end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13), there exists $C_5 := C_5(\underline{c}, \overline{\alpha}, \overline{C}, L_0, \Lambda, d) > 0$ such that

$$\mathbb{E}_x\left[h_r\left(Y_{\tau_{D\cap B(0,\lambda^{-1}R)}}\right)\right] \geqslant C_5\, \mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}R)}\right]\left(V(\lambda^{-1}R)^{-1} - V(R)^{-1}\right). \tag{4.14}$$

Using (3.4) again, $V(\lambda^{-1}R) \leqslant V(\lambda_0^{-1}R) \leqslant C_V\lambda_0^{-\underline{\alpha}/2}V(R)$ for any $\lambda \geqslant \lambda_0 \geqslant 4$. Let $\lambda_0 := (2C_V(C_5+C_{4.1})/C_5)^{2/\underline{\alpha}} \geqslant 1$. Then combining (4.10) and (4.14), we have that for $\lambda \geqslant \lambda_0$

$$V(\delta_D(x)) \geqslant \left(C_5 V(\lambda^{-1}R)^{-1} - (C_5 + C_{4.1})V(R)^{-1}\right)\mathbb{E}_x[\tau_{D\cap B(0,\lambda^{-1}R)}] \tag{4.15}$$

$$\geqslant (C_5/2)\, V(\lambda^{-1}R)^{-1}\mathbb{E}_x[\tau_{D\cap B(0,\lambda^{-1}R)}]. \tag{4.16}$$

Thus, we have proved (4.7) with $A_{4.2} = \lambda_0^{-1}$ and $s = \lambda^{-1}R$ where $\lambda \geqslant \lambda_0$.

From (4.11) and Corollary 2.4 with (3.3) for $\delta_D(x) < \lambda^{-1}R/2$ and $\lambda \geqslant 4$, we first note that

$$\begin{aligned}V(\delta_D(x)) &\leqslant V(R)\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}R)}} \in D\right) + C_{4.1}V(R)^{-1}\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}R)}\right] \\ &\leqslant \left(C_{2.4}V(R)\,V(\lambda^{-1}R)^{-2} + C_{4.1}V(R)^{-1}\right)\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}R)}\right] \\ &= c_6 V(R)\left(V(\lambda^{-1}R)^{-2} + V(R)^{-2}\right)\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}R)}\right].\end{aligned} \tag{4.17}$$

By (3.4) and the subadditivity of V, $V(s)^{-1} \geqslant 3^{-1}(3\lambda^{-1}R/s)^{\underline{\alpha}/2}C_V^{-1}V(\lambda^{-1}R)^{-1}$ for $s \leqslant 3\lambda^{-1}R$. Combining this with (3.3) and using the polar coordinate with $|y| = t$, we have that

$$\begin{aligned}\int_{A\cap(B(0,3\lambda^{-1}R)\setminus B(0,2\lambda^{-1}R))} \nu(|y|)dy &\geqslant c_7 \int_{2\lambda^{-1}R}^{3\lambda^{-1}R} \nu(t)t^{d-1}dt \\ &\geqslant c_8 \int_{2\lambda^{-1}R}^{3\lambda^{-1}R} V(t)^{-2}t^{-1}dt \geqslant c_9 V(\lambda^{-1}R)^{-2}.\end{aligned} \tag{4.18}$$

For any $z \in B(0, \lambda^{-1}R)$ and $y \in B(0, 3\lambda^{-1}R) \setminus B(0, 2\lambda^{-1}R)$, by (1.3) and the fact that ν is decreasing, $\nu(|y - z|) \geqslant \nu(|y| + |z|) \geqslant \nu(3|y|/2) \geqslant c_{10}\nu(|y|)$. So by (1.4), **(J1.1)** (4.1) and (4.18), we obtain

$$\begin{aligned}&\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}R)}} \in A \cap \left(B(0, 3\lambda^{-1}R) \setminus B(0, 2\lambda^{-1}R)\right)\right) \\ &\geqslant L_0^{-1}\mathbb{E}_x\left[\int_0^{\tau_{D\cap B(0,\lambda^{-1}R)}} \int_{A\cap(B(0,3\lambda^{-1}R)\setminus B(0,2\lambda^{-1}R))} \nu(|Y_s - y|)dy\, ds\right]\end{aligned}$$

$$\geqslant L_0^{-1} c_{10} \mathbb{E}_x \left[\int_0^{\tau_{D\cap B(0,\lambda^{-1}R)}} \int_{A\cap(B(0,3\lambda^{-1}R)\setminus B(0,2\lambda^{-1}R))} \nu(|y|) dy\, ds \right]$$
$$\geqslant c_9\, c_{10} L_0^{-1} V(\lambda^{-1}R)^{-2} \mathbb{E}\left[\tau_{D\cap B(0,\lambda^{-1}R)}\right]. \tag{4.19}$$

Hence combining (4.17), (4.19), and the fact that V is increasing, we conclude that for $\lambda \geqslant 4$,

$$\begin{aligned} V(\delta_D(x)) \leqslant & c_{11} V(R) \left(V(\lambda^{-1}r)^{-2} + V(R)^{-2}\right) V(\lambda^{-1}R)^2 \\ & \cdot \mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}R)}} \in A \cap \left(B(0,3\lambda^{-1}R) \setminus B(0,2\lambda^{-1}R)\right)\right) \\ \leqslant & 2c_{11} V(R) \mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}R)}} \in A \cap \left(B(0,R) \setminus B(0,\lambda^{-1}R)\right)\right). \end{aligned}$$

Thus, we have proved (4.8) with $s = R$. □

5. Upper bound estimates

In this section, we derive the upper bound estimate on $p_D(t,x,y)$ for $t \leqslant T$ in $C^{1,1}$ open set D with $C^{1,1}$ characteristics (R_0, Λ) . As before, we will assume that $R_0 < 1$ and $\Lambda > 1$ and fix such $C^{1,1}$ open set D throughout this section. We first introduce the next lemma which give a guideline to obtain the upper bound estimate on $p_D(t,x,y)$ (for its proof, see [13, Lemma 3.1] and [5, Lemma 1.10]). Applying (4.7) and Theorem 1.2 to (5.2), in Proposition 5.3 we will obtain the intermediate upper bound for $p_D(t,x,y)$ having one boundary decay. Applying this result, (4.7) and the upper bound of the survival probability of Y (Lemma 5.2) to (5.1), we can get the short time sharp upper bound estimate for $p_D(t,x,y)$.

Lemma 5.1. *Let Y be a symmetric pure jump Hunt process whose jumping intensity kernel J satisfies the conditions* **(J1.1)** *and* **(J1.2)**. *Suppose that $E \subset \mathbb{R}^d$ be an open set. Let $U_1, U_3 \subset E$ be disjoint open subsets and $U_2 := E \setminus (U_1 \cup U_3)$. If $x \in U_1$, $y \in U_3$ and $t > 0$, we have*

$$p_E(t,x,y) \leqslant \mathbb{P}_x\left(Y_{\tau_{U_1}} \in U_2\right) \cdot \sup_{s<t, z\in U_2} p_E(s,z,y) + \int_0^t \mathbb{P}(\tau_{U_1} > s)\mathbb{P}_y(\tau_E > t-s)\, ds \cdot \sup_{u\in U_1, z\in U_3} J(u,z) \tag{5.1}$$
$$\leqslant \mathbb{P}_x\left(Y_{\tau_{U_1}} \in U_2\right) \cdot \sup_{s<t, z\in U_2} p(s,z,y) + \left(t \wedge \mathbb{E}_x[\tau_{U_1}]\right) \cdot \sup_{u\in U_1, z\in U_3} J(u,z). \tag{5.2}$$

For the remainder of the section, we assume that Y is the symmetric pure jump Hunt process with the jumping intensity kernel J satisfying the conditions **(J1.1)**, **(J1.2)**, **(K$_\eta$)** and **(SD)**. Let

$$a_{T,R_0} := [V(A_{4.2}R_0/4)]^2 T^{-1}$$

where $A_{4.2}$ is the constant in Theorem 4.2(1). Denote V^{-1} be the inverse function of V, then $V^{-1}(\sqrt{a_{T,R_0} \cdot t}) \leqslant A_{4.2} R_0/4$ for any $t \leqslant T$.

Lemma 5.2. *There exists $C_{5.2} = C_{5.2}(\phi, L_0, L_3, \eta, R_0, \Lambda, T)$ such that for any $t \leqslant T$ and $x \in D$, we have that*

$$\mathbb{P}_x(\tau_D > t) \leqslant C_{5.2}\left(1 \wedge \frac{V(\delta_D(x))}{\sqrt{t}}\right).$$

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Proof. Let $r_t := V^{-1}(\sqrt{a_{T,R_0} \cdot t})$. We only consider the case $V(\delta_D(x)) < \sqrt{a_{T,R_0} \cdot t}$ which implies $\delta_D(x) < A_{4.2} R_0/4 < R_0/4$. Let $U_t := D \cap B(z_x, 2r_t) \subset D$ where $z_x \in \partial D$ with $\delta_D(x) = |x - z_x|$. Then, using Chebyshev's inequality and Corollary 2.4, we first obtain that

$$\begin{aligned}\mathbb{P}_x(\tau_D > t) &= \mathbb{P}_x(\tau_{U_t} > t, \tau_D = \tau_{U_t}) + \mathbb{P}_x(\tau_D > \tau_{U_t} > t)\\ &\leqslant \mathbb{P}_x(\tau_{U_t} > t) + \mathbb{P}_x(Y_{\tau_{U_t}} \in D) \leqslant \big(t^{-1} + C_{2.4}/\phi(2r_t)\big)\, \mathbb{E}_x \tau_{U_t}.\end{aligned}$$

From (3.3) and the fact that V is increasing and subadditive,

$$\phi(2r_t) \asymp [V(2r_t)]^2 \asymp [V(V^{-1}(\sqrt{a_{T,R_0} \cdot t}))]^2 = a_{T,R_0} \cdot t.$$

Therefore, using (4.7) in Theorem 4.2, we conclude that

$$\mathbb{P}_x(\tau_D > t) \leqslant c_1 \frac{1}{t} \mathbb{E}_x \tau_{U_t} \leqslant c_1 C_{4.2.1} \frac{1}{t} V(r_t) V(\delta_D(x)) \leqslant c_2 \frac{V(\delta_D(x))}{\sqrt{t}}. \qquad \square$$

We will use the following inequality several times, which follows from **(WS)**: there exist $C_I := (\overline{C}\underline{c}^{-1} \vee \underline{c}^{-2})^{1/\underline{\alpha}} > 1$ such that

$$C_I^{-1} \left(\frac{r}{R}\right)^{1/\underline{\alpha}} \leqslant \frac{\phi^{-1}(r)}{\phi^{-1}(R)} \leqslant C_I \left(\frac{r}{R}\right)^{1/\overline{\alpha}} \qquad \text{for } \ 0 < r \leqslant R. \tag{5.3}$$

Recall the functions $F_{a,\gamma,T}(t,r)$ and $\Psi(t,x)$ are defined in (1.6) and (1.7), respectively.

