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On weak convergence of stochastic heat equation with colored noise

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Abstract

In this work we are going to show weak convergence of probability measures. The measure corresponding to the solution of the following one dimensional nonlinear stochastic heat equation $\frac{\partial}{\partial t} u_t(x) = \frac{\kappa}{2}\frac{\partial^2}{\partial x^2} u_t(x) + \sigma(u_t(x))\eta_\alpha$ with colored noise η_α will converge to the measure corresponding to the solution of the same equation but with white noise η, as $\alpha \uparrow 1$. Function σ is taken to be Lipschitz and the Gaussian noise η_α is assumed to be colored in space and its covariance is given by $\mathrm{E}[\eta_\alpha(t,x)\eta_\alpha(s,y)] = \delta(t-s) f_\alpha(x-y)$ where f_α is the Riesz kernel $f_\alpha(x) \propto 1/|x|^\alpha$. We will work with the classical notion of weak convergence of measures, that is convergence of probability measures on a space of continuous function with compact domain and sup–norm topology. We will also state a result about continuity of measures in α, for $\alpha \in (0,1)$.

Keywords: The stochastic heat equation; Colored noise; Riesz kernel

1. Introduction

Throughout this work we will consider the following one-dimensional heat equation

$$\begin{aligned} \frac{\partial}{\partial t} u_{\alpha,t}(x) &= \frac{\kappa}{2}\frac{\partial^2}{\partial x^2} u_{\alpha,t}(x) + \sigma(u_{\alpha,t}(x))\eta_\alpha, \quad x \in \mathbb{R},\ t \geq 0, \\ u_{\alpha,0} &= w(x), \end{aligned} \tag{1}$$

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with $\kappa > 0$ and Gaussian space time colored noise η_α [6]. The noise η_α is assumed to have a particular covariance structure

$$\mathrm{E}\left[\eta_\alpha(t,x)\eta_\alpha(s,y)\right] = \delta(t-s) f_\alpha(x-y), \tag{2}$$

where [6, Ex. 1]

$$f_\alpha(x) = c_{1-\alpha} g_\alpha(x) = \hat{g}_{1-\alpha}(x), \qquad g_\alpha(x) = \frac{1}{|x|^\alpha} \quad \text{for } \alpha \in (0,1), \tag{3}$$

and the constant c_α is [9, (12) on pg. 173]

$$c_\alpha = 2\frac{\sin\left(\frac{\alpha\pi}{2}\right)\Gamma(1-\alpha)}{(2\pi)^{1-\alpha}}. \tag{4}$$

The function $\hat{g}_{1-\alpha}$ denotes the Fourier of function $g_{1-\alpha}$. For $F \in L^1(\mathbb{R})$, we will take $\hat{F}(\xi) = \int_{\mathbb{R}} e^{-2\pi\xi x} F(x)dx$. The initial condition, $w(x)$ is taken to be bounded and ϱ-Hölder continuous. We will also assume σ to be Lipschitz continuous, there exists $K \ge 0$ such that $|\sigma(x) - \sigma(y)| \le K|x-y|$ and $|\sigma(x)| \le K(1+|x|)$. Stochastic PDEs such as (1) have been studied in [6,14,2,13,5] and others.

The function f_α can be thought of as an *'approximation'* to the delta function in the following special sense, we know that one-dimensional Fourier transform of $g_{1-\alpha}$, denoted by $\hat{g}_{1-\alpha}$, is equal to f_α. We also know that the Fourier transform of a constant is δ distribution. Observe that $g_{1-\alpha}$ converges pointwise to 1 as $\alpha \uparrow 1$. We will study the solution of (1) as a function of α. This arises noticeably in [1, Sec. 7] where the authors have shown that $L^2(P)$ norm of $u_{\alpha,t}(x)$ converges to $L^2(P)$ norm of the solution to (5) as $\alpha \uparrow 1$ for every $t > 0$, $x \in \mathbb{R}$ and $\sigma(x) = x$.

The main question that has motivated this work, is whether the solution of (1) converges *in the appropriate sense* to the solution of the same equation, but with white noise η instead of colored noise η_α as $\alpha \uparrow 1$. By that we mean, the solution to

$$\begin{aligned} \frac{\partial}{\partial t} u_t(x) &= \frac{\kappa}{2}\frac{\partial^2}{\partial x^2} u_t(x) + \sigma(u_t(x))\eta, \quad x \in \mathbb{R},\ t \ge 0, \\ u_0(x) &= w(x), \end{aligned} \tag{5}$$

where η denotes white noise. We will state the main theorem in terms of measures corresponding to solutions. Let $\mathcal{C} = \mathcal{C}([0,T] \times [-N,N])$ be the space of continuous functions on $[0,T] \times [-N,N] \subset \mathbb{R}^+ \times \mathbb{R}$ with supremum norm. Denote by P_α, the measure corresponding to u_α restricted to $D = [0,T] \times [-N,N]$,

$$\mathrm{P}_\alpha(A) := \begin{cases} \mathrm{P}\left\{u_\alpha \in A^\circ\right\} & \text{for } \alpha \in (0,1), \\ \mathrm{P}\left\{u \in A^\circ\right\} & \text{for } \alpha = 1, \end{cases}$$

for any Borel set A of space $\mathcal{C}$. By A°, we denote the embedding of the set A in a larger space $\mathcal{C}(\mathbb{R}_+ \times \mathbb{R})$, that is

$$A^\circ = \{f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}) : f \text{ restricted to } [0,T] \times [-N,N] \text{ is in } A\}.$$

Here is the main theorem:

Theorem 1. *Measure* P_α *is continuous in* α, *for* $\alpha \in (0,1]$. *We precisely mean that* P_α *converges weakly to* P_1 *as* $\alpha \uparrow 1$ *and* P_α *converges weakly to* P_{α_0} *as* $\alpha \to \alpha_0$ *for any* $\alpha_0 \in (0,1)$.

The notion of weak convergence in Theorem 1 is the classical one [4]. Theorem 1 gives us a new way of thinking about the stochastic heat equation with white noise. Instead of studying the solution to (5) we can study the solution to (1) for $\alpha \approx 1$. Also note, that the noise with Riesz kernel spatial covariance produces noise which is less regular. We like to think that this *'roughness'* better captures properties of the stochastic heat equation with white noise.

Before we begin the proof of Theorem 1, let us recall the rigorous definition of the noises η_α, η as well as the form of [*mild*] solutions to (1) and (5). The noises η_α and η are $L^2(P)$ (random variables with finite second moment) valued set functions such that [6,7,12]

$$\eta_\alpha([0,t]\times B) \quad \text{and} \quad \eta_\alpha([0,t]\times B) \quad \text{for } t \geq 0,\ B \text{ bounded Borel set},$$

are mean zero Gaussian random variables. If A is another bounded Borel set, then the covariance of noises η_α and η will be

$$\mathrm{E}\left[\eta_\alpha([0,t]\times A)\eta_\alpha([0,t]\times B)\right] = t\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_A(x) f_\alpha(x-y)\mathbb{1}_B(y)dxdy,$$

$$\mathrm{E}\left[\eta([0,t]\times A)\eta([0,t]\times B)\right] = t\int_{\mathbb{R}} \mathbb{1}_A(x)\mathbb{1}_B(x)dx.$$

We often talk about Martingale valued measure, since $\eta_\alpha([0,t]\times A)$ and $\eta_\alpha([0,t]\times A)$ are set functions, which are $L^2(P)$ martingales in t. We refer the reader to [6,7,12] for more details.

