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Spectral estimation of continuous-time stationary processes from random sampling

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Abstract

Let $X = \{X(t),\ -\infty < t < \infty\}$ be a continuous-time stationary process with spectral density function $\phi_X(\lambda)$ and $\{\tau_k\}$ be a stationary point process independent of X. Estimates $\hat{\phi}_X(\lambda)$ of $\phi_X(\lambda)$ based on the discrete-time observation $\{X(\tau_k), \tau_k\}$ are considered. Asymptotic expressions for the bias and covariance of $\hat{\phi}_X(\lambda)$ are derived. A multivariate central limit theorem is established for the spectral estimators $\hat{\phi}_X(\lambda)$. Under mild conditions, it is shown that the bias is independent of the statistics of the sampling point process $\{\tau_k\}$ and that there exist sampling point processes such that the asymptotic variance is uniformly smaller than that of a Poisson sampling scheme for all spectral densities $\phi_X(\lambda)$ and all frequencies λ.

Key words: Spectral estimation of continuous-time processes; Point processes; Alias-free sampling; Asymptotic bias; Covariance; Normality

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1. Introduction

There is an extensive literature on the theory and applications of spectral analysis of time series. This paper is concerned with some theoretical properties of spectral density function estimation of continuous-time stationary processes when the observations are taken at discrete times. Let $X = \{X(t),\ -\infty < t < \infty\}$ be a real-valued stationary process with mean zero, continuous covariance function $R_X(t) \in L_1$ and spectral density function $\phi_X(\lambda)$. If the process is sampled at equally spaced times $\{\tau_k = k/\beta\}$, then it is well known that aliasing arises and consistent estimates of $R_X(t)$

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and $\phi_X(\lambda)$ from the samples $\{X(k/\beta)\}$ do not exist unless the process X is band limited, $\phi_X(\lambda) = 0$ for $|\lambda| > B$ and $B \le \beta\pi$. This motivated the consideration of irregularly spaced sampling point processes. Two notions of alias-free sampling are known (Shapiro and Silverman, 1960; Brillinger, 1972). The latter was further developed in Masry (1978b). The quadratic-mean consistency of various estimates of $\phi_X(\lambda)$, for the Poisson sampling process, is considered in Masry (1978a, 1980) while the consistency and asymptotic normality of estimates of $R_X(t)$ is established in Masry (1983) for general alias-free sampling schemes. For parametric (finite parameter) spectral estimation of continuous-time process from unequally spaced data, we refer to Robinson (1978) and Lii and Masry (1992a) and the references therein. There is a considerable interest in time-series analysis from irregularly observed data. See, for example, the collection of papers in Parzen (1983), and the simulation study in Moore et al. (1987). A recent application to the spectral analysis of ocean profiles from unequally spaced data can be found in Moore et al. (1988).

In this paper we consider a nonparametric spectral estimate of $\phi_X(\lambda)$ based on discrete-time observations $\{X(\tau_k), \tau_k\}$ where $\{\tau_k\}$ is a general alias-free stationary point process. We obtain asymptotic expressions for the bias and covariance of $\hat{\phi}_X(\lambda)$. We also establish a multivariate central limit theorem for $\hat{\phi}_X(\lambda)$. A key expression used in our derivations is the asymptotic cumulants of $d_{Z,T}(\lambda)$ of Eq. (2.11) given in Brillinger (1972, Theorem 4.1). Our general results are compared with those of the Poisson sampling. It is also shown that, under certain regularity conditions, there are alias-free sampling schemes such that the mean-squares error is uniformly smaller than that of the Poisson sampling for *all* spectral densities $\phi_X(\lambda)$ and all λ. These results may be of particular interest to oceanographic spectral analysis, Moore et al. (1988).

2. Preliminaries

We begin by setting the framework and notations.

Let $X = \{X(t), -\infty < t < \infty\}$ be a zero mean stationary process with finite second-order moments, continuous covariance function $R_X(t) \in L_1$ and spectral density function $\phi_X(\lambda)$. The point process $\{\tau_k\}_{k=-\infty}^{\infty}$ is stationary, orderly, independent of X, with finite second-order moments (Daley and Vere Jones, 1972). Let $N(\cdot)$ be the counting process associated with $\{\tau_k\}$ and $\beta = E[N((0, 1])]$ be the mean intensity of the point process, then

$$E[N((t, t + \mathrm{d}t])] = \beta\,\mathrm{d}t, \tag{2.1}$$

$$\mathrm{Cov}\{N((t, t + \mathrm{d}t]), N((t + u, t + u + \mathrm{d}u]) = C_N(\mathrm{d}u)\,\mathrm{d}t, \tag{2.2}$$

where C_N is the reduced covariance measure which is a σ-finite measure on the Borel sets $\mathscr{B}$ with an atom at the origin, $C_N(\{0\}) = \beta$. We assume that C_N is absolutely continuous, outside of the origin, with covariance density function $c_N(u)$ i.e.

$$C_N(B) = \beta\delta_0(B) + \int_B c_N(u)\,\mathrm{d}u, \quad B \in \mathscr{B} \tag{2.3}$$

and

$$\delta_0(B) = \begin{cases} 1 & \text{if } 0 \in B, \\ 0 & \text{otherwise.} \end{cases}$$

In the differential notation $\mathrm{d}N(t) = N((0, t + \mathrm{d}t]) - N((0, t])$, we can write

$$\mathrm{Cov}\{\mathrm{d}N(t_1), \mathrm{d}N(t_2)\} = c_N(t_2 - t_1)\,\mathrm{d}t_1\,\mathrm{d}t_2$$

for distinct t_j's.