Proposition 5.3. *Let $a \leqslant a_{T,R_0}$.*

(1) *Suppose that D is bounded and the jumping intensity kernel J satisfies the condition* **(J1)**. *Then there exists a positive constant $C_{5.3.1} = C_{5.3.1}(\phi, L_0, L_3, \eta, R_0, \Lambda, T, \mathrm{diam}(D), a)$ such that for any $(t,x,y) \in (0,T] \times D \times D$ with $V^{-1}(\sqrt{a \cdot t}) \leqslant |x-y|$, we have*

$$p_D(t,x,y) \leqslant C_{5.3.1}\, \Psi(t,x) \cdot \left([\phi^{-1}(t)]^{-d} \wedge t\nu(|x-y|)\right).$$

(2) *Suppose that the jumping intensity kernel J satisfies the condition* **(J2)**. *Then there exists a positive constant $C_{5.3} = C_{5.3}(\beta, \phi, L_0, L_3, \eta, R_0, \Lambda, T, a)$ such that for any $(t,x,y) \in (0,T] \times D \times D$ with $|x-y| \geqslant V^{-1}(\sqrt{a \cdot t}) \cdot \mathbf{1}_{\beta \in [0,1]} + 2 \cdot \mathbf{1}_{\beta \in (1,\infty)} + \big(2 + V^{-1}(\sqrt{a \cdot t})\big) \cdot \mathbf{1}_{\beta = \infty}$, we have*

$$p_D(t,x,y) \leqslant C_{5.3}\, \Psi(t,x) \cdot \begin{cases} F_{C_{1.2} \wedge \gamma_1, \gamma_1, T}\,(t, |x-y|/3) & \text{if } \beta \in [0,\infty), \\ (2t/T|x-y|)^{C_{1.2}|x-y|/2} & \text{if } \beta = \infty. \end{cases}$$

Proof. Since we assume that D is bounded in (1), by applying Theorem 1.1 instead of Theorem 1.2 the proof of (1) is similar to the that of (2), so we only give the proof of (2). Let $r_t := V^{-1}(\sqrt{a \cdot t})/9$. If $\delta_D(x) \geqslant r_t/2$, using subadditivity of V, we see that $\Psi(t,x) \asymp 1$. Thus, by Theorem 1.2, and the fact that $r \to F_{c,\gamma,T}(t,r)$ is decreasing, we obtain the conclusion.

Let $0 < \delta_D(x) \leqslant r_t/2$. Since $9r_t \leqslant V^{-1}(\sqrt{a_{T,R_0} \cdot T}) = A_{4.2} R_0/4 < 1$, $|x-y| \geqslant 9r_t$ for all $\beta \in [0,\infty]$. Choose a point $z_x \in \partial D$ with $\delta_D(x) = |x - z_x|$ and let $U_1 := B(z_x, r_t) \cap D$, $U_3 := \{z \in D : |z-x| \geqslant |x-y|/2\}$ and $U_2 := D \setminus (U_1 \cup U_3)$. Then $x \in U_1$, $y \in U_3$ and $U_1 \cap U_3 = \emptyset$. Note that $|x-y|/2 \leqslant |x-y| - |z-x| \leqslant |y-z|$ for any $z \in U_2$. Therefore, by virtue of Theorem 1.2, we obtain

$$\begin{aligned}\sup_{s<t, z \in U_2} p(s,z,y) &\leqslant C_{1.2} \sup_{s<t, |z-y|>|x-y|/2} F_{C_{1.2},\gamma_1,T}(s, |z-y|)\\ &\leqslant c_1 F_{C_{1.2},\gamma_1,T}(t, |x-y|/2).\end{aligned} \tag{5.4}$$

Please cite this article as: T. Grzywny, K.-Y. Kim and P. Kim, Estimates of Dirichlet heat kernel for symmetric Markov processes, Stochastic Processes and their Applications (2019), https://doi.org/10.1016/j.spa.2019.03.017.

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In fact, if $\beta \in (1,\infty]$, we have $|z-y| \geqslant |x-y|/2 > 1$ and so $F_{C_{1.2},\gamma_1,T}(s,|z-y|)$ is increasing in s. If $\beta \in [0,1]$, we have $|z-y| \geqslant |x-y|/2 \geqslant V^{-1}(\sqrt{a\cdot t})/2$ and so $\left([\phi^{-1}(s)]^{-d} \wedge s\nu(|z-y|)e^{-\gamma|z-y|^\beta}\right) \asymp s\nu(|z-y|)e^{-\gamma|z-y|^\beta}$ using (3.3), (3.4) and (5.3). Also, $s\nu(r)e^{-\gamma r^\beta}$ is increasing in s. Thus, combining there observations with the fact $r \to F_{C_{1.2},\gamma_1,T}(t,r)$ is decreasing, the second inequality above holds.

By Corollary 2.4 and (4.7) in Theorem 4.2, we obtain

$$\begin{aligned}\mathbb{P}_x(Y_{\tau_{U_1}} \in U_2) &\leqslant \mathbb{P}_x\left(Y_{\tau_{U_1}} \in D\right) \leqslant C_{2.4}\phi(r_t)^{-1}\mathbb{E}_x\tau_{U_1}\\ &\leqslant C_{2.4}\cdot C_{4.2.1}V(\delta_D(x))/V(r_t) \leqslant c_3 V(\delta_D(x))/\sqrt{t}.\end{aligned} \tag{5.5}$$

In the last inequality we use monotonicity and subadditivity of V, which imply $V(r_t) \asymp \sqrt{t}$.

Note that for $u \in U_1$ and $z \in U_3$ that

$$|u-z| \geqslant |z-x| - |x-z_x| - |u-z_x| \geqslant |x-y|/2 - 3r_t/2. \tag{5.6}$$

Let $\beta \in [0,\infty)$. Since $|x-y| \geqslant 9r_t$, from (5.6) we have $|u-z| \geqslant |x-y|/3$ for $(u,z) \in U_1 \times U_3$, therefore by (1.4), (1.5) and **(J2)**,

$$\left(\sup_{u\in U_1, z\in U_3} J(u,z)\right) \leqslant c_4 e^{-\gamma_1(|x-y|/3)^\beta}\nu(|x-y|) \leqslant c_5 t^{-1}F_{\gamma_1,\gamma_1,T}(t,|x-y|/3).$$

Combining this with (4.7) in Theorem 4.2, we conclude that

$$\begin{aligned}\mathbb{E}_x[\tau_{U_1}]\left(\sup_{u\in U_1, z\in U_3} J(u,z)\right) &\leqslant C_{4.2.1}V(r)V(\delta_D(x))\cdot c_5 t^{-1}F_{\gamma_1,\gamma_1,T}(t,|x-y|/3)\\ &\leqslant c_6\frac{V(\delta_D(x))}{\sqrt{t}}F_{\gamma_1,\gamma_1,T}(t,|x-y|/3).\end{aligned} \tag{5.7}$$

If $\beta = \infty$, since $|u-z| \geqslant |x-y|/2 - 3r_t/2 \geqslant (1+4r_t) - 3r_t/2 \geqslant 1$, we have $J(u,z) = 0$.

Hence, by applying (5.4)–(5.7) to (5.2) for the case $\beta \in [0,\infty)$ and by applying (5.7) to (5.2) for the case $\beta = \infty$, we reach the conclusion. □

We denote by X the process in the case $\beta = 0$ in **(J2)**, that is, X is a symmetric Hunt process whose jumping kernel is $J^X(x,y) := \kappa(x,y)\nu(|x-y|)$.

Proposition 5.4.

(1) *Suppose that the jumping intensity kernel J satisfying* **(J1)** *and D is bounded. There exists a positive constant $C_{5.4.1} = C_{5.4.1}(\phi, L_0, L_3, \eta, R_0, \Lambda, T, \mathrm{diam}(D))$ such that for any $(t,x,y) \in (0,T] \times D \times D$, we have*

$$p_D(t,x,y) \leqslant C_{5.4.1}\Psi(t,x)\Psi(t,y)\left([\phi^{-1}(t)]^{-d} \wedge t\nu(|x-y|)\right).$$

(2) *There exists a positive constant $C_{5.4} = C_{5.4}(\phi, L_0, L_3, \eta, R_0, \Lambda, T)$ such that for any $(t,x,y) \in (0,T] \times D \times D$, we have*

$$p^X_D(t,x,y) \leqslant C_{5.4}\Psi(t,x)\Psi(t,y)\left([\phi^{-1}(t)]^{-d} \wedge t\nu(|x-y|)\right).$$

Proof. Using Theorem 1.1 and Proposition 5.3 (1) instead of Theorem 1.2 and Proposition 5.3 (2) respectively, the proof of (1) is almost identical to the one of (2). So we only give the proof of (2).

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The semigroup property, Theorem 1.2 (for $\beta = 0$), (5.3) and Lemma 5.2 yield

$$
\begin{aligned}
p_D^X(t/2, x, y) &\leqslant \left(\sup_{z,w\in D} p_D^X(t/4, z, w) \right) \int_D p_D^X(t/4, x, z)dz \\
&\leqslant c_1[\phi^{-1}(t/4)]^{-d}\mathbb{P}_x(\tau_D > t/4) \leqslant c_2[\phi^{-1}(t)]^{-d}\,\Psi(t, x).
\end{aligned}
$$

Thus, by Proposition 5.3, (5.3) and Theorem 1.2 (for $\beta = 0$), we obtain

$$
p_D^X(t/2, x, y) \leqslant c_3\,\Psi(t/2, x) \left([\phi^{-1}(t/2)]^{-d} \wedge (t/2)\nu(|x-y|)\right) \leqslant c_4\,\Psi(t, x)p^X(t/2, x, y).
$$

Combining these with Theorem 1.2 (for $\beta = 0$), the symmetry p_D^X and the semigroup property of p^X, we conclude that

$$
\begin{aligned}
p_D^X(t, x, y) &= \int_D p_D^X(t/2, x, z)\cdot p_D^X(t/2, z, y)dz \leqslant c_4^2\,\Psi(t, x)\Psi(t, y)p^X(t, x, y) \\
&\leqslant c_5\,\Psi(t, x)\Psi(t, y) \left([\phi^{-1}(t)]^{-d} \wedge t\nu(|x-y|)\right). \quad \square
\end{aligned}
$$

Suppose that the jumping intensity kernel J satisfying **(J2)**. By Meyer's construction (e.g., see [19, §4.1]), when $\beta \in (0, \infty]$ the process Y can be constructed from X by removing jumps of size greater than 1 with suitable rate. Let $p_D^X(t, x, y)$ be the transition density function of X on D. For $\beta \in (0, \infty]$, we define

$$
\mathcal{J}(x) := \int_{\mathbb{R}^d} \kappa(x, y)\nu(|x-y|)\left(1 - \chi(|x-y|)^{-1}\right) dy
$$

where $\chi(|x-y|)$ is defined in (1.5). Then $\|\mathcal{J}\|_\infty \leqslant c_1 \int_{|z|\geqslant 1} \nu(|z|)dz < \infty$. By [1, Lemma 3.6] we have

$$
p_D(t, x, y) \leqslant e^{T\|\mathcal{J}\|_\infty} p_D^X(t, x, y) \quad \text{for any } (t, x, y) \in (0, T] \times D \times D. \tag{5.8}
$$

Thus, the sharp upper bound of $p_D(t, x, y)$ for $|x-y| < M$ for some $M > 0$ follows from the one of $p_D^X(t, x, y)$ and (5.8). Therefore Combining (5.8), Propositions 5.3(2) and 5.4(2), we have the following result.