The mild solutions are interpreted as solutions of the following integral equations [6,7,12]

$$u_{\alpha,t}(y) = (u_{\alpha,0} * p_t)(y) + \int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_{\alpha,s}(x))\eta_\alpha(ds,dx),$$

$$u_t(y) = (u_0 * p_t)(y) + \int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_s(x))\eta(ds,dx),$$

where p_t is the heat kernel

$$p_t(x) = \frac{1}{\sqrt{2\pi\kappa t}}\exp\left(-\frac{x^2}{2\kappa t}\right),$$

and $*$ denotes the convolution of two function ($f * g(x) = \int_{\mathbb{R}} f(y)g(x-y)dy$).

2. Proof of Theorem 1

We will only show the first part of the theorem, P_α converges weakly to P_1 as $\alpha \uparrow 1$. This is the worst case scenario. The second statement of Theorem 1 follows almost directly from the proof in this section.

The proof of the upcoming Theorem 2 uses coupling, which allows us to put both noises η_α and η on the same probability space. This idea was introduced in [5] and lets us write our noise η_α, for every $\alpha \in (0,1)$ in terms of one white noise η with covariance

$$\mathrm{E}\left[\eta(t,x)\eta(s,y)\right] = \delta(t-s)\delta(x-y).$$

The idea of coupling, or smoothing the noise in the spatial variable is not new. Authors in [3] smoothed the noise in the spatial variable by an infinitely differentiable function with compact support. They have showed that this kind of smoothing converges to the heat equation with white noise as our smoothing function converges to δ distribution. By coupling we mean that the martingale measure η_α will be defined as

$$\eta_\alpha([0,t]\times A) = \int_0^t\int_{\mathbb{R}} (\mathbb{1}_A * h_\alpha)(x)\eta(ds,dx), \tag{6}$$

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where

$$h_\alpha(x) = c_{\frac{1-\alpha}{2}} g_{\frac{1+\alpha}{2}}(x) = \hat{g}_{\frac{1-\alpha}{2}}(x).$$

This choice of h_α produces correct f_α in (2), that is

$$f_\alpha(x) = (h_\alpha * h_\alpha)(x), \tag{7}$$

since

$$g_{1-\alpha}(\xi) = g_{\frac{1-\alpha}{2}}(\xi) \cdot g_{\frac{1-\alpha}{2}}(\xi).$$

The typical stochastic Fubini theorem (see [5, pg. 492] or [12, pg. 14, pg. 50]) which would allow us to write (6), requires that $h_\alpha \in L^2(\mathbb{R})$. One might notice that $h_\alpha \notin L^2(\mathbb{R})$, but η_α is a well defined martingale measure, we refer the reader to [5] for more details. Before we state the main theorem of this section, let us state a technical lemma and define the following norms [12]

$$\mathcal{N}_{\gamma,k}(u) = \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \left(e^{-\gamma t} \|u_t(x)\|_{L^k(P)}\right), \quad \gamma > 1,\ k \geq 2.$$

The norm $\|\cdot\|_{L^k(P)}$ in the definition of $\mathcal{N}_{\gamma,k}$ stands for L^k norm on a probability space. For a random variable X it is defined as $\|X\|_{L^k(P)} = \mathrm{E}[|X|^k]^{1/k}$. We will often write $\|\cdot\|_k$ instead of $\|\cdot\|_{L^k(P)}$.

Lemma 1 (*[11, 3.478]*)**.** *The following equality holds for $s > 0$ and $\beta \in [0, 1)$*

$$\int_{\mathbb{R}} |x|^{-\beta} e^{-s4\pi^2x^2} dx = \left(\frac{1}{s4\pi^2}\right)^{-(\beta-1)/2} \Gamma(-\beta/2 + 1/2). \tag{8}$$

In the rest of this section, we will prove the following main theorem.

Theorem 2. *For every $k \geq 2$ we can find γ such that*

$$\lim_{\alpha\uparrow 1} \mathcal{N}_{\gamma,k}(u_\alpha - u) = 0.$$

Take the constant T that appears in Theorem 1 and the definition of the norm $\mathcal{N}_{\gamma,k}$ to be fixed throughout the whole proof. We will start our proof with Picard iterations for both noises η_α and η

$$u_t^{(n+1)}(y) = (u_0 * p_t)(y) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_s^{(n)}(x))\eta(ds, dx)$$
$$u_{\alpha,t}^{(n+1)}(y) = (u_0 * p_t)(y) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u_{\alpha,s}^{(n)}(x))\eta_\alpha(ds, dx),$$

which is equivalent to the following, thanks to [5, Sec. 3.2]:

$$u_t^{(n+1)}(y) = (u_0 * p_t)(y) + \int_0^t \int_{\mathbb{R}} (p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot)) * \delta)(x)\eta(ds, dx)$$
$$u_{\alpha,t}^{(n+1)}(y) = (u_0 * p_t)(y) + \int_0^t \int_{\mathbb{R}} (p_{t-s}(\cdot - y)\sigma(u_{\alpha,s}^{(n)}(\cdot)) * h_\alpha)(x)\eta(ds, dx).$$

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First, let us estimate the $L^k(P)$ norm of the difference of Picard iterates $u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y)$,

$$\mathrm{E}\left[\left|u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y)\right|^k\right] = \mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} \Big((p_{t-s}(\cdot - y)\sigma(u_{\alpha,s}^{(n)}) * h_\alpha)(x) - (p_{t-s}(\cdot - y)\sigma(u_s^{(n)}) * \delta)(x)\Big)\, \eta(ds, dx)\right|^k\right]. \tag{9}$$

Adding and subtracting the following term, $(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}) * h_\alpha)(x)$ inside the integral and using the inequality $|a-b|^k \le 2^k|a|^k + 2^k|b|^k$ yields

$$\begin{aligned}
&\mathrm{E}\left[\left|u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y)\right|^k\right] \\
&\le 2^k\, \mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} (p_{t-s}(\cdot - y)(\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)})) * h_\alpha)(x)\eta(ds, dx)\right|^k\right] \\
&\quad + 2^k\, \mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} (p_{t-s}(\cdot - y)\sigma(u_s^{(n)}) * (h_\alpha - \delta))(x)\eta(ds, dx)\right|^k\right].
\end{aligned}$$