We define the sampled process by

$$Z(B) = \sum_{\tau_i \in B} X(\tau_i), \quad B \in \mathscr{B} \tag{2.4}$$

or in differential from $\mathrm{d}Z(t) = X(t)\,\mathrm{d}N(t)$. The increment process Z has finite second-order moments and in particular $E[\mathrm{d}Z(t)] = 0$ and

$$\begin{aligned} \mu_Z(\mathrm{d}u)\,\mathrm{d}t &\triangleq E[\mathrm{d}Z(t)\,\mathrm{d}Z(t+u)] \\ &= R_X(u)\{\beta^2\,\mathrm{d}u + C_N(\mathrm{d}u)\}\,\mathrm{d}t \\ &= \mathrm{d}C_Z^{(2)}(u)\,\mathrm{d}t. \end{aligned} \tag{2.5}$$

If we define the σ-finite measure

$$\mu_N(B) = \int_B [\beta^2\,\mathrm{d}u + C_N(\mathrm{d}u)], \quad B \in \mathscr{B},$$

then

$$\begin{aligned} \mu_Z(B) &= \int_B R_X(u)\,\mu_N(\mathrm{d}u) \\ &= \beta R_X(0)\,\delta_0(B) + \int_B R_X(u)[\beta^2 + c_N(u)]\,\mathrm{d}u \end{aligned}$$

is a σ-finite measure on $\mathscr{B}$. We define the spectral density $\phi_Z(\lambda)$ of the increment process Z by

$$\begin{aligned} \phi_Z(\lambda) &\triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}u\lambda}\,\mu_Z(\mathrm{d}u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}u\lambda}\,\mathrm{d}C_Z^{(2)}(u) \\ &= \beta^2\phi_X(\lambda) + \frac{\beta R_X(0)}{2\pi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R_X(u)\,c_N(u)\,\mathrm{e}^{-\mathrm{i}u\lambda}\,\mathrm{d}u \\ &= \beta^2\phi_X(\lambda) + \frac{\beta R_X(0)}{2\pi} + \int_{-\infty}^{\infty} \phi_X(\lambda - u)\,\psi(u)\,\mathrm{d}u, \end{aligned} \tag{2.6}$$

where

$$\psi(\lambda) \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}u\lambda}\,c_N(u)\,\mathrm{d}u; \quad c_N(u) \in L_1 \tag{2.7}$$

is assumed. We note that $\phi_Z(\lambda)$ is bounded, uniformly continuous but not integrable in general. Define

$$\gamma(u) = \frac{c_N(u)}{\beta^2 + c_N(u)}. \tag{2.8}$$

Under the assumption that

$$\gamma(u) \in L_1 \quad \text{and} \quad \Gamma(\lambda) \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda u}\gamma(u)\,du \in L_1, \tag{2.9}$$

relationship (2.6) can be inverted and we have

$$\phi_X(\lambda) = \frac{1}{\beta^2}\left\{\phi_Z(\lambda) - \frac{\beta R_X(0)}{2\pi} - \int_{-\infty}^{\infty}\Gamma(\lambda - u)\left[\phi_Z(u) - \frac{\beta R_X(0)}{2\pi}\right]du\right\}. \tag{2.10}$$

We note that a sufficient condition for $\{\tau_k\}$ to be "alias-free" is $\beta^2 + c_N(u) > 0$ a.e.; however, this is not sufficient to invert relationship (2.6). A sufficient condition for (2.10) to hold is (2.9) (Masry, 1978b). Eq. (2.10) is formally given in Brillinger (1972). When (2.9) is satisfied, we call the point process $\{\tau_k\}$ admittable. Eq. (2.10) is the principal relationship which allows one to estimate $\phi_X(\lambda)$ from the discrete data $\{X(\tau_k), \tau_k\}$. Given the observations $\{X(\tau_k), \tau_k\}_{k=1}^{N(T)}$ with $T > 0$ and $N(T)$ the number of points in $[0, T]$, we estimate $\phi_X(\lambda)$ as follows. First let $I_T(\lambda)$ be the periodogram

$$\begin{aligned} I_T(\lambda) &= \frac{1}{2\pi T}\left|\int_0^T e^{-it\lambda}X(t)\,dN(t)\right|^2 \\ &\triangleq \frac{1}{2\pi T}|d_{Z,T}(\lambda)|^2 \end{aligned} \tag{2.11}$$

and we estimate $\phi_Z(\lambda)$ by

$$\hat{\phi}_Z(\lambda) = \int_{-\infty}^{\infty} W_T(\lambda - u) I_T(u)\,du, \tag{2.12}$$

where $W_T(\lambda) = (1/b_T)W(\lambda/b_T)$ is a spectral window where $b_T \to 0$ as $T \to \infty$ is the bandwidth parameter and $W(\lambda)$ is a real, even, weight function satisfying

$$W \in L_1 \cap L_\infty \quad \text{and} \quad \int_{-\infty}^{\infty} W(\lambda)\,d\lambda = 1. \tag{2.13}$$

Using (2.10) we now estimate $\phi_X(\lambda)$ by

$$\hat{\phi}_X(\lambda) = \frac{1}{\beta^2}\left\{\left[\hat{\phi}_Z(\lambda) - \frac{\beta\hat{R}_X(0)}{2\pi}\right] - \int_{-\infty}^{\infty}\Gamma(\lambda - u)\hat{\phi}_Z(u)\,du + \frac{\beta\gamma(0)\hat{R}_X(0)}{2\pi}\right\}, \tag{2.14}$$

where

$$\hat{R}_X(0) = \frac{1}{\beta T}\int_0^T X^2(t)\,dN(t). \tag{2.15}$$

Eq. (2.14) will be used in subsequent sections to establish the convergence properties of $\hat{\phi}_X(\lambda)$. In terms of actual data processing of the observations $\{X(\tau_k), \tau_k\}_{k=1}^{N(T)}$ one may prefer an alternative expression which is obtained as follows. Let $w(t)$ be the covariance averaging kernel

$$w(t) = \int_{-\infty}^{\infty} e^{it\lambda} W(\lambda)\, d\lambda. \tag{2.16}$$

Then it is seen from (2.11) that

$$I_T(\lambda) = \frac{1}{2\pi T} \sum_{j=1}^{N(T)} \sum_{k=1}^{N(T)} e^{-i(\tau_j - \tau_k)\lambda} X(\tau_j) X(\tau_k)$$

and

$$\hat{R}_X(0) = \frac{1}{\beta T} \sum_{j=1}^{N(T)} X^2(\tau_j).$$

Simple algebra then shows that

$$\hat{\phi}_X(\lambda) = \frac{1}{2\pi \beta^2 T} \sum_{j=1}^{N(T)} \sum_{\substack{k=1 \\ j \neq k}}^{N(T)} \exp[-i(\tau_j - \tau_k)\lambda]\, w[b_T(\tau_j - \tau_k)]$$
$$\times [1 - \gamma(\tau_j - \tau_k)] X(\tau_j) X(\tau_k), \tag{2.17}$$

which is an explicit expression in terms of the observations $\{X(\tau_j), \tau_j\}_{j=1}^{N(T)}$.

