

# Moment boundedness of linear stochastic delay differential equations with distributed delay<sup>☆</sup>

Zhen Wang<sup>a,b</sup>, Xiong Li<sup>a</sup>, Jinzhi Lei<sup>c,\*</sup>

<sup>a</sup> School of Mathematical Sciences, Beijing Normal University, Beijing 100875, PR China

<sup>b</sup> School of Mathematics, Hefei University of Technology, Hefei 230009, PR China

<sup>c</sup> Zhou Pei-Yuan Center for Applied Mathematics, Tsinghua University, Beijing 100084, PR China

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## Highlights

- This paper presents the characteristic function for moment stability of the linear stochastic delay differential equations with distributed delay.
- From the characteristic function, we obtain sufficient conditions for the second moment to be bounded or unbounded.

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## Abstract

This paper studies the moment boundedness of solutions of linear stochastic delay differential equations with distributed delay. For a linear stochastic delay differential equation, the first moment stability is known to be identical to that of the corresponding deterministic delay differential equation. However, boundedness of the second moment is complicated and depends on the stochastic terms. In this paper, the characteristic function of the equation is obtained through techniques of the Laplace transform. From the characteristic equation, sufficient conditions for the second moment to be bounded or unbounded are proposed.

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\* Corresponding author. Tel.: +86 10 62795156.

E-mail addresses: [jzlei@tsinghua.edu.cn](mailto:jzlei@tsinghua.edu.cn), [jinzhi.lei@gmail.com](mailto:jinzhi.lei@gmail.com) (J. Lei).

## 1. Introduction

Time delays are known to be involved in many processes in biology, chemistry, physics, engineering, *etc.*, and delay differential equations are widely used in describing these processes. Delay differential equations have been extensively developed in the past several decades (see [1,2,7]). Furthermore, stochastic perturbations are often introduced into these deterministic systems in order to describe the effects of fluctuations in the real environment, and thus yield stochastic delay differential equations. Mathematically, stochastic delay differential equations were first introduced by Itô and Nisio in the 1960s [8] in which the existence and uniqueness of the solutions have been investigated. In the last several decades, numerous studies have been developed toward the study of stochastic delay differential equations, such as stochastic stability, Lyapunov functional method, Lyapunov exponent, stochastic flow, invariant measure, invariant manifold, numerical approximation and attraction *etc.* (see [3–6,9,10,12,11,14,15,17,16,18, 20–22] and the references therein). However, many basic issues remain unsolved even for a simple linear equation with constant coefficients.

In this paper, we study the following linear stochastic differential equation with distributed delay

$$\begin{aligned} dx(t) = & \left( ax(t) + b \int_0^{+\infty} K(s)x(t-s)ds \right) dt \\ & + \left( \sigma_0 + \sigma_1 x(t) + \sigma_2 \int_0^{+\infty} K(s)x(t-s)ds \right) dW_t. \end{aligned} \quad (1.1)$$

Here  $a, b$  and  $\sigma_i$  ( $i = 0, 1, 2$ ) are constants,  $W_t$  is a one dimensional Wiener process, and  $K(s)$  represents the density function of the delay  $s$ . In this study, we always assume Itô interpretation for the stochastic integral. This paper studies the moment boundedness of the solutions of (1.1). Particularly, this paper gives the characteristic function of the equation, through which sufficient conditions for the second moment to be bounded or unbounded are obtained.

Despite the simplicity of (1.1), which is a linear equation with constant coefficients, current understanding for how the stability and moment boundedness depend on the equation coefficients is still incomplete. Most of known results are obtained through the method of Lyapunov functional. The Lyapunov functional method is useful for investigating the stability of differential equations, and has been well developed for delay differential equations [7], stochastic differential equations [15], and stochastic delay differential equations [10,12,14,15]. The Lyapunov functional method can usually give sufficient conditions for the stability of stochastic delay differential equations. For general results one can refer to the Razumikhin-type theorems on the exponential stability for the stochastic functional differential equation [15, Chapter 5]. However, these results often depend on the method of how the Lyapunov functional is constructed and are incomplete, not always applicable for all parameter regions. For example, sufficient conditions for the  $p$ th moment stability of the following stochastic differential delay equation

$$dx(t) = (ax(t) + bx(t - \tau)) dt + (\sigma_1 x(t) + \sigma_2 x(t - \tau)) dW_t, \quad (1.2)$$

can be obtained when  $a < 0$ , but not for  $a > 0$  [15, Example 6.9 in Chapter 5].

In 2007, Lei and Mackey [13] introduced the method of Laplace transform to study the stability and moment boundedness of Eq. (1.1) with discrete delay ( $K(s) = \delta(s - 1)$ ). In this particular case, the characteristic equation was proposed, which yields a sufficient (and is also necessary if not of the critical situation) condition for the boundedness of the second moment (see Theorem 3.6 in [13]). This result gives a complete description for the second moment stability

of Eq. (1.2) (the delay can be rescaled to  $\tau = 1$ ). Nevertheless, there is a disadvantage in the characteristic equation proposed in [13] in that the characteristic function is not explicitly given by the equation coefficients. Therefore, it is not convenient in applications.

The purposes of this paper are to study the stochastic delay differential equation (1.1) and to obtain a characteristic function that is given explicitly through the equation coefficients.

Rest of this paper is organized as follows. In Section 2 we briefly introduce basic results for the fundamental solutions of linear delay differential equations with distributed delay. Main results and proofs of this paper are given in Section 3. In Section 3.1 we discuss the first moment stability and show that it is identical to that of the unperturbed delay differential equation (2.1) (Theorem 3.3). Section 3.2 focuses on the second moment. When the stochastic perturbation is an additive noise, the result is simple and the bounded condition for the second moment is the same as the stability condition for the unperturbed delay differential equation (Theorem 3.4). However, in the presence of multiplicative white noises, boundedness for the second moment depends on the perturbation terms. We prove that the second moment is unbounded provided the zero solution of the unperturbed equation is unstable (Theorem 3.6). When the zero solution of the unperturbed equation is stable, we obtain a characteristic equation, and the boundedness of the second moments depends on the maximum real parts of all roots of the characteristic equation (Theorem 3.8). The characteristic function is given explicitly through the equation coefficients. In Section 4, as applications, we give several practical criteria for the boundedness of the second moment for some special situations (Theorem 4.1), and also sufficient conditions for the second moment to be unbounded (Theorem 4.2). An example is studied in Section 5.

## 2. Preliminaries

In this section, we first give some basic results for the fundamental solution of a linear differential equation with distributed delay

$$\frac{dx(t)}{dt} = ax(t) + b \int_0^{+\infty} K(s)x(t-s)ds. \quad (2.1)$$

We also give sufficient and necessary conditions for the stability of the zero solution of Eq. (2.1), which are useful for the rest of this paper. The linear delay differential equation has been studied extensively and the existence and uniqueness of the solution can be referred to [1,2,7].

First, we give some basic assumptions throughout this paper. We always assume that the initial functions of both (1.1) and (2.1) are  $x = \phi \in BC((-\infty, 0], \mathbb{R})$ . Here  $BC((-\infty, 0], \mathbb{R})$  means the space of all bounded and continuous functions  $\phi : (-\infty, 0] \rightarrow \mathbb{R}$  endowed with the norm

$$\|\phi\| = \sup_{\theta \in (-\infty, 0]} |\phi(\theta)|.$$

The delay kernel  $K$  is a nonnegative piecewise continuous function defined on  $[0, +\infty)$ , satisfying

$$\int_0^{+\infty} K(s)ds = 1 \quad (2.2)$$

and there is a positive constant  $\mu$  such that

$$\int_0^{+\infty} e^{\mu s} K(s)ds < +\infty. \quad (2.3)$$

We denote

$$\rho = \int_0^{+\infty} e^{\mu s} K(s) ds \quad (2.4)$$

for convenience. For example, if we have gamma distribution delays:

$$K(s) = \frac{r^j s^{j-1} e^{-rs}}{(j-1)!}, \quad s \geq 0, \quad r > 0, \quad j = 1, 2, 3, \dots \quad (2.5)$$

then (2.2) holds and (2.3) is satisfied for any  $\mu \in (0, r)$ .

For general linear functional differential equations, Lemmas 2.1 and 2.2 are known results (see Chapter 3 in [1]). However, for convenience and to emphasize the dependence on the delay kernel  $K$ , here we rewrite the lemmas.

**Lemma 2.1.** *Let  $x_\phi(t)$  be the solution of (2.1) with initial function  $\phi \in BC((-\infty, 0], \mathbb{R})$ . Then there exist positive constants  $A = A(b, \phi, K)$  and  $\gamma = \gamma(a, b)$  such that*

$$|x_\phi(t)| \leq A e^{\gamma t}, \quad t \geq 0. \quad (2.6)$$

The fundamental solution of the delay differential equation (2.1), denoted by  $X(t)$ , is defined as the solution of (2.1) with initial condition

$$X(t) = \begin{cases} 1, & t = 0, \\ 0, & t < 0. \end{cases}$$

Any solution of (2.1) with initial function  $\phi \in BC((-\infty, 0], \mathbb{R})$  can be represented through the fundamental solution  $X(t)$  as follows.

**Lemma 2.2.** *Let  $x_\phi(t)$  be the solution of (2.1) with initial function  $\phi \in BC((-\infty, 0], \mathbb{R})$ . Then*

$$x_\phi(t) = X(t)\phi(0) + b \int_0^t X(t-s) \int_s^{+\infty} K(\theta)\phi(s-\theta)d\theta ds, \quad t \geq 0. \quad (2.7)$$

Properties of the fundamental solution  $X(t)$  are closely related to the characteristic function of (2.1) defined below. For any function  $f(t) : [0, +\infty) \rightarrow \mathbb{R}$  which is measurable and satisfies

$$|f(t)| \leq a_1 e^{a_2 t}, \quad t \in [0, +\infty)$$

for some constants  $a_1, a_2$ , the Laplace transform

$$\mathcal{L}(f)(\lambda) = \int_0^{+\infty} e^{-\lambda t} f(t) dt, \quad \lambda \in \mathbb{C}$$

exists and is an analytic function of  $\lambda$  for  $\operatorname{Re}(\lambda) > a_2$ . Through the Laplace transform of the delay kernel  $K$ , the characteristic function of (2.1) is given by

$$h(\lambda) = \lambda - a - b\mathcal{L}(K)(\lambda). \quad (2.8)$$

It is easy to see that  $h(\lambda)$  is well defined and analytic when  $\operatorname{Re}(\lambda) \geq -\mu$ , and

$$\mathcal{L}(X)(\lambda) = 1/h(\lambda). \quad (2.9)$$

Now, we can obtain the precise exponential bound of the fundamental solution  $X(t)$  in terms of the supremum of the real parts of all roots of the characteristic function  $h(\lambda)$ .