Proposition 5.5. *Suppose that the jumping intensity kernel J satisfying* **(J2)**. *There exists a positive constant $C_{5.5} = C_{5.5}(\beta, \phi, L_0, L_3, \eta, R_0, \Lambda, T)$ such that for every $(t, x, y) \in (0, T] \times D \times D$ we have*

$$
p_D(t, x, y) \leqslant C_{5.5}\,\Psi(t, x) \cdot \begin{cases} F_{C_{1.2}\wedge\gamma_1,\gamma_1,T}(t, |x-y|/3) & \text{if } \beta \in [0, \infty), \\ F_{C_{1.2},\gamma_1,T}(t, |x-y|/2) & \text{if } \beta = \infty, \end{cases} \tag{5.9}
$$

where $C_{1.2}$ is the constant in Theorem 1.2 and γ_1 is the constant in (1.5).

Now we are ready to prove the upper bound of Theorem 1.3(1) and Theorem 1.5(1).

Proofs of the upper bounds of $p_D(t, x, y)$ in Theorems 1.3(1) and 1.5(1). In Proposition 5.5(1), we have proved the upper bound of $p_D(t, x, y)$ in Theorem 1.3(1). So we only give the proof of the upper bound of $p_D(t, x, y)$ in Theorem 1.5(1).

Let $r_t := V^{-1}(\sqrt{a_{T,R_0}\cdot t})$ so that $r_t \leqslant A_{4.2}R_0/4 < 1/4$. By Proposition 5.5 and the symmetry of $p_D(t, x, y)$, we only need to prove the upper bound of $p_D(t, x, y)$ for the case $\delta_D(x) \vee \delta_D(y) < r_t$, which we will assume throughout the proof.

If $\beta = \infty$ and $6 < |x-y| \leqslant 6(1 \vee C_{1.2}^{-1})$, by (5.8) and Proposition 5.4, we have

$$
p_D(t, x, y) \leqslant c_1\,\Psi(t, x)\Psi(t, y)(t/T) \leqslant c_1\,\Psi(t, x)\Psi(t, y)(t/T)^{(C_{1.2}\wedge 1)|x-y|/6}.
$$

Please cite this article as: T. Grzywny, K.-Y. Kim and P. Kim, Estimates of Dirichlet heat kernel for symmetric Markov processes, Stochastic Processes and their Applications (2019), https://doi.org/10.1016/j.spa.2019.03.017.

If either the case $\beta \in [0,\infty)$ and $|x-y| \leqslant 6(1\vee C_{1.2}^{-1})$ holds or the case $\beta=\infty$ and $|x-y|\leqslant 6$ holds, by (5.8) and Proposition 5.4(2), we have

$$p_D(t,x,y) \leqslant e^{T\|\mathcal{J}\|_\infty} p_D^X(t,x,y) \leqslant c_2 \Psi(t,x)\Psi(t,y)\left([\phi^{-1}(t)]^{-d} \wedge t\nu(|x-y|)\right).$$

Thus, the upper bound of $p_D(t,x,y)$ in Theorem 1.5(1) holds for $|x-y| \leqslant 6(1\vee C_{1.2}^{-1})$.

For the remainder of the proof, we assume that $\delta_D(x)\vee\delta_D(y) < r_t$ and $|x-y| > 6(1\vee C_{1.2}^{-1})$. For any x with $\delta_D(x) < r_t$, let $z_x \in \partial D$ such that $\delta_D(x) = |z_x - x|$. Let $U_1 := B(z_x, r_t)\cap D$, $U_3 := \{z\in D : |z-x| \geqslant |x-y|/2\}$, and $U_2 := D\setminus(U_1\cup U_3)$. Note that $x\in U_1$ and $y\in U_3$ and $|x-y|/2 \leqslant |z-y|$ for $z\in U_2$. Thus, by Proposition 5.5 we have

$$\begin{aligned}
&\sup_{s<t, z\in U_2} p_D(s,z,y)\\
&\leqslant \sup_{s<t, z\in U_2} C_{5.5}\frac{V(\delta_D(y))}{\sqrt{s}}\cdot\Big(F_{C_{1.2}\wedge\gamma_1,\gamma_1,T}(s,|z-y|/3)\cdot\mathbf{1}_{\beta\in[0,\infty)}\\
&\qquad + F_{C_{1.2},\gamma_1,T}(s,|z-y|/2)\cdot\mathbf{1}_{\beta=\infty}\Big)\\
&\leqslant C_{5.5}\, V(\delta_D(y)) \sup_{s<t, |x-y|/2\leqslant|z-y|}\frac{1}{\sqrt{s}}\\
&\qquad\cdot\Big(F_{C_{1.2}\wedge\gamma_1,\gamma_1,T}(s,|z-y|/3)\cdot\mathbf{1}_{\beta\in[0,\infty)} + F_{C_{1.2},\gamma_1,T}(s,|z-y|/2)\cdot\mathbf{1}_{\beta=\infty}\Big)\\
&\leqslant c_3\frac{V(\delta_D(y))}{\sqrt{t}}\cdot\Big(F_{C_{1.2}\wedge\gamma_1,\gamma_1,T}(t,|x-y|/6)\cdot\mathbf{1}_{\beta\in[0,\infty)}\\
&\qquad + F_{C_{1.2},\gamma_1,T}(t,|x-y|/4)\cdot\mathbf{1}_{\beta=\infty}\Big).
\end{aligned}\tag{5.10}$$

The last inequality is clear for $\beta\in[0,\infty)$ by the definition of $F_{C_{1.2}\wedge\gamma_1,\gamma_1,T}(t,r)$ and for $\beta=\infty$ we used the fact that $s\to s^{-1/2}(s/Tr)^{ar}$ is increasing if $ar\geqslant 1$. Hence from (5.5) and (5.10), we obtain

$$\begin{aligned}
&\mathbb{P}_x\left(Y_{\tau_{U_1}}\in U_2\right)\Big(\sup_{s<t,z\in U_2} p_D(s,z,y)\Big)\\
&\leqslant c_4\frac{V(\delta_D(x))}{\sqrt{t}}\frac{V(\delta_D(y))}{\sqrt{t}}\cdot\begin{cases}F_{C_{1.2}\wedge\gamma_1,\gamma_1,T}(t,|x-y|/6) & \text{if } \beta\in[0,\infty),\\ F_{C_{1.2},\gamma_1,T}(t,|x-y|/4) & \text{if } \beta=\infty.\end{cases}
\end{aligned}\tag{5.11}$$

Also from Lemma 5.2, we have

$$\begin{aligned}
&\int_0^t \mathbb{P}_x(\tau_{U_1}>s)\mathbb{P}_y(\tau_D>t-s)ds \leqslant \int_0^t \mathbb{P}_x(\tau_D>s)\mathbb{P}_y(\tau_D>t-s)ds\\
&\qquad\leqslant C_{5.2}^2\, V(\delta_D(x))V(\delta_D(y))\int_0^t s^{-1/2}(t-s)^{-1/2}ds \leqslant c_5\, t\,\frac{V(\delta_D(x))}{\sqrt{t}}\frac{V(\delta_D(y))}{\sqrt{t}}.
\end{aligned}\tag{5.12}$$

For $(u,z)\in U_1\times U_3$ and $|x-y|>6>6r_t$, note that $|u-z|\geqslant|x-y|-|x-u|-|z-y|\geqslant |x-y|/3$. Thus, if $\beta\in[0,\infty)$, by (1.5) and **(J2)**,

$$\left(\sup_{u\in U_1, z\in U_3} J(u,z)\right)\leqslant c_6 e^{-\gamma_1(|x-y|/3)^\beta}\nu(|x-y|/3)\leqslant c_7 t^{-1}F_{\gamma_1,\gamma_1,T}(t,|x-y|/3).$$

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Combining this with (5.12), we obtain

$$\int_0^t \mathbb{P}_x\left(\tau_{U_1} > s\right)\mathbb{P}_y\left(\tau_D > t-s\right)ds \cdot \left(\sup_{u\in U_1, z\in U_3} J(u,z)\right)$$
$$\leqslant c_8 \frac{V(\delta_D(x))}{\sqrt{t}} \frac{V(\delta_D(y))}{\sqrt{t}} F_{\gamma_1,\gamma_1,T}(t, |x-y|/3). \tag{5.13}$$

If $\beta = \infty$, since $|u-z| > 1$, $J(u,z) = 0$ on $U_1 \times U_3$.

Therefore by applying (5.11) and (5.13) for $\beta \in [0,\infty)$ and by applying (5.11) for $\beta = \infty$ in (5.1) of Lemma 5.1, we prove the upper bound of $p_D(t,x,y)$ in Theorem 1.5(1) for $\delta_D(x) \vee \delta_D(y) < r_t$ and $|x-y| > 6(1 \vee C_{1.2}^{-1})$.

6. Preliminary lower bound estimates

In this section, we discuss a preliminary lower bound for $p_D(t,x,y)$. In this section we will always assume that Y is the symmetric pure jump Hunt process with the jumping intensity kernel J satisfying either the conditions **(J1.2)** and **(J1.3)** or the condition **(J2)**.

Since Y satisfies conditions imposed in [11], using [11, Theorem 5.2 and Lemma 2.5], the proof of the next lemma is the same as that of [18, Lemma 3.2]. Thus, we omit the proof.

Lemma 6.1. *Let a, b and T be positive constants. Then there exists a constant $C_{6.1} = C_{6.1}(a,b,L_0,\phi,T) > 0$ such that for all $\lambda \in (0,T]$ we have*

$$\inf_{\substack{y\in\mathbb{R}^d \\ |y-z|\leqslant b\phi^{-1}(\lambda)}} \mathbb{P}_y\left(\tau_{B(z,2b\phi^{-1}(\lambda))} > a\lambda\right) \geqslant C_{6.1}. \tag{6.1}$$

Let D be an arbitrary non-empty open set, and a and T be positive constants. We use the convention that $\delta_D(\cdot) \equiv \infty$ when $D = \mathbb{R}^d$ to derive the lower bound of $p(t,x,y)$ in Theorems 1.1 and 1.2 simultaneously.

Using [11, Theorem 5.2] and Lemma 6.1, the proof of the following Proposition is similar to that of [18, Proposition 3.3]. Thus, we omit the proof.

Proposition 6.2. *Let D be an arbitrary open set and let a and T be positive constants. Suppose that $(t,x,y) \in (0,T] \times D \times D$, with $\delta_D(x) \geqslant a\phi^{-1}(t) \geqslant 2|x-y|$. Then there exists a positive constant $C_{6.2} = C_{6.2}(a,L_0,\phi,T)$ such that $p_D(t,x,y) \geqslant C_{6.2}[\phi^{-1}(t)]^{-d}$.*

From Lemma 6.1 and Proposition 6.2 we see that, under the condition **(WS)** on ϕ, the behavior of Y is locally stable in terms of ϕ.