The next series of steps will be used multiple times throughout this work. First we will use Burkholder–Davis–Gundy inequality and Minkowski integral inequality. Burkholder–Davis–Gundy (BDG) inequality (see for example [12, Thm. B.1]) states that for any continuous L^2 martingale M_t and $k \ge 2$ we have $\|M_t\|_k^2 \le 4k\|\langle M\rangle_t\|_{k/2}$, where $\langle M\rangle_t$ denotes the quadratic variation of M. Applying this inequality and evaluating the quadratic variation term [7, Thm. 5.26] on both terms gives us

$$\begin{aligned}
&\mathrm{E}\left[\left|u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y)\right|^k\right] \\
&\le \mathrm{const}\cdot \mathrm{E}\left[\left(\int_{[0,t]\times\mathbb{R}^2} p_{t-s}(x-y)\mathfrak{v}_s^{(n)}(x,z) f_\alpha(x-z) p_{t-s}(z-y)dsdxdz\right)^{k/2}\right] \\
&\quad + \mathrm{const}\cdot \mathrm{E}\Bigg[\Bigg(\int_{[0,t]\times\mathbb{R}^2} p_{t-s}(x-y)(z)\sigma(u_s^{(n)}(x))\left(f_\alpha - 2h_\alpha + \delta\right)(x-z) \\
&\quad \times\ p_{t-s}(z-y)\sigma(u_s^{(n)}(z))dsdxdz\Bigg)^{k/2}\Bigg],
\end{aligned}$$

where

$$\mathfrak{v}_s^{(n)}(x,z) = (\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(x)(\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(z).$$

Minkowski integral inequality states that $(\int(\int f d\mu)^k d\nu)^{1/k} \le \int(\int f^k d\nu)^{1/k} d\mu$ for any σ-finite measures μ, ν and jointly measurable $(\mu\times\nu)$ positive function f. We use this inequality on the first term in to order to obtain

$$\mathrm{E}\left[\left|u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y)\right|^k\right] \le \mathrm{const}\cdot\mathfrak{A}_{n,\alpha} + \mathrm{const}\cdot\mathfrak{B}_{n,\alpha},$$

where

$$\mathfrak{A}_{n,\alpha} = \left(\int_{[0,t]\times\mathbb{R}^2} p_{t-s}(x-y)v_s^{(n)}(x,z) f_\alpha(x-z) p_{t-s}(z-y)dxdzds\right)^{k/2}$$

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$$\mathfrak{B}_{n,\alpha} = \mathrm{E}\Bigg[\Bigg(\int_{[0,t]\times\mathbb{R}^2} p_{t-s}(x-y)(z)\sigma(u_s^{(n)}(x))\,(f_\alpha - 2h_\alpha + \delta)\,(x-z) \times p_{t-s}(z-y)\sigma(u_s^{(n)}(z))dxdzds\Bigg)^{k/2}\Bigg]$$

and $v_s^{(n)}$ denotes

$$v_s^{(n)}(x,z) = \mathrm{E}\left[\left|(\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(x)\right|^{k/2}\left|(\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(z)\right|^{k/2}\right]^{2/k}.$$

Ultimately, we would like to show that u_α is close to u as $\alpha \uparrow 1$.

For term $\mathfrak{A}_{n,\alpha}$ we use Cauchy–Schwarz inequality and take supremum over the term involving expectation, which yields

$$\mathfrak{A}_{n,\alpha} \le \Bigg(\int_0^t \sup_{x\in\mathbb{R}} \mathrm{E}\left[\left|(\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(x)\right|^k\right]^{2/k} \times \int_{\mathbb{R}^2} p_{t-s}(x-y)f_\alpha(x-z)p_{t-s}(z-y)dxdzds\Bigg)^{k/2}.$$

The following identity holds

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x)f_\alpha(x-y)\varphi(y)dxdy = \int_{\mathbb{R}} f_\alpha(x)(\varphi * \tilde{\varphi})(x)dx = \int_{\mathbb{R}} g_{1-\alpha}(\xi)|\mathcal{F}\varphi(\xi)|^2 d\xi, \tag{10}$$

for any φ from Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing test functions, where $\tilde{\varphi}(x) = \varphi(-x)$. This is a consequence of elementary properties of Fourier transform [6, pg. 6], [10, pg. 151,152]. We can further rewrite $\mathfrak{A}_{n,\alpha}$, using identity (10) and the assumption that σ is Lipschitz continuous (there exists $K \ge 0$, such that $|\sigma(x) - \sigma(y)| \le K\,|x-y|$ and $|\sigma(x)| \le K(1+|x|)$), as

$$\begin{aligned}\mathfrak{A}_{n,\alpha} &\le \left(\int_0^t \sup_{x\in\mathbb{R}} \mathrm{E}\left[\left|(\sigma(u_{\alpha,s}^{(n)}) - \sigma(u_s^{(n)}))(x)\right|^k\right]^{2/k} \int_{\mathbb{R}} g_{1-\alpha}(\xi)\left|\hat{p}_{t-s}(\xi)\right|^2 d\xi ds\right)^{k/2} \\ &\le K^k\left(\int_0^t \sup_{x\in\mathbb{R}} \mathrm{E}\left[\left|(u_{\alpha,s}^{(n)} - u_s^{(n)})(x)\right|^k\right]^{2/k} \int_{\mathbb{R}} g_{1-\alpha}(\xi)\left|\hat{p}_{t-s}(\xi)\right|^2 d\xi ds\right)^{k/2}.\end{aligned}$$

Multiply by term $e^{-k\gamma t}$ and obtain $\mathcal{N}_{\gamma,k}$ norm in the estimate

$$\begin{aligned}e^{-k\gamma t}\mathfrak{A}_{n,\alpha} &\le K^k\Bigg(\int_0^t e^{-2\gamma s}\sup_{x\in\mathbb{R}} \mathrm{E}\left[\left|(u_{\alpha,s}^{(n)} - u_s^{(n)})(x)\right|^k\right]^{2/k} e^{-2\gamma(t-s)} \times \int_{\mathbb{R}} g_{1-\alpha}(\xi)\left|\hat{p}_{t-s}(\xi)\right|^2 d\xi ds\Bigg)^{k/2} \\ &\le K^k\mathcal{N}_{\gamma,k}(u_\alpha^{(n)} - u^{(n)})^k\left(\int_0^t e^{-2\gamma(t-s)}\int_{\mathbb{R}} g_{1-\alpha}(\xi)\left|\hat{p}_{t-s}(\xi)\right|^2 d\xi ds\right)^{k/2} \\ &\le K^k\mathcal{N}_{\gamma,k}(u_\alpha^{(n)} - u^{(n)})^k\left(\int_0^t e^{-2\gamma(t-s)}\int_{\mathbb{R}} \frac{1}{|\xi|^{1-\alpha}} e^{-(t-s)\kappa 4\pi^2\xi^2} d\xi ds\right)^{k/2}.\end{aligned} \tag{11}$$