First, we note that  $h(\lambda)$  is analytic when  $\operatorname{Re}(\lambda) > -\mu$ , and therefore all zeros of  $h(\lambda)$  are isolated. Following the discussion in [7, Lemma 4.1 in Chapter 1] and (2.3), there is a real number  $\alpha_0$  such that all roots of  $h(\lambda) = 0$  satisfy  $\operatorname{Re}(\lambda) \leq \alpha_0$ . Thus,  $\alpha_0 = \sup\{\operatorname{Re}(\lambda) : h(\lambda) = 0\}$  is well defined. Furthermore, there are only a finite number of roots in any close subset in the complex plane.

**Theorem 2.3.** Let  $\alpha_0 = \sup\{\operatorname{Re}(\lambda) : h(\lambda) = 0, \lambda \in \mathbb{C}\}$ . Then

1. for any  $\alpha > \alpha_0$ , there exists a positive constant  $C_1 = C_1(\alpha)$  such that the fundamental solution  $X(t)$  satisfies

$$|X(t)| \leq C_1 e^{\alpha t}, \quad t \geq 0; \quad (2.10)$$

2. for any  $\alpha_1 < \alpha_0$ , there exist  $\bar{\alpha} \in (\alpha_1, \alpha_0)$  and a subset  $U \subset \mathbb{R}^+$  with measure  $m(U) = +\infty$  such that the fundamental solution  $X(t)$  satisfies

$$|X(t)| \geq e^{\bar{\alpha} t}, \quad \forall t \in U. \quad (2.11)$$

**Proof.** 1. The proof of (2.10) is the same as that of [7, Theorem 5.2 in Chapter 1] and is omitted here.

2. Let  $\alpha_1 < \alpha_0$ . Since all zeros of  $h(\lambda)$  are isolated, we can take  $\bar{\alpha} \in (\alpha_1, \alpha_0)$  such that the line  $\operatorname{Re}(\lambda) = \bar{\alpha}$  contains no root of the characteristic equation  $h(\lambda) = 0$ . Next, choose  $c > \alpha_0$ , then

$$X(t) = \frac{1}{2\pi i} \lim_{T \rightarrow +\infty} \int_{c-iT}^{c+iT} \frac{e^{\lambda t}}{h(\lambda)} d\lambda. \quad (2.12)$$

Following the proof of Theorem 5.2 in Chapter 1 in [7] and the Cauchy theorem of residues, we can rewrite (2.12) as

$$X(t) = X_{\bar{\alpha}}(t) + \sum_{j=1}^m P_j(t) e^{\lambda_j t},$$

where

$$X_{\bar{\alpha}}(t) = \frac{1}{2\pi i} \lim_{T \rightarrow +\infty} \int_{\bar{\alpha}-iT}^{\bar{\alpha}+iT} \frac{e^{\lambda t}}{h(\lambda)} d\lambda,$$

$\lambda_1, \lambda_2, \dots, \lambda_m$  are all roots of  $h(\lambda) = 0$  such that  $\bar{\alpha} < \operatorname{Re}(\lambda_j) \leq \alpha_0$  ( $j = 1, \dots, m$ ) ( $m \geq 1$  from the definition of  $\alpha_0$ , and  $m < +\infty$  since  $h(\lambda)$  is an analytic function), and  $P_j(t)$  is a nonzero polynomial of  $t$  with degree the multiplicity of  $\lambda_j$  minus 1. Here we assume

$$\bar{\alpha} < \operatorname{Re}(\lambda_1) \leq \operatorname{Re}(\lambda_2) \leq \dots \leq \operatorname{Re}(\lambda_m) \leq \alpha_0.$$

Similar to the proof of (2.10), there exists a positive constant  $\bar{C}_1 = \bar{C}_1(\bar{\alpha})$  such that  $X_{\bar{\alpha}}(t)$  satisfies

$$|X_{\bar{\alpha}}(t)| \leq \bar{C}_1 e^{\bar{\alpha} t} \quad (t \geq 0). \quad (2.13)$$

Thus

$$\begin{aligned} |X(t)| &\geq \left| \sum_{j=1}^m P_j(t) e^{\lambda_j t} \right| - |X_{\bar{\alpha}}(t)| \geq \left| \sum_{j=1}^m P_j(t) e^{\lambda_j t} \right| - \bar{C}_1 e^{\bar{\alpha} t} \\ &= e^{\bar{\alpha} t} (e^{(\operatorname{Re}(\lambda_1) - \bar{\alpha})t} f(t) - \bar{C}_1), \end{aligned}$$

where  $f(t) = \left| \sum_{j=1}^m P_j(t) e^{(\lambda_j - \operatorname{Re}(\lambda_1))t} \right|$ .

Let  $\lambda_j = \beta_j + i\omega_j$ , and assume  $k$  such that  $\beta_j < \beta_m$  when  $1 \leq j \leq k$ , and  $\beta_j = \beta_m$  when  $k+1 \leq j \leq m$ , then

$$\begin{aligned} f(t) &= e^{(\beta_m - \beta_1)t} \left| \sum_{j=1}^k e^{-(\beta_m - \beta_j)t} P_j(t) e^{i\omega_j t} + \sum_{j=k+1}^m P_j(t) e^{i\omega_j t} \right| \\ &\geq \left| \sum_{j=k+1}^m \operatorname{Re}(P_j(t) e^{i\omega_j t}) \right| - \sum_{j=1}^k e^{-(\beta_m - \beta_j)t} |P_j(t)|. \end{aligned}$$

Since  $P_j(t)$  are nonzero polynomials, there are a positive constant  $\varepsilon > 0$  and a subset  $U \subset \mathbb{R}^+$  with measure  $m(U) = +\infty$  such that for any  $t \in U$ ,<sup>1</sup>

$$\left| \sum_{j=k+1}^m \operatorname{Re}(P_j(t) e^{i\omega_j t}) \right| > 2\varepsilon.$$

Moreover, since  $\sum_{j=1}^k e^{-(\beta_m - \beta_j)t} |P_j(t)| \rightarrow 0$  as  $t \rightarrow +\infty$  and  $\operatorname{Re}(\lambda_1) - \bar{\alpha} > 0$ , we can further take  $U$  such that

$$e^{(\operatorname{Re}(\lambda_1) - \bar{\alpha})t} f(t) - \bar{C}_1 > 1, \quad \forall t \in U$$

and hence (2.11) is concluded.  $\square$

From Lemma 2.2 and Theorem 2.3, asymptotical behaviors of all solutions  $x_\phi(t)$  of Eq. (2.1) are determined by  $\alpha_0$ .

**Theorem 2.4.** Let  $\alpha_0$  be defined as in Theorem 2.3. For any  $\alpha > \max\{\alpha_0, -\mu\}$  there exists a positive constant  $K_1 = K_1(\alpha, \mu)$  such that

$$|x_\phi(t)| \leq K_1 \|\phi\| e^{\alpha t}, \quad t \geq 0, \quad (2.14)$$

where  $\mu$  is defined by (2.3). Therefore the zero solution of (2.1) is locally asymptotically stable if and only if  $\alpha_0 < 0$ .

**Proof.** For any initial function  $\phi \in BC((-\infty, 0], \mathbb{R})$

$$\begin{aligned} \left| \int_s^{+\infty} K(\theta) \phi(s - \theta) d\theta \right| &\leq e^{-\mu s} \int_s^{+\infty} e^{\mu \theta} K(\theta) |\phi(s - \theta)| d\theta \\ &\leq \|\phi\| e^{-\mu s} \int_s^{+\infty} e^{\mu \theta} K(\theta) d\theta \\ &\leq \rho \|\phi\| e^{-\mu s}. \end{aligned}$$

<sup>1</sup> It is easy to see that  $\sum_{j=k+1}^m \operatorname{Re}(P_j(t) e^{i\omega_j t}) = t^n [\sum_{j=k+1}^m (a_j \cos(\omega_j t) + b_j \sin(\omega_j t)) + O(t^{-1})]$  ( $t \rightarrow +\infty$ ) where  $a_j, b_j$  are constants, and  $n$  is the highest degree of the polynomials  $P_j(t)$  ( $j = k+1, \dots, m$ ). Thus, we can always find a subset  $U_0$  with measure  $m(U_0) = +\infty$  so that all functions  $a_j \cos(\omega_j t) + b_j \sin(\omega_j t) > \varepsilon$  ( $k+1 \leq j \leq m, \forall t \in U_0$ ) for some small positive constant  $\varepsilon$  (detailed proof is omitted here), and therefore the subset  $U$  is always possible by taking  $U = U_0 \cap (t_0, +\infty)$  with  $t_0$  large enough.

Thus from (2.7) and Theorem 2.3, for any  $\alpha > \alpha_0$ ,

$$\begin{aligned} |x_\phi(t)| &\leq |X(t)|\|\phi\| + \int_0^t |X(t-s)| \left| \int_s^{+\infty} K(\theta)\phi(s-\theta)d\theta \right| ds \\ &\leq C_1\|\phi\|e^{\alpha t} + C_1\rho\|\phi\|e^{\alpha t} \int_0^t e^{-(\alpha+\mu)s} ds \\ &\leq C_1 \left( 1 + \frac{2\rho}{|\alpha+\mu|} \right) \|\phi\|e^{\alpha t}. \end{aligned}$$

Thus, (2.14) is concluded with

$$K_1(\alpha, \mu) = C_1 \left( 1 + \frac{2\rho}{|\alpha+\mu|} \right). \quad (2.15)$$

The theorem is proved.  $\square$

For a general distribution density function  $K$ , it is not straightforward to obtain sufficient and necessary conditions for  $\alpha_0 < 0$  using equation coefficients. A sufficient condition (see Theorem A.1) is given in Appendix A.

Now we give several properties of the fundamental solution  $X(t)$  that are useful for our estimations of the second moment in the next section.

Obviously, both  $X^2(t)$  and  $X_s(t)X_l(t)$  have Laplace transforms (here  $X_s(t) = X(t-s)$ ). When  $\alpha_0 < 0$ , the explicit expression of the Laplace transform  $\mathcal{L}(X^2)$  is obtained below.

Since  $\mathcal{L}(X) = 1/h(\lambda)$  and  $\alpha_0 < 0$ , we have

$$X(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{h(i\omega)} d\omega. \quad (2.16)$$

Therefore, we obtain

$$\begin{aligned} \mathcal{L}(X^2)(\lambda) &= \int_0^\infty e^{-\lambda t} X^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{h(i\omega)} \int_0^\infty e^{-(\lambda-i\omega)t} X(t) dt d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{h(i\omega)h(\lambda-i\omega)} d\omega. \end{aligned} \quad (2.17)$$

Let

$$g(\lambda, s, l) = \frac{\mathcal{L}(X_s X_l)(\lambda)}{\mathcal{L}(X^2)(\lambda)}. \quad (2.18)$$

The function  $g(\lambda, s, l)$  is crucial for the characteristic function of (1.1). Similar to the above argument, an explicit expression of  $g(\lambda, s, l)$  is obtained in Lemma B.1.

