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Quadratic–exponential growth BSDEs with jumps and their Malliavin's differentiability[☆]

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Abstract

We investigate a class of quadratic–exponential growth BSDEs with jumps. The quadratic structure introduced by Barrieu & El Karoui (2013) yields the universal bounds on the possible solutions. With local Lipschitz continuity and the so-called A_Γ-condition for the comparison principle to hold, we prove the existence of a unique solution under the general quadratic–exponential structure. We have also shown that the strong convergence occurs under more general (not necessarily monotone) sequence of drivers, which is then applied to give the sufficient conditions for the Malliavin's differentiability.

Keywords: Jump; Random measure; Lévy; Malliavin derivative

1. Introduction

The backward stochastic differential equations (BSDEs) have been subjects of strong interest of many researchers since they were introduced by Bismut (1973) [6] and generalized later by Pardoux & Peng (1990) [36]. This is particularly because they provide a truly probabilistic approach to stochastic control problems, which has been soon recognized as a very powerful tool for both theoretical and numerical issues in many important applications.

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More recently, there has appeared an acute interest in quadratic-growth BSDEs because of their various fields of applications such as, risk sensitive control problems, dynamic risk measures and indifference pricing in an incomplete market. The first breakthrough was made by Kobylanski (2000) [29] in a Brownian filtration with a bounded terminal condition. The result was then extended by Briand & Hu (2006, 2008) [9,10] to unbounded solutions. Direct convergence based on a fixed-point theorem was proposed by Tevzadze (2008) [42]. Various extensions/applications can be found in, for example, Hu, Imkeller & Muller (2005) [20], Mania & Tevzadze (2006) [32], Morlais (2009) [33], Hu & Schweizer (2011) [21], Delbaen, Hu & Richou (2011) [13].

In contrast to the diffusion setup, the number of researches on quadratic BSDEs with jumps has been rather small. Morlais (2010) [34] deals with a particular BSDE appearing in the exponential utility optimization with jumps, and Antonelli & Mancini (2016) [2] study the setup with local Lipschitz continuity with different assumptions. Both of them adopt Kobylanski's approach making use of a weakly converging subsequence. Cohen & Elliott (2015) [11] and also Kazi-Tani, Possamai & Zhou (2015) [28] have adopted the fixed-point approach of Tevzadze [42]. See also Becherer (2006) [5] as an earlier attempt for utility optimization with different restrictions on the driver.

Recently, Barrieu & El Karoui (2013) [4] have proposed a new approach based on the stability of quadratic semimartingales by introducing a so-called quadratic structure condition. They have shown the existence of a solution, without the uniqueness, under the minimal assumption allowing the unbounded terminal condition in a continuous setup. Their result has been extended to the exponential utility optimization in a market with counterparty default risks by generalizing quadratic structure condition to a quadratic–exponential ($\mathcal{Q}_{\exp}$) structure condition in Ngoupeyou (2010) [35] (see also Jeanblanc, Matoussi & Ngoupeyou (2013) [23] and El Karoui, Matoussi & Ngoupeyou (2016) [24]).

The current work, with local Lipschitz continuity and the so-called A_Γ-condition for the comparison principle to hold, proves the existence of a unique bounded solution under the *general* $\mathcal{Q}_{\exp}$-structure condition. Let us emphasize that the assumptions are more general than those used [11,28,34,2] where the existence of a unique solution is proved. [11,28] additionally require the second-order differentiability of the driver. [34,2] are using a special form of the driver, in particular, it is bounded by a linear (not quadratic) function of $|z|$ from *below*, and the sign of the quadratic terms is prefixed. These features are inherited from the utility optimization problem in [34] and is explicitly assumed in [2]. These assumptions play an important role for constructing a monotone sequence of drivers by simply truncating the quadratic terms. In the current work, new regularization of the driver inspired by [30,12,24] provides a rather streamlined proof for the convergence under the general $\mathcal{Q}_{\exp}$-structure. Moreover, the uniqueness alone is proved without using the comparison principle by the new stability result.

The specific monotone sequence of drivers used in the proof for the existence is not useful for other purposes. By generalizing Theorem 2.8 [29], we prove the strong convergence under more general (not necessary monotone) sequence of drivers. The result is then used to achieve the convergence of globally-Lipschitz BSDEs constructed by a sequence of simply truncated drivers. The sufficient conditions for the Malliavin's differentiability of the $\mathcal{Q}_{\exp}$-growth BSDEs are then obtained by exploiting the properties of locally Lipschitz BSDEs with $\mathbb{H}^2_{BMO}$-coefficients. This extends the work of Ankirchner, Imkeller & Dos Reis (2007) [1] on the Malliavin's differentiability in the diffusion setup. The obtained representation theorem will be useful for the optimal hedging problems in financial applications, investigations on the path regularity

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necessary for numerical as well as analytical issues, and also for the development of an asymptotic expansion for the quadratic BSDEs.[1]

The organization of the paper is as follows: Section 2 gives preliminaries including some important results on the BMO martingales. Section 3 explains the setup of $Q_{\exp}$-growth BSDEs with jumps and gives the uniqueness result. Section 4 proves the existence of a solution by using the monotone sequence and the comparison principle. Section 5 deals with the Malliavin's differentiability of the $Q_{\exp}$-growth BSDEs, which is then applied to a forward–backward system to obtain a representation theorem on the martingale components in Section 6. Appendix A is a simple generalization of the results by Ankirchner, Imkeller & Dos Reis (2007) [1] and Briand & Confortola (2008) [8] on the locally Lipschitz BSDEs with BMO coefficients to the setup with jumps. Appendix B gives some results regarding the comparison principle. Appendix C gives a detailed proof for the Malliavin's differentiability of the Lipschitz BSDEs with jumps, which generalizes the result of Delong & Imkeller (2010) [15] and Delong (2013) [14] to local (instead of global) Lipschitz continuity for the Malliavin derivative of the driver, which becomes necessary to investigate a forward–backward system driven by a Markovian forward process. Finally, Appendix D gives the technical details of the proof for Theorem 5.1 omitted in the main text.

2. Preliminaries

2.1. General setting

Let us first state the general setting to be used throughout the paper. $T > 0$ is some bounded time horizon. The space $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ is the usual canonical space for a d-dimensional Brownian motion equipped with the Wiener measure $\mathbb{P}_W$. We also denote $(\Omega_\mu, \mathcal{F}_\mu, \mathbb{P}_\mu)$ as a product of canonical spaces $\Omega_\mu := \Omega^1_\mu \times \cdots \times \Omega^k_\mu$, $\mathcal{F}_\mu := \mathcal{F}^1_\mu \times \cdots \times \mathcal{F}^k_\mu$ and $\mathbb{P}^1_\mu \times \cdots \times \mathbb{P}^k_\mu$ with some constant $k \geq 1$, on which each μ^i is a Poisson measure with a compensator $\nu^i(dz)dt$. Here, $\nu^i(dz)$ is a σ-finite measure on $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}_0} |z|^2 \nu^i(dz) < \infty$. Throughout the paper, we work on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where the space $(\Omega, \mathcal{F}, \mathbb{P})$ is the product of the canonical spaces $(\Omega_W \times \Omega_\mu, \mathcal{F}_W \times \mathcal{F}_\mu, \mathbb{P}_W \times \mathbb{P}_\mu)$, and that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the canonical filtration completed for $\mathbb{P}$ and satisfying the usual conditions. In this construction, $(W, \mu^1, \ldots, \mu^k)$ are independent. We use a vector notation $\mu(\omega, dt, dz) := (\mu^1(\omega, dt, dz^1), \ldots, \mu^k(\omega, dt, dz^k))$ and denote the compensated Poisson measure as $\widetilde{\mu} := \mu - \nu$. We represent the $\mathbb{F}$-predictable σ-field on $\Omega \times [0, T]$ by $\mathcal{P}$.

Remark 2.1. We have chosen the above setting mainly because that it is known to guarantee the weak property of predictable representation and also because there exists an established Malliavin's differential rule. The contents up to Section 4 can be easily extendable to $\mathcal{P} \otimes \mathcal{E}$-measurable random compensator $\nu_t(dx)$ as long as $(W, \mu - \nu)$ is *assumed* to have *the weak property of predictable representation* (see Chapter XIII in [19]). For the general topics regarding stochastic calculus with random measures, see also [22].

[1] Recently, we have proposed an analytic approximation method of the Lipschitz BSDEs with jumps in Fujii & Takahashi (2015) [17], which is based on the small-variance asymptotic expansion (see, Takahashi (2015) [41] as a general review). Its extension to the $Q_{\exp}$-growth BSDEs is now ready to be investigated using the new results obtained here, which will be pursued in a different opportunity.

2.2. Notation

We denote a generic constant by C, which may change line by line, is sometimes associated with several subscripts (such as $C_{K,T}$) showing its dependence when necessary. $\mathcal{T}_0^T$ denotes the set of $\mathbb{F}$-stopping times $\tau \in [0,T]$.

Let us introduce a sup-norm for a $\mathbb{R}^r$-valued function $x:[0,T]\to\mathbb{R}^r$ as

$$\|x\|_{[a,b]} := \sup\{|x_t|, t\in[a,b]\}$$

and write $\|x\|_t := \|x\|_{[0,t]}$. We use the following spaces for stochastic processes for $p\geq 2$:

- $\mathbb{S}_r^p[s,t]$ is the set of $\mathbb{R}^r$-valued adapted càdlàg processes X such that

$$\|X\|_{\mathbb{S}_r^p[s,t]} := \mathbb{E}\left[\|X\|_{[s,t]}^p\right]^{1/p} < \infty\ .$$

- $\mathbb{S}_r^\infty$ is the set of $\mathbb{R}^r$-valued essentially bounded càdlàg processes X such that

$$\|X\|_{\mathbb{S}_r^\infty} := \Big\| \sup_{t\in[0,T]} |X_t| \Big\|_\infty < \infty.$$

- $\mathbb{H}^p[s,t]$ is the set of progressively measurable $\mathbb{R}^d$-valued processes Z such that

$$\|Z\|_{\mathbb{H}^p[s,t]} := \mathbb{E}\Big[\Big(\int_s^t |Z_u|^2 du\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} < \infty.$$

- $\mathbb{J}^p[s,t]$ is the set of k-dimensional functions $\psi=\{\psi^i, 1\leq i\leq k\}$, $\psi^i:\Omega\times[0,T]\times\mathbb{R}_0\to\mathbb{R}$ which are $\mathcal{P}\times\mathcal{B}(\mathbb{R}_0)$-measurable and satisfy

$$\|\psi\|_{\mathbb{J}^p[s,t]} := \mathbb{E}\Big[\Big(\sum_{i=1}^k \int_s^t \int_{\mathbb{R}_0} |\psi_u^i(x)|^2 \nu^i(dx)du\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} < \infty.$$

- $\mathbb{J}^\infty$ is the space of functions which are $d\mathbb{P}\otimes\nu(dz)$ essentially bounded i.e.,

$$\|\psi\|_{\mathbb{J}^\infty} := \Big\| \sup_{t\in[0,T]} \|\psi_t\|_{\mathbb{L}^\infty(\nu)} \Big\|_\infty < \infty,$$

where $\mathbb{L}^\infty(\nu)$ is the space of $\mathbb{R}^k$-valued measurable functions $\nu(dz)$-a.e. bounded endowed with the usual essential sup-norm.

- $\mathcal{K}^p[s,t]$ is the set of functions (Y,Z,ψ) in the space $\mathbb{S}^p[s,t]\times\mathbb{H}^p[s,t]\times\mathbb{J}^p[s,t]$ with the norm defined by

$$\|(Y,Z,\psi)\|_{\mathcal{K}^p[s,t]} := \big(\|Y\|_{\mathbb{S}^p[s,t]}^p + \|Z\|_{\mathbb{H}^p[s,t]}^p + \|\psi\|_{\mathbb{J}^p[s,t]}^p\big)^{\frac{1}{p}}.$$

For notational simplicity, we use $(E,\mathcal{E}) = (\mathbb{R}_0^k, \mathcal{B}(\mathbb{R}_0)^k)$ and denote the maps $\{\psi^i, 1\leq i\leq k\}$ defined above as $\psi:\Omega\times[0,T]\times E\to\mathbb{R}^k$ and say ψ is $\mathcal{P}\otimes\mathcal{E}$-measurable without referring to each component. We also use the notation such that

$$\int_s^t \int_E \psi_u(x)\widetilde{\mu}(du,dx) := \sum_{i=1}^k \int_s^t \int_{\mathbb{R}_0} \psi_u^i(x)\widetilde{\mu}^i(du,dx)$$

for simplicity. The similar abbreviation is used also for the integrals with respect to μ and ν. When we use E and $\mathcal{E}$, one should always interpret it in this way so that the integral with the k-dimensional Poisson measure does make sense. On the other hand, when we use the range $\mathbb{R}_0$ with the integrators $(\widetilde{\mu},\mu,\nu)$, for example,

$$\int_{\mathbb{R}_0} \psi_u(x)\nu(dx) := \Big(\int_{\mathbb{R}_0} \psi_u^i(x)\nu^i(dx)\Big)_{1\leq i\leq k}$$

we interpret it as a k-dimensional vector.

We frequently omit the subscripts specifying the dimension r and the time interval $[s, t]$ when they are unnecessary or obvious in the context. We use $\big(\Theta_s, s \in [0, T]\big)$ as a collective argument $\Theta_s = \big(Y_s, Z_s, \psi_s\big)$ to lighten the notation. We use the notation of partial derivatives such that for $x \in \mathbb{R}^d$

$$\partial_x = (\partial_{x_1}, \ldots, \partial_{x_d}) = \Big(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\Big)$$

and for Θ, $\partial_\Theta = \big(\partial_y, \partial_z, \partial_\psi\big)$. We use the similar notations for every higher order derivative without a detailed indexing. We suppress the obvious summation of indexes throughout the paper for notational simplicity.

2.3. BMO-martingale and its properties

The properties of the BMO-martingales play a crucial role throughout this work. This section summarizes the necessary facts used in the following discussions.

Definition 2.1. Let M be a square integrable martingale. When it satisfies

$$\|M\|^2_{BMO} := \sup_{\tau \in \mathcal{T}_0^T} \Big\| \mathbb{E}\Big[(M_T - M_{\tau-}\mathbf{1}_{\tau>0})^2 | \mathcal{F}_\tau\Big] \Big\|_\infty < \infty$$

then M is called a BMO-martingale and denoted by $M \in BMO$.

Lemma 2.1. *Suppose M is a square integrable martingale with initial value $M_0 = 0$. If M is a BMO-martingale, then its jump component is essentially bounded $\Delta M \in \mathbb{S}^\infty$. On the other hand, if $\Delta M \in \mathbb{S}^\infty$ and $\sup_{\tau \in \mathcal{T}_0^T} \Big\| \mathbb{E}\Big[\langle M\rangle_T - \langle M\rangle_\tau | \mathcal{F}_\tau\Big] \Big\|_\infty < \infty$, then M is a BMO-martingale.*

Proof. From Lemma 10.7 in [19], we have

$$\begin{aligned}\|M\|^2_{BMO} &= \sup_{\tau \in \mathcal{T}_0^T} \Big\| \mathbb{E}\Big[[M]_T - [M]_\tau | \mathcal{F}_\tau\Big] + M_0^2 \mathbf{1}_{\tau=0} + (\Delta M_\tau)^2 \Big\|_\infty \\ &= \sup_{\tau \in \mathcal{T}_0^T} \Big\| \mathbb{E}\Big[\langle M\rangle_T - \langle M\rangle_\tau | \mathcal{F}_\tau\Big] + (\Delta M_\tau)^2 \Big\|_\infty .\end{aligned}$$

Thus,

$$\begin{aligned}\sup_{\tau \in \mathcal{T}_0^T} \Big\| \mathbb{E}\Big[\langle M\rangle_T - \langle M\rangle_\tau | \mathcal{F}_\tau\Big] \Big\|_\infty \vee \|\Delta M\|^2_{\mathbb{S}^\infty} &\le \|M\|^2_{BMO} \\ &\le \sup_{\tau \in \mathcal{T}_0^T} \Big\| \mathbb{E}\Big[\langle M\rangle_T - \langle M\rangle_\tau | \mathcal{F}_\tau\Big] \Big\|_\infty + \|\Delta M\|^2_{\mathbb{S}^\infty}\end{aligned}$$

and hence the claim is proved. □

Let us introduce the following spaces. $\mathbb{H}^2_{BMO}$ is the set of progressively measurable $\mathbb{R}^d$-valued functions Z satisfying[2]

$$\|Z\|^2_{\mathbb{H}^2_{BMO}} := \Big\| \textstyle\int_0^\cdot Z_s dW_s \Big\|^2_{BMO} = \sup_{\tau \in \mathcal{T}_0^T} \Big\| \mathbb{E}\Big[\displaystyle\int_\tau^T |Z_s|^2 ds | \mathcal{F}_\tau\Big] \Big\|_\infty < \infty.$$

[2] We sometimes include a scalar function satisfying the rightmost inequality also in $\mathbb{H}^2_{BMO}$. By multiplying a d-dimensional unit vector, one can always connect to it the BMO norm if necessary.

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$\mathbb{J}^2_{BMO}$ and $\mathbb{J}^2_B$ are the sets of $\mathcal{P}\otimes\mathcal{E}$-measurable functions $\psi : \Omega\times[0,T]\times\mathbb{E}\to\mathbb{R}^k$ satisfying

$$\|\psi\|^2_{\mathbb{J}^2_{BMO}} := \Big\|\int_0^\cdot\int_E \psi_s(x)\widetilde{\mu}(ds,dx)\Big\|^2_{BMO}$$
$$= \sup_{\tau\in\mathcal{T}_0^T}\Big\|\mathbb{E}\Big[\int_\tau^T\int_E |\psi_s(x)|^2\mu(ds,dx)|\mathcal{F}_\tau\Big]+(\Delta M_\tau)^2\Big\|_\infty<\infty\,,$$

where ΔM_τ is a jump of $M=\int_0^\cdot\int_E\psi_s(x)\widetilde{\mu}(ds,dx)$ at time τ.

$$\|\psi\|^2_{\mathbb{J}^2_B} := \sup_{\tau\in\mathcal{T}_0^T}\Big\|\mathbb{E}\Big[\int_\tau^T\int_E|\psi_s(x)|^2\nu(dx)ds|\mathcal{F}_\tau\Big]\Big\|_\infty<\infty,$$

respectively. Note that $(\|\psi\|^2_{\mathbb{J}^2_B}\vee\|\psi\|^2_{\mathbb{J}^\infty})\le\|\psi\|^2_{\mathbb{J}^2_{BMO}}\le\|\psi\|^2_{\mathbb{J}^2_B}+\|\psi\|^2_{\mathbb{J}^\infty}$ from the proof of Lemma 2.1.

Lemma 2.2 (*Energy Inequality*). *Let $Z\in\mathbb{H}^2_{BMO}$ and $\psi\in\mathbb{J}^2_{BMO}$. Then, for any $n\in\mathbb{N}$,*

$$\mathbb{E}\Big[\Big(\int_0^T|Z_s|^2ds\Big)^n\Big]\le n!\big(\|Z\|^2_{\mathbb{H}^2_{BMO}}\big)^n,$$
$$\mathbb{E}\Big[\Big(\int_0^T\int_E|\psi_s(x)|^2\mu(ds,dx)\Big)^n\Big]\le n!\big(\|\psi\|^2_{\mathbb{J}^2_{BMO}}\big)^n,$$
$$\mathbb{E}\Big[\Big(\int_0^T\int_E|\psi_s(x)|^2\nu(dx)ds\Big)^n\Big]\le n!\big(\|\psi\|^2_{\mathbb{J}^2_B}\big)^n\le n!\big(\|\psi\|^2_{\mathbb{J}^2_{BMO}}\big)^n.$$

Proof. See proof of Lemma 9.6.5 in [12]. □

Let $\mathcal{E}(M)$ be a Doléan-Dade exponential of M.

Lemma 2.3 (*Reverse Hölder Inequality*). *Let $\delta>0$ be a positive constant and M be a BMO-martingale satisfying $\Delta M_t\ge-1+\delta$ $\mathbb{P}$-a.s. for all $t\in[0,T]$. Then, $\big(\mathcal{E}_t(M),t\in[0,T]\big)$ is a uniformly integrable martingale, and for every stopping time $\tau\in\mathcal{T}_0^T$, there exists some $p>1$ such that $\mathbb{E}\,[\mathcal{E}_T(M)^p|\mathcal{F}_\tau]\le C_{p,M}\mathcal{E}_\tau(M)^p$ with some positive constant $C_{p,M}$ depending only on p and $\|M\|_{BMO}$.*

Proof. See Kazamaki (1979) [26], and also Remark 3.1 of Kazamaki (1994) [27]. □

Note here that the condition $\Delta M_t\ge-1+\delta$ is the very reason why one needs a stronger assumption than the Lipschitz continuity for the comparison principle to hold for the BSDEs with jumps (see Proposition 2.6 in Barles et al. (1997) [3]). The following properties of the *continuous* BMO martingales by Kazamaki [27] are very useful.

Lemma 2.4. *Let M be a square integrable continuous martingale and $\hat{M}:=\langle M\rangle-M$. Then, $M\in BMO(\mathbb{P})$ if and only if $\hat{M}\in BMO(\mathbb{Q})$ with $d\mathbb{Q}/d\mathbb{P}=\mathcal{E}_T(M)$. Furthermore, $\|\hat{M}\|_{BMO(\mathbb{Q})}$ is determined by some function of $\|M\|_{BMO(\mathbb{P})}$ and vice versa.*

Proof. See Theorem 3.3 and Theorem 2.4 in [27]. □

Remark 2.2. For continuous martingales, Theorem 3.1 [27] also tells that there exists some decreasing function $\Phi(p)$ with $\Phi(1+)=\infty$ and $\Phi(\infty)=0$ such that if $\|M\|_{BMO(\mathbb{P})}$ satisfies

$\|M\|_{BMO(\mathbb{P})} < \Phi(p)$ then $\mathcal{E}(M)$ satisfies the reverse Hölder inequality with power p. This implies together with Lemma 2.4, one can take a common positive constant $\bar{r}$ satisfying $1 < \bar{r} \leq r^*$ such that both of the $\mathcal{E}(M)$ and $\mathcal{E}(\hat{M})$ satisfy the reverse Hölder inequality with power $\bar{r}$ under the respective probability measure $\mathbb{P}$ and $\mathbb{Q}$. Furthermore, the upper bound r^* is determined only by $\|M\|_{BMO(\mathbb{P})}$ (or equivalently by $\|M\|_{BMO(\mathbb{Q})}$).

3. $Q_{\exp}$-growth BSDEs with jumps

3.1. Universal bound

We now introduce, for $t \in [0, T]$, the quadratic–exponential ($Q_{\exp}$) growth BSDE;

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \psi_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E \psi_s(x)\widetilde{\mu}(ds, dx) , \tag{3.1}$$

where $\xi : \Omega \to \mathbb{R}$, $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k) \to \mathbb{R}$ and denote Z and ψ as row vectors for simplicity.

Let us introduce the quadratic–exponential structure condition proposed by Barrieu & El Karoui (2013) [4] and extended to a jump diffusion case by Ngoupeyou (2010) [35]. See also El Karoui et al. (2016) [24].

Assumption 3.1. (i)The map $(\omega, t) \mapsto f(\omega, t, \cdot)$ is $\mathbb{F}$-progressively measurable. For every $(y, z, \psi) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$, there exist two constants $\beta \geq 0$ and $\gamma > 0$ and a positive $\mathbb{F}$-progressively measurable process $(l_t, t \in [0, T])$ such that

$$\begin{aligned} &-l_t - \beta|y| - \frac{\gamma}{2}|z|^2 - \int_E j_\gamma(-\psi(x))\nu(dx) \\ &\qquad \leq\ f(t, y, z, \psi)\ \leq\ l_t + \beta|y| + \frac{\gamma}{2}|z|^2 + \int_E j_\gamma(\psi(x))\nu(dx) \end{aligned}$$

$dt \otimes d\mathbb{P}$-a.e. $(\omega, t) \in \Omega \times [0, T]$, where $j_\gamma(u) := \frac{1}{\gamma}\big(e^{\gamma u} - 1 - \gamma u\big)$.
(ii) $|\xi|$, $(l_t, t \in [0, T])$ are essentially bounded, i.e., $\|\xi\|_\infty, \|l\|_{S^\infty} < \infty$.

Assumption 3.1 yields useful universal bounds as Lemmas 3.1 and 3.2 for the possible solutions of (3.1).

Lemma 3.1. *Under Assumption 3.1, if there exists a solution $(Y, Z, \psi) \in \mathbb{S}^\infty \times \mathbb{H}^2 \times \mathbb{J}^2$ to the BSDE (3.1), then $Z \in \mathbb{H}^2_{BMO}$ and $\psi \in \mathbb{J}^2_{BMO}$ (and hence $\psi \in \mathbb{J}^\infty$) and $\|Z\|_{\mathbb{H}^2_{BMO}}, \|\psi\|_{\mathbb{J}^2_{BMO}}$ are bounded by some constant depending only on $(\gamma, \beta, T, \|\xi\|_\infty, \|l\|_{\mathbb{S}^\infty}, \|Y\|_{\mathbb{S}^\infty})$.*

Proof. Since $\|\psi\|_{J^\infty} \leq 2\|Y\|_{\mathbb{S}^\infty}$, it is clear that $\psi \in \mathbb{J}^\infty$. Applying Itô formula to $e^{2\gamma Y_t}$ and using the equality $2\gamma j_{2\gamma}(x) = (e^{\gamma x} - 1)^2 + 2\gamma j_\gamma(x)$, one obtains

$$\begin{aligned} &\int_\tau^T e^{2\gamma Y_s} 2\gamma^2 |Z_s|^2 ds + \int_\tau^T \int_E e^{2\gamma Y_s}\big(e^{\gamma\psi_s(x)} - 1\big)^2 \nu(dx)ds \\ &\quad = e^{2\gamma Y_T} - e^{2\gamma Y_\tau} + 2\gamma \int_\tau^T e^{2\gamma Y_s}\Big(f(s, Y_s, Z_s, \psi_s) - \int_E j_\gamma(\psi_s(x))\nu(dx)\Big)ds \\ &\qquad - \int_\tau^T e^{2\gamma Y_s} 2\gamma Z_s dW_s - \int_\tau^T \int_E e^{2\gamma Y_{s-}}\big(e^{2\gamma\psi_s(x)} - 1\big)\widetilde{\mu}(ds, dx) , \end{aligned}$$

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where $\tau \in \mathcal{T}_0^T$. Taking a conditional expectation and using Assumption 3.1, one obtains

$$\begin{aligned}
&\mathbb{E}\left[\int_\tau^T e^{2\gamma Y_s}\gamma^2|Z_s|^2 ds + \int_\tau^T \int_E e^{2\gamma Y_s}\left(e^{\gamma\psi_s(x)}-1\right)^2 \nu(dx)ds \Big| \mathcal{F}_\tau\right] \\
&\quad \le \mathbb{E}\left[e^{2\gamma Y_T} + 2\gamma\int_\tau^T e^{2\gamma Y_s}\left(l_s + \beta|Y_s|\right)ds\Big|\mathcal{F}_\tau\right] \\
&\quad \le e^{2\gamma\|Y\|_{\mathbb{S}^\infty}} + 2\gamma e^{2\gamma\|Y\|_{\mathbb{S}^\infty}} T\left(\beta\|Y\|_{\mathbb{S}^\infty} + \|l\|_{\mathbb{S}^\infty}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{E}\left[\int_\tau^T \gamma^2|Z_s|^2 ds + \int_\tau^T \int_E \left(e^{\gamma\psi_s(x)}-1\right)^2 \nu(dx)ds \Big| \mathcal{F}_\tau\right] \\
&\le e^{4\gamma\|Y\|_{\mathbb{S}^\infty}} + 2\gamma e^{4\gamma\|Y\|_{\mathbb{S}^\infty}} T\left(\beta\|Y\|_{\mathbb{S}^\infty} + \|l\|_{\mathbb{S}^\infty}\right).
\end{aligned} \tag{3.2}$$

Similar calculation for $e^{-2\gamma Y_t}$ yields

$$\begin{aligned}
&\mathbb{E}\left[\int_\tau^T \gamma^2|Z_s|^2 ds + \int_\tau^T \int_E \left(e^{-\gamma\psi_s(x)}-1\right)^2 \nu(dx)ds \Big| \mathcal{F}_\tau\right] \\
&\le e^{4\gamma\|Y\|_{\mathbb{S}^\infty}} + 2\gamma e^{4\gamma\|Y\|_{\mathbb{S}^\infty}} T\left(\beta\|Y\|_{\mathbb{S}^\infty} + \|l\|_{\mathbb{S}^\infty}\right).
\end{aligned} \tag{3.3}$$

Let us mention the fact that $(e^x-1)^2 + (e^{-x}-1)^2 \ge x^2$, $\forall x \in \mathbb{R}$. Indeed, for $g(x) := (e^x-1)^2 + (e^{-x}-1)^2 - x^2$, we have $g'(x) = 2(e^x-1)e^x + 2(1-e^{-x})e^{-x} - 2x$ which is an odd function. It is easy to see that $g'(x) \ge 0$ for $x \ge 0$ and $g'(0) = 0$. Thus $g(x) \ge g(0) = 0$. With the help of this relation, adding (3.2) and (3.3), and then taking $\sup_\tau \|\ \|_\infty$ separately for Z and ψ terms yields

$$\|Z\|^2_{\mathbb{H}^2_{BMO}} + \|\psi\|^2_{\mathbb{J}^2_B} \le \frac{e^{4\gamma\|Y\|_{\mathbb{S}^\infty}}}{\gamma^2}\left(3 + 6\gamma T\left(\beta\|Y\|_{\mathbb{S}^\infty} + \|l\|_{\mathbb{S}^\infty}\right)\right) < \infty.$$

Since $\|\psi\|_{\mathbb{J}^\infty} \le 2\|Y\|_{\mathbb{S}^\infty}$, one also sees $\|\psi\|_{\mathbb{J}^2_{BMO}} \le \|\psi\|_{\mathbb{J}^2_B} + \|\psi\|_{\mathbb{J}^\infty} < \infty$. □

The following result is an adaptation of Proposition 3.2 in [4] and Proposition 16 in [35] to our setting. Similar results can be found in [9] for a diffusion setup and in [34,2] with jumps.