In this paper it is assumed that the mean sampling rate β and the covariance density function $c_N(u)$ of the point process $\{\tau_k\}$ are known. Also, in case the process X has a nonzero mean m, the standard method is to estimate m by

$$\hat{m}_T = \frac{1}{\beta T} \int_0^T X(t)\, dN(t) = \frac{1}{\beta T} \sum_{j=1}^{N(T)} X(\tau_j) \tag{2.18}$$

and subtract it from the data $\{X(\tau_k)\}$. The results of this paper continue to hold since the contribution of the mean estimation is of a smaller order (as in classical spectral estimation).

In Section 3 we establish the quadratic-mean consistency of $\hat{\phi}_X(\lambda)$ as $T \to \infty$ along with explicit asymptotic expressions for the bias and covariance of $\hat{\phi}_X(\lambda)$. In Section 4 we derive the joint asymptotic normality of $\hat{\phi}_X(\lambda)$. Examples are given in Section 5.

3. Mean-squares consistency

Define

$$K_T(\lambda) = \int_{-\infty}^{\infty} \Gamma(\lambda - u) W_T(u)\, du \tag{3.1}$$

and note that

$$\int_{-\infty}^{\infty} |K_T(\lambda)|\,\mathrm{d}\lambda \le \int_{-\infty}^{\infty} |\Gamma(u)|\,\mathrm{d}u \int_{-\infty}^{\infty} |W(u)|\,\mathrm{d}u < \infty \tag{3.2a}$$

and

$$\int_{-\infty}^{\infty} K_T(\lambda)\,\mathrm{d}\lambda = \gamma(0). \tag{3.2b}$$

We can then write the spectral estimate $\hat{\phi}_X(\lambda)$ in the more compact form, using (2.12) and (2.14),

$$\hat{\phi}_X(\lambda) = \frac{1}{\beta^2}\int_{-\infty}^{\infty} I_T(u)[W_T(\lambda-u) - K_T(\lambda-u)]\,\mathrm{d}u - \frac{1-\gamma(0)}{2\pi\beta}\hat{R}_X(0). \tag{3.3}$$

We first show that, asymptotically, $E[\hat{\phi}_X(\lambda)]$ does not depend on the statistics of the point process $\{\tau_k\}$. We make use of the following result (Brillinger, 1972, Theorem 4.1): If

$$\int_{-\infty}^{\infty} (1+|u|)\,\mathrm{d}|C_Z^{(2)}(u)| < \infty, \tag{3.4}$$

then

$$\operatorname{cum}\{d_{Z,T}(\lambda), d_{Z,T}(\mu)\} = 2\pi D_T(\lambda+\mu)\phi_Z(\lambda) + \mathrm{O}(1),$$

where

$$D_T(\lambda) = \int_{0.}^{T} \mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}t = \frac{1-\mathrm{e}^{-\mathrm{i}\lambda T}}{\mathrm{i}\lambda} \tag{3.5}$$

and the $\mathrm{O}(1)$ term is uniform in λ. Hence,

$$\begin{aligned} E[I_T(\lambda)] &= \frac{1}{2\pi T}\operatorname{cum}\{d_{Z,T}(\lambda), d_{Z,T}(-\lambda)\} \\ &= \phi_Z(\lambda) + \mathrm{O}\left(\frac{1}{T}\right). \end{aligned}$$

Also,

$$E[\hat{R}_X(0)] = \frac{1}{\beta T}\int_0^T R_X(0)\beta\,\mathrm{d}t = R_X(0).$$

Hence, by (3.1) and $W_T(\lambda)\in L_1$, $K_T(\lambda)\in L_1$ we have

$$\begin{aligned} E[\hat{\phi}_X(\lambda)] = \frac{1}{\beta^2}\int_{-\infty}^{\infty} &\phi_Z(u)[W_T(\lambda-u) - K_T(\lambda-u)]\,\mathrm{d}u \\ &- \frac{1-\gamma(0)}{2\pi\beta}R_X(0) + \mathrm{O}\left(\frac{1}{T}\right). \end{aligned} \tag{3.6}$$

It is seen by (2.6), (2.16) and (3.1), using Fubini's theorem, that

$$J \triangleq \int_{-\infty}^{\infty} \phi_Z(u)[W_T(\lambda - u) - K_T(\lambda - u)]\,\mathrm{d}u$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}u\lambda}[1 - \gamma(u)]\,w(b_T u)\,\mathrm{d}C_Z^{(2)}(u)$$

and substituting (2.5) and using $w(0) = 1$ and $1 - \gamma(u) = \beta^2/(\beta^2 + c_N(u))$, we have

$$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}u\lambda}[1 - \gamma(u)]\,w(b_T u)\,R_X(u)\{\beta\delta(u) + \beta^2 + c_N(u)\}\,\mathrm{d}u$$
$$= \frac{1}{2\pi}\{\beta[1 - \gamma(0)]\,R_X(0) + \beta^2\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}u\lambda} R_X(u)\,w(b_T u)\,\mathrm{d}u\}.$$

It then follows by (3.6) that

$$E[\hat{\phi}_X(\lambda)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}u\lambda} w(b_T u)\,R_X(u)\,\mathrm{d}u + \mathrm{O}\left(\frac{1}{T}\right)$$
$$= \int_{-\infty}^{\infty} W_T(\lambda - \mu)\,\phi_X(\mu)\,\mathrm{d}\mu + \mathrm{O}\left(\frac{1}{T}\right).$$
$$\to \phi_X(\lambda) \quad \text{as } T \to \infty \tag{3.7}$$

by dominated convergence since W satisfies (2.13). We thus have the following theorem.

Theorem 3.1. *Let X be a continuous time, zero mean, stationary process with covariance function $R_X(t) \in L_1$ and spectral density $\phi_X(\lambda)$. Let the point process $\{\tau_k\}$ be stationary, independent of X, with mean intensity β, covariance density function $c_N(u) \in L_1$ such that (2.8) and (2.9) hold. If $W(\lambda)$ satisfies (2.13) then the spectral estimate $\hat{\phi}_X(\lambda)$ of (3.3) satisfies*

$$E[\hat{\phi}_X(\lambda)] = \int_{-\infty}^{\infty} W_T(\lambda - \mu)\,\phi_X(\mu)\,\mathrm{d}\mu + \mathrm{O}\left(\frac{1}{T}\right)$$
$$\to \phi_X(\lambda) \quad \text{as } T \to \infty\,,$$

where the term $\mathrm{O}(1/T)$ is uniform in λ.