Later on, we will see that we can make the integral on right hand side arbitrarily small. The estimate for $\mathfrak{B}_{n,\alpha}$ uses a similar technique as the estimate for $\mathfrak{A}_{n,\alpha}$, but some extra work is required because of the term $(f_\alpha - 2h_\alpha + \delta)$ inside the integral is not a positive function. Thanks to [6], [8, Cor. 3.4] identity (10) extends to a much broader class of functions. We will use this identity to bound term $\mathfrak{B}_{n,\alpha}$. Quantity $\sigma(u_s^{(n)}(\cdot))p_{t-s}(\cdot - y) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ almost surely, because

$$\mathrm{E}\left[\|\sigma(u_s^{(n)}(\cdot))p_{t-s}(\cdot - y)\|^2_{L^2(\mathbb{R})}\right] \le 2K^2(1 + \mathcal{N}_{\gamma,2}(u^{(n)})^2)\|p_{t-s}(\cdot)\|^2_{L^2(\mathbb{R})},$$

and $\mathcal{N}_{\gamma,2}(u^{(n)})$ is bounded uniformly (in n, γ) for every $n \in \mathbb{N}$ and all $\gamma > \gamma_1$ [12, proof of Thm. 5.5]. The constant γ_1 depends on K, κ and $\sup_x w(x)$ [12, Thm. 5.5]. A similar reasoning applies for $\|\cdot\|_{L^1(\mathbb{R})}$. We can write

$$\begin{aligned}
&\mathrm{E}\Bigg[\Bigg(\int_{[0,t]\times\mathbb{R}^2} p_{t-s}(x-y)\sigma(u_s^{(n)}(x))\,(f_\alpha - 2h_\alpha + \delta)\,(x-z) \\
&\quad \times\ p_{t-s}(z-y)\sigma(u_s^{(n)}(z))dxdzds\Bigg)^{k/2}\Bigg] \\
&\quad = \mathrm{E}\left[\left(\int_{[0,t]\times\mathbb{R}} (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi)\left|\mathcal{F}\left(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot))\right)(\xi)\right|^2 d\xi ds\right)^{k/2}\right].
\end{aligned}$$

Split this integral into two parts, and use inequality $|a-b|^{k/2} \le 2^{k/2}|a|^{k/2} + 2^{k/2}|b|^{k/2}$ to get

$$\begin{aligned}
&\mathrm{E}\left[\left(\int_{[0,t]\times\mathbb{R}} (g_{1-\alpha} - 2g_{(1-\alpha)/2} + 1)(\xi)\left|\mathcal{F}\left(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot))\right)(\xi)\right|^2 d\xi ds\right)^{k/2}\right] \\
&\quad \le \mathrm{const}\cdot(\mathfrak{C}_{n,\alpha} + \mathfrak{D}_{n,\alpha}),
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{C}_{n,\alpha} &= \mathrm{E}\Bigg[\Bigg(\int_0^t\int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) \\
&\qquad \times\ \left|\mathcal{F}\left(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot))\right)(\xi)\right|^2 d\xi ds\Bigg)^{k/2}\Bigg] \\
\mathfrak{D}_{n,\alpha} &= \mathrm{E}\Bigg[\Bigg(\int_0^t\int_{\mathbb{R}\setminus[-1,1]} (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) \\
&\qquad \times\ \left|\mathcal{F}\left(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot))\right)(\xi)\right|^2 d\xi ds\Bigg)^{k/2}\Bigg].
\end{aligned}$$

Properties of Fourier transform and Lipschitz continuity of $\sigma(x)$ give us

$$\begin{aligned}
&\left|\mathcal{F}(p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot)))(\xi)\right|^2 \le \|p_{t-s}(\cdot - y)\sigma(u_s^{(n)}(\cdot))\|^2_{L^1(\mathbb{R})} \\
&\quad \le K^2\|p_{t-s}(\cdot - y)(1 + |u_s^{(n)}(\cdot)|)\|^2_{L^1(\mathbb{R})} \le K^2(2 + 2\|p_{t-s}(\cdot - y)u_s^{(n)}(\cdot)\|^2_{L^1(\mathbb{R})}) \qquad (12)
\end{aligned}$$

for the term inside of $\mathfrak{C}_{n,\alpha}$. Splitting the term (12) inside of the integral into two yields

$$\begin{aligned}\mathfrak{C}_{n,\alpha} &\leq \mathrm{E}\Bigg[\Bigg(\int_0^t \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) K^2 \\ &\quad\times\ (2 + 2\|p_{t-s}(\cdot - y)u_s^{(n)}(\cdot)\|^2_{L^1(\mathbb{R})}) d\xi ds\Bigg)^{k/2}\Bigg] \\ &\leq C_\alpha + \mathrm{const} \cdot \mathrm{E}\Bigg[\Bigg(\int_0^t \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) \\ &\quad\times\ \|p_{t-s}(\cdot - y)u_s^{(n)}(\cdot)\|^2_{L^1(\mathbb{R})} d\xi ds\Bigg)^{k/2}\Bigg],\end{aligned}$$

where C_α denotes

$$C_\alpha = \mathrm{const}\left(\int_0^t \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) d\xi ds\right)^{k/2}.$$

Term C_α can be made as small as we like, due to the dominated convergence theorem. We use Minkowski integral inequality and get

$$\begin{aligned}\mathfrak{C}_{n,\alpha} &\leq C_\alpha + \mathrm{const}\left(\int_0^t \sup_{x\in\mathbb{R}} \mathrm{E}\left[\left|u_s^{(n)}(x)\right|^k\right]^{2/k} \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) d\xi ds\right)^{k/2} \\ &\leq C_\alpha + \mathrm{const}\Bigg(\int_0^t e^{-k\gamma s} \sup_{x\in\mathbb{R}} \mathrm{E}\left[\left|u_s^{(n)}(x)\right|^k\right]^{2/k} \\ &\quad\times\ e^{k\gamma s} \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) d\xi ds\Bigg)^{k/2} \\ &\leq C_\alpha + \mathrm{const} \cdot \mathcal{N}_{\gamma,k}(u^{(n)})^k \left(\int_0^t e^{k\gamma s} \int_{-1}^1 (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) d\xi ds\right)^{k/2}.\end{aligned}$$

From the general theory of stochastic partial differential equations [12, proof of Thm. 5.5] we know that the term $\mathcal{N}_{\gamma,k}(u^{(n)})$ is bounded uniformly in n and γ for every $n \in \mathbb{N}$ and $\gamma > \gamma_2$ where γ_2 again depends on K, κ and $\sup_{x\in\mathbb{R}} w(x)$. The integral term bounding $\mathfrak{C}_{n,\alpha}$ can be made arbitrarily small, again from the dominated convergence theorem. Overall, we get that $\mathfrak{C}_{n,\alpha}$ converges uniformly in n to zero as $\alpha \uparrow 1$.