Lemma 3.2. *Under Assumption 3.1, if there exists a solution $(Y, Z, \psi) \in \mathbb{S}^\infty \times \mathbb{H}^2 \times \mathbb{J}^2$ to the BSDE (3.1), it satisfies*

$$|Y_t| \le \frac{1}{\gamma}\ln\mathbb{E}\left[\exp\left(\gamma e^{\beta(T-t)}|\xi| + \gamma\int_t^T e^{\beta(s-t)} l_s ds\right)\Big|\mathcal{F}_t\right],$$

and in particular,

$$\|Y\|_{\mathbb{S}^\infty} \le e^{\beta T}\left(\|\xi\|_\infty + T\|l\|_{\mathbb{S}^\infty}\right).$$

Proof. An application of Meyer–Itô formula (Theorem 70 in [38]) yields

$$\begin{aligned}
d\left(e^{\beta s}|Y_s|\right) &= e^{\beta s}\left(\beta|Y_s|ds + d|Y_s|\right) \\
&= e^{\beta s}\left\{\beta|Y_s|ds + \mathrm{sign}(Y_{s-})\left(-f(s,\Theta_s)ds + Z_s dW_s + \int_E \psi_s(x)\widetilde{\mu}(ds,dx)\right) + dL_s^Y\right\}
\end{aligned}$$

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where L^Y is a non-decreasing process including a local time of Y at the origin. Let us define the process $(B_s, s \in [0,T])$ with $B_0 = 0$ by

$$dB_s = -\mathrm{sign}(Y_s) f(s, \Theta_s) ds + \Big(l_s + \beta |Y_s| + \frac{\gamma}{2}|Z_s|^2 + \int_E j_\gamma(\mathrm{sign}(Y_s)\psi_s(x))\nu(dx) \Big) ds$$

which is also a non-decreasing process by Assumption 3.1. Using this process,

$$\begin{aligned} d(e^{\beta s}|Y_s|) &= e^{\beta s}(dB_s + dL_s^Y) + e^{\beta s}\mathrm{sign}(Y_{s-})\Big(Z_s dW_s + \int_E \psi_s(x)\widetilde{\mu}(ds, dx) \Big) \\ &\quad - e^{\beta s}\Big(l_s + \frac{\gamma}{2}|Z_s|^2 + \int_E j_\gamma(\mathrm{sign}(Y_s)\psi_s(x))\nu(dx) \Big) ds , \end{aligned}$$

which is further transformed as

$$\begin{aligned} d(e^{\beta s}|Y_s|) &= e^{\beta s}\mathrm{sign}(Y_{s-})\Big(Z_s dW_s + \int_E \psi_s(x)\widetilde{\mu}(ds, dx) \Big) - \frac{\gamma}{2}\big|e^{\beta s}\mathrm{sign}(Y_s) Z_s\big|^2 ds \\ &\quad - \int_E j_\gamma(e^{\beta s}\mathrm{sign}(Y_s)\psi_s(x))\nu(dx) ds - e^{\beta s} l_s ds + \frac{\gamma}{2}\Big(e^{2\beta s}|Z_s|^2 - e^{\beta s}|Z_s|^2 \Big) ds \\ &\quad + \int_E \Big(j_\gamma(e^{\beta s}\mathrm{sign}(Y_s)\psi_s(x)) - e^{\beta s} j_\gamma(\mathrm{sign}(Y_s)\psi_s(x)) \Big)\nu(dx) ds + e^{\beta s}(dB_s + dL_s^Y) . \end{aligned}$$

It is easy to confirm that for $k \geq 1$,

$$j_\gamma(kx) - k j_\gamma(x) = \frac{1}{\gamma}(e^{k\gamma x} - k e^{\gamma x} - 1 + k) \geq 0 .$$

Thus we obtain

$$\begin{aligned} d(e^{\beta s}|Y_s|) &= e^{\beta s}\mathrm{sign}(Y_{s-})\Big(Z_s dW_s + \int_E \psi_s(x)\widetilde{\mu}(ds, dx) \Big) \\ &\quad - \frac{\gamma}{2}|e^{\beta s}\mathrm{sign}(Y_s) Z_s|^2 ds - \int_E j_\gamma(e^{\beta s}\mathrm{sign}(Y_s)\psi_s(x))\nu(dx) ds - e^{\beta s} l_s ds + dC_s, \end{aligned}$$

where C is a non-decreasing process.

Define the process P by $P_t := \exp\Big(\gamma e^{\beta t}|Y_t| + \gamma \int_0^t e^{\beta s} l_s ds \Big)$. Using another non-decreasing process C', one has

$$\begin{aligned} dP_t &= P_{t-}\Big(\gamma e^{\beta t}\mathrm{sign}(Y_t) Z_t dW_t + \int_E \Big(\exp\big(\gamma e^{\beta t}\mathrm{sign}(Y_{t-})\psi_t(x) \big) - 1 \Big) \\ &\qquad \times \widetilde{\mu}(dt, dx) + \gamma dC'_t \Big). \end{aligned} \tag{3.4}$$

The boundedness of P and Lemma 3.1 imply that the first two terms of (3.4) are true martingale and that the last term is an integrable increasing process. Therefore P is a submartingale and it follows that

$$\exp\Big(\gamma e^{\beta t}|Y_t| + \gamma \int_0^t e^{\beta s} l_s ds \Big) \leq \mathbb{E}\Big[\exp\Big(\gamma e^{\beta T}|\xi| + \gamma \int_0^T e^{\beta s} l_s ds \Big) \Big| \mathcal{F}_t \Big] ,$$

for $\forall t \in [0,T]$, and the claim is proved. □

3.2. Stability and uniqueness

We now introduce local Lipschitz conditions to derive the stability and uniqueness result for a bounded solution.

Assumption 3.2. For each $M > 0$, and for every $(y, z, \psi), (y', z', \psi') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$ satisfying

$$|y|, |y'|, \|\psi\|_{\mathbb{L}^\infty(\nu)}, \|\psi'\|_{\mathbb{L}^\infty(\nu)} \le M$$

there exists some positive constant K_M possibly depending on M such that

$$\begin{aligned}&\big|f(t, y, z, \psi) - f(t, y', z', \psi')\big| \le K_M\big(|y - y'| + \|\psi - \psi'\|_{\mathbb{L}^2(\nu)}\big)\\ &\quad + K_M\big(1 + |z| + |z'| + \|\psi\|_{\mathbb{L}^2(\nu)} + \|\psi'\|_{\mathbb{L}^2(\nu)}\big)|z - z'|\end{aligned}$$

$dt \otimes d\mathbb{P}$-a.e. $(\omega, t) \in \Omega \times [0, T]$.

Consider the two BSDEs with $i \in \{1, 2\}$ satisfying Assumptions 3.1 and 3.2;

$$Y_t^i = \xi^i + \int_t^T f^i(s, Y_s^i, Z_s^i, \psi_s^i)ds - \int_t^T Z_s^i dW_s - \int_t^T \int_E \psi_s^i(x)\widetilde{\mu}(ds, dx), \tag{3.5}$$

for $t \in [0, T]$ and let us denote

$$\begin{aligned}&\delta Y := Y^1 - Y^2, \quad \delta Z := Z^1 - Z^2, \quad \delta\psi := \psi^1 - \psi^2,\\ &\delta f(s) := (f^1 - f^2)(s, Y_s^1, Z_s^1, \psi_s^1)\,.\end{aligned}$$

Lemma 3.3. *Suppose Assumptions 3.1 and 3.2 hold for the two BSDEs (3.5) with $i \in \{1, 2\}$. Then, if there exists a solution $(Y^i, Z^i, \psi^i) \in \mathbb{S}^\infty \times \mathbb{H}^2 \times \mathbb{J}^2$, $i \in \{1, 2\}$ to the BSDEs, the following inequalities are satisfied;*

$$\begin{aligned}&(a)\ \|\delta Z\|_{\mathbb{H}^2_{BMO}} + \|\delta\psi\|_{\mathbb{J}^2_{BMO}}\\ &\quad\le C\Big(\|\delta Y\|_{\mathbb{S}^\infty} + \|\delta\xi\|_\infty + \sup_{\tau\in\mathcal{T}_0^T}\Big\|\mathbb{E}\Big[\int_\tau^T|\delta f(s)|ds\Big|\mathcal{F}_\tau\Big]\Big\|_\infty\Big)\\ &(b)\ \big\|(\delta Y, \delta Z, \delta\psi)\big\|^p_{\mathcal{K}^p[0,T]} \le C'\Big(\mathbb{E}\Big[|\delta\xi|^{p\bar{q}^2} + \Big(\int_0^T|\delta f(s)|ds\Big)^{p\bar{q}^2}\Big]\Big)^{\frac{1}{\bar{q}^2}},\\ &\quad\forall p \ge 2,\ \forall\bar{q} \ge q_*.\end{aligned}$$

Here, C and q_* (> 1) are positive constants depending only on $(K_M, \gamma, \beta, T, \|\xi\|_\infty, \|l\|_{\mathbb{S}^\infty})$ and the constant M is chosen such that $\|Y^i\|_{\mathbb{S}^\infty}, \|\psi^i\|_{\mathbb{J}^\infty} \le M$ for both $i \in \{1, 2\}$. C' is a positive constant depending only on $(p, \bar{q}, K_M, \gamma, \beta, T, \|\xi\|_\infty, \|l\|_{\mathbb{S}^\infty})$.

Proof. *Proof for* (a)
Firstly, due to the universal bounds, it is obvious that one can choose M such that $\|Y^i\|_{\mathbb{S}^\infty} \le M$ and $\|\psi^i\|_{\mathbb{J}^\infty} \le M$ for both $i \in \{1, 2\}$. For $\forall\tau \in \mathcal{T}_0^T$, one has

$$\begin{aligned}&|\delta Y_\tau|^2 + \int_\tau^T|\delta Z_s|^2ds + \int_\tau^T\int_E|\delta\psi_s(x)|^2\mu(ds, dx)\\ &= |\delta\xi|^2 + \int_\tau^T 2\delta Y_s\Big(\delta f(s) + f^2(s, \Theta_s^1) - f^2(s, \Theta_s^2)\Big)ds\\ &\quad - \int_\tau^T 2\delta Y_s\delta Z_s dW_s - \int_\tau^T\int_E 2\delta Y_{s-}\delta\psi_s(x)\widetilde{\mu}(ds, dx)\,.\end{aligned}$$

Taking the conditional expectation, one obtains

$$|\delta Y_\tau|^2+\mathbb{E}\Big[\int_\tau^T|\delta Z_s|^2ds|\mathcal{F}_\tau\Big]+\mathbb{E}\Big[\int_\tau^T\int_E|\delta\psi_s(x)|^2\mu(ds,dx)\Big|\mathcal{F}_\tau\Big]$$
$$=\mathbb{E}\left[|\delta\xi|^2+\int_\tau^T 2\delta Y_s\Big(\delta f(s)+f^2(s,\Theta_s^1)-f^2(s,\Theta_s^2)\Big)ds\Big|\mathcal{F}_\tau\right].$$

Taking $\sup_{\tau\in\mathcal{T}_0^T}$ for each term in the left gives

$$\|\delta Z\|^2_{\mathbb{H}^2_{BMO}}+\|\delta\psi\|^2_{\mathbb{J}^2_B}\le 2\|\delta\xi\|^2_\infty$$
$$+\,4\|\delta Y\|_{\mathbb{S}^\infty}\sup_{\tau\in\mathcal{T}_0^T}\left\|\mathbb{E}\left[\int_\tau^T\Big(|\delta f(s)|+K_M\big(|\delta Y_s|+\|\delta\psi_s\|_{\mathbb{L}^2(\nu)}+H_s|\delta Z_s|\big)\Big)ds\Big|\mathcal{F}_\tau\right]\right\|_\infty,$$

where the process H is defined by $H_s:=1+\sum_{i=1}^2\big(|Z_s^i|+\|\psi_s^i\|_{\mathbb{L}^2(\nu)}\big)$. It is clear that $H\in\mathbb{H}^2_{BMO}$ whose norm is dominated by the universal bounds given in Lemma 3.1. One can see

$$\sup_{\tau\in\mathcal{T}_0^T}\left\|\mathbb{E}\left[\int_\tau^T H_s|\delta Z_s|ds\Big|\mathcal{F}_\tau\right]\right\|_\infty$$
$$\le\sup_{\tau\in\mathcal{T}_0^T}\left\|\mathbb{E}\Big[\int_\tau^T|H_s|^2ds\Big|\mathcal{F}_\tau\Big]^{\frac12}\right\|_\infty\sup_{\tau\in\mathcal{T}_0^T}\left\|\mathbb{E}\Big[\int_\tau^T|\delta Z_s|^2ds\Big|\mathcal{F}_\tau\Big]^{\frac12}\right\|_\infty$$
$$\le\|H\|_{\mathbb{H}^2_{BMO}}\|\delta Z\|_{\mathbb{H}^2_{BMO}}.$$

Thus, with an arbitrary positive constant $\epsilon>0$,

$$\|\delta Z\|^2_{\mathbb{H}^2_{BMO}}+\|\delta\psi\|^2_{\mathbb{J}^2_B}\le 2\|\delta\xi\|^2_\infty+2\sup_{\tau\in\mathcal{T}_0^T}\left\|\mathbb{E}\left[\int_\tau^T|\delta f(s)|ds\Big|\mathcal{F}_\tau\right]\right\|^2_\infty$$
$$+\|\delta Y\|^2_{\mathbb{S}^\infty}\Big(2+4K_MT+\frac{4K_M^2}{\epsilon}+\frac{4K_M^2}{\epsilon}\|H\|^2_{\mathbb{H}^2_{BMO}}\Big)+\epsilon\Big(\|\delta Z\|^2_{\mathbb{H}^2_{BMO}}+\|\delta\psi\|^2_{\mathbb{J}^2_B}\Big).$$

Choosing $\epsilon<1$ and noticing the fact that $\|\delta\psi\|_{\mathbb{J}^2_{BMO}}\le\|\delta\psi\|_{\mathbb{J}^2_B}+2\|\delta Y\|_{\mathbb{S}^\infty}$, one obtains the desired result.

Proof for (b)

Define a d-dimensional $\mathbb{F}$-progressively measurable process $(b_s,s\in[0,T])$ by

$$b_s:=\frac{f^2(s,Y_s^1,Z_s^1,\psi_s^1)-f^2(s,Y_s^1,Z_s^2,\psi_s^1)}{|\delta Z_s|^2}\mathbf{1}_{\delta Z_s\neq0}\delta Z_s$$

and also the map $\widetilde{f}:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{L}^2(E,\nu;\mathbb{R}^k)\to\mathbb{R}$ by

$$\widetilde{f}(\omega,s,\widetilde{y},\widetilde{\psi}):=\delta f(\omega,s)-f^2(\omega,s,\Theta_s^2)+f^2\big(\omega,s,\widetilde{y}+Y_s^2,Z_s^2,\widetilde{\psi}+\psi_s^2\big).$$

Then, $(\delta Y,\delta Z,\delta\psi)$ can be interpreted as the solution to the BSDE

$$\delta Y_t=\delta\xi+\int_t^T\Big(\widetilde{f}(s,\delta Y_s,\delta\psi_s)+b_s\cdot\delta Z_s\Big)ds-\int_t^T\delta Z_sdW_s$$
$$-\int_t^T\int_E\delta\psi_s(x)\widetilde{\mu}(ds,dx).\tag{3.6}$$

Since $|b_s|\le K_M(1+|Z_s^1|+|Z_s^2|+2\|\psi_s^1\|_{\mathbb{L}^2(\nu)})$, the process b belongs to $\mathbb{H}^2_{BMO}$. Furthermore, $\widetilde{f}$ satisfies the linear growth property $|\widetilde{f}(s,\widetilde{y},\widetilde{\psi})|\le|\delta f(s)|+K_M(|\widetilde{y}|+\|\widetilde{\psi}\|_{\mathbb{L}^2(\nu)})$. Thus, the

BSDE (3.6) satisfies Assumption A.1 with $g = |\delta f|$. One obtains the desired result by applying Lemma A.1. The dependency of the constants C', q_* is obtained from the universal bounds in Lemmas 3.1 and 3.2, as well as the properties of the reverse Hölder inequality in Lemma 2.3 and the remarks that follow. □

We now gives the uniqueness result:

Proposition 3.1. *Suppose the BSDE (3.1) satisfies Assumptions 3.1* and *3.2. Then, if there exists a solution $(Y, Z, \psi) \in \mathbb{S}^\infty \times \mathbb{H}^2 \times \mathbb{J}^2$ to (3.1), it is unique in the space $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$.*

Proof. By Lemmas 3.2 and 3.1, if there exists such a solution it satisfies $(Y, Z, \psi) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$. Firstly, by Lemma 3.3(b), the solution is unique in the space $\mathcal{K}^p[0, T]$ for $\forall p \geq 2$. Since $Y \in \mathbb{S}^\infty$, the uniqueness of Y in $\mathbb{S}^p$ gives the uniqueness of Y also in the space $\mathbb{S}^\infty$. This can be easily shown from an argument of contradiction by assuming $\|Y^1 - Y^2\|^p_{\mathbb{S}^p} = 0$ but not equal in $\mathbb{S}^\infty$. □

4. Existence of solution to a $Q_{\exp}$-growth BSDE

In this section, we prove the existence of the solution to the BSDE (3.1). Although one may use the stability of quadratic semimartingales as [24], we provide a concrete, less abstract strategy similar to that of Kobylanski [29]. We need another assumption so that we can apply the comparison principle.

Assumption 4.1 (A_Γ*-condition*)**.** For all $t \in [0, T]$, $M > 0$, $y \in \mathbb{R}, z \in \mathbb{R}^d, \psi, \psi' \in \mathbb{L}^2(E, \nu; \mathbb{R}^k)$ with $|y|, \|\psi\|_{\mathbb{L}^\infty(\nu)}, \|\psi'\|_{\mathbb{L}^\infty(\nu)} \leq M$, there exists a $\mathcal{P} \otimes \mathcal{E}$-measurable process $\Gamma^{y,z,\psi,\psi'}$ satisfying $dt \otimes d\mathbb{P}$-a.e.

$$f(t, y, z, \psi) - f(t, y, z, \psi') \leq \int_E \Gamma_t^{y,z,\psi,\psi'}(x)\big[\psi(x) - \psi'(x)\big]\nu(dx) \tag{4.1}$$

and $C^1_M(1 \wedge |x|) \leq \Gamma_t^{y,z,\psi,\psi'}(x) \leq C^2_M(1 \wedge |x|)$ with two constants C^1_M, C^2_M. Here, $C^1_M > -1$ and $C^2_M > 0$ depend on M. (Hereafter, we frequently omit the superscripts y, z to lighten the notation.)[3]

Let us introduce a sequence of smooth truncation functions $\varphi_m : \mathbb{R} \to \mathbb{R}$ with $m \in \mathbb{N}$ with the following properties:

$$\varphi_m(x) = \begin{cases} -(m+1) & \text{for } x \leq -(m+2) \\ x & \text{for } |x| \leq m \\ m+1 & \text{for } x \geq m+2 \end{cases} \tag{4.2}$$

[3] A_Γ-condition implies M-dependent *local Lipschitz* continuity with respect to ψ, which is known to be satisfied in the case of the exponential utility optimization [34].

and $|\partial_x \varphi_m(x)| \leq 1$ uniformly in $x \in \mathbb{R}$.[4] We denote $\overline{f} := f \vee 0$, $\underline{f} := f \wedge 0$ and introduce the following regularization of the driver:

$$\begin{aligned}
\overline{f}^n(t,y,z,\psi) &:= \inf_{w\in\mathbb{R}^d}\{\overline{f}(t,y,w,\psi) + n|z-w|\} \\
\underline{f}^m(t,y,z,\psi) &:= \sup_{w\in\mathbb{R}^d}\{\underline{f}(t,y,w,\psi) - m|z-w|\} \\
\overline{f}^{n,k}(t,y,z,\psi) &:= \overline{f}^n(t,\varphi_k(y),z,\varphi_k(\psi)) \\
\underline{f}^{m,k}(t,y,z,\psi) &:= \underline{f}^m(t,\varphi_k(y),z,\varphi_k(\psi))
\end{aligned}$$

and $f^{n,m} := \overline{f}^n + \underline{f}^m$, $f^{n,m,k} := \overline{f}^{n,k} + \underline{f}^{m,k}$. For ψ, the mollifier φ_k should be applied component-wise.

Lemma 4.1. *For a driver f satisfying Assumptions 3.1, 3.2 and 4.1, we have*
(i) $\overline{f}^n, \underline{f}^m, \overline{f}^{n,k}, \underline{f}^{m,k}, f^{n,m}, f^{n,m,k}$ *satisfy the structure condition of Assumption 3.1 uniformly in $n,m,k \in \mathbb{N}$.*
(ii) $\overline{f}^n, \underline{f}^m$ *and $f^{n,m}$ satisfy A_Γ-condition (4.1) uniformly in $n,m \in \mathbb{N}$.*
(iii) $\overline{f}^{n,k}, \underline{f}^{m,k}, f^{n,m,k}$ *are globally Lipschitz continuous for each $n,m,k \in \mathbb{N}$.*

Proof. (i) One can easily confirm the assertion from the fact that $0 \leq \overline{f}^n \leq \overline{f}^{n+1} \leq \overline{f}$, $\underline{f} \leq \underline{f}^{m+1} \leq \underline{f}^m \leq 0$ and that $j_\gamma(\cdot)$ is convex. (ii) Firstly, let us check the condition for $\overline{f}^n$. Since

$$\begin{aligned}
&\overline{f}^n(t,y,z,\psi) - \overline{f}^n(t,y,z,\psi') \\
&= \inf_{w\in\mathbb{R}^d}\{\overline{f}(t,y,w,\psi) + n|z-w|\} - \inf_{w\in\mathbb{R}^d}\{\overline{f}(t,y,w,\psi') + n|z-w|\} \\
&\leq \sup_{w\in\mathbb{R}^d}\{\overline{f}(t,y,w,\psi) - \overline{f}(t,y,w,\psi')\}
\end{aligned}$$

one sees the desired result by considering the four cases of signs $(f(t,y,w,\psi), f(t,y,w,\psi')) = (+,+), (+,-), (-,+), (-,-)$. The first two cases are bounded by $f(\cdot,\psi) - f(\cdot,\psi')$. The last two cases are bounded by 0 and hence the condition is trivially satisfied. Similar analysis yields the same conclusion for $\underline{f}^m$. Finally, let us consider $f^{n,m}$. Based on the same categorization of signs $(f(\cdot,\psi), f(\cdot,\psi'))$, we have

$$f^{n,m}(\cdot,\psi) - f^{n,m}(\cdot,\psi') = \begin{cases} \overline{f}^n(\cdot,\psi) - \overline{f}^n(\cdot,\psi') & \text{if} \quad (+,+) \\ \underline{f}^m(\cdot,\psi) - \underline{f}^m(\cdot,\psi') & \text{if} \quad (-,-) \\ \underline{f}^m(\cdot,\psi) - \overline{f}^n(\cdot,\psi') & \text{if} \quad (-,+) \\ \overline{f}^n(\cdot,\psi) - \underline{f}^m(\cdot,\psi') & \text{if} \quad (+,-). \end{cases}$$

The first two cases satisfy A_Γ-condition by the previous discussion. The third case is trivial since it is bounded by 0. As for the last case, one sees $\overline{f}^n(\cdot,\psi) - \underline{f}^m(\cdot,\psi') \leq \overline{f}(\cdot,\psi) - \underline{f}(\cdot,\psi') = f(\cdot,\psi) - f(\cdot,\psi')$ and hence the conclusion follows. (iii) Lipschitz continuity with respect to y, ψ arguments can be shown similarly as (ii) above. Consider now the following obvious inequality $\overline{f}(t,\varphi_k(y),w,\varphi_k(\psi)) + n|z-w| \leq \overline{f}(t,\varphi_k(y),w,\varphi_k(\psi)) + n|z'-w| + n|z-z'|$. By taking $\inf_w$ in the both hands, we get $\overline{f}^{n,k}(t,y,z,\psi) \leq \overline{f}^{n,k}(t,y,z',\psi) + n|z-z'|$. The desired result

[4] The smoothness is introduced just for convenience so that one can use the same function later when proving Malliavin differentiability.

follows by flipping the role of z, z'. The same conclusion follows similarly for $\underline{f}^{m,k}$ and hence also $f^{n,m,k}$. □

The above regularization is inspired by [30,24,12] as an application to quadratic BSDEs. However, notice the differences from the one used in [24] regarding the arguments of y, ψ. The following result is an extension of Lemma 9.6.6 in [12] for our setting.

