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Nonlinear master equation of multitype particle systems

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Abstract

In this paper we prove the existence and uniqueness of solutions of the nonlinear martingale problems associated with the nonlinear master equations of multitype particle systems. Existence is shown to hold under some weak growth conditions and a finite range condition, while uniqueness is proved under a Lipschitz condition. Uniqueness is also shown to hold under a non-Lipschitz condition and a strong growth condition. The proof of uniqueness involves a coupling argument and the exponential martingale property. The results are then applied to some examples such as the Lotka-Volterra model and the Brusselator.

Keywords: Q-process; Pure jump Markov process; Nonlinear master equation; Coupling; Mean field interaction.

1. Introduction

The nonlinear master equation have been used to model a class of chemical reactions in Nicolis and Prigogine (1977). Rigorous studies of the equation in the case of particle systems of a single type have been carried out by several authors in the past few years. Feng and Zheng (1992) established the existence and uniqueness of a solution to the equation using an analytical method. Dawson and Zheng (1991) introduced a finite exchangeable particle system and proved that the empirical processes of the particle systems converge to the unique solution of the nonlinear master equation. Central limit theorems are also obtained in Dawson and Zheng (1991). Recently, Feng (1994, 1993) obtained results on large deviations for the same model.

The objective of present article is to extend the results of Feng and Zheng (1992) to multitype particle systems. The structure of multitype particle systems is more complicated and more interesting than that of single-type particle systems. One example that motivated this research is the Brusselator and its periodic behavior. Unfortunately, we

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cannot even prove the uniqueness of this model at this stage. In Section 2 we show the existence of solutions of the multitype nonlinear master equation under some weak growth conditions which are satisfied by a large class of models. In Section 3 stronger assumptions are then imposed to obtain the uniqueness.

We are interested in the nonlinear master equation because of the following:

(1) the existence of multiple equilibria and phase transitions;
(2) its role in the study of the exchangeable particle systems;
(3) the connection between mean field particle system and the lattice model;
(4) to get a better understanding of the structure and the stochastic calculus of nonlinear pure jump Markov processes.

Studies of this kind have been carried out for nonlinear diffusion processes by a number of authors such as Dawson (1983), Dawson and Gärtner (1988), Funaki (1984, 1985), Scheutzow (1986) and this list is in no way exhaustive.

Here is a detailed description of our setting, followed by the main results of this article.

Let Z_+ be the set of nonnegative integers, and for any integer $m > 1$, $S = Z_+^m$ denotes the m-fold product of Z_+ endowed with the metric ρ defined by:

$$\rho(X,Y) = \sum_{i=1}^{m} |x_i - y_i|, \quad \text{for every } X, Y \in S,\ X = (x_1, \dots, x_m),\ Y = (y_1, \dots, y_m).$$

For any $r \geqslant 1$, let $|X|^r = (\sum_{i=1}^m x_i)^r$, $|X| = \sum_{i=1}^m x_i$. We will use $M(S)$ to denote the space of probability measures on S equipped with the weak topology, and $M_r(S)$ will denote the space of all probability measures on S with finite moment of order r. The following two metrics will be used on $M(S)$: For any $u, v \in M(S)$,

$$\tilde{d}(u,v) = \sum_{X \in S} 2^{-|X|} \cdot \frac{|u(X) - v(X)|}{C_{|X|}^{m-1+|X|}},$$

$$d(u,v) = \sup_{P \in \mathscr{P}(u,v)} \left\{ \int_{S \times S} \rho(X,Y) P(\mathrm{d}X, \mathrm{d}Y) \right\},$$

where $C_{|X|}^{m-1+|X|} = (m-1+|X|)!/(|X|)!(m-1)!$, and $\mathscr{P}(u,v)$ is the set of all probability measures on $S^2 = S \times S$ with marginals u and v. It is known that $\tilde{d}$ metrizes the vague topology which is weaker than the weak topology, while d is associated with a topology which is stronger than the weak topology. In fact, in our particular case, d is equivalent to the total variation metric.

Let $A = (q_{X,Y})_{X,Y \in S}$ be a matrix satisfying:

$$\text{for any } X, Y \in S,\ X \neq Y,\ q_{X,Y} \geqslant 0, \tag{1.1}$$

$$\text{for any } X \in S,\ |q_{X,X}| < \infty,\ \sum_{Y \in S} q_{X,Y} = 0. \tag{1.2}$$

Such a matrix is called a totally stable, conservative Q-matrix.

Let $C_b(S)$ be the set of all bounded functions on S (note: all functions on S are continuous because of the discrete topology). For any $f \in C_b(S)$ and any vector-valued function

$$g : [0,+\infty) \to R^m, \quad t \mapsto g(t) = (g^1(t),\dots,g^m(t)),$$
$$g^i(t) \geqslant 0, \quad t \in [0,+\infty), \quad i = 1,\dots,m,$$

we define

$$Lf(X) = \sum_{Y \in S} q_{X,Y}(f(Y) - f(X)),$$
$$L_{g(t)}f(X) = Lf(X) + \sum_{i=1}^{m}[g^i(t)(f(X+e_i) - f(X)) + x_i(f(X-e_i) - f(X))],$$

where $e_i \in S$ with $(e_i)_i = 1, (e_i)_j = 0$ for $j \neq i$.

For any $u \in M(S)$, $f \in C_b(S)$, we define $\langle u, f\rangle = \int_S f(X)u(\mathrm{d}X)$. The nonlinear master equation has the form

$$\frac{\mathrm{d}\langle u(t), f\rangle}{\mathrm{d}t} = \langle u(t), L_{\|u(t)\|} f\rangle, \tag{1.3}$$

where $u(\cdot)$ is a measure-valued function from $[0,+\infty)$ to $M(S)$ and

$$\|u(t)\| = (\|u^1(t)\|,\dots,\|u^m(t)\|).$$

$u^i(t)$ and $\|u^i(t)\|$ denote the ith marginal of $u(t)$ and the first moment of $u^i(t)$, respectively.

Let $D = D([0,+\infty), S)$ denote the space of functions from $[0,+\infty)$ to S that are right continuous and have left-hand limits, equipped with the Skorohod topology. It is well-known that D is a Polish space in which the Borel σ-algebra coincides with $\mathscr{F} = \sigma\{X(t): t \geqslant 0\}$, the smallest σ-algebra generated by $\{X(t): t \geqslant 0\}$, where $X(t)(\omega) = \omega(t)$ for all $\omega \in D$. The σ-algebra $\mathscr{F}_t = \sigma\{X(s): s \in [0,t]\}$ is defined in the same way for any $t \geqslant 0$. $\mathscr{P}(D,\mathscr{F})$ denotes the set of all probability measures on $(D,\mathscr{F})$ equipped with the usual weak topology. For any $P \in \mathscr{P}(D,\mathscr{F})$ and $t \in [0,+\infty)$, $P \circ X^{-1}(t)$ is given by

$$P \circ X^{-1}(t)(A) = P\{X(t) \in A\}, \quad \text{for any } A \subset S.$$

Definition 1.1. For any $u \in M(S)$, $P_u \in \mathscr{P}(D,\mathscr{F})$ is called a solution of the martingale problem $[u, L]$ with initial distribution u if

$$P_u \circ X^{-1}(0) = u; \tag{1.4}$$

$$\forall\ Y \in S, \quad \left(I_{\{Y\}}(X(t)) - \int_0^t LI_{\{Y\}}(X(s))\,\mathrm{d}s, \mathscr{F}_t, P_u\right) \quad \text{is a martingale;} \tag{1.5}$$

where I_A represents the indicator function on set A.

Definition 1.2. Let $u \in M(S)$; and let $v : [0,+\infty) \to M(S)$ satisfy

$$\sup_{0 \leqslant t \leqslant T} \sum_{i=1}^{m} \|v^i(t)\| < \infty, \quad \text{for all } T > 0.$$

$P_{u,v(\cdot)} \in \mathscr{P}(D,\mathscr{F})$ is called a solution of the time inhomogeneous martingale problem $[u, L_{\|v(t)\|}]$ with initial distribution u if

$$P_{u,v(\cdot)} \circ X^{-1}(0) = u, \tag{1.6}$$

$$\forall\ Y \in S, (I_{\{Y\}}(X(t)) - \int_0^t L_{\|v(s)\|} I_{\{Y\}}(X(s))\,\mathrm{d}s, \mathscr{F}_t, P_{u,v(\cdot)}) \text{ is a martingale.} \tag{1.7}$$

Definition 1.3. For any $u \in M(S)$, $P_u \in \mathscr{P}(D,\mathscr{F})$ is called a solution of the nonlinear martingale problem $[u, L]$ with initial distribution u if

$$P_u \circ X^{-1}(0) = u; \tag{1.8}$$

$$P_u \circ X^{-1}(t) = u(t); \tag{1.9}$$

$$\forall\ Y \in S, (I_{\{Y\}}(X(t)) - \int_0^t L_{\|u(s)\|} I_{\{Y\}}(X(s))\,\mathrm{d}s, \mathscr{F}_t, P_u) \text{ is a martingale.} \tag{1.10}$$

The main results of this article are:

Theorem 1.1. *Let $A = (q_{X,Y})_{X,Y \in S}$ be a totally stable, conservative Q-matrix satisfying*

$$\text{there exists } \Lambda > 0, \text{ such that } q_{X,Y} = 0, \text{ for } \rho(X,Y) > \Lambda, \tag{1.11}$$

there exists $p > 1$, $C > 1$, such that

$$\left.\begin{array}{ll} \text{(i)} & \sum_{Y \in S} q_{X,Y}(|Y| - |X|) \leqslant C(1 + |X|), \\ \text{(ii)} & \sum_{Y \in S} q_{X,Y}(|Y|^p - |X|^p) \leqslant C(1 + |X|^p), \end{array}\right. \tag{1.12}$$

then for any $u \in M_p(S)$, there exists a solution of the nonlinear martingale problem $[u, L]$.

Note. 1.11 implies that $q_{X,Y} = 0$ for $||X| - |Y|| > \Lambda$.

Let $C_f(S^2)$ denote the set of all functions on S^2 with finite values. For any $(X,\tilde{X}) \in S^2$ and $G \in C_f(S^2)$ we introduce the following operator:

$$\begin{aligned} \Omega G(X,\tilde{X}) = \sum_{(Y,\tilde{Y}) \in S^2,\ Y - X = \tilde{Y} - \tilde{X}} & \{(q_{X,Y} - q_{\tilde{X},\tilde{Y}})^+ (G(Y,\tilde{X}) - G(X,\tilde{X})) \\ & + (q_{\tilde{X},\tilde{Y}} - q_{X,Y})^+ (G(X,\tilde{Y}) - G(X,\tilde{X})) \\ & + (q_{X,Y} \wedge q_{\tilde{X},\tilde{Y}})(G(Y,\tilde{Y}) - G(X,\tilde{X}))\} \\ & + \sum_{i=1}^m \{(x_i - \tilde{x}_i)^+ (G(X - e_i, \tilde{X}) - G(X,\tilde{X})) \\ & + (\tilde{x}_i - x_i)^+ (G(X, \tilde{X} - e_i) - G(X,\tilde{X})) \\ & + (x_i \wedge \tilde{x}_i)(G(X - e_i, \tilde{X} - e_i) - G(X,\tilde{X}))\}. \end{aligned}$$

Theorem 1.2. *Let $A = (q_{X,Y})_{X,Y \in S}$ be a totally stable, conservative Q-matrix satisfying* (1.11), (1.12) *and*

$$\text{there exists } h > 0, \text{ such that } \Omega\rho(X,\tilde{X}) \leqslant h\rho(X,\tilde{X}). \tag{1.13}$$

Then for any $u \in M_p(S)$, the nonlinear martingale problem $[u, L]$ is well-posed.

