

On the limit law of a random walk conditioned to reach a high level

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Abstract

We consider a random walk with a negative drift and with a jump distribution which under Cramér's change of measure belongs to the domain of attraction of a spectrally positive stable law. If conditioned to reach a high level and suitably scaled, this random walk converges in law to a nondecreasing Markov process which can be interpreted as a spectrally positive Lévy process conditioned not to overshoot level 1. © 2010 Elsevier B.V. All rights reserved.

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1. Introduction

Let ξ_1, ξ_2, \dots be i.i.d. random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with $\mathbf{E}\xi_1 < 0$ and $\mathbf{P}(\xi_1 > 0) > 0$. Then the random walk $S_n = \sum_{i=1}^n \xi_i$ tends to $-\infty$ with probability 1 as $n \rightarrow \infty$ and the event that it exceeds a high level has a small albeit positive probability. The asymptotics of this probability have been studied extensively. Let $\mathbf{E} \exp(\lambda \xi_1) < \infty$ for some $\lambda > 0$ and let

$$\gamma = \sup\{\lambda : \mathbf{E} \exp(\lambda \xi_1) \leq 1\}. \quad (1.1)$$

Clearly, $\gamma > 0$ and $\mathbf{E} \exp(\gamma \xi_1) \leq 1$.

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In the “classical” case where $\mathbf{E} \exp(\gamma \xi_1) = 1$ and $\beta = \mathbf{E} \xi_1 \exp(\gamma \xi_1) < \infty$, the celebrated Cramér–Lundberg theorem asserts that, for a certain constant C_1 ,

$$\mathbf{P}(\sup_n S_n > r) \sim C_1 e^{-\gamma r} \quad \text{as } r \rightarrow \infty, \quad (1.2)$$

where \sim stands for asymptotic equivalence. The limit is taken along all r if ξ_1 has a nonlattice distribution and along multiples of the lattice span if ξ_1 has a lattice distribution; see, for example, [1, XIII.5], [6, Section 22], or [13, XII].

If $\mathbf{E} \exp(\gamma \xi_1) < 1$, and so $\mathbf{E} \exp(\lambda \xi_1) = \infty$ for all $\lambda > \gamma$, then, under certain regularity assumptions on the distribution of ξ_1 (more specifically, provided it belongs to class \mathcal{S}_γ ; see [25]),

$$\mathbf{P}(\sup_n S_n > r) \sim C_2 \mathbf{P}(\xi_1 > r) \quad \text{as } r \rightarrow \infty,$$

where $C_2 = \mathbf{E} \exp(\gamma \sup_n S_n) / (1 - \mathbf{E} \xi_1 \exp(\gamma \xi_1))$; see, for example, [4] and references therein. For earlier results, see [6, Section 22]; recent developments can be found in [7,27].

The borderline case where $\mathbf{E} \exp(\gamma \xi_1) = 1$ and $\beta = \infty$ was first addressed in Borovkov’s monograph [6, Section 22]. More complete results have been obtained by Korshunov [19] who showed that if the distribution of ξ_1 is nonlattice and the distribution function F , where $F(dx) = \exp(\gamma x) \mathbf{P}(\xi_1 \in dx)$, has a regularly varying right-hand tail with index $-\alpha \in (-1, -1/2)$, then

$$\mathbf{P}(\sup_n S_n > r) \sim C_3 \frac{e^{-\gamma r}}{\gamma m(r)} \quad \text{as } r \rightarrow \infty, \quad (1.3)$$

for a certain universal constant C_3 and for $m(r) = \int_0^r (1 - F(u)) du$. A similar asymptotic relationship was shown to be valid for lattice distributions too, with a different constant C'_3 .

One can also clarify in what way the event of attaining a high level r is most likely to occur. In the classical case, the trajectory $(S_{\lfloor rt \rfloor} / r, t \in \mathbb{R}_+)$, where $\lfloor \cdot \rfloor$ stands for the integer part, stays with a probability close to 1 in a neighbourhood of the straight line with slope β ; see (1.9) for the exact formulation. Thus, the event of reaching level r is realised typically via multiple (of order r) jumps of order 1. If $\mathbf{E} \exp(\gamma \xi_1) < 1$, then the high level is most likely to be reached at the very beginning of the random walk, which occurs due to a single big jump of order r ; see [7] for the case of regular exponential distribution tails and Zachary and Foss [27] for the more general \mathcal{S}_γ distributions. In fact, as follows from the results of Zachary and Foss [27], the conditional distribution of the jump time converges weakly (no scaling is involved) to the geometric distribution with parameter $p = 1 - \mathbf{E} \exp(\gamma \xi_1)$.

The purpose of this paper is to study the most likely way for the random walk to attain a high level r in the setting considered by Korshunov [19]. Not unexpectedly, the results and intuitive explanations in the borderline situation are essentially more intriguing and complicated. On the one hand, as $\beta \uparrow \infty$ in the classical case, the trajectories $S_{\lfloor rt \rfloor} / r$ for large values of r increasingly look like a jump at time 0. On the other hand, as $\mathbf{E} \exp(\gamma \xi_1) \uparrow 1$, the time for the big jump that reaches (or almost reaches) r to occur tends to infinity. Therefore, the typical jump sizes in the case that we concern ourselves with here should, on the one hand, grow to infinity as $r \rightarrow \infty$, but, on the other hand, be of a smaller order of magnitude than r . The correct order of magnitude is captured by considering the random walk S_n with n tending to infinity at a slower rate than r . Typically, one takes $n = \lfloor r^\alpha t \rfloor$, which corresponds to the jumps of order $r^{1-\alpha}$.

To provide better insight into the kinds of results that we obtain, let us recall the argument underlying the asymptotics in the classical case. Its main points can be found in [2], which refers to Iglehart [17] and von Bahr [26]. Let \mathcal{F}_n denote the σ -algebra on Ω generated by

the ξ_i , $i = 1, 2, \dots, n$. We shall assume that the σ -algebra \mathcal{F} is generated by the σ -algebras \mathcal{F}_n , $n = 1, 2, \dots$. Let \mathbf{P}^* be the measure on (Ω, \mathcal{F}) defined by

$$\mathbf{P}^*(\Gamma) = \mathbf{E} \exp(\gamma S_n) \mathbf{1}_\Gamma \quad \text{for } \Gamma \in \mathcal{F}_n, \quad (1.4)$$

where $\mathbf{1}_\Gamma$ denotes the indicator function of event Γ . It is a probability measure by the assumption that $\mathbf{E} \exp(\gamma \xi_1) = 1$. The probability measures \mathbf{P} and \mathbf{P}^* are locally equivalent and $d\mathbf{P}/d\mathbf{P}^*|_{\mathcal{F}_n} = \exp(-\gamma S_n)$. We also note that under \mathbf{P}^* the ξ_k are i.i.d. with mean β .

For $r > 0$, let $\tau^{(r)}$ be the first time at which the random walk S_n attains level r , i.e.,

$$\tau^{(r)} = \min\{n : S_n \geq r\}. \quad (1.5)$$

Since $\{\tau^{(r)} = n\} \in \mathcal{F}_n$,

$$\mathbf{P}(\tau^{(r)} < \infty) = \sum_{n=1}^{\infty} \mathbf{P}(\tau^{(r)} = n) = \sum_{n=1}^{\infty} \mathbf{E}^* e^{-\gamma S_n} \mathbf{1}_{\{\tau^{(r)}=n\}} = \mathbf{E}^* e^{-\gamma S_{\tau^{(r)}}} \mathbf{1}_{\{\tau^{(r)} < \infty\}},$$

where \mathbf{E}^* denotes expectation with respect to \mathbf{P}^* . On noting that $\mathbf{P}^*(\tau^{(r)} < \infty) = 1$ as $\mathbf{E}^* \xi_1 > 0$, we conclude that

$$\mathbf{P}(\tau^{(r)} < \infty) = \mathbf{E}^* \exp(-\gamma S_{\tau^{(r)}}). \quad (1.6)$$

More generally, if $\Gamma \in \mathcal{F}_{\tau^{(r)}}$, $\mathcal{F}_{\tau^{(r)}}$ being the σ -algebra associated with the stopping time $\tau^{(r)}$, then by the fact that $\{\tau^{(r)} = n\} \cap \Gamma \in \mathcal{F}_n$,

$$\mathbf{P}(\Gamma \cap \{\tau^{(r)} < \infty\}) = \sum_{n=1}^{\infty} \mathbf{P}(\Gamma \cap \{\tau^{(r)} = n\}) = \sum_{n=1}^{\infty} \mathbf{E}^* e^{-\gamma S_n} \mathbf{1}_{\Gamma \cap \{\tau^{(r)}=n\}} = \mathbf{E}^* e^{-\gamma S_{\tau^{(r)}}} \mathbf{1}_\Gamma,$$

so

$$\mathbf{P}(\Gamma | \tau^{(r)} < \infty) = \frac{\mathbf{E}^* e^{-\gamma S_{\tau^{(r)}}} \mathbf{1}_\Gamma}{\mathbf{E}^* e^{-\gamma S_{\tau^{(r)}}}} = \frac{\mathbf{E}^* e^{-\gamma \chi^{(r)}} \mathbf{1}_\Gamma}{\mathbf{E}^* e^{-\gamma \chi^{(r)}}}, \quad (1.7)$$

where

$$\chi^{(r)} = S_{\tau^{(r)}} - r \quad (1.8)$$

is the overshoot of the random walk S_n over level r .

Suppose now that Γ is the event $\{\sup_{n \leq \tau^{(r)}} |S_n - \beta n| < \varepsilon r\}$, where $\varepsilon > 0$ is given. Since $\beta < \infty$, the \mathbf{P}^* -probability of this event tends to 1 as $r \rightarrow \infty$, by the strong law of large numbers. Also, the condition $\beta < \infty$ implies that the $\chi^{(r)}$ under \mathbf{P}^* tend in distribution, as $r \rightarrow \infty$, to a proper random variable, say $\chi^{(\infty)}$, provided the distribution of ξ_1 is nonlattice—see, e.g., [1, VIII.2], [16, III.10], or [13, XI.4]. Therefore, the quantities $\mathbf{E}^* \exp(-\gamma \chi^{(r)})$ converge to a positive limit as $r \rightarrow \infty$ and, by (1.7),

$$\lim_{r \rightarrow \infty} \mathbf{P}(\sup_{n \leq \tau^{(r)}} |S_n - \beta n| < \varepsilon r | \tau^{(r)} < \infty) = 1. \quad (1.9)$$

(In particular, (1.2) with $C_1 = \mathbf{E} e^{-\gamma \chi^{(\infty)}}$ follows by (1.6) and (1.8).) This argument breaks down in two places if $\beta = \infty$: we can no longer rely on the law of large numbers for the random walk and the $\chi^{(r)}$ might converge to infinity as $r \rightarrow \infty$.

In order to tackle these difficulties, we will assume, as in [19], that the distribution function F has a regularly varying right-hand tail with index $-\alpha$, where $\alpha \in (1/2, 1)$. We scale “time” by

$(1 - F(r))^{-1}$ so that the processes $(S_{\lfloor(1-F(r))^{-1}t\rfloor}/r, t \in \mathbb{R}_+)$ under \mathbf{P}^* converge in distribution as $r \rightarrow \infty$ to a stable subordinator $X = (X(t), t \in \mathbb{R}_+)$ with Lévy measure $\alpha x^{-\alpha-1} dx$; see [22] or Lemma A.3. Under the stated assumptions, the random variables $\chi^{(r)}/r$ converge in distribution to a proper random variable χ , which assumes values in $(0, 1)$ and has density $p_\alpha(x) = (\sin \pi \alpha / \pi) x^{-\alpha} (1+x)^{-1}$. (See [10, Theorem 2], or [13, XIV.3], for the case of renewal processes, Sinai [24] for the case of the sums of random variables with a stable distribution; the case in question follows by an application of Lemma 2 in [19].) It is then plausible that in (1.7) one should be able to replace $\chi^{(r)}$ with $r\chi$ and so $\exp(-\gamma \chi^{(r)})$ can be replaced with $\exp(-r\gamma\chi)$. For large values of r , the bulk of the contribution to $\mathbf{E}^* \exp(-r\gamma\chi)$ comes from the small values of χ , so the right-hand side of (1.7) should be asymptotically equivalent to $\mathbf{P}^*(\Gamma|\chi = 0)$. If Γ is an event associated with the process $(S_{\lfloor(1-F(r))^{-1}t\rfloor}/r, t \in \mathbb{R}_+)$, then it should translate in the limit into a similar event associated with X . Besides, since $\chi^{(r)}$ is the overshoot over level r by S_n , we have that $\chi^{(r)}/r$ is the overshoot over level 1 of the process $(S_{\lfloor(1-F(r))^{-1}t\rfloor}/r, t \in \mathbb{R}_+)$. That this process converges to X suggests the conjecture that χ should be the overshoot of X above level 1.