Proposition 6.3. *Let D be an arbitrary open set and let a and T be positive constants.*

(1) *Suppose that the jumping intensity kernel J satisfies the conditions* **(J1.2)** *and* **(J1.3)***. Then for every $M > 0$, there exists a constant $C_{6.3.1} = C_{6.3.1}(a,M,L_0,\phi,T) > 0$ such that for all $(t,x,y) \in (0,T] \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geqslant a\phi^{-1}(t)$ and $a\phi^{-1}(t) \leqslant 2|x-y| \leqslant 2M$ we have $p_D(t,x,y) \geqslant C_{6.3.1} t\nu(|x-y|)$.*

(2) *Suppose that the jumping intensity kernel J satisfies the condition* **(J2)***. Then there exists a constant $C_{6.3} = C_{6.3}(a,L_0,\phi,T) > 0$ such that for every $(t,x,y) \in (0,T] \times D \times D$, with $\delta_D(x) \wedge \delta_D(y) \geqslant a\phi^{-1}(t)$ and $a\phi^{-1}(t) \leqslant 2|x-y|$, we have $p_D(t,x,y) \geqslant C_{6.3} t\nu(|x-y|)/\chi(|x-y|)$.*

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Please cite this article as: T. Grzywny, K.-Y. Kim and P. Kim, Estimates of Dirichlet heat kernel for symmetric Markov processes, Stochastic Processes and their Applications (2019), https://doi.org/10.1016/j.spa.2019.03.017.

Proof. We first give the proof of (2). By Lemma 6.1, there exists $c_1 = c_1(a, L_0, \phi, T) > 0$ such that

$$\inf_{\{|z-y|\leqslant 4^{-1}a\phi^{-1}(t)\}} \mathbb{P}_z(\tau_{B(z,6^{-1}a\phi^{-1}(t))} > t) \geqslant c_1.$$

Thus by the strong Markov property

$$\mathbb{P}_x\left(Y_t^D \in B\big(y,\, 2^{-1}a\phi^{-1}(t)\big)\right) \geqslant c_1\mathbb{P}_x\left(Y^D \text{ hits the ball } B(y,\, 4^{-1}a\phi^{-1}(t)) \text{ by time } t\right).$$

Using this and the Lévy system in (4.1), we obtain

$$\begin{aligned}
&\mathbb{P}_x\left(Y_t^D \in B\big(y,\, 2^{-1}a\phi^{-1}(t)\big)\right) \\
&\geqslant c_1\mathbb{P}_x(Y^D_{t\wedge\tau_{B(x,6^{-1}a\phi^{-1}(t))}} \in B(y,\, 4^{-1}a\phi^{-1}(t)) \text{ and } t\wedge\tau_{B(x,6^{-1}a\phi^{-1}(t))} \text{ is a jumping time }) \\
&= c_1\mathbb{E}_x\left[\int_0^{t\wedge\tau_{B(x,6\cdot 2^{-5}a\phi^{-1}(t))}} \int_{B(y,\,4^{-1}a\phi^{-1}(t))} J(Y_s,u)\,du\,ds\right].
\end{aligned} \tag{6.2}$$

Lemma 6.1 also implies that

$$\mathbb{E}_x\left[t\wedge\tau_{B(x,6\cdot2^{-5}a\phi^{-1}(t))}\right] \geqslant t\,\mathbb{P}_x\left(\tau_{B(x,6\cdot2^{-5}a\phi^{-1}(t))}\geqslant t\right) \geqslant c_2\,t \qquad \text{for all } t\in(0,T]. \tag{6.3}$$

Let w be the point on the line connecting x and y (i.e., $|x-y| = |x-w| + |w-y|$) such that $|w-y| = 7\cdot 2^{-5}a\phi^{-1}(t)$, then $B(w, 2^{-5}a\phi^{-1}(t)) \subset B(y,\, 4^{-1}a\phi^{-1}(t))$. Moreover, for every $(z,u)\in B(x, 6\cdot 2^{-5}a\phi^{-1}(t))\times B(w, 2^{-5}a\phi^{-1}(t))$, we have

$$\begin{aligned}
|z-u| &< 6^{-1}a\phi^{-1}(t) + 2^{-5}a\phi^{-1}(t) + |x-w| \\
&= |x-y| + (6\cdot 2^{-5} + 2^{-5} - 7\cdot 2^{-5})a\phi^{-1}(t) = |x-y|
\end{aligned}$$

and thus $B(w, 2^{-5}a\phi^{-1}(t)) \subset \{u : |z-u| < |x-y|\}$. Combining this result with **(J2)**, (1.4) and (6.3), we obtain

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{t\wedge\tau_{B(x,6\cdot2^{-5}a\phi^{-1}(t))}} \int_{B(y,\,4^{-1}a\phi^{-1}(t))} J(Y_s,u)\,du\,ds\right] \\
&\geqslant \mathbb{E}_x\left[\int_0^{t\wedge\tau_{B(x,6\cdot2^{-5}a\phi^{-1}(t))}} \int_{B(w,2^{-5}a\phi^{-1}(t))} J(Y_s,u)\mathbf{1}_{\{|Y_s-u|<|x-y|\}}\,du\,ds\right] \\
&\geqslant L_0^{-1}\mathbb{E}_x\left[t\wedge\tau_{B(x,6\cdot2^{-5}a\phi^{-1}(t))}\right]|B(w, 2^{-5}a\phi^{-1}(t))|\nu(|x-y|)/\chi(|x-y|) \\
&> c_3 t[\phi^{-1}(t)]^d\nu(|x-y|)/\chi(|x-y|).
\end{aligned} \tag{6.4}$$

Then, using the semigroup property along with Proposition 6.2, (6.4) and (5.3), the proposition follows from the proof of [18, Proposition 3.5].

The proof of (1) is identical to the that of (2) except that we apply **(J1.3)** in (6.4) instead of **(J2)** and (1.4). □

Combining Propositions 6.2 and 6.3, we obtain the following preliminary lower bound of $p_D(t,x,y)$. Note that the lower bound in Proposition 6.4(1) is the sharp interior lower bound of $p_D(t,x,y)$ under the conditions **(J1.2)** and **(J1.3)**. Moreover, under the condition **(J2)**, the lower bound in Proposition 6.4(2) that yields the sharp interior lower bound of $p_D(t,x,y)$ for the case $\beta\in[0,1]$ and the case $\beta\in(1,\infty]$ with $|x-y|<1$.

Proposition 6.4. *Let D be an arbitrary open set and let a and T be positive constants.*

(1) *Suppose that the jumping intensity kernel J satisfies the conditions* **(J1.2)** *and* **(J1.3)**. *Then, for every $(t,x,y) \in (0,T] \times D \times D$ and $M > 0$, with $\delta_D(x) \wedge \delta_D(y) \geqslant a\phi^{-1}(t)$ and $|x-y| < M$, there exists a constant $C_{6.4.1} = C_{6.4.1}(a, M, L_0, \phi, T) > 0$ such that*

$$p_D(t,x,y) \geqslant C_{6.4.1}\Big([\phi^{-1}(t)]^{-d} \wedge t\nu(|x-y|)\Big).$$

(2) *Suppose that the jumping intensity kernel J satisfies the condition* **(J2)**. *Then, for every $(t,x,y) \in (0,T] \times D \times D$, with $\delta_D(x) \wedge \delta_D(y) \geqslant a\phi^{-1}(t)$, there exists a constant $C_{6.4} = C_{6.4}(a, L_0, \phi, T) > 0$ such that*

$$p_D(t,x,y) \geqslant C_{6.4}\Big([\phi^{-1}(t)]^{-d} \wedge t\nu(|x-y|)/\chi(|x-y|)\Big).$$

For the remainder of this section, assume that the jumping intensity kernel J satisfies the condition **(J2)** for $\beta \in (1,\infty]$ with $|x-y| \geqslant 1$. Also, we assume that D is a *connected* open set with the following property: there exist $\lambda_1 \in [1,\infty)$ and $\lambda_2 \in (0,1]$ such that for every $r \leqslant 1$ and x, y in the same component of D with $\delta_D(x) \wedge \delta_D(y) \geqslant r$ there exists in D a length parameterized rectifiable curve l connecting x to y with the length $|l|$ of l less than or equal to $\lambda_1|x-y|$ and $\delta_D(l(u)) \geqslant \lambda_2 r$ for $u \in [0,|l|]$.

Now we prove the preliminary lower bound of $p_D(t,x,y)$ separately for the case $\beta = \infty$ and the case $\beta \in (1,\infty)$. We will closely follow the proofs of [10, Theorem 3.6] and [12, Theorem 5.5].

Proposition 6.5. *Let $\beta = \infty$. Suppose that $T > 0$ and $a \in (0, \big(4\phi^{-1}(T)\big)^{-1}]$. Then there exist constants $C_{6.5.i} = C_{6.5.i}(a, L_0, \phi, T, \lambda_1, \lambda_2) > 0$, $i = 1,2$, such that for any $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geqslant a\phi^{-1}(t)$, $|x-y| \geqslant 1$, and $t \leqslant T$ we have*

$$p_D(t,x,y) \geqslant C_{6.5.1}\left(\frac{t}{T|x-y|}\right)^{C_{6.5.2}|x-y|}.$$

Proof. Let $R_1 := |x-y| \geqslant 1$, and by the assumption on D, there is a length parameterized curve $l \subset D$ connecting x and y such that the total length $|l| \leqslant \lambda_1 R_1$ and $\delta_D(l(u)) \geqslant \lambda_2 a\phi^{-1}(t)$ for every $u \in [0,|l|]$. Define k be the integer satisfying $(4 \leqslant)4\lambda_1 R_1 \leqslant k < 4\lambda_1 R_1 + 1 \leqslant 5\lambda_1 R_1$ and $r_t := 2^{-1}\lambda_2 a\phi^{-1}(t) \leqslant 8^{-1}$. For each $i = 0,1,2,\ldots,k$, let $x_i := l(i|l|/k)$ and $B_i := B(x_i, r_t)$, then $\delta_D(x_i) \geqslant 2r_t$ and $B_i \subset B(x_i, 2r_t) \subset D$. Since $4\lambda_1 R_1 \leqslant k$ for each $y_i \in B_i$, we have

$$|y_i - y_{i+1}| \leqslant |y_i - x_i| + |x_i - x_{i+1}| + |x_{i+1} - y_{i+1}| \leqslant \frac{1}{8} + \frac{|l|}{k} + \frac{1}{8} \leqslant \frac{\lambda_1 R_1}{4\lambda_1 R_1} + \frac{1}{4} = \frac{1}{2}. \tag{6.5}$$

Moreover $\delta_D(y_i) \geqslant \delta_D(x_i) - |y_i - x_i| \geqslant r_t \geqslant r_{t/k}$. Thus by Proposition 6.4(2), there are constants $c_i = c_i(a, L_0, \phi, T) > 0$, $i = 1,2$, such that for $(y_i, y_{i+1}) \in B_i \times B_{i+1}$ we have

$$p_D(t/k, y_i, y_{i+1}) \geqslant c_1\left(\frac{1}{[\phi^{-1}(t/k)]^d} \wedge \frac{t/k}{\phi(|y_i - y_{i+1}|)|y_i - y_{i+1}|^d}\right) \geqslant c_2 t/(Tk). \tag{6.6}$$

