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Wall repulsion and mutual interface repulsion: a harmonic crystal model in high dimensions

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Abstract

We consider two independent lattice harmonic crystals in dimension $d \geqslant 3$ constrained to live in the upper half-plane and to lie one above the other in a large region. We identify the leading order asymptotics of this model, both from the point of view of probability estimates and of pathwise behavior: this gives a rather complete picture of the phenomenon via a detailed analysis of the underlying entropy–energy competition. From the technical viewpoint, with respect to earlier work on sharp constants for harmonic entropic repulsion, this model is lacking certain monotonicity properties and the main tool that allows to overcome this difficulty is the comparison with suitable *rough substrate* models.

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1. Introduction

1.1. The model

A *harmonic crystal* or *lattice free field* on $\mathbb{Z}^d$ is the centered Gaussian field $\varphi = \{\varphi_x\}_{x\in\mathbb{Z}^d}$, $d \geqslant 3$, with covariance $\mathrm{cov}(\varphi_x, \varphi_y) = G(x, y)$, where $G(\cdot,\cdot) = \chi G^\star(\cdot,\cdot)$;

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χ is a positive number and $G^{\star}(\cdot,\cdot)$ is the Green function of a random walk $\{X_j\}_{j=0,1,\dots}$ on $\mathbb{Z}^d$ with 1-step transition probability $Q:\mathbb{Z}^d\times\mathbb{Z}^d\to[0,1]$. The following properties are satisfied:

- Q is symmetric and shift invariant: for every $x,y\in\mathbb{Z}^d$
$$Q(x,y)=Q(y,x) \quad \text{and} \quad Q(x,y)=Q(x-y,0), \tag{1.1}$$
- The walk has finite range: there exists a positive integer R such that $Q(0,x)=0$ if $|x|\geqslant R$;
- The walk is *irreducible* and $Q(0,0)=0$.

It is well known that under these conditions the walk is transient and the Green function is finite. Let us recall (notationally) different expressions of the Green function:

$$G^{\star}(x,y)=\sum_{j=0}^{\infty}\mathbf{P}_x(X_j=y)=\sum_{j=0}^{\infty}Q^j(x,y)=(-\Delta_Q)^{-1}(x,y), \tag{1.2}$$

where we introduce the notation $\mathbf{P}_x$ ($\mathbf{E}_x$) for the law (the expectation) of X when $X_0=x$. Moreover $\Delta_Q=Q-\mathbb{1}$. Given Q, we introduce an alternative to the Euclidean norm on $\mathbb{R}^d$: for $r\in\mathbb{R}^d$

$$|r|_Q=\sqrt{\sum_{x\in\mathbb{Z}^d}(x\cdot r)^2Q(0,x)}. \tag{1.3}$$

We associate to Q the symmetric $d\times d$ matrix M_Q defined by requiring $|r|_Q^2=(r,M_Qr)$ for every r. Of course, M_Q is invertible.

We work with two independent harmonic crystals: $\varphi=\{\varphi_x\}_x$ and $\psi=\{\psi_x\}_x$. The covariance of φ will be denoted by $G_1(\cdot,\cdot)$ and it is proportional to the Green function of a random walk with transition probability Q_1. The quantities referring to ψ will have instead the subscript 2. We may choose to represent (φ,ψ) on $\Omega=\mathbb{R}^{\mathbb{Z}^d}\times\mathbb{R}^{\mathbb{Z}^d}$: $\mathbb{R}$ is equipped with the standard Euclidean topology, Ω is equipped with the product topology and the σ-algebra that we choose for $\mathbb{R}$ and Ω, unless otherwise stated, is the Borel one, denoted as $\mathscr{B}(\mathbb{R})$ and $\mathscr{B}(\Omega)$. On $(\Omega,\mathscr{B}(\Omega))$ the probability measure will be simply denoted by $\mathbb{P}$ ($\mathbb{E}$ for the expectation). In particular $\mathbb{P}=\mathbb{P}_1\otimes\mathbb{P}_2$.

We define the event

$$\Omega_A^+=\{(\varphi,\psi):\ \psi_x\geqslant\varphi_x\geqslant 0 \text{ for every } x\in A\}. \tag{1.4}$$

Given D, an open bounded subset of $\mathbb{R}^d$ which contains the origin and whose boundary is piecewise smooth, we use the notation $\Omega_N^+=\Omega_{D_N}^+$, where $D_N=ND\cap\mathbb{Z}^d$.

We are interested in the asymptotic properties, as $N\to\infty$, of the trajectories of the coupled field (φ,ψ) under the measure $\mathbb{P}(\cdot|\Omega_N^+)$. This is a model for two interfaces that, in the region D_N, are forced not to intersect, with ψ on top of φ, and both cannot enter the negative half-space, which acts as a hard wall. It is well known (see Giacomin (2001) for a review of the literature on the subject) that a hard-wall of linear size N, pushes harmonic crystals, or more general interface models, to infinity as $N\to\infty$. For finite N, the interface is repelled to a typical height $\mathrm{O}(\sqrt{\log N})$ above the hard wall. This effect is purely of entropic nature and in fact it goes under the name of *entropic repulsion*. A sharp analysis of this phenomenon in the Gaussian setting (see Bolthausen (2000) and Giacomin (2001) for the most updated results) reveals a subtle *energy–entropy competition* that can be unraveled in great detail. Such completeness is of course due to the possibility of performing exact (Gaussian) computations, but an attentive analysis of the arguments reveals another crucial ingredient: the systematic exploitation of the monotonicity properties of the harmonic crystal.

The repulsion effects that we study in our model are more complex, due to the mutual action of the two fields that leads to loosing, partially, the desired monotonicity properties. Multi-interface phenomena is a topic of great interest for which mathematical results are rather limited, even in the $d=1$ case: one of the ultimate aims in the field is to study *a gas of non intersecting interfaces* confined by external forces or by hard walls in a large domain. We refer in particular to Bricmont et al. (1986), in which the authors consider the case of one interface constrained between two walls as a caricature of this challenging problem. We refer to Bricmont et al. (1986) also for further references on multi-interface phenomena: we stress however that, with respect to the work we just mentioned, our interest is in determining the precise leading asymptotics and not just rough bounds.

While we have chosen to introduce the harmonic crystals from a *strictly Gaussian standpoint*, it is certainly important to point out that harmonic crystals are Gibbs measures with respect to suitable quadratic Hamiltonians. For detailed accounts on the *Gibbsian* approach to these fields we refer for example to Bricmont et al. (1986) or Giacomin (2001) and references therein.

1.2. Main results

For $i=1,2$ define the i-capacity of D as

$$\begin{aligned}\mathrm{Cap}_i(D) &= \inf\{\chi_i^{-1}\| \, |\partial f|_{Q_i}\|_2^2/2\colon\ f\in C_0^\infty(\mathbb{R}^d;[0,\infty)),\ f(r)=1 \text{ for every } r\in D\}\\ &= \sup_{f\in L^\infty(D;[0,\infty))} \frac{(\int_D f(r)\,\mathrm{d}r)^2}{\int_D\int_D f(r)f(r')\chi_i g_Q(r-r')\,\mathrm{d}r\,\mathrm{d}r'},\end{aligned} \tag{1.5}$$

in which $\|\cdot\|_2$ is the $\mathbb{L}_2$ norm of $\cdot$, ∂ denotes the gradient in $\mathbb{R}^d$ and

$$g_{Q_i}(r) = \frac{\Gamma(d/2)}{(d-2)\pi^{d/2}(\det(M_{Q_i}))^{1/2}(r, M_{Q_i}^{-1}r)^{(d-2)/2}}. \tag{1.6}$$

We also stress that

$$\lim_{x\to\infty} \frac{G_i^\star(0,x)}{g_{Q_i}(x)} = 1. \tag{1.7}$$

The equivalence between the two formulas for the capacity in (1.5) can be found for example in Bolthausen and Deuschel (1994, Section 2) and for the existence of the limit in (1.7) we refer to Spitzer (1976). Set moreover $G_i = G_i(0,0)$.

Normally, it would be of course more natural to define the i-capacity of D without the χ_i^{-1} factor. We made the choice of introducing this factor in order to keep several formulas, starting already with (1.8) just below, uniform with analogous formulas appearing in related works and because the i-capacity, as we define it, appears naturally in evaluating the *lowest energy cost* for translating φ or ψ in D_N.