It thus appears as though the conditional distribution of the process $(S_{\lfloor(1-F(r))^{-1}t\rfloor \wedge \tau^{(r)}}/r, t \in \mathbb{R}_+)$ given that $\tau^{(r)} < \infty$ should converge to the conditional distribution of the process $(X(t \wedge \tau), t \in \mathbb{R}_+)$ given the event $X(\tau) = 1$, where

$$\tau = \inf\{t : X(t) \geq 1\}. \quad (1.10)$$

The main result of the paper (Theorem 2.2) confirms this conjecture. As a consequence, we have that if the distribution function F decays as $x^{-\alpha}$, then, assuming X is defined on a probability space $(\Omega', \mathcal{F}', \mathbf{P}')$, for $B > 0$ and $\varepsilon > 0$,

$$\lim_{r \rightarrow \infty} \mathbf{P}(\sup_{n \leq \tau^{(r)}} |S_n - Bn^{1/\alpha}| < \varepsilon r | \tau^{(r)} < \infty) = \mathbf{P}'(\sup_{t \leq \tau} |X(t) - Bt^{1/\alpha}| < \varepsilon | X(\tau) = 1),$$

which can be regarded as a counterpart of (1.9).

We denote the process X “conditioned not to overshoot level 1” by \tilde{X} . There are various ways to define this process rigorously. It can be obtained by an application of the Doob h -transform associated with the function $h(x) = (1-x)^{\alpha-1}$ to the process X . It is also the limit of conditional distributions of the processes $(X(t \wedge \tau), t \in \mathbb{R}_+)$ given $X(\tau) \leq 1 + \varepsilon$ as $\varepsilon \rightarrow 0$. We define it by the property that the process $(1 - \tilde{X}(t), t \in \mathbb{R}_+)$ is a positive self-similar Markov process, as in [20].

The main statement on the convergence of the conditional laws of $(S_{\lfloor(1-F(r))^{-1}t\rfloor \wedge \tau^{(r)}}/r, t \in \mathbb{R}_+)$, given $\tau^{(r)} < \infty$, to the law of \tilde{X} as $r \rightarrow \infty$ is formulated in Section 2 and then proved in Section 3. For the latter, we apply the general results on weak convergence of semimartingales as in [18,21]. A key is to compute the predictable jump measures of the processes in question under the “conditional” measures. This is done by applying the results on the transformations of the predictable triplets under absolutely continuous changes of the measure, as in [18,21]. Since the approaches of Korshunov [19] play an important part, we also have to treat the nonlattice and lattice cases separately. We also provide a representation for the limit process \tilde{X} which complements the Lamperti representation. Auxiliary and supplemental results are collected in the Appendix. In Appendix A.1 we present a more complete version of the proof of two theorems from [19] which are important for the developments in this paper, Appendix A.2 contains some properties of slowly and regularly varying functions, and Appendix A.3 contains a proof of the

convergence in distribution of the processes $(S_{\lfloor (1-F(r))^{-1}t \rfloor} / r, t \in \mathbb{R}_+)$ to X under measure \mathbf{P}^* based on the semimartingale weak convergence theory.

We conclude the Introduction with a list of notation and conventions adopted in the paper. We let \mathbb{N} denote the set of positive integers, \mathbb{R} the set of real numbers, $\mathcal{B}(\mathbb{R})$ the Borel σ -algebra on \mathbb{R} , and \mathbb{R}_+ the subset of \mathbb{R} of nonnegative reals. For $x \in \mathbb{R}$ and $y \in \mathbb{R}$, $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$, and $x^- = -x \wedge 0$. Recall also that $\lfloor x \rfloor$ denotes the integer part of x and $\mathbf{1}_\Gamma$ the indicator function of event Γ . Two positive functions $f(x)$ and $g(x)$ of a real-valued argument are said to be asymptotically equivalent as $x \rightarrow \infty$, which is written as $f(x) \sim g(x)$, if $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$. We write $f(x) = O(g(x))$ if $f(x) \leq Cg(x)$ for some $C > 0$ and for all x great enough. Integrals of the form \int_a^b are understood as $\int_{(a,b]}$ unless otherwise indicated.

We denote by \mathbb{D} the space of real-valued right-continuous functions on \mathbb{R}_+ with left-hand limits. Its elements are denoted by lower-case bold-face Roman characters, e.g., $\mathbf{x} = (\mathbf{x}(t), t \in \mathbb{R}_+)$; $\mathbf{x}(t-)$ is the left-hand limit of \mathbf{x} at t and $\Delta \mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}(t-)$ is the size of the jump at t . The space \mathbb{D} is assumed to be endowed with the Skorohod J_1 -topology, equipped with the Borel σ -algebra $\mathcal{B}(\mathbb{D})$, and metrised by a complete separable metric (see [12,18,21] for the definition and properties); \mathbb{D}_\uparrow denotes the subset of \mathbb{D} of increasing functions with $\mathbf{x}(0) = 0$ which is equipped with the subspace topology. All stochastic processes encountered in this paper have trajectories in \mathbb{D} and are considered as random elements of $(\mathbb{D}, \mathcal{B}(\mathbb{D}))$. Weak convergence of probability measures on \mathbb{D} and convergence in distribution of stochastic processes are understood with respect to the Skorohod topology.

We recall that a filtered probability space, or a stochastic basis, $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$ is defined as a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ endowed with an increasing right-continuous flow $\mathbf{F} = (\mathcal{F}(t), t \in \mathbb{R}_+)$ of sub- σ -algebras of \mathcal{F} . Such a flow is also referred to as a filtration. We will assume without further mention that all σ -algebras that we consider are complete with respect to the corresponding probability measure. For the background on the general theory of stochastic processes, the reader is referred to Jacod and Shiryaev [18] and Liptser and Shiryaev [21]. For the background on Lévy, see [3]. For the properties of regularly and slowly varying functions, see [5].

2. Convergence of the conditioned random walk

Given $\alpha \in (0, 1)$, let $Y = (Y(t), t \in \mathbb{R}_+)$ be the positive self-similar Markov process with the Lamperti representation (see [20])

$$Y(t) = \exp(-L(\sigma(t))), \quad (2.1)$$

where $L = (L(t), t \in \mathbb{R}_+)$ is a subordinator with the Lévy measure $\mathbf{1}_{\{x>0\}}\alpha e^{-\alpha x}(1 - e^{-x})^{-\alpha-1} dx$ and $L(0) = 0$, and $\sigma(t) = \inf\{s : \int_0^s \exp(-\alpha L(q)) dq > t\}$. Processes similar to Y have been investigated in [9,8] who variously refer to them as processes “conditioned to die at zero” and processes “conditioned to hit zero continuously”. Although the results of those papers do not apply directly to the process Y because the authors excluded the case of subordinators, their methods can be used to study its properties. We also mention that the Lamperti representation (2.1) can be complemented with a representation in terms of the Doléans exponential:

$$Y(t) = \prod_{0 < s \leq \sigma_1(t)} (1 - \Delta L_1(s)),$$

where $L_1 = (L_1(t), t \in \mathbb{R}_+)$ is a subordinator with the Lévy measure $\mathbf{1}_{\{x \in (0,1)\}}(1-x)^{\alpha-1} \alpha x^{-\alpha-1} dx$ and $\sigma_1(t) = \inf \left\{ s : \int_0^s \prod_{0 < s \leq q} (1 - \Delta L_1(s))^\alpha dq > t \right\}$. This is a consequence of the fact that the process $(-\sum_{0 \leq s \leq t} \ln(1 - \Delta L_1(s)), t \in \mathbb{R}_+)$ is distributed as the process L .

We define the process $\tilde{X} = (\tilde{X}(t), t \in \mathbb{R}_+)$ by letting $\tilde{X}(t) = 1 - Y(t)$. It is a pure jump process, with nondecreasing trajectories, and values in $[0, 1]$. The predictable jump measure of \tilde{X} is of the form $(\nu(\tilde{X}; dt, dx))$, where, for $G \in \mathcal{B}(\mathbb{R})$ and $t \in \mathbb{R}_+$,

$$\nu(\mathbf{x}; [0, t], G) = \int_0^t \int_G \mathbf{1}_{\{0 < x < 1 - \mathbf{x}(s)\}} \left(1 - \frac{x}{1 - \mathbf{x}(s)}\right)^{\alpha-1} \alpha x^{-\alpha-1} dx ds \quad (2.2)$$

if $\mathbf{x} \in \mathbb{D}_\uparrow$ and $\nu(\mathbf{x}; [0, t], G) = 0$ if $\mathbf{x} \in \mathbb{D} \setminus \mathbb{D}_\uparrow$. By reversing Lamperti's construction, one can show that the distribution of \tilde{X} is uniquely specified by its jump measure, meaning that there exists a unique semimartingale $\tilde{X} = (\tilde{X}(t), t \in \mathbb{R}_+)$ defined on a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{F}}, \tilde{\mathbf{P}})$ with the $\tilde{\mathbf{F}}$ -predictable jump measure $(\nu(\tilde{X}; dt, dx))$ and such that $\tilde{X}(t) = \sum_{0 \leq s \leq t} \Delta \tilde{X}(s)$. In addition, it can be shown that if $\tilde{\tau} = \inf\{t \geq 0 : \tilde{X}(t) = 1\}$, then $\tilde{\tau} < \infty$ a.s. and $\tilde{X}(\tilde{\tau}-) = 1$. Moreover, for $n \in \mathbb{N}$,

$$\tilde{\mathbf{E}} \tilde{\tau}^n = n! \prod_{k=1}^n \left(\int_0^1 (1 - x^{\alpha k}) \alpha x^{\alpha-1} (1-x)^{-\alpha-1} dx \right)^{-1}$$

and $\tilde{\mathbf{E}} e^{c\tilde{\tau}} \leq 1/(1 - c\tilde{\mathbf{E}}\tilde{\tau})$ when $c < 1/\tilde{\mathbf{E}}\tilde{\tau}$.

The next theorem, which is analogous to the results available in [9,8], confirms the interpretation of \tilde{X} as “a process conditioned not to overshoot”. Recall that $X = (X(t), t \in \mathbb{R}_+)$ represents the stable subordinator with Lévy measure $\alpha x^{-\alpha-1} dx$, where $\alpha \in (0, 1)$. We also let \hat{X} represent the process X stopped at τ : $\hat{X}(t) = X(t \wedge \tau)$.

Theorem 2.1. *The conditional laws of \hat{X} given the events $X(\tau) \leq 1 + \varepsilon$, considered as distributions on \mathbb{D} , weakly converge as $\varepsilon \downarrow 0$ to the law of \tilde{X} .*

The proof of this theorem is obtained by arguments modelled on the proof of Theorem 2.2 and will not be given here because of space constraint considerations. Instead, we outline the intuition behind the form of the jump measure of \tilde{X} . The intensity of jumps of size x of \tilde{X} at a point $\tilde{X}(t) = u$, where $u < u + x < 1$, can be obtained by “conditioning” the intensity of jumps of X on the event that the overshoot of X over 1 is not greater than ε . In other words, it should be given approximately by the product of the intensity of jumps of X from u to $u + x$, which is $\alpha x^{-\alpha-1}$, with the probability for X not to exceed level 1 by more than ε when starting at $u + x$ over the probability that X does not overshoot 1 by more than ε starting at u . As follows by the results of Dynkin [10] and Rogozin [23], the probability for the process X not to overshoot a level $y > 0$ by more than $\varepsilon > 0$ is asymptotically equivalent to $(\sin \pi \alpha / (\pi(1 - \alpha))) y^{\alpha-1} \varepsilon^{1-\alpha}$ as $\varepsilon \rightarrow 0$. Therefore, the intensity of jumps of \tilde{X} should be $\alpha x^{-\alpha-1} (1 - x/(1 - u))^{\alpha-1}$.

The readers who are interested in a complete and self-contained proof of the result of Theorem 2.1, may find it in an earlier version of this paper; see <http://arxiv.org/abs/0712.2637>. It also contains the proof of the main result, but under a slightly stronger version of condition 2.