The following lemma gives an important estimation of  $g(\lambda, s, l)$  with the proof given in Appendix B.

**Lemma 2.5.** *Let  $g(\lambda, s, l)$  defined as in (2.18). Then when  $\operatorname{Re}(\lambda) > \max\{2\alpha_0, a, a + \alpha_0, -\mu\}$ , for any  $\varepsilon > 0$ , there exists a constant  $T_0 = T_0(\varepsilon)$  independent of  $s$  and  $l$  such that*

- (1) when  $s = l = 0$ ,  $g(\lambda, s, l) = \frac{\mathcal{L}(X_s X_l)(\lambda)}{\mathcal{L}(X^2)(\lambda)} = 1$ ;
- (2) when  $s > l = 0$  (or  $l > s = 0$ ), there exists a constant  $G_s > 0$  (or  $G_l > 0$ ) such that for  $\operatorname{Re}(\lambda) > T_0$ ,  $|g(\lambda, s, 0)| \leq G_s$  (or  $|g(\lambda, 0, l)| \leq G_l$ );

(3) when  $s > 0$ ,  $l > 0$ , for  $\operatorname{Re}(\lambda) > T_0$ ,

$$|g(\lambda, s, l)| \leq \begin{cases} \frac{e^{-(T_0-a)l} e^{-as}}{1-\varepsilon} + \frac{\varepsilon e^{-T_0s}}{1-\varepsilon}, & \text{if } s \geq l > 0, \\ \frac{e^{-(T_0-a)s} e^{-al}}{1-\varepsilon} + \frac{\varepsilon e^{-T_0l}}{1-\varepsilon}, & \text{if } 0 < s < l, \end{cases} \quad (2.19)$$

and

$$\lim_{\operatorname{Re}(\lambda) \rightarrow +\infty} |g(\lambda, s, l)| = 0. \quad (2.20)$$

### 3. Moment boundedness of the equation with noise perturbation

Now we consider Eq. (1.1), i.e.,  $\sigma_i$  ( $i = 0, 1, 2$ ) are not all zeros. In this section, two main results are obtained: [Theorem 3.3](#) for the sufficient condition of the exponential stability of the first moment, and [Theorem 3.8](#) for the characteristic equation that implies the boundedness criteria for the second moments of solutions of Eq. (1.1).

The existence and uniqueness theorem for the stochastic differential delay equations has been established in [8,15,19]. Using the fundamental solution  $X(t)$  in the previous section, the solution  $x(t; \phi)$  of (1.1) with initial function  $\phi \in BC((-\infty, 0], \mathbb{R})$  is a 1-dimensional stochastic process given by the Itô integral as follows:

$$\begin{aligned} x(t; \phi) = & x_\phi(t) + \int_0^t X(t-s) \left( \sigma_0 + \sigma_1 x(s; \phi) \right. \\ & \left. + \sigma_2 \int_0^{+\infty} K(\theta) x(s-\theta; \phi) d\theta \right) dW_s, \quad t \geq 0, \end{aligned} \quad (3.1)$$

where  $x_\phi(t)$  is the solution of (2.1) defined by (2.7) and  $W_s$  is a 1-dimensional Wiener process.

The first and second moments of  $x(t; \phi)$  are very important for investigating the behavior of the solutions and are studied in this paper. Now we state definitions of the  $p$ th moment exponential stability and the  $p$ th moment boundedness. Here we denote by  $E$  the mathematical expectation.

**Definition 3.1.** Eq. (1.1) is said to be the first moment exponentially stable if there exist two positive constants  $\gamma$  and  $R$  such that

$$|E(x(t; \phi))| \leq R \|\phi\| e^{-\gamma t}, \quad t \geq 0,$$

for all  $\phi \in BC((-\infty, 0], \mathbb{R})$ . When  $p \geq 2$ , Eq. (1.1) is said to be the  $p$ th moment exponentially stable if there exist two positive constants  $\gamma$  and  $R$  such that

$$E(|x(t; \phi) - E(x(t; \phi))|^p) \leq R \|\phi\|^p e^{-\gamma t}, \quad t \geq 0,$$

for all  $\phi \in BC((-\infty, 0], \mathbb{R})$ .

**Definition 3.2.** For  $p \geq 2$ , Eq. (1.1) is said to be the  $p$ th moment bounded if there exists a positive constant  $\hat{R} = \hat{R}(\|\phi\|^p)$  such that

$$E(|x(t; \phi) - E(x(t; \phi))|^p) \leq \hat{R}, \quad t \geq 0,$$

for all  $\phi \in BC((-\infty, 0], \mathbb{R})$ . Otherwise, the  $p$ th moment is said to be unbounded.

We first investigate the exponential stability of the first moment.



### 3.1. First moment stability

From (3.1), it is easy to have  $Ex(t; \phi) = x_\phi(t)$  from the Itô integral, and therefore Theorem 2.4 yields the following result.

**Theorem 3.3.** *Let  $\alpha_0$  be defined as in Theorem 2.3. Then for any  $\alpha > \max\{\alpha_0, -\mu\}$  there exists a constant  $K_1 = K_1(\alpha, \mu)$  defined by (2.15) such that*

$$|Ex(t; \phi)| \leq K_1 \|\phi\| e^{\alpha t}, \quad t \geq 0. \quad (3.2)$$

Therefore if  $\alpha_0 < 0$  Eq. (1.1) is first moment exponentially stable.

Theorem 3.3 indicates that the stability condition of the first moment is the same as that of the deterministic equation (2.1). The stability is determined by coefficients  $a$  and  $b$  and is independent of the parameters  $\sigma_i$  ( $i = 0, 1, 2$ ).

### 3.2. Second moment boundedness

Now we study the second moment. Let  $x(t; \phi)$  be a solution of (1.1), and define

$$\begin{aligned} \tilde{x}(t; \phi) &= x(t; \phi) - Ex(t; \phi), \quad M(t) = E(\tilde{x}^2(t; \phi)), \\ N(t; s, l) &= E(\tilde{x}(t-s; \phi)\tilde{x}(t-l; \phi)) \quad (t, s, l \geq 0). \end{aligned}$$

Then  $M(t) = N(t; 0, 0)$  is the second moment of  $x(t; \phi)$ . Obviously, when  $t \leq 0$ ,  $\tilde{x}(t; \phi) = E\tilde{x}(t; \phi) = M(t) = 0$ , and when  $s \geq t$  or  $l \geq t$ ,  $N(t; s, l) = 0$ .

We introduce the following notations:

$$\begin{aligned} P(t) &= \left( \sigma_0 + \sigma_1 Ex(t; \phi) + \sigma_2 \int_0^{+\infty} K(\theta) Ex(t-\theta; \phi) d\theta \right)^2, \quad t \geq 0, \\ Q(t) &= \sigma_1^2 M(t) + 2\sigma_1\sigma_2 \int_0^t K(s) N(t; s, 0) ds \\ &\quad + \sigma_2^2 \int_0^t \int_0^t K(s) K(l) N(t; s, l) ds dl, \quad t \geq 0, \\ F(t) &= \int_0^t X^2(t-s) P(s) ds, \quad t \geq 0. \end{aligned}$$

Applying the Itô integral, a tedious calculation gives

$$N(t; s, l) = \int_0^{(t-s) \wedge (t-l)} X(t-s-\theta) X(t-l-\theta) (P(\theta) + Q(\theta)) d\theta, \quad (3.3)$$

where  $(t-s) \wedge (t-l) = \min\{t-s, t-l\} \geq 0$ . Therefore

$$N(t; s, 0) = \int_0^{t-s} X(t-\theta) X(t-s-\theta) (P(\theta) + Q(\theta)) d\theta, \quad t \geq s \quad (3.4)$$

and

$$M(t) = \int_0^t X^2(t-\theta) (P(\theta) + Q(\theta)) d\theta, \quad t \geq 0. \quad (3.5)$$

### 3.2.1. Additive noise

When  $\sigma_1 = \sigma_2 = 0$ , we have only the additive noise and the second moment becomes

$$M(t) = \sigma_0^2 \int_0^t X^2(s) ds. \quad (3.6)$$

In this case, from [Theorem 2.3](#), the sufficient conditions for the second moment  $M(t)$  to be bounded or unbounded are given as follows.

**Theorem 3.4.** *Let  $\alpha_0$  be defined as in [Theorem 2.3](#). When  $\sigma_1 = \sigma_2 = 0$ , then*

1. *if  $\alpha_0 < 0$ , the second moment of (3.1) is bounded. Moreover, for any  $\alpha \in (\alpha_0, 0)$ , there exists a constant  $C_1 = C_1(\alpha)$  (as in [Theorem 2.3](#)) such that*

$$|M(t) - M_1| \leq \frac{C_1^2 \sigma_0^2 e^{2\alpha t}}{|2\alpha|}, \quad t \geq 0,$$

where

$$M_1 = \lim_{t \rightarrow +\infty} M(t) = \sigma_0^2 \int_0^{+\infty} X^2(s) ds \leq \frac{C_1^2 \sigma_0^2}{|2\alpha|};$$

2. *if  $\alpha_0 > 0$ , the second moment of (3.1) is unbounded.*

**Proof.** 1. From (3.5) and [Theorem 2.3](#), the results are easy to be concluded.

2. If  $\alpha_0 > 0$ , from [Theorem 2.3](#), there exist  $\bar{\alpha} \in (0, \alpha_0)$  and a closed subset  $U \subset \mathbb{R}^+$  with  $m(U) = +\infty$  such that

$$|X(t)| \geq e^{\bar{\alpha}t}, \quad \forall t \in U.$$

Thus from (3.6),

$$\lim_{t \rightarrow +\infty} M(t) \geq \sigma_0^2 \int_U X^2(s) ds \geq \sigma_0^2 \int_U e^{2\bar{\alpha}s} ds \geq \sigma_0^2 m(U) = +\infty,$$

and hence the second moment is unbounded.  $\square$

**Remark 3.5.** The critical case  $\alpha_0 = 0$  is not discussed here and the stability issue remains open.

### 3.2.2. General cases ( $\sigma_1, \sigma_2$ are not all zeros)

First, we note a very special situation that  $\sigma_i$  ( $i = 0, 1, 2$ ) satisfy the following condition:

**H:**  $\sigma_0 = 0$ , and there is a constant  $\lambda$  such that  $h(\lambda) = 0$  and  $\sigma_1 + \sigma_2 \mathcal{L}(K)(\lambda) = 0$ .

In this situation, it is easy to verify that  $x(t) = e^{\lambda t}$  ( $t \in \mathbb{R}$ ) is a solution of (1.1) with initial function  $\phi(\theta) = e^{\lambda\theta}$  ( $\theta \leq 0$ ), and therefore the corresponding second moment  $M(t) = 0$ . This is a very rare situation to have a deterministic solution for a stochastic delay differential equation, and is excluded in the following discussions.

The following result gives a sufficient condition for the second moment of (3.1) to be unbounded when the trivial solution of (2.1) is unstable.