Remark. Condition (1.13) is just the Lipschitz condition.

In next theorem we prove the uniqueness of the solution to the nonlinear martingale problem under a particular non-Lipschitz condition and a strong growth condition.

Theorem 1.3. *Let $A = (q_{X,Y})_{X,Y \in S}$ be a totally stable, conservative Q-matrix satisfying 1.11 and*

$$\text{there exist } a > 0,\ b > 0, \text{ such that } \Omega\rho(X,\tilde{X}) \leqslant [a(|X| + |\tilde{X}|) + b]\rho(X,\tilde{X}), \tag{1.14}$$

$$\text{there exist } c > 0,\ \alpha > 0, \text{ such that } \textstyle\sum_{Y \in S} q_{X,Y}(e^{c|Y|} - e^{c|X|}) \leqslant \alpha e^{cX}. \tag{1.15}$$

Then for any $u \in M(S)$ satisfying $\int e^{c|X|} u\ (dX) < \infty$, the nonlinear martingale problem $[u, L]$ is well-posed.

Remark. If the state space is a subset of $\{0, 1, \ldots, N\}^m$ for some integer $N > 0$, then the *Q-matrix* satisfies conditions (1.11)–(1.15) and thus the corresponding martingale problems are well-posed.

The following are two interesting examples.

Example 1. (Lotka-Volterra model). This model involves the following chemical reactions between four chemical species or molecules A, B, X_1, and X_2:

$$A + X_1 \xrightarrow{\lambda_1} 2X_1,$$
$$X_1 + X_2 \xrightarrow{\lambda_2} 2X_2,$$
$$B + X_2 \xrightarrow{\lambda_3} E + B.$$

Each diagram represents a reaction. For example, the diagram $A + X_1 \xrightarrow{\lambda_1} 2X_1$ represents a reaction between one A molecule and one X_1 molecule resulting in two X_1 molecules; λ_1 is the reaction rate. The concentrations of A and B are held constant in time and space by appropriate feeding of the reactor. It is not hard to see that this is a nonequilibrium system. We are interested in the evolution of the concentrations of molecules of type X_1 and X_2 and the long-time behavior of the system such as the existence of phase transitions and metastability of the stationary states. Deterministic and stochastic differential equations have been used to model the reaction. In this article we consider a nonlinear jump Markov processes model with Q-matrix given by

$$q_{X,Y} = \begin{cases} \lambda_1 a x_1, & \text{for } Y = (x_1 + 1, x_2), \\ \lambda_3 b x_2, & \text{for } Y = (x_1, x_2 - 1), \\ \lambda_2 x_1 x_2, & \text{for } Y = (x_1 - 1, x_2 + 1), \\ 0, & \text{for other } Y \neq X, \\ -\sum_{Y \neq X} q_{X,Y}, & \text{for } Y = X. \end{cases} \tag{1.16}$$

It can be checked that (1.11), (1.12), and (1.14) are satisfied by this matrix. Condition (1.15) holds if $\lambda_1 \leqslant 1$.

Example 2. (The trimolecular model (the "Brusselator")). This model consists of following chemical reactions between reactants A, B, X_1, and X_2:

$$A \xrightarrow{\lambda_1} X_1,$$
$$B + X_1 \xrightarrow{\lambda_2} X_2 + C,$$
$$2X_1 + X_2 \xrightarrow{\lambda_3} 3X_1,$$
$$X_1 \xrightarrow{\lambda_4} E.$$

The concentrations of A and B are assumed to be constant in time and space. Differential equations and stochastic differential equations have been used to model these reactions. Both approches reveal periodic behavior of this model. For our jump model, the Q-matrix is given by

$$q_{X,Y} = \begin{cases} \lambda_1 a, & \text{for } Y = (x_1 + 1, x_2), \\ \lambda_4 x_1, & \text{for } Y = (x_1 - 1, x_2), \\ \lambda_2 b x_1, & \text{for } Y = (x_1 - 1, x_2 + 1), \\ \lambda_3 C_2^{x_1} x_2, & \text{for } Y = (x_1 + 1, x_2 - 1), \\ 0, & \text{for other } Y \neq X, \\ -\sum_{Y \neq X} q_{X,Y}, & \text{for } Y = X, \end{cases} \tag{1.17}$$

where $C_2^{x_1} = \frac{1}{2} x_1 (x_1 - 1)$. For this model, conditions (1.11), (1.12), (1.15) are satisfied but not (1.14). Thus, we are unable to prove the uniqueness for the jump model at this stage.

2. Existence

In this section we will prove Theorem 1.1 through a series of lemmas. The proof consists of two parts: First we construct a sequence of probability measures on $(D, \mathscr{F})$ and verify the tightness of this sequence; secondly, we will show that one of the limit points of this sequence is a solution of the nonlinear martingale problem $[u, L]$.

The following lemma appears as Lemma 2.1 in Feng and Zheng (1992).

Lemma 2.1. *Let $\mathscr{C} \subset \mathscr{P}(D, \mathscr{F})$. Assume for every $T > 0$, $0 < \gamma < 1$, there exists a compact subset K of S such that*

$$\inf_{P \in \mathscr{C}} P\{X(t) \in K; 0 \leqslant t \leqslant T\} > 1 - \gamma \tag{2.1}$$

and for every $T > 0$, $0 < \varepsilon < 1$,

$$\lim_{\delta \downarrow 0} \sup_{P \in \mathscr{C}} P\{\delta_\omega^T(\varepsilon) \leqslant \delta\} = 0, \tag{2.2}$$

where

$$\delta^T_\omega(\varepsilon) = \min\{\tau_n(\omega) - \tau_{n-1}(\omega); 1 \leqslant n \leqslant N_T(\omega)\},$$
$$N_T(\omega) = \min\{n; \tau_{n+1}(\omega) > T\},$$
$$\tau_0(\omega) = 0,\ \tau_n(\omega) = \inf\left\{t \geqslant \tau_{n-1}(\omega); \rho(X(t), X(\tau_{n-1}(\omega))) \geqslant \frac{1}{2}\varepsilon\right\}.$$

Then $\mathscr{C}$ is relatively compact in $\mathscr{P}(D,\mathscr{F})$ in the weak topology.

These conditions are not particularly practical when applied to specific situations. The following lemma gives a sufficient condition for (2.2) to be true for a class of stochastic processes on the space S.

Lemma 2.2. *Let $\{\Omega_n(t)\}_{n=1,2,\dots}$ be a sequence of time-dependent linear operators on $C_b(S)$ satisfying $\Omega_n(t)\mathbf{1} = 0$, where $\mathbf{1}$ is the function which is identically equal to one. The solutions of the associated martingale problems exist with initial distribution $u \in M_1(S)$. Let Q^n_u be a solution of the martingale problem associated with $\Omega_n(t)$ with initial distribution u, and let $C_k(S)$ denote the set of all bounded continuous functions on S with compact support. If*

for any $T > 0$, $\gamma \in (0,1)$, *there exists a* compact *set* $K \subset S$ such that (2.3)
$\inf_n Q^n_u\{X(t) \in K; 0 \leqslant t \leqslant T\} > 1 - \gamma$,

for any $f \in C_k(S)$, $T > 0$, there exists $A_f(T) > 0$, *such that* (2.4)
$\sup_{n, t\in[0,T]} |\Omega_n(t) f(X)| \leqslant A_f(T)$, *with $A_f(T)$ being a non-decreasing* function of T,

then $\{Q^n_u\}_{n=1,2,\dots}$ is relatively compact in $\mathscr{P}(D,\mathscr{F})$.

Proof. For any $L > 0$, let $K_L = \{X \in S : |X| \leqslant L\}$, $B_L = \{\sup_{t\in[0,T]} |X(t)| \leqslant L\}$. Then by (2.4) we have

$$\sup_n Q^n_u\{B^c_L\} \to 0 \quad \text{as } L \to \infty. \tag{2.5}$$

To prove the result it suffices to verify that $\{Q^n_u\}_{n=1,2,\dots}$ satisfies condition (2.2). This is verified through the following estimates. For any $0 < \delta < 1$ we have

$$Q^n_u\{\delta^T_\omega(\varepsilon) < \delta\} \leqslant Q^n_u\{\delta^T_\omega(\varepsilon) < \delta, B_L\} + Q^n_u\{B^c_L\}. \tag{2.6}$$

and

$$Q^n_u\{\delta^T_\omega(\varepsilon) < \delta, B_L\} \leqslant Q^n_u\{\delta^T_\omega(\varepsilon) < \delta, B_L, N_T(\omega) \leqslant k\} + Q^n_u\{B_L, N_T(\omega) > k\}. \tag{2.7}$$

For the first term on the right-hand side of (2.7), we have

$$
\begin{aligned}
&Q_u^n\{\delta_\omega^T(\varepsilon) < \delta, B_L, N_T(\omega) \leqslant k\} \\
&= \sum_{i=1}^{k} Q_u^n\{\delta_\omega^T(\varepsilon) < \delta, B_L, N_T(\omega) = i\} \\
&\leqslant \sum_{i=1}^{k}\sum_{j=1}^{i} Q_u^n\{\tau_j - \tau_{j-1} < \delta, B_L, N_T(\omega) = i\} \\
&\leqslant \sum_{i=1}^{k}\sum_{j=1}^{i} Q_u^n\left\{\tau_j - \tau_{j-1} < \delta, \vee_{l=1}^{j-1}|X(\tau_l)| \leqslant L, \tau_{j-1} < T\right\} \\
&\leqslant \sum_{i=1}^{k}\sum_{j=1}^{i} E^{Q_u^n}\Big\{Q_u^{n,j-1}[I_{\{X(\tau_{j-1})\}^c}(X(\tau_j \wedge (\tau_{j-1}+\delta))) = 1, \\
&\qquad\qquad \vee_{l=1}^{j-1}|X(\tau_l)| \leqslant L, \tau_{j-1} < T\Big\} \\
&\leqslant \sum_{i=1}^{k}\sum_{j=1}^{i} E^{Q_u^n}\Big\{E^{Q_u^{n,j-1}}\left[\int_{\tau_{j-1}}^{\tau_j\wedge(\tau_{j-1}+\delta)} \Omega_n(s) I_{\{X(\tau_{j-1})\}^c}(X(s))\,\mathrm{d}s\right]; \\
&\qquad\qquad \vee_{l=1}^{j-1}|X(\tau_l)| \leqslant L, \tau_{j-1} < T\Big\} \\
&\leqslant \sum_{i=1}^{k}\sum_{j=1}^{i} E^{Q_u^n}\Big\{E^{Q_u^{n,j-1}}\left[-\int_{\tau_{j-1}}^{\tau_j\wedge(\tau_{j-1}+\delta)} \Omega_n(s) I_{\{X(\tau_{j-1})\}}(X(s))\,\mathrm{d}s\right]; \\
&\qquad\qquad \vee_{l=1}^{j-1}|X(\tau_l)| \leqslant L, \tau_{j-1} < T\Big\} \\
&\leqslant \sum_{i=1}^{k}\sum_{j=1}^{i} C(L,T)\delta = \frac{k(k+1)}{2} C(L,T)\delta, \qquad (2.8)
\end{aligned}
$$

where $Q_u^{n,j-1}$ is the r.c.p.d. of Q_u^n given $\mathscr{F}_{\tau_{j-1}}, C(L,T) = \sup_{Y\in B_L} A_{I_{\{Y\}}}(T+1) < \infty$.