Proposition 4.1. *Suppose $\xi \in \mathcal{F}_T$ is bounded and the sequence $\{f^n, n \geq 1\}$ and f of the drivers are such that* (i) *They are continuous mappings and satisfy Assumptions* 3.1 and 4.1 *uniformly.* (ii) *$f^n \downarrow f$ (resp. $f^n \uparrow f$).* (iii) *If $y^n \to y$ in $\mathbb{R}$, $z^n \to z$ in $\mathbb{R}^d$ and $\psi^n \to \psi$ in $\mathbb{L}^2(\nu)$, then $f^n(\cdot, y^n, z^n, \psi^n) \to f(\cdot, y, z, \psi)$ in $\mathbb{R}$.* (iv) *There exists a solution $(Y^n, Z^n, \psi^n) \in \mathbb{S}^\infty \times \mathbb{H}^2 \times \mathbb{J}^2$ to the BSDE for each n*

$$Y_t^n = \xi + \int_t^T f^n(s, Y_s^n, Z_s^n, \psi_s^n)ds - \int_t^T Z_s^n dW_s - \int_t^T \int_E \psi_s^n(x)\widetilde{\mu}(ds, dx), \ t \in [0, T],$$

for which the comparison principle holds i.e. $Y_t^{n+1} \leq Y_t^n$ (resp. $Y_t^n \leq Y_t^{n+1}$) for $\forall t \in [0, T]$ a.s. Then, there exists $(Y, Z, \psi) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ such that $Y^n \to Y$ in $\mathbb{S}^\infty$, $Z^n \to Z$ in $\mathbb{H}^2$ and $\psi^n \to \psi$ in $\mathbb{J}^2$ and solves the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \psi_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E \psi_s(x)\widetilde{\mu}(ds, dx), \ t \in [0, T]. \tag{4.3}$$

Proof. It suffices to consider the case $f^n \downarrow f$ with monotonically decreasing sequence of Y^n. By condition (i), the solution (Y^n, Z^n, ψ^n) satisfies the universal bounds given in Lemmas 3.1 and 3.2 uniformly in n. By monotonicity, (Y^n) converges, for all $t \in [0, T]$, $Y_t^n \downarrow Y_t$ $\mathbb{P}$-a.s. to its limit process $Y := lim_n Y^n$. Furthermore, there exists (Z, ψ) satisfying the universal bounds, such that $Z^n \rightharpoonup Z$ weakly in $\mathbb{H}^2$ as well as $\psi^n \rightharpoonup \psi$ weakly in $\mathbb{J}^2$ under an appropriate subsequence (still denoted by the same n). By condition (i), each driver f^n satisfies, $dt \otimes d\mathbb{P}$-a.e., $f^n(t, Y_t^n, Z_t^n, \psi_t^n) \leq f^n(t, Y_t^n, Z_t^n, 0) + \int_E \Gamma_t^{\psi^n, 0}(x)\psi_t^n(x)\nu(dx) \leq l_t + \beta|Y_t^n| + \frac{\gamma}{2}|Z_t^n|^2 + C_M\|\psi_t^n\|_{\mathbb{L}^2(\nu)}$and similarly $-f^n(t, Y_t^n, Z_t^n, \psi_t^n) \leq l_t + \beta|Y_t^n| + \frac{\gamma}{2}|Z_t^n|^2 + C_M\|\psi_t^n\|_{\mathbb{L}^2(\nu)}$, where C_M is a constant depending only on the universal bounds.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth *convex* function such that $\phi(0) = 0$, $\phi'(0) = 0$, which will be specified later. We put $\delta Y^{n,m} := Y^n - Y^m$, $\delta Z^{n,m} := Z^n - Z^m$, $\delta\psi^{n,m} := \psi^n - \psi^m$, and assume $m \geq n$. Note that $\delta Y_T^{n,m} = 0$ and $\delta Y^{n,m} \geq 0$ for $m \geq n$. Itô formula gives

$$\begin{aligned}
&\phi(\delta Y_t^{n,m}) + \int_t^T \frac{1}{2}\phi''(\delta Y_s^{n,m})|\delta Z_s^{n,m}|^2 ds + \int_t^T \int_E \big[\phi(\delta Y_{s-}^{n,m} + \delta\psi_s^{n,m}(x)) - \phi(\delta Y_{s-}^{n,m}) \\
&- \phi'(\delta Y_{s-}^{n,m})\delta\psi_s^{n,m}(x)\big]\mu(ds, dx) = \int_t^T \phi'(\delta Y_s^{n,m})\big[f^n(s, \Theta_s^n) - f^m(s, \Theta_s^m)\big]ds \\
&- \int_t^T \phi'(\delta Y_s^{n,m})\delta Z_s^{n,m} dW_s - \int_t^T \int_E \phi'(\delta Y_{s-}^{n,m})\widetilde{\mu}(ds, dx) .
\end{aligned}$$

Using the previous driver's bound and noticing that $\phi'(y) \geq 0$ for $y \geq 0$, there exist constants C_M, C_0 independent of n, m satisfying

$$\mathbb{E}\int_0^T \frac{1}{2}\phi''(\delta Y_s^{n,m})|\delta Z_s^{n,m}|^2 ds + \mathbb{E}\int_0^T \int_E \big[\phi(\delta Y_{s-}^{n,m} + \delta\psi_s^{n,m}(x)) - \phi(\delta Y_{s-}^{n,m})$$
$$- \phi'(\delta Y_{s-}^{n,m})\delta\psi_s^{n,m}(x)\big]\mu(ds,dx) \leq \mathbb{E}\int_0^T C_M\phi'(\delta Y_s^{n,m})\Big(\frac{1}{\epsilon} + |Y_s^n| + |Y_s^m|$$
$$+ |Z_s^n|^2 + |Z_s^m|^2 + \epsilon\|\psi_s^n\|^2_{\mathbb{L}^2(\nu)} + \epsilon\|\psi_s^m\|^2_{\mathbb{L}^2(\nu)}\Big)ds$$
$$\leq \mathbb{E}\int_0^T C_0\phi'(\delta Y_s^{n,m})\Big(\frac{1}{\epsilon} + |\delta Z_s^{n,m}|^2 + |Z_s^n - Z_s|^2 + |Z_s|^2$$
$$+ \epsilon\|\delta\psi_s^{n,m}\|^2_{\mathbb{L}^2(\nu)} + \epsilon\|\psi_s^n - \psi_s\|^2_{\mathbb{L}^2(\nu)} + \epsilon\|\psi_s\|^2_{\mathbb{L}^2(\nu)}\Big)ds \tag{4.4}$$

for any constant $\epsilon > 0$. We now choose ϕ as

$$\phi(y) := \frac{1}{8C_0^2}\big[e^{4C_0 y} - 4C_0 y - 1\big], \quad \phi'(y) = \frac{1}{2C_0}\big[e^{4C_0 y} - 1\big], \quad \phi''(y) = 2e^{4C_0 y}.$$

By the mean-value theorem and the universal bound of Lemma 3.2 for $\delta Y_s^{n,m}, \delta Y_{s-}^{n,m}$,

$$c_M|\delta\psi_s^{n,m}(x)|^2 \leq \phi(\delta Y_{s-}^{n,m} + \delta\psi_s^{n,m}(x)) - \phi(\delta Y_{s-}^{n,m}) - \phi'(\delta Y_{s-}^{n,m})\delta\psi_s^{n,m}(x)$$

holds uniformly in (n,m) by choosing $c_M := \exp\big(-8C_0 e^{\beta T}(\|\xi\|_\infty + T\|l\|_{\mathbb{S}^\infty})\big)$. Similarly, one can choose the constant ϵ such that $C_0\phi'\big(2e^{\beta T}(\|\xi\|_\infty + T\|l\|_{\mathbb{S}^\infty})\big)\epsilon = c_M/4$. Then (4.4) implies (note that $\phi''(y) = 4C_0\phi'(y) + 2$),

$$\mathbb{E}\int_0^T \big[C_0\phi'(\delta Y_s^{n,m}) + 1\big]|\delta Z_s^{n,m}|^2 ds + \mathbb{E}\int_0^T \int_E \frac{3}{4}c_M|\delta\psi_s^{n,m}(x)|^2\nu(dx)ds$$
$$\leq \mathbb{E}\int_0^T C_0\phi'(\delta Y_s^{n,m})\Big(\frac{1}{\epsilon} + |Z_s^n - Z_s|^2 + |Z_s|^2 + \epsilon\|\psi_s^n - \psi_s\|^2_{\mathbb{L}^2(\nu)} + \epsilon\|\psi_s\|^2_{\mathbb{L}^2(\nu)}\Big)ds.$$

Let fix n. $\delta\psi^{n,m} \rightharpoonup \psi^n - \psi$ weakly in $\mathbb{J}^2$. Since $\delta Y^{n,m}$ is bounded and strongly converges $\forall t \in [0,T]$ $\delta Y_t^{n,m} \to Y_t^n - Y_t$ a.s., it is easy to see that $\sqrt{C_0\phi'(\delta Y^{n,m}) + 1}[\delta Z^{n,m}]$ converges weakly to $\sqrt{C_0\phi'(Y^n - Y) + 1}[Z^n - Z]$ in $\mathbb{H}^2$. From Proposition 3.5 (iii) [7], by passing to the limit $m \to \infty$,

$$\mathbb{E}\int_0^T \big[C_0\phi'(Y_s^n - Y_s) + 1\big]|Z_s^n - Z_s|^2 ds + \mathbb{E}\int_0^T \frac{3}{4}c_M\|\psi_s^n - \psi_s\|^2_{\mathbb{L}^2(\nu)}ds$$
$$\leq \liminf_{m\to\infty}\mathbb{E}\int_0^T \big[C_0\phi'(\delta Y_s^{n,m}) + 1\big]|\delta Z_s^{n,m}|^2 ds + \mathbb{E}\int_0^T\int_E \frac{3}{4}c_M|\delta\psi_s^{n,m}(x)|^2\nu(dx)ds$$
$$\leq \mathbb{E}\int_0^T C_0\phi'(Y_s^n - Y_s)\Big(\frac{1}{\epsilon} + |Z_s^n - Z_s|^2 + |Z_s|^2 + \epsilon\|\psi_s^n - \psi_s\|^2_{\mathbb{L}^2(\nu)} + \epsilon\|\psi_s\|^2_{\mathbb{L}^2(\nu)}\Big)ds,$$

which then yields

$$\mathbb{E}\int_0^T |Z_s^n - Z_s|^2 ds + \mathbb{E}\int_0^T \frac{c_M}{2}\|\psi_s^n - \psi_s\|^2_{\mathbb{L}^2(\nu)}ds$$
$$\leq \mathbb{E}\int_0^T C_0\phi'(Y_s^n - Y_s)\Big(\frac{1}{\epsilon} + |Z_s|^2 + \epsilon\|\psi_s\|^2_{\mathbb{L}^2(\nu)}\Big)ds \; . \tag{4.5}$$

Since $\phi'(Y_s^n - Y_s) \to 0$ a.s. as $n \to \infty$, one concludes $Z^n \to Z$ in $\mathbb{H}^2$ and $\psi^n \to \psi$ in $\mathbb{J}^2$ by the dominated convergence theorem.

Therefore, one can extract a subsequence such that $Z^n \to Z$ $dt \otimes d\mathbb{P}$-a.s. and $\psi^n \to \psi$ $\nu(dx)dt \otimes d\mathbb{P}$-a.s. Thus condition (iii) implies $f^n(t, Y_t^n, Z_t^n, \psi_t^n) \to f(t, Y_t, Z_t, \psi_t)$ $dt \otimes d\mathbb{P}$-a.s. Moreover, by extracting further subsequence if necessary, one sees from Lemma 2.5 of [29] that $G_z := \sup_n |Z^n|^2$, $G_\psi := \sup_n \|\psi^n\|^2_{\mathbb{L}^2(\nu)}$ are in $\mathbb{L}^1([0,T] \times \Omega)$. By assumption (i), for almost all ω, $|f^n(\cdot, Y^n, Z^n, \psi^n)|$ is dominated by $C_M\big(1 + G_z + G_\psi\big) \in \mathbb{L}^1([0,T])$ with some constant C_M depending only on the universal bounds. Note also that $f(\cdot, Y, Z, \psi) \in \mathbb{L}^1([0,T])$ a.s. Thus one obtains, for almost all ω, $\int_0^T |f^n(s, Y_s^n, Z_s^n, \psi_s^n) - f(s, Y_s, Z_s, \psi_s)|ds \to 0$ by Lebesgue's dominated convergence theorem. From (4.5) and the Burkholder–Davis–Gundy inequality,[5] one can also extract a subsequence in which $\sup_{t\in[0,T]}\Big|\int_t^T (Z_s^n - Z_s)dW_s\Big| \to 0$, $\sup_{t\in[0,T]}\Big|\int_t^T \int_E (\psi_s^n(x) - \psi_s(x))\widetilde{\mu}(ds, dx)\Big| \to 0$ a.s. By passing to the limit $m \to \infty$ and taking supremum over t in

$$|Y_t^n - Y_t^m| \le \int_t^T |f^n(s, \Theta_s^n) - f^m(s, \Theta_s^m)|ds + \Big|\int_t^T (Z_s^n - Z_s^m)dW_s\Big| + \Big|\int_t^T \int_E (\psi_s^n(x) - \psi_s^m(x))\widetilde{\mu}(ds, dx)\Big|,$$

one obtains

$$\sup_{t\in[0,T]} |Y_t^n - Y_t| \le \int_0^T |f^n(s, \Theta_s^n) - f(s, \Theta_s)|ds + \sup_{t\in[0,T]}\Big|\int_t^T (Z_s^n - Z_s)dW_s\Big| + \sup_{t\in[0,T]}\Big|\int_t^T \int_E (\psi_s^n(x) - \psi_s(x))\widetilde{\mu}(ds, dx)\Big|,$$

from which one concludes the uniform convergence $\sup_{t\in[0,T]}|Y_t^n - Y_t| \to 0$ a.s. (hence $\|Y^n - Y\|_{\mathbb{S}^\infty} \to 0$) under an appropriate subsequence and (Y, Z, ψ) solves (4.3). One can check that $\mathbb{S}^\infty$ convergence actually occurs in the entire sequence. If this is not the case, there exists a subsequence $(n_j) \subset (n)$ such that $\|Y^{n_j} - Y\|_{\mathbb{S}^\infty} > c$ with some $c > 0$ for all n_j, where $Y = \lim_n Y^n$ is independent of the choice of subsequence due to the monotonicity. However, one can extract a further subsequence $(n_{jk}) \subset (n_j)$ such that $\sup_{t\in[0,T]}|Y_t^{n_{jk}} - Y_t| \to 0$ a.s. by repeating the same discussion given above and hence $\|Y^{n_{jk}} - Y\|_{\mathbb{S}^\infty} \to 0$, which is a contradiction. □

Remark 4.1. By applying Itô-formula to $|Y^n - Y|^2$,

$$|Y_\tau^n - Y_\tau|^2 + \mathbb{E}\Big[\int_\tau^T |Z_s^n - Z_s|^2 ds\Big|\mathcal{F}_\tau\Big] + \mathbb{E}\Big[\int_\tau^T \int_E |\psi_s^n(x) - \psi_s(x)|^2 \mu(ds, dx)\Big|\mathcal{F}_\tau\Big] \le 2\|Y^n - Y\|_{\mathbb{S}^\infty}\mathbb{E}\Big[\int_\tau^T |f^n(s, Y_s^n, Z_s^n, \psi_s^n) - f(s, Y_s, Z_s, \psi_s)|ds\Big|\mathcal{F}_\tau\Big]$$

for any $\tau \in \mathcal{T}_0^T$. It follows that the uniform convergence of $Y^n \to Y$ implies $Z^n \to Z$ and $\psi^n \to \psi$ in $\mathbb{H}^2_{BMO}$ and $\mathbb{J}^2_{BMO}$ respectively, because $\sup_{\tau\in\mathcal{T}_0^T}\Big\|\mathbb{E}\Big[\int_\tau^T |f^n(s, \Theta_s^n) - f(s, \Theta_s)|ds\Big|\mathcal{F}_\tau\Big]\Big\|_\infty \le C\big(1 + \|Z^n\|^2_{\mathbb{H}^2_{BMO}} + \|\psi^n\|^2_{\mathbb{J}^2_{BMO}}\big) \le C$ with some constant C depending only on the universal bounds.[6]

[5] See, for example, Theorem 48 in IV.4. of [38].

[6] Convergence in the norm of $\mathcal{K}^p = \mathbb{S}^p \times \mathbb{H}^p \times \mathbb{J}^p$ with $\forall p \ge 2$ is actually enough for the discussions on Malliavin's differentiability.

Theorem 4.1. *Under Assumptions 3.1, 3.2 and 4.1, there exists a unique bounded solution $(Y, Z, \psi) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ of the BSDE (3.1).*

Proof. From Proposition 3.1, it suffices to prove the existence. Firstly, consider the BSDE with data $(\xi, f^{n,m,k})$. Since $f^{n,m,k}$ is globally Lipschitz, there exists a unique solution $(Y^{n,m,k}, Z^{n,m,k}, \psi^{n,m,k})$ for each n, m, k. One also sees $Y^{n,m,k} \in \mathbb{S}^\infty$ by Lemma B.1. Since the driver $f^{n,m,k}$ satisfies the $Q_{\exp}$-structure condition by Lemma 4.1, $(Y^{n,m,k}, Z^{n,m,k}, \psi^{n,m,k})$ satisfies the universal bounds of Lemmas 3.1 and 3.2 uniformly in n, m, k. In particular, since $\|Y^{n,m,k}\|_{\mathbb{S}^\infty}, \|\psi^{n,m,k}\|_{\mathbb{J}^\infty}$ are bounded uniformly, $(Y^{n,m,k}, Z^{n,m,k}, \psi^{n,m,k})$ also consists of a solution of the BSDE

$$Y_t^{n,m} = \xi + \int_t^T f^{n,m}(s, Y_s^{n,m}, Z_s^{n,m}, \psi_s^{n,m})ds - \int_t^T Z_s^{n,m} dW_s - \int_t^T \int_E \psi_s^{n,m}(x)\widetilde{\mu}(ds, dx) \tag{4.6}$$

for each n, m provided k is large enough. By Lemma B.2, this is actually the unique solution of (4.6) and satisfies the comparison principle $Y^{n,m+1} \le Y^{n,m} \le Y^{n+1,m}$ for every $n, m \in \mathbb{N}$. Thus, from Lemma 4.1, we can apply Proposition 4.1 with a fixed n. In particular, the condition (iii) follows from the continuity of the driver and the property of inf(sup)-convolution (see, Lemma 1 of [30]). We then obtain $Y^{n,m} \to \widetilde{Y}^n$ in $\mathbb{S}^\infty$, $Z^{n,m} \to \widetilde{Z}^n$ in $\mathbb{H}^2$ and $\psi^{n,m} \to \widetilde{\psi}^n$ in $\mathbb{J}^2$, which solves

$$\widetilde{Y}_t^n = \xi + \int_t^T \widetilde{f}^n(s, \widetilde{Y}_s^n, \widetilde{Z}_s^n, \widetilde{\psi}_s^n)ds - \int_t^T \widetilde{Z}_s^n dW_s - \int_t^T \int_E \widetilde{\psi}_s^n(x)\widetilde{\mu}(ds, dx)\,, \tag{4.7}$$

for each $n \in \mathbb{N}$, where $\widetilde{f}^n := \overline{f}^n + \underline{f}$. $\widetilde{f}^n$ satisfies the structure as well as A_Γ-conditions uniformly in n. By Lemma B.3, one can once again apply Proposition 4.1 to the monotone sequence $\widetilde{f}^n \uparrow f$. Then there exists $(Y, Z, \psi) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ with the convergence $\widetilde{Y}^n \to Y$ in $\mathbb{S}^\infty$, $\widetilde{Z}^n \to Z$ in $\mathbb{H}^2$, $\widetilde{\psi}^n \to \psi$ in $\mathbb{J}^2$, which solves the BSDE (3.1). By Remark 4.1, one also obtains the convergence in the stronger norms. □

Although we have used a specific regularization to obtain a monotone sequence of drivers, we can actually weaken the condition of monotonicity. The following result is the adaptation of Theorem 2.8 of [29] to our setting.

Proposition 4.2. *Suppose $\xi \in \mathcal{F}_T$ is bounded and the sequence $\{f^n, n \ge 1\}$ and f of the drivers are such that* (i) *They are continuous mappings and satisfy Assumption 3.1, 3.2, 4.1 uniformly in n.* (ii) *If $y^n \to y$ in $\mathbb{R}$, $z^n \to z$ in $\mathbb{R}^d$ and $\psi^n \to \psi$ in $\mathbb{L}^2(\nu)$, then $f^n(\cdot, y^n, z^n, \psi^n) \to f(\cdot, y, z, \psi)$ in $\mathbb{R}$.* (iii) *Let $(Y^n, Z^n, \psi^n) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ be the unique solution of the BSDE (which is guaranteed by Theorem 4.1)*

$$Y_t^n = \xi + \int_t^T f^n(s, Y_s^n, Z_s^n, \psi_s^n)ds - \int_t^T Z_s^n dW_s - \int_t^T \int_E \psi_s^n(x)\widetilde{\mu}(ds, dx),$$
$$t \in [0, T]$$

for each n. Then $Y^n \to Y$ in $\mathbb{S}^\infty$, $Z^n \to Z$ in $\mathbb{H}^2_{BMO}$ and $\psi^n \to \psi$ in $\mathbb{J}^2_{BMO}$ where (Y, Z, ψ) is a unique solution of (3.1) with data (ξ, f).

Proof. Let us define two drivers such that $G^n := \sup_{m \ge n} f^m$, $H^n := \inf_{m \ge n} f^m$. Then we have $G^n \downarrow f$, $H^n \uparrow f$ as $n \to \infty$. By condition (i), both G^n and H^n satisfy Assumptions 3.1 and 3.2

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uniformly in n. Moreover the relations $G^n(\cdot,\psi)-G^n(\cdot,\psi')\le \sup_{m\ge n}\big[f^m(\cdot,\psi)-f^m(\cdot,\psi')\big]$ and $H^n(\cdot,\psi)-H^n(\cdot,\psi')\le \sup_{m\ge n}\big[f^m(\cdot,\psi)-f^m(\cdot,\psi')\big]$ imply A_Γ-condition of Assumption 4.1 holds uniformly. Thus, by Theorem 4.1, there exists a unique solution $(Y^{n*},Z^{n*},\psi^{n*})$ (resp. (Y^n_*,Z^n_*,ψ^n_*)) in $S^\infty\times\mathbb{H}^2_{BMO}\times\mathbb{J}^2_{BMO}$ to the BSDEs with data (ξ,G^n) (resp. (ξ,H^n)) for each n. By the local Lipschitz continuity, A_Γ-condition, and the universal bounds of the solutions make the measure change used in the comparison principle well defined. Hence, by similar arguments of Lemma B.3, it is straightforward to confirm that the comparison principle holds among (Y^{n*},Y^n_*,Y^n). One has $Y^n_*\le Y^n\le Y^{n*}$ for every $n\in\mathbb{N}$. Furthermore, Proposition 4.1 also imply the convergence $Y^{n*}\downarrow Y$ and $Y^n_*\uparrow Y$ in $\mathbb{S}^\infty$. Thus we have $Y^n\to Y$ in $\mathbb{S}^\infty$. Remark 4.1 gives the convergence of Z^n,ψ^n in the desired norms. □

5. Malliavin differentiability

In the reminder of the paper, we study the Malliavin differentiability of the quadratic–exponential growth BSDEs. Among the various ways to develop Malliavin's calculus, we follow the conventions based on the chaos expansion used in Delong & Imkeller (2010) [15] and Delong (2013) [14], which were adopted from the work of Solé et al. (2007) [40]. See also Di Nunno et al. (2009) [16] for an extension to a multi-dimensional setup and other applications (with only a slight adjustment of conventions). For the detailed conventions, see Section 3 of [15]. Following the extension given in Section 17 of [16], we denote $(D^i_{t,0}, i\in\{1,\dots,d\})$ and $(D^i_{t,z}, i\in\{1,\dots,k\})$ as the Malliavin derivatives with respect to $(W_i(t), i\in\{1,\dots,d\})$ and $(\widetilde{\mu}^i(dt,dz), i\in\{1,\dots,k\})$, respectively.

Note that a random variable F is Malliavin differentiable if and only if $F\in\mathbb{D}^{1,2}$. Here, the space $\mathbb{D}^{1,2}\subset\mathbb{L}^2(\mathbb{P})$ is defined by the completion with respect to the norm $\|\cdot\|_{1,2}$ which is given by

$$\|F\|^2_{1,2}:=\mathbb{E}\Big[|F|^2\Big]+\sum_{i=1}^d\mathbb{E}\left[\int_0^T|D^i_{s,0}F|^2ds\right]+\sum_{i=1}^k\mathbb{E}\left[\int_0^T\int_{\mathbb{R}_0}|D^i_{s,z}F|^2z^2\nu^i(dz)ds\right].$$

For notational convenience, let us introduce two types of finite measures $m^i(dz)=\mathbf{1}_{z\neq 0}z^2\nu^i(dz)$ with $i\in\{1,\dots,k\}$ defined on whole $\mathbb{R}$, and q defined on $\widetilde{E}:=[0,T]\times\mathbb{R}^k$ by

$$q(dt,dz):=\mathbf{1}_{z=0}dt+\sum_{i=1}^k m^i(dz)dt\ .$$

We also introduce a space $\mathbb{L}^{1,2}(\mathbb{R}^n)$ of product measurable and $\mathbb{F}$-adapted processes $\chi:\Omega\times[0,T]\times\mathbb{R}^k\to\mathbb{R}^n$ satisfying

$$\begin{aligned}
&\mathbb{E}\Big[\int_{\widetilde{E}}|\chi(s,y)|^2q(ds,dy)\Big]<\infty,\\
&\chi(s,y)\in\mathbb{D}^{1,2}(\mathbb{R}^n),\ \text{ for } q\text{-a.e. } (s,y)\in\widetilde{E},\\
&\mathbb{E}\Big[\int_{\widetilde{E}}\int_{\widetilde{E}}|D_{t,z}\chi(s,y)|^2q(ds,dy)q(dt,dz)\Big]<\infty.
\end{aligned}$$

Note that the space $\mathbb{L}^{1,2}$ is a Hilbert space endowed with the norm

$$\|\chi\|^2_{\mathbb{L}^{1,2}}:=\mathbb{E}\Big[\int_{\widetilde{E}}|\chi(s,y)|^2q(ds,dy)\Big]+\mathbb{E}\Big[\int_{\widetilde{E}}\int_{\widetilde{E}}|D_{t,z}\chi(s,y)|^2q(ds,dy)q(dt,dz)\Big].$$

The fact that the Malliavin derivative is a closed operator in $\mathbb{L}^{1,2}$ (see, Theorem 12.6 in [16]) plays a crucial role later.

Suppose that (t,z) is a jump of size z at time t in a random measure μ^i. We denote by $\omega^{t,z}_{\mu^i}$ a transformed family of $\omega_{\mu^i} = ((t_1,z_1),(t_2,z_2),\ldots) \in \Omega_{\mu^i}$ into a new family with additional jump at (t,z); $\omega^{t,z}_{\mu^i} = ((t,z),(t_1,z_1),(t_2,z_2),\ldots) \in \Omega_{\mu^i}$. As for an element $\omega = (\omega_W, \omega_{\mu^1}, \omega_{\mu^2}, \ldots, \omega_{\mu^k}) \in \Omega$ in the full canonical product space, we denote $\omega^{t,z} \in \Omega$ as the above transformation only in the corresponding element, such as $\omega^{t,z} = (\omega_W, \omega_{\mu^1}, \ldots, \omega^{t,z}_{\mu^i}, \ldots, \omega_{\mu^k}) \in \Omega$ without specifying the relevant coordinate for notational simplicity. By the same reason, we also frequently omit i denoting the direction of derivative $D^i_{s,z}$ by assuming that we consider each Wiener ($z = 0, i \in \{1,\ldots,d\}$) and jump ($z \neq 0, i \in \{1,\ldots,k\}$)) direction separately (and summing them up whenever necessary, such as when considering integration on $\widetilde{E}$).