**Theorem 3.6.** *Let  $\alpha_0$  be defined as in [Theorem 2.3](#). If  $\alpha_0 > 0$  and the condition **H** is not satisfied, then the second moment of (3.1) is unbounded.*

**Proof.** We only need to show that there is a special solution  $x(t; \phi)$  such that the corresponding second moment is unbounded. First, we note

$$Q(t) = E \left( \sigma_1 \tilde{x}(t) + \sigma_2 \int_0^t K(s) \tilde{x}(t-s) ds \right)^2 \geq 0,$$

and therefore

$$M(t) \geq F(t) = \int_0^t X^2(t-s) P(s) ds.$$

Now, let  $\lambda = \alpha + i\beta$  be a solution of  $h(\lambda) = 0$  with  $0 < \alpha \leq \alpha_0$ , then  $x_\phi(t) = \operatorname{Re}(e^{\lambda t})$  is a solution of (2.1) with initial function  $\phi(\theta) = \operatorname{Re}(e^{\lambda\theta})$  ( $\theta \leq 0$ ). Hence, for the solution  $x(t; \phi)$  of (1.1) with this particular initial function, we have

$$\begin{aligned} P(t) &= (\operatorname{Re} [\sigma_0 + e^{\lambda t} (\sigma_1 + \sigma_2 \mathcal{L}(K)(\lambda))])^2 \\ &= (\sigma_0 + e^{\alpha t} \operatorname{Re}[e^{i\beta t} (\sigma_1 + \sigma_2 \mathcal{L}(K)(\lambda))])^2. \end{aligned}$$

Since the condition **H** is not satisfied, we have either  $\sigma_0 \neq 0$  or  $\sigma_1 + \sigma_2 \mathcal{L}(K)(\lambda) \neq 0$ . Thus, from  $\alpha > 0$ , and following the proof of Theorem 2.3, there is a subset  $U \subset \mathbb{R}^+$  with measure  $m(U) = +\infty$  and  $\varepsilon > 0$  such that

$$X^2(t) > e^{2\bar{\alpha}t}, \quad P(t) > \varepsilon, \quad \forall t \in U,$$

where  $0 < \bar{\alpha} < \alpha_0$ . Thus,

$$\lim_{t \rightarrow \infty} F(t) = \lim_{t \rightarrow +\infty} \int_0^t X^2(t-s) P(s) ds \geq \varepsilon m(U) = +\infty,$$

which implies that the second moment is unbounded.  $\square$

**Remark 3.7.** From the proof, for any  $\lambda$  with  $\operatorname{Re}(\lambda) > 0$  such that  $h(\lambda) = 0$ , if either  $\sigma_0 \neq 0$  or  $\sigma_1 + \sigma_2 \mathcal{L}(K)(\lambda) \neq 0$ , the second moment of the solution of (1.1) with initial function  $\phi(\theta) = \operatorname{Re}(e^{\lambda\theta})$  ( $\theta \leq 0$ ) is unbounded.

In the following discussions, we always assume  $\alpha_0 < 0$ .

Now we study the second moment through the method of Laplace transform. First we note that both  $M(t)$  and  $N(t; s, l)$  have Laplace transforms (for detailed proofs refer to Lemmas 3.9 and 3.10).

The following theorem presents the characteristic function of (1.1) and establishes the boundedness criteria for the second moment of the solutions of (1.1).

**Theorem 3.8.** Let  $\alpha_0$  be defined as in Theorem 2.3 and assume  $\alpha_0 < 0$ . Define

$$H(\lambda) = \lambda - \left(2a + \sigma_1^2\right) - 2(b + \sigma_1\sigma_2) f_1(\lambda) - \sigma_2^2 f_2(\lambda), \quad (3.7)$$

where

$$\begin{aligned} f_1(\lambda) &= \int_0^{+\infty} K(s) g(\lambda, s, 0) ds, \\ f_2(\lambda) &= \int_0^{+\infty} \int_0^{+\infty} K(s) K(l) g(\lambda, s, l) ds dl, \end{aligned} \quad (3.8)$$

and  $g(\lambda, s, l)$  is defined by (2.18). Then

1. if all roots of the characteristic equation  $H(\lambda) = 0$  have negative real parts, the second moment of any solution of (1.1) is bounded, and approaches a constant exponentially as  $t \rightarrow +\infty$ ;
2. if the characteristic equation  $H(\lambda) = 0$  has a root with positive real part, and the condition **H** is not satisfied, the second moment of (3.1) is unbounded.

From Theorem 3.8,  $H(\lambda)$  is the characteristic function for the second moment boundedness of the stochastic delay differential equation (1.1). We note that the characteristic function is independent of the coefficient  $\sigma_0$ . But as we can see in the proof below, when the second moment is bounded, the limit  $\lim_{t \rightarrow \infty} M(t)$  depends on  $\sigma_0$ . To prove Theorem 3.8, we first give some lemmas.

**Lemma 3.9.** For any  $\alpha \in (\alpha_0, 0)$ , there exists a positive constant  $K_2 = K_2(\alpha, \phi)$  such that

$$F(t) \leq K_2(1 - e^{2\alpha t}), \quad t \geq 0. \quad (3.9)$$

**Proof.** Since  $\mu > 0$ ,  $\alpha_0 < 0$  and (2.3), from Theorem 3.3, for any  $\alpha \in (\alpha_0, 0)$ , there exists a positive constant  $K_1$  such that

$$\begin{aligned} & \left| \int_0^{+\infty} K(\theta) Ex(s - \theta; \phi) d\theta \right| \\ & \leq \left| \int_0^s K(\theta) Ex(s - \theta; \phi) d\theta \right| + \left| \int_s^{+\infty} K(\theta) Ex(s - \theta; \phi) d\theta \right| \\ & \leq \int_0^s K(\theta) K_1 \|\phi\| e^{\alpha(s-\theta)} d\theta + \int_s^{+\infty} K(\theta) |\phi(s - \theta)| d\theta \\ & \leq (1 + K_1) \|\phi\|. \end{aligned} \quad (3.10)$$

Thus from Theorem 2.3, for any  $\alpha \in (\alpha_0, 0)$ ,

$$\begin{aligned} F(t) & \leq \int_0^t C_1^2 e^{2\alpha(t-s)} \left( \sigma_0 + \sigma_1 Ex(s; \phi) + \sigma_2 \int_0^{+\infty} K(s - \theta; \phi) Ex(\theta; \phi) d\theta \right)^2 ds \\ & \leq C_1^2 e^{2\alpha t} \int_0^t \left( |\sigma_0| + |\sigma_1| K_1 \|\phi\| + |\sigma_2| (1 + K_1) \|\phi\| \right)^2 e^{-2\alpha s} ds \\ & \leq K_2(1 - e^{2\alpha t}), \end{aligned}$$

where

$$K_2(\alpha, \mu) = \frac{C_1^2}{|2\alpha|} \left( |\sigma_0| + |\sigma_1| K_1 \|\phi\| + |\sigma_2| (1 + K_1) \|\phi\| \right)^2.$$

The lemma is proved.  $\square$

A direct consequence of Lemma 3.9 is that  $F(t)$  has the Laplace transform. In the following, we have similar estimation for  $M(t)$ .

**Lemma 3.10.** Let  $\alpha_0$  be defined as in Theorem 2.3 and assume  $\alpha_0 < 0$ , then

$$M(t) \leq K_2 e^{C_1^2(|\sigma_1| + |\sigma_2|^2)t}, \quad t \geq 0. \quad (3.11)$$

**Proof.** From (3.5), we have

$$M(t) = F(t) + \int_0^t X^2(t-s)Q(s)ds. \quad (3.12)$$

To estimate the integral, we note that

$$\begin{aligned} |N(t; s, l)| &= |E(\tilde{x}(t-s)\tilde{x}(t-l))| \leq \left(E(\tilde{x}^2(t-s))\right)^{\frac{1}{2}} \left(E(\tilde{x}^2(t-l))\right)^{\frac{1}{2}} \\ &\leq \frac{M(t-s) + M(t-l)}{2} \end{aligned} \quad (3.13)$$

by the Cauchy–Schwarz inequality. Therefore, we have

$$\begin{aligned} \left| \int_0^t K(s)N(t; s, 0)ds \right| &\leq \int_0^t K(s)|N(t; s, 0)|ds \\ &\leq \frac{1}{2}M(t) + \frac{1}{2} \int_0^t K(s)M(t-s)ds \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} \left| \int_0^t \int_0^t K(s)K(l)N(t; s, l)dsdl \right| &\leq \frac{1}{2} \int_0^t K(s)M(t-s)ds \int_0^t K(l)dl \\ &\quad + \frac{1}{2} \int_0^t K(l)M(t-l)dl \int_0^t K(s)ds \\ &\leq \int_0^t K(s)M(t-s)ds. \end{aligned} \quad (3.15)$$

It is easy to verify that  $M(t)$  is increasing on  $[0, +\infty)$ , and from (2.2),

$$\int_0^t K(s)M(t-s)ds \leq M(t). \quad (3.16)$$

Thus, from Lemma 3.9 and (3.12)–(3.16), for any  $\alpha \in (\alpha_0, 0)$ ,

$$\begin{aligned} M(t) &\leq K_2 + \int_0^t X^2(t-s) \left( \sigma_1^2 M(s) + 2|\sigma_1\sigma_2|M(s) + \sigma_2^2 M(s) \right) ds \\ &\leq K_2 + C_1^2 (|\sigma_1| + |\sigma_2|)^2 \int_0^t M(s)ds. \end{aligned}$$

Finally, applying the Gronwall inequality, we obtain

$$M(t) \leq K_2 e^{C_1^2 (|\sigma_1| + |\sigma_2|)^2 t}$$

and (3.11) is proved.  $\square$

Lemma 3.10 indicates that  $M(t)$  has the Laplace transform. Furthermore, from (3.11) and (3.13),  $N(t; s, l)$  ( $0 \leq s, l \leq t$ ) also has the Laplace transform.