We now state the main result of the paper. Let

$$X^{(r)}(t) = \frac{1}{r} \sum_{i=1}^{\lfloor t/(1-F(r)) \rfloor} \xi_i, \quad (2.3)$$

$$\widehat{\tau}^{(r)} = \inf\{t : X^{(r)}(t) \geq 1\}. \quad (2.4)$$

Note that $\widehat{\tau}^{(r)} = (1 - F(r))\tau^{(r)}$. We denote by $\widehat{X}^{(r)} = (\widehat{X}^{(r)}(t), t \in \mathbb{R}_+)$ the process $X^{(r)}$ stopped at $\widehat{\tau}^{(r)}$, i.e., $\widehat{X}^{(r)}(t) = X^{(r)}(t \wedge \widehat{\tau}^{(r)})$.

Theorem 2.2. *Let the following conditions hold:*

1. *the right-hand tail of the distribution function F is regularly varying at infinity with index $-\alpha$, where $\alpha \in (1/2, 1)$,*
2. *there exist $C > 0, B > 0$, and $\rho \in (0, 1)$ such that, for all z large enough and all $y \in [\rho, 1 - B/z]$,*

$$\frac{1 - F(yz)}{1 - F(z)} \leq 1 + C(1 - y).$$

If, in addition, F is nonlattice, then, as $r \rightarrow \infty$, the conditional distributions of the $\widehat{X}^{(r)}$ given $\tau^{(r)} < \infty$ weakly converge to the distribution of \widetilde{X} . If, instead, F is lattice with span h , then, as $n \rightarrow \infty$, where $n \in \mathbb{N}$, the conditional distributions of the $\widehat{X}^{(nh)}$ given $\tau^{(nh)} < \infty$ weakly converge to the distribution of \widetilde{X} .

Remark 2.1. The restriction on α to be greater than $1/2$ is due to the fact that a key element in the proof is a local renewal theorem with infinite (or non-existing) mean which has been established for $\alpha \in (1/2, 1)$ only; see [11]. The requirement that $\alpha \in (1/2, 1)$ is only used in the proof of Lemma A.1. The rest of the proof applies to any distribution F from the domain of attraction of a stable law with index $\alpha \in (0, 1)$.

Remark 2.2. Under condition 1, the function $\ell(x) = x^\alpha(1 - F(x))$ is slowly varying at infinity, i.e., $\lim_{x \rightarrow \infty} \ell(yx)/\ell(x) = 1$ for all $y > 0$. According to Karamata's theorem (see [5] or [13]), it admits the representation $\ell(x) = c(x) \exp(\int_1^x \varepsilon(u)/u \, du)$, where $c(x) \rightarrow c > 0$ and $\varepsilon(x) \rightarrow 0$ as $x \rightarrow \infty$. If $c(x)$ in this representation is a constant, or converges to the limit quickly enough, then condition 2 of the theorem holds. An example of a lattice distribution which meets the requirements of the theorem is provided by $F(x) = 1 - (\lfloor x \rfloor + 1)^{-\alpha}$ for $x \geq 0$.

Remark 2.3. The distribution F is lattice if and only if \widehat{F} is lattice with the same span, where \widehat{F} denotes the distribution function of ξ_1 under \mathbf{P} . We note that $\widehat{F}(dx) = \exp(-\gamma x)F(dx)$.

Remark 2.4. A sufficient (but not necessary) condition for F to have a regularly varying right-hand tail with index $-\alpha$ is for the function $e^{\gamma x} \mathbf{P}(\xi_1 > x)$ to be regularly varying with index $-\alpha - 1$; see [19] for further comments. Since the left-hand tail of F decays exponentially quickly, its right-hand tail is regularly varying with index $-\alpha$ if and only if F belongs to the domain of attraction of the spectrally positive stable law with index α ; cf. [15] or [13].

The proof of Theorem 2.2 will be obtained by an application of the following result, which is a special case of Theorem IX.3.21 on p. 546 in [18].

Lemma 2.1. *Consider a sequence $\Xi^{(n)}$ of \mathbb{R} -valued semimartingales with trajectories of locally bounded variation defined on filtered probability spaces $(\Omega^{(n)}, \mathcal{G}^{(n)}, \mathbf{G}^{(n)}, \mathbf{P}^{(n)})$. Suppose that the $\Xi^{(n)}$ are pure jump semimartingales meaning that $\Xi^{(n)}(t) = \sum_{0 < s \leq t} \Delta \Xi^{(n)}(s)$ for all $t > 0$ and let $\nu^{(n)}(dt, dx)$ denote their predictable jump measures. Let an \mathbb{R}_+ -valued function $K(y; G)$, where $y \in \mathbb{R}$ and $G \in \mathcal{B}(\mathbb{R})$, be Borel-measurable in y and be a σ -finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ in G such that $K(y; \{0\}) = 0$. Suppose that the following conditions hold:*

1. $\sup_{y \in \mathbb{R}} \int_{\mathbb{R}} |x| K(y; dx) < \infty$,
2. for an arbitrary \mathbb{R} -valued continuous function $g(x)$, $x \in \mathbb{R}$, such that $|g(x)| \leq M|x|$, $x \in \mathbb{R}$, with some $M > 0$, the function $\int_{\mathbb{R}} g(x) K(y; dx)$ is continuous in y ,
3. for arbitrary $\delta > 0$, $t > 0$, and an \mathbb{R} -valued continuous function $g(x)$, $x \in \mathbb{R}$, such that $|g(x)| \leq M|x|$, $x \in \mathbb{R}$, with some $M > 0$,

$$\lim_{n \rightarrow \infty} \mathbf{P}^{(n)} \left(\left| \int_0^t \int_{\mathbb{R}} g(x) v^{(n)}(ds, dx) - \int_0^t \int_{\mathbb{R}} g(x) K(\Xi^{(n)}(s); dx) ds \right| > \delta \right) = 0,$$

4. the $\Xi^{(n)}(0)$ converge in distribution to a random variable Ξ_0 as $n \rightarrow \infty$,
5. there exists at most one pure jump semimartingale $\Xi = (\Xi(t), t \in \mathbb{R}_+)$ of locally bounded variation with an initial condition Ξ_0 and with predictable jump measure $v(dt, dx) = K(\Xi(t); dx) dt$.

Then the $\Xi^{(n)}$ converge in distribution to Ξ .

3. Proof of Theorem 2.2

The proof of Theorem 2.2 follows from Lemmas 3.1–3.4. More detailed guidance is provided at the end of this section.

For the proof, Lemma 2.1 will be used with

$$K(y; G) = \int_{G \setminus \{0\}} \mathbf{1}_{\{0 < x < 1-y\}} \left(1 - \frac{x}{1-y}\right)^{\alpha-1} \alpha x^{-\alpha-1} dx \quad (3.1)$$

when $y \in (0, 1)$ and $K(y; G) = 0$ otherwise.

Lemma 3.1. The function K satisfies conditions 1 and 2 of Lemma 2.1.

Proof. For $g(x)$ as in condition 2 of Lemma 2.1 and for $y \in (0, 1)$, we have

$$\int_{\mathbb{R}} g(x) K(y; dx) = (1-y)^{-\alpha} \int_0^1 g(u(1-y)) (1-u)^{\alpha-1} \alpha u^{-\alpha-1} du. \quad (3.2)$$

Thus, $\int_{\mathbb{R}} |x| K(y; dx) \leq \alpha \int_0^1 u^{\alpha-1} (1-u)^{-\alpha} du$, so condition 1 of Lemma 2.1 holds. Condition 2 follows by the assumption that $|g(x)| \leq M|x|$, continuity of $g(x)$, and Lebesgue's bounded convergence theorem. \square

In order to apply Lemma 2.1, we need to compute the predictable triplets of the conditioned processes $\hat{X}^{(r)}$. We start with introducing a change of measure. Let the absolutely continuous, with respect to \mathbf{P}^* , probability measure $\mathbf{Q}^{(r)}$ on $(\Omega, \mathcal{F}_{\tau^{(r)}})$ be defined by

$$\frac{d\mathbf{Q}^{(r)}}{d\mathbf{P}^*} = \frac{e^{-\gamma \chi^{(r)}}}{\mathbf{E}^* e^{-\gamma \chi^{(r)}}}. \quad (3.3)$$

By (1.7), for $\Gamma \in \mathcal{F}_{\tau^{(r)}}$,

$$\mathbf{P}(\Gamma | \tau^{(r)} < \infty) = \frac{\mathbf{E}^* e^{-\gamma \chi^{(r)}} \mathbf{1}_{\Gamma}}{\mathbf{E}^* e^{-\gamma \chi^{(r)}}},$$

so, for a Borel subset U of \mathbb{D} ,

$$\mathbf{P}(\hat{X}^{(r)} \in U | \tau^{(r)} < \infty) = \mathbf{Q}^{(r)}(\hat{X}^{(r)} \in U). \quad (3.4)$$

Let $\mu^{(r)}$ denote the jump measure of $X^{(r)}$, so

$$\mu^{(r)}([0, t], G) = \sum_{i=1}^{\lfloor t/(1-F(r)) \rfloor} \mathbf{1}_{\{\xi_i/r \in G \setminus \{0\}\}}, \quad (3.5)$$

where G is a Borel subset of \mathbb{R} . Clearly,

$$X^{(r)}(t) = \int_0^t \int_{\mathbb{R}} x \mu^{(r)}(ds, dx). \quad (3.6)$$

Let $\mathcal{F}^{(r)}(t)$ represent the σ -algebra generated by the random variables $X^{(r)}(s)$ for $s \leq t$. Let $\mathbf{F}^{(r)} = (\mathcal{F}^{(r)}(t), t \in \mathbb{R}_+)$ be the filtration associated with $X^{(r)}$ and let $\widehat{\mathbf{F}}^{(r)} = (\widehat{\mathcal{F}}^{(r)}(t), t \in \mathbb{R}_+)$, with $\widehat{\mathcal{F}}^{(r)}(t) = \mathcal{F}^{(r)}(t \wedge \widehat{\tau}^{(r)})$, be the filtration associated with $\widehat{X}^{(r)}$. The $\mathbf{F}^{(r)}$ -predictable jump measure of $X^{(r)}$ under \mathbf{P}^* is of the form

$$\nu^{(r)}([0, t], G) = \left\lfloor \frac{t}{1-F(r)} \right\rfloor \int_{G \setminus \{0\}} F(r) dx. \quad (3.7)$$

The jump measure of $\widehat{X}^{(r)}$ is given by

$$\widehat{\mu}^{(r)}([0, t], G) = \sum_{i=1}^{\lfloor (t \wedge \widehat{\tau}^{(r)})/(1-F(r)) \rfloor} \mathbf{1}_{\{\xi_i/r \in G \setminus \{0\}\}}, \quad (3.8)$$

and the $\widehat{\mathbf{F}}^{(r)}$ -predictable jump measure of $\widehat{X}^{(r)}$ under \mathbf{P}^* is

$$\widehat{\nu}^{(r)}([0, t], G) = \nu^{(r)}([0, t \wedge \widehat{\tau}^{(r)}], G). \quad (3.9)$$

For $y > 0$, we let

$$u^{(r)}(y) = \mathbf{E}^* e^{-\gamma \chi^{(ry)}}. \quad (3.10)$$

Lemma 3.2. *There exists a version of the $\widehat{\mathbf{F}}^{(r)}$ -predictable jump measure of $\widehat{X}^{(r)}$ under $\mathbf{Q}^{(r)}$ of the form*

$$\begin{aligned} \widetilde{\nu}^{(r)}(dt, dx) = & \mathbf{1}_{\{\widehat{X}^{(r)}(t-) < 1\}} \left[\mathbf{1}_{\{\widehat{X}^{(r)}(t-) + x \geq 1\}} \frac{e^{-\gamma r(\widehat{X}^{(r)}(t-) + x - 1)}}{u^{(r)}(1 - \widehat{X}^{(r)}(t-))} \right. \\ & \left. + \mathbf{1}_{\{\widehat{X}^{(r)}(t-) + x < 1\}} \frac{u^{(r)}(1 - \widehat{X}^{(r)}(t-) - x)}{u^{(r)}(1 - \widehat{X}^{(r)}(t-))} \right] \nu^{(r)}(dt, dx). \end{aligned}$$