The last inequality comes from $t/k \leqslant T/4$ for the first part and (6.5) for the second part. Note that $r_t \geqslant c_3(t/kT)^{1/\underline{\alpha}}$ for some $c_3 = c_3(a, \phi, T, \lambda_2)$ by (5.3). Hence, combining these observations and the fact that $k \asymp R_1$, we conclude that

$$\begin{aligned} p_D(t,x,y) &\geqslant \int_{B_1} \dots \int_{B_{k-1}} p_D(t/k, x, y_1) \dots p_D(t/k, y_{k-1}, y) dy_{k-1} \dots dy_1 \\ &\geqslant (c_2 t (Tk)^{-1})^k \Pi_{i=1}^{k-1} |B_i| \geqslant (c_4 t (Tk)^{-1})^{c_5 k} \geqslant (c_6 t (TR_1)^{-1})^{c_7 R_1} \geqslant c_8 (t (TR_1)^{-1})^{c_9 R_1}. \qquad \square \end{aligned}$$

Proposition 6.6. *Let $\beta \in (1, \infty)$. Suppose that $T > 0$ and $a \in (0, \big(4\phi^{-1}(T)\big)^{-1}]$. Then there exist constants $C_{6.6.i} = C_{6.6.i}(a, \beta, \chi, L_0, \phi, T, \lambda_1, \lambda_2) > 0$, $i = 1, 2$ such that for any $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geqslant a\phi^{-1}(t)$, $|x - y| \geqslant 1$, and $t \leqslant T$ we have*

$$p_D(t,x,y) \geqslant C_{6.6.1} t \exp\left\{ -C_{6.6.2} \left(|x-y| \left(\log \frac{T|x-y|}{t} \right)^{\frac{\beta-1}{\beta}} \wedge (|x-y|)^\beta \right) \right\}.$$

Proof. Let $R_1 := |x - y|$. If either $1 \leqslant R_1 \leqslant 2$ or $R_1(\log(TR_1/t))^{(\beta-1)/\beta} \geqslant (R_1)^\beta$, the proposition holds by virtue of Proposition 6.4(2). Thus for the remainder of this proof we assume that $R_1 > 2$ and $R_1(\log(TR_1/t))^{(\beta-1)/\beta} < (R_1)^\beta$, which is equivalent to $1 < R_1 (\log TR_1/t)^{-1/\beta}$ and $R_1 \exp(-R_1^\beta) < t/T$.

Let $k \geqslant 2$ be a positive integer such that

$$R_1 \left(\log \frac{TR_1}{t} \right)^{-1/\beta} \leqslant k < R_1 \left(\log \frac{TR_1}{t} \right)^{-1/\beta} + 1 < 2R_1 \left(\log \frac{TR_1}{t} \right)^{-1/\beta} \tag{6.7}$$

then $R_1/k \geqslant 2^{-1}(\log(TR_1/t))^{1/\beta} \geqslant 2^{-1}(\log 2)^{1/\beta} =: c_0$.

By the assumption on D, there is a length parameterized curve $l \subset D$ connecting x and y such that $|l| \leqslant \lambda_1 R_1$ and $\delta_D(l(u)) \geqslant \lambda_2 a \phi^{-1}(t)$ for every $u \in [0, |l|]$. Let $r_t := (2^{-1}\lambda_2 a\phi^{-1}(t)) \wedge (c_0/2)$ and define $x_i := l(i|l|/k)$ and $B_i := B(x_i, r_t)$, with $i = 0, 1, \dots, k$. For every $y_i \in B_i$, $\delta_D(y_i) \geqslant 2^{-1}\lambda_2 a\phi^{-1}(t) > 2^{-1}\lambda_2 a \phi^{-1}(t/k)$ and

$$|y_i - y_{i+1}| \leqslant |x_i - x_{i+1}| + 2r_t \leqslant \frac{|l|}{k} + c_0 \leqslant (\lambda_1 + 1) \frac{R_1}{k}. \tag{6.8}$$

By Proposition 6.4(2) and (6.8), and using the facts that $t/k \leqslant T/2$ and $R_1/k \geqslant c_0$, we have that for any $(y_i, y_{i+1}) \in B_i \times B_{i+1}$,

$$\begin{aligned} p_D(t/k, y_i, y_{i+1}) &\geqslant c_1 \left(\frac{1}{[\phi^{-1}(t/k)]^d} \wedge \frac{t}{k} \cdot \nu(|y_i - y_{i+1}|)/\chi(|y_i - y_{i+1}|) \right) \\ &\geqslant c_2 \frac{t}{k} \cdot \frac{e^{-c_3 (R_1/k)^\beta}}{\phi(R_1/k)(R_1/k)^d} \end{aligned}$$

for some constants $c_i = c_i(a, L_0, \phi, \chi, \beta, T, \lambda_1) > 0$, $i = 2, 3$. Since $\phi(R_1/k) \leqslant c_4 (R_1/k)^{\overline{\alpha}}$ by **(WS)** with $R_1/k \geqslant c_0$, using (6.7), we have that

$$\begin{aligned} p_D(t/k, y_i, y_{i+1}) &\geqslant c_2 \cdot c_4 \frac{t}{TR_1} \left(\frac{k}{R_1} \right)^{\overline{\alpha}+d-1} e^{-c_3 (R_1/k)^\beta} \\ &\geqslant c_2 \cdot c_4 \frac{t}{TR_1} \left(\log \frac{TR_1}{t} \right)^{-\frac{\overline{\alpha}+d-1}{\beta}} \left(\frac{t}{TR_1} \right)^{c_3} \geqslant c_5 \left(\frac{t}{TR_1} \right)^{c_6} \end{aligned} \tag{6.9}$$

for some $c_i = c_i(a, L_0, \phi, \chi, \beta, T, \lambda_1)$, $i = 5, 6$. Note that $r_t \geqslant c_7 (t/TR_1)^{1/\underline{\alpha}}$ for some $c_7 = c_7(a, \beta, \phi, \lambda_2)$ by (5.3) and the fact that $t/TR_1 \leqslant 1/2$. Combining this with (6.9), (6.7)

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and by the semigroup property, we conclude that

$$\begin{aligned} p_D(t,x,y) &\geqslant \int_{B_1}\cdots\int_{B_{k-1}} p_D(t/k,x,y_1)\cdots p_D(t/k,y_{k-1},y)dy_1\cdots dy_{k-1} \\ &\geqslant c_8\exp\{-c_9 k\log(TR_1/t)\} \\ &\geqslant c_8\exp\left\{-c_9\left(2R_1\log\left(\frac{TR_1}{t}\right)^{-1/\beta}\right)\log\frac{TR_1}{t}\right\} \\ &\geqslant c_8\exp\left\{-2c_9\cdot R_1\log\left(\frac{TR_1}{t}\right)^{1-1/\beta}\right\}. \quad \square \end{aligned}$$

Proofs of the lower bounds in Theorem 1.1 and 1.2. The lower bound of $p(t,x,y)$ in Theorem 1.1 follows from Proposition 6.4(1) with $D=\mathbb{R}^d$. The lower bound of $p(t,x,y)$ in Theorem 1.2 for the case $\beta\in[0,1]$ and the case $\beta\in(1,\infty]$ with $|x-y|<1$ follows from Proposition 6.4(2) with $D=\mathbb{R}^d$ and the remaining cases of Theorem 1.2 follows from Propositions 6.5 and 6.6 with $D=\mathbb{R}^d$. $\square$

7. Lower bound estimates

In this section, we first obtain the boundary decay in Lemma 7.4 using (4.8), Lemma 6.1 and Lemmas 7.1 and 7.2. Using the semigroup property, and then applying Lemma 7.4 and the preliminary lower bound estimates in Section 6, we will derive the upper bound estimate on $p_D(t,x,y)$ with the boundary decay terms for $t\leqslant T$ in $C^{1,1}$ open set D with $C^{1,1}$ characteristics (R_0,Λ). As before, we assume that $R_0<1$ and $\Lambda>1$.

We first introduce the next lemma (for the proof see [17, Lemma 3.3]).

Lemma 7.1. *Suppose that $E\subset\mathbb{R}^d$ be an open set and $U_1,U_2\subset E$ be disjoint open subsets. If $x\in U_1$, $y\in U_2$ and $t>0$, we have*

$$p_E(t,x,y)\geqslant t\ \mathbb{P}_x(\tau_{U_1}>t)\ \mathbb{P}_y(\tau_{U_2}>t)\inf_{(u,w)\in U_1\times U_2} J(u,w).$$

For the remainder of the section, we assume that Y is the symmetric pure jump Hunt process with the jumping intensity kernel J satisfying the conditions **(J1.1)** , **(J1.2)** and **(K$_\eta$)**. For any $T>0$, let

$$\widehat{a}_T:=\widehat{a}_{T,R_0}:=\frac{R_0}{80\,\phi^{-1}(T)},$$

and for $x\in D$ we use z_x to denote a point on ∂D such that $|z_x-x|=\delta_D(x)$.

We first give the survival probability where x is near the boundary of D in the following lemma.

Lemma 7.2. *Let $a\leqslant\widehat{a}_T$. Then, there exists a constant $C_{7.2}=C_{7.2}(a,\phi,L_0,L_3,\eta,\Lambda,T)>0$ such that for every $t\leqslant T$ and $x\in D$ with $\delta_D(x)<a\phi^{-1}(t)$ we have*

$$\mathbb{P}_x(\tau_{B(z_x,10a\phi^{-1}(t))\cap D}>t/3)\geqslant C_{7.2}\frac{V(\delta_D(x))}{\sqrt{t}}. \tag{7.1}$$

Proof. Without loss of generality, we assume that $z_x=0$. Consider a coordinate system $CS:=CS_0$ such that $B(0,R_0)\cap D=\{y=(\widetilde{y},y_d)\in B(0,R_0)\text{ in }CS: y_d>\varphi(\widetilde{y})\}$, where φ is

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a $C^{1,1}$ function such that $\varphi(0) = 0$, $\nabla\varphi(0) = (0, \dots, 0)$, $\|\nabla\varphi\|_\infty \leqslant \Lambda$, and $|\nabla\varphi(\widetilde{y}) - \nabla\varphi(\widetilde{w})| \leqslant \Lambda|\widetilde{y} - \widetilde{w}|$. Define $\varphi_1(\widetilde{y}) := 2\Lambda|\widetilde{y}|$ and $V := \{y = (\widetilde{y}, y_d) \in B(0, R_0) \text{ in } CS : y_d > \varphi_1(\widetilde{y})\}$. Since $\varphi_1(\widetilde{y}) \geqslant 2\Lambda|\widetilde{y}|^2$ for $y \in B(0, R_0)$, the mean value theorem yields $V \subset B(0, R_0) \cap D$.