Thus, the mean of $\hat{\phi}_X(\lambda)$ is independent of the statistics of the sampling process $\{\tau_k\}$ (and in particular does not depend on the mean sampling rate β). The proof of the following corollary is given in the Appendix.

Corollary 3.1. *If, in addition, we have for some integer $n > 0$*

(i) *$u^n R_X(u) \in L_1$,*

(ii) *$w(t)$ is n-times differentiable with $w^{(n)}(t)$ being bounded and continuous.*

Then

$$E[\hat{\phi}_X(\lambda)] = \phi_X(\lambda) + \sum_{j=1}^{n} \frac{(\mathrm{i}b_T)^j w^{(j)}(0)}{j!} \phi_X^{(j)}(\lambda) + \mathrm{o}(b_T^n) + \mathrm{O}\left(\frac{1}{T}\right).$$

A special case of interest is $n = 2$ and noting that $w^{(1)}(0) = 0$, since $w(t)$ is even, we find

$$\text{bias}[\hat{\phi}_X(\lambda)] = -\frac{b_T^2 w^{(2)}(0)}{2} \phi_X^{(2)}(\lambda) + \mathrm{o}(b_T^2) + \mathrm{O}\left(\frac{1}{T}\right), \tag{3.8}$$

which is of course independent of the statistics of the sampling process.

Before we proceed to establish the covariances of $\hat{\phi}_X(\lambda)$ we make the following assumptions.

Assumption 3.1. For integer $K \geq 2$

(a) $\int_{R^{k-1}} (1 + |u_j|)|c_X^{(k)}(u_1, \ldots, u_{k-1})\mathrm{d}u_1 \cdots \mathrm{d}u_{k-1} < \infty$ for $j = 1, \ldots, k-1$; $k = 2, \ldots, K$, (3.9)
where $c_X^{(k)}$ is the kth order cumulant of $X = \{X(t), -\infty < t < \infty\}$ and we note that $c_X^{(2)}(u) = R_X(u)$.

(b) $c_X^{(4)}(0, u, u) \in L_1 \cap L_\infty$.

(c) $\int_{R^{k-1}} (1 + |u_j|)|c_N^{(k)}(u_1, \ldots, u_{k-1})| \mathrm{d}u_1 \cdots \mathrm{d}u_{k-1} < \infty$ for $j = 1, \ldots, k-1$; $k = 2, \ldots, K$, (3.10)
where $c_N^{(k)}$ is defined by

$$\mathrm{cum}\{\mathrm{d}N(t_1), \ldots, \mathrm{d}N(t_k)\} = c_N^{(k)}(t_2 - t_1, \ldots, t_k - t_1)\mathrm{d}t_1 \cdots \mathrm{d}t_k,$$

for distinct t_j's. Note that $c_N^{(2)}(u) = c_N(u)$.

Note that Assumption 3.1 requires that $E|X(t)|^K < \infty$ as well as $E|\mathrm{d}N(t)|^K < \infty$ for the existence of the cumulants $c_X^{(k)}, c_N^{(k)}, k = 2, \ldots, K$. We define the cumulants $C_Z^{(k)}(u_1, \ldots, u_{k-1})$ of the increment process Z (see (2.4)) by

$$\mathrm{cum}\{\mathrm{d}Z(t_1), \ldots, \mathrm{d}Z(t_k)\} = \mathrm{d}C_Z^{(k)}(t_2 - t_1, \ldots, t_k - t_1)\mathrm{d}t_1 \tag{3.11}$$

and $C_Z^{(k)}$ is of bounded variation over finite cubes. Then under Assumption 3.1(a) and (c) we have (Brillinger, 1972, p. 485)

$$\int_{R^{k-1}} (1 + |u_j|)\mathrm{d}|C_Z^{(k)}(u_1, \ldots, u_{k-1})| < \infty, \text{ for } j = 1, \ldots, k-1; k = 2, \ldots, K. \tag{3.12}$$

The kth-order cumulant spectrum $\phi_Z^{(k)}(\lambda_1, \ldots, \lambda_{k-1})$ is defined by

$$\phi_Z^{(k)}(\lambda_1, \ldots, \lambda_{k-1}) \triangleq \frac{1}{(2\pi)^{k-1}} \int_{R^{k-1}} \exp\left\{-\mathrm{i}\sum_{j=1}^{k-1} u_j \lambda_j\right\} \mathrm{d}C_Z^{(k)}(u_1, \ldots, u_{k-1}) \tag{3.13}$$

and we note that $\phi_Z^{(k)}$ is bounded, uniformly continuous but not integrable in general, and $\phi_Z^{(2)}(\lambda) = \phi_Z(\lambda)$. Under (3.12) we have (Brillinger, 1972, Theorem 4.1)

$$\operatorname{cum}(d_{Z,T}(\lambda_1), \ldots, d_{Z,T}(\lambda_k)) = (2\pi)^{k-1} D_T\left(\sum_{j=1}^{k} \lambda_j\right) \phi_Z^{(k)}(\lambda_1, \ldots, \lambda_{k-1}) + \mathrm{O}(1),$$

$$k = 2, \ldots, K, \tag{3.14}$$

where $d_{Z,T}(\lambda)$ is given by (2.11), $D_T(\lambda)$ is given by (3.5) and the O(1) term is uniform in λ's.

We need an Assumption on $W(\lambda)$.

Assumption 3.2. $W(\lambda)$ is a real even, uniformly continuous function on R^1 such that

$$W \in L_1 \cap L_\infty, \quad \int_{-\infty}^{\infty} W(\lambda)\,\mathrm{d}\lambda = 1.$$

We now state the asymptotic covariance of $\hat{\phi}_X(\lambda)$.

Theorem 3.2. *Under Assumptions* 3.1 *with* $K = 4$, *Assumption* 3.2, *and* $b_T \to 0$ *such that* $Tb_T \to \infty$, *we have for each* λ, μ,

$$Tb_T \operatorname{Cov}(\hat{\phi}_X(\lambda), \hat{\phi}_X(\mu)) = \frac{2\pi}{\beta^4} \phi_Z^2(\lambda)[\delta_{\lambda,\mu} + \delta_{\lambda,-\mu}]$$

$$\times \int_{-\infty}^{\infty} W^2(x)\,\mathrm{d}x + \mathrm{o}(1) + \mathrm{O}(\sqrt{b_T}). \tag{3.15}$$

A discussion of the implications of Theorems 3.1 and 3.2, along with some examples, will be given in Section 5.