**Lemma 3.11.** Let  $Q(t)$  and  $M(t)$  be defined as previous. Then

$$\mathcal{L}(Q)(\lambda) = (\sigma_1^2 + 2\sigma_1\sigma_2 f_1(\lambda) + \sigma_2^2 f_2(\lambda))\mathcal{L}(M)(\lambda). \quad (3.17)$$

**Proof.** First, from the expression of  $Q(t)$ , we have for  $t \geq 0$ ,

$$\begin{aligned}\mathcal{L}(Q)(\lambda) &= \sigma_1^2 \mathcal{L}(M)(\lambda) + 2\sigma_1\sigma_2 \int_0^{+\infty} e^{-\lambda t} \int_0^t K(s)N(t; s, 0)dsdt \\ &\quad + \sigma_2^2 \int_0^{+\infty} e^{-\lambda t} \int_0^t \int_0^t K(s)K(l)N(t; s, l)dsldt.\end{aligned}\quad (3.18)$$

A direct calculation yields

$$\begin{aligned}&\int_0^{+\infty} e^{-\lambda t} \int_0^t \int_0^t K(s)K(l)N(t; s, l)dsldt \\ &= \int_0^{+\infty} \int_l^{+\infty} e^{-\lambda t} \int_0^t K(s)K(l)N(t; s, l)dsdt dl \\ &= \int_0^{+\infty} K(l) \int_0^l K(s) \int_l^{+\infty} e^{-\lambda t} N(t; s, l)dt ds dl \\ &\quad + \int_0^{+\infty} K(l) \int_l^{+\infty} K(s) \int_s^{+\infty} e^{-\lambda t} N(t; s, l)dt ds dl \\ &= \int_0^{+\infty} K(l) \int_0^l K(s) \int_0^{+\infty} e^{-\lambda t} N(t; s, l)dt ds dl \\ &\quad + \int_0^{+\infty} K(l) \int_l^{+\infty} K(s) \int_0^{+\infty} e^{-\lambda t} N(t; s, l)dt ds dl \\ &= \int_0^{+\infty} K(l) \int_0^{+\infty} K(s) \mathcal{L}(N(t; s, l)) ds dl,\end{aligned}\quad (3.19)$$

and similarly

$$\int_0^{+\infty} e^{-\lambda t} \int_0^t K(s)N(t; s, 0)dsdt = \int_0^{+\infty} K(s) \mathcal{L}(N(t; s, 0))ds. \quad (3.20)$$

Since

$$N(t; s, l) = \int_0^{(t-s) \wedge (t-l)} X(t-s-\theta)X(t-l-\theta)(P(\theta) + Q(\theta))d\theta,$$

we have

$$\mathcal{L}(N(t; s, l)) = \mathcal{L}(X_s X_l)(\mathcal{L}(P) + \mathcal{L}(Q)). \quad (3.21)$$

We note

$$\mathcal{L}(M) = \mathcal{L}(X^2)(\mathcal{L}(P) + \mathcal{L}(Q)) \quad (3.22)$$

by applying the Laplace transform to both sides of (3.5). Therefore, for any  $s, l \in [0, t]$ , Eqs. (3.21) and (3.22) yield

$$\mathcal{L}(N(t; s, l)) = \frac{\mathcal{L}(X_s X_l)}{\mathcal{L}(X^2)} \mathcal{L}(M) = g(\lambda, s, l) \mathcal{L}(M)$$

and

$$\mathcal{L}(N(t; s, 0)) = \frac{\mathcal{L}(X X_s)}{\mathcal{L}(X^2)} \mathcal{L}(M) = g(\lambda, s, 0) \mathcal{L}(M).$$

Thus, from (3.19) and (3.20), we obtain

$$\int_0^{+\infty} e^{-\lambda t} \int_0^t \int_0^t K(s)K(l)N(t; s, l)dsdl dt = f_2(\lambda)\mathcal{L}(M) \quad (3.23)$$

and

$$\int_0^{+\infty} e^{-\lambda t} \int_0^t K(s)N(t; s, 0)ds dt = f_1(\lambda)\mathcal{L}(M). \quad (3.24)$$

Finally, (3.17) is concluded from (3.18), (3.23) and (3.24).  $\square$

Now, we are ready to prove Theorem 3.8.

**Proof of Theorem 3.8.** From (3.17) and (3.22), we obtain

$$\mathcal{L}(M) = \frac{1}{\mathcal{L}(X^2)^{-1} - (\sigma_1^2 + 2\sigma_1\sigma_2f_1(\lambda) + \sigma_2^2f_2(\lambda))}\mathcal{L}(P). \quad (3.25)$$

To obtain  $\mathcal{L}(X^2)^{-1}$ , multiplying  $2X(t)$  to both sides of (2.1), we have

$$\frac{dX^2(t)}{dt} = 2aX^2(t) + 2bX(t) \int_0^{+\infty} K(s)X(t-s)ds. \quad (3.26)$$

Taking the Laplace transform to both sides of (3.26) yields

$$-1 + \lambda\mathcal{L}(X^2) = 2a\mathcal{L}(X^2) + 2b \int_0^{+\infty} K(s)\mathcal{L}(XX_s)ds,$$

which gives

$$\frac{1}{\mathcal{L}(X^2)} = \lambda - 2a - 2bf_1(\lambda). \quad (3.27)$$

Now, from (3.25) and (3.27), we obtain

$$\mathcal{L}(M) = \frac{1}{H(\lambda)}\mathcal{L}(P). \quad (3.28)$$

Let  $Y(t) = \mathcal{L}^{-1}(H^{-1}(\lambda))$ , then (3.28) yields

$$M(t) = Y(t) * P(t) = \int_0^t Y(s)P(t-s)ds, \quad (3.29)$$

where  $*$  denotes the convolution product. Now, the bounds of  $M(t)$  can be obtained from (3.29).

From Lemma 2.5 and noting  $\alpha_0 < 0$ , we have

$$\lim_{\operatorname{Re}(\lambda) \rightarrow +\infty} |f_1(\lambda)| = \lim_{\operatorname{Re}(\lambda) \rightarrow +\infty} |f_2(\lambda)| = 0.$$

Furthermore,  $H(\lambda)$  is analytic when  $\operatorname{Re}(\lambda) > \max\{2\alpha_0, a\}$ . Thus, there is a real number  $\beta_0$  such that all roots of  $H(\lambda)$  satisfy  $\operatorname{Re}(\lambda) \leq \beta_0$  (refer to the discussion in [7, Lemma 4.1 in Chapter 1]), where

$$\beta_0 = \sup\{\operatorname{Re}(\lambda) : H(\lambda) = 0, \lambda \in \mathbb{C}\}.$$

Thus, for any  $\beta > \beta_0$  there exists a positive constant  $C_3 = C_3(\beta)$  such that

$$|Y(t)| \leq C_3 e^{\beta t}, \quad t \geq 0. \quad (3.30)$$

Now, we are ready to prove the conclusions.

1. First, from [Theorem 3.3](#) and (3.10), there are two positive constants  $K_3$  and  $K_4$  such that for  $t \geq 0$ ,

$$\begin{aligned} P(t) &= \left( \sigma_0 + \sigma_1 Ex(t; \phi) + \sigma_2 \int_0^{+\infty} K(s) Ex(t-s; \phi) ds \right)^2 \\ &= \left( \sigma_0 + \sigma_2 \int_0^{+\infty} K(s) Ex(t-s; \phi) ds \right)^2 + \sigma_1^2 (Ex(t; \phi))^2 \\ &\quad + 2\sigma_1 Ex(t; \phi) \left( \sigma_0 + \sigma_2 \int_0^{+\infty} K(s) Ex(t-s; \phi) ds \right) \\ &\leq (|\sigma_0| + |\sigma_2| (1 + K_1) \|\phi\|)^2 + \sigma_1^2 K_1^2 \|\phi\|^2 e^{2\alpha t} \\ &\quad + 2|\sigma_1| (|\sigma_0| + |\sigma_2| (1 + K_1) \|\phi\|) K_1 \|\phi\| e^{\alpha t} \\ &\leq K_3 + K_4 e^{\alpha t}, \end{aligned} \quad (3.31)$$

where

$$K_3 = (|\sigma_0| + |\sigma_2| (1 + K_1) \|\phi\|)^2$$

and

$$K_4 = \sigma_1^2 K_1^2 \|\phi\|^2 + 2|\sigma_1| K_1 \|\phi\| (|\sigma_0| + |\sigma_2| (1 + K_1) \|\phi\|).$$

We note that  $K_3$  and  $K_4$  are of order  $\|\phi\|^2$ .

If  $\beta_0 < 0$ , for any  $\beta \in (\beta_0, 0)$  there exists a constant  $C_3$  as in (3.30) such that for  $t \geq 0$ ,

$$\begin{aligned} |M(t)| &= \left| \int_0^t Y(s) P(t-s) ds \right| \\ &\leq C_3 K_3 \int_0^t e^{\beta s} ds + C_3 K_4 \int_0^t e^{\beta s} e^{\alpha(t-s)} ds \\ &\leq \frac{2C_3 K_3}{|\beta|} + \frac{2C_3 K_4}{|\alpha - \beta|}. \end{aligned}$$

Therefore, the second moment is bounded for any initial function  $\phi \in ((-\infty, 0], \mathbb{R})$ .

Now, let

$$M_\infty = K_3 \int_0^{+\infty} Y(t) dt,$$

then, by (3.30) and (3.31), we have

$$\begin{aligned} |M(t) - M_\infty| &= \left| \int_0^t Y(s) (P(t-s) - K_3) ds - K_3 \int_t^{+\infty} Y(s) ds \right| \\ &\leq K_4 \int_0^t |Y(s)| e^{\alpha(t-s)} ds + K_3 \int_t^{+\infty} |Y(t)| dt \end{aligned}$$



$$\begin{aligned}
&\leq K_3 \int_t^{+\infty} C_3 e^{\beta t} dt + C_3 K_4 \int_0^t e^{\beta s} e^{\alpha(t-s)} ds \\
&\leq \frac{C_3 K_3 e^{\beta t}}{|\beta|} + \frac{C_3 K_4 (e^{\beta t} - e^{\alpha t})}{\alpha - \beta} \\
&\leq C_3 \left( \frac{K_3}{|\beta|} + \frac{2K_4}{|\alpha - \beta|} \right) e^{t \max\{\alpha, \beta\}}.
\end{aligned}$$

Thus, there exists a positive constant  $C_4 = C_3 \left( \frac{K_3}{|\beta|} + \frac{2K_4}{|\alpha - \beta|} \right)$  such that

$$|M(t) - M_\infty| \leq C_4 e^{t \max\{\alpha, \beta\}} \rightarrow 0 \quad \text{as } t \rightarrow +\infty$$

since  $\max\{\alpha, \beta\} < 0$ , i.e.,  $M(t)$  approaches to  $M_\infty$  exponentially as  $t \rightarrow +\infty$ .

2. Assume  $\beta_0 > 0$ . We only need to show that there is a special solution  $x(t; \phi)$  of (1.1) of which the second moment is unbounded. Similar to the proof of Theorem 3.6, let  $\lambda = \alpha + i\beta$  be a solution of  $h(\lambda) = 0$ , then  $x_\phi(t) = \text{Re}(e^{\lambda t})$  is a solution of (2.1) with initial function  $\phi(\theta) = \text{Re}(e^{\lambda \theta})$  ( $\theta \leq 0$ ). Hence, for the solution  $x(t; \phi)$  of (1.1) with this particular initial function, we have

$$P(t) = (\sigma_0 + e^{\alpha t} \text{Re}[e^{i\beta t} (\sigma_1 + \sigma_2 \mathcal{L}(K)(\lambda))])^2.$$

Since the condition **H** is not satisfied, we have either  $\sigma_0 \neq 0$  or  $\sigma_1 + \sigma_2 \mathcal{L}(K)(\lambda) \neq 0$ . Thus, the function  $P(t) \not\equiv 0$ , and hence the Laplacian  $\mathcal{L}(P)(s)$  is nonzero.