Let $U_1 := B(0, 2a\phi^{-1}(t)) \cap D$, $U_2 := B(0, 10a\phi^{-1}(t)) \cap D$, and

$$W := \{y = (\widetilde{y}, y_d) \in B(0, 8a\phi^{-1}(t)) \setminus B(0, 2a\phi^{-1}(t)) \text{ in } CS : y_d > \varphi_1(\widetilde{y})\} \subset V. \tag{7.2}$$

Since $\Lambda|\widetilde{w}| = \varphi_1(\widetilde{w})/2 < w_d/2$ for $w \in W$, we have

$$\delta_D(w) > \frac{(w_d - \varphi(\widetilde{w}))}{(1+\Lambda)} > \frac{(w_d - \Lambda|\widetilde{w}|)}{(1+\Lambda)} > \frac{w_d}{2(1+\Lambda)} \qquad \text{for } w \in W. \tag{7.3}$$

Moreover, since $|\widetilde{w}| \leqslant (2\Lambda)^{-1}|w| \leqslant \Lambda^{-1}4a\phi^{-1}(t) \leqslant a\phi^{-1}(t)$ for $w \in W$, we have

$$w_d^2 = |w|^2 - |\widetilde{w}|^2 \geqslant (2a\phi^{-1}(t))^2 - (a\phi^{-1}(t))^2 \geqslant (a\phi^{-1}(t))^2 \qquad \text{for } w \in W. \tag{7.4}$$

Combining (7.3) and (7.4), we obtain $\delta_D(w) > 2^{-1}(1+\Lambda)^{-1}a\phi^{-1}(t)$ and $B(w, r_1 a\phi^{-1}(t)) \subset U_2$ for $w \in W$, where $r_1 := (2(1+\Lambda))^{-1}$. By virtue of the strong Markov property, Lemma 6.1, and (4.8), we have

$$\begin{aligned}
&\mathbb{P}_x(\tau_{U_2} > t/3) \geqslant \mathbb{P}_x(\tau_{U_2} > t/3, Y_{\tau_{U_1}} \in W) = \mathbb{E}_x[\mathbb{P}_{Y_{\tau_{U_1}}}(\tau_{U_2} > t/3) : Y_{\tau_{U_1}} \in W] \\
&\geqslant \mathbb{E}_x[\mathbb{P}_{Y_{\tau_{U_1}}}(\tau_{B(Y_{\tau_{U_1}}, r_1 a\phi^{-1}(t))} > t/3) : Y_{\tau_{U_1}} \in W] \\
&\geqslant \left(\inf_{z\in\mathbb{R}^d} \mathbb{P}_z(\tau_{B(z, r_1 a\phi^{-1}(t))} > t/3)\right) \mathbb{P}_x(Y_{\tau_{U_1}} \in W) \\
&\geqslant C_{6.1}\, \mathbb{P}_x(Y_{\tau_{U_1}} \in W) \geqslant C_{6.1} \cdot C_{4.2.2} \frac{V(\delta_D(x))}{V(8a\phi^{-1}(t))} \geqslant c_1 \frac{V(\delta_D(x))}{\sqrt{t}}.
\end{aligned}$$

By the subadditivity of V and (3.3), $V(8a\phi^{-1}(t)) \leqslant (8a+1)V(\phi^{-1}(t)) \asymp \sqrt{t}$, and therefore we obtain the last inequality. □

We introduce the following definition for the subsequent lemma.

Definition 7.3. Let $0 < \kappa_1 \leqslant 1/2$. We say that an open set D is κ_1-fat if there is $R_1 > 0$ such that for all $x \in \overline{D}$ and all $r \in (0, R_1]$ there is a ball $B(A_r(x), \kappa_1 r) \subset D \cap B(x, r)$. The pair (R_1, κ_1) are called the characteristics of the κ_1-fat open set D.

Note that a $C^{1,1}$ open set D with characteristics (R_0, Λ) is a κ_1-fat set with characteristics (R_1, κ_1) depending only on R_0, Λ, and d, and without loss of generality, we assume that $R_0 \leqslant R_1$ (by choosing R_0 smaller if necessary). Let $A_r(x)$ is always the point $A_r(x) \in D$ in Definition 7.3 for D.

Recall that the function Ψ is defined in (1.7).

Lemma 7.4. *There exists a constant $C_{7.4} = C_{7.4}(\phi, L_0, L_3, \eta, R_0, \Lambda, T) > 0$ such that, for every $t \leqslant T$ and $x \in D$, we can find x_1 with $\delta_D(x_1) \geqslant 2^{-1}\kappa_1\widehat{a}_T\phi^{-1}(t)$ and $|x_1 - x| \leqslant 6\widehat{a}_T\phi^{-1}(t)$ such that*

$$\int_{B(x_1, 4^{-1}\kappa_1\widehat{a}_T\phi^{-1}(t))} p_D(t/3, x, z)dz \geqslant C_{7.4}\, \Psi(t, x). \tag{7.5}$$

Proof. Let $r_t := \widehat{a}_T\phi^{-1}(t) \leqslant R_0/80 \leqslant 1/80$ and we consider the case $\delta_D(x) < 2^{-1}\kappa_1 r_t$ first. In this case we let $x_1 := A_{6r_t}(z_x)$ and denote $B_{x_1} := B(x_1, 4^{-1}\kappa_1 r_t)$ and $B_{z_x} := B(z_x, 5\kappa_1 r_t) \cap D$

so that $B_{x_1} \cap B_{z_x} = \emptyset$. For any $u \in B_{z_x}$ and $w \in B_{x_1}$,

$$|u - w| \leqslant |u - z_x| + |z_x - x_1| + |x_1 - w| \leqslant 12\kappa_1 r_t \leqslant 1.$$

Since $\phi(12\kappa_1 r_t) \asymp \phi(\phi^{-1}(t)) = t$ by **(WS)**, using **(J1.1)**, (1.2) and (1.4), we have that

$$\inf_{(u,w)\in B_{z_x}\times B_{x_1}} J(u,w) \geqslant L_0^{-1}\phi(12\kappa_1 r_t)^{-1}|12\kappa_1 r_t|^{-d} \geqslant c_1 t^{-1}[\phi^{-1}(t)]^{-d}$$

for some constant $c_1 := c_1(\phi, L_0, R_0, \Lambda, T) > 0$. Therefore, Lemma 7.1, 7.2, and 6.1 imply that

$$\begin{aligned}\int_{B_{x_1}} p_D(t/3,x,z)dz &\geqslant \frac{t}{3}\int_{B_{x_1}} \mathbb{P}_x(\tau_{B_{z_x}} > t/3)\,\mathbb{P}_z(\tau_{B_{x_1}} > t/3)\cdot \inf_{(u,w)\in B_{z_x}\times B_{x_1}} J(u,w)dz \\ &\geqslant \frac{1}{3}\,\mathbb{P}_x(\tau_{B_{z_x}} > t/3)\cdot C_{6.1}\int_{B_{x_1}} dz\cdot c_1\frac{1}{[\phi^{-1}(t)]^d} \\ &= c_2\mathbb{P}_x(\tau_{B_{z_x}} > t/3) \geqslant c_2\cdot C_{7.2}\,\frac{V(\delta_D(x))}{\sqrt{t}}.\end{aligned}$$

For $\delta_D(x) \geqslant 2^{-1}\kappa_1 r_t$, let $x_1 = x$ and $B_{x_1} := B(x_1, 4^{-1}\kappa_1 r_t)$. By Lemma 6.1,

$$\int_{B_{x_1}} p_D(t/3,x,z)dz \geqslant \int_{B_{x_1}} p_{B_{x_1}}(t/3,x,z)dz = \mathbb{P}_x(\tau_{B_{x_1}} > t/3) > C_{6.1},$$

and this proves the lemma. □

We are now ready to give the proof of the lower bound estimates for $p_D(t,x,y)$. Recall our assumption that D is a $C^{1,1}$ open set. When the jumping intensity J of Y satisfies **(J2)**, for the cases $\beta \in (1,\infty)$ with $|x-y| \geqslant 1$ and $\beta = \infty$ with $|x-y| \geqslant 4/5$, we assume in addition that the path distance in each connected component of D is comparable to the Euclidean distance with characteristic λ_1. Note that combining this assumption with $C^{1,1}$ assumption entails that D satisfies the assumption made before Proposition 6.5.

Proofs of the lower bound of $p_D(t,x,y)$ **in** Theorem 1.3(1), 1.5(2) **and** 1.5(3). Let $r_t := \widehat{a}_T\phi^{-1}(t) \leqslant R_0/80 \leqslant 1/80$. By Lemma 7.4, for any $x, y \in D$, there exists $x_1, y_1 \in D$ such that $\delta_D(x_1) \wedge \delta_D(y_1) \geqslant 2^{-1}\kappa_1 r_t$ and $|x_1 - x| \vee |y_1 - y| \leqslant 6r_t$, and

$$\int_{B_{x_1}} p_D(t/3,x,z)dz \int_{B_{y_1}} p_D(t/3,y,z)dz \geqslant C_{7.4}^2\,\Psi(t,x)\Psi(t,y), \tag{7.6}$$

where $B_{x_1} := B(x_1, 4^{-1}\kappa_1 r_t)$ and $B_{y_1} := B(y_1, 4^{-1}\kappa_1 r_t)$. Thus by the semigroup property,

$$\begin{aligned}p_D(t,x,y) &= \int_D\int_D p_D(t/3,x,u)p_D(t/3,u,w)p_D(t/3,w,y)dudw \\ &\geqslant \int_{B_{x_1}} p_D(t/3,x,u)du\int_{B_{y_1}} p_D(t/3,y,w)dw\left(\inf_{(u,w)\in B_{x_1}\times B_{y_1}} p_D(t/3,u,w)\right) \\ &\geqslant C_{7.4}^2\,\Psi(t,x)\Psi(t,y)\inf_{(u,w)\in B_{x_1}\times B_{y_1}} p_D(t/3,u,w). \end{aligned} \tag{7.7}$$

We now carefully calculate the lower bounds of $p_D(t/3,u,w)$ on $B_{x_1}\times B_{y_1}$. Since $|x-x_1| \vee |y-y_1| \leqslant 6r_t$, for $u \in B_{x_1}$ and $w \in B_{y_1}$ we have

$$\begin{aligned}&|x-y| - 6^{-1} \leqslant |x-y| - (12+(\kappa_1/2))r_t \\ &\leqslant |u-w| \leqslant |x-y| + (12+(\kappa_1/2))r_t \leqslant |x-y| + 6^{-1}\end{aligned} \tag{7.8}$$

and $\delta_D(u) \wedge \delta_D(w) \geqslant 4^{-1}\kappa_1 r_t$.

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We first assume that the jumping intensity kernel J satisfies the condition **(J2)**. Let $\beta \in [0,1]$. If $|x-y| \leqslant 15r_t$, then $|u-w| \leqslant 28r_t < 1$ and $\phi(|u-w|)|u-w|^d \leqslant c_1 t[\phi^{-1}(t)]^d$ since $\phi(|u-w|) \leqslant \phi(28\kappa_1 r_t) \asymp \phi(\phi^{-1}(t)) = t$ by **(WS)**. If $|x-y| > 15r_t$, then $|u-w| \leqslant |x-y| + 6^{-1}$ and $\phi(|u-w|)|u-w|^d \leqslant c_2\phi(|x-y|)|x-y|^d$ since $r \to \phi(r)$ is increasing and using **(WS)**. Combining these observations with Proposition 6.4(2), (1.2) and (1.5),

$$\begin{aligned} p_D(t/3,u,w) &\geqslant c_3\left([\phi^{-1}(t)]^{-d} \wedge te^{-\gamma_2|u-w|^\beta}\nu(|u-w|)\right) \\ &\geqslant c_4\left([\phi^{-1}(t)]^{-d} \wedge te^{-\gamma_2|x-y|^\beta}\nu(|x-y|)\right). \end{aligned}$$

If $\beta \in (1,\infty]$ and $|x-y| \leqslant 4/5$, then (7.8) yields $|u-w| \leqslant |x-y| + 6^{-1} < 1$. Similar to the above case, considering the cases $|x-y| \leqslant 15r_t$ and $|x-y| > 15r_t$ separately, we have $p_D(t/3,u,w) \geqslant c_5\left([\phi^{-1}(t)]^{-d} \wedge t\cdot\nu(|x-y|)\right)$. Moreover,

(1) if $\beta \in (1,\infty)$ and $4/5 \leqslant |x-y| < 2$, then $|u-w| \asymp 1$. Thus by Proposition 6.4(2), we have $p_D(t/3,u,w) \geqslant c_6 t$.