Proof. By (3.3) we rewrite $\hat{\phi}_X(\lambda)$ as

$$\hat{\phi}_X(\lambda) = \frac{1}{\beta^2}\left\{\int_{-\infty}^{\infty} I_T(u) Q_T(\lambda - u)\,\mathrm{d}u + \frac{\beta}{2\pi}[1 - \gamma(0)]\hat{R}_X(0)\right\}, \tag{3.16}$$

with

$$Q_T(\lambda) = W_T(\lambda) - K_T(\lambda). \tag{3.17}$$

Note that by (3.1) and (3.2)

$$\int_{-\infty}^{\infty} |Q_T(\lambda)|\,\mathrm{d}\lambda \le \text{const.} < \infty \tag{3.18a}$$

and

$$\int_{-\infty}^{\infty} Q_T(\lambda)\,\mathrm{d}\lambda = 1 - \gamma(0) \tag{3.18b}$$

where const. denotes a generic positive constant throughout the paper. Now,

$$\begin{aligned}\operatorname{Cov}(\hat{\phi}_X(\lambda), \hat{\phi}_X(\mu)) &= \frac{1}{\beta^2}\int_{R^2} \operatorname{Cov}(I_T(u_1), I_T(u_2)) Q_T(\lambda_1 - u_1) Q_T(\lambda_2 - u_2)\, \mathrm{d}u_1\, \mathrm{d}u_2 \\ &\quad + \frac{1}{(2\pi\beta)^2}[1-\gamma(0)]^2 \operatorname{Var}(\hat{R}_X(0)) \\ &\quad + \frac{1}{2\pi\beta}[1-\gamma(0)]\int_{-\infty}^{\infty} \operatorname{Cov}(I_T(u), \hat{R}_X(0)) Q_T(\lambda_1 - u)\, \mathrm{d}u \\ &\quad + \frac{1}{2\pi\beta}[1-\gamma(0)]\int_{-\infty}^{\infty} \operatorname{Cov}(I_T(u), \hat{R}_X(0)) Q_T(\lambda_2 - u)\, \mathrm{d}u \\ &\equiv I1 + I2 + I3 + I4. \end{aligned} \tag{3.19}$$

We will show that

(i) $\operatorname{Var}(\hat{R}_X(0)) = \mathrm{O}(1/T)$,

(ii) $I1 = \dfrac{2\pi}{\beta^4 T b_T}\phi_Z^2(\lambda)[\delta_{\lambda,\mu} + \delta_{\lambda,-\mu}]\int_{-\infty}^{\infty} W^2(u)\,\mathrm{d}u + \mathrm{o}\left(\dfrac{1}{Tb_T}\right)$.

Then by the Cauchy–Schwarz inequality and the fact that $Q_T(x) \in L_1$ we have $I3 + I4 = \mathrm{O}(1/(T\sqrt{b_T}))$ and Theorem 3.2 is proved. To show (i), we have by (2.15)

$$\begin{aligned}\operatorname{Var}(\hat{R}_X(0)) &= \frac{1}{(\beta T)^2}\int_0^T\int_0^T \operatorname{cum}(X^2(t)\,\mathrm{d}N(t), X^2(s)\,\mathrm{d}N(s)) \\ &= \frac{1}{(\beta T)^2}\int_0^T\int_0^T \operatorname{cum}(X^2(t), X^2(s)) E[\mathrm{d}N(t)\,\mathrm{d}N(s)] \\ &= \frac{1}{\beta^2 T}\int_{-T}^{T}\left(1 - \frac{|\tau|}{T}\right)\{2R_X^2(\tau) + c_X^{(4)}(0,\tau,\tau)\}\,\{\beta^2 + \beta\delta(\tau) + c_N(\tau)\}\,\mathrm{d}\tau \\ &\le \frac{1}{\beta^2 T}\left\{\int_{-T}^{T} |[2R_X^2(\tau) + c_X^{(4)}(0,\tau,\tau)]\,[\beta^2 + c_N(\tau)]|\,\mathrm{d}\tau \right. \\ &\qquad\qquad \left. + \beta[2R_X^2(0) + c_X^4(0,0,0)]\right\} \\ &= \mathrm{O}\left(\frac{1}{T}\right). \end{aligned} \tag{3.20}$$

For the last inequality in (3.20) we note that $R_X^2 \in L_1$ since $R_X \in L_1 \cap L_\infty$, $R_X^2 c_N \in L_1$ since $R_X \in L_\infty$, $c_N \in L_1$ and the inequality follows by using Assumption 3.1(b).

In order to show (ii) we note first that, using (3.14),

$$\begin{aligned}\operatorname{Cov}(I_T(u_1), I_T(u_2)) &= \frac{1}{(2\pi T)^2}\operatorname{cum}\{d_{Z,T}(u_1) d_{Z,T}(-u_1), d_{Z,T}(u_2) d_{Z,T}(-u_2)\} \\ &= \frac{1}{(2\pi T)^2}\{\operatorname{cum}[d_{Z,T}(u_1) d_{Z,T}(-u_1), d_{Z,T}(u_2) d_{Z,T}(-u_2)]\end{aligned}$$

$$\begin{aligned}
&+ \operatorname{cum}[d_{Z,T}(u_1), d_{Z,T}(u_2)] \\
&\times \operatorname{cum}[d_{Z,T}(-u_1) d_{Z,T}(u_2)] \\
&+ \operatorname{cum}[d_{Z,T}(u_1), d_{Z,T}(-u_2)] \\
&\times \operatorname{cum}[d_{Z,T}(-u_1) d_{Z,T}(u_2)]\} \\
= {} & \frac{1}{(2\pi T)^2} \{(2\pi)^3 T \phi_Z^{(4)}(u_1, -u_1, u_2) + \mathrm{O}(1) \\
&+ [2\pi D_T(u_1 + u_2) \phi_Z(u_1) + \mathrm{O}(1)] \\
&\times [2\pi D_T(-u_1 - u_2) \phi_Z(-u_1) + \mathrm{O}(1)] \\
&+ [2\pi D_T(u_1 - u_2) \phi_Z(u_1) + \mathrm{O}(1)] \\
&\times [2\pi D_T(-u_1 + u_2) \phi_Z(-u_1) + \mathrm{O}(1)]\} \\
= {} & \frac{2\pi}{T} \phi_Z^{(4)}(u_1, -u_1, u_2) + \frac{1}{T} \phi_Z^2(u_1)[\Delta_T(u_1 + u_2) \\
&+ \Delta_T(u_1 - u_2)] \\
&+ \frac{\mathrm{O}(1)}{T^2} \{\phi_Z(u_1)[D_T(u_1 + u_2) + D_T(u_1 - u_2)] \\
&+ \phi_Z(-u_1)[D_T(-u_1 - u_2) \\
&+ D_T(-u_1 + u_2)] + 1\},
\end{aligned} \tag{3.21}$$