We also have

$$
\begin{aligned}
&Q_u^n\{B_L, N_T(\omega) > k\} = Q_u^n\{\tau_k \leqslant T, B_L\} \\
&\leqslant E^{Q_u^n}[\mathrm{e}^{T-\tau_k}; \tau_k \leqslant T, B_L] \\
&\leqslant \mathrm{e}^T E^{Q_u^n}\{E^{Q_u^{n,k-1}}[\mathrm{e}^{\tau_k-\tau_{k-1}}; \tau_k \leqslant T, B_L]\mathrm{e}^{-\tau_{k-1}}\} \\
&\leqslant \mathrm{e}^T E^{Q_u^n}\{\mathrm{e}^{-\tau_{k-1}}[\mathrm{e}^{-t} Q_u^{n,k-1}(\tau_k - \tau_{k-1} \leqslant t, \tau_k \leqslant T, B_L) \\
&\quad + Q_u^{n,k-1}(\tau_k - \tau_{k-1} > t, \tau_k \leqslant T, B_L)]\} \\
&\leqslant \mathrm{e}^T E^{Q_u^n}\{\mathrm{e}^{-\tau_{k-1}}[\mathrm{e}^{-t} + (1-\mathrm{e}^{-t})Q_u^{n,k-1}(\tau_k - \tau_{k-1} \leqslant t)]; \\
&\quad \tau_{k-1} \leqslant T, \vee_{l=1}^{k-1}|X(\tau_l)| \leqslant L\} \\
&\leqslant \mathrm{e}^T E^{Q_u^n}\{\mathrm{e}^{-\tau_{k-1}}(\mathrm{e}^{-t} + (1-\mathrm{e}^{-t})C(L,T)t); \\
&\quad \tau_{k-1} \leqslant T, \vee_{l=1}^{k-1} x_{\tau_l} \leqslant L\}. \qquad (2.9)
\end{aligned}
$$

Letting $g(t) = \mathrm{e}^{-t} + (1-\mathrm{e}^{-t})C(L,T)t$, we have

$$g(0) = 1, \quad g'(0) = -1.$$

Thus, for fixed $L > 0$ there is a $t_0 \in (0,1)$ such that $0 < g(t_0) < 1$.

Denote $g(t_0)$ by $\Gamma(L,T)$. By induction we have

$$Q_u^n\{B_L, N_T(\omega) > k\} \leqslant \Gamma(L,T)^k \mathrm{e}^T. \tag{2.10}$$

Then (2.5), combined with (2.6)–(2.10), implies

$$\sup_n Q_u^n\{\delta_\omega^T(\varepsilon) < \delta\} \leqslant \frac{k(k+1)}{2} C(L,T)\delta + \Gamma(L,T)^k \mathrm{e}^T + \gamma_L, \tag{2.11}$$

where $\gamma_L = \sup_n Q_u^n\{B_L^c\}$.

Let $\delta \to 0$, then $k \to \infty$ and finally let $L \to \infty$, we get (2.2) and thus the result. $\square$

We will assume without explicitly mentioning that conditions (1.11) and (1.12) are satisfied in the remainder of this article.

By an argument similar to that used in the proof of Lemma 2.2 in Feng and Zheng (1992), we have

Lemma 2.3. *For every $n \geqslant 1$, $u \in M_1(S)$, there exists a $P_u^n \in \mathscr{P}(D,\mathscr{F})$ such that*

$$P_u^n \circ X^{-1}(0) = u, \tag{2.12}$$

$$\text{for any } Y \in S, (I_{\{Y\}}(X(t)) - \int_0^t L_{\|u_n(s)\|} I_{\{Y\}}(X(s))\,\mathrm{d}s,\ \mathscr{F}_t,\ P_u^n) \text{ is a martingale,} \tag{2.13}$$

where $u_n(s) = \tilde{u}_n([ns]/n)$, $\tilde{u}_n(s) = P_u^n \circ X^{-1}(s)$ and $[nt]$ is the integer part of nt.

Lemma 2.4. *Let $|||\tilde{u}_n(t)|||_p = \int_S |X|^p \tilde{u}_n(t)(\mathrm{d}X)$. Then for every $T > 0$, $u \in M_p(S)$, we have*

$$\sup_{n \geqslant 1,\ t \in [0,T]} |||\tilde{u}_n(t)|||_p < \infty. \tag{2.14}$$

Proof. For every $N \geqslant 1$, $1 \leqslant k \leqslant m$, $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m$, let

$$f_N(X) = \sum_{i=1}^m (x_i \wedge N), \quad S_N = \{0,1,\ldots,N\}^m,$$

$$S_N^{i_1,\ldots,i_k} = \{X \in S : x_{i_r} > N, \text{for } r = 1,\ldots,k;\ x_i \leqslant N \text{ for all other coordinates}\}.$$

Then it is not hard to check that

$$f_N(X) = \sum_{Y \in S_N} |Y| I_{\{Y\}}(X) + \sum_{k=1}^m \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m} \sum_{Y \in S_N^{i_1,\ldots,i_k}} (|Y| - \sum_{l=1}^k y_{i_l} + kN) I_{\{Y\}}(X).$$

This, combined with Lemma 2.3, implies that $f_N(X(t)) - \int_0^t L_{\|u_n(s)\|} f_N(X(s))\,\mathrm{d}s$ is a $(\mathscr{F}_t, P_u^n)$-martingale. For any $\beta \leqslant N - \Lambda$, we introduce a stopping time

$$\sigma_\beta = \inf\{t \geqslant 0: |X(t)| \geqslant \beta\}.$$

Then by Theorem 6.1 of Ikeda and Watanabe (1981, p. 26) we have for any $0 \leqslant s < t < \infty$,

$$\begin{aligned}
E^{P^n_u} f_N(X(t\wedge\sigma_\beta)) &= E^{P^n_u} f_N(X(s\wedge\sigma_\beta)) + E^{P^n_u}\int_{s\wedge\sigma_\beta}^{t\wedge\sigma_\beta} L_{||u_n(r)||} f_N(X(r))\,\mathrm{d}r \\
&\leqslant |||\tilde{u}_n(s\wedge\sigma_\beta)||| + E^{P^n_u}\int_{s\wedge\sigma_\beta}^{t\wedge\sigma_\beta}\{\sum_{Y\in S} q_{X(r),Y}(f_N(Y) - f_N(X(r))) \\
&\quad + \sum_{i=1}^{m}[||u_i(r)||(f_N(X(r)+e_i) - f_N(X(r))) \\
&\quad + x_i(r)(f_N(X(r)-e_i) - f_N(X(r)))]\}\,\mathrm{d}r \\
&\leqslant |||\tilde{u}_n(s\wedge\sigma_\beta)||| + E^{P^n_u}\int_s^t \{C(1+|X(r)|) + |||u_n(r)|||\}\,\mathrm{d}r \\
&\leqslant \{|||\tilde{u}_n(s\wedge\sigma_\beta)||| + C(t-s) \\
&\quad + \int_s^t |||u_n(r)|||\,\mathrm{d}r\} + \int_s^t E^{P^n_u} C|X(r)|\,\mathrm{d}r. \qquad (2.15)
\end{aligned}$$

Letting $N \to \infty$ and then $\beta \to \infty$, we get

$$E^{P^n_u}(|X(t)|) \leqslant \{|||\tilde{u}_n(s)||| + C(t-s) + \int_s^t |||u_n(r)|||\,\mathrm{d}r\} + \int_s^t E^{P^n_u} C|X(r)|\,\mathrm{d}r. \quad (2.16)$$

By Gronwall's inequality, we have

$$E^{P^n_u}(|X(t)|) \leqslant \{|||\tilde{u}_n(s)||| + C(t-s) + \int_s^t |||u_n(r)|||\,\mathrm{d}r\}\mathrm{e}^{C(t-s)}. \qquad (2.17)$$

For any $1 \leqslant k \leqslant n$, if we choose $s = k/n$, $t = (k+1)/n$, then (2.17) implies

$$E^{P^n_u}(|X((k+1)/n)|) \leqslant \{|||u_n(k/n)||| + C/n + \int_{k/n}^{(k+1)/n} |||u_n(r)|||\,\mathrm{d}r\}\mathrm{e}^{C/n} \qquad (2.18)$$

or equivalently

$$\begin{aligned}
|||u_n((k+1)/n)||| &\leqslant \{(1+1/n)|||u_n(k/n)||| + C/n\}\mathrm{e}^{C/n} \\
&\leqslant |||u_n(k/n)|||\mathrm{e}^{2C/n} + \frac{C}{n}\mathrm{e}^{C/n}. \qquad (2.19)
\end{aligned}$$

By induction (see the proof of Lemma 2.3 of Feng and Zheng (1992) for details), we get

$$\sup_{n\geqslant 1, t\in[0,T]} |||u_n(t)||| = C(T) < \infty. \qquad (2.20)$$

This, combined with condition (1.12), implies that

$$L_{||u_n(t)||}|X|^p \leqslant [C + C(T)2^p] + [C + C(T)(2^p - 1)]|X|^p. \qquad (2.21)$$

For any $N \geqslant 1$, let $g_N(X) = [\sum_{i=1}^m (x_i \wedge N)]^p$. Then (2.21), combined with an argument similar to that used between (2.15)–(2.17), implies that for any $t \in [0,T]$ and $n \geqslant 1$

$$|||\tilde{u}_n(t)|||_p \leqslant [|||u|||_p + (C + C(T)2^p)T]\exp[C + C(T)(2^p - 1)]T$$

which implies (2.14). □

For any $N \geqslant 1$ let $\varsigma_N = \inf\{t \geqslant 0: |X(t)| \geqslant N\}$. We have