Hence combining (7.7) with these observations, we have proved the lower bound of $p_D(t,x,y)$ in Theorem 1.5(2).

Suppose the jumping intensity kernel J satisfies the condition **(J1)** and $M > 0$. Let $|x-y| < M$. Similar to the $\beta \in [0,1]$ case, applying Proposition 6.4(1) instead of Proposition 6.4(2) and considering $|x-y| \leqslant 15r_t \wedge M$ and $15r_t \wedge M < |x-y| \leqslant M$ separately, we have $p_D(t/3,u,w) \geqslant c_7\left([\phi^{-1}(t)]^{-d} \wedge t\cdot\nu(|x-y|)\right)$. Hence combining (7.7) with this, we have proved the lower bound of $p_D(t,x,y)$ in Theorem 1.3(1).

We now return to the assumption that the jumping intensity kernel J satisfies the condition **(J2)**, further assume that the path distance in D is comparable to the Euclidean distance. If $4/5 \leqslant |x-y|$, then (7.8) yields $|u-w| \asymp |x-y|$. Recall that we have already discuss the case $\beta \in (1,\infty)$ and $4/5 \leqslant |x-y| < 2$ in (1). We now consider $p_D(t/3,u,w)$ in each of the remaining cases.

(2) If $\beta = \infty$ and $4/5 \leqslant |x-y| < 2$, then by Propositions 6.4 and 6.5, we have

$$p_D(t/3,u,w) \geqslant c_8\frac{4t}{5T|x-y|} \geqslant c_8\left(\frac{4t}{5T|x-y|}\right)^{5|x-y|/4}.$$

(3) If $\beta \in (1,\infty)$ and $2 \leqslant |x-y|$, then $1 < |u-w|$ and from Proposition 6.6 and (7.8) we obtain

$$\begin{aligned} p_D(t/3,u,w) &\geqslant c_9 t\exp\left\{-c_{10}\left(|u-w|\left(\log\frac{T|u-w|}{t}\right)^{\frac{\beta-1}{\beta}} \wedge |u-w|^\beta\right)\right\} \\ &\geqslant c_9 t\exp\left\{-c_{10}\left((5|x-y|/4)\left(\log\left(\frac{T(|x-y|+6^{-1})}{t}\right)\right)^{\frac{\beta-1}{\beta}} \wedge (5|x-y|/4)^\beta\right)\right\} \\ &\geqslant c_9 t\exp\left\{-c_{11}\left(|x-y|\left(\log\frac{T|x-y|}{t}\right)^{\frac{\beta-1}{\beta}} \wedge |x-y|^\beta\right)\right\}. \end{aligned}$$

The last inequality comes from the inequality $\log r \leqslant \log(r+b) \leqslant 2\log r$ for $r \geqslant 2 \vee b > 0$.

(4) If $\beta = \infty$ and $2 \leqslant |x-y|$, then $1 < |u-w|$ and from Proposition 6.5 and (7.8) we have

$$p_D(t/3,u,w) \geqslant c_{12}\left(\frac{t}{T|u-w|}\right)^{c_{13}|u-w|} \geqslant c_{12}\left(\frac{t}{T(|x-y|+6^{-1})}\right)^{c_{13}5|x-y|/4}$$
$$\geqslant c_{12}\left(\frac{t}{T|x-y|}\right)^{c_{13}5|x-y|/2} \geqslant c_{12}\left(\frac{4t}{5T|x-y|}\right)^{c_{13}5|x-y|/2}.$$

The second last inequality holds by virtue of the inequality $r^2 \geqslant r+b$ for $r \geqslant 2\vee b > 0$.

Hence combining (7.7) with the above observations $(1)-(4)$ on the lower bound of $p_D(t/3,u,w)$, we have proved the lower bound of $p_D(t,x,y)$ in Theorem 1.5(3). □

Proof of Theorem 1.5(4). Let $D(x)$ and $D(y)$ be connected components containing x and y, respectively. By definition of a $C^{1,1}$ open set, the distance between x and y is at least R_0. Using Lemma 7.4, we find that $x_1 \in D(x)$ and $y_1 \in D(y)$. Define B_{x_1} and B_{y_1} in the same way as when beginning the proof of Theorems 1.5(2) and 1.5(3) so that (7.6) holds.

For any $u \in B_{x_1}$ and $w \in B_{y_1}$, since $3R_0/4 \leqslant 3|x-y|/4 \leqslant |u-w| \leqslant 5|x-y|/4$, by Proposition 6.4 (2) and (5.3),

$$p_D(t/3,u,w) \geqslant c_1 t\nu(|u-w|)e^{-\gamma_2|u-w|^\beta} \geqslant c_2 t\nu(|x-y|)e^{-\gamma_2(5|x-y|/4)^\beta}.$$

By the semigroup property, combining (7.6) and this observation, we conclude that

$$\begin{aligned} p_D(t,x,y) &\geqslant \int_{B_{x_1}}\int_{B_{y_1}} p_D(t/3,x,w)p_D(t/3,u,w)p_D(t/3,w,y)dwdv \\ &\geqslant \int_{B_{x_1}} p_D(t/3,x,u)du \int_{B_{y_1}} p_D(t/3,y,w)dw \cdot \inf_{(u,w)\in B_{x_1}\times B_{y_1}} p_D(t/3,u,w) \\ &\geqslant c_3\Psi(t,x)\Psi(t,y)\cdot t\nu(|x-y|)e^{-\gamma_2(5|x-y|/4)^\beta}. \quad \square \end{aligned}$$

Proofs of Theorem 1.3(2) and 1.5(5). Using Lemmas 5.2 and 7.2 instead of [18, (5.1) and (5.10)], and by the fact that (3.3) and D is bounded (and connected when J satisfies the condition **(J2)** and $\beta = \infty$), we can obtain the large time heat kernel estimates for $p_D(t,x,y)$ following the proofs of [18, Theorems 1.3(iii) and 1.5(iii)], so we omit the proofs.

8. Green function and boundary Harnack inequality

In this section we give the Green function estimates and establish the boundary Harnack inequality as applications of the Dirichlet heat kernel estimates.

Proof of Theorem 1.6. When $d \geqslant 2$, the proof of Green function estimates is almost identical to the one of [18, Section 7]. Thus we skip the proof.

Suppose $d = 1$. Note that by the inequality in Proposition 3.1, we have

$$V'(r) \leqslant c\frac{V(r)}{r} \qquad \text{for } 0 < r \leqslant M, \tag{8.1}$$

Using (8.1) instead of [18, (7.3)], one can obtain the Green function estimates by following the proofs in [18, Section 7] line by line. Indeed, for any $T > 0$, let

$$K_T(a,r) := a + \phi(r)\int_{\phi(r)/T}^{1}\left(1\wedge\frac{ua}{\phi(r)}\right)\frac{1}{u^2\phi^{-1}(u^{-1}\phi(r))}du + \frac{\phi(r)}{r}\left(1\wedge\frac{a}{\phi(r)}\right)$$

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which is defined in [18, (7.4)]. By the same proof of [18, Theorem 7.3(iii)], we have that

$$G_D(x,y) \asymp K_{T_1}(a(x,y), |x-y|)$$

where $a(x,y) = \sqrt{\phi(\delta_D(x))}\sqrt{\phi(\delta_D(y))}$. Recall that C_I is the constant in (5.3). Let $T_1 := (2 \vee (2C_I)^{\overline{\alpha}})\phi(\text{diam}(D))$. Since $0 < a(x,y) \leqslant \phi(\text{diam}(D)) = (2^{-1} \wedge (2C_I)^{-\overline{\alpha}})T_1$ and $\phi(|x-y|) \leqslant \phi(\text{diam}(D)) \leqslant T_1/2$, it is enough to show that for any $T > 0$ and for any $0 < a \leqslant (2^{-1} \wedge (2C_I)^{-\overline{\alpha}})T$ and $0 < \phi(r) \leqslant T/2$,

$$K_T(a,r) \asymp \frac{a}{r} \wedge \left(\frac{a}{\phi^{-1}(a)} + \left(\int_r^{\phi^{-1}(a)} \frac{\phi(s)}{s^2}\right)^+\right) \tag{8.2}$$

where $x^+ := x \vee 0$.

When $0 < a < \phi(r) \leqslant T/2$, the proof of (8.2) is the same as that of [18, Lemma 7.2]. Now we assume that $\phi(r) \leqslant a \leqslant (2^{-1} \wedge (2C_I)^{-\overline{\alpha}})T$. Using (3.3), we have $c_1^{-1}V(r)^2 \leqslant \phi(r) \leqslant c_1 V(r)^2$ for some constant $c_1 > 1$. Thus by the change of variable $u = V(r)^2/V(s)^2$, we have that

$$\begin{aligned}\int_{\phi(r)/T}^{\phi(r)/a} \frac{a du}{u\phi^{-1}(u^{-1}\phi(r))} &\leqslant \int_{V(r)^2/(c_1T)}^{c_1V(r)^2/a} \frac{a du}{u\phi^{-1}(u^{-1}\phi(r))} \\ &\leqslant \int_{1/V(a/c_1)^2}^{1/V(c_1T)^2} \frac{2a}{\phi^{-1}(c_1^{-1}V(s)^2)} \frac{V'(s)}{V(s)} ds.\end{aligned}$$

Since $\phi^{-1}(c_1^{-1}V(s)^2) \geqslant \phi^{-1}(c_1^{-2}\phi(s)) \geqslant c_2 s$ by (3.3) and (5.3), combining this with (8.1), we have that

$$\int_{\phi(r)/T}^{\phi(r)/a} \frac{a \cdot du}{u\phi^{-1}(u^{-1}\phi(r))} \leqslant c_3 a \int_{1/V(a/c_1)^2}^{1/V(c_1T)^2} \frac{1}{s^2} ds \leqslant c_4 \frac{a}{\phi^{-1}(a)}. \tag{8.3}$$

For the last inequality, we again used (3.3) and (5.3). Applying (8.3) to the proof of the upper bound for $K_T(a,r)$ in [18, (7.6)], and following the rest of the proof of [18, Theorem 7.3(iii)] for the $\phi(r) \leqslant a \leqslant (2^{-1} \wedge (2C_I)^{-\overline{\alpha}})T$ case, we obtain (8.2) and hence we prove Theorem 1.6 for all dimension. □

To prove Theorem 1.8 we use the above estimates of Green function and the following scale and translate invariant boundary Harnack inequality.