Theorem 1.1. *The following two results hold*:

(1) *The Laplace asymptotics of the probability of* $\mathbb{P}(\Omega_N^+)$ *is identified*:

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathbb{P}(\Omega_N^+) = -2\left[\left(\sqrt{G_1+G_2}+\sqrt{G_1}\right)^2 \mathrm{Cap}_2(D) + G_1 \mathrm{Cap}_1(D)\right]. \tag{1.8}$$

(2) *For every* $\underline{\alpha}$, $\bar{\alpha}$, $\underline{\beta}$ *and* $\bar{\beta}$ *such that* $\underline{\alpha} < \sqrt{4G_1} < \bar{\alpha}$ *and* $\underline{\beta} < \sqrt{4(G_1+G_2)} + \sqrt{4G_1} < \bar{\beta}$ *and for every* $\varepsilon > 0$ *we have*

$$\lim_{N\to\infty} \mathbb{P}\left(\left|\left\{x \in D_N\colon \frac{\varphi_x}{\sqrt{\log N}} \in (\underline{\alpha}, \bar{\alpha}), \frac{\psi_x}{\sqrt{\log N}} \in (\underline{\beta}, \bar{\beta})\right\}\right| \leqslant (1-\varepsilon)|D_N| \,\middle|\, \Omega_N^+\right) = 0. \tag{1.9}$$

1.3. About the results and the strategy of the proof

We have already remarked the importance of *monotonicity* properties of the field and the fact that results are for now mostly restricted to the harmonic crystal. The first of these properties is due to the fact that the covariance of the harmonic field is pointwise positive, so the field is *positively correlated* or, in statistical mechanics language, it satisfies the Fortuin–Kasteleyn–Ginibre (FKG) property. Moreover, both mean and covariance have a random walk representation that allows more comparison arguments (and, of course, sharp estimates). For the moment, there exists no general technique to get sharp constants for more general models and, as a matter of fact, very interesting cases, like the case of the *membrane* fields (Giacomin, 2001, A.12), are out of reach for the moment (see Sakagawa (2003) for some estimates in a broad Gaussian class).

The model we present enjoys some monotonicities, but not all the ones that would allow a direct application of the ideas developed up to now. If we concentrate on probability estimates, cf. part (1) of Theorem 1.1, the most serious obstacles appear in proving the lower bound. This is a specificity of our model, for which random walk representation tools are still applicable, and one should not be led to think that it is a typical situation.

In order to explain the strategy of the proof let us recall that a measure μ on $\mathbb{R}^A$, $A \subseteq \mathbb{Z}^d$, *stochastically dominates* another measure $\tilde{\mu}$, defined on the same space, if for every nondecreasing bounded measurable function f on the partially ordered set $\mathbb{R}^A$ we have $\int f\,\mathrm{d}\mu \geqslant \int f\,\mathrm{d}\tilde{\mu}$ (notation: $\mu \succ \tilde{\mu}$). We say moreover that μ satisfies the FKG property if $\int fg\,\mathrm{d}\mu \geqslant \int f\,\mathrm{d}\mu \cdot \int g\,\mathrm{d}\mu$, g a non decreasing function too. Of course, the notion of monotonicity of an event E is simply the monotonicity of its indicator function $\mathbf{1}_E$.

Harmonic crystals enjoy the FKG properties, but Ω_N^+, as well as other events associated to the lower bounds arguments that yield sharp results in the cases solved up to now, is not a monotonic event. We overcome this difficulty by writing $\mathbb{P}(\Omega_N^+) = \mathbb{P}(\Omega_N^+|\Omega_{1,N}^+)\mathbb{P}(\Omega_{1,N}^+)$, $\Omega_{1,N}^+ = \{(\varphi,\psi): \varphi_x \geqslant 0$ for every $x \in D_N\}$, and then we estimate $\mathbb{P}(\Omega_N^+|\Omega_{1,N}^+)$ by conditioning on φ (sharp estimates on $\mathbb{P}(\Omega_{1,N}^+)$ are already known, Bolthausen et al., 1995). Conditioning on φ leads to a single interface repulsion problem, but this time the hard wall is the fixed configuration φ, typical with respect to $\mathbb{P}_1(\cdot|\varphi_x \geqslant 0$ for $x \in D_N)$ acting on the random field ψ. This is what we call a *rough substrate* repulsion problem: the model enjoys all the monotonicity properties we desire, but it lacks translation invariance and it adds the problem of understanding the effect of rare large excursions of the substrate φ, a strongly correlated field in entropic repulsion, on the field ψ. It turns out that combining on one side the precise estimates available on $\mathbb{P}_1(\mathrm{d}\varphi|\varphi_x \geqslant 0$ for $x \in D_N)$ (Bolthausen et al., 1995; Deuschel and Giacomin, 1999), Brascamp–Lieb inequalities (Brascamp and Lieb, 1976; Giacomin, 2003) and the rough substrate estimates of Bertacchi and Giacomin (2002), one obtains the optimal lower bound. Note that this a priori is not obvious since Bertacchi and Giacomin (2002) deals with the case of a substrate modeled by independent random variables. We refer also to Bertacchi and Giacomin (2003) where a related rough substrate problem is considered.

One way of understanding why such a procedure yields optimal results is hidden in the pathwise behavior of $\mathbb{P}(\cdot|\Omega_N^+)$, that is part (2) of Theorem 1.1. This pathwise behavior is at first somewhat surprising and it can be informally read as follows:

- There is no *push down* effect of ψ on $\varphi : \varphi$ is repelled to the *same* (in the sense of leading asymptotic) height as when ψ is absent;
- We could replace the field φ with Gaussian independent random variables of variance G_1 centered around $\sqrt{4G_1 \log N}$ and the pathwise behavior, in the sense of the result in part (2) of Theorem 1.1, would be unchanged.

Notice that the role of the two underlying walks is somewhat downscaled: while in the probability estimates the asymptotics of the walks are still relevant, the pathwise behavior turns out to be rather universal, since it depends only on the variances.

We can now interpret the probability estimate in part (1) of Theorem 1.1: it is immediate to recognize in the right-hand side of (1.8) the (squares of the) repulsion heights of the two fields in a linear combination weighted by the capacity terms. The appearance of capacity terms in evaluating rare events of entropic repulsion type is not a novelty, cf. Bolthausen and Deuschel (1994) and Bolthausen et al. (1995): the N^{d-2} scaling factor in the left-hand side of (1.8) is just the discrete to continuum scaling factor of capacities and the $\log N$ correction is directly related to the square of the repulsion heights.

Still about pathwise estimates, we stress that in Section 4 we prove estimates on the conditioned field which go beyond what we report in part (2) of Theorem 1.1. In general however we have tried to present concise arguments and we did not try to get results that are uniform in x, except for the upper bound on φ, see Proposition 4.2, where the statement is a direct consequence of known results. We stress that getting a uniform estimate on ψ analogous to the one for φ would automatically yield the extension of the results to the case of three interfaces and, by iteration, of any finite number of interfaces. We believe that such a result can be extracted by combining the ideas of Bolthausen et al. (1995), of Deuschel and Giacomin (1999) and of what we present here, but it does not appear to be straightforward.

It is important to note that this difficulty is not present in the two-dimensional case. This has been recently pointed out by Sakagawa (2004), who dealt with the case of $Q_i = Q$ for every i and $Q(x,y) = 1/4$ for x and y nearest neighbors. In the $d = 2$ case Gaussian interfaces are *rough*, that is to say that the harmonic crystal does not exist when considered on the whole of $\mathbb{Z}^2$, simply because the Green function does not exist. One has therefore to state the repulsion problem in a slightly different way, starting from the field defined over finite subsets of $\mathbb{Z}^2$, in a way analogous to what is done in Bolthausen et al. (2001). We do not enter the details of this case, also because it involves different scaling factors, and we refer directly to Sakagawa (2004). However we point out that there is a strong similarity between the $d = 2$ and the $d \geqslant 3$ case and direct analogs of the two •-items above hold (in particular there is no push down effect). The reason why in the $d = 2$ case one can directly consider an arbitrary (finite) number of interfaces is intimately connected to the rough nature of two-dimensional interfaces, that leads to a simpler repulsion mechanism (overall harder to establish though). In particular upper bounds on repulsion heights are rather straightforward (the difficulties lie in the lower bounds) and they rely much less on monotonicity properties, so the main difficulty we encounter in $d \geqslant 3$ is not present in $d = 2$.

1.4. More preliminaries, notations and organization of the paper

We will repeatedly use the following entropy inequality: if P and $\tilde{P}$ are two probability measures, $\tilde{P} \ll P$, and E is a positive $\tilde{P}$-probability event then if we set $H(\tilde{P}|P) = \tilde{E}[\log(\mathrm{d}\tilde{P}/\mathrm{d}P)]$ by Jensen's inequality we have (see e.g. Bolthausen et al., 1995 or Giacomin, 2001, B.3)

$$\log\left(\frac{P(E)}{\tilde{P}(E)}\right) \geqslant -\frac{1}{\tilde{P}(E)}\,[H(\tilde{P}|P) + \mathrm{e}^{-1}]. \tag{1.10}$$

By $\mathscr{F}^{\varphi}$, respectively $\mathscr{F}^{\psi}$, we denote the σ-algebra generated by the field φ, respectively ψ. When an event E is in $\mathscr{F}^{\varphi}$ or in $\mathscr{F}^{\psi}$, then we will commit frequent abuse of notation by considering it at the same time as a subset of $\mathbb{R}^{\mathbb{Z}^d}$. For example, $\Omega^+_{1,N}$ may mean, according to the context, $\{\varphi : \varphi_x \geqslant 0$ for every $x \in D_N\}$ or $\{(\varphi,\psi)$: $\varphi_x \geqslant 0$ for every $x \in D_N\}$. Abuse of notation will also be committed in systematically not distinguishing between random and numerical variables.

