

Smoothed periodogram asymptotics and estimation for processes and fields with possible long-range dependence

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In this paper we establish central limit theorems for the smoothed unbiased periodogram $\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \mathbf{g}(\boldsymbol{\omega}, \boldsymbol{\theta}) \{I_{T,X}^*(\boldsymbol{\omega}) - E I_{T,X}^*(\boldsymbol{\omega})\} d\omega_1 \cdots d\omega_r$, where $\{X_t\}$ is a stationary r -dimensional random process or random field, possibly with long-range dependence, which is not necessarily Gaussian. Here $I_{T,X}^*(\boldsymbol{\omega})$ is the unbiased periodogram and $\mathbf{g}(\boldsymbol{\omega}, \boldsymbol{\theta})$ is a smoothing function satisfying modest regularity conditions. This result implies asymptotic normality of the asymptotic quasi-likelihood estimator of a distributional characteristic $\boldsymbol{\theta}$ of the process $\{X_t\}$ under very general conditions. In particular, these results show the asymptotic optimality of the Whittle estimation procedure for both short and long-range dependence in the absence of the Gaussian assumption, and extend those of Giraitis and Surgailis (1990) for the case $r = 1$.

central limit theorem * estimating function * asymptotic quasi-likelihood * smoothed periodogram * random process * random field * long-range dependence

1. Introduction

In this paper, a class of estimating functions consisting of smoothed periodograms of random processes or random fields, possibly exhibiting long-range dependence, is shown to be asymptotically multivariate normal. Asymptotic quasi-likelihood methods are then used to establish an optimality result for the Whittle estimation procedure for parameters of such stationary random functions, without Gaussian assumptions. Several authors, but notably Fox and Taquq (1986, 1987), and Dahlhaus (1989) earlier investigated the univariate case for Gaussian processes, and Giraitis and Surgailis (1990) the univariate case without Gaussian assumptions.

Parameter estimation for random fields and random processes based on smoothed periodograms has a long history. The idea derives from suggestive forms of *Whittle's*

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estimation procedure, which has formed the backbone of asymptotic estimation since its discovery (Whittle, 1951). The general methodology, developed by Whittle (1951, 1952, 1953, 1954), with many subsequent contributions, such as from Hannan (1970, 1973), Dunsmuir and Hannan (1976), Kabaila (1980, 1983), Kulpberger (1985), Fox and Taquq (1986), Giraitis and Surgailis (1990) for random processes and by Guyon (1982) and Rosenblatt (1985), for random fields, has evolved steadily towards a unified theory which this discussion substantially augments.

Let $\{X_t\}$ be a real, stationary, ergodic, purely non-deterministic random process or random field with finite dimensional distribution indexed by $\theta \in \Theta \subset \mathbb{R}^p$, Θ being open. Optimality results of Heyde and Gay (1989), for a broad class of estimating functions are applied to a class \mathcal{G} of smoothed $p \times 1$ vector periodograms

$$G_{T,X}(\theta) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} g(\omega, \theta) \{I_{T,X}^*(\omega) - EI_{T,X}^*(\omega)\} d\omega_1 \cdots d\omega_r, \quad (1)$$

for a suitably restricted class \mathcal{G} of weighting functions $g(\omega, \theta)$. Here $I_{T,X}^*(\omega)$ is the unbiased periodogram. Theorem 1 deals with the case of an r -dimensional random field and supplies a central limit theorem for $T^{r/2}G_{T,X}(\theta) \in \mathcal{G}$ and also for an asymptotic quasi-score estimating function $T^{r/2}G_{T,X}^*(\theta) \in \mathcal{G}$ obtained from (1) by choosing

$$g^*(\omega, \theta) = \frac{\partial}{\partial \theta} f_X(\omega, \theta) / f_X^2(\omega, \theta),$$

$f_X(\omega, \theta)$ being the spectral density of $\{X_t\}$, for the $g(\omega, \theta)$, which produces minimum size asymptotic confidence bounds for θ within \mathcal{G} . The estimator θ^* obtained from the estimating equation

$$G_{T,X}^*(\theta) = 0$$

is easily seen to be asymptotically equivalent to the Whittle estimator $\bar{\theta}$ obtained by choosing θ to minimize

$$\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \{\log f_X(\omega, \theta) + I_{T,X}^*(\omega)(f_X(\omega, \theta))^{-1}\} d\omega_1 \cdots d\omega_r,$$

i.e., to solve

$$\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\partial}{\partial \theta} f_X(\omega, \theta)(f_X(\omega, \theta))^{-2} \{I_{T,X}^*(\omega) - f_X(\omega, \theta)\} d\omega_1 \cdots d\omega_r = 0.$$

Asymptotic results for θ^* (and $\bar{\theta}$) naturally follow via the usual Taylor expansion methods, and these have already been discussed in some detail in the literature, for example by Rosenblatt (1985) Fox and Taquq (1986) and Dahlhaus (1989). This will not be repeated here as no new principles are involved. The forms of the general asymptotic results for $\theta^* - \theta$ are precisely those of the well-known ones for weakly dependent processes and fields. It should be remarked that the asymptotic results for $\theta^* - \theta$ are the same as those for the maximum likelihood estimator in the Gaussian case (e.g., Kabaila, 1983; Dahlhaus, 1989).

Extension of the results of Theorem 1 to multivariate processes (the restricted parameter version investigated by Dunsmuir and Hannan, 1976; see also Dunsmuir, 1979; Deistler, Dunsmuir and Hannan, 1978; and Hosoya and Taniguchi, 1982) is discussed in Theorem 2. Here the setting is more general; the innovations satisfy a martingale difference condition rather than being assumed independent. For further background to these results and their application to situations of modelling and identification see Heyde and Gay (1991).