Proposition 8.1. *Suppose that D is an open set in $\mathbb{R}^d$. Let Y be a symmetric pure jump Hunt process whose jumping intensity kernel J satisfies the conditions* **(J1)**, **(L)**, **(C)** *and* $(\mathbf{K}_\eta)$. *Then, there exists $c = c(\phi, \eta, L_0, L_3, d)$ such that for any $0 < r < 1$, $z \in \partial D$ and any non-negative functions f, g in $\mathbb{R}^d$ which are regular harmonic in $D \cap B(z,r)$ with respect to Y, and vanish in $D^c \cap B(z,r)$, we have*

$$\frac{f(x)}{f(y)} \leqslant c\frac{g(x)}{g(y)} \qquad \text{for any } x, y \in D \cap B(z, 2r/3).$$

Proof. To prove the claim we use [7]. We only have to check assumptions stated therein. Note that **(C)** is a uniform version of [7, Assumption C]. Thus there is a constant $c_{(2.7)}$ in [7] satisfies $c_{(2.7)}(x_0, R_1, R_2) = C^*(\phi, d, R_1/R_2)$ for any $x_0 \in \mathbb{R}^d$ and $0 < R_1 < R_2 \leqslant 2$. Let $0 < r < 1$ and $2/3 < a \leqslant 2$.

We first check the bounds on the constants $c_{(2.8)}$ and $c_{(2.9)}$ in [7]. In our case, the constants $c_{(2.8)}$ and $c_{(2.9)}$ in [7] can be taken as

$$c_{(2.8)}(x_0, ar, 2r) := \inf_{\{ar\leqslant|x_0-y|\leqslant 2r\}} J(x_0, y) \quad \text{and}$$
$$c_{(2.9)}(x_0, r) \leqslant C^* \left(\int_{\mathbb{R}^d\setminus B(x_0,2r)} J(x_0, y)dy\right)^{-1}$$

where $C^* = C^*(\phi, d, 1/2)$ is the constant in **(C)**. (see [7, (2.8) and (2.9)] and the last display of [7, Proposition 2.9]). Then by **(J1.3)** and **(WS)** for $c_{(2.8)}$, and by **(J1.1)**, **(J1.2)** (1.4) and **(WS)** for $c_{(2.9)}$, we have that

$$c_{(2.8)}(x_0, ar, 2r) \geqslant c_1\phi(r)^{-1}r^{-d} \qquad \text{and} \qquad c_{(2.9)}(x_0, r) \leqslant c_2\phi(r) \tag{8.4}$$

where the constant $c_1 > 0$ depends on ϕ, a and d, and the constant $c_2 > 0$ depends on ϕ, L_0 and d.

We now check [7, Assumption A–D] (and its scale and translate invariant version) holds. First of all, since $p(t, x, y)$ is continuous, clearly the transition operators T_t of Y is strong Feller. Recall that we assume that T_t is Feller , that is, T_t maps $C_0(\mathbb{R}^d)$ into $C_0(\mathbb{R}^d)$. Since Y is symmetric, [7, Assumption A] holds.

Let $\widehat{A}$ be the corresponding generator on $C_0(\mathbb{R}^d)$ defined as

$$\widehat{A}u := \lim_{t\to 0}\frac{T_tu - u}{t} \qquad \text{(strong limit) and}$$
$$D(\widehat{A}) := \{u \in C_0(\mathbb{R}^d) : \widehat{A}u < \infty\}.$$

Recall the operator $\mathcal{L}g(x) = P.V. \int(g(y) - g(x))J(x, y)dy$ defined in (2.1). Then

$$C_c^2(\mathbb{R}^d) \subset D(\widehat{A}) \qquad \text{and} \qquad \widehat{A}u = \mathcal{L}u \qquad \text{for any } u \in C_c^2(\mathbb{R}^d). \tag{8.5}$$

Indeed, we first obtain that for any $u \in C_c^2(\mathbb{R}^d)$, $\mathcal{L}u \in C_0(\mathbb{R}^d)$ by **(L)** and so,

$$\|T_t(\mathcal{L}u) - \mathcal{L}u\|_\infty \to 0 \qquad \text{as } t \to 0. \tag{8.6}$$

Since, from Lemma 2.2, $M_t^u = u(Y_t) - u(Y_0) - \int_0^t \mathcal{L}u(Y_s)ds$ is $\mathbb{P}_x$-martingale with respect to the filtration of Y, we have that

$$\frac{T_tu(x) - u(x)}{t} = \frac{1}{t}\mathbb{E}_x\left[\int_0^t \mathcal{L}u(Y_s)ds\right].$$

Thus we obtain that for any $u \in C_c^2(\mathbb{R}^d)$,

$$\begin{aligned}\sup_x\left|\frac{T_tu(x) - u(x)}{t} - \mathcal{L}u(x)\right| &= \sup_x\left|\frac{1}{t}\int_0^t T_s\mathcal{L}u(x) - \mathcal{L}u(x)ds\right|\\ &\leqslant \frac{1}{t}\int_0^t \|T_t(\mathcal{L}u) - \mathcal{L}u\|_\infty ds,\end{aligned}$$

and combining this with (8.6), we conclude (8.5). Therefore, [7, Assumption B] holds with $\mathcal{D} = C_c^2(\mathbb{R}^d)$.

For $0 < R_1 < R_2$, let $A(x, R_1, R_2) = \{y \in \mathbb{R}^d : R_1 < |x - y| < R_2\}$ be the open annulus around x, and $\overline{A}(x, R_1, R_2)$ the closure of $A(x, R_1, R_2)$. For every compact set K and open set U satisfying $K \subset U \subset \mathbb{R}^d$, let

$$\mathcal{F}_{K,U} := \{f \in C_c^2(\mathbb{R}^d) : f \equiv 1 \text{ in } K,\ \ f \equiv 0 \text{ in } U^c, \quad \text{and } 0 \leqslant f(x) \leqslant 1\},$$

and $\varrho(K,U) := \inf_{f\in\mathcal{F}_{K,U}} \sup_x \mathcal{L}f(x)$. Then by Lemma 2.1 and **(WS)**, for $2/3 < a < b \leqslant 1$ there exist $c_3 = c_3(\phi, \eta, L_0, L_3, a, b)$ such that for any $x_0 \in \mathbb{R}^d$ and $0 < r < 1$,

$$\widehat{\varrho}(x_0, ar, br) := \varrho(\overline{A}(x_0, ar, br), A(x_0, 2r/3, 2r)) + \varrho(\overline{B}(x_0, ar), B(x_0, br)) \leqslant c_3\phi(r)^{-1}. \tag{8.7}$$

Let $B_u := B(x_0, u)$ be a ball centered at x_0 with radius $u > 0$. Let $d \geqslant 1$, $0 < r < 1$ and $x, y \in B_r$. By Theorem 1.1 (with $M = 2$ and $T = \phi(2)$) and the semigroup property we have, for $t_0 = \phi(|x-y|)$,

$$\begin{aligned} G_{B_r}(x,y) &\leqslant \int_0^{t_0} p(s,x,y)ds + \int_0^\infty p_{B_r}(s+t_0,x,y)ds \\ &\leqslant C_{1.1}\int_0^{t_0} s\nu(|x-y|)ds + \int_{B_r} p(t_0,z,y)G_{B_r}(x,z)dz \\ &\leqslant C_{1.1}\left(t_0^2\nu(|x-y|) + [\phi^{-1}(t_0)]^{-d}\mathbb{E}_x\tau_{B_r}\right) \\ &= \frac{C_{1.1}}{|x-y|^d}\left(\phi(|x-y|) + \mathbb{E}_x\tau_{B_r}\right). \end{aligned} \tag{8.8}$$

Let $5/6 < a < 1$. For $x \in B_{5r/6}$ and $y \in B_r \setminus B_{ar}$, $(a-5/6)r \leqslant |x-y| \leqslant 2r$. Hence, by (8.4) and **(WS)** we obtain

$$c_{(2.10)}(x_0, 5r/6, ar, r) := \sup_{\substack{x\in B_{5r/6}\\ y\in B_r\setminus B_{ar}}} G_{B_r}(x,y) \leqslant c_4\frac{\phi(r)}{r^d}. \tag{8.9}$$

where the constant c_4 depends on ϕ, a and d. Hence [7, Assumption D] holds.

We have observed that [7, Assumption A–Assumption D] hold. In addition, by (8.4), (8.7) and (8.9), the upper bound of the constants $c_{(3.9)}$, $c_{(3.11)}$ and $c_{(1.1)}$ in [7] from the expressions of the constants $c_{(3.9)}$, $c_{(3.11)}$ and $c_{(1.1)}$ in [7, (3.9)–(3.11)] so that for any $x_0 \in \mathbb{R}^d$ and $0 < r < 1$,

$$\begin{aligned} &c_{(3.9)}(x_0, 5r/6, 11r/12, r) \leqslant c_6\frac{\phi(r)}{r^d}, \\ &c_{(3.11)}(x_0, 5r/6, r) \leqslant 2c_{(3.9)}(x_0, 5r/6, 11r/12, r) \\ &\qquad \cdot \max\left(\frac{\widehat{\varrho}(x_0, 11r/12, r)}{c_{(2.8)}(x_0, 11r/12, 2r)}, |B(0,1)|C^*(\phi, d, 1/2)r^d\right) \leqslant c_7\phi(r), \qquad \text{and} \\ &c_{(1.1)}(x_0, 2r/3, r) \leqslant \left(\widehat{\varrho}(x_0, 3r/4, 5r/6)\cdot c_{(3.11)}(x_0, 5r/6, r) + C^*(\phi, d, 9/10)\right)^4 \leqslant c_8 \end{aligned}$$

where the constants c_i, $i = 6, 7, 8$ are depending only on ϕ, η, L_0, L_3 and d. Therefore, we obtain the scaling and translation invariant version of [7, (BHI)] for $r < 1$, with the constant $c_{(1.1)} = c_{(1.1)}(x_0, 2r/3, r)$ which is independent of $r < 1$ and $x_0 \in \mathbb{R}^d$. $\square$

Alternatively, one can check the conditions in [35, Section 4], which also provides [35, Corollary 4.2], the scaling and translation invariant version of (BHI).

We now use the above proposition to prove Theorem 1.8.

Proof of Theorem 1.8. Suppose that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ). Since D is a $C^{1,1}$ open set, it is easy to see that for any $z \in \partial D$ there exists a bounded $C^{1,1}$ open set U in $\mathbb{R}^d$ whose characteristics depend only on R_0 and Λ (independent of $z \in \partial D$) such that $B(z, 7R_0/8) \cap D \subset U \subset B(z, R_0) \cap D$ (if $d = 1$ we can take $U = (z, z+R_0)$ or $U = (z-R_0, z)$). Choose a point $z_0 \in U \setminus \overline{B(z, 3R_0/4)}$ and let $g_1(x) = G_U(x, z_0)$. Since g_1 is

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Please cite this article as: T. Grzywny, K.-Y. Kim and P. Kim, Estimates of Dirichlet heat kernel for symmetric Markov processes, Stochastic Processes and their Applications (2019), https://doi.org/10.1016/j.spa.2019.03.017.

regular harmonic in $D \cap B(z, 3R_0/4)$, applying Proposition 8.1 we obtain

$$\frac{f(x)}{f(y)} \leqslant c_1 \frac{g_1(x)}{g_1(y)}, \quad x, y \in D \cap B(z, r/2).$$

Theorem 1.6 implies the claim of the theorem.

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SPA: 3487

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