All we have left to do is find the estimate for $\mathfrak{D}_{n,\alpha}$. Add and subtract the term $\sigma(u_s(x))$ inside the Fourier transform, split into two integrals and obtain

$$\mathfrak{D}_{n,\alpha} \leq \mathrm{const} \cdot \mathfrak{D}^{(1)}_{n,\alpha} + \mathrm{const} \cdot \mathfrak{D}^{(2)}_{n,\alpha},$$

where

$$\begin{aligned}\mathfrak{D}^{(1)}_{n,\alpha} \ =\ &\mathrm{E}\Bigg[\Bigg(\int_0^t \int_{\mathbb{R}\setminus[-1,1]} (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi)\cdot \\ &\left|\mathcal{F}\left(p_{t-s}(\cdot - y)\left(\sigma(u_s^{(n)}(\cdot)) - \sigma(u_s(\cdot))\right)\right)(\xi)\right|^2 d\xi ds\Bigg)^{k/2}\Bigg],\end{aligned}$$

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$$\mathfrak{D}_{n,\alpha}^{(2)} = \mathrm{E}\left[\left(\int_0^t \int_{\mathbb{R}\setminus[-1,1]} (g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi) \times |\mathcal{F}(p_{t-s}(\cdot - y)\sigma(u_s(\cdot)))(\xi)|^2 \, d\xi ds\right)^{k/2}\right].$$

Term $\mathfrak{D}_{n,\alpha}^{(1)}$ converges to zero as $n \uparrow \infty$, uniformly in $\alpha \in (0, 1)$. Use Plancherel's theorem and the fact that $(g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)$ is bounded by a constant on $\mathbb{R} \setminus [-1, 1]$, uniformly for all $\alpha \in (0, 1)$ to write

$$\mathfrak{D}_{n,\alpha}^{(1)} \leq \mathrm{E}\left[\left(\int_0^t \int_{\mathbb{R}} \left(p_{t-s}(x-y)\left(\sigma(u_s^{(n)}(x)) - \sigma(u_s(x))\right)\right)^2 dxdt\right)^{k/2}\right].$$

From the convergence of Picard's iterations and theory of SPDEs [12, proof of Thm. 5.5] we get convergence of $\mathfrak{D}_{n,\alpha}^{(1)}$ to zero as $n \to \infty$, uniformly in $\alpha \in (0, 1)$. For $\epsilon > 0$, we can find $n_0(\epsilon)$ such that for every $n > n_0$ we have $\mathfrak{D}_{n,\alpha}^{(1)} < \epsilon/2$. If $n \leq n_0$, we can find α_n such that $\mathfrak{D}_{n,\alpha}^{(1)} < \epsilon/2$ for $\alpha \in (\alpha_n, 1)$, by dominated convergence theorem. Do not forget that $(g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1)(\xi)$ converges pointwise to zero. If we apply dominated convergence theorem to $\mathfrak{D}_{n,\alpha}^{(2)}$, we get that for $\alpha \in (\alpha_0, 1)$, $\mathfrak{D}_{n,\alpha}^{(2)} < \epsilon/2$. Altogether we have that for every $\epsilon > 0$ there is $\alpha_\epsilon := \max_{0\leq n\leq n_0} \alpha_n$ such that $\mathfrak{D}_{n,\alpha} < \epsilon$ for $\alpha \in (\alpha_\epsilon, 1)$ and every $n \in \mathbb{N}$. Therefore $\mathfrak{D}_{n,\alpha}$ converges uniformly in n to zero as $\alpha \uparrow 1$.

We have shown that

$$\mathrm{E}\left[\left|u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y)\right|^k\right] \leq \mathrm{const} \cdot \mathfrak{A}_{n,\alpha} + \mathrm{const} \cdot \mathfrak{C}_{n,\alpha} + \mathrm{const} \cdot \mathfrak{D}_{n,\alpha},$$

where $\mathfrak{C}_{n,\alpha} + \mathfrak{D}_{n,\alpha}$ converges to zero as $\alpha \uparrow 1$, uniformly in n. For $\epsilon > 0$, we can pick α_ϵ such that

$$\mathrm{E}\left[\left|u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y)\right|^k\right] \leq \mathrm{const} \cdot \mathfrak{A}_{n,\alpha} + \epsilon,$$

for $\alpha \in (\alpha_\epsilon, 1)$. Multiply the previous line by $e^{-k\gamma t}$ and use (11) to arrive at

$$\begin{aligned} &e^{-k\gamma t}\,\mathrm{E}\left[\left|u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y)\right|^k\right] \\ &\leq \mathrm{const} \cdot \mathcal{N}_{\gamma,k}(u_\alpha^{(n)} - u^{(n)})^k \left(\int_0^t e^{-2\gamma(t-s)} \int_{\mathbb{R}} \frac{1}{|\xi|^{1-\alpha}} e^{-(t-s)\kappa 4\pi^2\xi^2} d\xi ds\right)^{k/2} + \epsilon, \end{aligned}$$

where $\gamma \geq 1$. We use Lemma 1 to evaluate the term inside the integral. A straightforward calculation yields, for $1 > \alpha > \alpha_\epsilon > 0$,

$$\begin{aligned} &e^{-k\gamma t}\,\mathrm{E}\left[\left|u_{\alpha,t}^{(n+1)}(y) - u_t^{(n+1)}(y)\right|^k\right] \\ &\leq \mathrm{const} \cdot \mathcal{N}_{\gamma,k}(u_\alpha^{(n)} - u^{(n)})^k \left(\int_0^t e^{-2\gamma(t-s)}(t-s)^{-\alpha/2} ds\right)^{k/2} + \epsilon \\ &\leq \mathrm{const} \cdot \mathcal{N}_{\gamma,k}(u_\alpha^{(n)} - u^{(n)})^k \left(\left(\frac{1}{\gamma}\right)^{1-\alpha/2}\right)^{k/2} + \epsilon \\ &\leq \mathrm{const} \cdot \mathcal{N}_{\gamma,k}(u_\alpha^{(n)} - u^{(n)})^k \left(\frac{1}{\gamma}\right)^{k/4} + \epsilon. \end{aligned}$$

We can take supremum over $y \in \mathbb{R}$ and $t \in [0, T]$ to get

$$\mathcal{N}_{\gamma,k}(u_\alpha^{(n+1)} - u^{(n+1)})^k \leq \mathfrak{a} \cdot \mathcal{N}_{\gamma,k}(u_\alpha^{(n)} - u^{(n)})^k + \epsilon, \tag{13}$$

where $\mathfrak{a} = \text{const}\left(\frac{1}{\gamma}\right)^{k/4}$. The constant in $\mathfrak{a}$ depends only on K, k and choice of α_ϵ. It can be made explicit by tracking constants in front of $\mathfrak{A}_{n,\alpha}$ together with a constant dependent on α_ϵ, which comes from Lemma 1. The dependency on α_ϵ comes only from constant that appears in Lemma 1 and can be bounded from above as long as α_ϵ is bounded away from zero. We can always pick $\alpha_\epsilon > 1/2$ and get rid of dependency in α_ϵ. Eq. (13) defines a convergent geometric series assuming that the coefficient $\mathfrak{a} < 1$ and $\gamma > \max(\gamma_1, \gamma_2)$. We have

$$\mathcal{N}_{\gamma,k}(u_\alpha^{(n+1)} - u^{(n+1)})^k \leq \mathfrak{a}^n \mathcal{N}_{\gamma,k}(u_\alpha^{(1)} - u^{(1)})^k + \sum_{i=1}^{n-1} \mathfrak{a}^i \epsilon,$$

and

$$\mathcal{N}_{\gamma,k}(u_\alpha^{(n+1)} - u^{(n+1)})^k \leq \frac{\epsilon}{1-\mathfrak{a}}.$$

Let n go to infinity and conclude the proof.