2. Main results

We shall provide central limit theorems for $\mathbf{G}_{T,X}(\boldsymbol{\theta})$ of (1) for random processes and random fields, possibly with long-range dependence, without Gaussian assumptions. Separate discussions are given for random fields (Section 2.1) and random processes (Section 2.2) although the methodology is similar and involves a unilateral model in each case. The martingale assumption (B1)(ii) for example, which means that the best linear predictor is the best predictor, is only relevant for the random process case where time provides a natural ordering. For the random field case, the innovations ε_t are taken to be i.i.d. In the case $r=1$ the results of Theorem 2 extend those of Theorem 1 (and the corresponding results of Giraitis and Surgailis, 1990, where i.i.d. innovations are assumed). The results of Theorems 1(i) and 2(i) appear nonparametric but this is not really the case since $EI_{T,X}^* \approx f_x(\boldsymbol{\omega}, \boldsymbol{\theta})$. These results are applied in parts (ii) to a parametric situation.

For the purposes of a unified discussion our results are based on the use of the unbiased periodogram rather than the usual version. Following Guyon (1982) this must be used for random fields ($r > 1$) if edge effects are to be made negligible.

The main idea in the proofs is to filter the original field or process, which may not have a square integrable spectral density, to produce a related (unobservable) one for which the spectral density is square integrable, and then to apply standard results for the asymptotic distribution of unbiased serial covariances. All proofs are relegated to the Appendix.

2.1. Multivariate random fields

We first note that, without essential loss of generality, it is possible to confine attention to random fields for which a unilateral innovations representation exists. Tjøstheim (1978, 1983) has given a detailed account of this theory, which is analogous to that of the Wold decomposition for stationary random processes.

Conditions (A).

(A1) X_t has one-sided representation

$$X_t = \sum_{j_1=0}^{\infty} \cdots \sum_{j_r=0}^{\infty} \alpha(\boldsymbol{\theta})_{j_1 j_2 \cdots j_r} e_{t_1-j_1, \dots, t_r-j_r}, \quad (2)$$

for $\theta \in \Theta$, an open set of \mathbb{R}^p . This representation is possible (Helson and Lowdenslager, 1958) if

$$\int_{[-\pi, \pi]^r} \log f(\omega, \theta) d\omega > -\infty.$$

The $\{e_t\}$ are i.i.d. r.v.'s with $Ee_t^4 < \infty$, and spectral density $f(\omega, \theta)$ in $L_1[-\pi, \pi]^r$ $\forall \theta \in \Theta$, and X_t is real, stationary and ergodic.

(A2) (i) $g(\omega, \theta)$ is symmetric about $\omega = 0$ for $\theta \in [-\pi, \pi]^r$ $\forall \theta \in \Theta$.

(ii) $g(\omega, \theta) \in L_1[-\pi, \pi]^r$ $\forall \theta \in \Theta$.

(iii) $f(\omega, \theta)g(\omega, \theta)$ is in L_1 and L_2 over $[-\pi, \pi]^r$ $\forall \theta \in \Theta$.

(iv) There exist d_1, \dots, d_r with $|d_i| \leq 1$, $1 \leq i \leq r$, such that $|\omega_1|^{d_1} \cdots |\omega_r|^{d_r} \times f(\omega, \theta)$ is bounded and $g(\omega, \theta)/|\omega_1|^{d_1} \cdots |\omega_r|^{d_r}$ is in $L_2[-\pi, \pi]^r$ $\forall \theta \in \Theta$.

(v) $\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(\omega, \theta)g(\omega, \theta) d\omega = 0$ $\forall \theta \in \Theta$, and/or the fourth cumulant of the $\{e_t\}$, κ_4 , is zero.

(A3) $(\partial f(\omega, \theta)/\partial \theta)g'(\omega, \theta)$ ($r \times r$) is in $L_1[-\pi, \pi]^r \times \Theta_0$, $\Theta_0 \in \Theta$.

Then we have the following result.

Theorem 1. Suppose that the unbiased periodogram $I_{T,X}^*(\omega)$ is taken over a cube P_T of lattice points and for convenience, $P_T = [1, T]^r$. Write

$$C_T(k) = \prod_{i=1}^r (T - |k_i|)^{-1} \sum_{j, j+k \in P_T} X_j X_{j+k}, \quad (3)$$

where all $|k_i| \leq T-1$, and

$$I_{T,X}^*(\omega) = (2\pi)^{-r} \sum C_T(j) e^{ij'\omega},$$

where $j = (j_1, j_2, \dots, j_r)$, and the sum is taken over all $|j_i| \leq T-1$.

Then

(i) Under Conditions (A1)-(A2),

$$T^{r/2} G_{T,X}(\theta) = T^{r/2} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} g(\omega, \theta) \{I_{T,X}^*(\omega) - EI_{T,X}^*(\omega)\} d\omega_1 d\omega_2 \cdots d\omega_r,$$

is asymptotically multivariate normal with dispersion matrix $W(\theta)$ as $T \rightarrow \infty$, where

$$W(\theta) = 2(2\pi)^r \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f^2(\omega, \theta) g(\omega, \theta) g'(\omega, \theta) d\omega_1 d\omega_2 \cdots d\omega_r, \quad \theta \in \Theta.$$

(ii) Under Conditions (A1)-(A3) the asymptotic quasi-score function $G_{T,X}^*(\theta)$ has

$$g^*(\omega, \theta) = -\frac{\partial f^{-1}}{\partial \theta}(\omega, \theta),$$

so that $T^{r/2} G_{T,X}^*(\theta)$ tends in distribution to $MVN(0, W^*(\theta))$, $\theta \in \Theta_0$, where

$$W^*(\theta) = 2(2\pi)^r \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left\{ \frac{\partial}{\partial \theta} \log f(\omega, \theta) \right\} \left\{ \frac{\partial}{\partial \theta} \log f(\omega, \theta) \right\}' d\omega_1 d\omega_2 \cdots d\omega_r.$$

Remark 1. For a univariate stationary process, if $\{X_t\}$ has long-range dependence, it has a spectral density (s.d.) $f_X(\omega, \theta)$ which is in L_1 but not in L_2 . One standard model has for instance

$$f_X(\omega, H) = L(\omega, H)|\omega|^{1-2H},$$

with $H \in (\frac{1}{2}, 1)$, and with $L(\omega, H)$ slowly varying at zero. Theorem 1 readily applies to the estimation of H .