We have

$$M(t) = \mathcal{L}^{-1} \left( \frac{\mathcal{L}(P)(s)}{H(s)} \right) = \frac{1}{2\pi i} \lim_{T \rightarrow +\infty} \int_{c-iT}^{c+iT} e^{st} \frac{\mathcal{L}(P)(s)}{H(s)} ds$$

with  $c > \beta_0$ . Similar to the proof of Theorem 2.3, and noting that  $\mathcal{L}(P)(s)$  is analytic when  $\text{Re}(s) = c > 0$ , there are  $\bar{\beta} \in (0, \beta_0)$  and a sequence  $\{t_k\}$  with  $t_k \rightarrow +\infty$  such that  $M(t_k) > e^{\bar{\beta} t_k}$ , which implies that the second moment is unbounded.  $\square$

**Remark 3.12.** The critical case when  $\beta_0 = 0$  is not considered here, and the issue of boundedness criteria remains open.

#### 4. Applications

The functions  $f_1(\lambda)$  and  $f_2(\lambda)$  in  $H(\lambda)$  depend not only on the coefficients of Eq. (1.1), but also on the Laplace transforms of  $X^2(t)$ ,  $X_s(t)X_l(t)$  and the delay kernel  $K$ . Though it is possible to calculate these two functions numerically according to Lemma B.1, it is not trivial to obtain  $\beta_0 = \sup\{\text{Re}(\lambda) : H(\lambda) = 0\}$  for a given equation. Hence further studies are required for practical applications of the boundedness criteria established in Theorem 3.8. Here, we give several practical conditions, according to Theorem 3.8, for applications.

First, in the case of discrete delay ( $K(s) = \delta(s - 1)$ ), we have

$$f_1(\lambda) = \frac{\mathcal{L}(XX_1)(\lambda)}{\mathcal{L}(X^2)(\lambda)} = g(\lambda, 1, 0)$$

and

$$f_2(\lambda) = \frac{\mathcal{L}(X_1^2)(\lambda)}{\mathcal{L}(X^2)(\lambda)} = e^{-\lambda}.$$

Thus

$$H(\lambda) = \lambda - (2a + \sigma_1^2) - 2(b + \sigma_1\sigma_2)g(\lambda, 1, 0) - \sigma_2^2 e^{-\lambda},$$

which gives the same characteristic function  $H(s)$  as in [13, Theorem 3.6]. In fact, we have improved the result in [13] by providing explicit formulations for the functions  $f(s)$  and  $g(s)$  in [13] for defining the characteristic function.

Next, if  $b = 0$ , the fundamental solution  $X(t)$  is known. Hence it is possible to obtain explicit sufficient conditions for the boundedness. Here we give a sufficient condition for the second moment to be bounded when  $b = 0$  and  $K(s) = re^{-rs}$  ( $j = 1$  in the gamma distribution (2.5)).

**Theorem 4.1.** Let  $b = 0$  and  $K(s) = re^{-rs}$  ( $r > 0$ ). If  $a < 0$  and

$$a_1 > 0, \quad a_3 > 0, \quad a_1 a_2 - a_3 > 0 \quad (4.1)$$

where

$$\begin{aligned} a_1 &= 3(r - a) - \sigma_1^2, \\ a_2 &= 2r(r - a) - (2a + \sigma_1^2)(3r - a) - 2r\sigma_1\sigma_2, \\ a_3 &= -2ar(2(r - a) - \sigma_1^2) - 2r^2(\sigma_1 + \sigma_2)^2, \end{aligned}$$

the second moment is bounded.

**Proof.** When  $b = 0$ , the fundamental solution of (2.1) is given by

$$X(t) = \begin{cases} e^{at}, & t \geq 0, \\ 0, & t < 0 \end{cases}$$

and  $\alpha_0 = a$ . Hence Eq. (2.1) is of first moment asymptotically stable if and only if  $a < 0$ .

From the fundamental solution, we have for  $\text{Re}(\lambda) > 2a$ ,

$$\begin{aligned} \mathcal{L}(X^2)(\lambda) &= \int_0^\infty e^{-\lambda t} X^2(t) dt = \frac{1}{\lambda - 2a}, \\ \mathcal{L}(XX_s)(\lambda) &= \int_0^\infty e^{-\lambda t} X(t)X(t-s) dt = \frac{e^{-(\lambda-a)s}}{\lambda - 2a} \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}(X_s X_l)(\lambda) &= \int_0^\infty e^{-\lambda t} X(t-s)X(t-l) dt = e^{-a(s+l)} \int_{s \vee l}^\infty e^{-(\lambda-2a)t} dt \\ &= \begin{cases} \frac{e^{-(\lambda-a)s} e^{-al}}{\lambda - 2a}, & s \geq l \geq 0, \\ \frac{e^{-(\lambda-a)l} e^{-as}}{\lambda - 2a}, & 0 \leq s < l, \end{cases} \end{aligned}$$

where  $s \vee l = \max\{s, l\}$ . Therefore

$$g(\lambda, s, l) = \frac{\mathcal{L}(X_s X_l)(\lambda)}{\mathcal{L}(X^2)(\lambda)} = \begin{cases} e^{-(\lambda-a)s} e^{-al}, & s \geq l \geq 0, \\ e^{-(\lambda-a)l} e^{-as}, & 0 \leq s < l. \end{cases} \quad (4.2)$$

Let  $K(s) = re^{-rs}$  ( $r > 0$ ), then (4.2) yields

$$f_1(\lambda) = \frac{r}{\lambda + r - a}, \quad f_2(\lambda) = \frac{2r^2}{(\lambda + r - a)(\lambda + 2r)}.$$

Thus from (3.7),

$$H(\lambda) = \lambda - \left(2a + \sigma_1^2\right) - \frac{2r\sigma_1\sigma_2}{\lambda + r - a} - \frac{2r^2\sigma_2^2}{(\lambda + r - a)(\lambda + 2r)}. \quad (4.3)$$

Hence  $H(\lambda) = 0$  if and only if

$$\bar{H}(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \quad (4.4)$$

where

$$\begin{aligned} a_1 &= 3(r - a) - \sigma_1^2, & a_2 &= 2r(r - a) - (2a + \sigma_1^2)(3r - a) - 2r\sigma_1\sigma_2, \\ a_3 &= -2ar(2(r - a) - \sigma_1^2) - 2r^2(\sigma_1 + \sigma_2)^2. \end{aligned}$$

From the Routh–Hurwitz criterion, all roots of  $\bar{H}(\lambda) = 0$  have negative real parts if and only if

$$a_1 > 0, \quad a_3 > 0 \quad \text{and} \quad a_1a_2 - a_3 > 0.$$

Thus, the theorem is proved.  $\square$

The following theorem gives a sufficient condition for the unboundedness of the second moment for general situations.

**Theorem 4.2.** *If either*

$$b + \sigma_1\sigma_2 \leq 0, \quad \sigma_2^2 - 2\sigma_1\sigma_2 - \sigma_1^2 < 2(a + b) \quad (4.5)$$

*or*

$$b + \sigma_1\sigma_2 \geq 0, \quad \sigma_2^2 + 2\sigma_1\sigma_2 - \sigma_1^2 < 2(a - b), \quad (4.6)$$

*then the second moment is unbounded.*

**Proof.** From (B.1), we have

$$|g(0, s, I)| \leq 1,$$

and therefore

$$|f_1(0)| \leq \int_0^{+\infty} K(s)ds \leq 1, \quad |f_2(\lambda)| \leq \left(\int_0^{+\infty} K(s)ds\right)^2 \leq 1.$$

Thus, when either (4.5) or (4.6) is satisfied,

$$H(0) = -(2a + \sigma_1^2) - 2(b + \sigma_1\sigma_2)f_1(0) - \sigma_2^2 f_2(0) < 0.$$

Moreover, it is easy to have  $H(\lambda) > 0$  when  $\lambda \in \mathbb{R}$  is large enough. Thus the equation  $H(\lambda) = 0$  ( $\lambda \in \mathbb{R}$ ) has at least one positive solution, which implies  $\beta_0 > 0$ , and the second moment is unbounded by Theorem 3.8.  $\square$

## 5. An example

Here, we consider an example of the linear stochastic delay differential equation

$$dx(t) = -x(t)dt + \left(\sigma_1 x(t) + \sigma_2 \int_0^{+\infty} K(s)x(t-s)ds\right)dW_t, \quad (5.1)$$

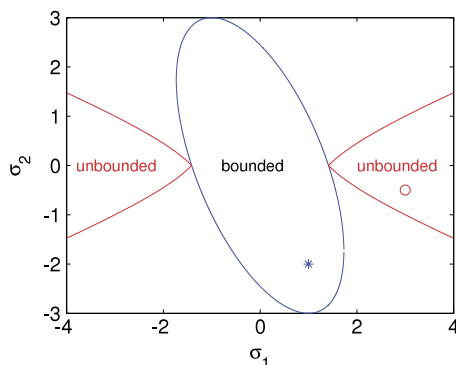


Fig. 1. The bounded and unbounded regions of the second moment obtained from Theorems 4.1 and 4.2, where  $r = 0.5$ .

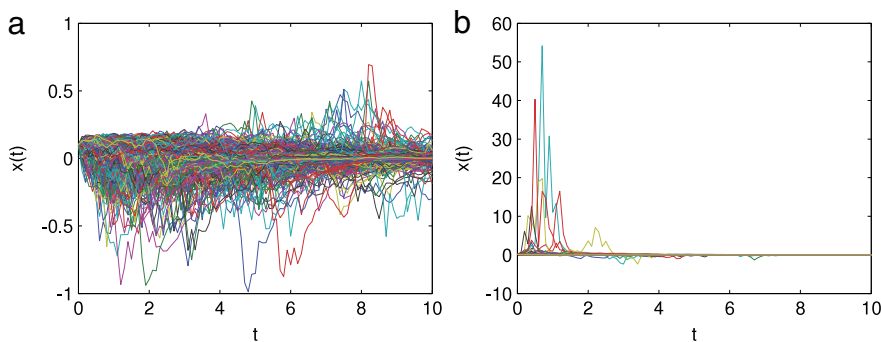


Fig. 2. Numerical results of 1000 sample solutions of (4.1). Parameters used are (a)  $(\sigma_1, \sigma_2) = (1, -2)$  (the star in Fig. 1), and (b)  $(\sigma_1, \sigma_2) = (3, -0.5)$  (the circle in Fig. 1). All initial functions are taken as  $x(t) = 0.1$  for  $t < 0$ .

where  $K(s) = re^{-rs}$  ( $r > 0$ ). Fig. 1 shows regions in the  $(\sigma_1, \sigma_2)$  plane to have bounded and unbounded second moments according to Theorems 4.1 and 4.2, respectively. Fig. 2 shows sample solutions with  $(\sigma_1, \sigma_2) = (1, -2)$  (the star in Fig. 1) and with  $(\sigma_1, \sigma_2) = (3, -0.5)$  (the circle in Fig. 1), respectively. Simulations show that when  $(\sigma_1, \sigma_2) = (1, -2)$ , all 1000 sample solutions are bounded from  $-1$  to  $1$ . But when  $(\sigma_1, \sigma_2) = (3, -0.5)$ , the sample solutions have a positive probability to reach a large value. These numerical results show agreement with our theoretical analysis.