2.1. Continuity in $\mathcal{N}_{\gamma,k}$ norm

Our proof of Theorem 2 also implies continuity in α for $\alpha \in (0, 1)$. We will only comment on how the proof would change in Section 2.

Theorem 3. *For every $k \geq 2$ and $\alpha_0 \in (0, 1)$ we can find γ such that*

$$\lim_{\alpha \to \alpha_0} \mathcal{N}_{\gamma,k}(u_\alpha - u_{\alpha_0}) = 0.$$

The proof of Theorem 3 follows the same general direction of the proof of Theorem 2 with the following changes. We need to replace $u_t(x)$ with $u_{\alpha_0,t}(x)$ and change $(f_\alpha - 2h_\alpha + \delta)$ in the estimate for $\mathfrak{B}_{n,\alpha}$ to $\left(f_\alpha - 2f_{\frac{\alpha+\alpha_0}{2}} + f_{\alpha_0}\right)$ and change $\left(g_{1-\alpha} - 2g_{\frac{1-\alpha}{2}} + 1\right)$ to $\left(g_{1-\alpha} - 2g_{1-\frac{\alpha+\alpha_0}{2}} + g_{1-\alpha_0}\right)$. We will also need an existence of γ_0 such that for every $\gamma > \gamma_0$ norms $\mathcal{N}_{\gamma,k}(u_{\alpha_0})$, $\mathcal{N}_{\gamma,k}(u_{\alpha_0}^{(n)})$, $\mathcal{N}_{\gamma,2}(u_{\alpha_0})$, $\mathcal{N}_{\gamma,2}(u_{\alpha_0}^{(n)})$ are finite, uniformly in $n \in \mathbb{N}$ and γ. This can be obtained from [5, Prep. 9.1].

One can notice in both Theorems 2 and 3, the coefficient $e^{-\gamma t}$ in $\mathcal{N}_{\gamma,k}$ norm served only as a helping hand to make part of the term $\mathfrak{A}_{n,\alpha}$ in (11) small. Let us summarize our effort in the following corollary.

Corollary 1. *Define $u_{1,t}(x) \equiv u_t(x)$, then for all $\alpha_0 \in (0, 1]$ we have*

$$\lim_{\alpha \to \alpha_0} \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} \mathrm{E}\left[\left|u_{\alpha,t}(x) - u_{\alpha_0,t}(x)\right|^k\right] = 0.$$

2.2. Convergence of finite dimensional distributions

Theorem 2 also states that the solution u_α converges to u in $L^2(P)$ norm for every $t \in [0, T]$ and $x \in \mathbb{R}$. This implies weak convergence of finite dimensional distributions of u_α to finite dimensional distributions of u. The easiest way to see that is to show convergence in probability for a finite number of pairs $(t_i, x_i) \in [0, T] \times [-N, N]$, which implies weak convergence of finite dimensional distribution. By Chebyshev's inequality, we have for $x_i \in [-N, N]$ and $t_i \in [0, T]$

$$\mathrm{P}\left(\sum_{i=1}^{l}(u_{\alpha,t_i}(x_i) - u_{t_i}(x_i))^2 > \epsilon^2\right) \le \frac{\sum_{i=1}^{l} \mathrm{E}\left[(u_{\alpha,t_i}(x_i) - u_{t_i}(x_i))^2\right]}{\epsilon^2}. \tag{14}$$

Right hand side of (14) converges to zero by Theorem 2, which also means that we have a convergence in probability of random vectors

$$(u_{\alpha,t_1}(x_1), \dots, u_{\alpha,t_l}(x_l)) \quad \text{to} \quad (u_{t_1}(x_1), \dots, u_{t_l}(x_l)).$$

From convergence in probability of finite dimensional distributions, we can conclude the weak convergence of finite dimensional distributions [4, pg. 27, pg. 207].

2.3. Estimates for Kolmogorov's continuity theorem and tightness

We will prove tightness (and thus weak convergence) from Kolmogorov's continuity theorem [12, pg. 107]. Before we begin the proof, we will need the following two lemmas.

Lemma 2 (*[5, Lemma 6.4]*)**.** *For all $t > 0$ and $x \in \mathbb{R}$*

$$\int_{\mathbb{R}} |p_t(y - x) - p_t(y)|\, dy \le \text{const} \cdot \left(\frac{|x|}{\sqrt{\kappa t}} \wedge 1\right),$$

where the implied constant does not depend on (t, x).

Lemma 3. *For all $t, \epsilon > 0$ we have*

$$\int_{\mathbb{R}} |p_{t+\epsilon}(y) - p_t(y)|\, dy \le \text{const} \cdot ((\log(t + \epsilon) - \log(t)) \wedge 1).$$

Proof. Direct computation gives us

$$\begin{aligned}
\int_{\mathbb{R}} |p_{t+\epsilon}(y) - p_t(y)|\, dy &= \int_{\mathbb{R}} \left| \int_t^{t+\epsilon} \dot{p}_s(y) ds \right| dy \\
&= \int_{\mathbb{R}} \left| \int_t^{t+\epsilon} \left(-\frac{1}{2s} + \frac{y^2}{2s^2\kappa}\right) p_s(y) ds \right| dy \\
&\le \int_t^{t+\epsilon} \int_{\mathbb{R}} \left(\frac{1}{2s} + \frac{y^2}{2s^2\kappa}\right) p_s(y) dy ds \\
&= \int_t^{t+\epsilon} \frac{1}{s} ds = (\log(t + \epsilon) - \log(t)).
\end{aligned}$$

In addition, we have that

$$\int_{\mathbb{R}} |p_{t+\epsilon}(y) - p_t(y)|\, dy \le 2. \quad \square$$

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Lemma 4 (*[15, pg. 314]*). *Let w be a bounded ϱ-Hölder continuous function, then there exists $C > 0$ such that for every $t > 0, \delta > 0, x \in \mathbb{R}, z \in \mathbb{R}$ we have*

$$\int_{\mathbb{R}} (p_t(x-y) - p_t(z-y))\, w(y)dy \le C \cdot |x-z|^{\varrho},$$
$$\int_{\mathbb{R}} (p_{t+\delta}(x-y) - p_t(x-y))\, w(y)dy \le C \cdot \delta^{\varrho}.$$

2.3.1. Difference in the spatial variable

Denote

$$I_{\alpha,t}(x) = \int_0^t \int_{\mathbb{R}} (p_{t-s}(x-z) - p_{t-s}(y-z))\, \sigma(u_{\alpha,s}(z))\eta_\alpha(ds, dz), \tag{15}$$

which is the stochastic integral for the mild solution. We will estimate the spatial and the time difference of the stochastic integral I in this and next subsection. The estimates of differences of the solution u_α will be obtained by combining Lemma 4 and estimates on I.