Lemma 2.5. *For every $0 \leqslant T < \infty$, there exists a constant $M(T)$ such that*

$$\sup_{n\geqslant 1} P_u^n\{\varsigma_N < T\} \leqslant \frac{M(T)}{N} \tag{2.22}$$

Proof. By (1.12) and (2.20), we have for any $t \in [0, T]$

$$L_{\|u_n(t)\|}|X| \leqslant [C + C(T)] + C|X|.$$

Let $\phi(X) = |X| + (C + C(T))/C$ and $\phi_N(X) = \sum_{i=1}^m (x_i) \wedge N + (C + C(T))/C$ for $N \geqslant 1$. Then $M_t = \phi_N(X(t)) - \int_0^t L_{\|u_n(r)\|}\phi_N(X(r))\,\mathrm{d}r$ is a $(\mathscr{F}_t, P_u^n)$ martingale. By Corollary 1.2.7 of Stroock and Varadhan (1979), $(M_{t\wedge\sigma_\beta}, \mathscr{F}_t, P_u^n)$ is also a martingale. Thus, for any $0 \leqslant s < t \leqslant T$, $A \in \mathscr{F}_s$ we have

$$\begin{aligned} E^{P_u^n}[\phi_N(X(t\wedge\sigma_\beta)); A] &= E^{P_u^n}[\phi_N(X(s\wedge\sigma_\beta)); A] \\ &\quad + E^{P_u^n}\Big[\int_{s\wedge\sigma_\beta)}^{t\wedge\sigma_\beta)} L_{\|u_n(r)\|}\phi_N(X(r))\,\mathrm{d}r; A\Big] \\ &\leqslant E^{P_u^n}[\phi_N(X(s)); A] + E^{P_u^n}\left[\int_s^t C\phi_N(X(r))\,\mathrm{d}r; A\right]. \end{aligned}$$

Letting $N \to \infty$ and then $\beta \to \infty$, we get

$$E^{P_u^n}[\phi(X(t)); A] \leqslant E^{P_u^n}[\phi(X(s)); A] + E^{P_u^n}\left[\int_s^t C\phi(X(r))\,\mathrm{d}r; A\right]. \tag{2.23}$$

This, combined with Gronwall's inequality, implies

$$E^{P_u^n}[\phi(X(t)); A] \leqslant E^{P_u^n}[\phi(X(s)); A]\mathrm{e}^{C(t-s)}. \tag{2.24}$$

In other words, on the time interval $[0, T]$, $Y(t) = \phi(X(t))\mathrm{e}^{-Ct}$ is a supermartingale with respect to $(\mathscr{F}_t, P_u^n)$. Hence,

$$P_u^n\{\varsigma_N < T\} \leqslant \frac{\mathrm{e}^{CT}}{N} E^{P_u^n} Y(T \wedge \varsigma_N) \leqslant \frac{\mathrm{e}^{CT}}{N} E^{P_u^n} Y(0)$$

which implies (2.22) with $M(T) = [|\,||u||\,| + C + C(T)/C]\mathrm{e}^{CT}$. $\square$

Remark. In the proof of Lemmas 2.4 and 2.5 we use the assumption (1.11) and a stopping time argument. This differs from the proofs of Lemmas 2.3 and 2.4 in Feng and Zheng (1992). It is expected that condition (1.11) can be weakened.

Lemma 2.6. *For every $u \in M_1(S)$, $\{P_u^n\}_{n=1,2,\dots}$ is relatively compact in $\mathscr{P}(D,\mathscr{F})$ in the weak topology.*

Proof. It is not hard to see that (2.22) implies condition (2.1) in Lemma 2.1, while (1.11) and (2.20) together imply condition (2.4). Thus, by Lemma 2.2 we get the result. □

Lemma 2.7. *Let $u \in M_p(S)$. There exist subsequences $\{u_{n_k}(\cdot)\}$ of $\{u_n(\cdot)\}$ and $\{P_u^{n_k}\}$ of $\{P_u^n\}$ such that $P_u^{n_k}$ converges weakly to some $P_u \in \mathscr{P}(D,\mathscr{F})$, and for any $t \in [0,+\infty)$,*

$$d(u_{n_k}(t),u(t)) \to 0 \quad (k \to \infty)$$

where $u(t) = P_u \circ X^{-1}(t)$.

Proof. It is known (see Rachev, 1982) that a subset $\mathscr{C}$ of $M_1(S)$ is relatively compact in the topology generated by the metric d, defined in Section 1, if

$$\mathscr{C} \text{ is relatively compact in the weak topology,} \tag{2.25}$$

$$\sup_{P\in\mathscr{C}} \int_{|X|>L} |X|P(\mathrm{d}X) \to 0, \quad \text{when } L \to \infty. \tag{2.26}$$

If we choose $\mathscr{C} = \{\tilde{u}_n(t)\}_{n\geqslant 1, t\in[0,T]}$, then (2.14), combined with Hölder's inequality, implies (2.25) and (2.26). Thus, $\mathscr{C}$ is relatively compact in the spaces $(M_1(S),\tilde{d})$ and $(M_1(S),d)$.

On the other hand, we have

$$\tilde{d}(\tilde{u}_n(t),\tilde{u}_n(s)) = \sum_{Y\in S}\left[2^{-|Y|}\cdot\frac{|E^{P_u^n}(I_{\{Y\}}(X(t)) - I_{\{Y\}}(X(s)))|}{C_{|Y|}^{m-1+|Y|}}\right],$$

while

$$\begin{aligned}
&E^{P_u^n}(I_{\{Y\}}(X(t)) - I_{\{Y\}}(X(s)))\\
&= E^{P_u^n}\left\{\int_s^t L_{\|u_n(r)\|}I_{\{Y\}}(X(r))\,\mathrm{d}r\right\}\\
&= E^{P_u^n}\int_s^t\left\{\sum_{Z\in S} q_{X(r),Z}[I_{\{Y\}}(Z) - I_{\{Y\}}(X(r))]\right.\\
&\quad +\sum_{i=1}^m\left\{\|u_n^i(r)\|[I_{\{Y\}}(X(r)+e_i) - I_{\{Y\}}(X(r))]\right.\\
&\quad \left.\left.+x_i(r)[I_{\{Y\}}(X(r)-e_i) - I_{\{Y\}}(X(r))]\right\}\right\}\mathrm{d}r\\
&= \int_s^t\left\{\sum\nolimits_{Z\in S}[q_{Z,Y}P_u^n(X(r)=Z) - q_{Y,Z}P_u^n(X(r)=Y)]\right.\\
&\quad +\sum_{i=1}^m\left\{\|u_n^i(r)\|[P_u^n(X(r)=Y-e_i) - P_u^n(X(r)=Y)]\right.\\
&\quad \left.\left.+[(y_i(r)+1)P_u^n(X(r)=Y+e_i) - y_i(r)P_u^n(X(r)=Y)]\right\}\right\}\mathrm{d}r.
\end{aligned}$$

For any $\varepsilon > 0$ choose $L > 0$ such that $\sum_{i=L+1}^{\infty} 2^{-i} < \varepsilon/2$. Let

$$C(L,T) = \sup_{|Y| \leqslant L} q_{Y,Y} + \sup_{t\in[0,T]} |||u(t)||| + L + m.$$

Then we have

$$\sup_{Y\in S} |E^{P_u^n}(I_{\{Y\}}(X(t)) - I_{\{Y\}}(X(s)))| \leqslant C(L,T)(t-s), \quad t, s \in [0,T],$$

and for any $\delta \leqslant \varepsilon/(2C(L,T))$,

$$\sup_{s,t\in[0,T],\ |t-s|<\delta} \tilde{d}(\tilde{u}_n(t), \tilde{u}_n(s)) < \varepsilon. \tag{2.27}$$

Let $C([0,+\infty),(M_1(S),\tilde{d}))$ denote the space of all $(M_1(S),\tilde{d})$-valued continuous functions on $[0,+\infty)$ equipped with the topology of uniformly convergence on compact subintervals of $[0,+\infty)$. Then (2.27) implies that $\{\tilde{u}_n(t)\}$ is a $\tilde{d}$-equicontinuous subset of $C([0,+\infty),(M_1(S),\tilde{d}))$. This, combined with the fact that $\{\tilde{u}_n(t): n\geqslant 1, t \in [0,T]\}$ is relatively compact in $(M_1(S),\tilde{d})$, implies, by using the theorem of Arzela–Ascoli's that $\{\tilde{u}_n(\cdot)\}_{n\geqslant 1}$ is in fact a relatively compact subset of $C([0,+\infty),(M_1(S),\tilde{d}))$. By Lemma 2.6 there exist $P_u \in \mathscr{P}(D,\mathscr{F})$ and subsequence $\{P_u^{n_k}\}$ of $\{P_u^n\}$ such that $P_u^{n_k}$ converges weakly to P_u as $k \to \infty$. Without loss of generality, we may assume that there is a $u_0(\cdot) \in C([0,+\infty),(M_1(S),\tilde{d}))$ such that $\tilde{u}_{n_k}(\cdot)$ converges to $u_0(\cdot)$ in $C([0,+\infty),(M_1(S),\tilde{d}))$ as $k \to \infty$.

Let $u(t) = P_u \circ X^{-1}(t)$, $t\geqslant 0$, and $T_J = \{t\geqslant 0: P_u(X(t) \neq X(t-)) = 0\}$. Then by Lemma 7.7 of Ethier and Kurtz (1986, Ch. 3), T_J is dense in $[0,+\infty)$ and T_J^c (the complement of T_J in $[0,+\infty)$) is at most countable. This, combined with the right continuity of $u(\cdot)$ in the space D, implies that $u_0(\cdot) = u(\cdot)$. Moreover, for each $t\geqslant 0$,

$$\begin{aligned}\tilde{d}(u_{n_k}(t), u(t)) &= \tilde{d}(\tilde{u}_{n_k}([n_k t]/n_k), u(t)) \\ &\leqslant \sup_{0\leqslant s\leqslant t} \tilde{d}(\tilde{u}_{n_k}(s), u(s)) + \tilde{d}(u([n_k t]/n_k), u(t)) \to 0 \quad \text{as } n_k \to \infty.\end{aligned}$$

Thus, $u_{n_k}(t)$ converges to $u(t)$ in the vague topology. Noting that $\{u_{n_k}(t)\}$ is a relatively d-compact subset of $M_1(S)$, we conclude that $d(u_{n_k}(t), u(t)) \to 0$ as $k \to \infty$. □

Proof of Theorem 1.1. Let P_u be as in Lemma 2.7. For any $Y \in S$, let

$$\Psi_t(Y) = \int_0^t L_{||u(s)||} I_{\{Y\}}(X(s))\,\mathrm{d}s, \qquad \Psi_t^n(Y) = \int_0^t L_{||u_n(s)||} I_{\{Y\}}(X(s))\,\mathrm{d}s.$$