where $\Delta_T(\lambda)$ is the Fejer kernel

$$\Delta_T(\lambda) = \frac{1}{T} |D_T(\lambda)|^2. \tag{3.22}$$

Therefore, $I1$ of (3.19) is equal to

$$\begin{aligned}
I1 = {} & \frac{2\pi}{\beta^4 T} \int_{R^2} \phi_Z^{(4)}(u_1, -u_1, u_2) Q_T(\lambda_1 - u_1) Q_T(\lambda_2 - u_2) \mathrm{d}u_1 \mathrm{d}u_2 \\
&+ \frac{1}{\beta^4 T} \int_{R^2} \phi_Z^2(u_1) Q_T(\lambda_1 - u_1) Q_T(\lambda_2 - u_2) [\Delta_T(u_1 + u_2) \\
&+ \Delta_T(u_1 - u_2)] \mathrm{d}u_1 \mathrm{d}u_2 \\
&+ \frac{\mathrm{O}(1)}{\beta^4 T^2} \int_{R^2} \{\phi_Z(u_1)[D_T(u_1 + u_2) + D_T(u_1 - u_2)] \\
&\qquad + \phi_Z(-u_1)([D_T(-u_1 - u_2) + D_T(-u_1 + u_2)]\} \\
&\qquad \times Q_T(\lambda_1 - u_1) Q_T(\lambda_2 - u_2) \mathrm{d}u_1 \mathrm{d}u_2 \\
&+ \mathrm{O}(1/T^2) \\
\equiv {} & J1 + J2 + J3 + \mathrm{O}\left(\frac{1}{T^2}\right).
\end{aligned} \tag{3.23}$$

Since $\phi_Z^{(4)}$ is bounded and $Q_T \in L_1$ we have

$$J1 = \mathrm{O}\left(\frac{1}{T}\right) \tag{3.24}$$

uniformly in λ_1, λ_2. Also,

$$J3 = \mathrm{O}\left(\frac{1}{T}\right) \tag{3.25}$$

uniformly in λ_1, λ_2 because $\phi_Z(u)$ and $D_T(u)/T$ are uniformly bounded and $Q_T \in L_1$ by (3.18). To evaluate $J2$, we need the following lemma whose proof is given in the Appendix.

Lemma 3.1. *Under Assumption* 3.2, $\Gamma(\lambda) \in L_1$, *and* $b_T \to 0$ *such that* $Tb_T \to \infty$ *as* $T \to \infty$ *we have*

$$\int_{-\infty}^{\infty} Q_T(\lambda - u)\Delta_T(u)\,\mathrm{d}u = 2\pi Q_T(\lambda) + \mathrm{o}(1/b_T),$$

where the term $\mathrm{o}(1/b_T)$ *is uniform in* λ.

By (3.23) and Lemma 3.1 we have

$$\begin{aligned} J2 &= \frac{1}{\beta^4 T}\int_{-\infty}^{\infty} \phi_Z^2(u_1) Q_T(\lambda_1 - u_1)\,\mathrm{d}u_1 \\ &\quad \times \int_{-\infty}^{\infty} Q_T(\lambda_2 - u_2)[\Delta_T(u_1 + u_2) + \Delta_T(u_1 - u_2)]\,\mathrm{d}u_2 \\ &= \frac{2\pi}{\beta^4 T}\int_{-\infty}^{\infty} \phi_Z^2(u_1) Q_T(\lambda_1 - u_1)[Q_T(\lambda_2 + u_1) + Q_T(\lambda_2 - u_1)]\,\mathrm{d}u_1 \\ &\quad + \mathrm{o}\left(\frac{1}{Tb_T}\right)\int_{-\infty}^{\infty} \phi_Z^2(u_1) Q_T(\lambda_1 - u_1)\,\mathrm{d}u_1 \\ &\equiv J_{21} + J_{22}. \end{aligned} \tag{3.26}$$

Since ϕ_Z is bounded and $\int_{-\infty}^{\infty} |Q_T(\lambda)|\,\mathrm{d}\lambda \le$ const. by (3.18) we have

$$J_{22} = \mathrm{o}\left(\frac{1}{Tb_T}\right) \quad \text{uniformly in } \lambda_1, \lambda_2. \tag{3.27}$$

To evaluate J_{21} we first note that, from (3.17) and (3.1),

$$\begin{aligned} Q_T(\lambda_1 - u_1)Q_T(\lambda_2 \pm u_1) &= W_T(\lambda_1 - u_1)W_T(\lambda_2 \pm u_1) - W_T(\lambda_1 - u_1) \\ &\quad \times \int_{-\infty}^{\infty} \Gamma(\lambda_2 \pm u_1 - v_2)W_T(v_2)\,\mathrm{d}v_2 \end{aligned}$$

$$- W_T(\lambda_2 \pm u_1) \int_{-\infty}^{\infty} \Gamma(\lambda_1 - u_1 - v_1) W_T(v_1) \, dv_1$$

$$+ \int_{-\infty}^{\infty} \Gamma(\lambda_1 - u_1 - v_1) \Gamma(\lambda_2 \pm u_1 - v_2)$$

$$\times W_T(v_1) W_T(v_2) \, dv_1 \, dv_2$$

$$\equiv F_1 + F_2 + F_3 + F_4. \tag{3.28}$$

Thus, J_{21} of (3.26) has four terms

$$J_{21} = J'_{21} + J''_{21} + J'''_{21} + J''''_{21} \tag{3.29}$$

corresponding to (3.28). We show that J'_{21} is the dominant term: we have with $v = (\lambda_1 - u_1)/b_T$

$$J'_{21} = \frac{2\pi}{\beta^4 T} \int_{-\infty}^{\infty} \phi_Z^2(u_1) W_T(\lambda_1 - u_1)[W_T(\lambda_2 + u_1) + W_T(\lambda_2 - u_1)] \, du_1$$

$$= \frac{2\pi}{\beta^4 T b_T} \int_{-\infty}^{\infty} \phi_Z^2(\lambda_1 - b_T v) W(v) \left[W\left(\frac{\lambda_2 + \lambda_1}{b_T} - v\right) + W\left(\frac{\lambda_2 - \lambda_1}{b_T} + v\right) \right] dv.$$