Proof. Let us introduce

$$\kappa^{(r)}(y) = \inf\{t \in \mathbb{R}_+ : X^{(r)}(t) \geq y\} \quad (3.11)$$

and

$$\eta^{(r)}(y) = X^{(r)}(\kappa^{(r)}(y)) - y. \quad (3.12)$$

Note that $\widehat{\tau}^{(r)} = \kappa^{(r)}(1)$ and $\chi^{(ry)} = r y \eta^{(ry)}(1) = r \eta^{(r)}(y)$, so by (3.10),

$$u^{(r)}(y) = \mathbf{E}^* e^{-\gamma r \eta^{(r)}(y)}. \quad (3.13)$$

Also, by (3.3) the density process of $\mathbf{Q}^{(r)}$ with respect to \mathbf{P}^* , which we denote by $Z^{(r)} = (Z^{(r)}(t), t \in \mathbb{R}_+)$, can be written as

$$Z^{(r)}(t) = \mathbf{E}^* \left[\frac{e^{-\gamma r \eta^{(r)}(1)}}{\mathbf{E}^* e^{-\gamma r \eta^{(r)}(1)}} \middle| \mathcal{F}^{(r)}(t \wedge \kappa^{(r)}(1)) \right].$$

Since $\eta^{(r)}(1)$ is an $\mathcal{F}^{(r)}(\kappa^{(r)}(1))$ -measurable random variable and $X^{(r)}$ has independent stationary increments,

$$\begin{aligned} Z^{(r)}(t) &= \mathbf{1}_{\{\kappa^{(r)}(1) \leq t\}} \mathbf{E}^* \left[\frac{e^{-\gamma r \eta^{(r)}(1)}}{\mathbf{E}^* e^{-\gamma r \eta^{(r)}(1)}} \middle| \mathcal{F}^{(r)}(\kappa^{(r)}(1)) \right] \\ &\quad + \mathbf{1}_{\{\kappa^{(r)}(1) > t\}} \mathbf{E}^* \left[\frac{e^{-\gamma r \eta^{(r)}(1)}}{\mathbf{E}^* e^{-\gamma r \eta^{(r)}(1)}} \middle| \mathcal{F}^{(r)}(t) \right] \\ &= \mathbf{1}_{\{\kappa^{(r)}(1) \leq t\}} \frac{e^{-\gamma r \eta^{(r)}(1)}}{\mathbf{E}^* e^{-\gamma r \eta^{(r)}(1)}} + \mathbf{1}_{\{\kappa^{(r)}(1) > t\}} \frac{\mathbf{E}^* e^{-\gamma r \eta^{(r)}(y)}|_{y=1-X^{(r)}(t)}}{\mathbf{E}^* e^{-\gamma r \eta^{(r)}(1)}}. \end{aligned}$$

Thus, on recalling (3.13) and taking into account that $\widehat{X}^{(r)}(t) = X^{(r)}(t)$ for $t < \kappa^{(r)}(1)$,

$$Z^{(r)}(t) = \mathbf{1}_{\{\kappa^{(r)}(1) \leq t\}} \frac{e^{-\gamma r \eta^{(r)}(1)}}{u^{(r)}(1)} + \mathbf{1}_{\{\kappa^{(r)}(1) > t\}} \frac{u^{(r)}(1 - \widehat{X}^{(r)}(t))}{u^{(r)}(1)}. \quad (3.14)$$

By Liptser and Shiryaev [21, p. 223] or Jacod and Shiryaev [18, p. 170], the $\widehat{\mathbf{F}}^{(r)}$ -predictable jump measure of $\widehat{X}^{(r)}$ under $\mathbf{Q}^{(r)}$ admits a version of the form

$$\widehat{\nu}^{(r)}(dt, dx) = Y^{(r)}(t, x) \widehat{\nu}^{(r)}(dt, dx), \quad (3.15)$$

where

$$Y^{(r)}(t, x) = \frac{\mathbf{1}_{\{Z^{(r)}(t-) > 0\}}}{Z^{(r)}(t-)} \mathbf{M}_{\widehat{\mu}^{(r)}}^{\mathbf{P}^*}(Z^{(r)} | \widetilde{\mathcal{P}})(t, x). \quad (3.16)$$

For the reader's convenience, we recall that $\widetilde{\mathcal{P}}$ denotes the σ -algebra on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ that is the product of the predictable σ -algebra on $\Omega \times \mathbb{R}_+$ associated with $\widehat{\mathbf{F}}^{(r)}$ and the Borel σ -algebra on \mathbb{R} , and $\mathbf{M}_{\widehat{\mu}^{(r)}}^{\mathbf{P}^*}$ denotes the measure on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ defined by the equality

$$\mathbf{M}_{\widehat{\mu}^{(r)}}^{\mathbf{P}^*} f = \mathbf{E}^* \int_0^\infty \int_{\mathbb{R}} f(\omega, t, x) \widehat{\mu}^{(r)}(dt, dx)$$

for $f \geq 0$, where $\widehat{\mu}^{(r)}$ is the jump measure of $\widehat{X}^{(r)}$, i.e.,

$$\widehat{\mu}([0, t], G) = \sum_{0 < s \leq t \wedge \tau} \mathbf{1}_{\{\Delta X(s) \in G \setminus \{0\}\}}. \quad (3.17)$$

Accordingly, $\mathbf{M}_{\widehat{\mu}^{(r)}}^{\mathbf{P}^*}(Z^{(r)} | \widetilde{\mathcal{P}})(t, x)$ is the conditional expectation of $Z^{(r)}$ with respect to $\widetilde{\mathcal{P}}$, i.e., it is a $\widetilde{\mathcal{P}}$ -measurable function $g(\omega, t, x)$ such that $\mathbf{M}_{\widehat{\mu}^{(r)}}^{\mathbf{P}^*} h Z^{(r)} = \mathbf{M}_{\widehat{\mu}^{(r)}}^{\mathbf{P}^*} h g$ for all nonnegative $\widetilde{\mathcal{P}}$ -measurable functions h .

In order to find $\mathbf{M}_{\hat{\mu}^{(r)}}^{\mathbf{P}^*}(Z^{(r)}|\tilde{\mathcal{P}})(t, x)$, we write by (3.8), (3.14), and the definition of $\eta^{(r)}(1)$ in (3.12), for a $\tilde{\mathcal{P}}$ -measurable nonnegative function h ,

$$\begin{aligned} \mathbf{M}_{\hat{\mu}^{(r)}}^{\mathbf{P}^*} h Z^{(r)} &= \mathbf{E}^* \int_0^\infty \int_{\mathbb{R}} h(t, x) \mathbf{1}_{\{\hat{X}^{(r)}(t-) < 1\}} \left(\mathbf{1}_{\{\hat{X}^{(r)}(t-) + x \geq 1\}} \frac{e^{-\gamma r(\hat{X}^{(r)}(t-) + x - 1)}}{u^{(r)}(1)} \right. \\ &\quad \left. + \mathbf{1}_{\{\hat{X}^{(r)}(t-) + x < 1\}} \frac{u^{(r)}(1 - \hat{X}^{(r)}(t-) - x)}{u^{(r)}(1)} \right) \hat{\mu}^{(r)}(dt, dx). \end{aligned}$$

The expression in parentheses is $\tilde{\mathcal{P}}$ -measurable. Hence,

$$\begin{aligned} \mathbf{M}_{\hat{\mu}^{(r)}}^{\mathbf{P}^*}(Z^{(r)}|\tilde{\mathcal{P}})(t, x) &= \frac{\mathbf{1}_{\{\hat{X}^{(r)}(t-) < 1\}}}{u^{(r)}(1)} \left(\mathbf{1}_{\{\hat{X}^{(r)}(t-) + x \geq 1\}} e^{-\gamma r(\hat{X}^{(r)}(t-) + x - 1)} \right. \\ &\quad \left. + \mathbf{1}_{\{\hat{X}^{(r)}(t-) + x < 1\}} u^{(r)}(1 - \hat{X}^{(r)}(t-) - x) \right). \end{aligned}$$

The expression for $\tilde{\nu}^{(r)}(dt, dx)$ in the statement of the lemma follows now by (3.9), (3.14)–(3.16). \square

In what follows, we work with the version of the $\hat{\mathbf{F}}^{(r)}$ -predictable jump measure of $\hat{X}^{(r)}$ given in the statement of Lemma 3.2. We study the properties of $\tilde{\nu}^{(r)}$. Let

$$\tilde{\nu}_1^{(r)}(dt, dx) = \mathbf{1}_{\{\hat{X}^{(r)}(t-) < 1\}} \mathbf{1}_{\{\hat{X}^{(r)}(t-) + x \geq 1\}} \frac{e^{-\gamma r(\hat{X}^{(r)}(t-) + x - 1)}}{u^{(r)}(1 - \hat{X}^{(r)}(t-))} \nu^{(r)}(dt, dx), \quad (3.18a)$$

$$\tilde{\nu}_2^{(r)}(dt, dx) = \mathbf{1}_{\{\hat{X}^{(r)}(t-) < 1\}} \mathbf{1}_{\{x < 0\}} \frac{u^{(r)}(1 - \hat{X}^{(r)}(t-) - x)}{u^{(r)}(1 - \hat{X}^{(r)}(t-))} \nu^{(r)}(dt, dx), \quad (3.18b)$$

$$\tilde{\nu}_3^{(r)}(dt, dx) = \mathbf{1}_{\{x > 0\}} \mathbf{1}_{\{\hat{X}^{(r)}(t-) + x < 1\}} \frac{u^{(r)}(1 - \hat{X}^{(r)}(t-) - x)}{u^{(r)}(1 - \hat{X}^{(r)}(t-))} \nu^{(r)}(dt, dx). \quad (3.18c)$$

Note that

$$\tilde{\nu}^{(r)} = \tilde{\nu}_1^{(r)} + \tilde{\nu}_2^{(r)} + \tilde{\nu}_3^{(r)}. \quad (3.19)$$

One can see that $\tilde{\nu}_1^{(r)}$ describes the intensity of jumps of $\hat{X}^{(r)}$ reaching level 1, that $\tilde{\nu}_2^{(r)}$ concerns the intensity of downward jumps, and that $\tilde{\nu}_3^{(r)}$ characterises the intensity of upward jumps. Not unexpectedly, the first two measures are inconsequential, as the next lemma shows.

Lemma 3.3. *Let the hypotheses of Theorem 2.2 hold. Then, for $i = 1, 2$,*

$$\lim_{r \rightarrow \infty} \sup_{\omega \in \Omega} \int_0^t \int_{\mathbb{R}} |x| \tilde{\nu}_i^{(r)}(ds, dx) = 0$$

if F is nonlattice and

$$\lim_{n \rightarrow \infty} \sup_{\omega \in \Omega} \int_0^t \int_{\mathbb{R}} |x| \tilde{\nu}_i^{(nh)}(ds, dx) = 0$$

if F is lattice with span h .

Proof. Suppose that F is nonlattice. We start with $i = 1$. Using (3.7) and integrating on x yields, on taking into account that $\hat{X}^{(r)}(s)$ is constant on the intervals $[i(1 - F(r)), (i + 1)(1 - F(r))]$,

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}} |x| \tilde{v}_1^{(r)}(ds, dx) \\
& \leq \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{\widehat{X}^{(r)}(s-) < 1, \widehat{X}^{(r)}(s-) + x \geq 1\}} \frac{e^{-\gamma r(\widehat{X}^{(r)}(s-) + x - 1)}}{u^{(r)}(1 - \widehat{X}^{(r)}(s-))} |x| v^{(r)}(ds, dx) \\
& = \int_0^t \frac{\mathbf{1}_{\{\widehat{X}^{(r)}(s-) < 1\}}}{u^{(r)}(1 - \widehat{X}^{(r)}(s-))} \int_{1 - \widehat{X}^{(r)}(s-)}^{\infty} x e^{-\gamma r(\widehat{X}^{(r)}(s-) + x - 1)} F(r dx) d[(1 - F(r))^{-1} s] \\
& = \frac{1}{1 - F(r)} \int_0^{\lfloor t/(1 - F(r)) \rfloor (1 - F(r))} \frac{\mathbf{1}_{\{\widehat{X}^{(r)}(s) < 1\}}}{u^{(r)}(1 - \widehat{X}^{(r)}(s))} \\
& \quad \times \int_{1 - \widehat{X}^{(r)}(s)}^{\infty} x e^{-\gamma r(\widehat{X}^{(r)}(s) + x - 1)} F(r dx) ds \\
& \leq \frac{t}{1 - F(r)} \sup_{y \in [0, 1]} \frac{1}{u^{(r)}(y)} \int_y^{\infty} x e^{-\gamma r(x - y)} F(r dx). \tag{3.20}
\end{aligned}$$

For $y \in [0, 1]$ and $A \in (0, r)$, on recalling (3.13), employing a change of variables and integrating by parts,

$$\begin{aligned}
& \frac{1}{(1 - F(r))u^{(r)}(y)} \int_y^{\infty} x e^{-\gamma r(x - y)} F(r dx) \\
& = \frac{1}{r(1 - F(r))\mathbf{E}^* e^{-\gamma \chi^{(ry)}}} \int_{ry}^{\infty} x e^{-\gamma(x - ry)} F(dx) \\
& \leq \mathbf{1}_{\{ry \leq A\}} \frac{1}{r(1 - F(r)) \inf_{z \leq A} \mathbf{E}^* e^{-\gamma \chi^{(z)}}} \sup_{z \leq A} \int_A^{\infty} x e^{-\gamma(x - z)} F(dx) \\
& \quad + \mathbf{1}_{\{ry > A\}} \frac{1}{ry(1 - F(ry))\mathbf{E}^* e^{-\gamma \chi^{(ry)}}} \frac{y(1 - F(ry))}{1 - F(r)} \gamma \int_0^{\infty} e^{-\gamma z} \int_{ry}^{z + ry} x F(dx) dz. \tag{3.21}
\end{aligned}$$

We first work with the second term on the rightmost side. By the first assertion of part 2 of Lemma A.2, given arbitrary $\varepsilon > 0$, $\sup_{y \in [A/r, 1]} y(1 - F(ry))/(1 - F(r)) \leq 1 + \varepsilon$ for all A and r large enough. By Lemma A.1, $(ry)(1 - F(ry))\mathbf{E}^* \exp(-\gamma \chi^{(ry)})$ converges to a positive limit as $ry \rightarrow \infty$.