By Lemma 2.7, for each $t \in T_J$, $\Phi \in C_b(D)$ (the set of all bounded continuous functions on D), there exists $\{n_k\} \subset \{0,1,2,\ldots\}$ such that

$$E^{P_u^{n_k}}[I_{\{Y\}}(X(t))\Phi] \to E^{P_u}[I_{\{Y\}}(X(t))\Phi], \tag{2.28}$$

$$|E^{P_u^{n_k}}[(\Psi_t^{n_k}(Y) - \Psi_t(Y))\Phi]| \leqslant \|\Phi\| \int_0^t d(u_{n_k}(s), u(s))\,\mathrm{d}s \to 0 \quad (n_k \to \infty), \tag{2.29}$$

where $\|\Phi\|$ is the supremum of Φ on D. The boundedness of Φ and condition (1.11) imply that $[L_{\|u(s)\|}I_{\{Y\}}(X(s))]\Phi \in C_b(D)$ for $s \in T_J$. By Fubini's theorem we get that for every $t \geqslant 0$,

$$E^{P_u^{n_k}}[\Psi_t(Y)\Phi] \to E^{P_u}[\Psi_t(Y)\Phi].$$

Thus for any $s, t \in T_J$, $0 \leqslant s < t < \infty$, $\Phi \in C_b(D)$,

$$E^{P_u}[(I_{\{Y\}}(X(t)) - \Psi_t(Y))\Phi] = E^{P_u}[(I_{\{Y\}}(X(s)) - \Psi_s(Y))\Phi].$$

Finally, by the right continuity of $I_{\{Y\}}(X(t)) - \Psi_t(Y)$ and the dominated convergence theorem, we get the result. □

From the proof of Lemma 2.4 and Lemma 2.7, we obtain the following:

Theorem 2.8. *Assume* (1.11) *and* (1.12) *are satisfied. Let $u \in M_p(S)$, and let $P_u \in \mathscr{P}(D, \mathscr{F})$ be any solution of the nonlinear martingale problem $[u, L]$ with initial distribution u. Then $u(t) = P_u \circ X^{-1}(t)$ is a $(M_1(S), d)$-valued continuous function on* $[0, +\infty)$.

Proof. It is not hard to check, by using the martingale property, that the following is true.

$$\sup_{t \in [0,T]} |||u(t)|||_p < \infty,$$

$$\lim_{L \to \infty} \sup_{t \in [0,T]} \int_{|X| > L} |X| u(t)(\mathrm{d}X) = 0,$$

$$\lim_{\delta \to 0} \sup\{\tilde{d}(u(t), u(s)); t, s \in [0, T],\ |t - s| < \delta\} = 0.$$

These, combined with an argument similar to that used in the last part of the proof of Lemma 2.7, imply the result. □

3. Uniqueness

In this section we prove the uniqueness of solutions of the nonlinear martingale problems.

In the sequel we will always assume that $u \in M_p(S)$, $T > 0$ and $v : [0, +\infty) \to (M_1(S), d)$ is continuous and satisfies $\sup_{0 \leqslant t \leqslant T} \sum_{i=1}^m \langle x^p, v^i(t) \rangle < \infty$. Let $(q_{X,Y}(t))_{X,Y \in S}$ denote the Q-matrix determined by $L_{\|v(t)\|}$. Then we have

Lemma 3.1. *For all $s, t \in [0, +\infty)$, $X, Y \in S$, the minimal nonnegative solutions of the following two equations*

$$\phi(s, X; t, Y) = \delta_{\{Y\}}(X) \mathrm{e}^{\int_s^t q_{X,X}(\tau) d\tau} + \int_s^t \sum_{Z \neq X} q_{X,Z}(\sigma) \phi(\sigma, Z; t, Y) \mathrm{e}^{\int_s^\sigma q_{X,X}(\tau)\,\mathrm{d}\tau} \mathrm{d}\sigma, \tag{3.1}$$

$$\phi(s,X;t,Y)=\delta_{\{Y\}}(X)\mathrm{e}^{\int_s^t q_{Y,Y}(\tau)\,\mathrm{d}\tau}$$
$$+\int_s^t \sum_{Z\neq Y}\phi(s,X;\sigma,Z)q_{Z,Y}(\sigma)\mathrm{e}^{\int_\sigma^t q_{Y,Y}(\tau)\,\mathrm{d}\tau}\mathrm{d}\sigma, \tag{3.2}$$

exist and are, in fact, the same.

Proof. First we show the existence of minimal nonnegative solutions of (3.1) and (3.2). For every $n\geqslant 1$, let

$$P^1(s,X;t,Y)=\delta_{\{Y\}}(X)\mathrm{e}^{\int_s^t q_{X,X}(\tau)\,\mathrm{d}\tau},$$
$$P^{n+1}(s,X;t,Y)=\int_s^t \sum_{Z\neq X}q_{X,Z}(\sigma)P^n(\sigma,Z;t,Y)\mathrm{e}^{\int_s^\sigma q_{X,X}(\tau)\,\mathrm{d}\tau}\mathrm{d}\sigma;$$
$$\bar{P}^1(s,X;t,Y)=\delta_{\{Y\}}(X)\mathrm{e}^{\int_s^t q_{Y,Y}(\tau)\,\mathrm{d}\tau},$$
$$\bar{P}^{n+1}(s,X;t,Y)=\int_s^t \sum_{Z\neq Y}\bar{P}^n(s,X;\sigma,Z)q_{Z,Y}(\sigma)\mathrm{e}^{\int_\sigma^t q_{Y,Y}(\tau)\,\mathrm{d}\tau}\mathrm{d}\sigma;$$
$$P(s,X;t,Y)=\sum_{n=1}^{\infty}P^n(s,X;t,Y),$$
$$\bar{P}(s,X;t,Y)=\sum_{n=1}^{\infty}\bar{P}^n(s,X;t,Y).$$

Then $P(s,X;t,Y)$ and $\bar{P}(s,X;t,Y)$ are nonnegative solutions of Eqs. (3.1) and (3.2), respectively.

Let $\phi(s,X;t,Y)$ and $\bar{\phi}(s,X;t,Y)$ be any two nonnegative solutions of (3.1) and (3.2), respectively. Then it is obvious that

$$\phi(s,X;t,Y)\geqslant P^1(s,X;t,Y),\quad \bar{\phi}(s,X;t,Y)\geqslant \bar{P}^1(s,X;t,Y).$$

By induction it is not hard to check that

$$\phi(s,X;t,Y)\geqslant P(s,X;t,Y),\quad \bar{\phi}(s,X;t,Y)\geqslant \bar{P}(s,X;t,Y).$$

Thus, $P(s,X;t,Y)$ and $\bar{P}(s,X;t,Y)$ are in fact minimal nonnegative solutions of Eqs. (3.1) and (3.2), respectively.

To finish the proof it suffices to verify that for each $n\geqslant 1$,

$$P^n(s,X;t,Y)=\bar{P}^n(s,X;t,Y). \tag{3.3}$$

This can be done by induction. For $n=1$ the result is obvious. For $n=2$ we have that for $X=Y$, $P^2(s,X;t,X)=\bar{P}^2(s,X;t,X)=0$; for $X\neq Y$

$$P^2(s,X;t,Y)=\int_s^t \sum_{Z\neq X}q_{X,Z}(\sigma)P^1(\sigma,Z;t,Y)\mathrm{e}^{\int_s^\sigma q_{X,X}(\tau)\,\mathrm{d}\tau}\mathrm{d}\sigma,$$
$$=\int_s^t \sum_{Z\neq X}q_{X,Z}(\sigma)\delta_{\{Y\}}(Z)\mathrm{e}^{\int_\sigma^t q_{Z,Z}(\tau)\,\mathrm{d}\tau}\mathrm{e}^{\int_s^\sigma q_{X,X}(\tau)\,\mathrm{d}\tau}\mathrm{d}\sigma,$$
$$=\int_s^t q_{X,Y}(\sigma)\mathrm{e}^{\int_\sigma^t q_{Y,Y}(\tau)\,\mathrm{d}\tau}\mathrm{e}^{\int_s^\sigma q_{X,X}(\tau)\,\mathrm{d}\tau}\mathrm{d}\sigma$$

and

$$\bar{P}^2(s,X;t,Y) = \int_s^t \sum_{Z\neq Y} P^1(s,X;\sigma,Z) q_{Z,Y}(\sigma) e^{\int_\sigma^t q_{Y,Y}(\tau)\,d\tau} d\sigma,$$

$$= \int_s^t \sum_{Z\neq Y} \delta_{\{X\}}(Z) e^{\int_s^\sigma q_{Z,Z}(\tau)\,d\tau} q_{Z,Y}(\sigma) e^{\int_\sigma^t q_{Y,Y}(\tau)\,d\tau} d\sigma,$$

$$= \int_s^t q_{X,Y}(\sigma) e^{\int_\sigma^t q_{Y,Y}(\tau)\,d\tau} e^{\int_s^\sigma q_{X,X}(\tau)\,d\tau} d\sigma,$$

which implies that (3.3) is true for $n = 2$.

Now we assume that (3.3) holds for $n-1$ and n. For $n+1$ we have

$$P^{n+1}(s,X;t,Y) = \int_s^t \sum_{Z\neq X} q_{X,Z}(\sigma) P^n(\sigma,Z;t,Y) e^{\int_s^\sigma q_{X,X}(\tau)\,d\tau} d\sigma,$$

$$= \int_s^t \sum_{Z\neq X} q_{X,Z}(\sigma) \bar{P}^n(\sigma,Z;t,Y) e^{\int_s^\sigma q_{X,X}(\tau)\,d\tau} d\sigma,$$

$$= \int_s^t \sum_{Z\neq X} q_{X,Z}(\sigma) e^{\int_s^\sigma q_{X,X}(\tau)\,d\tau} d\sigma$$

$$\times \int_\sigma^t \sum_{Z'\neq Y} \bar{P}^{n-1}(\sigma,Z;\sigma',Z') q_{Z',Y}(\sigma') e^{\int_{\sigma'}^t q_{Y,Y}(\tau)\,d\tau} d\sigma'$$

$$= \int_s^t d\sigma \left\{ \int_\sigma^t \sum_{Z\neq X, Z'\neq Y} q_{X,Z}(\sigma) q_{Z',Y}(\sigma') e^{\int_s^\sigma q_{X,X}(\tau)\,d\tau} \right.$$

$$\left. e^{\int_{\sigma'}^t q_{Y,Y}(\tau)\,d\tau} P^{n-1}(\sigma,Z;\sigma',Z')\,d\sigma' \right\}$$

and

$$\bar{P}^{n+1}(s,X;t,Y) = \int_s^t \sum_{Z'\neq Y} \bar{P}^n(s,X;\sigma',Z') q_{Z',Y}(\sigma') e^{\int_{\sigma'}^t q_{Y,Y}(\tau)\,d\tau} d\sigma'$$

$$= \int_s^t \sum_{Z'\neq Y} P^n(s,X;\sigma',Z') q_{Z',Y}(\sigma') e^{\int_{\sigma'}^t q_{Y,Y}(\tau)\,d\tau} d\sigma'$$

$$= \int_s^t \sum_{Z'\neq Y} q_{Z',Y}(\sigma') e^{\int_{\sigma'}^t q_{Y,Y}(\tau)\,d\tau} d\sigma'$$

$$\times \int_s^{\sigma'} \sum_{Z\neq X} q_{X,Z}(\sigma) e^{\int_s^\sigma q_{X,X}(\tau)\,d\tau} P^{n-1}(\sigma,Z;\sigma',Z')\,d\sigma$$

$$= \int_s^t d\sigma' \left\{ \int_s^{\sigma'} \sum_{Z\neq X, Z'\neq Y} q_{X,Z}(\sigma) q_{Z',Y}(\sigma') e^{\int_s^\sigma q_{X,X}(\tau)\,d\tau} \right.$$

$$\left. e^{\int_{\sigma'}^t q_{Y,Y}(\tau)\,d\tau} P^{n-1}(\sigma,Z;\sigma',Z')\,d\sigma \right\}$$

$$= \int_s^t d\sigma \left\{ \int_\sigma^t \sum\nolimits_{Z\neq X, Z'\neq Y} q_{X,Z}(\sigma) q_{Z',Y}(\sigma') e^{\int_s^\sigma q_{X,X}(\tau)\,d\tau} \right.$$

$$\left. e^{\int_{\sigma'}^t q_{Y,Y}(\tau)\,d\tau} P^{n-1}(\sigma,Z;\sigma',Z')\,d\sigma' \right\},$$

where the last step is obtained by changing the order of integration. Thus, we get the result. $\square$