Note that since $W \in L_1$ and is uniformly continuous we have $W(u) \to 0$ as $|u| \to \infty$. Thus, as $T \to \infty$ the integrand tends to

$$\phi_Z^2(\lambda_1) W^2(v) [\delta_{\lambda_1, \lambda_2} + \delta_{\lambda_1, -\lambda_2}]$$

and is bounded by const. $W(v) \in L_1$ since $\phi_Z(\lambda)$ and $W(\mu)$ are bounded. Thus, by dominated convergence

$$T b_T J'_{21} \to \frac{2\pi}{\beta^4} \phi_Z^2(\lambda_1)[\delta_{\lambda_1, \lambda_2} + \delta_{\lambda_1, -\lambda_2}] \int_{-\infty}^{\infty} W^2(u) \, du, \tag{3.30}$$

where $\delta_{\lambda, \mu}$ is the Kronecker delta.

Next

$$J''_{21} = -\frac{2\pi}{\beta^4 T} \int_{R^2} \phi_Z^2(u_1) W_T(\lambda_1 - u_1)[\Gamma(\lambda_2 + u_1 - v_2) + \Gamma(\lambda_2 - u - v_2)] \, dv_2 \, du_1$$

and since $\phi_Z(u)$ is bounded

$$|J''_{21}| \leq \frac{\text{const.}}{T} \int_{-\infty}^{\infty} |W_T(u)| \, du \int_{-\infty}^{\infty} |\Gamma(v)| \, dv$$

$$= O\left(\frac{1}{T}\right) \quad \text{uniformly in } \lambda_1, \lambda_2. \tag{3.31}$$

Similarly,

$$J_{21}''' = O\left(\frac{1}{T}\right) \quad \text{uniformly in } \lambda_1, \lambda_2. \tag{3.32}$$

Finally,

$$\begin{aligned} |J_{21}''''| &= \frac{2\pi}{\beta^4 T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\phi_Z^2(u_1) \Gamma(\lambda_1 - u_1 - v_1) W_T(v_1) W_T(v_2) \\ &\quad \times [\Gamma(\lambda_2 + u_1 - v_2) + \Gamma(\lambda_2 - u_1 - v_2)]| \, du_1 \, dv_1 \, dv_2 \\ &\le \frac{\text{const.}}{T} \int_{-\infty}^{\infty} |W_T(v_1)| \, dv_1 \int_{-\infty}^{\infty} |W_T(v_2)| \, dv_2 \int_{-\infty}^{\infty} |\Gamma(u)| \, du \\ &= O\left(\frac{1}{T}\right) \quad \text{uniformly in } \lambda_1, \lambda_2, \end{aligned} \tag{3.33}$$

since ϕ_Z and $\Gamma \in L_1$ are bounded function. It follows by (3.26), (3.27) and (3.29)–(3.33) that

$$J_2 = \frac{2\pi(1 + o(1))}{\beta^4 (Tb_T)} \phi_Z^2(\lambda_1) [\delta_{\lambda_1, \lambda_2} + \delta_{\lambda_1, -\lambda_2}] \int_{-\infty}^{\infty} W^2(u) \, du + O\left(\frac{1}{T}\right), \tag{3.34}$$

where the term $O(1/T)$ is uniform in λ_1, λ_2. This completes the proof of (ii) and the theorem is proved. □

4. Joint asymptotic normality

In this section we prove the following result.

Theorem 4.1. *Let Assumption* 3.1 (*for all* $K \geq 2$) *and Assumption* 3.2 *hold. If* $b_T = O(T^{-\alpha})$, *for some* $0 < \alpha < 1$, *then the standardized variables* $\{(Tb_T)^{1/2}\{\hat{\phi}_X(\lambda_i) - E[\hat{\phi}_X(\lambda_i)]\}_{i=1}^n\}$ *are jointly asymptotically normally distributed with mean* 0 *and covariances given by* (3.15).

Proof. Recall from (3.3) and (3.17) that

$$\hat{\phi}_X(\lambda) = \frac{1}{\beta^2} \int_{-\infty}^{\infty} I_T(u) Q_T(\lambda - u) \, du - \frac{\beta}{2\pi} (1 - \gamma(0)) \hat{R}_X(0).$$

We have already seen in Section 3 that $\mathrm{Var}(\hat{R}_X(0)) = O(1/T)$ which is of smaller order of magnitude than that of $G(\lambda_i) \triangleq \int_{-\infty}^{\infty} I_T(u) Q_T(\lambda_i - u) \, du$ which is $O(1/(Tb_T))$. Therefore, to prove the theorem, we only need to establish the joint asymptotic normality of $\{\sqrt{Tb_T} G(\lambda_i)\}$. For this it suffices to show that all joint cumulants of $\{\sqrt{Tb_T} G(\lambda_i)\}$ of order $k \geq 3$ tend to zero as $T \to \infty$. For $k \geq 3$, consider

$$(Tb_T)^{k/2} \operatorname{cum}\{G(\lambda_1), \dots, G(\lambda_k)\} = (Tb_T)^{k/2} \int_{R^k} \left[\prod_{j=1}^{k} Q_T(\lambda_j - u_j) \right] \times h_k(u_1 \dots u_k) \, du_1, \dots, du_k, \tag{4.1}$$

where

$$h_k(u_1, \dots, u_k) = \operatorname{cum}(I_T(u_1), \dots, I_T(u_k)) = (2\pi T)^{-k} \sum_{p=1}^{k} \sum_{v} \operatorname{cum}(\{d_{Z,T}(u_l); l \in v_1\}) \cdots \operatorname{cum}(\{d_{Z,T}(u_l); l \in v_p\}) \tag{4.2}$$

and the inner sum is over all indecomposable partition $v = v_1 \cup \cdots \cup v_p$ of the transformed table

$$\begin{matrix} u_1 & -u_1 \\ u_2 & -u_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ u_k & -u_k \end{matrix} \quad \to \quad \begin{matrix} 1 & -1 \\ 2 & -2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ k & -k \end{matrix}$$