Remark 2. Notice that when $r = 1$, and the more usual biased periodogram

$$I_{T,X}(\omega) = (2\pi T)^{-1} \left| \sum_{j=1}^T X_j e^{-i\omega j} \right|^2$$

is used, then

$$T^{1/2} \mathbf{G}_{T,X}(\theta) \quad \text{and} \quad T^{-1/2} \left\{ \sum_{i=1}^T \sum_{j=1}^T \mathbf{g}_{i-j} X_i X_j - E \sum_{i=1}^T \sum_{j=1}^T \mathbf{g}_{i-j} X_i X_j \right\}$$

have the same limit distribution under any additional conditions which ensure that

$$T^{-1/2} \sum_{-T}^T |t| \mathbf{g}_t \gamma_t \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

Remark 3. It should be noted that the condition

$$\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \mathbf{g}(\omega, \theta) f(\omega, \theta) d\omega = 0$$

of (A2)(v) precludes the innovation variance as one of the components of θ to be estimated.

2.2. Multivariate random processes

Suppose that $\mathbf{X}(t)$, $t = 1, 2, \dots, T$, is part of a realization of a real, stationary, discrete time, ergodic, $r \times 1$ vector process with components X_a , $a = 1, 2, \dots, r$ which is purely non-deterministic, having representation

$$\mathbf{X}(t) = \sum_{j=0}^{\infty} \alpha_j(\theta) \mathbf{e}(t-j), \quad \theta \in \mathbb{R}^p,$$

where $\mathbf{e}(t)$ is a vector satisfying $E\mathbf{e}(t) = \mathbf{0}$, $E\mathbf{e}(s)\mathbf{e}'(t) = \delta_{st} \mathbf{V}$, t and all $\theta \in \Theta \subset \mathbb{R}^p$. Here δ_{st} is Kronecker's delta. The $\alpha_j(\theta)$ are $r \times r$ matrices satisfying $\text{tr} \sum_{j=0}^{\infty} \alpha_j(\theta) \mathbf{V} \alpha_j'(\theta) < \infty$. We use zero means and

$$C_{ab}(t) = (T-t)^{-1} \sum_{j=1}^{T-t} X_a(j) X_b(j+t), \quad t \geq 0,$$

$$\gamma(t)_{ab} = EX_a(t) X_b(0),$$

for simplicity, but introduction of mean corrections does not effect our results. Define

$$A(\omega, \theta) = \sum_{j=0}^{\infty} \alpha_j(\theta) e^{-i\omega j}.$$

Then

$$f(\omega, \boldsymbol{\theta}) = (2\pi)^{-1} A(\omega, \boldsymbol{\theta}) V A'(\omega, \boldsymbol{\theta}).$$

We investigate the restricted case as discussed by Dunsmuir and Hannan (1976) and treated as part of a more general estimation framework by Dunsmuir (1979), and Hosoya and Taniguchi (1982). As with Conditions (A) we make assumptions which ensure that fourth-order cumulant terms in the asymptotic dispersion matrix of the estimating function vanish. The regularity conditions are formulated below as Conditions (B). In particular, long-range dependence indices of component series can be estimated efficiently, with estimates enjoying joint asymptotic normality and a convenient form of the dispersion matrix. For the optimal estimating function, we require that V does not depend on $\boldsymbol{\theta}$. Hosoya and Taniguchi refer to $\boldsymbol{\theta}$ as 'the innovation-free parameter'.

$$E\mathbf{e}(s)\mathbf{e}'(t) = \delta_{st} V(\boldsymbol{\mu}),$$

say, where $\boldsymbol{\theta}$ and $\boldsymbol{\mu}$ are functionally independent.

Formulation. A $p \times 1$ vector estimating function is to be selected from \mathcal{G} comprising $\{\mathbf{G}_T(\mathbf{X}, \boldsymbol{\theta})\}$, where $\mathbf{G}_T(\mathbf{X}, \boldsymbol{\theta})$ has j th component

$$G_{T,j}(\mathbf{X}, \boldsymbol{\theta}) = 2\pi \int_{-\pi}^{\pi} \text{tr}[g_j(\omega, \boldsymbol{\theta})\{I_{T,\mathbf{X}}^*(\omega) - EI_{T,\mathbf{X}}^*(\omega)\}] d\omega,$$

for suitably restricted $r \times r$ matrices $g_j(\omega, \boldsymbol{\theta})$, $j = 1, 2, \dots, p$, and $I_{T,\mathbf{X}}^*(\omega)$ is the unbiased periodogram matrix.

Conditions analogous to Conditions (A) are provided by the following set.

Conditions (B).