Remark. The uniqueness of the minimal nonnegative solutions of Eqs. (3.1) and (3.2) follows from the definition of minimal nonnegative solution.

Lemma 3.2. *For any* $(s,X) \in [0,+\infty) \times S$, *let* $P^{\min}(s,X;\cdot,\cdot)$ *denote the minimal nonnegative solution of Eqs.* (3.1) *and* (3.2). *Then for every* $t \geqslant s$ *we have*

$$\sum_{Y\in S} P^{\min}(s,X;t,Y) = 1. \tag{3.4}$$

Proof. For any $n \geqslant 1$, $u \geqslant 0$ and $X, Y \in S$, let $S_n = \{0,1,\ldots,n\}^m$ and $q^n_{X,Y}(u) = I_{S_n}(X) q_{X,Y}(u)$. Then $q^n_{X,Y}(u)$ is uniformly bounded on any finite interval $[0,T]$ for any n. Let $P^{\min}_n(s,X;t,Y)$ denote the minimal nonnegative solution of Eqs. (3.1) and (3.2) with $q_{X,Y}(u)$ being replaced by $q^n_{X,Y}(u)$, $P^{\min}_n(s,X;t,S) = \sum_{Y\in S} P^{\min}_n(s,X;t,Y)$. Then a direct calculation, combined with the uniformly boundedness of $q^n_{X,Y}(u)$ and Fubini's theorem, gives

$$\begin{aligned} P^{\min}_n(s,X;t,S) &= \mathrm{e}^{\int_s^t q^n_{X,X}(\tau)\,d\tau} \\ &\quad + \sum_{Y\in S} \int_s^t \sum_{Z\neq Y} P^{\min}_n(s,X;\sigma,Z) q^n_{Z,Y}(\sigma) \mathrm{e}^{\int_\sigma^t q^n_{Y,Y}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma, \end{aligned} \tag{3.5}$$

which implies that

$$\begin{aligned} \frac{\mathrm{d}P^{\min}_n(s,X;t,S)}{dt} &= q^n_{X,X}(t)\mathrm{e}^{\int_s^t q^n_{X,X}(\tau)\,\mathrm{d}\tau} + \sum_{Y\in S}\sum_{Z\neq Y} P^{\min}_n(s,X;t,Z) q^n_{Z,Y}(t) \\ &\quad + \sum_{Y\in S} \int_s^t \sum_{Z\neq Y} P^{\min}_n(s,X;\sigma,Z) q^n_{Z,Y}(\sigma)\mathrm{e}^{\int_\sigma^t q^n_{Y,Y}(\tau)\,\mathrm{d}\tau} q^n_{Y,Y}(t)\,\mathrm{d}\sigma \\ &= \sum_{Z\in S}\sum_{Y\neq Z} P^{\min}_n(s,X;t,Z) q^n_{Z,Y}(t) \\ &\quad + \textstyle\sum_{Y\in S} q^n_{Y,Y}(t) \Big\{ \delta_{\{Y\}}(X) \mathrm{e}^{\int_s^t q^n_{Y,Y}(\tau)\,\mathrm{d}\tau} \\ &\quad + \displaystyle\int_s^t \sum_{Z\neq Y} P^{\min}_n(s,X;\sigma,Z) q^n_{Z,Y}(\sigma) \mathrm{e}^{\int_\sigma^t q^n_{Y,Y}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\sigma \Big\} \\ &= \sum_{Y\in S} P^{\min}_n(s,X;t,Y) \sum_{Z\neq Y} q^n_{Y,Z}(t) + \sum_{Y\in S} P^{\min}_n(s,X;t,Y) q^n_{Y,Y}(t) \\ &= 0, \quad \textit{a.e.}\text{ in } t. \end{aligned} \tag{3.6}$$

The last equality holds because of the conservativity of the Q-matrix $(q^n_{X,Y}(t))_{X,Y\in S}$.
Since $\mathrm{d}P^{\min}_n(s,X;t,S)/\mathrm{d}t$ is continuous in t, we conclude that

$$\sum_{Y\in S} P^{\min}_n(s,X;t,Y) = 1. \tag{3.7}$$

By Fatou's lemma and Theorem 2.6 in Chen (1992) we have

$$\lim_{n\to\infty} P^{\min}_n(s,X;t,S_n) = P^{\min}(s,X;t,S). \tag{3.8}$$

Hence, to finish the proof it suffices to verify that

$$\lim_{n\to\infty} P_n^{\min}(s,X;t,S_n^c)=0. \tag{3.9}$$

By condition (1.12), we have that for any $s,t\in[0,T]$

$$\sum_{Y\neq X} q^n_{X,Y}(t)|Y|^p-(C(T)-q^n_{X,X}(t))|X|^p\leqslant 0,$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t}[|X|^p\mathrm{e}^{-\int_s^t(C(T)-q^n_{X,X}(\tau))\mathrm{d}\tau}]+\sum_{Y\neq X} q^n_{X,Y}(t)|Y|^p\mathrm{e}^{-\int_s^t(C(T)-q^n_{X,X}(\tau))\mathrm{d}\tau}\leqslant 0.$$

Integrating from s to t and then multiplying on both sides by $\mathrm{e}^{C(T)(t-s)}$, we get

$$\mathrm{e}^{C(T)(t-s)}|X|^p\geqslant |X|^p\mathrm{e}^{\int_s^t q^n_{X,X}(\tau)\mathrm{d}\tau}+\int_s^t\sum_{Z\neq X} q^n_{X,Z}(\sigma)|Z|^p\mathrm{e}^{C(T)(t-\sigma)}\mathrm{e}^{\int_s^\sigma q^n_{X,X}(\tau)\mathrm{d}\tau}\mathrm{d}\sigma.$$

Noting that $P_n^{\min}(s,X;t,Y)$ is the minimal nonnegative solution of Eq. (3.1) with $q_{X,Y}(u)$ being replaced by $q^n_{X,Y}(u)$, then Theorem 2.12 of Chen (1992) implies that the minimal nonnegative solution of equation

$$\begin{aligned}h(s,X,t)&=|X|^p\mathrm{e}^{\int_s^t q^n_{X,X}(\tau)\mathrm{d}\tau}\\&\quad+\int_s^t\sum_{Z\neq X} q^n_{X,Z}(\sigma)h(\sigma,Z,t))\mathrm{e}^{\int_s^\sigma q^n_{X,X}(\tau)\mathrm{d}\tau}\mathrm{d}\sigma\end{aligned}$$

is $\sum_{Y\in S} P_n^{\min}(s,X;t,Y)|Y|^p$. Using the comparison theorem again we get

$$\sum_{Y\in S} P_n^{\min}(s,X;t,Y)|Y|^p\leqslant \mathrm{e}^{C(T)(t-s)}|X|^p$$

which implies (3.9) and thus the result. □

Lemma 3.3. *Under assumptions* (1.11) *and* (1.12)(i), *the time inhomogeneous martingale problem* $[u,L_{\|v(t)\|}]$ *with initial distribution* u *is well-posed.*

Proof. For any integer $n>1$, let $v_n(t)=v([nt]/n)$. Under assumptions (1.11) and (1.12), the sequence of operators $L_{\|v_n(t)\|}$ satisfies condition (2.4) in Lemma 2.2. By applying Theorem 2.25 of Chen (1992) and Theorem 3.2 of Zheng and Zheng (1987), we conclude that for every $X\in S$ the martingale problem $[\delta_{\{X\}},L_{\|v(0)\|}]$ is well-posed. We denote the unique solution by $P_{X,v(0)}$. For $s\in[0,+\infty)$, let $\Phi_s:D\to D$ be the standard shift operator defined by $\Phi_s(\omega)=\omega(s+\cdot)$. Let $P^s_{X,v(0)}\in\mathscr{P}(D,\mathscr{F})$ be the unique probability measure satisfying:

$$\begin{aligned}&P^s_{X,v(0)}\circ X^{-1}(t)=\delta_{\{X\}},\quad 0\leqslant t\leqslant s,\\&P^s_{X,v(0)}\circ\Phi_s^{-1}=P_{X,v(0)}.\end{aligned}$$

It is easy to check that $P^s_{X,v(0)}$ is a solution of the martingale problem for $L_{\|v(0)\|}$ starting from (s,X). In fact, $v(0)$ can be replaced by any probability measure in $M_1(S)$ and the above result still hold.

Let $P_{u,v(0)}$ be defined as

$$P_{u,v(0)}(\cdot) = \int_S P_{X,v(0)} u(\mathrm{d}X).$$

By induction we define $P_{u,v((k+1)/n)} \in \mathscr{P}(D,\mathscr{F})$ to be the unique probability measure satisfying

(i) $P_{u,v((k+1)/n)} = P_{u,v(k/n)}$ on $\mathscr{F}_{k/n}$,
(ii) A regular conditional probability of $P_{u,v((k+1)/n)}$ given $\mathscr{F}_{k/n}$ is $P^{k/n}_{X(k/n),v((k+1)/n)}$.