In this section, we consider Malliavin's differentiability of the following BSDE;

$$Y_t = \xi + \int_t^T f\Big(s, Y_s, Z_s, \int_{\mathbb{R}_0} \rho(x) G(s, \psi_s(x))\nu(dx)\Big) ds - \int_t^T Z_s dW_s \\ - \int_t^T \int_E \psi_x(x) \widetilde{\mu}(ds, dx), \tag{5.1}$$

for $t \in [0,T]$ where $\xi : \Omega \to \mathbb{R}$, $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$, and $\rho^i : \mathbb{R} \to \mathbb{R}$, $G^i : [0,T] \times \mathbb{R} \to \mathbb{R}$ for each $i \in \{1,\ldots,k\}$. The last arguments of the driver denote a k-dimensional vector whose ith element is given by $\int_{\mathbb{R}_0} \rho^i(x) G^i(s, \psi^i_s(x))\nu^i(dx)$. With slight abuse of notation, we adopt $\Theta_r := \Big(Y_r, Z_r, \int_{\mathbb{R}_0} \rho(z) G(r, \psi_r(z))\nu(dz)\Big)$, $r \in [0,T]$ as a collective argument in this section.

Remark 5.1. In Solé et al. [40] and Delong & Imkeller [15], the conventions

$$\psi(x) \to \psi(x)/x, \quad \widetilde{\mu}(dt,dx) \to x\widetilde{\mu}(dt,dx) \quad x \in \mathbb{R}_0$$

are used. For the convenience when discussing the $\mathbb{L}^{1,2}$-norm, we introduce the notation $\overline{\phi}(x) := \phi(x)/x, x \in \mathbb{R}_0$ for the control variables of the random measure, $\phi = \psi, \psi^m$ etc. See, in particular, Section 3.5 of [14].

Assumption 5.1. (i) For every $i \in \{1,\ldots,k\}$, ρ^i is a continuous function satisfying $\int_{\mathbb{R}_0} |\rho^i(x)|^2 \nu^i(dx) < \infty$. (ii) For every $i \in \{1,\ldots,k\}$, $G^i(s,v)$ is a continuous function in the both arguments and one-time continuously differentiable with respect to v with continuous derivative. Moreover, for every $R > 0$,

$$G_R := \sup_{(s,v)\in[0,T]\times(|v|\le R)} \sum_{i=1}^k |G^i(s,v)| < \infty,$$

$$G'_R := \sup_{(s,v)\in[0,T]\times(|v|\le R)} \sum_{i=1}^k |\partial_v G^i(s,v)| < \infty\,.$$

We put without loss of generality that $G^i(\cdot,0) = 0$ for every $i \in \{1,\ldots,k\}$.

Assumption 5.2. The driver F defined by $F(s,y,z,\psi) := f(s,y,z,\int_{\mathbb{R}_0} \rho(x) G(s,\psi(x))\nu(dx))$ for $s \in [0,T], y \in \mathbb{R}, z \in \mathbb{R}^d, \psi \in \mathbb{L}^2(E,\nu;\mathbb{R}^k)$ and the data (ξ, l) satisfies both Assumptions 3.1 and 4.1.

Assumption 5.3. For each $M > 0$, and for every $(y, z, \psi), (y', z', \psi') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$ satisfying $|y|, |y'|, \|\psi\|_{\mathbb{L}^\infty(\nu)}, \|\psi'\|_{\mathbb{L}^\infty(\nu)} \le M$, there exists some positive constant K_M possibly depending on M such that

$$\begin{aligned}\left|f\left(t, y, z, u_t\right) - f\left(t, y', z', u'_t\right)\right| &\le K_M\left(|y - y'| + |u_t - u'_t|\right)\\ &\quad + K_M\left(1 + |z| + |z'| + |u_t| + |u'_t|\right)|z - z'|\end{aligned}$$

$d\mathbb{P} \otimes dt$-a.e. $(\omega, t) \in \Omega \times [0, T]$, where we have used $u_t := \int_{\mathbb{R}_0} \rho(x)G(t, \psi(x))\nu(dx)$ and $u'_t := \int_{\mathbb{R}_0} \rho(x)G(t, \psi'(x))\nu(dx)$ for notational simplicity.

Remark 5.2. In the above assumption, using the fact that

$$|u_t| \le \|\rho\|_{\mathbb{L}^2(\nu)} G'_M \|\psi\|_{\mathbb{L}^2(\nu)}, \quad |u_t - u'_t| \le \|\rho\|_{\mathbb{L}^2(\nu)} G'_M \|\psi - \psi'\|_{\mathbb{L}^2(\nu)} ,$$

one can see the consistency with Assumption 3.2. Therefore, under Assumptions 5.1–5.3, there exists a unique solution $(Y, Z, \psi) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ to the BSDE (5.1) by Theorem 4.1.

For Malliavin differentiability, we need the following additional assumptions:

Assumption 5.4. With the notation $u_t = \int_{\mathbb{R}_0} \rho(x)G(t, \psi(x))\nu(dx)$, $u'_t = \int_{\mathbb{R}_0} \rho(x)G(t, \psi'(x))\nu(dx)$,

(i) The terminal value is Malliavin differentiable; $\xi \in \mathbb{D}^{1,2}$.

(ii) For each $M > 0$, and for every $(y, z, \psi) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$ satisfying $|y|, \|\psi\|_{\mathbb{L}^\infty(\nu)} \le M$, the driver $\left(f(t, y, z, u_t), t \in [0, T]\right)$ belongs to $\mathbb{L}^{1,2}(\mathbb{R})$ and its Malliavin derivative is denoted by $(D_{s,z} f)(t, y, z, u_t)$. Furthermore, the driver f is one-time continuously differentiable with respect to its spacial variables with continuous derivatives.

(iii) For every Wiener as well as jump direction, for every $M > 0$ and $d\mathbb{P} \otimes dt$-a.e. $(\omega, t) \in \Omega \times [0, T]$, and for every $(y, z, \psi), (y', z', \psi') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$ satisfying $|y|, |y'|, \|\psi\|_{\mathbb{L}^\infty(\nu)}, \|\psi'\|_{\mathbb{L}^\infty(\nu)} \le M$, the Malliavin derivative of the driver satisfies the following local Lipschitz conditions;

$$\begin{aligned}&\left|(D^i_{s,0} f)(t, y, z, u_t) - (D^i_{s,0} f)(t, y', z', u'_t)\right|\\ &\le K^{M,i}_{s,0}(t)\left(|y - y'| + |u_t - u'_t| + (1 + |z| + |z'| + |u_t| + |u'_t|)|z - z'|\right)\end{aligned}$$

for ds-a.e. $s \in [0, T]$ with $i \in \{1, \ldots, d\}$, and

$$\begin{aligned}&\left|(D^i_{s,z} f)(t, y, z, u_t) - (D^i_{s,z} f)(t, y', z', u'_t)\right|\\ &\le K^{M,i}_{s,z}(t)\left(|y - y'| + |u_t - u'_t| + (1 + |z| + |z'| + |u_t| + |u'_t|)|z - z'|\right)\end{aligned}$$

for $m^i(dz)ds$-a.e. $(s, z) \in [0, T] \times \mathbb{R}_0$ with $i \in \{1, \ldots, k\}$. For every $M > 0$ and (s, z), $\left(K^{M,i}_{s,0}(t), t \in [0, T]\right)_{i\in\{1,\ldots,d\}}$ and $\left(K^{M,i}_{s,z}(t), t \in [0, T]\right)_{i\in\{1,\ldots,k\}}$ are $\mathbb{R}_+$-valued $\mathbb{F}$-progressively measurable processes.

(iv) There exists some positive constant $p \ge 2$ such that

$$\int_{\widetilde{E}} \left(\mathbb{E}\left[|D_{s,z}\xi|^{pq} + \left(\int_0^T |(D_{s,z} f)(r, 0)|dr\right)^{pq} + \|K^M_{s,z}\|^{2pq}_T\right]\right)^{\frac{1}{q}} q(ds, dz) < \infty$$

hold for $\forall q \ge 1$ and $\forall M > 0$.

Remark 5.3. Assumption 5.4(iv) implies, for each (s, z) in $\widetilde{E}$ $q(ds, dz)$-a.e.,

$$\mathbb{E}\left[|D_{s,z}\xi|^{p'} + \left(\int_0^T |(D_{s,z} f)(r, 0)|dr\right)^{p'} + \|K^M_{s,z}\|^{2p'}_T\right] < \infty$$

for $\forall p' \geq 2$. In particular, $K^M_{s,0} \in \mathbb{S}^{p'}$ for ds-a.e. $s \in [0,T]$ and $K^M_{s,z} \in \mathbb{S}^{p'}$ for $z^2\nu(dz)ds$-a.e. $(s,z) \in [0,T] \times \mathbb{R}_0$ for $\forall p' \geq 2$.

We now give the main result of this section.

Theorem 5.1. *Suppose that Assumptions 5.1–5.4 hold true and denote the solution to the BSDE (5.1) as $(Y,Z,\psi) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$. Then, the following statements hold:* (a) *For each Wiener direction $i \in \{1,\ldots,d\}$ and ds-a.e. $s \in [0,T]$, there exists a unique solution $(Y^{s,0,i}, Z^{s,0,i}, \psi^{s,0,i}) \in \mathcal{K}^{p'}[0,T]$ with $\forall p' \geq 2$ to the BSDE*

$$Y^{s,0,i}_t = D^i_{s,0}\xi + \int_t^T f^{s,0,i}(r)dr - \int_t^T Z^{s,0,i}_r dW_r - \int_t^T \int_E \psi^{s,0,i}_r(x)\widetilde{\mu}(dr,dx) \tag{5.2}$$

for $0 \leq s \leq t \leq T$, where

$$\begin{aligned} f^{s,0,i}(r) &:= (D^i_{s,0}f)(r,\Theta_r) + \partial_\Theta f(r,\Theta_r)\Theta^{s,0,i}_r \\ &:= (D^i_{s,0}f)(r,\Theta_r) + \partial_y f(r,\Theta_r)Y^{s,0,i}_r + \partial_z f(r,\Theta_r)Z^{s,0,i}_r \\ &\quad + \partial_u f(r,\Theta_r)\int_E \rho(x)\partial_v G(r,\psi_r(x))\psi^{s,0,i}_r(x)\nu(dx)\,. \end{aligned}$$

The solution also satisfies $\int_0^T \|(Y^{s,0,i}, Z^{s,0,i}, \psi^{s,0,i})\|^p_{\mathcal{K}^p[0,T]}ds < \infty$.
(b) *For each jump direction $i \in \{1,\ldots,k\}$ and $m^i(dz)ds$-a.e $(s,z) \in [0,T] \times \mathbb{R}_0$, there exists a unique solution $(Y^{s,z,i}, Z^{s,z,i}, \psi^{s,z,i}) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ to the BSDE*

$$Y^{s,z,i}_t = D^i_{s,z}\xi + \int_t^T f^{s,z,i}(r)dr - \int_t^T Z^{s,z,i}_r dW_r - \int_t^T \int_E \psi^{s,z,i}_r(x)\widetilde{\mu}(dr,dx) \tag{5.3}$$

for $0 \leq s \leq t \leq T$ and $z \neq 0$, where

$$\begin{aligned} f^{s,z,i}(r) &:= \frac{1}{z}\Big(f(\omega^{s,z}, r, \Theta_r + z\Theta^{s,z,i}_r) - f(\omega, r, \Theta_r)\Big) := \frac{1}{z}\Big\{f\Big(\omega^{s,z}, r, Y_r + zY^{s,z,i}_r \\ &\quad , Z_r + zZ^{s,z,i}_r, \int_{\mathbb{R}_0}\rho(x)G\big(r, \psi_r(x) + z\psi^{s,z,i}_r(x)\big)\nu(dx)\Big) - f(\omega, r, \Theta_r)\Big\}\,. \end{aligned}$$

The solution also satisfies $\int_0^T \int_{\mathbb{R}} \|(Y^{s,z,i}, Z^{s,z,i}, \psi^{s,z,i})\|^p_{\mathcal{K}^p[0,T]} m^i(dz)ds < \infty$.
(c) *The solution of the BSDE (5.1) is Malliavin differentiable $(Y, Z, \overline{\psi}) \in \mathbb{L}^{1,2} \times \mathbb{L}^{1,2} \times \mathbb{L}^{1,2}$. Put, for every i, $Y^{s,\cdot,i}_t = Z^{s,\cdot,i}_t = \psi^{s,\cdot,i}_t(\cdot) \equiv 0$ for $t < s \leq T$, then $\big((Y^{s,z,i}_t, Z^{s,z,i}_t, \psi^{s,z,i}_t(x)), 0 \leq s,t \leq T, x \in \mathbb{R}_0, z \in \mathbb{R}\big)$ is a version of the Malliavin derivative $\big((D^i_{s,z}Y_t, D^i_{s,z}Z_t, D^i_{s,z}\psi_t(x)), 0 \leq s,t \leq T, x \in \mathbb{R}_0, z \in \mathbb{R}\big)$ for every Wiener and jump direction.*

Proof. Firstly, from Assumptions 5.1–5.3, Theorem 4.1 tells us that there exists a unique solution $(Y,Z,\psi) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ to the BSDE (5.1). Since $\|Y\|_{\mathbb{S}^\infty}, \|\psi\|_{\mathbb{J}^\infty}$ are bounded by the universal bounds, one can choose a constant $M > 0$ big enough so that the local Lipschitz conditions hold true for the whole relevant range. We choose one such M and fix it throughout the proof. We also omit the superscript i denoting the direction of derivative by assuming that we always discuss each direction separately.

Proof for (a): Firstly, the continuous differentiability of f and the local Lipschitz conditions imply that, for the relevant range of variables,

$$\begin{aligned} &|\partial_y f(t,y,z,u_t)| \leq K_M, \quad |\partial_u f(t,y,z,u_t)| \leq K_M, \\ &|\partial_z f(t,y,z,u_t)| \leq K_M(1 + 2|z| + 2|u_t|)\,. \end{aligned} \tag{5.4}$$

It is easy to check that the BSDE (5.2) satisfies Assumption A.2. Indeed, its second condition follows from the relation

$$|(D_{s,0}f)(r,\Theta_r)| \le |(D_{s,0}f)(r,0)| + K^M_{s,0}(|Y_r| + \|\rho\|_{\mathbb{L}^2(\nu)}G'_M\|\psi_r\|_{\mathbb{L}^2(\nu)}) \\ + K^M_{s,0}(1+|Z_r| + \|\rho\|_{\mathbb{L}^2(\nu)}G'_M\|\psi_r\|_{\mathbb{L}^2(\nu)})|Z_r|\,,$$

Lemma 2.2 and Remark 5.3. Thus, Theorem A.1 implies that there exists a unique solution $(Y^{s,0}, Z^{s,0}, \psi^{s,0}) \in \mathcal{K}^{p'}_{[0,T]}$ to the BSDE (5.2) satisfying

$$\|(Y^{s,0}, Z^{s,0}, \psi^{s,0})\|^{p'}_{\mathcal{K}^{p'}} \le C_{p'}\Big(1+\mathbb{E}\Big[|D_{s,0}\xi|^{p'\bar{q}^2} + \Big(\int_0^T |(D_{s,0}f)(r,0)|dr\Big)^{p'\bar{q}^2} \\ + \|K^M_{s,0}\|_T^{2p'\bar{q}^2} + \|Y\|_T^{2p'\bar{q}^2} + \Big(\int_0^T |Z_r|^2dr\Big)^{2p'\bar{q}^2} + \Big(\int_0^T \|\psi_r\|^2_{\mathbb{L}^2(\nu)}dr\Big)^{2p'\bar{q}^2}\Big]\Big)^{\frac{1}{\bar{q}^2}} < \infty,$$

for $\forall p' \ge 2$, where $C_{p'}$ and $\bar{q} > 1$ are positive constants. Assumption 5.4(iv) also gives the 2nd claim $\int_0^T \|(Y^{s,0}, Z^{s,0}, \psi^{s,0})\|^p_{\mathcal{K}^p[0,T]}ds < \infty$.

Proof for (b): Let us first consider the BSDE

$$\mathcal{Y}^{s,z}_t = \xi(\omega^{s,z}) + \int_t^T f\Big(\omega^{s,z}, r, \mathcal{Y}^{s,z}_r, \mathcal{Z}^{s,z}_r, \int_{\mathbb{R}_0}\rho(x)G(r,\Psi^{s,z}_r(x))\nu(dx)\Big)dr \\ - \int_t^T \mathcal{Z}^{s,z}_r dW_r - \int_t^T\int_E \Psi^{s,z}_r(x)\widetilde{\mu}(dr,dx)\,. \tag{5.5}$$

For every $(s,z) \in [0,T]\times\mathbb{R}_0$, $m(dz)ds$-a.e, Assumption 5.1, 5.2, 5.3 are all satisfied. Thus, by Theorem 4.1, there exists a unique solution $(\mathcal{Y}^{s,z}, \mathcal{Z}^{s,z}, \Psi^{s,z}) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ to the BSDE (5.5) satisfying the universal bounds. Now, let us define for $z \in \mathbb{R}_0$,

$$Y^{s,z} := \frac{\mathcal{Y}^{s,z}-Y}{z}, \quad Z^{s,z} := \frac{\mathcal{Z}^{s,z}-Z}{z}, \quad \psi^{s,z} := \frac{\Psi^{s,z}-\psi}{z}\,,$$

and then $(Y^{s,z}, Z^{s,z}, \psi^{s,z}) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ is the unique solution to the BSDE (5.3). Note that $D_{s,z}\xi := \frac{1}{z}(\xi(\omega^{s,z}) - \xi(\omega))$.

We use a new collective argument $\Xi^{s,z}_r := \big(\mathcal{Y}^{s,z}_r, \mathcal{Z}^{s,z}_r, \int_{\mathbb{R}_0}\rho(x)G(r,\Psi^{s,z}_r(x))\nu(dx)\big)$. Let us introduce

$$f^{s,z}(r) := \frac{1}{z}\big(f(\omega^{s,z}, r, \Xi^{s,z}_r) - f(\omega, r, \Theta_r)\big) \\ = (D_{s,z}f)(r,\Theta_r) + \frac{f(\omega^{s,z}, r, \Xi^{s,z}_r) - f(\omega^{s,z}, r, \Theta_r)}{z}\,,$$

a d-dimensional $\mathbb{F}$-progressively measurable process $(b^{s,z}_r, r \in [0,T])$,

$$b^{s,z}_r(\omega) := \frac{1}{|\mathcal{Z}^{s,z}_r - Z_r|^2}\Big\{f\Big(\omega^{s,z}, r, Y_r, \mathcal{Z}^{s,z}_r, \int_{\mathbb{R}_0}\rho(x)G(r,\psi_r(x))\nu(dx)\Big) \\ - f\Big(\omega^{s,z}, r, Y_r, Z_r, \int_{\mathbb{R}_0}\rho(x)G(r,\psi_r(x))\nu(dx)\Big)\Big\}\mathbf{1}_{\mathcal{Z}^{s,z}_r - Z_r \neq 0}(\mathcal{Z}^{s,z}_r - Z_r)$$

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and also the map $\widetilde{f}^{s,z} : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{L}^2(E,\nu;\mathbb{R}_k) \to \mathbb{R}$,

$$\begin{aligned}\widetilde{f}^{s,z}(\omega, r, \widetilde{y}, \widetilde{\psi}) := (D_{s,z}f)(r, \Theta_r) \\ + \frac{1}{z}\Big\{ f\Big(\omega^{s,z}, r, z\widetilde{y} + Y_r, \mathcal{Z}_r^{s,z}, \int_{\mathbb{R}_0} \rho(x) G(r, z\widetilde{\psi}(x) + \psi_r(x))\nu(dx)\Big) \\ - f\Big(\omega^{s,z}, r, Y_r, \mathcal{Z}_r^{s,z}, \int_{\mathbb{R}_0} \rho(x) G(r, \psi_r(x))\nu(dx)\Big)\Big\}.\end{aligned}$$

Then, $(Y^{s,z}, Z^{s,z}, \psi^{s,z})$ can also be expressed as a solution to the BSDE

$$\begin{aligned}Y_t^{s,z} &= D_{s,z}\xi + \int_t^T \Big(\widetilde{f}^{s,z}(r, Y_r^{s,z}, \psi_r^{s,z}) + b_r^{s,z} \cdot Z_r^{s,z}\Big)dr \\ &\quad - \int_t^T Z_r^{s,z} dW_r - \int_t^T \int_E \psi_r^{s,z}(x)\widetilde{\mu}(dr, dx).\end{aligned}$$

It is straightforward to check that Assumption A.1 is satisfied. Thus, Lemma A.1 gives

$$\begin{aligned}&\|(Y^{s,z}, Z^{s,z}, \psi^{s,z})\|_{\mathcal{K}^{p'}}^{p'} \\ &\le C_{p'}\Big(1 + \mathbb{E}\Big[|D_{s,z}\xi|^{p'\bar{q}^2} + \Big(\int_0^T |(D_{s,z}f)(r,0)|dr\Big)^{p'\bar{q}^2} + \|K_{s,z}^M\|_T^{2p'\bar{q}^2} \\ &\quad + \|Y\|_T^{2p'\bar{q}^2} + \Big(\int_0^T |Z_r|^2 dr\Big)^{2p'\bar{q}^2} + \Big(\int_0^T \|\psi_r\|_{\mathbb{L}^2(\nu)}^2 dr\Big)^{2p'\bar{q}^2}\Big]\Big)^{\frac{1}{\bar{q}^2}} < \infty\end{aligned}$$

for $\forall p' \ge 2$, where $C_{p'}$ and $\bar{q} > 1$ are the positive constants. Choosing $p' = p$, one can show $\int_0^T \int_{\mathbb{R}} \|(Y^{s,z}, Z^{s,z}, \psi^{s,z})\|_{\mathcal{K}^p}^p m(dz)ds < \infty$ from Assumption 5.4(iv), which proves the second claim of (b). Note that, we also have $\int_{\widetilde{E}} \|(Y^{s,z}, Z^{s,z}, \psi^{s,z})\|_{\mathcal{K}^p}^p q(ds, dz) < \infty$ by combining the results (a) and (b).

Proof for (c): *First step (Approximating sequence of globally Lipschitz BSDEs)*
We finally proceed to the proof for (c). Firstly, let us define for each $m \in \mathbb{N}$

$$G_m(s, \psi(x)) := G(s, \varphi_m(\psi \circ \zeta_m(x))), \qquad f_m(s, y, z, u) := f(s, \varphi_m(y), \varphi_m(z), u)$$

where φ_m is the smooth truncation function defined in (4.2), and $\psi \circ \zeta_m(x) := \psi(x)\mathbf{1}_{|x|\ge 1/m}$, which are applied component-wise for z and ψ. Let us now define a sequence of regularized drivers $(F_m, m \in \mathbb{N})$ by $F_m(s, y, z, \psi) := f_m\big(s, y, z, \int_{\mathbb{R}_0} \rho(x) G_m(s, \psi(x))\nu(dx)\big)$ for $s \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $\psi \in \mathbb{L}^2(E, \nu; \mathbb{R}^k)$. Note that

$$\|\varphi_m(\psi \circ \zeta_m)\|_{\mathbb{L}^2(\nu)}^2 = \int_E |\varphi_m(\psi \circ \zeta_m(x))|^2 \nu(dx) \le (m+1)^2 C_m$$

where $C_m := k\max_{1\le i\le k} \int_{\mathbb{R}_0} \mathbf{1}_{|x|\ge 1/m}\nu^i(dx)$. Combined with Assumption 5.3 and Remarks 5.2, one sees F_m is globally Lipschitz for each $m \in \mathbb{N}$. One can also check $|F_m|$ is bounded. Thus, for each $m \in \mathbb{N}$, there exists a unique solution (Y^m, Z^m, ψ^m) of the BSDE

$$Y^m = \xi + \int_t^T F_m(s, Y_s^m, Z_s^m, \psi_s^m)ds - \int_t^T Z_s^m dW_s - \int_t^T \int_E \psi_s^m(x)\widetilde{\mu}(ds, dx), \qquad (5.6)$$

with $Y^m \in \mathbb{S}^\infty$. Moreover, the convexity of positive function $j_\gamma(\cdot)$ and Assumption 5.2 imply that F_m satisfy the $Q_{\exp}$-structure condition *uniformly* in m. Therefore, (Y^m, Z^m, ψ^m) satisfies the universal bounds of Lemmas 3.1 and 3.2. Since $\|Y^m\|_{\mathbb{S}^\infty}$ and $\|\psi^m\|_{\mathbb{J}^\infty}$ are bounded uniformly in m, the truncation φ_m for (y, ψ) becomes irrelevant provided m is large enough. Thus, for large

m, (Y^m, Z^m, ψ^m) also consists of a unique bounded solution[7] to the BSDE with data $(\xi, \widetilde{F}_m)$ where

$$\widetilde{F}_m(s,y,z,\psi) := f\Big(s, y, \varphi_m(z), \int_{\mathbb{R}_0} \rho(x) G(s, \psi \circ \zeta_m(x))\nu(dx)\Big) .$$

Since $(\widetilde{F}_m)$ satisfies A_Γ-condition uniformly in m, and also $\widetilde{F}_m \to F$ locally uniformly in the spacial variables, Proposition 4.2 implies $Y^m \to Y$ in $\mathbb{S}^\infty$, $Z^m \to Z$ in $\mathbb{H}^2_{BMO}$ and $\psi^m \to \psi$ in $\mathbb{J}^2_{BMO}$ where (Y, Z, ψ) is a unique solution of the BSDE (5.1). One can also check that, for each $m \in \mathbb{N}$, the BSDE (5.6) satisfies Assumptions C.1 as well as C.2. Therefore Theorem C.1 implies that the approximating BSDEs are Malliavin differentiable and $(Y^m, Z^m, \overline{\psi}^m) \in (\mathbb{L}^{1,2})^3$ for $\forall m \in \mathbb{N}$.

Second step (Uniform boundedness of $\mathbb{L}^{1,2}$-norm of the approximating BSDEs)
From the *first step*, one can define the Malliavin derivatives of (Y^m, Z^m, ψ^m) for every $m \in \mathbb{N}$ as the solution to the following BSDEs: For every Wiener direction $i \in \{1, \dots, d\}$, ds-a.e. $s \in [0,T]$ and $s \le t \le T$,

$$\begin{aligned} &D^i_{s,0}Y^m_t = D^i_{s,0}\xi + \int_t^T D^i_{s,0} f_m(r)dr - \int_t^T D^i_{s,0}Z^m_r dW_r - \int_t^T\int_E D^i_{s,0}\psi^m_r(x)\widetilde{\mu}(dr,dx), \\ &D^i_{s,0} f_m(r) := (D_{s,0} f_m)(r, \Theta^m_r) + \partial_\Theta f_m(r, \Theta^m_r) D^i_{s,0}\Theta^m_r, \end{aligned} \tag{5.7}$$

and for jump direction $i \in \{1, \dots, k\}$, $m^i(dz)ds$-a.e. $(s,z) \in [0,T] \times \mathbb{R}_0$ and $s \le t \le T$,

$$\begin{aligned} &D^i_{s,z}Y^m_t = D^i_{s,z}\xi + \int_t^T D^i_{s,z} f_m(r)dr - \int_t^T D^i_{s,z}Z^m_r dW_r - \int_t^T\int_E \psi^m_r(x)\widetilde{\mu}(dr,dx), \\ &D^i_{s,z} f_m(r) := \frac{1}{z}\big(f_m(\omega^{s,z}, r, \Theta^m_r + zD^i_{s,z}\Theta^m_r) - f_m(\omega, r, \Theta^m_r)\big) \\ &\quad = (D^i_{s,z} f_m)(r, \Theta^m_r) + \frac{1}{z}\big(f_m(\omega^{s,z}, r, \Theta^m_r + zD^i_{s,z}\Theta^m_r) - f_m(\omega^{s,z}, r, \Theta^m_r)\big) . \end{aligned} \tag{5.8}$$

Here, we have defined $\Theta^m_r := \big(Y^m_r, Z^m_r, \int_{\mathbb{R}_0}\rho(x)G_m(r, \psi^m_r(x))\nu(dx)\big)$ for $r \in [0,T]$ and slightly abused its notation in such a way that $f_m(\omega^{s,z}, r, \Theta^m_r + zD^i_{s,z}\Theta^m_r) := f_m\big(\omega^{s,z}, r, Y^m_r + zD^i_{s,z}Y^m_r,\ Z^m_r + zD^i_{s,z}Z^m_r, \int_{\mathbb{R}_0}\rho(x)G_m\big(r, \psi^m_r(x) + zD^i_{s,z}\psi^m_r(x)\big)\nu(dx)\big)$ to save the space. For $0 \le t < s$, one has $D_{s,z}\Theta^m_t \equiv 0$.