(B1) (i) $\mathbf{X}(t) = \sum_{j=0}^{\infty} \alpha_j(\boldsymbol{\theta})\mathbf{e}(t-j)$ is real, stationary, ergodic and purely non-deterministic;

(ii) $E(\mathbf{e}(n)|F_{n-1}) = \mathbf{0}$ a.s.;

(iii) $E(e_a(n)e_b(n)|F_{n-1}) = v_{ab}$ a.s.;

(iv) $E(e_a(n)e_b(n)e_c(n)|F_{n-1}) = \beta_{abc}$ a.s.;

(v) $|E(e_a(n)e_b(n)e_c(n)e_d(n)|F_{n-1})| < \infty$ a.s.;

for all n , and $1 \leq a, b, c, d \leq r$ where F_{n-1} is the σ -field induced by realisations up to time $n-1$, and v_{ab} and β_{abc} are constant. We will denote the fourth-order cumulant between e_a , e_b , e_c and e_d by κ_{abcd} ;

(vi) $A(\omega, \boldsymbol{\theta})$ has elements $A_{ab}(\omega, \boldsymbol{\theta})$ which are twice differentiable for $\boldsymbol{\theta} \in \boldsymbol{\Theta}_0 \subset \boldsymbol{\Theta}$ and $\omega \in [-\pi, \pi]$ a.e.; these second derivatives are continuous for $\omega \in [-\pi, \pi]$ a.e.;

(vii) $f(\omega, \boldsymbol{\theta}) \in L_1[-\pi, \pi]$, $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subset \mathbb{R}^p$.

(B2) (i) $g_j(\omega, \boldsymbol{\theta}) \in L_1[-\pi, \pi]$;

(ii) $\text{tr} f(\omega, \boldsymbol{\theta})g_j(\omega, \boldsymbol{\theta})$ and $\text{tr} f(\omega, \boldsymbol{\theta})g_j(\omega, \boldsymbol{\theta})f(\omega, \boldsymbol{\theta})g_k(\omega, \boldsymbol{\theta}) \in L_1[-\pi, \pi]$ $\forall \boldsymbol{\theta} \in \boldsymbol{\Theta}$, and $j, k = 1, 2, \dots, p$;

(iii) $g_{j,ab}(\omega, \boldsymbol{\theta}) = g_{j,ab}(-\omega, \boldsymbol{\theta})$, $j = 1, 2, \dots, p$, $a, b = 1, 2, \dots, r$;

- (iv) There are constants d_i with $|d_i| \leq 1$, $1 \leq i \leq r$, such that writing $P(\omega) = \text{diag}(|\omega|^{d_1}, \dots, |\omega|^{d_r})$, we have
- (a) $P(\omega)f(\omega, \theta)P(\omega)$ is bounded, and
- (b) $(P(\omega))^{-1}g_j(\omega, \theta)P^{-1}(\omega)$ is in $L_2[-\pi, \pi]$, for $j = 1, 2, \dots, p$;
- (v)
$$\sum_{a,b,c,d}^r \kappa_{abcd} \left[\int_{-\pi}^{\pi} A'(\omega, \theta) g_i(\omega, \theta) A(\omega, \theta) d\omega \right]_{ab} \times \left[\int_{-\pi}^{\pi} A'(\omega) g_j(\omega, \theta) A(\omega) d\omega \right]_{cd} = 0 \quad \forall i, j = 1, 2, \dots, p.$$
- (B3) $\text{tr}(\partial/\partial\theta_i)f(\omega, \theta)g_j(\omega, \theta)$ and $\text{tr}(\partial/\partial\theta_i)f(\omega, \theta)g_j(\omega, \theta)(\partial/\partial\theta_i)f(\omega, \theta)g_l(\omega, \theta)$ $\in L_1[-\pi, \pi] \quad \forall \theta \in \Theta_0$ and $i, j, k, l = 1, 2, \dots, p$.
- (B4) $\partial V/\partial\theta_j = 0 \quad \forall j = 1, 2, \dots, p$.

Then we have the following result.

Theorem 2. (i) Under Conditions (B1)–(B2), the estimating function $T^{1/2}G_T(\mathbf{X}, \theta)$ with

$$G_{T,j}(\mathbf{X}, \theta) = 2\pi \int_{-\pi}^{\pi} \text{tr}[g_j(\omega, \theta)\{I_{T,X}^*(\omega) - EI_{T,X}^*(\omega)\}] d\omega$$

is asymptotically MVN with asymptotic dispersion matrix $W(\theta)$ where

$$W_{ij}(\theta) = 4\pi \text{tr} \int_{-\pi}^{\pi} f(\omega, \theta) g_i(\omega, \theta) f(\omega, \theta) g_j(\omega, \theta) d\omega.$$

(ii) Under Conditions (B1)–(B4), the quasi-score estimating function $G_T^*(\mathbf{X}, \theta)$ has j th component with

$$g_j^*(\omega, \theta) = -f^{-1}(\omega, \theta) \frac{\partial}{\partial\theta_j} f(\omega, \theta) f^{-1}(\omega, \theta)$$

and

$$T^{1/2}G_T^*(\mathbf{X}, \theta) \xrightarrow{d} \text{MVN}(\mathbf{0}, W^*(\theta))$$

as $T \rightarrow \infty$, $\theta \in \Theta_0$, where

$$W_{ij}^*(\theta) = 4\pi \int_{-\pi}^{\pi} \text{tr} \left\{ f^{-1}(\omega, \theta) \frac{\partial}{\partial\theta_i} f(\omega, \theta) f^{-1}(\omega, \theta) \frac{\partial}{\partial\theta_j} f(\omega, \theta) \right\} d\omega.$$

Appendix

Proof of Theorem 1. We will prove the theorem for the univariate case $r = 1$, and then outline modifications necessary for $r > 1$.