It is obvious that $P_{u,v(k/n)}\{k/n \leqslant t\} \to 0$ as $k \to \infty$. Thus, by Theorem 1.3.5 of Stroock and Varadhan (1979), there exists a unique probability measure $P^n_u \in \mathscr{P}(D,\mathscr{F})$ such that

$$P^n_u = P_{u,v(k/n)} \text{ on } \mathscr{F}_{k/n}, \quad k = 1,2,\dots .$$

From Theorem 6.1.2 of Stroock and Varadhan (1979) we know that P^n_u is the solution of the martingale problem $[u, L_{\|v_n(t)\|}]$. By applying Lemma 2.2 we can show that $\{P^n_u\}_{n=1,2,\dots}$ is relatively compact and, by the construction, all the limit points solve the martingale problem $[u, L_{\|v(t)\|}]$. By a shifting argument we can prove the existence of solutions of the martingale problem for $L_{\|v_n(t)\|}$ starting from time s and with initial distribution u.

Now we turn to the proof of uniqueness. In order to do this it suffices to verify that the solution of the martingale problem for $L_{\|v(t)\|}$ starting from (s,X) is unique. By Theorem 6.2.3 of Stroock and Varadhan (1979), this is equivalent to showing that any two solutions of the martingale problem have the same one-dimensional distributions.

Let $P_{s,X}$ be any solution of the martingale problem for $L_{\|v(t)\|}$ starting from (s,X). Then for any $t > s$ and $Y \in S$ we have

$$E^{P_{s,X}}\left[I_{\{Y\}}(X(t)) - I_{\{Y\}}(X(s)) - \int_s^t L_{\|v(\tau)\|} I_{\{Y\}}(X(\tau))\,\mathrm{d}\tau\right] = 0.$$

Let $P(s,X;t,Y) = P_{s,X}\{X(t) = Y\}$. Then we have

$$P(s,X;t,Y) - \delta_{\{Y\}}(X) - E^{P_{X,s}}\left[\int_s^t L_{\|v(\tau)\|} I_{\{Y\}}(X(\tau))\,\mathrm{d}\tau\right] = 0.$$

By assumption (1.11), we have that $|L_{\|v(\tau)\|} I_{\{Y\}}(X(\tau))|$ is bounded by a constant depending on Y. This, combined with Fubini's theorem, implies

$$P(s,X;t,Y) - \delta_{\{Y\}}(X) = \int_s^t \sum_{Z\in S} P(s,X;\tau,Z) q_{Z,Y}(\tau)\,\mathrm{d}\tau. \tag{3.10}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} P(s,X;t,Y) = \sum_{Z\in S} P(s,X;t,Z) q_{Z,Y}(t), \quad \text{a.e. in } t. \tag{3.11}$$

Now, multiplying both sides of (3.11) by $e^{-\int_s^t q_{Y,Y}(\tau)\,d\tau}$ and then integrating from s to t we get

$$P(s,X;t,Y) = \delta_{\{Y\}}(X)e^{\int_s^t q_{Y,Y}(\tau)} + \int_s^t \sum_{Z\neq Y} P(s,X;\sigma,Z)q_{Z,Y}(\sigma)e^{\int_\sigma^t q_{Y,Y}(\tau)\,d\tau}d\sigma. \tag{3.12}$$

By Lemmas 3.1 and 3.2, the minimal nonnegative solution $P^{\min}(s,X;t,Y)$ of Eq. (3.2) exists and satisfies (3.4). Hence, we get that $\forall\ s,\ t \in [0,+\infty)$ and $X,\ Y \in S$,

$$P(s,X;t,Y) = P^{\min}(s,X;t,Y).$$

Since $P_{s,X}$ was arbitrarily chosen, we get the result. □

For any $u \in M_p(S)$, let P_u and $\bar{P}_u$ be any two solutions of the nonlinear martingale problem $[u,L]$ with initial distribution u. Let $u(t) = P_u \circ X^{-1}(t)$, $\bar{u}(t) = \bar{P}_u \circ X^{-1}(t)$. Then, by Theorem 2.8, both $u(t)$ and $\bar{u}(t)$ are $(M_1(S),d)$-valued continuous functions. By Lemma 3.3, in order to prove the equality $P_u = \bar{P}_u$ it suffices to verify that for every $t \geqslant 0$, $\|u(t)\| = \|\bar{u}(t)\|$.

For any $(X,\tilde{X}) \in S^2$ and $G \in C_f(S^2)$ we introduce the following time-inhomogeneous coupling operator:

$$\begin{aligned}\Omega_t G(X,\tilde{X}) = \Omega G(X,\tilde{X}) + \sum_{i=1}^m \{&(\|u^i(t)\| - \|\bar{u}^i(t)\|)^+(G(X+e_i,\tilde{X}) - G(X,\tilde{X}))\\ &+ (\|\bar{u}^i(t)\| - \|u^i(t)\|)^+(G(X,\tilde{X}+e_i) - G(X,\tilde{X}))\\ &+ (\|u^i(t)\| \wedge \|\bar{u}^i(t)\|)(G(X+e_i,\tilde{X}+e_i) - G(X,\tilde{X}))\},\end{aligned}$$

where

$$\begin{aligned}\Omega G(X,\tilde{X}) = &\sum_{(Y,\tilde{Y})\in S^2,\ Y-X=\tilde{Y}-\tilde{X}} \{(q_{X,Y} - q_{\tilde{X},\tilde{Y}})^+(G(Y,\tilde{X}) - G(X,\tilde{X}))\\ &+(q_{\tilde{X},\tilde{Y}} - q_{X,Y})^+(G(X,\tilde{Y}) - G(X,\tilde{X}))\\ &+(q_{X,Y} \wedge q_{\tilde{X},\tilde{Y}})(G(Y,\tilde{Y}) - G(X,\tilde{X}))\}\\ &+\sum_{i=1}^m \{(x_i - \tilde{x}_i)^+(G(X-e_i,\tilde{X}) - G(X,\tilde{X}))\\ &+(\tilde{x}_i - x_i)^+(G(X,\tilde{X}-e_i) - G(X,\tilde{X}))\\ &+(x_i \wedge \tilde{x}_i)(G(X-e_i,\tilde{X}-e_i) - G(X,\tilde{X}))\}.\end{aligned}$$

Now we are ready to prove the uniqueness.

Proof of Theorem 1.2. It is not hard to check that the Q-matrix associated with operator Ω satisfies assumptions (1.11) and (1.12)(i), and the time inhomogeneous jump rates in Ω_t are continuous functions of time t. Thus, the existence of solutions of the time-inhomogeneous martingale problem associated with Ω_t follows from an argument similar to that used in the first part of the proof of Lemma 3.3. Let $\boldsymbol{Q}_u$ be a solution of the time inhomogeneous martingale problem associated with Ω_t and with initial distribution Λ_u defined by

$$\Lambda_u(X,\tilde{X}) = \begin{cases} 0, & \text{for } X \neq \tilde{X}, \\ u(X), & \text{for } X = \tilde{X}. \end{cases}$$

It is obvious that $\boldsymbol{Q}_u$ has marginal distributions P_u and $\bar{P}_u$.

For every $N \geqslant 1$, let $F_N(X,\tilde{X}) = \rho(X,\tilde{X}) I_{\{|Y| \leqslant N, |\tilde{Y}| \leqslant N\}}(X,\tilde{X})$. For $n \leqslant N - \Lambda$ define

$$\eta_n = \inf\{t \geqslant 0 : |X(t)| \geqslant n,\ |\tilde{X}(t)| \geqslant n\}.$$

Then we have

$$\begin{aligned} & E^{\boldsymbol{Q}_u} F_N(X(t \wedge \eta_n), \tilde{X}(t \wedge \eta_n)) \\ &= E^{\Lambda_u} F_N(X(0), \tilde{X}(0)) + E^{\boldsymbol{Q}_u} \int_0^{t \wedge \eta_n} \Omega_s F_N(X(s), \tilde{X}(s))\,\mathrm{d}s \\ &= E^{\boldsymbol{Q}_u} \int_0^{t \wedge \eta_n} \Omega F_N(X(s), \tilde{X}(s))\,\mathrm{d}s + E^{\boldsymbol{Q}_u} \int_0^{t \wedge \eta_n} (\Omega_s - \Omega) F_N(X(s), \tilde{X}(s))\,\mathrm{d}s \\ &\leqslant hE^{\boldsymbol{Q}_u} \int_0^t \rho(X(s), \tilde{X}(s))\,\mathrm{d}s + 2 \int_0^t \sum_{i=1}^m [|\|u^i(s)\| - \|\bar{u}^i(s)\||]\,\mathrm{d}s \\ &\leqslant (h+2) \int_0^t E^{\boldsymbol{Q}_u} \rho(X(s), \tilde{X}(s))\,\mathrm{d}s. \end{aligned}$$

In the last inequality we used the following fact:

$$\sum_{i=1}^m |\|u^i(t)\| - \|\bar{u}^i(t)\|| \leqslant E^{\boldsymbol{Q}_u} \rho(X(t), \tilde{X}(t)). \tag{3.13}$$

Letting $N \to \infty$ and then $n \to \infty$, we have

$$E^{\boldsymbol{Q}_u} \rho(X(t), \tilde{X}(t)) \leqslant (h+2) \int_0^t E^{\boldsymbol{Q}_u} \rho(X(s), \tilde{X}(s))\,\mathrm{d}s,$$

which, by Gronwall's inequality, implies that $E^{\boldsymbol{Q}_u} \rho(X(t), \tilde{X}(t)) = 0,\ \forall\ t \geqslant 0$.

Hence $\|u(t)\| = \|\bar{u}(t)\|$ for all $t \geqslant 0$, and thus the result follows. □

In order to prove Theorem 1.3, we introduce the following stopping times:

$$\tau_N = \inf\{t \geqslant 0 : |X(t)| \geqslant N\}, \tag{3.14}$$

$$\sigma_N = \inf\{t \geqslant 0 : |\tilde{X}(t)| \geqslant N\}, \quad \xi_N = \tau_N \wedge \sigma_N. \tag{3.15}$$

Let $(q_{X,Y}(t))$ and $(\bar{q}_{X,Y}(t))$ denote the Q-matrices associated with $L_{\|u(t)\|}$ and $L_{\|\bar{u}(t)\|}$, respectively. For any finite subset A of S and $X \in S$, let

$$L_{\|u(t)\|}(X,A) = \sum_{Y \in A, Y \neq X} q_{X,Y}(t),\ L_{\|\bar{u}(t)\|}(X,A) = \sum_{Y \in A, Y \neq X} \bar{q}_{X,Y}(t) \tag{3.16}$$

and

$$N(t,A;X(\cdot)) = \#\{s: X(s) \in A, X(s) \neq X(s-), s \leqslant t\} \tag{3.17}$$

$$\tilde{N}(t,A;X(\cdot)) = N(t,A;X(\cdot)) - \int_0^t L_{\|u(s)\|}(X(s),A)\,\mathrm{d}s \tag{3.18}$$

$$\tilde{\tilde{N}}(t,A;X(\cdot)) = N(t,A;X(\cdot)) - \int_0^t L_{\|\bar{u}(s)\|}(X(s),A)\,\mathrm{d}s. \tag{3.19}$$