Next, for $u > 0$,

$$\int_u^{z+u} x F(dx) \leq (z + u)(F(z + u) - F(u)). \tag{3.22}$$

Condition 2 of Theorem 2.2 implies that, for all $a > B$, $\lim_{x \rightarrow \infty} x(F(x + a) - F(x)) = 0$, so

$$\lim_{u \rightarrow \infty} (z + u)(F(z + u) - F(u)) = 0.$$

Also, for $u \geq 1$,

$$\begin{aligned}
e^{-\gamma z}(z + u)(F(z + u) - F(u)) & \leq e^{-\gamma z} z + e^{-\gamma z} u(F(z + u) - F(u)) \mathbf{1}_{\{z \leq 2 \ln u / \gamma\}} \\
& \quad + e^{-\gamma z} u \mathbf{1}_{\{z > 2 \ln u / \gamma\}} \\
& \leq e^{-\gamma z} z + e^{-\gamma z} u \left(F\left(\frac{2 \ln u}{\gamma} + u\right) - F(u) \right) + e^{-\gamma z/2}.
\end{aligned}$$

Another application of condition 2 shows that, for all $a > 0$, $\lim_{x \rightarrow \infty} x(F(x + a \ln x) - F(x)) = 0$. Thus, the rightmost side of the latter display is bounded from above by a function of z with a finite $(0, \infty)$ -integral with respect to Lebesgue measure. Hence, by the dominated convergence theorem,

$$\lim_{u \rightarrow \infty} \gamma \int_0^\infty e^{-\gamma z} \int_u^{u+z} x F(dx) dz = 0.$$

Thus, given arbitrary $\varepsilon > 0$, if A is large enough, then the second term on the rightmost side of (3.21) is less than ε . Since the first term on the rightmost side of (3.21) converges to zero as $r \rightarrow \infty$ for fixed A , we conclude that

$$\lim_{r \rightarrow \infty} \sup_{y \in [0, 1]} \frac{1}{(1 - F(r))u^{(r)}(y)} \int_y^\infty x e^{-\gamma r(x-y)} F(r dx) = 0.$$

By (3.20), $\lim_{r \rightarrow \infty} \sup_{\omega \in \Omega} \int_0^t \int_{\mathbb{R}} |x| \tilde{v}_1^{(r)}(ds, dx) = 0$.

Let $i = 2$. By analogy with (3.20) and (3.21) and by the bound $u^{(r)}(y) \leq 1$, we have that for $A > 0$,

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}} |x| \tilde{v}_2^{(r)}(ds, dx) \\ & \leq \frac{1}{1 - F(r)} \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{\hat{X}^{(r)}(s) < 1\}} x^- \frac{u^{(r)}(1 - \hat{X}^{(r)}(s) - x)}{u^{(r)}(1 - \hat{X}^{(r)}(s))} F(r dx) ds \\ & \leq \frac{1}{r(1 - F(r)) \inf_{z \leq A} \mathbf{E}^* e^{-\gamma \chi(z)}} \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{\hat{X}^{(r)}(s) < 1\}} \mathbf{1}_{\{r(1 - \hat{X}^{(r)}(s)) \leq A\}} x^- F(dx) ds \\ & \quad + \frac{1}{r(1 - F(r)) \inf_{z \geq A} \mathbf{E}^* \exp(-\gamma \chi(z))} \int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{\hat{X}^{(r)}(s) < 1\}} \mathbf{1}_{\{r(1 - \hat{X}^{(r)}(s)) > A\}} x^- F(dx) ds. \end{aligned}$$

Thus, by Lemma A.1, given $\varepsilon > 0$, for all A large enough and all $r > A$,

$$\int_0^t \int_{\mathbb{R}} |x| \tilde{v}_2^{(r)}(ds, dx) \leq \frac{t}{r(1 - F(r))} \left(\frac{1}{\inf_{z \leq A} \mathbf{E}^* e^{-\gamma \chi(z)}} + 1 + \varepsilon \right) \int_{\mathbb{R}} x^- F(dx).$$

Since $\int_{\mathbb{R}} x^- F(dx) = \int_{-\infty}^0 (-x) \exp(\gamma x) \hat{F}(dx) < \infty$, the function $r^\alpha(1 - F(r))$ is slowly varying at infinity, and $\alpha < 1$, we conclude that $\sup_{\omega \in \Omega} \int_0^t \int_{\mathbb{R}} |x| \tilde{v}_1^{(r)}(ds, dx)$ converges to 0 as $r \rightarrow \infty$.

The assertion of the lemma for nonlattice F has been proved. The proof for the case where F is lattice with span h proceeds analogously. The only changes consist in assuming that $r = nh$, that y is of the form k/n for $k = 1, 2, \dots, n$, that z is of the form kh for $k = 1, 2, \dots$, and that $n \rightarrow \infty$, and in using the part of Lemma A.1 that concerns the lattice case. \square

We are now in a position to verify condition 3 of Lemma 2.1.

Lemma 3.4. *Let the hypotheses of Theorem 2.2 hold. Let K be defined by (3.1), and $g(x)$ be a bounded continuous function such that $|g(x)| \leq M|x|$ for some $M > 0$. Take any $t > 0$ and $\varepsilon > 0$.*

If F is nonlattice, then

$$\lim_{r \rightarrow \infty} \mathbf{Q}^{(r)} \left(\left| \int_0^t \int_{\mathbb{R}} g(x) \tilde{v}^{(r)}(ds, dx) - \int_0^t \int_{\mathbb{R}} g(x) K(\widehat{X}^{(r)}(s); dx) ds \right| > \varepsilon \right) = 0.$$

If, instead, F is lattice with span h , then

$$\lim_{n \rightarrow \infty} \mathbf{Q}^{(nh)} \left(\left| \int_0^t \int_{\mathbb{R}} g(x) \tilde{v}^{(nh)}(ds, dx) - \int_0^t \int_{\mathbb{R}} g(x) K(\widehat{X}^{(nh)}(s); dx) ds \right| > \varepsilon \right) = 0.$$

Proof. Suppose F is nonlattice. We will prove that $\lim_{r \rightarrow \infty} \sup_{\omega \in \Omega} M_2 = 0$ where

$$M_2 = \left| \int_0^t \int_{\mathbb{R}} g(x) \tilde{v}^{(r)}(ds, dx) - \int_0^t \int_{\mathbb{R}} g(x) K(\widehat{X}^{(r)}(s); dx) ds \right|. \quad (3.23)$$

By Lemma 3.3,

$$\lim_{r \rightarrow \infty} \sup_{\omega \in \Omega} \int_0^t \int_{\mathbb{R}} |g(x)| \tilde{v}_i^{(r)}(ds, dx) = 0, \quad i = 1, 2. \quad (3.24)$$

We turn our attention to $\tilde{v}_3^{(r)}$. By (3.18c),

$$\tilde{v}_3^{(r)}(dt, dx) = \mathbf{1}_{\{0 < x < 1 - \widehat{X}^{(r)}(t-)\}} \frac{u^{(r)}(1 - \widehat{X}^{(r)}(t-) - x)}{u^{(r)}(1 - \widehat{X}^{(r)}(t-))} v^{(r)}(dt, dx).$$

Let $h(t, r) = \lfloor t/(1 - F(r)) \rfloor (1 - F(r))$. By (3.7),

$$\int_0^t \int_{\mathbb{R}} g(x) \tilde{v}_3^{(r)}(ds, dx) = \int_0^{h(t, r)} \int_0^1 V_1^{(r)}(\widehat{X}^{(r)}(s), x) \frac{F(r(1 - \widehat{X}^{(r)}(s)) dx)}{1 - F(r)} ds$$

where

$$V_1^{(r)}(z, x) = g((1 - z)x) \mathbf{1}_{\{z < 1\}} \frac{u^{(r)}((1 - z)(1 - x))}{u^{(r)}(1 - z)}.$$

Let also

$$V_2(z, x) = g((1 - z)x) \mathbf{1}_{\{z < 1\}} (1 - z)^{-\alpha} (1 - x)^{\alpha-1} \alpha x^{-\alpha-1}$$

and

$$V_3(z, x) = g((1 - z)x) \mathbf{1}_{\{z < 1\}} (1 - x)^{\alpha-1}.$$

Then, for $\eta \in (0, 1)$, recalling (3.2),

$$\begin{aligned} M_2 &\leq \int_0^t \int_{1-\eta}^1 |V_2(\widehat{X}^{(r)}(s), x)| dx ds \\ &\quad + \int_0^{h(t, r)} \int_{1-\eta}^1 |V_1^{(r)}(\widehat{X}^{(r)}(s), x)| \frac{F(r(1 - \widehat{X}^{(r)}(s)) dx)}{1 - F(r)} ds \\ &\quad + \int_{h(t, r)}^t \int_0^{1-\eta} |V_3(\widehat{X}^{(r)}(s), x)| \frac{F(r(1 - \widehat{X}^{(r)}(s)) dx)}{1 - F(r)} ds \\ &\quad + \int_0^{h(t, x)} \int_0^{1-\eta} |V_1^{(r)}(\widehat{X}^{(r)}(s), x) - V_3(\widehat{X}^{(r)}(s), x)| \frac{F(r(1 - \widehat{X}^{(r)}(s)) dx)}{1 - F(r)} ds \end{aligned}$$

$$+ \left| \int_0^t \int_0^{1-\eta} V_3(\widehat{X}^{(r)}(s), x), \frac{F(r(1 - \widehat{X}^{(r)}(s)) \, dx}{1 - F(r)} \, ds \right. \\ \left. - \int_0^t \int_0^{1-\eta} V_2(\widehat{X}^{(r)}(s), x) \, dx \, ds \right|.$$

We denote the terms on the right-hand side by I_1 , I_2 , I_3 , I_4 , and I_5 , respectively. We treat them successively. We have

$$I_1 \leq M \int_0^t (1 - \widehat{X}^{(r)}(s))^{1-\alpha} \mathbf{1}_{\{\widehat{X}^{(r)}(s) < 1\}} \int_{1-\eta}^1 (1-x)^{\alpha-1} \alpha x^{-\alpha} \, dx \, ds \\ \leq Mt \int_{1-\eta}^1 (1-x)^{\alpha-1} \alpha x^{-\alpha} \, dx,$$

so

$$\lim_{\eta \rightarrow 0} \limsup_{r \rightarrow \infty} \sup_{\omega \in \Omega} I_1 = 0. \quad (3.25)$$

We now prove that

$$\lim_{\eta \rightarrow 0} \limsup_{r \rightarrow \infty} \sup_{\omega \in \Omega} I_2 = 0. \quad (3.26)$$