Consider

$$G_{T,X}(\theta) = \int_{-\pi}^{\pi} g(\omega, \theta) \{I_{T,X}^*(\omega) - EI_{T,X}^*(\omega)\} d\omega = \sum_{-T+1}^{T-1} g_t(\theta) \{\hat{\gamma}_t - \gamma_t\}$$

where $g_t(\boldsymbol{\theta})$ is the Fourier coefficient of $g(\omega, \boldsymbol{\theta})$ and

$$\hat{\gamma}_t = (T-t)^{-1} \sum_{j=1}^{T-t} X_j X_{j+t}, \quad \gamma_t = EX_j X_{j+t}.$$

To establish part (i) of Theorem 1 it is necessary to evaluate the asymptotic dispersion matrix of $T^{1/2} \mathbf{G}_{T,X}(\boldsymbol{\theta})$ by any convenient method which does not assume that $f_X(\omega, \boldsymbol{\theta}) \in L_2$, and with asymptotic quasi-likelihood and mean square convergence developments in mind, we prove the following lemma.

Lemma 1. Suppose $g(\omega, \boldsymbol{\theta})$ and $h(\omega, \boldsymbol{\theta})$ satisfy (A2) in relation to $f_X(\omega, \boldsymbol{\theta})$ and $f_Y(\omega, \boldsymbol{\theta})$ respectively, and $g(\omega, \boldsymbol{\theta}) = |L(e^{i\omega})|^2 h(\omega, \boldsymbol{\theta})$ where $|L(e^{i\omega})|^2 / |\omega|^d \rightarrow K$ (a constant) as $\omega \rightarrow 0$. Let $Y_t = \sum_{j=0}^{\infty} \beta_{t-j}(\boldsymbol{\theta}) e_j$ with spectral density $f_Y(\omega, \boldsymbol{\theta})$, be another linear process which depends upon $\boldsymbol{\theta}$ with innovation sequence $\{e_t\}$ satisfying Condition (A1)(i), such that $Y_t = L(B)X_t$ where $L(B) = \sum_{j=0}^{\infty} l_j B^j$, B denoting the backward shift operator. Let

$$\begin{aligned} \mathbf{G}_{T,X}(\boldsymbol{\theta}) &= \int_{-\pi}^{\pi} g(\omega, \boldsymbol{\theta}) \{I_{T,X}^*(\omega) - EI_{T,X}^*(\omega)\} d\omega, \\ \mathbf{H}_{T,Y}(\boldsymbol{\theta}) &= \int_{-\pi}^{\pi} h(\omega, \boldsymbol{\theta}) \{I_{T,Y}^*(\omega) - EI_{T,Y}^*(\omega)\} d\omega, \\ \mathbf{g}_t &= (2\pi)^{-1} \int_{-\pi}^{\pi} e^{i\omega t} g(\omega, \boldsymbol{\theta}) d\omega, \\ \mathbf{h}_t &= (2\pi)^{-1} \int_{-\pi}^{\pi} e^{i\omega t} h(\omega, \boldsymbol{\theta}) d\omega, \end{aligned}$$

then

$$\lim_{T \rightarrow \infty} TEG_{T,X} \mathbf{H}'_{T,Y} = 4\pi \int_{-\pi}^{\pi} g(\omega, \boldsymbol{\theta}) h'(\omega, \boldsymbol{\theta}) f_X(\omega, \boldsymbol{\theta}) f_Y(\omega, \boldsymbol{\theta}) d\omega.$$

Proof. Denote by $\{\gamma_t\}$ and $\{\delta_t\}$ the covariances for the processes $\{X_t\}$ and $\{Y_t\}$ respectively, $\{C_{X,s}\}$ and $\{C_{Y,t}\}$ are their unbiased sample estimators. Let $\{\chi_t\}$ be the cross-correlation function of the processes, and $f_{XY}(\omega, \boldsymbol{\theta})$ the cross spectral density. A straightforward calculation gives $f_{XY}(\omega, \boldsymbol{\theta}) = L(e^{-i\omega}) f_X(\omega, \boldsymbol{\theta})$ which belongs to $L_2[-\pi, \pi]$ and we already have the relationship $f_Y(\omega, \boldsymbol{\theta}) = |L(e^{i\omega})|^2 f_X(\omega, \boldsymbol{\theta})$.

Suppressing $\boldsymbol{\theta}$ in the notation, we have

$$\begin{aligned} & \lim_{T \rightarrow \infty} TEG_{T,X} \mathbf{H}'_{T,Y} \\ &= \lim_{T \rightarrow \infty} T \sum_{s=-T+1}^{T-1} \sum_{t=-T+1}^{T-1} \mathbf{g}_s \mathbf{h}'_t \text{Cov}(C_{X,s}, C_{Y,t}) \\ &= \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} \mathbf{g}_s \mathbf{h}'_t \left[\sum_{j=-\infty}^{\infty} \{\chi_{j+s} \chi_{j-t} + \chi_{j-s} \chi_{j-t}\} + \text{const} \cdot \gamma_s \delta_t \right], \end{aligned} \quad (4)$$

using expression (5) of Hannan (1976), suitably adjusted to deal with unbiased sample covariances, for $\text{Cov}(C_{X,s}, C_{Y,t})$. It should be noted that for processes with long-range dependence the overall limit exists because of the presence of the attenuating coefficients $\{\mathbf{g}_s\}$ and $\{\mathbf{h}_t\}$. The result needed then comes from observing

that (4) can be written as

$$\sum_{j=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \mathbf{g}_s \chi_{j+s} \sum_{t=-\infty}^{\infty} \mathbf{h}'_t \chi_{j-t} + \sum_{j=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \mathbf{g}_s \chi_{j-s} \sum_{t=-\infty}^{\infty} \mathbf{h}'_t \chi_{j-t} \\ + \text{const} \cdot \sum_{s=-\infty}^{\infty} \mathbf{g}_s \gamma_s \sum_{t=-\infty}^{\infty} \mathbf{h}'_t \delta_t.$$