By applying Lemma 3.4(i) in Dawson and Zheng (1991) we can show that for any $f \in C_k(S)$,

$$\int_S f(Y)\tilde{N}(t,\mathrm{d}Y;X(\cdot)) = \sum_{Y\in S} f(Y)\tilde{N}(t,Y;X(\cdot))$$

and

$$\int_S f(Y)\tilde{\tilde{N}}(t,\mathrm{d}Y;X(\cdot)) = \sum_{Y\in S} f(Y)\tilde{\tilde{N}}(t,Y;X(\cdot))$$

are P_u- and $\bar{P}_u$-martingales, respectively. We also have

Lemma 3.4. *For any $T>0$, $N \geqslant 1$, and $u \in M(S)$ satisfying $\int_S \mathrm{e}^{c|X|}u(\mathrm{d}X) < \infty$, we have*

$$P_u(\sup_{0\leqslant t\leqslant T} |X(t)| \geqslant N) \leqslant C(c,u,T)\mathrm{e}^{-cN}, \tag{3.20}$$

$$\bar{P}_u(\sup_{0\leqslant t\leqslant T} |X(t)| \geqslant N) \leqslant C(c,u,T)\mathrm{e}^{-cN}, \tag{3.21}$$

where c is the constant in (1.15), *and $C(c,u,T)$ is a constant depending on c, u, T.*

Proof. For any $N \geqslant 1$, let $f_N(X) = c|X|I_{\{|X|\leqslant N\}}(X)$, $f(X) = c|X|$. Let

$$\begin{aligned} A_N(t) &= \int_0^t \int_S \{\exp[f_N(Y) - f_N(X(s-))] - 1\} L_{\|u(s)\|}(X(s-),\mathrm{d}Y)\,\mathrm{d}s \\ &= \int_0^t \sum_{Y\in S} \{\exp[f_N(Y) - f_N(X(s-))] - 1\} q_{X(s-),Y}(s)\,\mathrm{d}s, \\ A(t) &= \int_0^t \int_S \{\exp[f(Y) - f(X(s-))] - 1\} L_{\|u(s)\|}(X(s-),\mathrm{d}Y)\,\mathrm{d}s \\ &= \int_0^t \sum_{Y\in S} \{\exp[f(Y) - f(X(s-))] - 1\} q_{X(s-),Y}(s)\,\mathrm{d}s. \end{aligned}$$

By Ito's formula in Ikeda and Watanabe (1981) and (3.17)–(3.19), we can show that $\exp[f_N(X(t)) - A_N(t)]$ is a P_u-martingale which implies that $\exp[f(X(t)) - A(t)]$ is a P_u-local martingale and thus a supermartingale. By definition for any $t \geqslant 0$ and $X \in S$

$$q_{X,Y}(t) = \begin{cases} q_{X,Y} & \text{if } Y - X \notin \{\pm e_i; i = 1,\dots,m\}, \\ q_{X,Y} + \|u^i(t)\| & \text{if } Y = X + e_i, \text{ for } i \in \{1,\dots,m\}, \\ q_{X,Y} + x_i & \text{if } Y = X - e_i, \text{ for } i \in \{1,\dots,m\}. \end{cases}$$

By condition (1.15) and the facts that $\|u(t)\|$ is uniformly bounded on $[0,T]$ and x_i is the death rate for $i=1,\ldots,m$, we get

$$\sup_{t\in[0,T]} |A(t)| \leqslant \alpha T.$$

This, combined with Doob's inequality for supermartingales, implies

$$\begin{aligned} P_u\Big(\sup_{0\leqslant t\leqslant T} |X(t)| \geqslant N\Big) &= P_u\Big(\sup_{0\leqslant t\leqslant T} \mathrm{e}^{c|X(t)|} \geqslant \mathrm{e}^{cN}\Big) \\ &\leqslant P_u\Big(\sup_{0\leqslant t\leqslant T} \exp[f(X(t)) - A(t)] \geqslant \exp[cN - \alpha T]\Big) \\ &\leqslant E^{P_u}\exp[f(X(0))]\exp[-cN+\alpha T]) \\ &= C(c,u,T)\mathrm{e}^{-cN}, \end{aligned}$$

where $C(c,u,T) = \mathrm{e}^{\alpha T}\int_S \mathrm{e}^{c|X|}u(\mathrm{d}X) < \infty$. The inequality (3.21) can be obtained in the same way. $\square$

Proof of Theorem 1.3. Let $\boldsymbol{R}_u$ be the solution of the time- inhomogeneous martingale problem for Ω_t with initial distribution Λ_u defined as above. Assume $t \leqslant 1$ in the sequel. By the martingale property we have

$$\begin{aligned} E^{\boldsymbol{R}_u}\rho(X(t\wedge\xi_N),\tilde{X}(t\wedge\xi_N)) &= E^{\Lambda_u}\rho(X(0),\tilde{X}(0)) \\ &\quad + E^{\boldsymbol{R}_u}\int_0^{t\wedge\xi_N} \Omega_s\rho(X(s),\tilde{X}(s))\,\mathrm{d}s \\ &= E^{\boldsymbol{R}_u}\int_0^{t\wedge\xi_N} \Omega\rho(X(s\wedge\xi_N),\tilde{X}(s\wedge\xi_N))\,\mathrm{d}s \\ &\quad + E^{\boldsymbol{R}_u}\int_0^{t\wedge\xi_N} (\Omega_s-\Omega)\rho(X(s\wedge\xi_N),\tilde{X}(s\wedge\xi_N))\,\mathrm{d}s \\ &\leqslant (2aN+b)E^{\boldsymbol{R}_u}\int_0^t \rho(X(s\wedge\xi_N),\tilde{X}(s\wedge\xi_N))\,\mathrm{d}s \\ &\quad + E^{\boldsymbol{R}_u}\int_0^t \sum_{i=1}^m [|\,\|u^i(s)\| - \|\bar{u}^i(s)\|\,|]\,\mathrm{d}s. \end{aligned} \tag{3.22}$$

while

$$\begin{aligned} &\int_0^t \sum_{i=1}^m [|\,\|u^i(s)\| - \|\bar{u}^i(s)\|\,|]\,\mathrm{d}s \\ &\leqslant \int_0^t E^{\boldsymbol{R}_u}\rho(X(s),\tilde{X}(s))\,\mathrm{d}s \leqslant \int_0^t E^{\boldsymbol{R}_u}\rho(X(s\wedge\xi_N),\tilde{X}(s\wedge\xi_N))\,\mathrm{d}s \\ &\quad + \int_0^t \sum_{i=1}^m [E^{\boldsymbol{R}_u}(x_i;\xi_N\leqslant s) + E^{\boldsymbol{R}_u}(\tilde{x}_i;\xi_N\leqslant s)]\,\mathrm{d}s. \end{aligned} \tag{3.23}$$

By Hölder's inequality, we have for each $1\leqslant i\leqslant m$

$$\begin{aligned} E^{\boldsymbol{R}_u}(x_i;\xi_N\leqslant s) &\leqslant \{E^{u^i(s)}(x_i^p)\}^{1/p}\boldsymbol{R}_u^{1/q}\{\xi_N\leqslant s\} \\ &\leqslant \{E^{u^i(s)}(x_i^p)\}^{1/p}[P_u(\tau_N\leqslant s) + \bar{P}_u(\sigma_N\leqslant s)]^{1/q}, \end{aligned} \tag{3.24}$$

$$E^{R_u}(\tilde{x}_i;\xi_N \leqslant s) \leqslant \{E^{\bar{u}^i(s)}(\tilde{x}_i^p)\}^{1/p} R_u^{1/q}\{\xi_N \leqslant s\}$$
$$\leqslant \{E^{\bar{u}^i(s)}(\tilde{x}_i^p)\}^{1/p}[P_u(\tau_N \leqslant s) + \bar{P}_u(\sigma_N \leqslant s)]^{1/q}, \tag{3.25}$$

where $1/p + 1/q = 1$.

Let $M = \sup_{s\in[0,1], 1\leqslant i\leqslant m}\{\{E^{u^i(s)}(x_i^p)\}^{1/p}, \{E^{\bar{u}^i(s)}(\tilde{x}_i^p)\}^{1/p}\}$. Then by (3.20) and (3.21) we have

$$\int_0^t \sum_{i=1}^m [|\,\|u^i(s)\| - \|\bar{u}^i(s)\|\,|]\,\mathrm{d}s$$
$$\leqslant \int_0^t E^{R_u}\rho(X(s\wedge\xi_N),\tilde{X}(s\wedge\xi_N))\,\mathrm{d}s + 2mM[2C(c,u,1)]^{1/q}t\mathrm{e}^{-(c/q)N}.$$

which, combined with (3.22), implies

$$E^{R_u}\rho(X(t\wedge\xi_N),\tilde{X}(t\wedge\xi_N)) \leqslant 2mM[2C(c,u,1)]^{1/q}e^{-(c/q)N}$$
$$+(2aN+b+1)\int_0^t E^{R_u}\rho(X(s\wedge\xi_N),\tilde{X}(s\wedge\xi_N))\,\mathrm{d}s.$$

By Gronwall's inequality we get

$$E^{R_u}\rho(X(t\wedge\xi_N),\tilde{X}(t\wedge\xi_N)) \leqslant 2mM[2C(c,u,1)]^{1/q}\mathrm{e}^{-(c/q)N}\mathrm{e}^{(2aN+b+1)t} \tag{3.26}$$

If we choose t small enough such that $c > 2taq$ and let $N \to \infty$, then (3.26) implies that for $t \in [0, c/2aq)$ we have

$$E^{R_u}\rho(X(t),\tilde{X}(t)) = 0.$$

This implies that

$$u(t) = \bar{u}(t), \quad t \in \left[0, \frac{c}{2aq}\right). \tag{3.27}$$

By a standard argument we conclude that (3.27) is true for all t, which gives the result. □

Remark. The proof above does not imply the uniqueness of the trimolecular model because of the competition between the nonlinear term and the cubic term in generator Ω_t. Noting that the cubic term does not change the total number of particles in the system, we can introduce the following nonlinear model:

$$Qf(X) = \lambda_1 a(f(X+e_1) - f(X)) + \lambda_4 x_1(f(X-e_1) - f(X))$$
$$+\lambda_2 x_1(f(X-e_1+e_2) - f(X))$$
$$+\lambda_3 C_2^{x_1} x_2(f(X+e_1-e_2) - f(X)),$$
$$Q_t f(X) = Lf(X) + (\|u^1(t)\| + \|u^2(t)\|)(f(X+e_1) - f(X))$$
$$+(x_1+x_2)(f(X-e_1) - f(X))$$
$$+(\|u^1(t)\| + \|u^2(t)\|)(f(X+e_2) - f(X))$$
$$+(x_1+x_2)(f(X-e_2) - f(X)).$$

In this case the cubic term does not change the rates in the nonlinear term. The model looks like the nonlinear master equation a single type of particle systems and the uniqueness follows from an argument similar to that used in Feng and Zheng (1992). But this modified model is essentially different from the trimolecular model.

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