One can check that the unique solution of (5.7) satisfies $(D_{s,0}Y^m, D_{s,0}Z^m, D_{s,0}\psi^m) \in \mathcal{K}^{p'}[0,T]$ for $\forall p' \ge 2$ by Theorem A.1. Let us also define (for each direction $i \in \{1, \dots, k\}$)

$$\begin{aligned} &\mathcal{Y}^m_{s,z}(t) := Y^m_t + zD_{s,z}Y^m_t, \quad \mathcal{Z}^m_{s,z}(t) := Z^m_t + zD_{s,z}Z^m_t, \\ &\Psi^m_{s,z}(t,\cdot) := \psi^m_t(\cdot) + zD_{s,z}\psi^m_t(\cdot) , \end{aligned}$$

for $(s,z) \in [0,T] \times \mathbb{R}_0$ and $t \in [0,T]$, and denote its collective argument as $\Xi^m_{s,z}(t) := \big(\mathcal{Y}^m_{s,z}(t), \mathcal{Z}^m_{s,z}(t), \int_{\mathbb{R}_0}\rho(x)G_m(t, \Psi^m_{s,z}(t,x))\nu(dx)\big)$. Note that $(\mathcal{Y}^m_{s,z}, \mathcal{Z}^m_{s,z}, \Psi^m_{s,z})$ is a solution to a Lipschitz BSDE (5.5) with f, G replaced by f_m, G_m. Since it satisfies the structure condition uniformly in m, $(\mathcal{Y}^m_{s,z}, \mathcal{Z}^m_{s,z}, \Psi^m_{s,z})$ satisfies the universal bounds. It then shows $(D_{s,z}Y^m, D_{s,z}Z^m, D_{s,z}\psi^m) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ for $z \neq 0$. Moreover, by the same analysis given in the *first step*, one observes the convergence $(\mathcal{Y}^m_{s,z}, \mathcal{Z}^m_{s,z}, \Psi^m_{s,z}) \to (\mathcal{Y}^{s,z}, \mathcal{Z}^{s,z}, \Psi^{s,z})$ in the space $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$.

[7] Using the universal bounds, uniqueness is checked similarly as in the standard Lipschitz BSDE.

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By the same arguments used in the proofs for (a) and (b), one can apply Theorem A.1 to the BSDE (5.7) and Lemma A.1 to the BSDE (5.8) to obtain

$$\begin{aligned}
&\left\|(D_{s,z}Y^m, D_{s,z}Z^m, D_{s,z}\psi^m)\right\|^{p'}_{\mathcal{K}^{p'}[0,T]} \\
&\quad\le C_{p'}\Big(1+\mathbb{E}\Big[|D_{s,z}\xi|^{p'\bar{q}^2}+\Big(\int_0^T|(D_{s,z}f)(r,0)|dr\Big)^{p'\bar{q}^2}+\|K_{s,z}\|_T^{2p'\bar{q}^2} \\
&\qquad+\|Y^m\|_T^{2p'\bar{q}^2}+\Big(\int_0^T|Z^m_r|^2dr\Big)^{2p'\bar{q}^2}+\Big(\int_0^T\|\psi^m_r\|^2_{\mathbb{L}^2(\nu)}dr\Big)^{2p'\bar{q}^2}\Big]\Big)^{\frac{1}{\bar{q}^2}}
\end{aligned}$$

with $\forall p'\ge 2$, for the Wiener ($z=0$) as well as the jump ($z\neq 0$) directions. Here, $C_{p'}$ and $\bar{q}>1$ are positive constants independent of m. Assumption 5.4(iv), the universal bounds for Θ^m and the energy inequality give

$$\int_{\widetilde{E}}\sup_{m\in\mathbb{N}}\left\|(D_{s,z}Y^m, D_{s,z}Z^m, D_{s,z}\psi^m)\right\|^p_{\mathcal{K}^p[0,T]}q(ds,dz)<\infty\,. \tag{5.9}$$

It then easily follows that $\mathbb{L}^{1,2}$-norm of $(Y^m, Z^m, \overline{\psi}^m)$ is bounded uniformly in m. The estimate (5.9) also gives

$$\begin{aligned}
&\sum_{i=1}^k\int_0^T\int_{|z|>\epsilon}\left\|(D^i_{s,z}Y^m, D^i_{s,z}Z^m, D^i_{s,z}\psi^m)\right\|^p_{\mathcal{K}^p[0,T]}m^i(dz)ds \\
&\to\sum_{i=1}^k\int_0^T\int_{\mathbb{R}_0}\left\|(D^i_{s,z}Y^m, D^i_{s,z}Z^m, D^i_{s,z}\psi^m)\right\|^p_{\mathcal{K}^p[0,T]}m^i(dz)ds
\end{aligned}\tag{5.10}$$

as $\epsilon\downarrow 0$ uniformly in $m\in\mathbb{N}$ by the Lebesgue's dominated convergence theorem.

Third step (Convergence of $D_{s,0}\Theta^m\to\Theta^{s,0}$)
For ds-a.e. $s\in[0,T]$ and $m\in\mathbb{N}$, set

$$\Delta^{s,0}Y^m:=Y^{s,0}-D_{s,0}Y^m,\quad \Delta^{s,0}Z^m:=Z^{s,0}-D_{s,0}Z^m,\quad \Delta^{s,0}\psi^m:=\psi^{s,0}-D_{s,0}\psi^m$$

and then $(\Delta^{s,0}Y^m, \Delta^{s,0}Z^m, \Delta^{s,0}\psi^m)\in\mathcal{K}^{p'}[0,T]$ with $\forall p'\ge 2$ is the unique solution to the BSDE

$$\begin{aligned}
\Delta^{s,0}Y^m_t&=\int_t^T\Big(f^{s,0}(r)-D_{s,0}f_m(r)\Big)dr-\int_t^T\Delta^{s,0}Z^m_r dW_r \\
&\quad-\int_t^T\int_E\Delta^{s,0}\psi^m_r(x)\widetilde{\mu}(dr,dx)\,.
\end{aligned}$$

We claim

$$\lim_{m\to\infty}\int_0^T\left\|(\Delta^{s,0}Y^m, \Delta^{s,0}Z^m, \Delta^{s,0}\psi^m)\right\|^p_{\mathcal{K}^p[0,T]}ds=0\,. \tag{5.11}$$

The proof is straightforward and we give the details in Appendix D.1.

Fourth step (Convergence of $D_{s,z}\Theta^m\to\Theta^{s,z}$ ($z\neq 0$))
For each direction of jump, let us put

$$\Delta^{s,z}Y^m:=Y^{s,z}-D_{s,z}Y^m,\quad \Delta^{s,z}Z^m=Z^{s,z}-D_{s,z}Z^m,\quad \Delta^{s,z}\psi^m=\psi^{s,z}-D_{s,z}\psi^m\,.$$

Then, $(\Delta^{s,z}Y^m, \Delta^{s,z}Z^m, \Delta^{s,z}\psi^m) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ is the unique solution to

$$\Delta^{s,z}Y^m_t = \int_t^T \Big(f^{s,z}(r) - D_{s,z}f_m(r)\Big)dr - \int_t^T \Delta^{s,z}Z^m_r dW_r \\ - \int_t^T \int_E \Delta^{s,z}\psi^m_r(x)\widetilde{\mu}(dr,dx)\,,$$

with $t \in [0,T]$. As in the third step, we claim

$$\lim_{m\to 0}\int_0^T \int_{\mathbb{R}_0} \|(\Delta^{s,z}Y^m, \Delta^{s,z}Z^m, \Delta^{s,z}\psi^m)\|^p_{\mathcal{K}^p[0,T]} m(dz)ds = 0. \tag{5.12}$$

The proof is tedious but straightforward and we give the details in Appendix D.2.

Final step
From the previous steps, one sees $(Y^m, Z^m, \overline{\psi}^m)$ converges to $\big((Y,Z,\overline{\psi}),(Y^{s,z},Z^{s,z},\overline{\psi}^{s,z})\big)$ in $\mathbb{L}^2(0,T;\mathbb{D}^{1,2}) = \mathbb{L}^{1,2}$. The closability of the Malliavin derivatives in $\mathbb{L}^{1,2}$ (see Theorem 12.6 in [16]), one concludes $(Y,Z,\overline{\psi}) \in \mathbb{L}^{1,2}$ and that $(Y^{s,z},Z^{s,z},\psi^{s,z})$ is a version of $(D_{s,z}Y, D_{s,z}Z, D_{s,z}\psi)$. □

Corollary 5.1. *Under the assumptions of Theorem 5.1, we have*
(i) $\Big((D^i_{t,0}Y_t)^{\mathcal{P}}, t \in [0,T]\Big)$ *is a version of* $\Big(Z^i_t, t\in[0,T]\Big)$ *for* $i \in \{1,\dots,d\}$,
(ii) $\Big((zD^i_{t,z}Y_t)^{\mathcal{P}}, (t,z) \in [0,T]\times\mathbb{R}_0\Big)$ *is a version of* $\Big(\psi^i_t(z), (t,z) \in [0,T]\times\mathbb{R}_0\Big)$ *for* $i \in \{1,\dots,k\}$,
where $(\cdot)^{\mathcal{P}}$ *denotes the predictable projection of a process.*

Proof. See Corollary 4.1 in [15]. □

6. An application: Markovian forward–backward system

6.1. Forward SDE

As an important application, we consider a $Q_{\exp}$-growth BSDE driven by an n-dimensional Markovian process $\big(X^{t,x}_s, s\in[0,T]\big)$ defined by the next SDE:

$$X^{t,x}_s = x + \int_t^s b(r,X^{t,x}_r)dr + \int_t^s \sigma(r,X^{t,x}_r)dW_r + \int_t^s\int_E \gamma(r,X^{t,x}_{r-},e)\widetilde{\mu}(dr,de) \tag{6.1}$$

for $s \in [t,T]$ and put $X^{t,x}_s \equiv x$ for $s < t$. Here, $x \in \mathbb{R}^n$, $b : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times d}$ and $\gamma : [0,T]\times\mathbb{R}^n\times E \to \mathbb{R}^{n\times k}$. Let us introduce $\eta : \mathbb{R}\to\mathbb{R}_+$ by $\eta(e) = 1\wedge|e|$.

Assumption 6.1. The functions $b(t,x)$, $\sigma(t,x)$ and $\gamma(t,x,e)$ are continuous in all their arguments and one-time continuously differentiable with respect to x with continuous derivatives. Furthermore, there exists some positive constant K such that
(i) $|b(t,0)| + |\sigma(t,0)| \le K$ uniformly in $t \in [0,T]$.
(ii) $|\partial_x b(t,x)| + |\partial_x\sigma(t,x)| \le K$ uniformly in $(t,x) \in [0,T]\times\mathbb{R}^n$.
(iii) For each column vector $i \in \{1,\dots,k\}$, $|\gamma^i(t,0,e)| \le K\eta(e)$ uniformly in $(t,e) \in [0,T]\times\mathbb{R}_0$.
(iv) For each column vector $i \in \{1,\dots,k\}$, $|\partial_x\gamma^i(t,x,e)| \le K\eta(e)$ uniformly in $(t,x,e) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}_0$.

We have the following result:

Proposition 6.1. *Under Assumption 6.1, there exists a unique solution $X^{t,x} \in \mathbb{S}^p[0,T]$ with $\forall p \geq 2$ for every initial data $(t,x) \in [0,T] \times \mathbb{R}^n$. Furthermore, the process $X^{t,x}$ is Malliavin differentiable $X^{t,x} \in \mathbb{L}^{1,2}$ and satisfies, for $\forall p \geq 2$,*

$$\int_{\widetilde{E}} \mathbb{E}\Big[\|D_{u,z}X^{t,x}\|_T^p\Big] q(du,dz) \leq C(1+|x|^p)$$

with some positive constant C depending only on (p,T,K).

Proof. The fact that $X^{t,x} \in \mathbb{S}^p[0,T]$ with $\forall p \geq 2$ is rather standard. See, for example, Lemma A.3 in [17]. The existence of Malliavin derivative follows from Theorem 3 of Petrou (2008) [37]. This implies, for $u \in [t,s]$ and $i \in \{1,\ldots,d\}$,

$$\begin{aligned} D^i_{u,0}X^{t,x}_s &= \sigma^i(u, X^{t,x}_u) + \int_u^s \partial_x b(r, X^{t,x}_r) D^i_{u,0}X^{t,x}_r dr + \int_u^s \partial_x \sigma(r, X^{t,x}_r) D^i_{u,0}X^{t,x}_r dW_r \\ &\quad + \int_u^s \int_E \partial_x \gamma(r, X^{t,x}_{r-}, e) D^i_{u,0}X^{t,x}_r \widetilde{\mu}(dr,de) , \end{aligned}$$

and for $(u,z) \in [t,s] \times \mathbb{R}_0$ and $i \in \{1,\ldots,k\}$,

$$\begin{aligned} D^i_{u,z}X^{t,x}_s &= \frac{\gamma^i(u, X^{t,x}_{u-}, z)}{z} + \int_u^s D^i_{u,z} b(r, X^{t,x}_r) dr + \int_u^s D^i_{u,z}\sigma(r, X^{t,x}_r) dW_r \\ &\quad + \int_u^s \int_E D^i_{u,z}\gamma(r, X^{t,x}_{r-}, e) \widetilde{\mu}(dr,de) , \end{aligned}$$

where both σ^i and γ^i denote the ith column vectors of dimension n, and for $\varphi = b, \sigma, \gamma$,

$$D^i_{u,z}\varphi(r, X^{t,x}_r) := \frac{\varphi(r, X^{t,x}_r + z D^i_{u,z}X^{t,x}_r) - \varphi(r, X^{t,x}_r)}{z}.$$

By Lemma A.3 [17], the above SDEs satisfy the a priori estimates

$$\begin{aligned} \mathbb{E}\Big[\|D_{u,0}X^{t,x}\|_T^p\Big] &\leq C_{p,T,K} \mathbb{E}\Big[|\sigma(u, X^{t,x}_u)|^p\Big] \\ &\leq C_{p,T,K} \mathbb{E}\Big[|\sigma(u,0)|^p + \|X^{t,x}\|_T^p\Big] \leq C_{p,T,K}(1+|x|^p) \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\Big[\|D_{u,z}X^{t,x}\|_T^p\Big] &\leq C_{p,T,K} \mathbb{E}\Big[\Big|\frac{\gamma(u, X^{t,x}_{u-}, z)}{z}\Big|^p\Big] \\ &\leq C_{p,T,K} \mathbb{E}\Big[\Big|\frac{\gamma(u,0,z)}{z}\Big|^p + \|X^{t,x}\|_T^p\Big] \leq C_{p,T,K}(1+|x|^p) . \end{aligned}$$

Since $q(du,dz)$ on $\widetilde{E}$ is a finite measure, the claim is proved. □

6.2. $Q_{\exp}$-growth BSDE driven by $X^{t,x}$

In many applications, there appears a BSDE driven by a Markovian forward process. Let us consider a $Q_{\exp}$-BSDE driven by the process $\left(X_s^{t,x}, s \in [0,T]\right)$ introduced in the last section;

$$Y_s^{t,x} = \xi(X_T^{t,x}) + \int_s^T f\Big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \int_{\mathbb{R}_0} \rho(e)G(r,\psi_r(e))\nu(de)\Big)dr$$
$$- \int_s^t Z_r^{t,x} dW_r - \int_s^T \int_E \psi_r^{t,x}(e)\widetilde{\mu}(dr,de) \tag{6.2}$$

for $s \in [t,T]$ and put $(Y_s^{t,x}, Z_s^{t,x}, \psi_s^{t,x}) \equiv (Y_t^{t,x}, 0, 0)$ for $s < t$. Here, $\xi : \mathbb{R}^n \to \mathbb{R}$, $f : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ are measurable functions. We treat Z and ψ as row vectors for notational simplicity. In this setup, the driver f is deterministic without explicit dependence on ω, which is now provided by the dependence on $X^{t,x}$.

Assumption 6.2. (i) For every $(x,y,z,\psi) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E,\nu;\mathbb{R}^k)$, there exist two positive constants $\beta \geq 0, \gamma > 0$ and the non-negative measurable function $l : [0,T] \to \mathbb{R}_+$ such that the measurable function f satisfies

$$-l_t - \beta|y| - \frac{\gamma}{2}|z|^2 - \int_E j_\gamma\big(-\psi(e)\big)\nu(de) \leq f\Big(t,x,y,z,\int_{\mathbb{R}_0} \rho(e)G(t,\psi(e))\nu(de)\Big)$$
$$\leq l_t + \beta|y| + \frac{\gamma}{2}|z|^2 + \int_E j_\gamma\big(\psi(e)\big)\nu(de)$$

dt-a.e. $t \in [0,T]$, where $j_\gamma(u) := \frac{1}{\gamma}\big(e^{\gamma u} - 1 - \gamma u\big)$. (ii) $|\xi(x)| + l_t$ is bounded uniformly in $(t,x) \in [0,T] \times \mathbb{R}^n$. (iii) $F(t,x,y,z,\psi) := f\big(t,x,y,z,\int_{\mathbb{R}_0} \rho(e)G(t,\psi(e))\nu(de)\big)$ satisfies the A_Γ-condition (Assumption 4.1).

Assumption 6.3. For each $M > 0$, for every $x \in \mathbb{R}^n$ and $(y,z,\psi), (y',z',\psi') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E,\nu;\mathbb{R}^k)$ satisfying

$$|y|, |y'|, \|\psi\|_{\mathbb{L}^\infty(\nu)}, \|\psi'\|_{\mathbb{L}^\infty(\nu)} \leq M,$$

there exists some positive constant K_M (possibly dependent on M) such that

$$\big|f(t,x,y,z,u_t) - f(t,x,y',z',u_t')\big|$$
$$\leq K_M\big(|y-y'| + |u_t - u_t'|\big) + K_M\big(1 + |z| + |z'| + |u_t| + |u_t'|\big)|z - z'|$$

with the short-hand notation $u_t := \int_{\mathbb{R}_0} \rho(e)G(t,\psi(e))\nu(de)$ and $u_t' := \int_{\mathbb{R}_0} \rho(e)G(t,\psi'(e))\nu(de)$.

Lemma 6.1. *Under Assumptions 5.1 and 6.1–6.3, there exists a unique solution* $(Y^{t,x}, Z^{t,x}, \psi^{t,x}) \in \mathbb{S}^\infty_{[0,T]} \times \mathbb{H}^2_{BMO[0,T]} \times \mathbb{J}^2_{BMO[0,T]}$ *to the BSDE (6.2) for every* $(t,x) \in [0,T] \times \mathbb{R}^n$.

Proof. This is a special case of Theorem 4.1. □

We denote $\Theta_r^{t,x} := \big(Y_r^{t,x}, Z_r^{t,x}, \int_{\mathbb{R}_0} \rho(e)G(r,\psi_r^{t,x}(e))\nu(de)\big)$ as a collective argument of the solution indexed by the initial data (t,x).

Assumption 6.4. (i) ξ and the driver f are one-time continuously differentiable with respect to the spacial variables with continuous derivatives.

(ii) There exists some positive constant K such that $|\partial_x \xi(x)| \leq K$ as well as $|\partial_x f(t,x,0,0,0)| \leq K$ uniformly in $(t,x) \in [0,T] \times \mathbb{R}^n$.
(iii) For each $M > 0$, for every $x \in \mathbb{R}^n$ and $(y,z,\psi), (y',z',\psi') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E,\nu;\mathbb{R}^k)$ satisfying

$$|y|, |y'|, \|\psi\|_{\mathbb{L}^\infty(\nu)}, \|\psi'\|_{\mathbb{L}^\infty(\nu)} \leq M,$$

there exists some positive constant K_M (possibly dependent on M) such that

$$\begin{aligned}&\left|\partial_x f(t,x,y,z,u_t) - \partial_x f(t,x,y',z',u'_t)\right| \\ &\leq K_M\big(|y-y'| + |u_t - u'_t|\big) + K_M\big(1 + |z| + |z'| + |u_t| + |u'_t|\big)|z - z'|\end{aligned}$$

with the short-hand notation $u_t := \int_{\mathbb{R}_0} \rho(e) G(t,\psi(e))\nu(de)$ and $u'_t := \int_{\mathbb{R}_0} \rho(e) G(t,\psi'(e))\nu(de)$.

One sees that Assumption 6.4, together with Assumption 6.3, implies

$$\begin{aligned}&|\partial_x f(t,x,y,z,u_t)| \leq C K_M\big(1 + |y| + |z|^2 + |u_t|^2\big), \quad |\partial_y f(t,x,y,z,u_t)| \leq K_M, \\ &|\partial_z f(t,x,y,z,u_t)| \leq K_M\big(1 + 2|z| + 2|u_t|\big), \quad |\partial_u f(t,x,y,z,u_t)| \leq K_M ,\end{aligned}$$

where C is some positive constant.

Theorem 6.1. *Under Assumptions 5.1 and 6.1–6.4, the solution of the BSDE (6.2) is Malliavin differentiable* $(Y^{t,x}, Z^{t,x}, \overline{\psi}^{t,x}) \in \mathbb{L}^{1,2} \times \mathbb{L}^{1,2} \times \mathbb{L}^{1,2}$ *for every initial data* $(t,x) \in [0,T] \times \mathbb{R}^n$.
(i) A version of $\big((D^i_{s,0} Y^{t,x}_r, D^i_{s,0} Z^{t,x}_r, D^i_{s,0} \psi^{t,x}_r(e)), 0 \leq s,r \leq T, e \in \mathbb{R}_0\big)_{i\in\{1,\ldots,d\}}$ *is the unique solution to the BSDE*

$$\begin{aligned}&D^i_{s,0} Y^{t,x}_u = D^i_{s,0} Z^{t,x}_u = D^i_{s,0} \psi^{t,x}_u(\cdot) = 0, \qquad 0 \leq u < s \leq T, \\ &D^i_{s,0} Y^{t,x}_u = \partial_x \xi(X^{t,x}_T) D^i_{s,0} X^{t,x}_T + \int_u^T f^{s,0,i}(r)dr - \int_u^T D^i_{s,0} Z^{t,x}_r dW_r \\ &\qquad - \int_u^T \int_E D^i_{s,0} \psi^{t,x}_r \widetilde{\mu}(dr,de), \quad u \in [s,T]\end{aligned}$$

where $f^{s,0,i}(r) := \partial_x f(r, X^{t,x}_r, \Theta^{t,x}_r) D_{s,0} X^{t,x}_r + \partial_\Theta f(r, X^{t,x}_r, \Theta^{t,x}_r) D_{s,0} \Theta^{t,x}_r$. *Moreover, for a given ds-a.e.* $s \in [0,T]$, $(D^i_{s,0} Y^{t,x}, D^i_{s,0} Z^{t,x}, D^i_{s,0} \psi^{t,x}) \in \mathcal{K}^p[0,T]$ *with* $\forall p \geq 2$.
(ii) A version of $\big((D^i_{s,z} Y^{t,x}_r, D^i_{s,z} Z^{t,x}_r, D^i_{s,z} \psi^{t,x}_r(e)), 0 \leq s,r \leq T, e,z \in \mathbb{R}_0\big)_{i\in\{1,\ldots,k\}}$ *is the unique solution to the BSDE*

$$\begin{aligned}&D^i_{s,z} Y^{t,x}_u = D^i_{s,z} Z^{t,x}_u = D^i_{s,z} \psi^{t,x}_u(\cdot) = 0, \qquad 0 \leq u < s \leq T, \\ &D^i_{s,z} Y^{t,x}_u = \xi^{s,z,i} + \int_u^T f^{s,z,i}(r)dr - \int_u^T D^i_{s,z} Z^{t,x}_r dW_r - \int_u^T \int_E D^i_{s,z} \psi^{t,x}_r(e) \widetilde{\mu}(dr,de),\end{aligned}$$

for $u \in [s,T]$ *where*

$$\begin{aligned}&\xi^{s,z,i} := \frac{\xi(X^{t,x}_T + z D^i_{s,z} X^{t,x}_T) - \xi(X^{t,x}_T)}{z}, \\ &f^{s,z,i}(r) := \frac{1}{z}\Big\{ f\Big(r, X^{t,x}_r + z D^i_{s,z} X^{t,x}_r,\ Y^{t,x}_r + z D^i_{s,z} Y^{t,x}_r,\ Z^{t,x}_r + z D^i_{s,z} Z^{t,x}_r \\ &\qquad , \int_{\mathbb{R}_0} \rho(e) G(r, \psi^{t,x}_r(e) + z D^i_{s,z} \psi^{t,x}_r(e))\nu(e)de\Big) - f(r, X^{t,x}_r, \Theta^{t,x}_r)\Big\} .\end{aligned}$$

Moreover, for a given $m^i(dz)ds$-a.e. $(s,z) \in [0,T] \times \mathbb{R}_0$, $(D^i_{s,z} Y^{t,x}, D^i_{s,z} Z^{t,x}, D^i_{s,z} \psi^{t,x}) \in \mathbb{S}^\infty[0,T] \times \mathbb{H}^2_{BMO}[0,T] \times \mathbb{J}^2_{BMO}[0,T]$.

Proof. It suffices to check Assumption 5.4 to hold so that Theorem 5.1 can be applied. (i), (ii) are obviously satisfied due to the Malliavin's differential rule (Theorem 3.5 and Theorem 12.8 in [16]). The local Lipschitz condition (iii) is satisfied if we replace $K^M_{s,z}(r)$ by $K_M|D_{s,z}X^{t,x}_r|$. This is easy to see for a Wiener direction ($z=0$). For a jump direction ($z\neq 0$), notice that

$$\begin{aligned}(D_{s,z}f)(r,y,z,u_r) &= \frac{1}{z}\Big[f(r,X^{t,x}_r+zD_{s,z}X^{t,x}_r,y,z,u_r)-f(r,X^{t,x}_r,y,z,u_r)\Big]\\ &= \left(\int_0^1 \partial_x f\Big(r,X^{t,x}_r+\theta zD_{s,z}X^{t,x}_r,y,z,u_r\Big)d\theta\right)D_{s,z}X^{t,x}_r\ ,\end{aligned}$$

which implies

$$\begin{aligned}&\big|(D_{s,z}f)(r,y,z,u_r)-(D_{s,z}f)(r,y',z',u'_r)\big|\\ &\le |D_{s,z}X^{t,x}_r|\int_0^1\Big|\partial_x f(r,X^{t,x}_r+\theta zD_{s,z}X^{t,x}_r,y,z,u_r)\\ &\quad -\partial_x f(r,X^{t,x}_r+\theta zD_{s,z}X^{t,x}_r,y',z',u'_r)\Big|d\theta\\ &\le K_M|D_{s,z}X^{t,x}_r|\big(|y-y'|+|u_r-u'_r|+(1+|z|+|z'|+|u_r|+|u'_r|)|z-z'|\big)\ .\end{aligned}$$

Since $|D_{s,z}\xi|\le K|D_{s,z}X^{t,x}_T|$ and $|(D_{s,z}f)(r,0,0,0)|\le K|D_{s,z}X^{t,x}_r|$, one can confirm that the condition (iv) are satisfied from an inequality

$$\begin{aligned}&\mathbb{E}\left[|D_{s,z}\xi|^p+\left(\int_0^T|(D_{s,z}f)(r,0,0,0)|dr\right)^p+K_M^{2p}\|D_{s,z}X^{t,x}\|^{2p}_T\right]\\ &\le C_{p,K,K_M,T}\mathbb{E}\Big[1+\|D_{s,z}X^{t,x}\|^{2p}_T\Big]\le C_{p,K,K_M,T}(1+|x|^{2p})\end{aligned}$$

uniformly in $(s,z)\in[0,T]\times\mathbb{R}$ for $\forall p\ge 2$ (see, proof of Proposition 6.1). □

Corollary 6.1. *Under the assumptions of Theorem 6.1, let us define the deterministic function $u:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ by $u(t,x):=Y^{t,x}_t$. Then, $u(t,x)$ is continuous in (t,x), one-time continuously differentiable with respect to x with continuous derivative. Moreover,*

$$\begin{aligned}&\big(Z^{t,x}(s)\big)^i=\partial_x u(s,X^{t,x}_{s-})\sigma^i(s,X^{t,x}_{s-}),\quad t\le s\le T\ ,i\in\{1,\dots,d\}\\ &\big(\psi^{t,x}_s(z)\big)^i=u(s,X^{t,x}_{s-}+\gamma^i(s,X^{t,x}_{s-},z))-u(s,X^{t,x}_{s-}),\quad t\le s\le T\ ,i\in\{1,\dots,k\}\end{aligned}$$

where σ^i and γ^i denote the ith column vectors.