The plan of the paper is straightforward: in Section 2 we prove the lower bound corresponding to part (1) of Theorem 1.1, while the upper bound may be found in Section 3. The proof of part (2) is instead in Section 4, split in four propositions.

2. Probability lower bounds

In this section we prove the lower bound for the limit in part (1) of Theorem 1.1. The proof consists of two parts:

(1) First (Lemma 2.1) we exploit the sharp results available for $\mathbb{P}^+_{1,N}(\cdot) := \mathbb{P}_1(\cdot|\Omega^+_{1,N})$ and a version of the Brascamp–Lieb inequalities to find an upper bound on the upward excursions. The average height of the trajectories of $\mathbb{P}^+_{1,N}(\cdot)$ in D_N is $\approx \sqrt{4G_1 \log N}$: the Brascamp–Lieb inequality provides a sharp concentration of the measure and gives a good upper bound on the number of points in which the field is above $\sqrt{a \log N}$, $a > 4G_1$. In short, this step identifies a set $E_N \subset \Omega$ whose probability is close to 1 for N large and for which we have suitable upper bounds on high level $(\mathrm{O}(\sqrt{\log N}))$ excursions of φ.

(2) Then, in Proposition 2.2, we obtain the desired lower bound on $\mathbb{P}(\Omega^+_N)$ by separating the problem into estimating from below $\mathbb{P}(\Omega^+_{1,N})$, a problem already solved in Bolthausen et al. (1995), and $\mathbb{P}_2(\psi_x \geqslant \varphi_x$ for every $x \in D_N)$ for $\varphi \in E_N$. This last term is clearly an entropic repulsion problem in presence of an inhomogeneous wall or, in other words, in presence of a random quenched substrate: we dealt with this kind of estimate in Bertacchi and Giacomin (2002, Proposition 2.1) and there are only minor modifications in this case: since a priori it may not be clear to everybody that the problem is the same and since the notations are necessarily rather different we choose to detail these steps.

2.1. Upper bound on the high excursions of $\mathbb{P}^+_{1,N}$

We will prove the following result:

Lemma 2.1. *For every* $\alpha > 0$ *and* $\eta > 0$

$$\lim_{N\to\infty} \mathbb{P}^+_{1,N}\left(\left|\left\{x \in D_N:\ \varphi_x \geqslant \left(\sqrt{\alpha} + \sqrt{4G_1 + \eta}\right)\sqrt{\log N}\right\}\right| \geqslant |D_N|\, N^{-\alpha/2G_1}\right) = 0. \tag{2.1}$$

Proof. Let us recall the following two results:

- A uniform asymptotic control on $\mathbb{E}^+_{1,N}[\varphi_x]$ is known:

$$\lim_{N\to\infty} \sup_{x\in D_N} \left| \frac{\mathbb{E}^+_{1,N}[\varphi_x] - \sqrt{4G_1 \log N}}{\sqrt{\log N}} \right| = 0. \tag{2.2}$$

This result is proven in Deuschel and Giacomin (1999, Lemma 3.3) for the basic case of $Q_1(x, y) = 1/2d$ if $|y - x| = 1$. The generalization to the finite range case considered here is a lengthy book-keeping exercise that we leave to the interested reader. Here we will make use only of the upper bound on $\mathbb{E}^+_{1,N}[\varphi_x]$ corresponding to the full estimate in (2.2).

- Set $M^+_N(x) = \min\{t\colon \mathbb{P}^+_{1,N}(\varphi_x \leqslant t) \geqslant 1/2\}$, that is $M^+_N(x)$ is the (unique) median of φ_x when the latter is distributed according to $\mathbb{P}^+_{1,N}$. By Giacomin (2003, Theorem 1.1) we have that for every positive β

$$\mathbb{P}^+_{1,N}(\varphi_x - M^+_N(x) \geqslant \beta) \leqslant \mathbb{P}(\varphi_x \geqslant \beta), \tag{2.3}$$

and by Giacomin (2003, Remark 2.3) we know that mean and median of log-concave perturbations of Gaussian measures cannot be too far from each other:

$$\sup_N |\mathbb{E}^+_{1,N}[\varphi_x] - M^+_N(x)| \leqslant \sqrt{2G_1/\pi}. \tag{2.4}$$

By combining the results we just stated and an elementary upper bound on the tail of a Gaussian random variable we directly obtain that for every $\eta > 0$ we can find N_0 such that if $N \geqslant N_0$

$$\begin{aligned}
&\sup_{x\in D_N} \mathbb{P}^+_{1,N}\left(\varphi_x \geqslant (\sqrt{\alpha} + \sqrt{4G_1+\eta})\sqrt{\log N}\right)\\
&\quad \leqslant \mathbb{P}(\varphi_0 \geqslant \sqrt{\alpha \log N}) \leqslant \left(\frac{G_1}{2\pi\alpha\log N}\right)^{1/2} N^{-\alpha/2G_1},
\end{aligned} \tag{2.5}$$

for every $\alpha > 0$. It is therefore clear that, with obvious definition of c, we have

$$\begin{aligned}
&\mathbb{E}^+_{1,N}\left[\left|\left\{x \in D_N:\ \varphi_x \geqslant \left(\sqrt{\alpha} + \sqrt{4G_1+\eta}\right)\sqrt{\log N}\right\}\right|\right]\\
&\quad \leqslant \frac{c}{\sqrt{\alpha \log N}} |D_N|\, N^{-\alpha/2G_1},
\end{aligned} \tag{2.6}$$

for N sufficiently large. We now apply the Markov inequality and the proof is complete. □

2.2. The lower bound via quenched estimates

We are now ready to prove the lower bound for the limit in part (1) of Theorem 1.1.

Proposition 2.2.

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathbb{P}(\Omega^+_N)\\
&\quad \geqslant -2\left[\left(\sqrt{G_1+G_2} + \sqrt{G_1}\right)^2 \mathrm{Cap}_2(D) + G_1 \mathrm{Cap}_1(D)\right].
\end{aligned} \tag{2.7}$$

Proof. We define an auxiliary field $\tilde{\varphi}$ as a function of φ. Fix a large integer $\bar{k}$ and define $\theta = \sqrt{4G_1}(1+(1/2\bar{k}))/\bar{k}$, $\tilde{k} = [(\sqrt{3dG_1})/\theta]$, $\gamma = \sqrt{4G_1+\eta}$ $(\eta > 0)$ and

$$\tilde{\varphi}_x = \begin{cases} (\gamma+\theta)\sqrt{\log N} & \text{if } \varphi_x \leqslant (\gamma+\theta)\sqrt{\log N}, \\ (\gamma+k\theta)\sqrt{\log N} & \text{if } \varphi_x \in ((\gamma+(k-1)\theta)\sqrt{\log N}, (\gamma+k\theta)\sqrt{\log N}] \\ & \text{for } k = 2,3,\ldots,\bar{k}, \\ (\gamma+\tilde{k}\theta)\sqrt{\log N} & \text{if } \varphi_x \in ((\gamma+\bar{k}\theta)\sqrt{\log N}, (\gamma+\tilde{k}\theta)\sqrt{\log N}], \\ \infty & \text{if } \varphi_x > (\gamma+\tilde{k}\theta)\sqrt{\log N}, \end{cases} \tag{2.8}$$

and set $L_N(k) = \{x \in D_N : \tilde{\varphi}_x = (\gamma+k\theta)\sqrt{\log N}\}$, $N_k = |L_N(k)|$, for $k \in \{1,2,\ldots,\bar{k},\tilde{k},\infty\}$.

Let us introduce the set $E_N \in \mathscr{F}^{\varphi}$ specified by the following three conditions:

$$\begin{aligned} &N_k \leqslant cN^{d-((k-1)^2\theta^2/2G_1)} \quad \text{for all } k = 1,\ldots,\bar{k}, \\ &N_{\tilde{k}} \leqslant cN^{d-(\bar{k}^2\theta^2/2G_1)}, \\ &N_\infty = 0. \end{aligned} \tag{2.9}$$

By Lemma 2.1 we can choose $c = c(D)$ so that $\mathbb{P}^+_{1,N}(E_N)$ tends to 1 as N tends to infinity. From now on we choose N such that $\mathbb{P}^+_{1,N}(E_N) \geqslant \frac{1}{2}$.