**Final Remark.** All results in this paper are obtained under the Itô interpretation. Analogous results can be obtained for the Stratonovich interpretation.

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## Appendix A. A sufficient condition for $\alpha_0 < 0$

**Theorem A.1.** If  $(a, b) \in S$  with

$$S = \left\{ (a, b) \in \mathbb{R}^2 : a < 0, \min \left\{ a, - \left( \int_0^{+\infty} t K(t) dt \right)^{-1} \right\} < b < -a \right\}, \quad (\text{A.1})$$

then  $\alpha_0 < 0$ .

**Proof.** Assume  $(a, b) \in S$ , and let  $\lambda = \xi + i\eta$  ( $\eta > 0$ ) be a solution of  $h(\lambda) = 0$ . Separating the real and imaginary parts, we have

$$\begin{cases} \xi - a - b \int_0^{+\infty} e^{-\xi t} K(t) \cos(\eta t) dt = 0, \\ \eta + b \int_0^{+\infty} e^{-\xi t} K(t) \sin(\eta t) dt = 0. \end{cases}$$

When  $0 < b < -a$ . If  $\xi \geq 0$ , then

$$\begin{aligned} \xi - a - b \int_0^{+\infty} e^{-\xi t} K(t) \cos(\eta t) dt &> \xi + b - b \int_0^{+\infty} K(t) dt \\ &= \xi + b - b = \xi \geq 0. \end{aligned}$$

When  $b \leq 0$ . If  $a < b \leq 0$  and  $\xi \geq 0$ , then

$$\begin{aligned} \xi - a - b \int_0^{+\infty} e^{-\xi t} K(t) \cos(\eta t) dt &\geq \xi - a - |b| \int_0^{+\infty} K(t) dt \\ &= \xi - (a + |b|) > \xi \geq 0. \end{aligned}$$

If  $-(\int_0^{+\infty} t K(t) dt)^{-1} < b \leq 0$  and  $\xi \geq 0$ , then

$$\begin{aligned} \eta + b \int_0^{+\infty} e^{-\xi t} K(t) \sin(\eta t) dt &\geq \eta + b \int_0^{+\infty} K(t) \eta t dt \\ &= \eta \left( 1 + b \int_0^{+\infty} t K(t) dt \right) > 0. \end{aligned}$$

Thus, the above discusses indicate that  $h(\lambda)$  cannot be zero when  $\text{Re}(\lambda) \geq 0$ . Hence, all roots of  $h(\lambda)$  must have negative real parts, and  $\alpha_0 < 0$ .  $\square$

## Appendix B. Proof of Lemma 2.5

**Lemma B.1.** Let  $g(\lambda, s, l)$  be defined as in (2.18), then

$$g(\lambda, s, l) = \begin{cases} \frac{\int_{-\infty}^{+\infty} \frac{e^{-\lambda s} e^{i\omega(s-l)}}{h(i\omega)h(\lambda-i\omega)} d\omega}{\int_{-\infty}^{+\infty} \frac{1}{h(i\omega)h(\lambda-i\omega)} d\omega}, & s \geq l > 0, \\ \frac{\int_{-\infty}^{+\infty} \frac{e^{-\lambda l} e^{i\omega(l-s)}}{h(i\omega)h(\lambda-i\omega)} d\omega}{\int_{-\infty}^{+\infty} \frac{1}{h(i\omega)h(\lambda-i\omega)} d\omega}, & 0 \leq s < l. \end{cases} \quad (\text{B.1})$$

**Proof.** From (2.16), we have

$$\begin{aligned}\mathcal{L}(X_s X_l) &= \int_{s \vee l}^{+\infty} e^{-\lambda t} X(t-s) X(t-l) dt \quad (s \vee l = \max\{s, l\}) \\ &= \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega l} e^{-(\lambda-i\omega)s}}{h(i\omega)} \int_s^{+\infty} e^{-(\lambda-i\omega)(t-s)} X(t-s) dt d\omega & (s \geq l \geq 0), \\ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega s} e^{-(\lambda-i\omega)l}}{h(i\omega)} \int_l^{+\infty} e^{-(\lambda-i\omega)(t-l)} X(t-l) dt d\omega & (0 \leq s \leq l) \end{cases} \\ &= \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-\lambda s} e^{i\omega(s-l)}}{h(i\omega)h(\lambda-i\omega)} d\omega & (s \geq l \geq 0) \\ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-\lambda l} e^{i\omega(l-s)}}{h(i\omega)h(\lambda-i\omega)} d\omega & (0 \leq s \leq l). \end{cases} \quad (\text{B.2})\end{aligned}$$

Thus (B.1) is followed from (2.17), (2.18) and (B.2).  $\square$

From (B.1), we have

$$g(\lambda, s, 0) = \frac{\int_{-\infty}^{+\infty} \frac{e^{-\lambda s} e^{i\omega s}}{h(i\omega)h(\lambda-i\omega)} d\omega}{\int_{-\infty}^{+\infty} \frac{1}{h(i\omega)h(\lambda-i\omega)} d\omega}. \quad (\text{B.3})$$

Now, we give an estimation of  $\mathcal{L}(X^2)(\lambda)$ . When  $\text{Re}(\lambda) > \max\{2\alpha_0, a, a + \alpha_0\}$ , since

$$\begin{aligned}\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{h(i\omega)h(\lambda-i\omega-a)} d\omega &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{h(i\omega)} \int_0^{+\infty} e^{-(\lambda-i\omega-a)t} dt d\omega \\ &= \int_0^{+\infty} e^{-(\lambda-a)t} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{h(i\omega)} d\omega dt \\ &= \int_0^{+\infty} e^{-(\lambda-a)t} X(t) dt \\ &= \frac{1}{h(\lambda-a)}, \quad (\text{B.4})\end{aligned}$$

then

$$\begin{aligned}\mathcal{L}(X^2)(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{h(i\omega)(\lambda-i\omega-a)} d\omega \\ &\quad + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{h(i\omega)} \left( \frac{1}{h(\lambda-i\omega)} - \frac{1}{\lambda-i\omega-a} \right) d\omega \\ &= \frac{1}{h(\lambda-a)} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{b\mathcal{L}(K)(\lambda-i\omega)}{h(i\omega)h(\lambda-i\omega)(\lambda-i\omega-a)} d\omega \\ &= \frac{1}{h(\lambda-a)} (1 + g(\lambda)), \quad (\text{B.5})\end{aligned}$$

where

$$g(\lambda) = \frac{h(\lambda-a)}{2\pi} \int_{-\infty}^{+\infty} \frac{b\mathcal{L}(K)(\lambda-i\omega)}{h(i\omega)h(\lambda-i\omega)(\lambda-i\omega-a)} d\omega \quad (\text{B.6})$$

is convergent for  $\text{Re}(\lambda) > \max\{2\alpha_0, a, a + \alpha_0\}$ . We have the following result for  $g(\lambda)$ .

**Lemma B.2.** Let  $g(\lambda)$  be defined as in (B.6). Then for any  $\operatorname{Re}(\lambda) > \max\{2\alpha_0, a, a + \alpha_0, -\mu\}$ ,

$$\lim_{|\lambda| \rightarrow +\infty} |g(\lambda)| = 0.$$

**Proof.** First, when  $\operatorname{Re}(\lambda) > -\mu$ ,

$$\begin{aligned} |\mathcal{L}(K)(\lambda - i\omega)| &= \left| \int_0^{+\infty} e^{-(\lambda - i\omega)t} K(t) dt \right| \leq \int_0^{+\infty} e^{-\operatorname{Re}(\lambda)t} K(t) dt \\ &\leq \int_0^{+\infty} e^{\mu t} K(t) dt = \rho, \end{aligned} \quad (\text{B.7})$$

and hence,

$$|g(\lambda)| \leq \frac{|b|\rho}{2\pi} \int_{-\infty}^{\infty} \frac{|h(\lambda - a)|}{|h(i\omega)h(\lambda - i\omega)(\lambda - i\omega - a)|} d\omega.$$

Given a positive constant  $\omega_0$  such that  $\omega_0 > |\lambda| + |a| + |b|$  ( $\lambda \in \mathbb{C}$ ). Then for any  $|\omega| > \omega_0$ ,

$$\frac{1}{|h(\lambda - i\omega)|} \leq \frac{1}{|\omega| - |\lambda| - |a| - |b|}.$$

Thus for any  $|\omega| > \omega_0$ ,

$$\begin{aligned} &\left| \frac{1}{h(i\omega)h(\lambda - i\omega)(\lambda - i\omega - a)} \right| \\ &\leq \frac{1}{(|\omega| - |a| - |b|)(|\omega| - |\lambda| - |a| - |b|)(|\omega| - |\lambda| - |a|)} \\ &\leq \frac{1}{(|\omega| - |\lambda| - |a| - |b|)^3}. \end{aligned}$$

Therefore when  $\operatorname{Re}(\lambda) > -\mu$ ,

$$\begin{aligned} |g(\lambda)| &\leq \frac{|b|\rho}{2\pi} \int_{-\infty}^{-\omega_0} \frac{|h(\lambda - a)|}{|h(i\omega)h(\lambda - i\omega)(\lambda - i\omega - a)|} d\omega \\ &\quad + \frac{|b|\rho}{2\pi} \int_{-\omega_0}^{\omega_0} \frac{|h(\lambda - a)|}{|h(i\omega)h(\lambda - i\omega)(\lambda - i\omega - a)|} d\omega \\ &\quad + \frac{|b|\rho}{2\pi} \int_{\omega_0}^{\infty} \frac{|h(\lambda - a)|}{|h(i\omega)h(\lambda - i\omega)(\lambda - i\omega - a)|} d\omega \\ &\leq \frac{|b|\rho}{2\pi} \times 2 \int_{\omega_0}^{\infty} \frac{|h(\lambda - a)|}{(\omega - |\lambda| - |a| - |b|)^3} d\omega \\ &\quad + \frac{|b|\rho}{2\pi} \int_{-\omega_0}^{\omega_0} \frac{|h(\lambda - a)|}{|h(i\omega)h(\lambda - i\omega)(\lambda - i\omega - a)|} d\omega \\ &= \frac{|b|\rho}{\pi} \left( \frac{2|h(\lambda - a)|}{(\omega_0 - |\lambda| - |a| - |b|)^2} + \int_{-\omega_0}^{\omega_0} \frac{|h(\lambda - a)|}{|h(i\omega)h(\lambda - i\omega)(\lambda - i\omega - a)|} d\omega \right). \end{aligned}$$