Proof. By replacing *a priori* estimates for the Lipschitz BSDEs of Lemma 5.1 in [17] with the local Lipschitz ones given in Theorem A.1 and Lemma A.2, one can follow the same arguments in Theorem 3.1 in [31] to show that the function $u(t,x)$ is continuous in the both arguments and one-time continuously differentiable with respect to x with continuous derivatives. Then the fact that

$$D^i_{s,0}X^{t,x}_s=\sigma^i(s,X^{t,x}_s),\quad zD^i_{s,z}X^{t,x}_s=\gamma^i(s,X^{t,x}_s,z)\ ,$$

Corollary 5.1, and the Malliavin differential rule for a continuously differentiable function give the desired result. □

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Appendix A. An a priori estimate and BMO-Lipschitz BSDEs

A.1. An a priori estimate

Firstly, we establish a priori estimate which plays a crucial role throughout the paper. Although it is similar to that of BMO-Lipschitz BSDEs, which will be discussed in the next section, it has a much wider range of applications. See discussion in Section 3 of Ankirchner et al. [1] for a diffusion setup. Let us consider the BSDE, for $t \in [0, T]$,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \psi_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E \psi_s(x)\widetilde{\mu}(ds, dx) , \tag{A.1}$$

where $\xi : \Omega \to \mathbb{R}$, $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k) \to \mathbb{R}$. We treat Z, ψ are row vectors for simplicity. We introduce another driver $\widetilde{f} : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k) \to \mathbb{R}$. The crucial point of the next assumption is that the process $(H_t)_{t\in[0,T]}$ is not forbidden to be a function of $(Y_t, Z_t, \psi_t)_{t\in[0,T]}$.

Assumption A.1. (i) The maps $(\omega, t) \mapsto f(\omega, t, \cdot)$, $\widetilde{f}(\omega, t, \cdot)$ are $\mathbb{F}$-progressively measurable. ξ is an $\mathcal{F}_T$-measurable random variable.
(ii) There exists a solution (Y, Z, ψ) to the BSDE (A.1) satisfying $Y \in \mathbb{S}^p$ for $\forall p \geq 2$.
(iii) For every $(y, z, \psi) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$, the driver $\widetilde{f}$ satisfies with some positive constant K such that[8]

$$|\widetilde{f}(\omega, t, y, z, \psi)| \leq g_t + K\big(|y| + |z| + \|\psi\|_{\mathbb{L}^2(\nu)}\big)$$

$d\mathbb{P}\otimes dt$-a.e. $(\omega, t) \in \Omega \times [0, T]$, where $(g_t, t \in [0, T])$ is an $\mathbb{F}$-progressively measurable positive process. Moreover, ξ and g satisfy, for $\forall p \geq 2$, $\mathbb{E}\Big[|\xi|^p + \Big(\int_0^T g_s ds\Big)^p\Big] < \infty$.
(iv) With the solution (Y, Z, ψ) to the BSDE (A.1), there exists an $\mathbb{F}$-progressively measurable positive process $(H_t, t \in [0, T])$, $H \in \mathbb{H}^2_{BMO}$ such that

$$|f(s, Y_s, Z_s, \psi_s) - \widetilde{f}(s, Y_s, Z_s, \psi_s)| \leq H_s|Z_s|$$

for $d\mathbb{P} \otimes ds$-a.e. $(\omega, s) \in \Omega \times [0, T]$.

Lemma A.1. *Suppose Assumption A.1 holds true. Then the solution (Y, Z, ψ) to the BSDE (A.1) satisfies, for $\forall p \geq 2$,*

$$\|(Y, Z, \psi)\|^p_{\mathcal{K}^p[0,T]} \leq C\Big(\mathbb{E}\Big[|\xi|^{p\bar{q}^2} + \Big(\int_0^T g_s ds\Big)^{p\bar{q}^2}\Big]\Big)^{\frac{1}{\bar{q}^2}}$$

with a positive constant $\bar{q}$ satisfying $q_* \leq \bar{q} < \infty$ whose lower bound $q_* > 1$ is controlled only by $\|H\|_{\mathbb{H}^2_{BMO}}$, and some positive constant C depending only on $(p, \bar{q}, T, K, \|H\|_{\mathbb{H}^2_{BMO}})$.

Proof. Define a d-dimensional progressively measurable process $(b_s, s \in [0, T])$ by

$$b_s := \frac{f(s, Y_s, Z_s, \psi_s) - \widetilde{f}(s, Y_s, Z_s, \psi_s)}{|Z_s|^2}\mathbf{1}_{Z_s\neq 0}Z_s,$$

which satisfies $|b_s| \leq H_s$ and hence $b \in \mathbb{H}^2_{BMO}$ whose norm is bounded by $\|H\|_{\mathbb{H}^2_{BMO}}$. Using the process b, (A.1) can be written as

$$Y_t = \xi + \int_t^T \Big(\widetilde{f}(s, Y_s, Z_s, \psi_s) + b_s \cdot Z_s\Big)ds - \int_t^T Z_s dW_s - \int_t^T \int_E \psi_s(x)\widetilde{\mu}(ds, dx)$$

[8] This can be generalized to a monotone condition.

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and hence under the new measure $\mathbb{Q}$ defined by $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}_T(b * W)$, one obtains

$$Y_t = \xi + \int_t^T \widetilde{f}(s, Y_s, Z_s, \psi_s)ds - \int_t^T Z_s dW_s^{\mathbb{Q}} - \int_t^T \int_E \psi_s(x)\widetilde{\mu}^{\mathbb{Q}}(ds, dx) \tag{A.2}$$

where $W^{\mathbb{Q}} := W - \int_0^\cdot b_s ds$ and $\widetilde{\mu}^{\mathbb{Q}} = \widetilde{\mu}$ due to the independence of $(W, \widetilde{\mu})$. By the linear growth property of $\widetilde{f}$, one has

$$Y_s \widetilde{f}(s, Y_s, Z_s, \psi_s) \leq |Y_s|\big(g_s + K(|Y_s| + |Z_s| + \|\psi_s\|_{\mathbb{L}^2(\nu)})\big) ,$$

and hence for $\forall \lambda > 0$

$$Y_s \widetilde{f}(s, Y_s, Z_s, \psi) \leq |Y_s|^2\big(K + K^2/(2\lambda)\big) + |Y_s| g_s + \lambda(|Z_s|^2 + \|\psi_s\|^2_{\mathbb{L}^2(\nu)}) .$$

Thus by choosing $V_t^\lambda := \big(K + \frac{K^2}{2\lambda}\big)t$ and $N_t^\lambda = \int_0^t g_s ds$, the BSDE (A.2) satisfies Assumption B.1 in [17]. Then Lemma B.1 in [17] of an a prior estimate for the BSDEs with a monotone driver implies, for $\forall p \geq 2$,

$$\big\|(Y, Z, \psi)\big\|^p_{\mathcal{K}^p(\mathbb{Q})[0,T]} \leq C\mathbb{E}^{\mathbb{Q}}\Big[|\xi|^p + \Big(\int_0^T g_s ds\Big)^p\Big]$$

with some positive constant $C = C_{p,K,T}$ depending only on (p, K, T).

By the properties of the BMO martingales, one can choose $\bar{r} > 1$ with which both $\mathcal{E}_T(b * W)$ and $\mathcal{E}_T(-b * W^{\mathbb{Q}})$ satisfy the reverse Hölder inequality (see Lemma 2.4 and the following remark). Define $\bar{q} = \frac{\bar{r}}{\bar{r}-1}$ as its dual. Let us put $D := \max\big(\|\mathcal{E}_T(b * W)\|_{\mathbb{L}^{\bar{r}}(\mathbb{P})}, \|\mathcal{E}_T(-b * W^{\mathbb{Q}})\|_{\mathbb{L}^{\bar{r}}(\mathbb{Q})}\big)$, which is dominated by some constant depending only on $\|H\|_{\mathbb{H}^2_{BMO}(\mathbb{P})}$. Then one obtains

$$\begin{aligned}
&\big\|(Y, Z, \psi)\big\|^p_{\mathcal{K}^p(\mathbb{P})_{[0,T]}} \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\mathcal{E}_T(-b * W^{\mathbb{Q}})\Big(\|Y\|_T^p + \Big(\int_0^T |Z_s|^2 ds\Big)^{\frac{p}{2}} + \Big(\int_0^T \|\psi_s\|^2_{\mathbb{L}^2(\nu)} ds\Big)^{\frac{p}{2}}\Big)\Big] \\
&\leq D\big\|(Y, Z, \psi)\big\|^p_{\mathcal{K}^{p\bar{q}}(\mathbb{Q})[0,T]} \leq C_{p,\bar{q},K,T} D\Big(\mathbb{E}^{\mathbb{Q}}\Big[|\xi|^{p\bar{q}} + \Big(\int_0^T g_s ds\Big)^{p\bar{q}}\Big]\Big)^{\frac{1}{\bar{q}}} \\
&\leq C_{p,\bar{q},K,T} D^{1+\frac{1}{\bar{q}}}\Big(\mathbb{E}\Big[|\xi|^{p\bar{q}^2} + \Big(\int_0^T g_s ds\Big)^{p\bar{q}^2}\Big]\Big)^{\frac{1}{\bar{q}^2}} ,
\end{aligned}$$

which proves the desired result. □

A.2. BMO-Lipschitz BSDE

In this subsection, we study the properties of the BSDE with a locally Lipschitz driver where the Lipschitz coefficient for the control variable belongs to $\mathbb{H}^2_{BMO}$. In the diffusion setup, the details have been discussed by Briand & Confortola (2008) [8]. As we have announced before, we keep the reverse Hölder property only to the continuous part and assume only the standard Lipschitz continuity for the jump coefficient.

Assumption A.2. The map $(\omega, t) \mapsto f(\omega, t, \cdot)$ is $\mathbb{F}$-progressively measurable.
(i) There exist a positive constant K and a positive $\mathbb{F}$-progressively measurable process $(H_t, t \in [0, T]) \in \mathbb{H}^2_{BMO}$ such that, for every $(y, z, \psi), (y', z', \psi') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$,

$$|f(\omega, t, y, z, \psi) - f(\omega, t, y', z', \psi')| \leq K\big(|y - y'| + \|\psi - \psi'\|_{\mathbb{L}^2(\nu)}\big) + H_t(\omega)|z - z'|$$

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$d\mathbb{P}\otimes dt$-a.e. $(\omega,t)\in\Omega\times[0,T]$.
(ii) ξ is $\mathcal{F}_T$-measurable and, for $\forall p\geq 2$,

$$\mathbb{E}\Big[|\xi|^p+\Big(\int_0^T|f(s,0,0,0)|ds\Big)^p\Big]<\infty\,.$$

Theorem A.1. *Under Assumption A.2, there exists a unique solution (Y,Z,ψ) to the BSDE (A.1) and it satisfies, for $\forall p\geq 2$,*

$$\big\|(Y,Z,\psi)\big\|^p_{\mathcal{K}^p[0,T]}\leq C\Big(\mathbb{E}\Big[|\xi|^{p\bar{q}^2}+\Big(\int_0^T|f(s,0,0,0)|ds\Big)^{p\bar{q}^2}\Big]\Big)^{\frac{1}{\bar{q}^2}}$$

with a positive constant $\bar{q}$ satisfying $q_*\leq\bar{q}<\infty$ whose lower bound $q_*>1$ is controlled only by $\|H\|_{\mathbb{H}^2_{BMO}}$, and some positive constant C depending only on $(p,\bar{q},T,K,\|H\|_{\mathbb{H}^2_{BMO}})$.

Proof. Define a progressively measurable process $(b_s,s\in[0,T])$ taking values in $\mathbb{R}^d$ by

$$b_s:=\frac{f(s,Y_s,Z_s,\psi_s)-f(s,Y_s,0,\psi_s)}{|Z_s|^2}\mathbf{1}_{Z_s\neq 0}Z_s$$

then $|b_s|\leq H_s$ and hence $b\in\mathbb{H}^2_{BMO}$ and its norm is dominated by $\|H\|_{\mathbb{H}^2_{BMO}}$. Under the measure $\mathbb{Q}$ defined by $d\mathbb{Q}/d\mathbb{P}=\mathcal{E}_T(b*W)$,

$$Y_t=\xi+\int_t^T f(s,Y_s,0,\psi_s)ds-\int_t^T Z_s dW^{\mathbb{Q}}_s-\int_t^T\psi_s(x)\widetilde{\mu}^{\mathbb{Q}}(ds,dx) \tag{A.3}$$

where $W^{\mathbb{Q}}=W-\int_0^\cdot b_s ds$ and $\widetilde{\mu}^{\mathbb{Q}}=\widetilde{\mu}$. As discussed in Lemma A.1, one can choose $\bar{r}>1$ with which both of $\mathcal{E}_T(b*W)$ and $\mathcal{E}_T(-b*W^{\mathbb{Q}})$ satisfy the reverse Hölder inequality and $\bar{q}=\frac{\bar{r}}{\bar{r}-1}$ as its dual. Let us put $D:=\max\big(\|\mathcal{E}_T(b*W)\|_{\mathbb{L}^{\bar{r}}(\mathbb{P})},\|\mathcal{E}_T(-b*W^{\mathbb{Q}})\|_{\mathbb{L}^{\bar{r}}(\mathbb{Q})}\big)$, which is dominated by some constant depending only on $\|H\|_{\mathbb{H}^2_{BMO}(\mathbb{P})}$.

It is clear that the BSDE satisfies the global Lipschitz properties under the measure $\mathbb{Q}$. Furthermore, the following inequality is satisfied due to (reverse) Hölder inequalities:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big[|\xi|^p+\Big(\int_0^T|f(s,0,0,0)|ds\Big)^p\Big]&=\mathbb{E}\Big[\mathcal{E}_T(b*W)\Big(|\xi|^p+\Big(\int_0^T|f(s,0,0,0)|ds\Big)^p\Big)\Big]\\&\leq C_{\bar{q}}D\mathbb{E}\Big[|\xi|^{p\bar{q}}+\Big(\int_0^T|f(s,0,0,0)|ds\Big)^{p\bar{q}}\Big]^{\frac{1}{\bar{q}}}<\infty\,,\end{aligned}$$

with some positive constant $C_{\bar{q}}$. Thus, by Lemma B.2 in [17], one concludes that there exists a unique solution (Y,Z,ψ) to (A.3) in $\mathbb{Q}$ and hence also to (A.1) in $\mathbb{P}$. Furthermore, it also satisfies by the same lemma,

$$\|(Y,Z,\psi)\|^p_{\mathcal{K}^p(\mathbb{Q})}\leq C_{p,K,T}\mathbb{E}^{\mathbb{Q}}\Big[|\xi|^p+\Big(\int_0^T|f(s,0,0,0)|ds\Big)^p\Big]\,.$$

We thus have

$$\begin{aligned}&\big\|(Y,Z,\psi)\big\|^p_{\mathcal{K}^p(\mathbb{P})}\leq C_{\bar{q}}D\,\big\|(Y,Z,\psi)\big\|^p_{\mathcal{K}^{p\bar{q}}(\mathbb{Q})}\\&\leq C_{p,\bar{q},K,T}D^{1+\frac{1}{\bar{q}}}\Big(\mathbb{E}\Big[|\xi|^{p\bar{q}^2}+\Big(\int_0^T|f(s,0,0,0)|ds\Big)^{p\bar{q}^2}\Big]\Big)^{\frac{1}{\bar{q}^2}}\,,\end{aligned}$$

which proves the second part of the claim. □

Now, we gives the stability result which is required to show the uniqueness of the quadratic–exponential growth BSDE. Consider the two BSDEs with $i \in \{1,2\}$ satisfying Assumption A.2;

$$Y_t^i = \xi^i + \int_t^T f^i(s, Y_s^i, Z_s^i, \psi_s^i)ds - \int_t^T Z_s^i dW_s - \int_t^T \int_E \psi_s^i(x)\widetilde{\mu}(ds,dx) \tag{A.4}$$

and put $\delta Y := Y^1 - Y^2$, $\delta Z := Z^1 - Z^2$, $\delta\psi := \psi^1 - \psi^2$, $\delta f(s) := (f^1 - f^2)(s, Y_s^1, Z_s^1, \psi_s^1)$.

Lemma A.2. *The unique solutions $(Y^i, Z^i, \psi^i), i \in \{1,2\}$ to the BSDEs (A.4) under Assumption A.2 satisfy*

$$\|(\delta Y, \delta Z, \delta\psi)\|_{\mathcal{K}^p[0,T]}^p \le C\Big(\mathbb{E}\Big[|\delta\xi|^{p\bar{q}^2} + \Big(\int_0^T |\delta f(s)|ds\Big)^{p\bar{q}^2}\Big]\Big)^{\frac{1}{\bar{q}^2}}$$

with a positive constant $q_* \le \bar{q} < \infty$ whose lower bound $q_* > 1$ is controlled only by $\|H\|_{\mathbb{H}^2_{BMO}}$, and some positive constant C depending only on $(p, \bar{q}, T, K, \|H\|_{\mathbb{H}^2_{BMO}})$.

Proof. Let us introduce a process $(b_s, s \in [0,T])$ defined by

$$b_s := \frac{f^2(s, Y_s^1, Z_s^1, \psi_s^1) - f^2(s, Y_s^1, Z_s^2, \psi_s^1)}{|\delta Z_s|^2}\mathbf{1}_{\delta Z_s \neq 0}\delta Z_s$$

and also a map $\widetilde{f} : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{L}^2(E, \nu; \mathbb{R}^k) \to \mathbb{R}$ by

$$\widetilde{f}(\omega, s, \widetilde{y}, \widetilde{\psi}) := \delta f(\omega, s) + f^2(\omega, s, \widetilde{y} + Y_s^2, Z_s^2, \widetilde{\psi} + \psi_s^2) - f^2(\omega, s, Y_s^2, Z_s^2, \psi_s^2)\,.$$

Then, $(\delta Y, \delta Z, \delta\psi)$ can be interpreted as the solution to the BSDE

$$\delta Y_t = \delta\xi + \int_t^T \Big(\widetilde{f}(s, \delta Y_s, \delta\psi_s) + b_s \cdot \delta Z_s\Big)ds - \int_t^T \delta Z_s dW_s - \int_t^T \int_E \delta\psi_s(x)\widetilde{\mu}(ds,dx)\,.$$

Since $|b_s| \le H_s \in \mathbb{H}^2_{BMO}$ and $\widetilde{f}$ has the linear-growth property with respect to $(\widetilde{y}, \widetilde{\psi})$, Lemma A.1 with $g = |\delta f|$ gives the desired result. □

Appendix B. Some remarks on the comparison principle

Lemma B.1. *If (Y, Z, ψ) is the square integrable solution of the BSDE with data $(\xi, f^{n,m,k})$, then $Y \in \mathbb{S}^\infty$.*

Proof. Consider a sequence of the BSDEs with $l \in \mathbb{N}$,

$$Y_t^l = \xi + \int_t^T F^l(s, Y_s^l, Z_s^l, \psi_s^l)ds - \int_t^T Z_s^l dW_s - \int_t^T \int_E \psi_s^l(x)\widetilde{\mu}(ds,dx), t \in [0,T] \tag{B.1}$$

where $F^l(s, y, z, \psi) := f^{n,m,k}(s, y, z, \psi \circ \zeta_l)$ and $(\psi_s \circ \zeta_l)(x) := \psi_s(x)\mathbf{1}_{\{|x| \ge 1/l\}}$. F^l is globally Lipschitz and satisfy $\mathcal{Q}_{\exp}$-structure condition uniformly in l. Since $|f^{n,m}| \le |\overline{f}^n| \vee |\underline{f}^m| \le |f|$, one sees that $|F^l(s, y, 0, \psi)| \le |f(s, \varphi_k(y), 0, \varphi_k(\psi \circ \zeta_l))|$, which is clearly bounded for all s, y, ψ. Thus, by absorbing the Z argument by the measure change, one sees $Y^l \in \mathbb{S}^\infty$. One can now apply the universal bounds of Lemmas 3.1 and 3.2 to conclude $\|Y^l\|_{\mathbb{S}^\infty}, \|Z^l\|_{\mathbb{H}^2_{BMO}}, \|\psi^l\|_{\mathbb{J}^2_{BMO}}$ are bounded uniformly in l. It now suffices to prove (Y^l, Z^l, ψ^l) converges to the solution (Y, Z, ψ) of the BSDE with data $(\xi, f^{n,m,k})$.

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Since (B.1) is globally Lipschitz uniformly in l, the standard stability formula gives

$$\|(Y^l - Y^{l'}, Z^l - Z^{l'}, \psi^l - \psi^{l'})\|^2_{\mathcal{K}^2} \le C\mathbb{E}\Big[\Big(\int_0^T |\delta f(s)|ds\Big)^2\Big] \le CT\mathbb{E}\Big[\int_0^T |\delta f(s)|^2 ds\Big]$$

where C is independent of l and $\delta f(s) := (F^l - F^{l'})(s, Y^l, Z^l, \psi^l)$. Let us suppose $l \le l'$. For any $(s, y, z, \psi) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$, A_Γ-condition for $f^{n,m}$ gives

$$\begin{aligned}
&|F^l(s, y, z, \psi) - F^{l'}(s, y, z, \psi)| \\
&\quad = |f^{n,m}(s, \varphi_k(y), z, \varphi_k(\psi \circ \zeta_l)) - f^{n,m}(s, \varphi_k(y), z, \varphi_k(\psi \circ \zeta_{l'}))| \\
&\quad \le \int_E \Gamma^{l,l'}_s(x)|\varphi_k(\psi(x))|\mathbf{1}_{\{|x|<1/l\}}\nu(dx)
\end{aligned}$$

with some non-negative $\mathcal{P} \otimes \mathcal{E}$-measurable process $\Gamma^{l,l'}$ satisfying $\Gamma^{l,l'}(x) \le C(1 \wedge |x|)$. Here, the constant C depends only on k. Noticing the fact that $\|\psi^l\|_{\mathbb{J}^2}$ is bounded uniformly in l, the dominated convergence theorem gives

$$\mathbb{E}\Big[\int_0^T |\delta f(s)|^2 ds\Big] \le C\Big(\int_E |x|^2\mathbf{1}_{\{|x|<1/l\}}\nu(dx)\Big)\mathbb{E}\Big[\int_0^T \int_E |\psi^l_s(x)|^2 \nu(dx)ds\Big] \to 0$$

as l (and hence also l')$\to \infty$. This proves $(Y^l, Z^l, \psi^l)_{l\ge1}$ converges to some $(\widetilde{Y}, \widetilde{Z}, \widetilde{\psi})$ in $\mathcal{K}^2$. Since $(Y^l)_{l\ge1}$ are uniformly bounded, so is $\widetilde{Y}$. It is straightforward to check $(\widetilde{Y}, \widetilde{Z}, \widetilde{\psi})$ actually gives a solution to the BSDE with data $(\xi, f^{n,m,k})$, but it is unique and hence equal to (Y, Z, ψ) due to the global Lipschitz continuity. □

The remaining two lemmas are on the comparison principle.

Lemma B.2. *With Assumptions 3.1, 3.2 and 4.1, if there exists a solution $(Y^{n,m}, Z^{n,m}, \psi^{n,m}) \in \mathbb{S}^\infty \times \mathbb{H}^2 \times \mathbb{J}^2$ to the BSDE*

$$\begin{aligned}
Y^{n,m}_t &= \xi + \int_t^T f^{n,m}(s, Y^{n,m}_s, Z^{n,m}_s, \psi^{n,m}_s)ds - \int_t^T Z^{n,m}_s dW_s \\
&\quad - \int_t^T \int_E \psi^{n,m}_s(x)\widetilde{\mu}(ds, dx)\,,
\end{aligned}$$

then it is unique. Moreover, if the relevant solutions exist for the pairs of (n, m), they satisfy $Y^{n,m+1}_t \le Y^{n,m}_t \le Y^{n+1,m}_t$ for $\forall t \in [0, T]$ a.s.

Proof. Since $f^{n,m}$ satisfies the structure condition in Assumption 3.1 uniformly in (n, m), if there exists a bounded solution, then we have $(Y^{n,m}, Z^{n,m}, \psi^{n,m}) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ and the same universal bounds in Lemmas 3.1 and 3.2. Hence, from Assumption 3.2, one can choose a constant K_M as the Lipschitz constant with regard to y, ψ arguments. Since the driver is $(n \vee m)$-Lipschitz with respect to z, one obtains the same stability condition as the globally Lipschitz BSDE. The uniqueness of the solution then follows. Since the driver $f^{n,m}$ satisfies Assumption 4.1, one has for bounded solutions (ψ, ψ'),

$$\mathbb{E}\Big[\int_\tau^T |\Gamma^{\psi,\psi'}_t(x)|^2 \nu(dx)ds\Big|\mathcal{F}_\tau\Big] \le (C^2_M \vee |C^1_M|)\int_\tau^T |x|^2\nu(dx)ds \le C_0 T \tag{B.2}$$

for any $\tau \in \mathcal{T}^T_0$ with some constant C_0 depending only on the universal bounds. This implies $\Gamma^{\psi,\psi'}.\widetilde{\mu}$ is a BMO-martingale. Moreover $\mathcal{E}(\Gamma^{\psi,\psi'}.\widetilde{\mu})$ is a uniformly integrable martingale by Lemma 2.3. The comparison principles now follows in the same way as the Lipschitz case. See, for example, Theorem 2.5 of Royer (2006) [39]. □

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Lemma B.3. *With Assumptions 3.1, 3.2 and 4.1, if there exists a solution* $(\widetilde{Y}^n, \widetilde{Z}^n, \widetilde{\psi}^n) \in \mathbb{S}^\infty \times \mathbb{H}^2 \times \mathbb{J}^2$ *to the BSDE with* $\widetilde{f}^n = \overline{f}^n + \underline{f}$

$$\widetilde{Y}^n_t = \xi + \int_t^T \widetilde{f}^n(s, \widetilde{Y}^n_s, \widetilde{Z}^n_s, \widetilde{\psi}^n_s)ds - \int_t^T \widetilde{Z}^n_s dW_s - \int_t^T \int_E \widetilde{\psi}^n_s(x)\widetilde{\mu}(ds, dx)\,,$$

then it is unique. Moreover, if the relevant solutions exist for $n, n+1$*, they satisfy* $\widetilde{Y}^n_t \leq \widetilde{Y}^{n+1}_t$ *for* $\forall t \in [0, T]$ *a.s.*

Proof. Since $\widetilde{f}^n$ satisfies the structure condition in Assumption 3.1, if there exists a bounded solution it satisfies the universal bounds. Thus the driver is K_M-Lipschitz continuous with respect to y, ψ as in the previous lemma. For z argument, the driver is local Lipschitz continuous whose coefficient is given by the sum of n and that given in Assumption 3.2. Thanks to the universal bounds, it has a bounded H^2_{BMO}-norm for each n. It is also easy to confirm that $\widetilde{f}^n$ satisfies A_Γ-condition uniformly in n as in the proof of Lemma 4.1. Thus the measure change used in Theorem 2.5 of Royer [39] is still valid and hence the comparison principle follows. The uniqueness follows from Proposition 3.1 or from the comparison principle as [39]. □

Appendix C. Malliavin differentiability for Lipschitz BSDEs with jumps

In order to show Malliavin's differentiability of $Q_{\exp}$-growth BSDEs, we have to establish the differentiability for Lipschitz BSDEs with slightly more general setup than what was proved in [15,14]. For convenience of the readers, we give the detailed proof in this section. We closely follow the arguments used in El Karoui et al. (1997) [25]. The complication relative to a diffusion case is the treatment of small jumps. The difference from the work [15] is a local Lipschitz condition instead of the global Lipschitz condition for the Malliavin derivative of the driver.