Now observe that

$$\begin{aligned} \mathbb{P}(\psi_x \geqslant \varphi_x \geqslant 0 \text{ for } x \in D_N) &\geqslant \mathbb{P}(\{\psi_x \geqslant \varphi_x \geqslant 0 \text{ for } x \in D_N\} \cap E_N) \\ &\geqslant \mathbb{P}(\{\psi_x \geqslant \tilde{\varphi}_x, \varphi_x \geqslant 0 \text{ for } x \in D_N\} \cap E_N) \\ &= \mathbb{E}(\mathbb{P}_2(\psi_x \geqslant \tilde{\varphi}_x \text{ for } x \in D_N)(\varphi); E_N | \Omega^+_{1,N})\mathbb{P}(\Omega^+_{1,N}) \\ &\geqslant \frac{1}{2} \inf_{\varphi \in E_N} \mathbb{P}_2(\psi_x \geqslant \tilde{\varphi}_x \text{ for } x \in D_N)\mathbb{P}(\Omega^+_{1,N}). \end{aligned} \tag{2.10}$$

Let us simplify a bit the notation by setting $\tilde{E}_N = \{\tilde{\varphi}(\varphi) \colon \varphi \in E_N\}$. Since, by Bolthausen et al. (1995, Theorem 1.1), $\log \mathbb{P}(\Omega^+_{1,N})$ is asymptotic to $-2G_1 \mathrm{Cap}_1(D) N^{d-2} \log N$, it is enough to show that for every $\tilde{\varphi} \in \tilde{E}_N$ we have

$$\begin{aligned} &\liminf_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathbb{P}_2(\psi_x \geqslant \tilde{\varphi}_x, \text{ for } x \in D_N) \\ &\quad \geqslant -2\left(\sqrt{G_1} + \sqrt{G_1+G_2}\right)^2 \mathrm{Cap}_2(D). \end{aligned} \tag{2.11}$$

Note that, by the FKG inequality, we have that

$$\begin{aligned} &\liminf_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathbb{P}_2(\psi_x \geqslant \tilde{\varphi}_x, \text{ for } x \in D_N) \\ &\quad \geqslant \liminf_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathbb{P}_2(F) \end{aligned}$$

$$+\liminf_{N\to\infty}\frac{1}{N^{d-2}\log N}\log\mathbb{P}_2(\psi_x \geqslant \tilde\varphi_x \text{ for } x\in D_N^-)$$

$$=: \mathbf{T}_1+\mathbf{T}_2, \tag{2.12}$$

where $F := \{\psi\colon \psi_x \geqslant (\gamma+\tilde k\theta)\sqrt{\log N}, x\in L_N(\tilde k)\}$, $D_N^- = D_N\setminus L_N(\tilde k)$.

We first show that $\mathbf{T}_1=0$. Let T_σ be defined by $(T_\sigma\varphi)_x=\varphi_x+\sigma_x$, for every $x\in\mathbb{Z}^d$, with $\sigma_x=\sqrt{2G_2(d+2)\log N}+(\gamma+\tilde k\theta)\sqrt{\log N}$ for $x\in L_N(\tilde k)$ and $\sigma_x=0$ otherwise. Then by direct computation (recall that $(G_2(\cdot,\cdot))^{-1}=-\chi_2^{-1}\Delta_{Q_2}(\cdot,\cdot)$) one shows that

$$\begin{aligned}H(\mathbb{P}_2T_\sigma^{-1}|\mathbb{P}_2) &= \frac{1}{4\chi_2}\sum_{x,y}(\sigma_x-\sigma_y)^2Q_2(x,y)\\ &\leqslant \chi_2^{-1}(\sqrt{2G_2(d+2)}+\gamma+\tilde k\theta)^2N_{\tilde k}\log N.\end{aligned}\tag{2.13}$$

By the definition of θ and (2.9) we have that for N and $\bar k$ sufficiently large

$$H(\mathbb{P}_2T_\sigma^{-1}|\mathbb{P}_2)\leqslant N^{d-2(1+(1/3\bar k))^2}. \tag{2.14}$$

Moreover by using the FKG inequality we obtain that for N sufficiently large

$$\begin{aligned}\mathbb{P}_2T_\sigma^{-1}(F) &= \mathbb{P}_2\left(\psi_x\geqslant -\sqrt{2G_2(d+2)\log N}\text{ for every } x\in L_N(\tilde k)\right)\\ &\geqslant \prod_{x\in L_N(\tilde k)}\mathbb{P}_2\left(\psi_x\geqslant -\sqrt{2G_2(d+2)\log N}\right)\\ &\geqslant (1-(1/N^{d+1}))^{N^d}\geqslant 1/2,\end{aligned}\tag{2.15}$$

and therefore by applying the entropy inequality (1.10) we obtain

$$\mathbb{P}_2(F)\geqslant \exp(-N^{d-2(1+(1/4\bar k))^2}), \tag{2.16}$$

for sufficiently large N, which shows that $\mathbf{T}_1=0$.

We are left with evaluating $\mathbf{T}_2$. Set $\alpha_N=\alpha\sqrt{\log N}$, $\alpha>0$, $(\sigma_N)_x=\alpha_N f(x/N)$, $f\in C_0^\infty(\mathbb{R}^d;[0,\infty))$ and $f(r)=1$ if $r\in D$. Let $\mathbb{P}_{2,N}=\mathbb{P}_2T_{\sigma_N}^{-1}$ and $\tilde{\mathbb{P}}_{2,N}(\cdot)=\mathbb{P}_{2,N}(\cdot|\tilde F)$, where $\tilde F=\{\psi_x\geqslant\tilde\varphi_x,\ x\in D_N^-\}$. Therefore $\mathrm{d}\tilde{\mathbb{P}}_{2,N}/\mathrm{d}\mathbb{P}_2=(\mathrm{d}\tilde{\mathbb{P}}_{2,N}/\mathrm{d}\mathbb{P}_{2,N})(\mathrm{d}\mathbb{P}_{2,N}/\mathrm{d}\mathbb{P}_2)$ and by the entropy inequality (1.10) we have

$$\begin{aligned}\log\mathbb{P}_2(\tilde F) &\geqslant -H(\tilde{\mathbb{P}}_{2,N}|\mathbb{P}_2)-\mathrm{e}^{-1}\\ &= -H(\tilde{\mathbb{P}}_{2,N}|\mathbb{P}_{2,N})-\tilde{\mathbb{E}}_{2,N}\left(\log\left(\frac{\mathrm{d}\mathbb{P}_{2,N}}{\mathrm{d}\mathbb{P}_2}\right)\right)-\mathrm{e}^{-1}\\ &=: -H_1-H_2-\mathrm{e}^{-1}.\end{aligned}\tag{2.17}$$

First of all by direct evaluation and FKG we have

$$\begin{aligned}H_1 &= -\log\mathbb{P}_2T_{\sigma_N}^{-1}(\tilde F)\\ &\leqslant -\sum_{k=1}^{\bar k}N_k\log\left(1-\mathbb{P}_{2,N}\left(\psi_0<(\gamma+k\theta-\alpha)\sqrt{\log N}\right)\right).\end{aligned}\tag{2.18}$$

One checks directly that if

$$\alpha > \gamma + \bar{k}\theta = \sqrt{4G_1 + \eta} + \sqrt{4G_1}\left(1 + \frac{1}{2\bar{k}}\right) \tag{2.19}$$

and

$$\frac{(k-1)^2\theta^2}{2G_1} + \frac{(\sqrt{4G_1+\eta} + k\theta - \alpha)^2}{2G_2} > 2, \tag{2.20}$$

for all $k \leqslant \bar{k}$, then each of the $\bar{k}$ summands in (2.18) is $o(N^{d-2})$, and therefore negligible:

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} H_1 = 0. \tag{2.21}$$

Observe that, by (2.19) and (2.20), a more explicit assumption that implies (2.21) is

$$\alpha > \sqrt{4G_1 + \eta} + k\theta + \sqrt{4G_2 - (k-1)^2\theta^2\left(\frac{G_2}{G_1}\right)} \quad \text{for every } k \leqslant \bar{k}. \tag{2.22}$$

It is then easy to see that (2.19) and (2.22) are satisfied if

$$\alpha > \sqrt{4G_1 + \eta} + \sqrt{4(G_1 + G_2)} + \theta. \tag{2.23}$$

Therefore, under this hypothesis on α estimate (2.21) holds.

Let us consider H_2: observe that

$$\begin{aligned}\log\left(\frac{\mathrm{d}\mathbb{P}_{2,N}}{\mathrm{d}\mathbb{P}_2}(T_{\sigma_N}\psi)\right) &= \frac{\alpha_N^2}{4}\sum_{x,y}(f(x/N) - f(y/N))^2\chi_2^{-1}Q_2(x,y)\\ &\quad + \alpha_N \sum_x f(x/N)\sum_y(\psi_x - \psi_y)\chi_2^{-1}Q_2(x,y),\end{aligned} \tag{2.24}$$

and therefore

$$\begin{aligned}\frac{H_2}{N^{d-2}\log N} &= \frac{\alpha_N^2}{4N^d\log N}\sum_{x,y}N^2(f(x/N) - f(y/N))^2\chi_2^{-1}Q_2(x,y)\\ &\quad + \frac{\alpha_N}{N^{d-2}\log N}\,\mathbb{E}_2\left[\sum_x f(x/N)\sum_y(\psi_x - \psi_y)\chi_2^{-1}Q_2(x,y)\,\middle|\, T_{\sigma_N}^{-1}\tilde{F}\right]\\ &=: C_N + R_N.\end{aligned} \tag{2.25}$$