Now, since

$$0 \leq \lim_{|\lambda| \rightarrow +\infty} \frac{|h(\lambda - a)|}{(\omega_0 - |\lambda| - |a| - |b|)^2} \leq \lim_{|\lambda| \rightarrow +\infty} \frac{|\lambda| + 2|a| + |b|\rho}{(\omega_0 - |\lambda| - |a| - |b|)^2} = 0,$$

and when  $\operatorname{Re}(\lambda) > \max\{2\alpha_0, a, -\mu\}$ ,

$$\begin{aligned} 0 &\leq \lim_{|\lambda| \rightarrow +\infty} \int_{-\omega_0}^{\omega_0} \frac{|h(\lambda - a)|}{|h(i\omega)h(\lambda - i\omega)(\lambda - i\omega - a)|} d\omega \\ &= 2 \int_0^{\omega_0} \lim_{|\lambda| \rightarrow +\infty} \frac{|h(\lambda - a)|}{|h(i\omega)| |h(\lambda - i\omega)| |(\lambda - i\omega - a)|} d\omega = 0, \end{aligned}$$

we have

$$\lim_{|\lambda| \rightarrow +\infty} |g(\lambda)| = 0, \quad \operatorname{Re}(\lambda) > \max\{2\alpha_0, a, a + \alpha_0, -\mu\}.$$

The lemma is proved.  $\square$

**Proof of Lemma 2.5.** Similar to (B.4), we have, when  $s \geq l$ ,

$$\begin{aligned} \frac{e^{-\lambda s}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega(s-l)}}{h(i\omega)(\lambda - i\omega - a)} d\omega &= \frac{e^{-\lambda s}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega(s-l)}}{h(i\omega)} \int_0^{+\infty} e^{-(\lambda - i\omega - a)t} dt d\omega \\ &= \int_0^{+\infty} e^{-(\lambda - a)t} e^{-\lambda s} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega(t+s-l)}}{h(i\omega)} d\omega dt \\ &= \int_0^{+\infty} e^{-(\lambda - a)t} e^{-\lambda s} X(t + s - l) dt \\ &= e^{-(\lambda - a)l} e^{-as} \int_0^{+\infty} e^{-(\lambda - a)(t+s-l)} X(t + s - l) dt \\ &= \frac{e^{-(\lambda - a)l} e^{-as}}{h(\lambda - a)} - e^{-(\lambda - a)l} e^{-as} \int_0^{s-l} e^{-(\lambda - a)t} X(t) dt. \end{aligned}$$

Thus, from (B.2), when  $\operatorname{Re}(\lambda) > \max\{2\alpha_0, a\}$ , we obtain

$$\begin{aligned} \mathcal{L}(X_s X_l) &= \frac{e^{-\lambda s}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega(s-l)}}{h(i\omega)(\lambda - i\omega - a)} d\omega \\ &\quad + \frac{e^{-\lambda s}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega(s-l)}}{h(i\omega)} \left( \frac{1}{h(\lambda - i\omega)} - \frac{1}{\lambda - i\omega - a} \right) d\omega \\ &= \frac{e^{-(\lambda - a)l} e^{-as}}{h(\lambda - a)} - e^{-(\lambda - a)l} e^{-as} \int_0^{s-l} e^{-(\lambda - a)t} X(t) dt \\ &\quad + \frac{e^{-\lambda s}}{2\pi} \int_{-\infty}^{\infty} \frac{be^{i\omega(s-l)} \mathcal{L}(K)(\lambda - i\omega)}{h(i\omega)h(\lambda - i\omega)(\lambda - i\omega - a)} d\omega \\ &= \frac{1}{h(\lambda - a)} \left[ e^{-(\lambda - a)l} e^{-as} - I(\lambda, s, l) + e^{-\lambda s} \hat{g}(\lambda, s, l) \right], \end{aligned}$$

where

$$I(\lambda, s, l) = h(\lambda - a) e^{-(\lambda - a)l} e^{-as} \int_0^{s-l} e^{-(\lambda - a)t} X(t) dt, \quad (\text{B.8})$$

$$\hat{g}(\lambda, s, l) = \frac{h(\lambda - a)}{2\pi} \int_{-\infty}^{\infty} \frac{be^{i\omega(s-l)} \mathcal{L}(K)(\lambda - i\omega)}{h(i\omega)h(\lambda - i\omega)(\lambda - i\omega - a)} d\omega. \quad (\text{B.9})$$



Similarly, when  $0 \leq s < l$ ,

$$\mathcal{L}(X_s X_l) = \frac{1}{h(\lambda - a)} \left[ e^{-(\lambda-a)s} e^{-al} - I(\lambda, l, s) + e^{-\lambda l} \hat{g}(\lambda, l, s) \right].$$

From (2.18) and (B.5), we have for  $\operatorname{Re}(\lambda) > \max\{2\alpha_0, a, a + \alpha_0, -\mu\}$ ,

$$\begin{aligned} g(\lambda, s, l) &= \frac{\mathcal{L}(X_s X_l)(\lambda)}{\mathcal{L}(X^2)(\lambda)} \\ &= \begin{cases} \frac{e^{-(\lambda-a)l} e^{-as} - I(\lambda, s, l) + e^{-\lambda s} \hat{g}(\lambda, s, l)}{1 + g(\lambda)}, & s \geq l \geq 0, \\ \frac{e^{-(\lambda-a)s} e^{-al} - I(\lambda, l, s) + e^{-\lambda l} \hat{g}(\lambda, l, s)}{1 + g(\lambda)}, & 0 \leq s \leq l. \end{cases} \end{aligned} \quad (\text{B.10})$$

From (B.8) and (2.10), we have for  $\operatorname{Re}(\lambda) > \alpha + a$  ( $\forall \alpha \in (\alpha_0, 0)$ )

$$\begin{aligned} |I(\lambda, s, l)| &\leq |h(\lambda - a)| e^{-(\operatorname{Re}(\lambda)-a)l} e^{-as} \int_0^{s-l} e^{-(\operatorname{Re}(\lambda)-a)t} C_1 e^{\alpha t} dt \\ &\leq \frac{e^{-\operatorname{Re}(\lambda)l} |h(\lambda - a)| e^{a(l-s)} C_1}{\operatorname{Re}(\lambda) - a - \alpha}. \end{aligned} \quad (\text{B.11})$$

Thus, for  $I(\lambda, s, l)$ , by (B.11), we obtain the following results.

- (i) When  $s = l = 0$ ,  $I(\lambda, s, l) = 0$ .
- (ii) When  $s > l = 0$ ,

$$I(\lambda, s, 0) = h(\lambda - a) e^{-as} \int_0^s e^{-(\lambda-a)t} X(t) dt,$$

and there exists a constant  $C_s > 0$  such that

$$\lim_{\operatorname{Re}(\lambda) \rightarrow +\infty} |I(\lambda, s, 0)| = C_s.$$

Similarly, when  $l > s = 0$ , there exists a constant  $C_l > 0$  such that

$$\lim_{\operatorname{Re}(\lambda) \rightarrow +\infty} |I(\lambda, 0, l)| = C_l.$$

- (iii) When  $l > s > 0$ , we have

$$\lim_{\operatorname{Re}(\lambda) \rightarrow +\infty} |I(\lambda, s, l)| = 0 \quad (\text{uniformly for } s, l),$$

from (B.8).

From (B.9) and (B.7), we get

$$|\hat{g}(\lambda, s, l)| \leq \frac{|h(\lambda - a)|}{2\pi} \int_{-\infty}^{\infty} \frac{|b| \rho}{|h(i\omega)h(\lambda - i\omega)(\lambda - i\omega)|} d\omega.$$

Thus, similar to the proof of Lemma B.2, for  $\operatorname{Re}(\lambda) > \max\{2\alpha_0, a, -\mu\}$ , we have

$$\lim_{|\lambda| \rightarrow +\infty} |\hat{g}(\lambda, s, l)| = 0 \quad (\text{uniformly for } s, l). \quad (\text{B.12})$$

From Lemma B.2 and (B.12), when  $\operatorname{Re}(\lambda) > \max\{2\alpha_0, a, a + \alpha_0, -\mu\}$ , for any  $\varepsilon > 0$ , there exists a constant  $T_0 = T_0(\varepsilon)$ , independent of  $s$  and  $l$  such that for  $\operatorname{Re}(\lambda) > T_0$ ,

$$|g(\lambda)| < \varepsilon, \quad |I(\lambda, s, l)| < \varepsilon \quad (s, l > 0), \quad |\hat{g}(\lambda, s, l)| < \varepsilon.$$

Now we prove the results of Lemma 2.5.

(1) When  $s = l = 0$ ,  $g(\lambda, s, l) = \frac{\mathcal{L}(X_s X_l)(\lambda)}{\mathcal{L}(X^2)(\lambda)} = 1$ .

(2) When  $s > l = 0$ , by (B.10), there exists a constant  $G_s > 0$  such that for the above  $\varepsilon$  and  $T_0$ , for  $\text{Re}(\lambda) > T_0$ ,

$$\begin{aligned} |g(\lambda, s, 0)| &= \left| \frac{\mathcal{L}(X_s X)(\lambda)}{\mathcal{L}(X^2)(\lambda)} \right| \\ &= \frac{e^{-as} - I(\lambda, s, 0) + e^{-\lambda s} \hat{g}(\lambda, s, 0)}{1 + g(\lambda)} \\ &\leq \frac{e^{-as} + C_s + \varepsilon e^{-T_0 s}}{1 - \varepsilon} \leq G_s. \end{aligned}$$

When  $l > s = 0$ , similarly, from (B.10), there exists a constant  $G_l > 0$  such that for the above  $\varepsilon$  and  $T_0$  and  $\text{Re}(\lambda) > T_0$ ,  $|g(\lambda, 0, l)| \leq G_l$ .

(3) When  $s, l > 0$ , by (B.10), for the above  $\varepsilon$  and  $T_0$  and  $\text{Re}(\lambda) > T_0$ ,

$$\begin{aligned} |g(\lambda, s, l)| &= \left| \frac{\mathcal{L}(X_s X_l)(\lambda)}{\mathcal{L}(X^2)(\lambda)} \right| \\ &\leq \begin{cases} \frac{e^{-(T_0-a)l} e^{-as}}{1 - \varepsilon} + \frac{\varepsilon + \varepsilon e^{-T_0 s}}{1 - \varepsilon}, & s \geq l > 0, \\ \frac{e^{-(T_0-a)s} e^{-al}}{1 - \varepsilon} + \frac{\varepsilon + \varepsilon e^{-T_0 l}}{1 - \varepsilon}, & 0 < s < l \end{cases} \end{aligned}$$

and

$$\lim_{\text{Re}(\lambda) \rightarrow +\infty} |g(\lambda, s, l)| = 0.$$

The proof is complete.  $\square$

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