We consider a BSDE defined by

$$\begin{aligned} Y_t &= \xi + \int_t^T f\Big(s, Y_s, Z_s, \int_{\mathbb{R}_0} \rho(x)G(s, \psi_s(x))\nu(dx)\Big)ds - \int_t^T Z_s dW_s \\ &\quad - \int_t^T \int_E \psi_s(x)\widetilde{\mu}(ds, dx), \end{aligned} \tag{C.1}$$

where $\xi : \Omega \to \mathbb{R}$, $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$. Here, $\int_{\mathbb{R}_0} \rho(x)G(s, \psi_s(x))\nu(dx)$ denotes a k-dimensional vector whose ith element is given by $\int_{\mathbb{R}_0} \rho^i(x)G^i(s, \psi^i_s(x))\nu^i(dx)$ where $\rho^i : \mathbb{R} \to \mathbb{R}$, $G^i : [0, T] \times \mathbb{R} \to \mathbb{R}$. With slight abuse of notation, we use $\Theta_r := \Big(Y_r, Z_r, \int_{\mathbb{R}_0} \rho(x)G(r, \psi_r(x))\nu(dx)\Big)$ as a collective argument in this section. The results in this section can be straightforwardly extended to multi-dimensional Lipschitz BSDEs.

Assumption C.1. (i) For every $i \in \{1, \dots, k\}$, $\rho^i(s)$ and $G^i(s, v)$ are continuous functions in $s \in [0, T]$ and $(s, v) \in [0, T] \times \mathbb{R}$, respectively. We set without loss of generality that $G^i(\cdot, 0) = 0$. In addition $\int_{\mathbb{R}_0} |\rho^i(x)|^2 \nu^i(dx) < \infty$, and with some positive constant K, G^i satisfies

$$|G^i(s, v) - G^i(s, v')| \leq K|v - v'|, \text{ for every } s \in [0, T] \text{ and } v, v' \in \mathbb{R}.$$

(ii) The map $(\omega, t) \mapsto f(\omega, t, \cdot)$ is $\mathbb{F}$-progressively measurable, and for every $(y, z, u), (y', z', u') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^k$, there exists some positive constant K such that

$$|f(\omega, t, y, z, u) - f(\omega, t, y', z', u')| \leq K(|y - y'| + |z - z'| + |u - u'|)$$

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$d\mathbb{P}\otimes dt$-a.e. $(\omega,t)\in\Omega\times[0,T]$.
(iii) $\xi\in\mathbb{L}^4(\Omega,\mathcal{F}_T,\mathbb{P})$ and $\big(f(t,0),t\in[0,T]\big)\in\mathbb{H}^4[0,T]$.

Remark C.1. Due to the property of G and ρ, it is easy to see that

$$\Big|\int_{\mathbb{R}_0}\rho(x)G(s,\psi_s(x))\nu(dx)-\int_{\mathbb{R}_0}\rho(x)G(s,\psi_s'(x))\nu(dx)\Big|\le K'\|\psi_s-\psi_s'\|_{\mathbb{L}^2(\nu)}$$

with some constant $K'>0$. Thus, Assumption C.1 yields the standard global Lipschitz conditions. By Lemma B.2 in [17], the BSDE (C.1) has a unique solution $(Y,Z,\psi)\in\mathcal{K}^4[0,T]$. In order to show the Malliavin's differentiability, we need additional assumptions.

Assumption C.2. (i) For every $i\in\{1,\dots,k\}$, G^i is one-time continuously differentiable with respect to its spacial variable v with a uniformly bounded and continuous derivative.
(ii) The terminal value is Malliavin differentiable $\xi\in\mathbb{D}^{1,2}$ and satisfies

$$\mathbb{E}\Big[\int_{\widetilde{E}}|D_{s,z}\xi|^2q(ds,dz)\Big]<\infty.$$

(iii) The driver $f(\cdot,y,z,u)$ is one-time continuously differentiable with respect to (y,z,u) with uniformly bounded and continuous derivatives. For every $(y,z,u)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^k$, the driver $\big(f(t,y,z,u),t\in[0,T]\big)$ belongs to $\mathbb{L}^{1,2}$ and its Malliavin derivative is denoted by $(D_{s,z}f)(t,y,z,u)$.
(iv) For every Wiener as well as jump direction, and for every $(y,z,u),(y',z',u')\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^k$ and $d\mathbb{P}\otimes dt$-a.e. $(\omega,t)\in\Omega\times[0,T]$, the Malliavin derivative of the driver satisfies the following local Lipschitz conditions[9] ;

$$|(D^i_{s,0}f)(t,y,z,u)-(D^i_{s,0}f)(t,y',z',u')|\le K^i_{s,0}(t)\big(|y-y'|+|z-z'|+|u-u'|\big),$$

for ds-a.e. $s\in[0,T]$ with $i\in\{1,\dots,d\}$, and

$$|(D^i_{s,z}f)(t,y,z,u)-(D^i_{s,z}f)(t,y',z',u')|\le K^i_{s,z}(t)\big(|y-y'|+|z-z'|+|u-u'|\big),$$

for $m^i(dz)ds$-a.e. $(s,z)\in[0,T]\times\mathbb{R}_0$ with $i\in\{1,\dots,k\}$. Here, $\big(K^i_{s,0}(t),t\in[0,T]\big)_{i\in\{1,\dots,d\}}$ and $\big(K^i_{s,z}(t),t\in[0,T]\big)_{i\in\{1,\dots,k\}}$ are $\mathbb{R}_+$-valued $\mathbb{F}$-progressively measurable processes satisfying $\int_{\widetilde{E}}\|K_{s,z}(\cdot)\|^4_{\mathbb{S}^4[0,T]}q(ds,dz)<\infty$.

Remark C.2. It follows from the conditions (ii), (iii) and (iv) that

$$\sum_{i=1}^k\int_0^T\int_{|z|\le\epsilon}\mathbb{E}\Big[|D^i_{s,z}\xi|^2+\Big(\int_0^T|(D^i_{s,z}f)(r,0)|dr\Big)^2+\|K^i_{s,z}\|_T^4\Big]m^i(dz)ds\to0$$

as $\epsilon\downarrow0$ by the dominated convergence.

Theorem C.1. *Suppose that Assumptions C.1 and C.2 hold true and denote the solution to the BSDE (C.1) as $(Y,Z,\psi)\in\mathcal{K}^4[0,T]$. Then, the following statements hold:*
(a) *For each Wiener direction $i\in\{1,\dots,d\}$ and ds-a.e. $s\in[0,T]$, there exists a unique solution $(Y^{s,0,i},Z^{s,0,i},\psi^{s,0,i})\in\mathcal{K}^2[0,T]$ to the BSDE*

$$Y_t^{s,0,i}=D^i_{s,0}\xi+\int_t^Tf^{s,0,i}(r)dr-\int_t^TZ_r^{s,0,i}dW_r-\int_t^T\int_E\psi_r^{s,0,i}(x)\widetilde{\mu}(dr,dx)\qquad\text{(C.2)}$$

[9] Delong & Imkeller (2010) [15] have treated a special case where $(K_{s,0},\ K_{s,z})$ are positive constants. The current generalization is necessary when one introduces a Markovian process X driven by a FSDE to create a forward–backward SDE system, which is the subject of interests in many applications.

for $0 \le s \le t \le T$, *where*

$$\begin{aligned} f^{s,0,i}(r) &:= (D^i_{s,0} f)(r, \Theta_r) + \partial_\Theta f(r, \Theta_r)\Theta^{s,0,i}_r \\ &= (D^i_{s,0} f)(r, \Theta_r) + \partial_y f(r, \Theta_r) Y^{s,0,i}_r + \partial_z f(r, \Theta_r) Z^{s,0,i}_r \\ &\quad + \partial_u f(r, \Theta_r) \int_{\mathbb{R}_0} \rho(x) \partial_v G(r, \psi_r(x)) \psi^{s,0,i}_r(x) \nu(dx) . \end{aligned}$$

(b) *For each jump direction* $i \in \{1, \dots, k\}$ *and* $m^i(dz)ds$*-a.e.* $(s, z) \in [0, T] \times \mathbb{R}_0$*, there exists a unique solution* $(Y^{s,z,i}, Z^{s,z,i}, \psi^{s,z,i}) \in \mathcal{K}^2[0, T]$ *to the BSDE*

$$Y^{s,z,i}_t = D^i_{s,z}\xi + \int_t^T f^{s,z,i}(r)dr - \int_t^T Z^{s,z,i}_r dW_r - \int_t^T \int_E \psi^{s,z,i}_r(x) \widetilde{\mu}(dr, dx) \tag{C.3}$$

for $0 \le s \le t \le T$ *and* $z \ne 0$, *where*

$$\begin{aligned} f^{s,z,i}(r) &:= \frac{1}{z}\Big(f(\omega^{s,z}, r, \Theta_r + z\Theta^{s,z,i}_r) - f(\omega, r, \Theta_r) \Big) \\ &= \frac{1}{z}\Big\{ f\Big(\omega^{s,z}, r, Y_r + zY^{s,z,i}_r, Z_r + zZ^{s,z,i}_r \\ &\qquad , \int_{\mathbb{R}_0} \rho(x) G\big(r, \psi_r(x) + z\psi^{s,z,i}_r(x)\big)\nu(dx) - f(\omega, r, \Theta_r)\Big)\Big\}. \end{aligned}$$

(c) *Solution of the BSDE* (C.1) *is Malliavin differentiable* $(Y, Z, \overline{\psi}) \in \mathbb{L}^{1,2} \times \mathbb{L}^{1,2} \times \mathbb{L}^{1,2}$. *Put, for every* i, $Y^{s,\cdot,i}_t = Z^{s,\cdot,i}_t = \psi^{s,\cdot,i}_t(\cdot) \equiv 0$ *for* $t < s \le T$, *then* $\big((Y^{s,z,i}_t, Z^{s,z,i}_t, \psi^{s,z,i}_t(x)), 0 \le s, t \le T, x \in \mathbb{R}_0, z \in \mathbb{R}\big)$ *is a version of the Malliavin derivative* $\big((D^i_{s,z}Y_t, D^i_{s,z}Z_t, D^i_{s,z}\psi_t(x)), 0 \le s, t \le T, x \in \mathbb{R}_0, z \in \mathbb{R}\big)$ *for every Wiener and jump direction.*

Proof. For notational simplicity, we omit i denoting the direction of derivative by assuming that we consider each direction separately.

Proof for (a) *and* (b)

It is easy to see that both of the BSDEs (C.2) and (C.3) satisfy the standard global Lipschitz conditions. We have $|f^{s,0}(r)| \le |(D_{s,0}f)(r, 0)| + K_{s,0}(r)|\Theta_r| + K|\Theta^{s,0}_r|$. Since

$$\begin{aligned} f^{s,z}(r) &= \frac{f(\omega^{s,z}, r, \Theta_r) - f(\omega, r, \Theta_r)}{z} + \frac{f(\omega^{s,z}, r, \Theta_r + z\Theta^{s,z}_r) - f(\omega^{s,z}, r, \Theta_r)}{z} \\ &= (D^{s,z} f)(r, \Theta_r) + \frac{f(\omega^{s,z}, r, \Theta_r + z\Theta^{s,z}_r) - f(\omega^{s,z}, r, \Theta_r)}{z}, \end{aligned}$$

we also have $|f^{s,z}(r)| \le |(D_{s,z}f)(r, 0)| + K_{s,z}(r)|\Theta_r| + K|\Theta^{s,z}_r|$ for $z \in \mathbb{R}_0$. Thus, Lemma B.2 in [17] tells us that for all $(s, z) \in [0, T] \times \mathbb{R}$ (thus including $\Theta^{s,0}$) there exists a unique solution $\Theta^{s,z} \in \mathcal{K}^2[0, T]$ satisfying

$$\begin{aligned} &\|(Y^{s,z}, Z^{s,z}, \psi^{s,z})\|^2_{\mathcal{K}^2[0,T]} \\ &\le C_{K,T}\mathbb{E}\Big[|D_{s,z}\xi|^2 + \Big(\int_0^T \Big[|(D_{s,z}f)(r, 0)| + K_{s,z}(r)|\Theta_r|\Big]dr\Big)^2\Big] \\ &\le C_{K,T}\mathbb{E}\Big[|D_{s,z}\xi|^2 + \Big(\int_0^T |(D_{s,z}f)(r, 0)|dr\Big)^2 + \|K_{s,z}\|^4_T + \Big(\int_0^T |\Theta_r|^2 dr\Big)^2\Big] < \infty. \end{aligned}$$

Note here that $\Theta \in \mathcal{K}^4[0, T]$. By Assumption C.2(ii), (iii) and (iv), it also follows that

$$\int_{\widetilde{E}} \|(Y^{s,z}, Z^{s,z}, \psi^{s,z})\|^2_{\mathcal{K}^2[0,T]} q(ds, dz) < \infty .$$

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Proof for (c)
We consider a sequence of solution $(Y^n, Z^n, \psi^n)_{n\geq 1}$ of the following BSDEs that converges to (Y, Z, ψ) of (C.1) in $\mathcal{K}^4[0, T]$;

$$Y_t^{n+1} = \xi + \int_t^T f^n(r) - \int_t^T Z_r^{n+1} dW_r - \int_t^T \int_E \psi_r^{n+1}(x)\widetilde{\mu}(dr, dx), \tag{C.4}$$

for $t \in [0, T]$ and $n \in \mathbb{N}$, where $f^n(r) := f\Big(r, Y_r^n, Z_r^n, \int_{\mathbb{R}_0} \rho(x)G(r, \psi_r^n(x))\nu(dx)\Big)$. The convergence can be proven by the standard arguments of contraction mapping for the Lipschitz BSDEs. See, for example, Lemma B.2 in [17] and its proof.

[**First step: Showing** $(Y^{n+1}, Z^{n+1}, \overline{\psi}^{n+1}) \in (\mathbb{L}^{1,2})^3$]
We first suppose that $(Y^n, Z^n, \overline{\psi}^n) \in (\mathbb{L}^{1,2})^3$ and are going to prove that $(Y^{n+1}, Z^{n+1}, \overline{\psi}^{n+1}) \in (\mathbb{L}^{1,2})^3$. Then, we can inductively show $(Y^n, Z^n, \overline{\psi}^n) \in (\mathbb{L}^{1,2})^3$ for every $n \in \mathbb{N}$. Firstly, the *chain* rules (Theorem 3.5 and Theorem 12.8 in [16] with the division by the jump size in the current convention) and Lemma 3.2 in [15] show that

$$\int_{\mathbb{R}_0} \rho(x)G(r, \psi_r^n(x))\nu(dx)dr \in \mathbb{D}^{1,2} . \tag{C.5}$$

In particular, this is because

$$\int_{\widetilde{E}} \|D_{t,z}G(\cdot, \psi_\cdot^n)\|^2_{\mathbb{J}^2[0,T]} q(dt, dz) \leq K^2 \int_{\widetilde{E}} \|D_{t,z}\psi_\cdot^n\|^2_{\mathbb{J}^2[0,T]} q(dt, dz) < \infty,$$

where we have used the bounded derivative and the Lipschitz condition for G and the assumption that $\overline{\psi}^n \in \mathbb{L}^{1,2}$. This also shows that $G(\cdot, \psi_\cdot^n) \in \mathbb{L}^{1,2}$.

By (C.5) and by the general chain rule for random functions (see, Theorem 3.12 [18] for Wiener directions and Proposition 5.5 [40] for jump directions in a canonical Levy space, respectively), we see $f^n(r) = f(r, \Theta_r^n) \in \mathbb{D}^{1,2}$ for every $r \in [0, T]$. It is easy to check $\|f^n(\cdot)\|^2_{\mathbb{H}^2[0,T]} < \infty$. Next, Assumption C.2, the hypothesis $(Y^n, Z^n, \overline{\psi}^n) \in \mathcal{K}^4[0, T] \cap (\mathbb{L}^{1,2})^3$ and the estimate $|D_{s,z} f^n(r)| \leq |(D_{s,z} f)(r, 0)| + K_{s,z}(r)|\Theta_r^n| + K|D_{s,z}\Theta_r^n|$ imply

$$\begin{aligned}
&\int_{\widetilde{E}} \|D_{t,z} f^n(\cdot)\|^2_{\mathbb{H}^2[0,T]} q(dt, dz) \\
&\quad \leq C_K \int_{\widetilde{E}} \mathbb{E}\Big[\int_0^T \big(|(D_{t,z} f)(r, 0)|^2 + |D_{t,z}\Theta_r^n|^2\big)dr + \|K_{t,z}\|_T^4 \\
&\qquad + \Big(\int_0^T |\Theta_r^n|^2 dr\Big)^2\Big] q(dt, dz) < \infty
\end{aligned}$$

with some positive constant C_K. Thus, Lemma 3.2 [15] shows that $\int_t^T f^n(r)dr \in \mathbb{D}^{1,2}$ for every $t \in [0, T]$. As a result, we have $\xi + \int_t^T f^n(r) \in \mathbb{D}^{1,2}$ for each $t \in [0, T]$. Thus, by Lemma 3.1 [15], we conclude that $Y_t^{n+1} = \mathbb{E}\left[\xi + \int_t^T f^n(r)\Big|\mathcal{F}_t\right] \in \mathbb{D}^{1,2}$, which then implies

$$\int_t^T Z_r^{n+1} dW_r + \int_t^T \int_E \psi_r^{n+1}(x)\widetilde{\mu}(dr, dx) = -Y_t^{n+1} + \xi + \int_t^T f^n(r)dr \in \mathbb{D}^{1,2} ,$$

which, together with Lemma 3.3 [15], shows $Z^{n+1}, \overline{\psi}^{n+1} \in \mathbb{L}^{1,2}$.

We are now going to prove $Y^{n+1} \in \mathbb{L}^{1,2}$. For a Wiener ($z = 0$) as well as a jump ($z \neq 0$) direction, we have,

$$D_{s,z}Y_t^{n+1} = D_{s,z}\xi + \int_t^T D_{s,z}f^n(r)dr - \int_t^T D_{s,z}Z_r^{n+1}dW_r$$
$$- \int_t^T \int_E D_{s,z}\psi_r^{n+1}(x)\widetilde{\mu}(dr,dx),$$
$$\text{for } 0 \le s \le t \le T \quad \text{and} \quad z \in \mathbb{R}^k,$$

by Lemma 3.3 [15]. By Lemmas B.2 in [17], one obtains

$$\int_{\widetilde{E}} \|D_{s,z}Y^{n+1}\|^2_{\mathbb{S}^2[0,T]}q(ds,dz) \le C_{K,T}\int_{\widetilde{E}} \mathbb{E}\Big[|D_{s,z}\xi|^2 + \Big(\int_0^T |D_{s,z}f^n(r)|dr\Big)^2\Big]q(ds,dz)$$
$$\le C_{K,T}\int_{\widetilde{E}} \mathbb{E}\Big[|D_{s,z}\xi|^2 + \Big(\int_0^T |(D_{s,z}f)(r,0)| + |D_{s,z}\Theta_r^n|dr\Big)^2$$
$$+ \|K_{s,z}\|_T^4 + \Big(\int_0^T |\Theta_r^n|^2dr\Big)^2\Big]q(ds,dz) < \infty\,, \tag{C.6}$$

where $D_{s,z}Y_t^{n+1} \equiv 0$ for $t < s$ is used. Hence $(Y^{n+1}, Z^{n+1}, \overline{\psi}^{n+1}) \in (\mathbb{L}^{1,2})^3$ is proved.

[**Second step: convergence of** $D_{s,0}\Theta^n \to \Theta^{s,0}$]
Let us set the difference process as follows:

$$\Delta^{s,0}Y^n := Y^{s,0} - D_{s,0}Y^n, \quad \Delta^{s,0}Z^n := Z^{s,0} - D_{s,0}Z^n, \quad \Delta^{s,0}\psi^n := \psi^{s,0} - D_{s,0}\psi^n,$$

and denote $\Delta^{s,0}\Theta^n := (\Delta^{s,0}Y^n, \Delta^{s,0}Z^n, \Delta^{s,0}\psi^n)$ for every $n \in \mathbb{N}$. We claim

$$\lim_{n\to\infty}\int_0^T \|(\Delta^{s,0}\Theta^n)\|^2_{\mathcal{K}^2[0,T]}ds = 0\,. \tag{C.7}$$

Since $|f^{s,0}(r) - D_{s,0}f^n(r)| \le K_{s,0}(r)|\Theta_r - \Theta_r^n| + |\partial_\Theta f(r,\Theta_r) - \partial_\Theta f(r,\Theta_r^n)||\Theta_r^{s,0}| + K|\Delta^{s,0}\Theta_r^n|$, the a priori estimate given in Lemma B.2 [17] gives

$$\int_0^T \big\|(\Delta^{s,0}Y^{n+1}, \Delta^{s,0}Z^{n+1}, \Delta^{s,0}\psi^{n+1})\big\|^2_{\mathcal{K}^2[0,T]}ds$$
$$\le C_T\int_0^T \mathbb{E}\Big[\Big(\int_0^T |f^{s,0}(r) - D_{s,0}f^n(r)|dr\Big)^2\Big]ds$$
$$\le C_T\int_0^T \mathbb{E}\Big[\Big(\int_0^T \Big[K_{s,0}(r)|\Theta_r - \Theta_r^n| + |\partial_\Theta f(r,\Theta_r) - \partial_\Theta f(r,\Theta_r^n)||\Theta_r^{s,0}|\Big]dr\Big)^2\Big]ds$$
$$+ C_{T,K}\int_0^T \mathbb{E}\Big[\Big(\int_0^T |\Delta^{s,0}\Theta_r^n|dr\Big)^2\Big]ds\,.$$

One sees that the first line converges to zero because $\Theta^n \to \Theta \in \mathcal{K}^4[0,T]$. Thus, by using a sequence of small positive constants $(\epsilon_n)_{n\ge1}$ converging to zero, one can write

$$\int_0^T \big\|(\Delta^{s,0}Y^{n+1}, \Delta^{s,0}Z^{n+1}, \Delta^{s,0}\psi^{n+1})\big\|^2_{\mathcal{K}^2[0,T]}ds$$
$$\le \epsilon_n + C_{T,K}\int_0^T \mathbb{E}\left[\Big(\int_0^T |\Delta^{s,0}\Theta_r^n|dr\Big)^2\right]ds$$
$$\le \epsilon_n + C'_{T,K}\max(T^2,T)\int_0^T \big\|(\Delta^{s,0}Y^n, \Delta^{s,0}Z^n, \Delta^{s,0}\psi^n)\big\|^2_{\mathcal{K}^2[0,T]}ds.$$

For a sufficiently **small** $T(>0)$ so that $\alpha := C'_{T,K}\max(T^2,T) < 1$, one obtains $\int_0^T \|(\Delta^{s,0}\Theta^{n+1})\|^2_{\mathcal{K}^2[0,T]}ds \le \epsilon_n + \alpha\int_0^T \|(\Delta^{s,0}\Theta^n)\|^2_{\mathcal{K}^2[0,T]}ds$. Then, by fixing some $n_0 \in \mathbb{N}$,

$$\int_0^T \|(\Delta^{s,0}\Theta^{n+n_0})\|^2_{\mathcal{K}^2[0,T]}ds \le \frac{\epsilon_{n_0}}{1-\alpha} + \alpha^n \int_0^T \|(\Delta^{s,0}\Theta^{n_0})\|^2_{\mathcal{K}^2[0,T]}ds.$$

Thus, by passing n and then n_0 to ∞, (C.7) is proved for small T.

For **general** $T>0$, one can use a time partition $0=T_0<T_1<\cdots<T_N=T$ that is fine enough so that $\alpha<1$ in every time interval. Due to the uniqueness of the solution, by setting $Y^{s,0}_{T_i}$ as the terminal condition for the interval $[T_{i-1},T_i]$, one can prove (C.7) for the interval. Repeating the procedures from $i=N$ to $i=1$ proves the claim.