It is easy to see that C_N converges for $N \to \infty$ to $\alpha^2\chi_2^{-1}\| \, |\partial f|_{Q_2}\|^2/4$. We show now that $\lim_{N\to\infty} R_N = 0$ if (2.23) holds. Observe in fact that

$$\begin{aligned}&2\sum_x f(x/N)\sum_y(\psi_x - \psi_y)\chi_2^{-1}Q_2(x,y)\\ &\quad = -\chi_2^{-1}\sum_x(\Delta_{Q_2} f(\cdot/N))(x)\psi_x \sim \mathcal{N}(0, s_N^2),\end{aligned} \tag{2.26}$$

where $s_N^2 = \chi_2^{-2} \| \, |\partial f|_{Q_2} \|_2^2 N^{d-2}(1+o(1))$. We use now the following consequence of Jensen inequality (Y a random variable, E a positive probability event, $t>0$)

$$\mathbb{E}[Y|E] \leqslant \frac{1}{t} \log \mathbb{E}[\exp(tY)] - \frac{1}{t} \log \mathbb{P}(E), \tag{2.27}$$

to obtain with $t = N^{d-2}$ that

$$\begin{aligned} |R_N| &\leqslant \frac{1}{t} \log \mathbb{E}_2 \left[\exp\left(\frac{t\alpha_N}{N^{d-2}\log N} \sum_x \sum_y f(x/N)(\psi_x - \psi_y)\chi_2^{-1} Q_2(x,y) \right) \right] \\ &\quad - \frac{1}{t} \log \mathbb{P}_2(T_{\sigma_N}^{-1} \tilde{F}) \leqslant \frac{\alpha^2 s_N^2}{8N^{d-2}\log N} + \frac{H_1}{N^{d-2}} = o(1). \end{aligned} \tag{2.28}$$

This shows that under hypothesis (2.23) on α

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log N} H_2 = \frac{\alpha^2}{4} \chi_2^{-1} \| \, |\partial f|_{Q_2} \|_2^2. \tag{2.29}$$

The thesis is obtained by optimizing the choices of f and α, by the definition of the capacity (1.5) and using the fact that θ and η can be taken arbitrarily small. □

3. Probability upper bounds

Proposition 3.1.

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathbb{P}(\Omega_N^+) \\ &\quad \leqslant -2 \left[\left(\sqrt{G_1+G_2} + \sqrt{G_1} \right)^2 \mathrm{Cap}_2(D) + G_1 \mathrm{Cap}_1(D) \right]. \end{aligned} \tag{3.1}$$

Proof. Let us choose an even natural number L larger than $2R$, recall that R is larger than the ranges of the random walks, and for $y \in 2L\mathbb{Z}^d$ let us set

$$B(y) = B_L(y) = \left\{ x\colon \max_{i=1,\dots,d} |x_i - y_i| \in [L/2, L/2+R] \right\}, \tag{3.2}$$

and Λ_c is the set of $y \in 2L\mathbb{Z}^d$ such that $B(y) \subset D_N$. Set also $\Lambda = \bigcup_{y\in\Lambda_c} B(y)$. We denote by $\mathscr{F}_A^{\varphi,\psi}$ the σ-algebra generated by φ_x and ψ_x, $x \in A$. We have that

$$\begin{aligned} \mathbb{P}(\Omega_N^+) &\leqslant \mathbb{P}(\Omega_{\Lambda\cup\Lambda_c}^+) \\ &= \mathbb{E}\left[\prod_{y\in\Lambda_c} \mathbb{P}(\psi_y \geqslant \varphi_y \geqslant 0 | \mathscr{F}_{B(y)}^{\varphi,\psi}); \Omega_\Lambda^+ \right], \end{aligned} \tag{3.3}$$

in which we used the Markov property of the fields φ and ψ.

Observe that, if $y \in \Lambda_c$ is fixed, $q^{(i)}(z) = q_L^{(i)}(z)$ is the probability that the Q_i–random walk leaving at y hits $B(y)$ at z and $M_y^{(i)}(\sigma) = \sum_{z\in B(y)} q^{(i)}(z)\sigma_z$, $\sigma \in \mathbb{R}^{\mathbb{Z}^d}$, the law of

φ_y and ψ_y, conditioned to $\mathscr{F}^{\varphi,\psi}_{B(y)}$, is the law of two independent Gaussian random variables of mean $M^{(1)}_y(\varphi)$ and $M^{(2)}_y(\psi)$, respectively, and of variance $G_{1,L}$ and $G_{2,L}$ respectively (positive numbers with the property that $G_{i,L} \nearrow G_i$ as $L \nearrow \infty$).

We now choose a small positive number κ and consider the *inner κ–discretization* of D: that is for $r \in \kappa\mathbb{Z}^d$, set $A_r = r + [0,\kappa)^d$ and define $I = \{r \in \kappa\mathbb{Z}^d : A_r \subset D\}$ (assume $I \neq \emptyset$). We are interested in this decomposition at the lattice level or, more precisely, on the $2L$–rarefied lattice level (the sublattice Λ_c of centers): so define $C_r = NA_r \cap \Lambda_c$ and remark that $c_{\kappa,N} = |C_r|/(N\kappa/2L)^d$ tends to a finite non zero limit as $N \to \infty$ and then $\kappa \to 0$.

For $\eta, \alpha, \beta > 0$ let us now consider the events

$$
\begin{aligned}
E_{\eta,\alpha} &= \{\text{there exists } r \in I \text{ such that } |\{y \in C_r\colon M^{(1)}_y(\varphi) \leqslant \sqrt{\alpha \log N}\}| \geqslant \eta |C_r|\},\\
F_{\eta,\beta} &= \{\text{there exists } r \in I \text{ such that } |\{y \in C_r\colon M^{(2)}_y(\psi) - M^{(1)}_y(\varphi)\\
&\qquad \leqslant \sqrt{\beta \log N}\}| \geqslant \eta |C_r|\}.
\end{aligned} \tag{3.4}
$$

For what follows η will be chosen smaller than 1/4.

Observe that on $E_{\eta,\alpha}$

$$
\begin{aligned}
\prod_{y\in\Lambda_c} \mathbb{P}(\psi_y \geqslant \varphi_y \geqslant 0 | \mathscr{F}^{\varphi,\psi}_{B(y)}) &\leqslant \prod_{y\in\Lambda_c} \mathbb{P}(\varphi_y \geqslant 0 | \mathscr{F}^{\varphi,\psi}_{B(y)})\\
&= \prod_{y\in\Lambda_c} \left(1 - \Phi\left(-\frac{M^{(1)}_y(\varphi)}{\sqrt{G_{1,L}}}\right)\right)\\
&\leqslant \left(1 - \Phi\left(-\sqrt{\frac{\alpha \log N}{G_{1,L}}}\right)\right)^{\eta|C_r|},
\end{aligned} \tag{3.5}
$$

where r is an arbitrary element in I. Then for N sufficiently large and suitable choices of positive constants c' and c'' we have that

$$
\begin{aligned}
\mathbb{E}\left[\prod_{y\in\Lambda_c} \mathbb{P}(\varphi_y \geqslant 0 | \mathscr{F}_{B(y)}); E_{\eta,\alpha}\right] &\leqslant \left(1 - \frac{c'}{\sqrt{\alpha \log N/G_{1,L}}} N^{-\alpha/2G_{1,L}}\right)^{c_{\kappa,N}\eta(N\kappa/2L)^d}\\
&\leqslant \exp(-c'' N^{d-(\alpha/2G_{1,L})}),
\end{aligned} \tag{3.6}
$$

which is negligible (recall that we want to prove (3.1)) if $\alpha < 4G_{1,L}$. In an analogous way, one proves that one may choose $c > 0$ such that

$$
\begin{aligned}
\mathbb{E}\left[\prod_{y\in\Lambda_c} \mathbb{P}(\psi_y \geqslant \varphi_y \geqslant 0 | \mathscr{F}_{B(y)}); F_{\eta,\beta}\right] &\leqslant \left(1 - \Phi\left(-\sqrt{\frac{\beta \log N}{G_{1,L} + G_{2,L}}}\right)\right)^{\eta|C_r|}\\
&\leqslant \exp(-cN^{d-(\beta/(G_{1,L}+G_{2,L}))}),
\end{aligned} \tag{3.7}
$$

which is negligible if $\beta < 4(G_{1,L} + G_{2,L})$.

Let us then assume that $\alpha < 4G_{1,L}$ and $\beta < 4(G_{1,L}+G_{2,L})$. We may therefore replace the event Ω_Λ^+ with $\Omega_\Lambda^+ \cap E_{\eta,\alpha}^{\complement} \cap F_{\eta,\beta}^{\complement}$ in the rightmost expression of (3.3).