The first two sums are of the form $\sum_{j=-\infty}^{\infty} \mathbf{m}_j \mathbf{n}'_{-j}$ where \mathbf{m}_j and \mathbf{n}_j are Fourier coefficients of $\mathbf{g}(\omega, \boldsymbol{\theta}) f_{XY}(\omega, \boldsymbol{\theta})$ and $\mathbf{h}(\omega, \boldsymbol{\theta}) f_{XY}(\omega, \boldsymbol{\theta})$ respectively. Each of these terms has sum

$$2\pi \int_{-\pi}^{\pi} \mathbf{g}(\omega, \boldsymbol{\theta}) \mathbf{h}'(\omega, \boldsymbol{\theta}) f_X(\omega, \boldsymbol{\theta}) f_Y(\omega, \boldsymbol{\theta}) d\omega.$$

The third term of the sum is zero because it is

$$\text{const} \cdot \int_{-\pi}^{\pi} \mathbf{g}(\omega, \boldsymbol{\theta}) f_X(\omega, \boldsymbol{\theta}) d\omega \int_{-\pi}^{\pi} \mathbf{h}'(\omega, \boldsymbol{\theta}) f_{XY}(\omega, \boldsymbol{\theta}) d\omega,$$

which is zero by (A2)(v), so that we have established the lemma. \square

Proof of Theorem 1 (continued). We let d be as in Condition (A2)(iv), and introduce the (unobservable) process $Y_t = \nabla^{d/2} X_t$ where $\nabla = 1 - B$, B denoting the backward shift operator $BX_t = X_{t-1}$ and $\nabla^{d/2}$ being defined by

$$\nabla^{d/2} = (1 - B)^{d/2} = \sum_{j=0}^{\infty} \binom{d/2}{j} (-B)^j.$$

Note that $\{Y_t\}$ has spectral density

$$f_Y(\omega, \boldsymbol{\theta}) = (2 \sin \frac{1}{2} \omega)^d f_X(\omega, \boldsymbol{\theta}).$$

Now let

$$T^{1/2} \mathbf{H}_{T,Y}(\boldsymbol{\theta}) = T^{1/2} \int_{-\pi}^{\pi} \mathbf{h}(\omega, \boldsymbol{\theta}) \{I_{T,Y}^*(\omega) - EI_{T,Y}^*(\omega)\} d\omega$$

be the estimating function corresponding to $\{Y_t\}$ where $\mathbf{h}(\omega, \boldsymbol{\theta}) = \mathbf{g}(\omega, \boldsymbol{\theta}) / (2 \sin \frac{1}{2} \omega)^d$. We shall show that $T^{1/2} \mathbf{G}_{T,X}(\boldsymbol{\theta})$ is asymptotically mean-square equivalent to $T^{1/2} \mathbf{H}_{T,Y}(\boldsymbol{\theta})$. Indeed,

$$\lim_{T \rightarrow \infty} TE |\mathbf{G}_{T,X}(\boldsymbol{\theta}) - \mathbf{H}_{T,Y}(\boldsymbol{\theta})|^2 \\ = \lim_{T \rightarrow \infty} T \{E \mathbf{G}_{T,X}(\boldsymbol{\theta}) \mathbf{G}'_{T,X}(\boldsymbol{\theta}) - E \mathbf{G}_{T,X}(\boldsymbol{\theta}) \mathbf{H}'_{T,Y}(\boldsymbol{\theta}) \\ - E \mathbf{H}_{T,Y}(\boldsymbol{\theta}) \mathbf{G}'_{T,X}(\boldsymbol{\theta}) + E \mathbf{H}_{T,Y}(\boldsymbol{\theta}) \mathbf{H}'_{T,Y}(\boldsymbol{\theta})\} \\ = 4\pi \left\{ \int_{-\pi}^{\pi} f_X^2(\omega, \boldsymbol{\theta}) \mathbf{g}(\omega, \boldsymbol{\theta}) \mathbf{g}'(\omega, \boldsymbol{\theta}) d\omega \right. \\ \left. - 2 \int_{-\pi}^{\pi} f_X(\omega, \boldsymbol{\theta}) f_Y(\omega, \boldsymbol{\theta}) \mathbf{g}(\omega, \boldsymbol{\theta}) \mathbf{h}'(\omega, \boldsymbol{\theta}) d\omega \right. \\ \left. + \int_{-\pi}^{\pi} f_Y^2(\omega, \boldsymbol{\theta}) \mathbf{h}(\omega, \boldsymbol{\theta}) \mathbf{h}'(\omega, \boldsymbol{\theta}) d\omega \right\} = 0,$$

using four applications of Lemma 1, and the relationship

$$g(\omega, \theta)f_X(\omega, \theta) = h(\omega, \theta)f_Y(\omega, \theta).$$

Next, for fixed integer M , write

$$h_M^{(1)}(\omega, \theta) = \sum_{t=0}^{M-1} h_t \cos t\omega, \quad h_M^{(2)}(\omega, \theta) = \sum_{t=M}^{\infty} h_t \cos t\omega,$$

where $h_t = h_t(\theta)$ are the Fourier coefficients of $h(\omega, \theta)$. Then note that

$$\begin{aligned} T^{1/2}H_{T,Y}(\theta) &= T^{1/2} \int_{-\pi}^{\pi} h_M^{(1)}(\omega, \theta) \{I_{T,Y}^*(\omega) - EI_{T,Y}^*(\omega)\} d\omega \\ &\quad + T^{1/2} \int_{-\pi}^{\pi} h_M^{(2)}(\omega, \theta) \{I_{T,Y}^*(\omega) - EI_{T,Y}^*(\omega)\} d\omega \\ &= T^{1/2}H_{M,T,Y}^{(1)}(\theta) + T^{1/2}H_{M,T,Y}^{(2)}(\theta), \end{aligned}$$

say, and using Lemma 1,

$$\begin{aligned} \lim_{T \rightarrow \infty} TE(H_{M,T,Y}^{(2)}(\theta)(H_{M,T,Y}^{(2)}(\theta))') \\ = 4\pi \int_{-\pi}^{\pi} f_Y^2(\omega, \theta) h_M^{(2)}(\omega, \theta) (h_M^{(2)}(\omega, \theta))' d\omega \rightarrow 0, \end{aligned}$$

as $M \rightarrow \infty$ since $f_Y^2(\omega, \theta)$ is bounded by (A2)(iv) and $h_M^{(2)}(\omega, \theta)$, being the Fourier series of a function in $L_2[-\pi, \pi]$ converges in the L_2 norm to the function.