[**Third step: convergence of** $D_{s,z}\Theta^n \to \Theta^{s,z}$ $(z\neq 0)$]
Choosing one direction of jump (omit i for simplicity) and put

$$\Delta^{s,z}Y^n := Y^{s,z} - D_{s,z}Y^n, \quad \Delta^{s,z}Z^n := Z^{s,z} - D_{s,z}Z^n, \quad \Delta^{s,z}\psi^n := \psi^{s,z} - D_{s,z}\psi^n.$$

and denote $\Delta^{s,z}\Theta^n := (\Delta^{s,z}Y^n, \Delta^{s,z}Z^n, \Delta^{s,z}\psi^n)$ for every $n\in\mathbb{N}$. In this step, our final goal is to show the convergence

$$\lim_{n\to\infty}\int_0^T\int_{\mathbb{R}_0} \|(\Delta^{s,z}\Theta^n)\|^2_{\mathcal{K}^2[0,T]}m(dz)ds = 0\,. \tag{C.8}$$

Before discussing (C.8), we have to prove first that the convergence

$$\begin{aligned}&\lim_{\epsilon\downarrow 0}\int_0^T\int_{|z|>\epsilon} \left\|(\Delta^{s,z}\Theta^{n+1})\right\|^2_{\mathcal{K}^2[0,T]}m(dz)ds \\ &= \int_0^T\int_{\mathbb{R}_0} \left\|(\Delta^{s,z}\Theta^{n+1})\right\|^2_{\mathcal{K}^2[0,T]}m(dz)ds\end{aligned} \tag{C.9}$$

occurs *uniformly* in (sufficiently large) n. As the proof of Theorem 4.1 [15], it suffices to show that, for each $\epsilon>0$, there exists a positive constant C and $\bar\epsilon>0$ independent of n such that

$$\int_0^T\int_{|z|\le\bar\epsilon} \left\|(\Delta^{s,z}\Theta^{n+1})\right\|^2_{\mathcal{K}^2[0,T]}m(dz)ds < C\epsilon\,.$$

By Remark C.2, for a given arbitrary $\epsilon>0$, there exists $\bar\epsilon>0$ such that

- $$\int_0^T\int_{|z|\le\bar\epsilon}\mathbb{E}\Big[|D_{s,z}\xi|^2 + \Big(\int_0^T |(D_{s,z}f)(r,0)|dr\Big)^2 + \|K_{s,z}\|^4_T\Big]m(dz)ds < \epsilon \tag{C.10}$$
- $$\int_0^T\int_{|z|\le\bar\epsilon} m(dz)ds < \epsilon. \tag{C.11}$$

Let us fix $\bar\epsilon>0$ as above. By Lemma B.2 [17], we have $\|(\Delta^{s,z}\Theta^{n+1})\|^2_{\mathcal{K}^2[0,T]} \le C_T\mathbb{E}\Big[\Big(\int_0^T |f^{s,z}(r) - D_{s,z}f^n(r)|dr\Big)^2\Big]$. Using the (local) Lipschitz properties, it is easy to show that

$$|f^{s,z}(r) - D_{s,z}f^n(r)| \le K_{s,z}(r)|\Theta_r - \Theta^n_r| + K|\Theta^{s,z}_r| + K|D_{s,z}\Theta^n_r|$$

and hence

$$\int_0^T \int_{|z|\le\bar{\epsilon}} \|(\Delta^{s,z}\Theta^{n+1})\|^2_{\mathcal{K}^2[0,T]} m(dz)ds$$
$$\le C_{T,K}\int_0^T \int_{|z|\le\bar{\epsilon}} \mathbb{E}\Big[\Big(\int_0^T K_{s,z}(r)|\Theta_r-\Theta_r^n|dr\Big)^2$$
$$+\Big(\int_0^T |\Theta_r^{s,z}|dr\Big)^2+\Big(\int_0^T |D_{s,z}\Theta_r^n|dr\Big)^2\Big]m(dz)ds\ . \tag{C.12}$$

We are now going to discuss each term of (C.12). For the first term, it is straightforward to see that there exists n independent constant C such that

$$C_{T,K}\int_0^T \int_{|z|\le\bar{\epsilon}} \mathbb{E}\Big[\Big(\int_0^T K_{s,z}(r)|\Theta_r-\Theta_r^n|dr\Big)^2\Big]$$
$$\le C_{T,K}\int_0^T \int_{|z|\le\bar{\epsilon}} \mathbb{E}\Big[\|K_{s,z}\|_T^4+\Big(\int_0^T |\Theta_r-\Theta_r^n|^2dr\Big)^2\Big]m(dz)ds < C\epsilon$$

where the last inequality follows from (C.10), (C.11) and the fact that $\|\Theta-\Theta^n\|^4_{\mathbb{H}^4[0,T]}$ is bounded due to the convergence $\Theta^n \to \Theta$ in $\mathcal{K}^4[0,T]$. For the second term of (C.12), one can show

$$C_{T,K}\int_0^T \int_{|z|\le\bar{\epsilon}} \mathbb{E}\Big[\Big(\int_0^T |\Theta_r^{s,z}|dr\Big)^2\Big]m(dz)ds \le C_{T,K}\int_0^T \int_{|z|\le\bar{\epsilon}} \|(\Theta^{s,z})\|^2_{\mathcal{K}^2[0,T]}m(dz)ds$$
$$\le C_{T,K}\int_0^T \int_{|z|\le\bar{\epsilon}} \mathbb{E}\Big[|D_{s,z}\xi|^2+\Big(\int_0^T |(D_{s,z}f)(r,0)|dr\Big)^2$$
$$+\|K_{s,z}\|_T^4+\Big(\int_0^T |\Theta_r|^2dr\Big)^2\Big]m(dz)ds$$
$$< C\epsilon \tag{C.13}$$

where the last inequality follows from (C.10), (C.11) and the fact that $\Theta \in \mathcal{K}^4[0,T]$. Finally, the third term of (C.12) can be evaluated as

$$C_{T,K}\int_0^T \int_{|z|\le\bar{\epsilon}} \mathbb{E}\Big[\Big(\int_0^T |D_{s,z}\Theta_r^n|dr\Big)^2\Big]m(dz)ds$$
$$\le C_{T,K}\int_0^T \int_{|z|\le\bar{\epsilon}} \|(D_{s,z}\Theta^n)\|^2_{\mathcal{K}^2[0,T]}m(dz)ds\ .$$

Here, by the same a priori estimate used in (C.6),

$$C_{T,K}\|(D_{s,z}\Theta^n)\|^2_{\mathcal{K}^2[0,T]} \le C_{K,T}\mathbb{E}\Big[|D_{s,z}\xi|^2+\Big(\int_0^T |(D_{s,z}f)(r,0)|dr\Big)^2+\|K_{s,z}\|_T^4$$
$$+\Big(\int_0^T |\Theta_r^{n-1}|^2dr\Big)^2\Big]+C_{K,T}\mathbb{E}\Big[\Big(\int_0^T |D_{s,z}\Theta_r^{n-1}|dr\Big)^2\Big]$$

$$\le C_{K,T}\mathbb{E}\Big[\epsilon_{n-1}+|D_{s,z}\xi|^2+\Big(\int_0^T |(D_{s,z}f)(r,0)|dr\Big)^2+\|K_{s,z}\|_T^4+\Big(\int_0^T |\Theta_r|^2dr\Big)^2\Big]$$
$$+C_{K,T}\max(T^2,T)\|(D_{s,z}\Theta^{n-1})\|^2_{\mathcal{K}^2[0,T]}\ , \tag{C.14}$$

where $(\epsilon_n)_{n\geq 1}$ is a sequence of positive constants with $\epsilon_n := \left| \|\Theta^n\|^4_{\mathbb{H}^4[0,T]} - \|\Theta\|^4_{\mathbb{H}^4[0,T]} \right|$. It is bounded ($\sup_{n\in\mathbb{N}}(\epsilon_n) \leq \delta$) with some n-independent constant δ due to the convergence of $\Theta^n \to \Theta$ in $\mathcal{K}^4[0,T]$. Choosing the terminal time T small enough so that $\alpha := C_{K,T}\max(T^2,T) < 1$, (C.14) yields

$$
\begin{aligned}
&C_{T,K}\int_0^T\int_{|z|\leq\bar\epsilon}\|(D_{s,z}\Theta^n)\|^2_{\mathcal{K}^2[0,T]}m(dz)ds\\
&\leq \frac{C_{K,T}}{1-\alpha}\int_0^T\int_{|z|\leq\bar\epsilon}\mathbb{E}\Bigg[\delta+|D_{s,z}\xi|^2+\left(\int_0^T|(D_{s,z}f)(r,0)|dr\right)^2+\|K_{s,z}\|_T^4\\
&+\left(\int_0^T|\Theta_r|^2dr\right)^2\Bigg]m(dz)ds+\alpha^{n-1}\int_0^T\int_{|z|\leq\bar\epsilon}\|(D_{s,z}\Theta^1)\|^2_{\mathcal{K}^2[0,T]}m(dz)ds.
\end{aligned}
$$

It is free to choose $\Theta^1 \equiv 0$ in the fixed point iteration (C.4). Thus, the right hand side is dominated by $C\epsilon$ with some n independent constant C due to (C.10) and (C.11).

By the previous arguments, we have shown that the convergence of (C.9) is uniform in n, at least for **sufficiently small** T. In this case, one can exchange the order of limit operations;

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{\epsilon\downarrow 0}\int_0^T\int_{|z|>\epsilon}\left\|(\Delta^{s,z}\Theta^{n+1})\right\|^2_{\mathcal{K}^2}m(dz)ds\\
&=\lim_{\epsilon\downarrow 0}\lim_{n\to\infty}\int_0^T\int_{|z|>\epsilon}\left\|(\Delta^{s,z}\Theta^{n+1})\right\|^2_{\mathcal{K}^2}m(dz)ds\ .
\end{aligned}
$$

Therefore, in order to show the convergence (C.8), it is enough to prove

$$
\lim_{n\to\infty}\int_0^T\int_{|z|>\epsilon}\left\|(\Delta^{s,z}\Theta^{n+1})\right\|^2_{\mathcal{K}^2[0,T]}m(dz)ds=0
$$

for each $\epsilon > 0$. An inequality from the Lipschitz property of the driver

$$
\begin{aligned}
|f^{s,z}(r)-D_{s,z}f^n(r)| &\leq \frac{1}{|z|}\left|f(\omega^{s,z},r,\Theta_r+z\Theta_r^{s,z})-f(\omega^{s,z},r,\Theta_r^n+zD_{s,z}\Theta_r^n)\right|\\
&+\frac{1}{|z|}\left|f(\omega,r,\Theta_r)-f(\omega,r,\Theta_r^n)\right|\leq\frac{2K}{|z|}|\Theta_r-\Theta_r^n|+K|\Delta^{s,z}\Theta_r^n|
\end{aligned}
$$

implies

$$
\begin{aligned}
&\int_0^T\int_{|z|>\epsilon}\left\|(\Delta^{s,z}\Theta^{n+1})\right\|^2_{\mathcal{K}^2[0,T]}m(dz)ds\\
&\leq C_{T,K}\int_0^T\int_{|z|>\epsilon}\mathbb{E}\left[\frac{1}{|z|^2}\left(\int_0^T|\Theta_r-\Theta_r^n|dr\right)^2+\left(\int_0^T|\Delta^{s,z}\Theta_r^n|dr\right)^2\right]m(dz)ds\\
&\leq \epsilon_n+C_{T,K}\max(T^2,T)\int_0^T\int_{|z|>\epsilon}\|(\Delta^{s,z}\Theta^n)\|^2_{\mathcal{K}^2[0,T]}m(dz)ds
\end{aligned}
$$

where $\epsilon_n \to 0$ as $n \to 0$ due to the convergence of $\Theta^n \to \Theta$. If necessary by re-choosing T small enough so that $\alpha := C_{T,K}\max(T^2,T) < 1$, one gets

$$
\begin{aligned}
&\int_0^T\int_{|z|>\epsilon}\left\|(\Delta^{s,z}\Theta^{n+n_0})\right\|^2_{\mathcal{K}^2[0,T]}m(dz)ds\\
&\leq\frac{\epsilon_{n_0}}{1-\alpha}+\alpha^n\int_0^T\int_{|z|>\epsilon}\left\|(\Delta^{s,z}\Theta^{n_0})\right\|^2_{\mathcal{K}^2[0,T]}m(dz)ds.
\end{aligned}
$$

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By passing to the limit $n, n_0 \to \infty$, (C.8) is proved for small T.

For **general** $T > 0$, one can construct a partition $0 = T_0 < T_1 < \cdots < T_N = T$ fine enough so that one can conclude by the previous arguments

$$\lim_{n\to 0} \int_{T_{N-1}}^{T} \int_{|z|>\epsilon} \|(\Delta^{s,z}\Theta^n)\|^2_{\mathcal{K}^2[0,T]} m(dz)ds = 0\,.$$

Note that (C.13) implies $\lim_{\epsilon\downarrow 0}\int_0^T\int_{|z|<\epsilon}\mathbb{E}|Y^{s,z}_{T_{N-1}}|^2 m(dz)ds = 0$, in particular. Therefore, by the same procedures with a new terminal value $Y^{s,z}_{T_{N-1}}$ instead of $D_{s,z}\xi$, the convergence (C.8) in $[T_{N-2}, T_{N-1}]$ is proved. Repeating the same arguments proves (C.8) for general T. Hence, one can conclude $(Y^n, Z^n, \overline{\psi}^n)$ converges to $\big((Y, Z, \overline{\psi}), (Y^{s,z}, Z^{s,z}, \overline{\psi}^{s,z})\big)$ in $(\mathbb{L}^{1,2})^3$. Finally, thanks to the closability of the Malliavin derivatives in $\mathbb{L}^{1,2}$ (see Theorem 12.6 in [16]), one concludes $(Y, Z, \overline{\psi}) \in \mathbb{L}^{1,2}$ and that $(Y^{s,z}, Z^{s,z}, \psi^{s,z})$ is a version of $(D_{s,z}Y, D_{s,z}Z, D_{s,z}\psi)$. $\square$

Appendix D. Technical details omitted in the proof of Theorem 5.1

D.1. Proof for (5.11)

By (5.9) and the dominated convergence theorem, it suffices to show

$$\lim_{m\to\infty} \|(\Delta^{s,0}Y^m, \Delta^{s,0}Z^m, \Delta^{s,0}\psi^m)\|^p_{\mathcal{K}^p[0,T]} = 0$$

for ds-a.e. $s \in [0, T]$. Since

- $$f^{s,0}(r) - D_{s,0}f_m(r) = f^{s,0}(r) - \big((D_{s,0}f_m)(r, \Theta^m_r) + \partial_\Theta f_m(r, \Theta^m_r)\Theta^{s,0}_r\big) + \partial_\Theta f_m(r, \Theta^m_r)(\Theta^{s,0}_r - D_{s,0}\Theta^m_r)\,,$$

and

- $$\Big|f^{s,0}(r) - \big((D_{s,0}f_m)(r, \Theta^m_r) + \partial_\Theta f_m(r, \Theta^m_r)\Theta^{s,0}_r\big)\Big| \le |(D_{s,0}f)(r, \Theta_r) - (D_{s,0}f)(r, \Theta^m_r)| + |(D_{s,0}f)(r, \Theta^m_r) - (D_{s,0}f_m)(r, \Theta^m_r)| + |\partial_\Theta f(r, \Theta_r) - \partial_\Theta f_m(r, \Theta^m_r)||\Theta^{s,0}_r|\,,$$

Lemma A.2 implies that

$$\begin{aligned}
&\|(\Delta^{s,0}Y^m, \Delta^{s,0}Z^m, \Delta^{s,0}\psi^m)\|^p_{\mathcal{K}^p[0,T]} \\
&\le C\mathbb{E}\Big[\Big(\int_0^T |(D_{s,0}f)(r, \Theta_r) - (D_{s,0}f)(r, \Theta^m_r)|dr\Big)^{p\bar{q}^2} \\
&\quad + \Big(\int_0^T |(D_{s,0}f)(r, \Theta^m_r) - (D_{s,0}f_m)(r, \Theta^m_r)|dr\Big)^{p\bar{q}^2} \\
&\quad + \Big(\int_0^T |\partial_\Theta f(r, \Theta_r) - \partial_\Theta f_m(r, \Theta^m_r)||\Theta^{s,0}_r|dr\Big)^{p\bar{q}^2}\Big]^{\frac{1}{\bar{q}^2}}
\end{aligned}$$

where, as before, $C > 0$ and $\bar{q} > 1$ are constants independent of m.

Let us check each term. By the local Lipschitz property, the first term yields

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^T |(D_{s,0}f)(r, \Theta_r) - (D_{s,0}f)(r, \Theta^m_r)|dr\Big)^{p\bar{q}^2}\Big] \\
&\quad \le C\mathbb{E}\Big[\|K^M_{s,0}\|_T^{2p\bar{q}^2}\Big]^{\frac{1}{2}}\mathbb{E}\Big[\|\delta Y^m\|_T^{2p\bar{q}^2} + \Big(\int_0^T \|\delta\psi^m_r\|^2_{\mathbb{L}^2(\nu)}dr\Big)^{p\bar{q}^2}\Big]^{\frac{1}{2}} \\
&\quad + C\mathbb{E}\Big[\|K^M_{s,0}\|_T^{2p\bar{q}^2}\Big(\int_0^T |H^m(r)|^2dr\Big)^{p\bar{q}^2}\Big]^{\frac{1}{2}}\mathbb{E}\Big[\Big(\int_0^T |\delta Z^m_r|^2dr\Big)^{p\bar{q}^2}\Big]^{\frac{1}{2}},
\end{aligned} \tag{D.1}$$

where the process H^m is defined by $H^m(r) := 1 + |Z_r| + |Z_r^m| + \|\psi_r\|_{\mathbb{L}^2(\nu)} + \|\psi_r^m\|_{\mathbb{L}^2(\nu)}$ and $(\delta Y^m, \delta Z^m, \delta\psi^m) := (Y - Y^m, Z - Z^m, \psi - \psi^m)$. Since $H^m \in \mathbb{H}^2_{BMO}$ with the norm dominated by constant independent of m, the convergence of $\Theta^m \to \Theta$ in $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ implies that (D.1) converges to zero as $m \to \infty$.

Secondly, by definition of the truncated driver, $(D_{s,0} f_m)(r, \Theta_r^m) = (D_{s,0} f)\big(r, \varphi_m(\Theta_r^m)\big)$. Since both Θ^m and $\varphi_m(\Theta^m)$ converge to Θ in $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$, the convergence of the second term can be shown in the same way as the first term.

Finally, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^T |\partial_\Theta f(r,\Theta_r) - \partial_\Theta f_m(r,\Theta_r^m)||\Theta_r^{s,0}|dr\Big)^{p\bar{q}^2}\Big] \\
&\quad \le \mathbb{E}\Big[\Big(\int_0^T |\partial_\Theta f(r,\Theta_r) - \partial_\Theta f_m(r,\Theta_r^m)|^2 dr\Big)^{p\bar{q}^2}\Big]^{\frac{1}{2}} \mathbb{E}\Big[\Big(\int_0^T |\Theta_r^{s,0}|^2 dr\Big)^{p\bar{q}^2}\Big]^{\frac{1}{2}}. \qquad \text{(D.2)}
\end{aligned}$$

By (5.4), there exists a constant C_M depends only on the universal bounds such that $|\partial_\Theta f_m(r,\Theta_r^m)| \le C_M(1 + |Z_r^m| + \|\psi_r^m\|_{\mathbb{L}^2(\nu)})$. Since $Z^m \to Z$ (resp. $\psi^m \to \psi$) in $\mathbb{H}^2_{BMO}$ (resp. $\mathbb{J}^2_{BMO}$), the energy inequality of Lemma 2.2 gives the convergence in $\mathbb{H}^{p'}$ (resp. $\mathbb{J}^{p'}$) with $\forall p' \ge 2$. Thus, by extracting subsequence if necessary, one sees $\sup_m |Z^m|$, $\sup_m \|\psi^m\|_{\mathbb{L}^2(\nu)}$ are in $\mathbb{H}^{p'}$ for any $p' \ge 2$ from Lemma 2.5 of [29]. Since $\partial_\Theta f_m(r,\Theta_r^m) \to \partial_\Theta f(r,\Theta_r)\, dt \otimes d\mathbb{P}$-a.e., the dominated convergence shows the RHS of (D.2) tends to 0 as $m \to \infty$. One can confirm the convergence actually occurs in the entire sequence, since otherwise there exists a subsequence (m_j) such that the RHS must be bounded from below by some positive constant. However, one can once again choose a further subsequence from (m_j) so that the RHS converges to zero by the dominated convergence as the last discussion, which is a contradiction. This proves (5.11).

D.2. Proof for (5.12)

Let us define a d-dimensional $\mathbb{F}$-progressively measurable process $(b_{s,z}^m(r), r \in [0,T])$ by

$$b_{s,z}^m(\omega, r) := \frac{f_m(\omega^{s,z}, r, \check{\Xi}_{s,z}^m(r)) - f_m(\omega^{s,z}, r, \Xi_{s,z}^m(r))}{z|\Delta^{s,z} Z_r^m|^2} \mathbf{1}_{\Delta^{s,z} Z_r^m \neq 0} \Delta^{s,z} Z_r^m$$

where $\check{\Xi}_{s,z}^m := (\mathcal{Y}_{s,z}^m, Z^m + zZ^{s,z}, \int_{\mathbb{R}_0} \rho(x) G_m(\cdot, \Psi_{s,z}^m(\cdot,x))\nu(dx))$ and $\Xi_{s,z}^m := (\mathcal{Y}_{s,z}^m, \mathcal{Z}_{s,z}^m, \int_{\mathbb{R}_0} \rho(x) G_m(\cdot, \Psi_{s,z}^m(\cdot,x))\nu(dx))$. Noticing the fact $\mathcal{Z}^{s,z} = Z + zZ^{s,z}$, one sees $(\mathcal{Y}_{s,z}^m, Z^m + zZ^{s,z}, \Psi_{s,z}^m) \to (\mathcal{Y}^{s,z}, \mathcal{Z}^{s,z}, \Psi^{s,z})$ in $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$. Let us also introduce a map $\widetilde{f}_{s,z}^m : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{L}^2(E,\nu;\mathbb{R}^k) \to \mathbb{R}$ by

$$\begin{aligned}
&\widetilde{f}_{s,z}^m(\omega, r, \widetilde{y}, \widetilde{\psi}) \\
&\quad := (D_{s,z} f)(r,\Theta_r) - (D_{s,z} f_m)(r,\Theta_r^m) - \frac{1}{z}\big[f(\omega^{s,z}, r, \Theta_r) - f_m(\omega^{s,z}, r, \Theta_r^m)\big] \\
&\qquad + \frac{1}{z}\Big\{f\Big(\omega^{s,z}, r, z\widetilde{y} + \mathcal{Y}_{s,z}^m(r) + \delta Y_r^m, \mathcal{Z}_r^{s,z} \\
&\qquad , \int_{\mathbb{R}_0} \rho(x) G(r, z\widetilde{\psi}(x) + \Psi_{s,z}^m(r,x) + \delta\psi_r^m(x))\nu(dx)\Big) - f_m(\omega^{s,z}, r, \check{\Xi}_{s,z}^m(r))\Big\}.
\end{aligned}$$

Then, $(\Delta^{s,z} Y^m, \Delta^{s,z} Z^m, \Delta^{s,z} \psi^m)$ is the solution to the BSDE

$$\begin{aligned}
\Delta^{s,z} Y_t^m &= \int_t^T \Big(\widetilde{f}_{s,z}^m(r, \Delta^{s,z} Y_r^m, \Delta^{s,z}\psi_r^m) + b_{s,z}^m(r) \cdot \Delta^{s,z} Z_r^m\Big) dr \\
&\quad - \int_t^T \Delta^{s,z} Z_r^m dW_r - \int_t^T \int_E \Delta^{s,z}\psi_r^m(x) \widetilde{\mu}(dr, dx).
\end{aligned}$$

By denoting an $\mathbb{F}$-progressively measurable process $H^m_{s,z}$ as

$$H^m_{s,z}(r) := K_M\Big(1 + |\mathcal{Z}^m_{s,z}(r)| + |\mathcal{Z}^{s,z}_r| + |\delta Z^m_r| + 2\|\rho\|_{\mathbb{L}^2(\nu)} G'_M \|\Psi^m_{s,z}(r,\cdot)\|_{\mathbb{L}^2(\nu)}\Big),$$

one obtains $|b^m_{s,z}(r)| \le H^m_{s,z}(r)$ for $\forall r \in [0,T]$. Here, $H^m_{s,z} \in \mathbb{H}^2_{BMO}$ and for $m(dz)ds$-a.e. $(s,z) \in [0,T]\times\mathbb{R}_0$, its norm $\|H^m_{s,z}\|_{\mathbb{H}^2_{BMO}}$ is bounded by some m-independent constant thanks to the universal bounds. Furthermore, the new driver satisfies the linear growth property $|\widetilde{f}^m_{s,z}(r,\widetilde{y},\widetilde{\psi})| \le |\widetilde{f}^m_{s,z}(r,0,0)| + K_M\big(|\widetilde{y}| + \|\rho\|_{\mathbb{L}^2(\nu)} G'_M \|\widetilde{\psi}\|_{\mathbb{L}^2(\nu)}\big)$ and

$$\begin{aligned}
|\widetilde{f}^m(s,z)(r,0,0)| \le{}& |(D_{s,z}f)(r,\Theta_r) - (D_{s,z}f_m)(r,\Theta^m_r)| \\
&+ \frac{1}{|z|}|f(\omega^{s,z},r,\Theta_r) - f_m(\omega^{s,z},r,\Theta^m_r)| \\
&+ \frac{1}{|z|}|f(\omega^{s,z},r,\check{\Xi}^m_{s,z}(r)) - f_m(\omega^{s,z},r,\check{\Xi}^m_{s,z}(r))| \\
&+ CK_M\frac{1}{|z|}\Big(|\delta Y^m_r| + \|\delta\psi^m_r\|_{\mathbb{L}^2(\nu)} + \mathcal{H}^m_{s,z}(r)|\delta Z^m_r|\Big)
\end{aligned}$$

where C is a positive constant depending only on $\|\rho\|_{\mathbb{L}^2(\nu)}$, G'_M and

$$\mathcal{H}^m_{s,z}(r) := 1 + 2|\mathcal{Z}^{s,z}_r| + |\delta Z^m_r| + 2\|\Psi^m_{s,z}(r,\cdot)\|_{\mathbb{L}^2(\nu)} + \|\delta\psi^m_r\|_{\mathbb{L}^2(\nu)}\,.$$

$\mathcal{H}^m_{s,z} \in \mathbb{H}^2_{BMO}$ and its norm is bounded by some m-independent constant $m(dz)ds$-a.e. $(s,z) \in [0,T]\times\mathbb{R}_0$. By applying Lemma A.1, one obtains

$$\begin{aligned}
&\big\|(\Delta^{s,z}Y^m, \Delta^{s,z}Z^m, \Delta^{s,z}\psi^m)\big\|^p_{\mathcal{K}^p[0,T]} \\
&\le C\mathbb{E}\Big[\Big(\int_0^T |(D_{s,z}f)(r,\Theta_r) - (D_{s,z}f_m)(r,\Theta^m_r)|dr\Big)^{p\bar{q}^2}\Big]^{\frac{1}{\bar{q}^2}} \\
&+ \frac{C}{|z|^p}\mathbb{E}\Big[\Big(\int_0^T |f(\omega^{s,z},r,\Theta_r) - f_m(\omega^{s,z},r,\Theta^m_r)|dr\Big)^{p\bar{q}^2}\Big]^{\frac{1}{\bar{q}^2}} \\
&+ \frac{C}{|z|^p}\mathbb{E}\Big[\Big(\int_0^T |f(\omega^{s,z},r,\check{\Xi}^m_{s,z}(r)) - f_m(\omega^{s,z},r,\check{\Xi}^m_{s,z}(r))|dr\Big)^{p\bar{q}^2}\Big]^{\frac{1}{\bar{q}^2}} \\
&+ \frac{C}{|z|^p}\mathbb{E}\Big[\Big(\int_0^T \big[|\delta Y^m_r| + \|\delta\psi^m_r\|_{\mathbb{L}^2(\nu)} + \mathcal{H}^m_{s,z}(r)|\delta Z^m_r|\big]dr\Big)^{p\bar{q}^2}\Big]^{\frac{1}{\bar{q}^2}},
\end{aligned} \tag{D.3}$$

where the positive constants C and $\bar{q} > 1$ are m-independent as before.

Due to (5.9) and (5.10), the convergence in $\lim_{\epsilon\downarrow 0}$ is uniform in m and hence the order of limit operations can be exchanged. By the dominated convergence from (5.9),

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{\epsilon\downarrow 0}\int_0^T\int_{|z|>\epsilon}\big\|(\Delta^{s,z}Y^m, \Delta^{s,z}Z^m, \Delta^{s,z}\psi^m)\big\|^p_{\mathcal{K}^p[0,T]} m(dz)ds \\
&= \lim_{\epsilon\downarrow 0}\int_0^T\int_{|z|>\epsilon}\lim_{m\to\infty}\big\|(\Delta^{s,z}Y^m, \Delta^{s,z}Z^m, \Delta^{s,z}\psi^m)\big\|^p_{\mathcal{K}^p[0,T]} m(dz)ds.
\end{aligned}$$

Therefore, in order to prove the convergence (5.12) it suffices to show, for $m(dz)ds$-a.e. $(s,z) \in [0,T]\times\mathbb{R}_0$, $\lim_{m\to\infty}\big\|(\Delta^{s,z}Y^m, \Delta^{s,z}Z^m, \Delta^{s,z}\psi^m)\big\|^p_{\mathcal{K}^p[0,T]} = 0$. This can be easily confirmed from (D.3) by using the local Lipschitz continuity and the fact that Θ^m and $\varphi_m(\Theta^m) \to \Theta$ and $\check{\Xi}^m_{s,z}$ and $\varphi_m(\check{\Xi}^m_{s,z}) \to \Xi^{s,z}$ converge in $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$. This finishes the proof for (5.12).

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