If $(\varphi,\psi) \in \Omega_\Lambda^+ \cap E_{\eta,\alpha}^{\complement} \cap F_{\eta,\beta}^{\complement}$ then for every $r \in I$ there are at least $(1-2\eta)|C_r|$ sites $y \in C_r$ such that $M_y^{(1)}(\varphi) > \sqrt{\alpha \log N}$ and $M_y^{(2)}(\psi) - M_y^{(1)}(\varphi) > \sqrt{\beta \log N}$ and in the remaining sites $M_y^{(1)}(\varphi) \geqslant 0$ and $M_y^{(2)}(\psi) - M_y^{(1)}(\varphi) \geqslant 0$. This in turn implies that

$$M_y^{(1)}(\varphi) > \sqrt{\alpha \log N} \text{ and } M_y^{(2)}(\psi) > (\sqrt{\beta} + \sqrt{\alpha})\sqrt{\log N}, \tag{3.8}$$

for at least $(1-4\eta)|C_r|$ sites $y \in C_r$, and $M_y^{(1)}(\varphi) \geqslant 0$ and $M_y^{(2)}(\psi) \geqslant 0$ elsewhere. Therefore for every choice of $f_r \geqslant 0$, $\tilde{f}_r \geqslant 0$, $r \in I$, (on $\Omega_\Lambda^+ \cap E_{\eta,\alpha}^{\complement} \cap F_{\eta,\beta}^{\complement}$) we have that

$$\sum_{r \in I} f_r \frac{1}{|C_r|} \sum_{y \in C_r} M_y^{(1)}(\varphi) \geqslant (1-4\eta)\sqrt{\alpha \log N} \sum_{r \in I} f_r, \tag{3.9}$$

$$\sum_{r \in I} \tilde{f}_r \frac{1}{|C_r|} \sum_{y \in C_r} M_y^{(2)}(\psi) \geqslant (1-4\eta)\sqrt{\beta \log N} \sum_{r \in I} \tilde{f}_r. \tag{3.10}$$

Therefore, it suffices to find an upper bound on the probability that (3.9) and (3.10) happen together. Two observations are in order: first it suffices to treat the probability of (3.9) independently of (3.10) (φ and ψ are independent!) and, secondly, these two problems are effectively only one problem (a Gaussian computation), that has been already treated in detail in Giacomin (2001, Section 3.6), see also Bertacchi and Giacomin (2002) and Bolthausen et al (1995). We sum up the net result: if we set for $f \in L^\infty(D;[0,\infty))$

$$\mathbf{C}_i(f) = \frac{(\int_D f(r)\,\mathrm{d}r)^2}{\int_D \int_D f(r) f(r') \chi_i g_Q(r-r')\,\mathrm{d}r\,\mathrm{d}r'}, \tag{3.11}$$

then for every $\alpha < 4G_{1,L}$, $\beta < 4(G_{1,L}+G_{2,L})$, we have that

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathbb{P}(\Omega_N^+)$$
$$\leqslant -(1-4\eta)^2 \left(\frac{\left(\sqrt{\beta}+\sqrt{\alpha}\right)^2}{2} \mathbf{C}_2(\tilde{f}) + \frac{\alpha}{2}\mathbf{C}_1(f) \right), \tag{3.12}$$

for every $f, \tilde{f} \in L^\infty(D;[0,\infty))$. Let then $\alpha \nearrow 4G_{1,L}$, $\beta \nearrow 4(G_{1,L}+G_{2,L})$, $L \nearrow \infty$, $\eta \searrow 0$, $\kappa \searrow 0$, optimize over the choice of f and $\tilde{f}$ to recover the i-capacities of D (recall (1.5)) and the proof of Proposition 3.1 is complete. □

4. Repulsion phenomena

In this section we prove part (2) of Theorem 1.1, along with some further (and sharper) estimates. We will use the compact notation $\mathbb{P}_N^+(\cdot)$ for $\mathbb{P}(\cdot|\Omega_N^+)$.

4.1. Lower and upper bounds for φ

Both the lower and the upper bound for φ claimed in part (2) of Theorem 1.1 are obtained by reducing the problem to known pathwise estimates on the repulsion action of a wall on a single interface.

Proposition 4.1. *For all $\delta > 0$ and $a < \sqrt{4G_1}$,*

$$\lim_{N\to\infty} \mathbb{P}_N^+\left(\left|\left\{x\in D_N:\ \varphi_x < \sqrt{a\log N}\right\}\right| \geqslant \delta|D_N|\right) = 0. \tag{4.1}$$

Proof. Set

$$A_N = \left\{(\varphi,\psi):\ \left|\left\{x\in D_N : \varphi_x < \sqrt{a\log N}\right\}\right| \geqslant \delta|D_N|\right\}, \tag{4.2}$$

and observe that

$$\mathbb{P}_N^+(A_N) \leqslant \frac{\mathbb{P}(A_N\cap\Omega_{1,N}^+)}{\mathbb{P}(\Omega_N^+)}. \tag{4.3}$$

From part (1) of Theorem 1.1 we know that

$$\mathbb{P}(\Omega_N^+) \geqslant \exp(-cN^{d-2}\log N), \tag{4.4}$$

for $c > 2[(\sqrt{G_1+G_2}+\sqrt{G_1})^2\mathrm{Cap}_2(D)+G_1\mathrm{Cap}_1(D)]$ and N sufficiently large. On the other hand, since $a<\sqrt{4G_1}$, there exists $\epsilon>0$ such that

$$\mathbb{P}(A_N\cap\Omega_{1,N}^+) \leqslant \exp(-N^{d-2+\epsilon}), \tag{4.5}$$

for N sufficiently large. This result is a statement involving φ alone and, while not explicitly stated, it has been already established in Bolthausen et al. (1995, Section 4), see Giacomin (2001, Proposition 3.2) for a more explicit and concise treatment. We also remark that, strictly speaking, the needed argument is also included in this work, but in the slightly more involved context of proving a lower bound for ψ (see footnote in the proof of Proposition 4.3).

By inserting (4.4) and (4.5) into (4.3), we see that $\mathbb{P}_N^+(A_N)$ vanishes as N tends to infinity and the proof is complete. □

Proposition 4.2. *We have that*

$$\limsup_{N\to\infty}\ \sup_{x\in D_N} \frac{\mathbb{E}_N^+[\varphi_x]}{\sqrt{\log N}} \leqslant \sqrt{4G_1}, \tag{4.6}$$

and for all $\delta>0$ and $b>\sqrt{4G_1}$,

$$\lim_{N\to\infty} \mathbb{P}_N^+(|\{x\in D_N:\ \varphi_x > \sqrt{b\log N}\}| \geqslant \delta|D_N|) = 0. \tag{4.7}$$

Proof. Here we simply observe that for every ψ such that $\psi_x\geqslant 0$ and every increasing event $B\in\mathscr{F}^\psi$

$$\mathbb{P}_N^+(B|\mathscr{F}^\psi)(\psi) \leqslant \mathbb{P}_1(B|\Omega_{1,N}^+). \tag{4.8}$$

The term on the left is in fact equal to $\mathbb{P}_1(\cdot|0 \leqslant \varphi_x \leqslant \psi_x$ for $x \in D_N)$, with ψ a fixed configuration and φ random: what (4.8) is saying is therefore that the field φ constrained between two walls is dominated by φ constrained just from below, see Giacomin (2001, B.1-1) for a proof. Therefore (4.6) follows directly from Bolthausen et al. (1995, Lemma 4.7) and (4.7) from Bolthausen et al. (1995, Proposition 4.1). As a side remark, (4.7) follows from (4.6) by using Brascamp–Lieb inequalities, cf. Deuschel and Giacomin (1999) and Giacomin (2001, 2003). □

4.2. Lower bounds for ψ

Proposition 4.3. *For all $\delta > 0$ and $a < \sqrt{4G_1} + \sqrt{4(G_1+G_2)}$,*

$$\lim_{N\to\infty} \mathbb{P}_N^+(|\{x \in D_N\colon\ \psi_x < a\sqrt{\log N}\}| \geqslant \delta|D_N|) = 0. \tag{4.9}$$

Proof. Let Λ_c be the set of vertices of a $2L$-subgrid and Λ be the set of *walls* as defined in the proof of Theorem 3.1 The thesis is proven once we show that

$$\lim_{N\to\infty} \mathbb{P}_N^+(|\{x \in \Lambda_c\colon\ \psi_x < a\sqrt{\log N}\}| \geqslant \delta|\Lambda_c|) = 0, \tag{4.10}$$

since it suffices to repeat a finite number of times, proportional to L^d, the same estimate by shifting the $2L$-subgrid and the corresponding set of centers.