Let us estimate the difference in the spatial variable is

$$\begin{aligned} &\mathrm{E}\left[\left|I_{\alpha,t}(x) - I_{\alpha,t}(y)\right|^k\right] \\ &= \mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}} (p_{t-s}(x-z) - p_{t-s}(y-z))\, \sigma(u_{\alpha,s}(z))\eta_\alpha(ds, dz)\right|^k\right], \end{aligned}$$

and denote

$$B_s(z) = (p_{t-s}(x-z) - p_{t-s}(y-z)),$$
$$A_s(x, y) = \sigma(u_{\alpha,s}(x))\sigma(u_{\alpha,s}(y)).$$

We will proceed just as in Section 2. We use Burkholder–Davis–Gundy inequality, Minkowski integral inequality, Cauchy–Schwarz inequality and take the absolute value inside the integral and get

$$\begin{aligned} &\mathrm{E}\left[\left|I_{\alpha,t}(x) - I_{\alpha,t}(y)\right|^k\right] \\ &\le \mathrm{const} \cdot \mathrm{E}\left[\left|\int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} f_\alpha(z-w) B_s(z) B_s(w) A_s(x,y) dsdzdw\right|^{k/2}\right] \\ &\le \mathrm{const} \left|\int_0^t \sup_{x\in\mathbb{R}} \|\sigma(u_{\alpha,s}(x))\|_k^2 \int_{\mathbb{R}}\int_{\mathbb{R}} f_\alpha(z-w)\, |B_s(z)|\, |B_s(w)|\, dsdzdw\right|^{k/2} \\ &\le \mathrm{const} \left|\int_0^t \sup_{x\in\mathbb{R}} \|\sigma(u_{\alpha,s}(x))\|_k^2 (f_\alpha * p_{t-s})(0) \int_{\mathbb{R}} |p_{t-s}(x-z) - p_{t-s}(y-z)|\, dsdz\right|^{k/2} \\ &\le \mathrm{const}(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k) \left|\int_0^t (f_\alpha * p_{t-s})(0) \int_{\mathbb{R}} |p_{t-s}(x-z) - p_{t-s}(y-z)|\, dzds\right|^{k/2} \\ &\le \mathrm{const}(1 + \mathcal{N}_{\gamma,k}(u_\alpha)^k) \left|\int_0^t (f_\alpha * p_{t-s})(0) \left(\frac{|x-y|}{\sqrt{\kappa(t-s)}} \wedge 1\right)\right|^{k/2}, \end{aligned}$$

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where the last inequality is due to Lemma 2. We also used the fact that

$$\int_{\mathbb{R}} f_\alpha(z-w)\,|p_{t-s}(x-w)-p_{t-s}(y-w)|\,dw \le 2\,(f_\alpha * p_{t-s})\,(0). \tag{16}$$

The previous line (16) can be easily checked by the use of the triangle inequality and maximization over variables. The inequality $r \wedge 1 \le r^{2a}$ for $a \in (0, 1/2)$ gives us

$$\begin{aligned} &\mathrm{E}\left[\left|I_{\alpha,t}(x) - I_{\alpha,t}(y)\right|^k\right] \\ &\quad \le \mathrm{const}(1+\mathcal{N}_{\gamma,k}(u_\alpha)^k)\,|x-y|^{ak}\left|\int_0^t (f_\alpha * p_{t-s})(0)\cdot(t-s)^{-a}ds\right|^{k/2}. \end{aligned} \tag{17}$$

It remains to show that the integral on the right hand side is bounded for all $\alpha \in (\alpha_0, 1)$, $\alpha_0 > 0$. To show this, we will need an explicit form of f_α. The result is stated in the next lemma.

Lemma 5. *For every* $1 > \alpha > \alpha_0 > 0$ *we have*

$$f_\alpha * p_s(0) \le \mathrm{const}\cdot s^{-\alpha/2},$$

where the constant depends only on our choice of α_0.

Proof. By direct computation and (8) we get

$$\begin{aligned} (f_\alpha * p_s)(0) &= c_{1-\alpha}\int_{\mathbb{R}} \frac{1}{|x|^\alpha} p_s(x)dx \\ &= 2\frac{\sin\left(\frac{(1-\alpha)\pi}{2}\right)\Gamma(\alpha)}{(2\pi)^\alpha} 2^{-\alpha/2}\Gamma\left(\frac{1-\alpha}{2}\right)s^{-\alpha/2}\pi^{-1/2}. \end{aligned}$$

The boundedness of constant

$$2\frac{\sin\left(\frac{(1-\alpha)\pi}{2}\right)\Gamma(\alpha)}{(2\pi)^\alpha} 2^{-\alpha/2}\Gamma\left(\frac{1-\alpha}{2}\right)\pi^{-1/2}$$

can be concluded from Euler's reflection formula ($\Gamma(1-z)\Gamma(z) = \pi/\sin(\pi z)$) for $z = (1-\alpha)/2$. □

Because of the lemma above, the integral on the right hand side of (17) is finite as long as $\alpha/2 + a < 1$. Since $\alpha \in (0, 1)$, we can always take $a \in (0, 1/2)$. In [5, proof of Prep. 6.5], authors also get (17), but some extra effort is required to show that (17) holds with one constant on right hand side for $\alpha \in (\alpha_0, 1)$, $\alpha_0 > 0$.

2.3.2. Difference in the time variable

The difference in the time variable is going to be, for $\delta > 0$

$$\begin{aligned} &\mathrm{E}\left[\left|I_{\alpha,t+\delta}(x) - I_{\alpha,t}(x)\right|^k\right] \\ &\quad = \mathrm{const}\cdot\mathrm{E}\left[\left|\int_0^t\int_{\mathbb{R}} (p_{t+\delta-s}(x-z)-p_{t-s}(x-z))\,\sigma(u_{\alpha,s}(z))\eta_\alpha(ds,dz)\right|^k\right] \\ &\qquad + \mathrm{const}\cdot\mathrm{E}\left[\left|\int_t^{t+\delta}\int_{\mathbb{R}} p_{t+\delta-s}(x-z)\sigma(u_{\alpha,s}(z))\eta_\alpha(ds,dz)\right|^k\right]. \end{aligned}$$

Let us estimate the second integral, we can use the same technique as in the case of the spatial variable and write

$$\begin{aligned}
&\mathrm{E}\left[\left|\int_t^{t+\delta}\int_{\mathbb{R}} p_{t+\delta-s}(x-z)\sigma(u_{\alpha,s}(z))\eta_\alpha(ds,dz)\right|^k\right]\\
&\leq \mathrm{const}\left(\int_t^{t+\delta}\sup_x \mathrm{E}\left[\left|\sigma(u_{\alpha,s}(x))\right|^k\right]^{2/k}\int_{\mathbb{R}}\frac{1}{|\xi|^{1-\alpha}}\left|\hat{p}_{t+\delta-s}(\xi)\right|^2 d\xi ds\right)^{k/2}\\
&\leq \mathrm{const}(1+\mathcal{N}_{\gamma,k}(u_\alpha)^k)\left(\int_t^{t+\delta}(t+\delta-s)^{-\alpha/2}ds\right)^{k/2}\\
&\leq \mathrm{const}(1+\mathcal{N}_{\gamma,k}(u_\alpha)^k)\,|\delta|^{k(2-\alpha)/4}\leq \mathrm{const}(1+\mathcal{N}_{\gamma,k}(u_\alpha)^k)\,|\delta|^{k/4}.
\end{aligned}$$