Furthermore,

$$\begin{aligned} \lim_{T \rightarrow \infty} TE(H_{M,T,Y}^{(1)}(\theta)(H_{M,T,Y}^{(1)}(\theta))') \\ = 4\pi \int_{-\pi}^{\pi} f_Y^2(\omega, \theta) h_M^{(1)}(\omega, \theta) (h_M^{(1)}(\omega, \theta))' d\omega = W_M(\theta) \\ \rightarrow 4\pi \int_{-\pi}^{\pi} f_Y^2(\omega, \theta) h(\omega, \theta) h'(\omega, \theta) d\omega = W(\theta), \end{aligned}$$

as $M \rightarrow \infty$ and also, using Schwarz' inequality,

$$\lim_{T \rightarrow \infty} TE(H_{M,T,Y}^{(1)}(\theta)(H_{M,T,Y}^{(2)}(\theta))') \rightarrow 0.$$

But, since the process $\{Y_t\}$ has a spectral density in $L_2[-\pi, \pi]$ by construction, the theorem of Hannan (1976), modified to deal with unbiased sample covariances, gives

$$T^{1/2}H_{M,T,Y}^{(1)}(\theta) \xrightarrow{d} \text{MVN}(\mathbf{0}, W_M(\theta))$$

and letting $M \rightarrow \infty$, the result of part (i) of the theorem follows via a simple application of Theorem 4.2, p. 25 of Billingsley (1968).

For part (ii), under Conditions (A1)–(A3),

$$\lim_{T \rightarrow \infty} T E \mathbf{G}_T \mathbf{G}_T^{*'} = 4\pi \int_{-\pi}^{\pi} \mathbf{g}(\omega) \mathbf{g}^{*'}(\omega) f_X^2(\omega) d\omega,$$

$$\lim_{T \rightarrow \infty} E \dot{\mathbf{G}}_T = -(2\pi)^{-1} \int_{-\pi}^{\pi} \frac{\partial}{\partial \boldsymbol{\theta}} f_X(\omega, \boldsymbol{\theta}) \mathbf{g}'(\omega, \boldsymbol{\theta}) d\omega,$$

since if $(\partial/\partial \theta_j) f_X(\omega, \boldsymbol{\theta}) \in L_1$,

$$\frac{\partial}{\partial \theta_j} E I_{T,X}^*(\omega) \rightarrow (2\pi)^{-1} \left(\frac{\partial}{\partial \boldsymbol{\theta}} f_X(\omega, \boldsymbol{\theta}) \right)_j,$$

$j = 1, 2, \dots, p$ in the L_1 norm as $T \rightarrow \infty$ (see Hannan, 1970, pp. 507–508). Then

$$\lim_{T \rightarrow \infty} T (E \dot{\mathbf{G}}_T)^{-1} E \mathbf{G}_T \mathbf{G}_T^{*'} = 8\pi^2 I_p,$$

I_p being the $p \times p$ unit matrix, if

$$\mathbf{g}^*(\omega, \boldsymbol{\theta}) = -\frac{\partial}{\partial \boldsymbol{\theta}} f_X(\omega, \boldsymbol{\theta}) / f_X^2(\omega, \boldsymbol{\theta}) = \frac{\partial}{\partial \boldsymbol{\theta}} f_X^{-1}(\omega, \boldsymbol{\theta}),$$

and the condition of the theorem of Heyde and Gay (1989) is satisfied. Furthermore,

$$\mathbf{W}^*(\boldsymbol{\theta}) = 4\pi \int_{-\pi}^{\pi} \left\{ \frac{\partial}{\partial \boldsymbol{\theta}} \log f_X(\omega, \boldsymbol{\theta}) \right\} \left\{ \frac{\partial}{\partial \boldsymbol{\theta}} \log f_X(\omega, \boldsymbol{\theta}) \right\}' d\omega,$$

for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_0 \subset \boldsymbol{\Theta} \subset \mathbb{R}^p$. This completes the proof for $r = 1$.

The proof of Theorem 1 for $r > 1$ follows the univariate case, but invokes an extended version of Hannan's (1976) theorem for multivariate normality of covariances adjusted to deal with unbiased covariances, whose proof parallels the original, with essentially only notational changes, since \mathbf{X}_t has a one-sided representation. For the filtered process we use

$$Y_{t_1, \dots, t_r} = \nabla_1^{d_1/2} \cdots \nabla_r^{d_r/2} X_{t_1, \dots, t_r} = \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} \left\{ \prod_{i=1}^r \binom{d_i/2}{k_i} \right\} X_{t_1-k_1, \dots, t_r-k_r},$$

with spectral density

$$f_Y(\omega_1, \dots, \omega_r, \boldsymbol{\theta}) = 2^{d_1 + \dots + d_r} \left\{ \prod_{i=1}^r |\sin \frac{1}{2} \omega_i|^{d_i} \right\} f_X(\omega_1, \dots, \omega_r, \boldsymbol{\theta}).$$

The expression for the asymptotic dispersion matrix $\mathbf{W}(\boldsymbol{\theta})$ for $r > 1$ is well-known (see for example Guyon, 1982). \square