By Lemma A.1, $ry(1 - F(ry))u^{(r)}(y) \rightarrow C_0 > 0$ as $ry \rightarrow \infty$ and $\sup_{ry \geq 0} ry(1 - F(ry))u^{(r)}(y) < \infty$. On denoting the latter supremum by N , we have that, for ry large enough and for $x \in (0, 1)$,

$$\frac{u^{(r)}(y(1-x))}{u^{(r)}(y)} \leq \frac{2N}{C_0} \frac{1 - F(ry)}{(1-x)(1 - F(ry(1-x)))}. \quad (3.27)$$

Since $u^{(r)}(y(1-x)) \leq 1$, we can write, for large enough $A \in (0, r)$ and for $A' \in (B, A)$,

$$I_2 \leq M \int_0^t \int_{1-\eta}^1 \mathbf{1}_{\{\widehat{X}^{(r)}(s) < 1\}} x (1 - \widehat{X}^{(r)}(s)) \\ \times \frac{u^{(r)}((1 - \widehat{X}^{(r)}(s))(1-x))}{u^{(r)}(1 - \widehat{X}^{(r)}(s))} \frac{F(r(1 - \widehat{X}^{(r)}(s)) \, dx}{1 - F(r)} \, ds \\ \leq Mt \sup_{y \in [0, 1]} \int_{1-\eta}^1 xy \frac{u^{(r)}(y(1-x))}{u^{(r)}(y)} \frac{F(ry \, dx)}{1 - F(r)} \, ds \\ \leq Mt \frac{A}{r(1 - F(r))} \frac{F(A)}{\inf_{z \leq A} \mathbf{E}^* \exp(-\gamma \chi^{(z)})} \\ + Mt \sup_{y \in (A/r, 1]} \int_{1-\eta}^1 \mathbf{1}_{\{ry(1-x) \leq A'\}} \frac{xy}{u^{(r)}(y)} \frac{F(ry \, dx)}{1 - F(r)} \\ + Mt \frac{2N}{C_0} \sup_{y \in (A/r, 1]} \int_{1-\eta}^1 \mathbf{1}_{\{ry(1-x) > A'\}} xy \frac{1 - F(ry)}{(1-x)(1 - F(ry(1-x)))} \frac{F(ry \, dx)}{1 - F(r)}. \quad (3.28)$$

The first term on the rightmost side of (3.28) tends to zero as $r \rightarrow \infty$. By the fact that $1 - x \leq A'/(ry)$ when the integrand of the second term is positive and by (3.27), that term

is not greater than

$$Mt \frac{2}{C_0} \sup_{y \in (A/r, 1]} \frac{y(1 - F(r y))}{1 - F(r)} \sup_{v > A} v(F(v) - F(v - A')).$$

By the first assertion of part 2 of [Lemma A.2](#), the supremum in the middle is bounded from above for all large enough A and r . By condition 2 of [Theorem 2.2](#), the other supremum tends to zero as $A \rightarrow \infty$. Thus, the second term on the rightmost side of (3.28) tends to zero as $A \rightarrow \infty$ and $r \rightarrow \infty$.

Let us consider the third term on the rightmost side of (3.28). Pick arbitrary $\varepsilon \in (0, \alpha \wedge (1 - \alpha))$. The function $\ell(x) = x^\alpha(1 - F(x))$ is slowly varying at infinity, so by part 1 of [Lemma A.2](#), for A' large enough and for $ry(1 - x) > A'$,

$$\frac{1 - F(r y)}{1 - F(r y(1 - x))} = (1 - x)^\alpha \frac{\ell(r y)}{\ell(r y(1 - x))} \leq (1 + \varepsilon)(1 - x)^{\alpha - \varepsilon}.$$

We have

$$\begin{aligned} & \sup_{y \in (A/r, 1]} \int_{1-\eta}^1 \mathbf{1}_{\{ry(1-x) > A'\}} x y \frac{1 - F(r y)}{(1 - x)(1 - F(r y(1 - x)))} \frac{F(r y) dx}{1 - F(r)} \\ & \leq (1 + \varepsilon) \sup_{y \in (A/r, 1]} \frac{y(1 - F(r y))}{1 - F(r)} \sup_{z > A} \frac{1}{1 - F(z)} \int_{1-\eta}^{1-A'/z} (1 - x)^{\alpha - \varepsilon - 1} F(z dx). \end{aligned} \quad (3.29)$$

By the first assertion of part 2 of [Lemma A.2](#), the first supremum on the right of (3.29) is bounded from above uniformly in all A (and r) large enough.

For the second supremum, the integration by parts yields

$$\begin{aligned} & \frac{1}{1 - F(z)} \int_{1-\eta}^{1-A'/z} (1 - x)^{\alpha - \varepsilon - 1} F(z dx) \\ & = (1 + \varepsilon - \alpha) \int_{1-\eta}^{1-A'/z} (1 - x)^{\alpha - \varepsilon - 2} \frac{F(z) - F(zx)}{1 - F(z)} dx \\ & \quad + \eta^{\alpha - \varepsilon - 1} \frac{F(z - A') - F(z(1 - \eta))}{1 - F(z)}. \end{aligned}$$

By condition 2 of [Theorem 2.2](#), we have that, for z large enough, for η small enough, and for $x \in [1 - \eta, 1 - A'/z]$,

$$\frac{F(z) - F(zx)}{1 - F(z)} \leq C(1 - x).$$

Therefore,

$$\int_{1-\eta}^{1-A'/z} (1 - x)^{\alpha - 2 - \varepsilon} \frac{F(z) - F(zx)}{1 - F(z)} dx \leq C \int_{1-\eta}^1 (1 - x)^{\alpha - \varepsilon - 1} dx.$$

It follows that

$$\lim_{\eta \rightarrow 0} \lim_{A \rightarrow \infty} \sup_{z > A} \frac{1}{1 - F(z)} \int_{1-\eta}^1 (1 - x)^{\alpha - \varepsilon - 1} F(z dx) = 0.$$

By (3.29), we conclude that the third term on the rightmost side of (3.28) tends to zero as $A \rightarrow \infty$ and $\eta \rightarrow 0$. By letting successively $r \rightarrow \infty$, $A \rightarrow \infty$, and $\eta \rightarrow 0$ in (3.28) and picking large enough A' , we obtain (3.26).

Now consider I_3 . We have, by a change of variables,

$$\begin{aligned} I_3 &\leq M \int_{h(t,r)}^t \int_0^{1-\eta} (1 - \widehat{X}^{(r)}(s)) x \mathbf{1}_{\{\widehat{X}^{(r)}(s) < 1\}} (1-x)^{\alpha-1} \frac{F(r(1 - \widehat{X}^{(r)}(s))) dx}{1 - F(r)} ds \\ &\leq M(t - h(t, r)) \frac{\eta^{\alpha-1}}{1 - F(r)} \int_0^{1-\eta} x F(r) dx. \end{aligned}$$

By the second assertion of part 2 of [Lemma A.2](#),

$$\limsup_{r \rightarrow \infty} \frac{1}{1 - F(r)} \int_0^{1-\eta} x F(r) dx \leq \frac{1}{1 - \alpha}. \quad (3.30)$$

Since $t - h(t, r) \rightarrow 0$ as $r \rightarrow \infty$, it follows that

$$\lim_{r \rightarrow \infty} \sup_{\omega \in \Omega} I_3 = 0. \quad (3.31)$$

For I_4 , we reuse the earlier arguments to obtain that, for $A > 0$,

$$\begin{aligned} I_4 &\leq Mt \frac{A}{r(1 - F(r))} \left(\frac{1}{\inf_{z \leq A} \mathbf{E}^* \exp(-\gamma \chi^{(z)})} + \eta^{\alpha-1} \right) F(A) + Mt \eta^{-1} \sup_{z > A} \sup_{x \in [0, 1-\eta]} \\ &\quad \times \left| \frac{(1-x) \mathbf{E}^* \exp(-\gamma \chi^{(z(1-x))})}{\mathbf{E}^* \exp(-\gamma \chi^{(z)})} - (1-x)^\alpha \right| \sup_{y \in [0, 1]} \frac{y}{1 - F(r)} \int_0^{1-\eta} x F(r y) dx. \end{aligned} \quad (3.32)$$

Take an arbitrary $\varepsilon \in (0, C_0)$. By [Lemma A.1](#), $\mathbf{E}^* \exp(-\gamma \chi^{(z)}) \sim C_0/(z(1 - F(z)))$ as $z \rightarrow \infty$. Therefore, for all z large enough and uniformly over $x \in [0, 1 - \eta]$, we have

$$\begin{aligned} \left| \frac{(1-x) \mathbf{E}^* \exp(-\gamma \chi^{(z(1-x))})}{\mathbf{E}^* \exp(-\gamma \chi^{(z)})} - (1-x)^\alpha \right| &\leq \varepsilon \frac{1 - F(z)}{1 - F(z(1-x))} \\ &\quad + \left| \frac{1 - F(z)}{1 - F(z(1-x))} - (1-x)^\alpha \right|. \end{aligned}$$

The first summand on the right is not greater than ε . The second summand tends to zero as $z \rightarrow \infty$ uniformly over $x \in [0, 1 - \eta]$ by the Uniform Convergence Theorem for slowly varying functions. Thus,

$$\lim_{z \rightarrow \infty} \sup_{x \in [0, 1-\eta]} \left| \frac{(1-x) \mathbf{E}^* \exp(-\gamma \chi^{(z(1-x))})}{\mathbf{E}^* \exp(-\gamma \chi^{(z)})} - (1-x)^\alpha \right| = 0.$$

Since also (3.30) holds, the second summand on the right of (3.32) tends to zero as A and r tend to infinity. Since the first summand tends to zero as $r \rightarrow \infty$, we arrive at the convergence

$$\lim_{r \rightarrow \infty} \sup_{\omega \in \Omega} I_4 = 0. \quad (3.33)$$

We now consider I_5 . For $\delta \in (0, 1 - \eta)$, we employ a change of variables in the inside integral of the first term on the left-hand side of the first inequality and get

$$I_5 \leq \int_0^t \int_0^{1-\eta} |V_3(\widehat{X}^{(r)}(s), x)| \frac{F(r(1 - \widehat{X}^{(r)}(s))) dx}{1 - F(r)} \mathbf{1}_{\{\widehat{X}^{(r)}(s) \in (1-\delta, 1)\}} ds$$

$$\begin{aligned}
& + \int_0^t \int_0^{1-\eta} |V_2(\widehat{X}^{(r)}(s), x)| \mathbf{1}_{\{\widehat{X}^{(r)}(s) \in (1-\delta, 1)\}} dx ds \\
& + \left| \int_0^t \int_0^{1-\eta} V_3(\widehat{X}^{(r)}(s), x) \frac{F(r(1-\widehat{X}^{(r)}(s)) dx}{1-F(r)} \mathbf{1}_{\{\widehat{X}^{(r)}(s) \leq 1-\delta\}} ds \right. \\
& \left. - \int_0^t \int_0^{1-\eta} V_2(\widehat{X}^{(r)}(s), x) \mathbf{1}_{\{\widehat{X}^{(r)}(s) \leq 1-\delta\}} dx ds \right| \\
& \leq M t \eta^{\alpha-1} \left(\int_0^{\delta(1-\eta)} x \frac{F(r dx)}{1-F(r)} + \delta^{1-\alpha} \int_0^{1-\eta} \alpha x^{-\alpha} dx \right) \\
& + t \sup_{y \in [\delta, 1]} \left| \int_0^{1-\eta} g(yx)(1-x)^{\alpha-1} \frac{F(ry dx)}{1-F(r)} \right. \\
& \left. - \int_0^{1-\eta} g(yx)y^{-\alpha}(1-x)^{\alpha-1} \alpha x^{-\alpha-1} dx \right|. \tag{3.34}
\end{aligned}$$

By the second assertion of part 2 of [Lemma A.2](#), we conclude that the term in the parentheses on the rightmost side of (3.34) tends to zero as $r \rightarrow \infty$ and $\delta \rightarrow 0$, i.e.,

$$\lim_{\delta \rightarrow 0} \limsup_{r \rightarrow \infty} M t \eta^{\alpha-1} \left(\int_0^{\delta(1-\eta)} x \frac{F(r dx)}{1-F(r)} + \delta^{1-\alpha} \int_0^{1-\eta} \alpha x^{-\alpha} dx \right) = 0. \tag{3.35}$$