Call B_N the event whose $\mathbb{P}_N^+$-probability is evaluated in (4.10): in view of the lower bound on the probability of Ω_N^+ (cf. part (1) of Theorem 1.1), it suffices to show that for every $a < \sqrt{4G_1} + \sqrt{4(G_1+G_2)}$

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathbb{P}(B_N \cap \Omega_N^+) \leqslant -C, \tag{4.11}$$

for a sufficiently large C. We are going to prove (4.11) with $C = +\infty$.[1]

The next step is to remark that

$$\mathbb{P}(B_N \cap \Omega_N^+) \leqslant \mathbb{P}(B_N \cap \Omega_{\Lambda\cup\Lambda_c}^+), \tag{4.12}$$

and proceed with an estimate that is just a rough version of the first part of the proof of Proposition 3.1: for any fixed positive η, α and β, define

$$\tilde{E}_{\eta,\alpha} = \{|\{y \in \Lambda_c\colon\ M_y^{(1)}(\varphi) \leqslant \sqrt{\alpha \log N}\}| \geqslant \eta|\Lambda_c|\},$$

$$\tilde{F}_{\eta,\beta} = \{|\{y \in \Lambda_c\colon\ M_y^{(2)}(\psi) - M_y^{(1)}(\varphi) \leqslant \sqrt{\beta \log N}\}| \geqslant \eta|\Lambda_c|\}. \tag{4.13}$$

Let us observe that

$$\mathbb{P}(\Omega_{\Lambda\cup\Lambda_c}^+ \cap \tilde{E}_{\eta,\alpha}) \leqslant \mathbb{E}\left[\prod_{y\in\Lambda_c} \mathbb{P}(\varphi_y \geqslant 0|\mathscr{F}_{B(y)}^{\varphi,\psi}); \tilde{E}_{\eta,\alpha}\right] \tag{4.14}$$

[1] With reference to formula (4.5), the explicit estimates that we exhibit show that the probability in (4.10), as well as the one in (4.11), are bounded above by $\exp(-N^{d-2+\epsilon})$ for some $\epsilon > 0$ and N sufficiently large.

and, on $\tilde{E}_{\eta,\alpha}$, if $\alpha < 4G_{1,L}$ for some positive constants c and ϵ

$$\prod_{y\in\Lambda_c} \mathbb{P}(\varphi_y \geqslant 0 | \mathscr{F}^{\varphi,\psi}_{B(y)}) \leqslant \left(1 - \frac{1}{\sqrt{c\log N}} N^{-\alpha/(2G_{1,L})}\right)^{cN^d L^{-d}}$$
$$\leqslant \exp(-N^{d-2+\epsilon}). \tag{4.15}$$

Analogously one shows that $\mathbb{P}(\Omega^+_{\Lambda\cup\Lambda_c} \cap \tilde{F}_{\eta,\beta})$ is bounded by $\exp(-N^{d-2+\epsilon})$ if $\beta < 4(G_{1,L} + G_{2,L})$. In view of (4.11), we are left with estimating $\mathbb{P}(B_N \cap \Omega^+_{\Lambda\cup\Lambda_c} \cap \tilde{E}^{\complement}_{\eta,\alpha} \cap \tilde{F}^{\complement}_{\eta,\beta})$, for α and β in the range that we have just chosen. Then we observe that, by definition of $\tilde{E}_{\eta,\alpha}$ and of $\tilde{F}_{\eta,\beta}$,

$$\tilde{E}^{\complement}_{\eta,\alpha} \cap \tilde{F}^{\complement}_{\eta,\beta} \subset B'_N$$
$$:= \left\{ |\{ y\in\Lambda_c \colon M^{(2)}_y(\psi) > (\sqrt{\alpha} + \sqrt{\beta})\sqrt{\log N}\}| > (1-2\eta)|\Lambda_c| \right\}, \tag{4.16}$$

thus $\mathbb{P}(B_N \cap \Omega^+_{\Lambda\cup\Lambda_c} \cap \tilde{E}^{\complement}_{\eta,\alpha} \cap \tilde{F}^{\complement}_{\eta,\beta}) \leqslant \mathbb{P}(B_N \cap B'_N)$. Now we choose $\eta < \delta/2$ (recall that δ is fixed) and α and β sufficiently close respectively to $4G_1$ and $4(G_1 + G_2)$ so that $\varepsilon := \sqrt{\alpha} + \sqrt{\beta} - a > 0$ (of course, to do this we have to choose L sufficiently large). With these choices we obtain

$$B_N \cap B'_N \subset \Big\{ |\{ y\in\Lambda_c : \psi_y < \sqrt{a\log N},$$
$$M^{(2)}_y(\psi) > (\sqrt{\alpha} + \sqrt{\beta})\sqrt{\log N}\}| > (\delta - 2\eta)|\Lambda_c| \Big\}$$
$$\subset B''_N := \left\{ |\{ y\in\Lambda_c : |\psi_y - M^{(2)}_y(\psi)| > \epsilon\sqrt{\log N}\}| > (\delta-2\eta)|\Lambda_c| \right\}. \tag{4.17}$$

Since $\{\psi_y - M^{(2)}_y(\psi)\}_{y\in\Lambda_c}$ are Gaussian IID variables we have that for some $c > 0$

$$\mathbb{P}(B''_N) \leqslant \mathbb{P}\left(\frac{1}{|\Lambda_c|} \sum_{y\in\Lambda_c} |\psi_y - M^{(2)}_y(\psi)| > \epsilon(\delta - 2\eta)\sqrt{\log N} \right)$$
$$\leqslant \exp(-cN^d \log N). \tag{4.18}$$

Therefore $\mathbb{P}(B_N \cap \Omega^+_{\Lambda\cup\Lambda_c} \cap \tilde{E}^{\complement}_{\eta,\alpha} \cap \tilde{F}^{\complement}_{\eta,\beta}) \leqslant \exp(-cN^d \log N)$ and the proof is complete. □

4.3. *Upper bounds for* ψ

We first prove a result for $\mathbb{E}^+_N(M^\Lambda_N(\psi))$, where $M^\Lambda_N(\psi) = \sum_{x\in\Lambda_N} \psi_x/|\Lambda_N|$, Λ subset of $\mathbb{R}^d$ with piecewise smooth boundary and $\Lambda_N = N\Lambda$.

Proposition 4.4. *For every* Λ *we have that*

$$\limsup_{N\to\infty} \frac{\mathbb{E}^+_N[M^\Lambda_N(\psi)]}{\sqrt{\log N}} \leqslant \sqrt{4G_1} + \sqrt{4(G_1+G_2)}. \tag{4.19}$$

Proof. As for the arguments in Section 2, this proof relies on estimates for the *rough substrate* model that arises when one conditions with respect to $\mathscr{F}^{\varphi}$. More precisely we will make use of

$$\mathbb{E}_N^+(u(\psi)|\mathscr{F}^{\varphi})(\varphi) = \mathbb{E}_2(u(\psi)|\psi_x \geqslant \varphi_x \text{ for } x \in D_N), \tag{4.20}$$

that holds for every measurable $u : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ such that $\mathbb{E}[u(\psi)] < \infty$. To avoid possible misunderstanding due to abuse of notation, we stress that in (4.20) φ on both sides is a numerical variable, while ψ is a random variable. Therefore, the ψ–marginal of $\mathbb{P}_N^+(\cdot|\mathscr{F}^{\varphi})(\varphi)$ is a repulsion model for the field ψ constrained above the fixed rough substrate φ.

In evaluating $\mathbb{E}_N^+[M_N^{\Lambda}(\psi)]$ via conditioning on $\mathscr{F}^{\varphi}$, the random substrate φ is distributed according to $\mathbb{P}_{1,N}^+(\mathrm{d}\varphi)$. We may therefore use again the fact that in the proof of Proposition 2.2 we have identified a set $E_N \in \mathscr{F}^{\varphi}$ such that $\mathbb{P}_{1,N}^+(E_N) \stackrel{N\to\infty}{\to} 1$. Since we are evaluating the expectation of the unbounded random variable $M_N^{\Lambda}(\psi)$, we have to take care of what happens also for *atypical* substrates, that is what happens on $E_N^{\complement}$. Both in considering $\varphi \in E_N$ and $\varphi \in E_N^{\complement}$, the following procedure, already employed in Bertacchi and Giacomin (2002, Proposition 4.6) and in Giacomin (2001, Proposition 3.2) turns out to be helpful: for every $\alpha \geqslant 0$

$$\mathbb{P}_2(\cdot|\psi_x \geqslant \varphi_x \text{ for } x \in D_N) \prec \mathbb{P}_2 T_{\alpha}^{-1}(\cdot|\psi_x \geqslant \varphi_x \text{ for } x \in D_N), \tag{4.21}$$

where T_α is a short-cut notation for T_σ, $\sigma \in \mathbb{R}^{\mathbb{Z}^d}$ and $\sigma_x = \alpha$ for every x, so that

$$\mathbb{E}_2[M_N^{\Lambda}(\psi)|\psi_x \geqslant \varphi_x \text{ for } x \in D_N] \leqslant \alpha + \mathbb{E}_2[M_N^{\Lambda}(\psi)|\psi_x \geqslant \varphi_x - \alpha \text{ for } x \in D_N]. \tag{4.22}$$

The proof of (4.21) may be found for example in Giacomin (2001, Appendix B). We observe moreover that, by using equation (2.27) with $Y = M_N^{\Lambda}(\psi)$, $t = \rho N^{d-2}$ $(\rho > 0)$ and $E = \{\psi_x \geqslant \varphi_x - \alpha \text{ for } x \in D_N\}$, one obtains that there exists a constant $c(\Lambda)$ such that

$$\begin{aligned}&\mathbb{E}_2[M_N^{\Lambda}(\psi)|\psi_x \geqslant \varphi_x - \alpha \text{ for } x \in D_N]\\ &\quad\leqslant c(\Lambda) - \frac{1}{\rho N^{d-2}} \log \mathbb{P}_2(\psi_x \geqslant \varphi_x - \alpha \text{ for } x \in D_N).\end{aligned} \tag{4.23}$$

Let us now consider the case $\varphi \in E_N$ and choose $\alpha = (\sqrt{4G_1} + \sqrt{4(G_1+G_2)} + \delta)\sqrt{\log N}$, $\delta > 0$. We claim that in this case