Proof of Theorem 2 (The multiple time series case). Observe that for $j = 1, 2, \dots, p$,

$$\begin{aligned} T^{1/2} G_{T,j}(\mathbf{X}, \boldsymbol{\theta}) &= T^{1/2} \int_{-\pi}^{\pi} \text{tr}[g_j(\omega, \boldsymbol{\theta}) \{I_{T,X}^*(\omega) - E I_{T,X}^*(\omega)\}] d\omega \\ &= T^{1/2} \sum_{-T+1}^{T-1} \sum_{a=1}^r \sum_{b=1}^r \{ \hat{\gamma}_{ab}(s) - \gamma_{ab}(s) \} g_{j,ab}(s), \end{aligned}$$

where

$$g_{j,ab}(s) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-i\omega s} g_{j,ab}(\omega, \boldsymbol{\theta}) d\omega,$$

and asymptotic normality follows as in the case $r=1$. Asymptotic multivariate normality of the vector $T^{1/2} \mathbf{G}_T(\mathbf{X}, \boldsymbol{\theta})$ follows similarly using the Cramér-Wold device. To calculate the covariance matrix of the limit distribution we consider

$$\begin{aligned} & \lim_{T \rightarrow \infty} \text{Cov}\{T^{1/2} \mathbf{G}_{T,j}(\mathbf{X}, \boldsymbol{\theta}), T^{1/2} \mathbf{G}_{T,k}(\mathbf{X}, \boldsymbol{\theta})\} \\ &= \lim_{T \rightarrow \infty} \text{Cov} \left[\sum_{-T+1}^{T-1} \sum_{a=1}^r \sum_{b=1}^r T^{1/2} \{\hat{\gamma}_{ab}(s) - \gamma_{ab}(s)\} g_{j,ab}(s) \right. \\ & \quad \left. \times \sum_{-T+1}^{T-1} \sum_{c=1}^r \sum_{d=1}^r T^{1/2} \{\hat{\gamma}_{cd}(t) - \gamma_{cd}(t)\} g_{k,cd}(t) \right], \end{aligned}$$

which, using expression (5) of Hannan (1976) and expression (3.6) of Hannan (1970), with suitable adjustment for unbiased covariances, can be written as

$$\begin{aligned} & \sum_{a=1}^r \sum_{b=1}^r \sum_{c=1}^r \sum_{d=1}^r \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} g_{j,ab}(s) g_{k,cd}(t) \\ & \times \left[\sum_{l=-\infty}^{\infty} \{\gamma_{ac}(l+s) \gamma_{bd}(l-t) + \gamma_{ad}(l+s) \gamma_{bc}(l-t)\} \right. \\ & \quad \left. + \sum_n \sum_o \sum_p \sum_q \kappa_{nopq} \cdot \sum_{m=-\infty}^{\infty} \alpha_{an}(m) \alpha_{bo}(m+s) \sum_{l=-\infty}^{\infty} \alpha_{cp}(l) \alpha_{dq}(l+t) \right]. \end{aligned}$$

The two terms containing covariance products sum as in Theorem 1, or for example Dunsmuir (1979, p. 497) to

$$W_{jk} = 2(2\pi) \text{tr} \int_{-\pi}^{\pi} f(\omega, \boldsymbol{\theta}) g_j(\omega, \boldsymbol{\theta}) f(\omega, \boldsymbol{\theta}) g_k(\omega, \boldsymbol{\theta}) d\omega, \quad (5)$$

while the fourth-order cumulant term is

$$\begin{aligned} & \lim_{T \rightarrow \infty} 2 \sum_{a=1}^r \sum_{b=1}^r \sum_{c=1}^r \sum_{d=1}^r \kappa_{abcd} \left[\int_{-\pi}^{\pi} A'(\omega, \boldsymbol{\theta}) g_j(\omega, \boldsymbol{\theta}) A(\omega, \boldsymbol{\theta}) d\omega \right]_{ab} \\ & \quad \times \left[\int_{-\pi}^{\pi} A'(\omega, \boldsymbol{\theta}) g_j(\omega, \boldsymbol{\theta}) A(\omega, \boldsymbol{\theta}) d\omega \right]_{cd}, \end{aligned}$$

which is zero, by Condition (B2)(v).

In this context, Remark 3 of Dunsmuir (1979, pp. 498–499) is particularly relevant to selection of the asymptotic quasi-score function. If Condition (B2)(v) obtains, then the condition of the theorem of Heyde and Gay (1989) translates into the condition

$$2(2\pi) \text{tr} \int_{-\pi}^{\pi} f(\omega, \boldsymbol{\theta}) g_j(\omega, \boldsymbol{\theta}) f(\omega, \boldsymbol{\theta}) g_k^*(\omega, \boldsymbol{\theta}) d\omega \sim T \cdot \text{tr} \int_{-\pi}^{\pi} g_j(\omega, \boldsymbol{\theta}) \frac{\partial}{\partial \theta_k} f(\omega, \boldsymbol{\theta}) d\omega,$$

as $T \rightarrow \infty$, which holds under Conditions (B1)–(B3). This results in the identification

$$g_k^*(\omega, \boldsymbol{\theta}) = -f^{-1}(\omega, \boldsymbol{\theta}) \frac{\partial}{\partial \theta_k} f(\omega, \boldsymbol{\theta}) f^{-1}(\omega, \boldsymbol{\theta}),$$

and with Condition (B4), $T^{1/2} \mathbf{G}_T^*(\mathbf{X}, \boldsymbol{\theta})$ converges in distribution to $\text{MVN}(\mathbf{0}, \mathbf{W}^*(\boldsymbol{\theta}))$ as asserted by Theorem 2. \square

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