The other summand is not greater than

$$\begin{aligned}
& t \sup_{y \in [\delta, 1]} \int_0^{1-\eta} |g(yx)| \left| \frac{1-F(ry)}{1-F(r)} - y^{-\alpha} \right| (1-x)^{\alpha-1} \frac{F(ry dx)}{1-F(ry)} \\
& + t \sup_{z \geq r\delta} \sup_{y \in [\delta, 1]} W(0, y, z) \\
& \leq M t \eta^{\alpha-1} \sup_{y \in [\delta, 1]} \left| \frac{1-F(ry)}{1-F(r)} - y^{-\alpha} \right| \sup_{y \in [\delta, 1]} \int_0^{1-\eta} yx \frac{F(ry dx)}{1-F(ry)} \\
& + M t \eta^{\alpha-1} \sup_{z \geq r\delta} \int_0^{\delta} x \frac{F(z dx)}{1-F(z)} + M t \eta^{\alpha-1} \int_0^{\delta} \alpha x^{-\alpha} dx \\
& + t \sup_{z \geq r\delta} \sup_{y \in [\delta, 1]} W(\delta, y, z) \tag{3.36}
\end{aligned}$$

where, for $0 \leq a \leq 1$,

$$\begin{aligned}
W(a, y, z) = & \left| \int_a^{1-\eta} g(yx)y^{-\alpha}(1-x)^{\alpha-1} \frac{F(z dx)}{1-F(z)} \right. \\
& \left. - \int_a^{1-\eta} g(yx)y^{-\alpha}(1-x)^{\alpha-1} \alpha x^{-\alpha-1} dx \right|.
\end{aligned}$$

Since $1-F(y)$ is regularly varying with index $-\alpha$,

$$\sup_{y \in [\delta, 1]} \left| \frac{1-F(ry)}{1-F(r)} - y^{-\alpha} \right| \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

Also,

$$\sup_{y \in [\delta, 1]} \int_0^{1-\eta} yx \frac{F(ry \, dx)}{1 - F(ry)} \leq \int_0^{1-\eta} x \frac{F(r \, dx)}{1 - F(r)}.$$

By the second assertion of part 2 of Lemma A.2, we get

$$\lim_{r \rightarrow \infty} Mt\eta^{\alpha-1} \sup_{y \in [\delta, 1]} \left| \frac{1 - F(ry)}{1 - F(r)} - y^{-\alpha} \right| \sup_{y \in [\delta, 1]} \int_0^{1-\eta} yx \frac{F(ry \, dx)}{1 - F(ry)} = 0. \quad (3.37)$$

Another application of the second assertion of part 2 of Lemma A.2 yields

$$\limsup_{z \rightarrow \infty} \int_0^\delta x \frac{F(z \, dx)}{1 - F(z)} \leq \frac{\delta^{1-\alpha}}{1-\alpha},$$

so

$$\lim_{\delta \rightarrow 0} \lim_{r \rightarrow \infty} \sup_{z \geq r\delta} \int_0^\delta x \frac{F(z \, dx)}{1 - F(z)} = 0. \quad (3.38)$$

Also, clearly,

$$\lim_{\delta \rightarrow 0} Mt\eta^{\alpha-1} \int_0^\delta \alpha x^{-\alpha} \, dx = 0. \quad (3.39)$$

Now consider the last term on the right-hand side of (3.36). Since $1 - F$ is regularly varying at infinity with index $-\alpha$, the $F(z \, dx)/(1 - F(z))$, considered as measures on $[\delta, 1 - \eta]$, weakly converge as $z \rightarrow \infty$ to $\alpha x^{-\alpha-1} \, dx$. Since $g(x)$ is a continuous function, the functions $(g(yx)y^{-\alpha}(1-x)^{\alpha-1}, x \in [\delta, 1 - \eta])$ are uniformly bounded and equicontinuous over $y \in [\delta, 1]$. Therefore,

$$\lim_{z \rightarrow \infty} \sup_{y \in [\delta, 1]} W(\delta, y.z) = 0. \quad (3.40)$$

Thus, by (3.34)–(3.40),

$$\lim_{r \rightarrow \infty} \sup_{\omega \in \Omega} I_5 = 0. \quad (3.41)$$

Putting together (3.25), (3.26), (3.31), (3.33) and (3.41), we conclude that

$$\lim_{r \rightarrow \infty} \sup_{\omega \in \Omega} \left| \int_0^t \int_{\mathbb{R}} g(x) \tilde{v}_3^{(r)}(ds, dx) - \int_0^t \int_{\mathbb{R}} g(x) K(\widehat{X}^{(r)}(s); dx) \, ds \right| = 0.$$

On recalling (3.19) and (3.24), we arrive at (3.23).

The assertion of the lemma for nonlattice F has been proved. If F is lattice with span h , the proof proceeds analogously provided one assumes that $r = nh$, where $n \in \mathbb{N}$, that the y 's are of the form k/n for $k = 1, 2, \dots, n$, that the z 's are of the form kh for $k \in \mathbb{N}$, and that $n \rightarrow \infty$. \square

Proof of Theorem 2.2. We apply Lemma 2.1 to the processes $\widehat{X}^{(r)}$ under the measures $\mathbf{Q}^{(r)}$ if F is a nonlattice distribution and to the processes $\widehat{X}^{(nh)}$ under the measures $\mathbf{Q}^{(nh)}$ if F is a lattice distribution with span h . Conditions 1 and 2 of the lemma follow by Lemma 3.1. Condition 3 holds by Lemma 3.4. Condition 4 is obviously met. The validity of condition 5 had been established earlier. \square

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Appendix. Auxiliary results

A.1. A proof of two results by Korshunov [19]

The result formulated in this subsection is contained in Theorems 1 and 2 in [19]. However, some details of the proof are omitted there (especially for the lattice case), so we fill in the gaps in our proof below. We recall that \mathbf{E}^* denotes expectation with respect to measure \mathbf{P}^* defined by (1.4).

Lemma A.1. *Let condition 1 of Theorem 2.2 hold.*

1. *If, in addition, F is nonlattice, then, for some $C_0 > 0$, as $r \rightarrow \infty$,*

$$\mathbf{E}^* e^{-\gamma \chi^{(r)}} \sim \frac{C_0}{r(1 - F(r))}. \quad (\text{A.1})$$

2. *If, instead, F is lattice with span h , then, for some $C'_0 > 0$, as $n \rightarrow \infty$,*

$$\mathbf{E}^* e^{-\gamma \chi^{(nh)}} \sim \frac{C'_0}{nh(1 - F(nh))}.$$

Remark A.1. Note that, by Karamata's theorem, the coefficients C_3 in (1.3) and C_0 in (A.1) are related by $C_0 = C_3(1 - \alpha)/\gamma$.

Proof. We introduce strict ascending ladder indices T_1, T_2, \dots by letting $T_0 = 0$ and $T_n = \min\{k > T_{n-1} : S_k - S_{T_{n-1}} > 0\}$ for $n \in \mathbb{N}$. Let $\zeta_k = S_{T_k} - S_{T_{k-1}}$ for $k \in \mathbb{N}$. Under \mathbf{P}^* , the ζ_k are a.s. finite and i.i.d., and $\mathbf{E}^* T_1 < \infty$, see [1, VIII.2]. We let F_+ denote the common distribution function of the ζ_k (under \mathbf{P}^*). Adapting the argument of the proof of Lemma 2 in [19], we write, for $x \geq 0$,

$$\frac{1 - F_+(x)}{1 - F(x)} = \int_{-\infty}^0 \frac{1 - F(x - u)}{1 - F(x)} H(du),$$

where $H(u) = \mathbf{1}_{\{u=0\}} + \sum_{k=1}^{\infty} \mathbf{P}^*(S_1 \leq 0, S_2 \leq 0, \dots, S_k \leq 0, S_k \leq u)$ for $u \leq 0$. Under condition 1 of Theorem 2.2, $\lim_{x \rightarrow \infty} (1 - F(x - u))/(1 - F(x)) = 1$, so by Lebesgue's bounded convergence theorem,

$$\lim_{x \rightarrow \infty} \frac{1 - F_+(x)}{1 - F(x)} = H(0).$$

Since $H(0) = 1 + \sum_{k=1}^{\infty} \mathbf{P}^*(T_1 > k) = \mathbf{E}^* T_1$, we conclude that

$$1 - F_+(x) \sim (1 - F(x)) \mathbf{E}^* T_1 \quad \text{as } x \rightarrow \infty. \quad (\text{A.2})$$

Thus, $1 - F_+(x)$ is regularly varying at infinity with index $-\alpha$.

Since $\chi^{(r)}$ is the overshoot over level r of the random walk S_n , it is also the overshoot over r of the random walk with steps ζ_k . Denoting by $H_+(x)$ the corresponding renewal function, i.e., $H_+(x) = \mathbf{1}_{\{x \geq 0\}} + \sum_{n=1}^{\infty} \mathbf{P}^*(\sum_{k=1}^n \zeta_k \leq x)$, we have

$$\mathbf{E}^* e^{-\gamma \chi^{(r)}} = \int_{[0,r)} \int_{[r-x,\infty)} e^{-\gamma(y-(r-x))} F_+(dy) H_+(dx).$$

On introducing

$$z(x) = \int_{[x,\infty)} e^{-\gamma(y-x)} F_+(dy), \quad (\text{A.3})$$

we obtain that

$$\mathbf{E}^* e^{-\gamma \chi^{(r)}} = \int_{[0,r]} z(r-x) H_+(dx) - z(0) \Delta H_+(r). \quad (\text{A.4})$$

Note that $z(x) = O(1/x)$ as $x \rightarrow \infty$, which follows from the following calculations:

$$\begin{aligned} z(x) &= \int_{[x, x+\ln x/\gamma]} + \int_{(x+\ln x/\gamma, \infty)} \leq (F_+(x + \frac{\ln x}{\gamma}) - F_+(x-1)) + \frac{1}{x}, \\ F_+(x + \frac{\ln x}{\gamma}) - F_+(x-1) &\sim \frac{\alpha}{\gamma} \frac{(1 - F_+(x)) \ln x}{x}, \end{aligned}$$

where for the equivalence we used the fact that $(1 - F_+(x + \ln x/\gamma))/(1 - F_+(x-1)) \sim (1 + \ln x/(\gamma x))^{-\alpha}$ by the uniform convergence theorem for regularly varying functions; see [5, Theorem 1.5.2].

Suppose now that F is nonlattice. Then F_+ is nonlattice too; see [1, VIII.1]. By (A.3), the function $z(x)$ is directly Riemann integrable, as defined in [13, XI.1], and

$$\int_0^\infty z(x) dx = \int_0^\infty e^{-\gamma x} (1 - F_+(x)) dx.$$

Thus, by Theorem 3 of Erickson [11], as $r \rightarrow \infty$,

$$\int_{[0,r]} z(r-x) H_+(dx) \sim \left(\int_0^r (1 - F_+(x)) dx \right)^{-1} \frac{\sin \pi \alpha}{\pi(1-\alpha)} \int_0^\infty e^{-\gamma x} (1 - F_+(x)) dx.$$

In addition, by Theorem 1 of Erickson [11], as $r \rightarrow \infty$,

$$\Delta H_+(r) \int_0^r (1 - F_+(x)) dx \rightarrow 0.$$

If we also recall (A.2) and the fact that, according to Karamata's theorem (see Proposition 1.5.8 in [5]), $\int_0^r (1 - F_+(x)) dx \sim r(1 - F_+(r))/(1-\alpha)$, we obtain the asymptotic equivalence asserted in part 1 with

$$C_0 = \frac{1}{\mathbf{E}^* T_1} \frac{\sin \pi \alpha}{\pi} \int_0^\infty e^{-\gamma x} (1 - F_+(x)) dx.$$

For lattice distributions, we haven't been able to find in the literature an analogue of Erickson's Theorem 3. Therefore, we, in effect, deduce it from the local renewal theorem of Garsia and Lamperti [14] for our particular case by using the approach of Erickson [11]. As a matter of fact, we improve on Erickson's argument so that we can give a streamlined proof of his Theorem 3.