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}_2(\psi_x \geqslant \varphi_x - \alpha \text{ for } x \in D_N) = 0, \tag{4.24}$$

so that, by (4.22) and (4.23), we conclude that

$$\limsup_{N\to\infty} \frac{\mathbb{E}_N^+[M_N^{\Lambda}(\psi); E_N]}{\sqrt{\log N}} \leqslant \sqrt{4G_1} + \sqrt{4(G_1+G_2)}. \tag{4.25}$$

We prove claim (4.24) by exploiting the tools developed in the Proof of Proposition 2.2. We have

$$\begin{aligned}
&\frac{1}{N^{d-2}} \log \mathbb{P}_2(\psi_x \geqslant \varphi_x - \alpha \text{ for } x \in D_N) \\
&\quad \geqslant \frac{1}{N^{d-2}} \log \mathbb{P}_2(\psi_x \geqslant \tilde{\varphi}_x - \alpha \text{ for } x \in D_N) \\
&\quad \geqslant \frac{1}{N^{d-2}} \log \mathbb{P}_2(\psi_x \geqslant \tilde{\varphi}_x - \alpha \text{ for } x \in D_N^-) \\
&\qquad + \frac{1}{N^{d-2}} \log \mathbb{P}_2(\psi_x \geqslant (\gamma + \tilde{k}\theta)\sqrt{\log N} - \alpha \text{ for } x \in L_N(\tilde{k})) \\
&\quad =: \mathbf{T}_2(N) + \mathbf{T}_1(N),
\end{aligned} \tag{4.26}$$

where $\gamma = \sqrt{4G_1 + \eta}$ and $\tilde{\varphi}$, D_N^- and $L_N(\tilde{k})$ are defined in Section 2. Of course, the fact that $\mathbf{T}_2$, defined in (2.12), is equal to zero also implies that $\lim_{N\to\infty} \mathbf{T}_2(N) = 0$. To get to the same conclusion for $\mathbf{T}_1(N)$ we apply the FKG inequality:

$$\begin{aligned}
&\mathbf{T}_1(N) \\
&\quad \geqslant \sum_{k=1}^{\bar{k}} \frac{N_k}{N^{d-2}} \log\left(1 - \mathbb{P}_2\left(\psi_0 < (k\theta - \sqrt{4(G_1+G_2)} + \eta - \delta)\sqrt{\log N}\right)\right),
\end{aligned} \tag{4.27}$$

for which, since $N_k \leqslant cN^{d-((k-1)^2\theta^2/2G_1)+\epsilon}$, we get the following lower bound:

$$-\frac{c}{\sqrt{\log N}} \sum_{k=1}^{\bar{k}} N^{2-((k-1)^2\theta^2/2G_1)+\epsilon-((k\theta-2\sqrt{G_1+G_2}+\eta-\delta)^2/2G_2)}. \tag{4.28}$$

This term is negligible if

$$\frac{(k-1)^2\theta^2}{2G_1} + \frac{(k\theta - 2\sqrt{G_1+G_2} + \eta - \delta)^2}{2G_2} > 2 \tag{4.29}$$

for all $k \leqslant \bar{k}$ and this is true whenever $|\eta - \delta| > \theta$, that is for θ sufficiently small, which is achieved by choosing $\bar{k}$ sufficiently large. Therefore (4.24) is proven.

In view of (4.25) we are therefore left with showing

$$\limsup_{N\to\infty} \frac{\mathbb{E}_N^+[M_N^A(\psi); E_N^\complement]}{\sqrt{\log N}} \leqslant 0. \tag{4.30}$$

Configurations in $E_N^\complement$ will be split according to the decomposition $\mathbb{R}^{\mathbb{Z}^d} = \bigcup_{j=0}^{\infty} \Omega_j$, where

$$\begin{aligned}
\Omega_0 &= \left\{\varphi : \max_{x\in D_N} \varphi_x \leqslant K\sqrt{\log N}\right\}, \\
\Omega_j &= \left\{\varphi : \max_{x\in D_N} \varphi_x \in (K+j-1, K+j]\sqrt{\log N}\right\} \quad \text{for } j = 1,2,\ldots,
\end{aligned} \tag{4.31}$$

where K is a (sufficiently large) constant that will be chosen below. If $\varphi \in \Omega_0$ then $\varphi_x - (K + \sqrt{4G_2} + \delta)\sqrt{\log N} \leqslant -(\sqrt{4G_2} + \delta)\sqrt{\log N}$ for every $x \in D_N$ and by applying the FKG inequality one obtains that

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}_2(\psi_x > -(\sqrt{4G_2} + \delta)\sqrt{\log N} \text{ for } x \in D_N) = 0, \tag{4.32}$$

for every $\delta > 0$. Once we apply these considerations to (4.22) and (4.23) with $\alpha = (K + \sqrt{4G_2} + \delta)\sqrt{\log N}$ we find

$$\frac{\mathbb{E}_2[M_N^\Lambda(\psi)|\psi_x \geqslant \varphi_x \text{ for } x \in D_N]}{\sqrt{\log N}} \leqslant 2K, \tag{4.33}$$

where we have chosen $K > \sqrt{4G_2} + \delta$ (and N sufficiently large). Therefore

$$\frac{\mathbb{E}_N^+[M_N^\Lambda(\psi); E_N^\complement \cap \Omega_0]}{\sqrt{\log N}} \leqslant 2K\mathbb{P}(E_N^\complement) \stackrel{N\to\infty}{\to} 0. \tag{4.34}$$

We are left with focusing on $\varphi \in \Omega_j$, $j = 1, 2, \ldots$: the estimate is technically almost identical to the case of Ω_0. It is a matter of using again (4.22) and (4.23): this time we choose $\alpha = (K + j + \sqrt{4G_2} + \delta)\sqrt{\log N}$ and we obtain

$$\begin{aligned}
&\mathbb{E}_2[M_N^\Lambda(\psi)|\psi_x \geqslant \varphi_x \text{ for } x \in D_N] \\
&\quad \leqslant K + j + \sqrt{4G_2} + \delta + \frac{c(\Lambda)}{\sqrt{\log N}} \\
&\quad\quad - \frac{1}{N^{d-2}\sqrt{\log N}} \log \mathbb{P}_2\left(\psi_x \geqslant -\left(\sqrt{4G_2} + \delta\right)\sqrt{\log N} \text{ for } x \in D_N\right) \\
&\quad \leqslant 3K + j.
\end{aligned} \tag{4.35}$$

We have spelled out these steps to stress that the estimate is not asymptotic in N: it holds for $N \geqslant N_0$ and N_0 is independent of j. We finally have

$$\begin{aligned}
\mathbb{E}_N^+\left[M_N^\Lambda(\psi); E_N^\complement \cap \bigcup_{j=1}^\infty \Omega_j\right] &\leqslant \mathbb{E}_N^+\left[M_N^\Lambda(\psi); \bigcup_{j=1}^\infty \Omega_j\right] \\
&\leqslant \sum_{j=1}^\infty (3K + j)\mathbb{P}_{1,N}^+(\Omega_j).
\end{aligned} \tag{4.36}$$

By (2.6) (applying the Markov inequality) we obtain that $\mathbb{P}_{1,N}^+(\Omega_j) \leqslant cN^{d-((K+j-1-\sqrt{4G_1}-\delta)^2/2G_1)}$, $\delta > 0$. Therefore the series in the right-hand side of (4.36) is summable and, if $K > \sqrt{2dG_1} + \sqrt{4G_2}$, it vanishes as $N \to \infty$. Therefore

$$\lim_{N\to\infty} \mathbb{E}_N^+\left[M_N^\Lambda(\psi); E_N^\complement \cap \bigcup_{j=1}^\infty \Omega_j\right] = 0. \tag{4.37}$$

Putting together (4.34) and (4.37) we obtain (4.30) and the proof is complete. □

Remark 4.5. It is immediate to see that Proposition 4.3 and the fact that $\psi_x \geqslant 0$ for every $x \in D_N$ if $(\varphi, \psi) \in \Omega_N^+$ imply the lower bound corresponding to (4.19), so that we have that for every $\Lambda \subseteq D$ with piecewise smooth boundary

$$\lim_{N\to\infty} \frac{\mathbb{E}_N^+[M_N^\Lambda(\psi)]}{\sqrt{\log N}} = \sqrt{4G_1} + \sqrt{4(G_1+G_2)}. \tag{4.38}$$

Finally it is not too difficult to see that Proposition 4.4 is compatible with Proposition 4.3 only if the density of sites in which ψ is above $\sqrt{4G_1} + \sqrt{4(G_1+G_2)} + \epsilon$, any $\epsilon > 0$, is negligible. Namely:

Proposition 4.6. *For all $\delta > 0$ and $b > \sqrt{4G_1} + \sqrt{4(G_1+G_2)}$,*

$$\lim_{N\to\infty} \mathbb{P}_N^+(|\{x \in D_N : \psi_x > b\sqrt{\log N}\}| \geqslant \delta |D_N|) = 0. \tag{4.39}$$

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