The estimate for the second integral will be

$$\begin{aligned}
&\mathrm{E}\left[\left|\int_0^t\int_{\mathbb{R}}\left(p_{t+\delta-s}(x-z)-p_{t-s}(x-z)\right)\sigma(u_{\alpha,s}(z))\eta_\alpha(ds,dz)\right|^k\right]\\
&\leq \mathrm{const}\left(\int_0^t\sup_{x\in R}\mathrm{E}\left[\left|\sigma(u_{\alpha,s}(x))\right|^k\right]^{2/k}(f_\alpha * p_{t-s})(0)\right.\\
&\qquad\left.\times\int_{\mathbb{R}}\left|p_{t+\delta-s}(z)-p_{t-s}(z)\right|dzds\right)^{k/2}\\
&\leq \mathrm{const}(1+\mathcal{N}_{\gamma,k}(u_\alpha)^k)\left(\int_0^t(f_\alpha * p_{t-s})(0)\int_{\mathbb{R}}\left|p_{t+\delta-s}(z)-p_{t-s}(z)\right|dzds\right)^{k/2}\\
&\leq \mathrm{const}(1+\mathcal{N}_{\gamma,k}(u_\alpha)^k)\left(\int_0^t s^{-1/2}\left(\log(s+\delta)-\log(s)\right)ds\right)^{k/2}\\
&\leq \mathrm{const}(1+\mathcal{N}_{\gamma,k}(u_\alpha)^k)\left(4\sqrt{\delta}\,\mathrm{atan}\left(\sqrt{\frac{t}{\delta}}\right)+2\sqrt{t}\log(1+\delta/t)\right)^{k/2},
\end{aligned}\tag{18}$$

by using a similar technique as in the case for the spatial variable and Lemma 3. The first step in (18) uses that $(f_\alpha * |p_{t+\delta-s}-p_{t-s}|)(z)\leq f_\alpha * p_{t+\delta-s}(0)+f_\alpha * p_{t-s}(0)\leq 2f_\alpha * p_{t-s}(0)$. The last step in (18) can be verified by differentiating function $4\sqrt{\delta}\,\mathrm{atan}(\sqrt{t/\delta})+2\sqrt{t}\log(1+\delta/t)$. The inequality $\log(1+\zeta)<\sqrt{\zeta}$ for all $\zeta>0$ gives us

$$\begin{aligned}
&\mathrm{E}\left[\left|\int_0^t\int_{\mathbb{R}}\left(p_{t+\delta-s}(x-z)-p_{t-s}(x-z)\right)\sigma(u_{\alpha,s}(z))\eta(ds,dz)\right|^k\right]\\
&\leq \mathrm{const}\left(1+\mathcal{N}_{\gamma,k}(u_\alpha)^k\right)\delta^{k/4}.
\end{aligned}\tag{19}$$

We can combine both estimates (18) and (19) to finally get

$$\mathrm{E}\left[\left|I_{\alpha,t+\delta}(x)-I_{\alpha,t}(x)\right|^k\right]=\mathrm{const}\cdot\left(1+\mathcal{N}_{\gamma,k}(u_\alpha)^k\right)\delta^{k/4}.\tag{20}$$

Estimates similar to Sections 2.3.2 and 2.3.1 can be found in numerous places in the literature, for example [16]. Authors in [16] use a different technique and investigate noise with more general covariance structure. We do not know of continuity estimates which take into account α as a variable, thus our estimates in Sections 2.3.2 and 2.3.1 are novel in that sense.

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2.4. Kolmogorov's continuity theorem and tightness

Let us mention that $\mathcal{N}_{k,\gamma}(u_\alpha)$ is uniformly bounded in $\gamma, \gamma > 0$ and $\alpha \in (\alpha_0, 1)$ where $\alpha_0 > 0$. This follows from Corollary 1 and it is important for bounds on differences in Sections 2.3.2 and 2.3.1. We have that for every $1 > \alpha > \alpha_0 > 0$ and $(s, x), (t, y)$ from $D := [0, T] \times [-N, N] \subset \mathbb{R}_0^+ \times \mathbb{R}$ the following holds for $k \geq 2$

$$\mathrm{E}\left[\left|u_{\alpha,s}(x) - u_{\alpha,t}(y)\right|^k\right] \leq \text{const}\,|x-y|^{ka} + \text{const}\,|t-s|^{kb},$$

where $a \in (0, \frac{1}{2} \wedge \varrho)$ and $b \in (0, \frac{1}{4} \wedge \varrho)$, thanks to our estimates from Sections 2.3.1, 2.3.2 and Lemma 4. Denote $\rho(t, x) = |x|^a + |t|^b$, then Kolmogorov's continuity theorem states that there is a modification of $u_{\alpha,s}(x)$ such that (see for example [12, pg. 107])

$$\mathrm{E}\left[\sup_{(s,x),(t,y)\in D}\left|\frac{u_{\alpha,s}(x) - u_{\alpha,t}(y)}{\rho(s-t, x-y)^q}\right|^k\right] < \Lambda < +\infty \tag{21}$$

for every $\alpha \in (\alpha_0, 1)$ and $q \in (0, 1 - H/k)$ where $H = 1/a + 1/b$. By Markov's inequality and (21), we can write

$$\mathrm{P}\left\{\sup_{\substack{(s,x),(t,y)\in D\\ \rho(s-t,x-y)<\delta}}\left|u_{\alpha,s}(x) - u_{\alpha,t}(y)\right| > \epsilon\right\} < \frac{\Lambda}{\epsilon^k}\delta^{kq},$$

which implies

$$\lim_{\delta\to 0}\sup_{\alpha\in(\alpha_0,1)}\mathrm{P}\left\{\sup_{\substack{(s,x),(t,y)\in D\\ \rho(s-t,x-y)<\delta}}\left|u_{\alpha,s}(x) - u_{\alpha,t}(y)\right| > \epsilon\right\} = 0, \tag{22}$$

for every $\epsilon > 0$. We established both convergence of finite dimensional distributions [17, Thm. 2 (i)] and tightness (22) for measures P_α on $\mathcal{C}$ [17, Thm. 2 (ii)]. The tightness of P_α can be seen by adapting [4, Thm. 7.3] to a setting of two dimensional continuous functions with compact domain and supremum norm. We can conclude [17, Thm. 2] that the measure P_α corresponding to u_α restricted to D converges weakly as $\alpha \uparrow 1$ to a measure P_1 corresponding to u restricted to D. We can also conclude weak convergence of P_α to P_1 as $\alpha \uparrow 1$ by adapting [4, Chapter 2] to the two dimensional setting.

The second part of the main theorem, that is weak convergence of P_α to P_{α_0} as $\alpha \to \alpha_0 \in (0, 1)$ also follows from the current section. We have tightness for this claim, but we are missing convergence of finite dimensional distributions. Little modification of Section 2.2 would give us that.

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