Let F be lattice with span h . Then F_+ is also lattice with span h . We can write for $\theta \in (0, 1)$ and suitable $A > 0$, on recalling that $z(x) = O(1/x)$ as $x \rightarrow \infty$,

$$\begin{aligned} \int_{[0, nh]} z(nh - x) H_+(dx) &= \int_{[0, \theta nh]} z(nh - x) H_+(dx) + \int_{(\theta nh, nh]} z(nh - x) H_+(dx) \\ &\leq \frac{A}{(1 - \theta)nh} H_+(\theta nh) + \int_{(\theta nh, nh]} z(nh - x) H_+(dx). \end{aligned} \quad (\text{A.5})$$

By the fact that the tail of F_+ is regularly varying at infinity with index $-\alpha$, we have (see [13, XIV.3] or [5, 8.6]) that $H_+(x) \sim (\sin \pi \alpha / \pi \alpha) (1 - F_+(x))^{-1}$ as $x \rightarrow \infty$, so

$$\lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} nh(1 - F_+(nh)) \frac{A}{(1 - \theta)nh} H_+(\theta nh) = 0. \quad (\text{A.6})$$

Next,

$$\begin{aligned} nh(1 - F_+(nh)) \int_{(\theta nh, nh]} z(nh - x) H_+(dx) \\ &= nh(1 - F_+(nh)) \sum_{k=\lfloor \theta n \rfloor}^n z((n - k)h) \Delta H_+(kh) \\ &= h \sum_{k=0}^{\infty} z(kh) \frac{n}{n - k} \frac{1 - F_+(nh)}{1 - F_+((n - k)h)} m(n - k) \mathbf{1}_{\{n - k \geq \lfloor \theta n \rfloor\}}, \end{aligned} \quad (\text{A.7})$$

where we used the notation $m(k) = k(1 - F_+(kh)) \Delta H_+(kh)$ for $k > 0$ and $m(k) = 0$ for $k \leq 0$. Since the ζ_i assume values kh , $k \in \mathbb{N}$, it follows, by Garsia and Lamperti [14], that

$$\lim_{k \rightarrow \infty} m(k) = \frac{\sin \pi \alpha}{\pi}. \quad (\text{A.8})$$

Hence,

$$\lim_{n \rightarrow \infty} \frac{n}{n - k} \frac{1 - F_+(nh)}{1 - F_+((n - k)h)} m(n - k) \mathbf{1}_{\{n - k \geq \lfloor \theta n \rfloor\}} = \frac{\sin \pi \alpha}{\pi}. \quad (\text{A.9})$$

Also by the uniform convergence theorem for regularly varying functions,

$$\lim_{n \rightarrow \infty} \sup_{k \leq n - \lfloor \theta n \rfloor} \left| \frac{1 - F_+(nh)}{1 - F_+((n - k)h)} - \frac{n^{-\alpha}}{(n - k)^{-\alpha}} \right| = 0.$$

Thus,

$$\limsup_{n \rightarrow \infty} \sup_{k=0, 1, 2, \dots} \frac{n}{n - k} \frac{1 - F_+(nh)}{1 - F_+((n - k)h)} m(n - k) \mathbf{1}_{\{n - k \geq \lfloor \theta n \rfloor\}} < \infty,$$

so by (A.9), Fatou's lemma, and Lebesgue's bounded convergence theorem,

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} z(kh) \frac{n}{n - k} \frac{1 - F_+(nh)}{1 - F_+((n - k)h)} m(n - k) \mathbf{1}_{\{n - k \geq \lfloor \theta n \rfloor\}} = \frac{\sin \pi \alpha}{\pi} \sum_{k=0}^{\infty} z(kh). \quad (\text{A.10})$$

Putting together (A.2), (A.5)–(A.7) and (A.10), we conclude that

$$\lim_{n \rightarrow \infty} nh(1 - F(nh)) \int_{[0, nh]} z(nh - x) H_+(dx) = \frac{1}{\mathbf{E}^* T_1} \frac{\sin \pi \alpha}{\pi} \sum_{k=0}^{\infty} z(kh) h. \quad (\text{A.11})$$

By Garsia and Lamperti [14], (A.2) and (A.8), $nh(1 - F(nh))z(0)\Delta H_+(nh) \rightarrow hz(0)(\mathbf{E}^*T_1)^{-1} \sin(\pi\alpha)/\pi$ as $n \rightarrow \infty$, so the second assertion of the lemma follows by (A.4) and (A.11) with

$$C'_0 = \frac{1}{\mathbf{E}^*T_1} \frac{\sin \pi\alpha}{\pi} \sum_{k=1}^{\infty} z(kh)h = \frac{1}{\mathbf{E}^*T_1} \frac{\sin \pi\alpha}{\pi} \sum_{k=0}^{\infty} e^{-\gamma kh} (1 - F_+(kh))h. \quad \square$$

A.2. Some properties of slowly and regularly varying functions

The following lemma comes in useful in the proof of Theorem 2.2. Note that the first part is Potter's theorem (see [5, Theorem 1.5.6]).

Lemma A.2. 1. Let $L(x)$ be a slowly varying at infinity function. Then, given an arbitrary $\varepsilon > 0$, there exists $x_0 > 0$ such that $L(x)/L(y) \leq (1+\varepsilon)((x/y) \vee (y/x))^\varepsilon$ for all $x \geq x_0$ and $y \geq x_0$.
2. If F is regularly varying at infinity with index $-\alpha$, where $\alpha \in (0, 1)$, then

$$\limsup_{\substack{r \rightarrow \infty \\ A \rightarrow \infty}} \sup_{y \in [A/r, 1]} \frac{y(1 - F(r y))}{1 - F(r)} \leq 1$$

and, for $y \in [0, 1]$,

$$\limsup_{r \rightarrow \infty} \frac{1}{1 - F(r)} \int_0^y x F(r dx) \leq \frac{y^{1-\alpha}}{1 - \alpha}.$$

Proof. In order to prove the first inequality of part 2, note that the function $\ell(x) = x^\alpha(1 - F(x))$ is slowly varying at infinity. Hence, for given arbitrary $\varepsilon \in (0, 1 - \alpha)$, we have by part 1 for all $y \in (0, 1]$ and r such that ry is large enough

$$\frac{y(1 - F(r y))}{1 - F(r)} = \frac{y^{1-\alpha} \ell(r y)}{\ell(r)} \leq y^{1-\alpha} (1 + \varepsilon) y^{-\varepsilon} \leq 1 + \varepsilon.$$

We prove the second inequality. Integration by parts yields

$$\int_0^y x F(r dx) = \int_0^y (F(r y) - F(r x)) dx.$$

On picking $A \in (0, ry)$ and partitioning the integration interval $[0, y]$ into two pieces $[0, A/r]$ and $(A/r, y]$, we have

$$\frac{1}{1 - F(r)} \int_0^y x F(r dx) \leq \frac{A}{r(1 - F(r))} + \frac{1}{1 - F(r)} \int_{A/r}^y (1 - F(r x)) dx. \quad (\text{A.12})$$

Let $\varepsilon \in (0, 1 - \alpha)$ be otherwise arbitrary. If A is large enough, then by part 1, on recalling that the function $x^\alpha(1 - F(x))$ is slowly varying at infinity and $y \leq 1$, we have for all $x \in [A/r, y]$,

$$\frac{1 - F(r x)}{1 - F(r)} \leq (1 + \varepsilon) x^{-\alpha - \varepsilon}.$$

Therefore, for these A and r ,

$$\frac{1}{1 - F(r)} \int_{A/r}^y (1 - F(r x)) dx \leq (1 + \varepsilon) \int_0^y x^{-\alpha - \varepsilon} dx = \frac{(1 + \varepsilon) y^{1 - \alpha - \varepsilon}}{1 - \alpha - \varepsilon}.$$

The required bound follows now by (A.12) and the fact that the first term on the right of (A.12) tends to zero as $r \rightarrow \infty$ (and as A is kept fixed large enough). \square

A.3. Convergence of the unconstrained random walk

We recall that X is a stable subordinator with Lévy measure $\alpha x^{-\alpha-1} dx$, $x > 0$, and the processes $X^{(r)}$ are defined by (2.3).

Lemma A.3. *Let the right-hand tail of F be regularly varying at infinity with index $-\alpha$, where $\alpha \in (0, 1)$. Then the $X^{(r)}$ under measure \mathbf{P}^* converge in distribution to X .*

The proof will use the following implication of Lemma 2.1. One can also apply Theorem VII.3.4 on p. 414 in [18] or Theorem 7.3.1 on p. 592 in [21].

Lemma A.4. *Consider a sequence $\Xi^{(n)} = (\Xi^{(n)}(t), t \in \mathbb{R}_+)$ of \mathbb{R} -valued pure jump semimartingales of locally bounded variation with independent increments defined on filtered probability spaces $(\Omega^{(n)}, \mathcal{G}^{(n)}, \mathbf{G}^{(n)}, \mathbf{P}^{(n)})$. Let Ξ be a Lévy process on a probability space $(\mathcal{Y}, \mathcal{G}, \mathbf{\Pi})$, with Lévy measure K such that $\int_{\mathbb{R}} 1 \wedge |x| K(dx) < \infty$. Suppose that $\Xi^{(n)}(0) = 0 \mathbf{P}^{(n)}$ -a.s. and $\Xi(0) = 0 \mathbf{\Pi}$ -a.s. If for an arbitrary \mathbb{R} -valued bounded continuous function $g(x)$, $x \in \mathbb{R}$, such that $|g(x)| \leq M|x|$ in a neighbourhood of the origin with some $M > 0$,*

$$\lim_{n \rightarrow \infty} \int_0^t \int_{\mathbb{R}} g(x) v^{(n)}(ds, dx) = t \int_{\mathbb{R}} g(x) K(dx),$$

where $v^{(n)}(dt, dx)$ denotes the $\mathbf{G}^{(n)}$ -predictable jump measure of $\Xi^{(n)}$, then the $\Xi^{(n)}$ converge in distribution to Ξ .

Proof of Lemma A.3. The processes $X^{(r)}$ under \mathbf{P}^* are pure jump semimartingales of locally bounded variation with independent increments. Their $\mathbf{F}^{(r)}$ -predictable jump measures under \mathbf{P}^* are given by (3.7), so for $g(x)$ as in the hypotheses,

$$\int_0^t \int_{\mathbb{R}} g(x) v^{(r)}(ds, dx) = \lfloor (1 - F(r))^{-1} t \rfloor \int_{\mathbb{R}} g(x) F(rdx). \quad (\text{A.13})$$

We may assume that, for a suitable M' , $|g(x)| \leq M'|x|$ for all x . We have on writing $1 - F(x) = x^{-\alpha} \ell(x)$, where ℓ is slowly varying at infinity,

$$\lfloor (1 - F(r))^{-1} t \rfloor \left| \int_{-\infty}^0 g(x) F(rdx) \right| \leq M' \frac{r^{\alpha-1}}{\ell(r)} t \int_{-\infty}^0 |x| F(dx).$$

Since $\int_{-\infty}^0 |x| F(dx) = \int_{-\infty}^0 |x| \exp(\gamma x) \widehat{F}(dx) < \infty$ and $\alpha < 1$, we obtain that

$$\lim_{r \rightarrow \infty} \lfloor (1 - F(r))^{-1} t \rfloor \int_{-\infty}^0 g(x) F(rdx) = 0. \quad (\text{A.14})$$

For $\varepsilon > 0$,

$$\lfloor (1 - F(r))^{-1} t \rfloor \int_0^\varepsilon |g(x)| F(rdx) \leq t M' \int_0^\varepsilon x \frac{F(rdx)}{1 - F(r)}.$$

Hence, by the second assertion of part 2 of Lemma A.2,

$$\lim_{\varepsilon \rightarrow 0} \limsup_{r \rightarrow \infty} \lfloor (1 - F(r))^{-1} t \rfloor \int_0^\varepsilon g(x) F(rdx) = 0. \quad (\text{A.15})$$

The hypotheses on F imply that, for $x > \varepsilon > 0$, $(F(rx) - F(r\varepsilon))/(1 - F(r)) \rightarrow \varepsilon^{-\alpha} - x^{-\alpha}$ as $r \rightarrow \infty$, so the $F(dx)/(1 - F(r))$, as measures on $[\varepsilon, \infty)$, weakly converge to the measure $\alpha x^{-\alpha-1} dx$. On recalling that $g(x)$ is a bounded and continuous function, we conclude that

$$\lim_{r \rightarrow \infty} [(1 - F(r))^{-1} t] \int_{\varepsilon}^{\infty} g(x) F(dx) = t \int_{\varepsilon}^{\infty} g(x) \alpha x^{-\alpha-1} dx. \quad (\text{A.16})$$

By (A.13)–(A.16),

$$\lim_{r \rightarrow \infty} \int_0^t \int_{\mathbb{R}} g(x) v^{(r)}(ds, dx) = t \int_0^{\infty} g(x) \alpha x^{-\alpha-1} dx,$$

which completes the proof by Lemma A.4. \square

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