(see Rosenblatt, 1985). We note that in using the transformed table, $\operatorname{cum}(\{d_{Z,T}(u_l); l \in v_j\})$ in (4.2) denotes the joint cumulant of all random variables in $\{d_{Z,T}(u_l); l \in v_j\}$ with the convention $\lambda_l = -\lambda_{|l|}$ if $l < 0$. Also, any partition v which has a single element in a subset v_j contributes zero since $\operatorname{cum}(d_{Z,T}(\lambda)) = E(d_{Z,T}(\lambda)) = 0$. Hence, in any partition v, all subsets v_j must have at least two elements from the transformed table in order to contribute and thus the sum in (4.2) over p has an upper limit k. Denote by $\#(v_j)$ the number of elements in the subset v_j. We now evaluate (4.2) in detail. First consider the case $p = 1$. Then by (4.2), its contribution to (4.1), using (3.14), is

$$\frac{(Tb_T)^{k/2}}{(2\pi T)^k} \int_{R^k} [(2\pi)^{2k-1} D_T(0) \phi_Z^{(2k)}(u_l; l = \pm 1, \dots, \pm (k-1), k) + \mathrm{O}(1)] \times \prod_{j=1}^{k} Q_T(\lambda_j - u_j) \, du_1 \cdots du_k$$
$$= \mathrm{O}(T^{1-k/2}) \to 0 \quad \text{as } T \to \infty \text{ for } k \geq 3.$$

The last equality is obtained by the boundedness of $\phi_Z^{(2k)}$, the integrability (3.18a) of the Q_T's and $D_T(0) = T$. Now for $1 < p \leq k$, given an indecomposable partition $v = v_1 \cup \cdots \cup v_p$ of the transformed table, there exists at least one j such that $j \in v_m$ and $-j \in v_n$ with $m \neq n$; otherwise the partition is not indecomposable. Without loss of generality, we can assume $j = 1$, $m = 1$, and $n = 2$ for convenience. The contribution of such a partition in (4.2) to (4.1) is

$$\frac{(Tb_T)^{k/2}}{(2\pi T)^k}\int_{R^k}\prod_{j=1}^{k} Q_T(\lambda_j - u_j)\Bigg[(2\pi)^{\#(v_1)-1} D_T \times\Bigg(u_1 + \sum_{l\in v_1, l\neq 1} u_l\Bigg)\phi_Z(u_l, \{u_l, l\in v_1'\}) + \mathrm{O}(1)\Bigg]$$
$$\times\Bigg[(2\pi)^{\#(v_2)-1} D_T\Bigg(-u_1 + \sum_{l\in v_2, l\neq -1} u_l\Bigg)\phi_Z(-u_l, \{u_l, l\in v_2'\}) + \mathrm{O}(1)\Bigg]$$
$$\times\prod_{j=3}^{p}\Bigg[(2\pi)^{\#(v_j)-1} D_T\Bigg(\sum_{l\in v_j} u_l\Bigg)\phi_Z(\{u_l\in v_j\}) + \mathrm{O}(1)\Bigg]\mathrm{d}u_1\cdots\mathrm{d}u_k, \tag{4.3}$$

with the expanded notation that $\phi_Z(\{u_l, l\in v_j\})$ being the cumulant spectrum of order $\#(v_j)$ with $\#(v_j) - 1$ arguments. Also, v_j' is implicitly defined in

$$\phi_Z(u_i, \{u_l\in v_j'\}) = \phi_Z(u_i, u_l\in v_j \text{ but } l\neq i).$$

The expansion of the product of the form $\prod_{j=1}^{p}[a_j + \mathrm{O}(1)]$ in the integrand of (4.3) has many terms. The most significant term is $\prod_{j=1}^{p} a_j$ which involves the product of all the Dirichlet kernels D_T's. This dominant term contribution to (4.1) is

$$B_1 = \frac{(b_T/T)^{k/2}}{(2\pi)^k}(2\pi)^{2k-p}\int_{R^{k-1}}\Bigg\{\int_{-\infty}^{\infty} D_T\Bigg(u_1 + \sum_{l\in v_1, l\neq 1} u_l\Bigg)\phi_Z(u_1, \{u_l, l\in v_1'\})$$
$$\times D_T\Bigg(-u_1 + \sum_{l\in v_2, l\neq -1} u_l\Bigg)\phi_Z(-u_1, \{u_l, l\in v_2'\})\, Q_T(\lambda_1 - u_1)\mathrm{d}u_1\Bigg\}$$
$$\times\prod_{j=3}^{p} D_T\Bigg(\sum_{l\in v_j} u_l\Bigg)\phi_Z(u_l, l\in v_j)\prod_{j=2}^{k} Q_T(\lambda_j - u_j)\mathrm{d}u_2\cdots\mathrm{d}u_k. \tag{4.4}$$

We note that by (3.17) and (3.1) we have $Q_T(u) = \mathrm{O}(1/b_T)$ since W is bounded. Applying the Cauchy–Schwarz inequality to the inner integral with respect to u_1, using the boundedness of ϕ_Z, $\phi_Z \leq M$, we find that the inner integral is bounded by

$$M^2\,\mathrm{O}\left(\frac{1}{b_T}\right)\left[\int_{-\infty}^{\infty}|D_T(u_1+\eta)|^2\,\mathrm{d}u_1\int_{-\infty}^{\infty}|D_T(-u_1+\xi)]^2\,\mathrm{d}u_1\right]^{1/2}$$
$$= M^2\mathrm{O}\left(\frac{1}{b_T}\right)T\left[\int_{-\infty}^{\infty}\frac{1}{T}|D_T(u)|^2\,\mathrm{d}u\right] = \mathrm{O}\left(\frac{T}{b_T}\right).$$

Then

$$|B_1| \leq \mathrm{O}(b_T/T)^{k/2-1}\int_{R^{k-1}}\Bigg|\prod_{j=3}^{p} D_T\Bigg(\sum_{l\in v_j} u_l\Bigg)$$
$$\times\phi_Z(u_l, l\in v_j)\prod_{j=2}^{k} Q_T(\lambda_j - u_j)\Bigg|\,\mathrm{d}u_2\cdots\mathrm{d}u_k.$$

If $p = 2$ there is no Dirichlet kernel D_T left in the integrand and by the integrability of Q_T's, we have $|B_1| = O(b_T/T)^{k/2-1} \to 0$ as $T \to \infty$ for $k \geq 3$. If $p > 2$ then there exists an $l \neq 1$ such that $l \in v_j$ for $j \geq 3$ and $-l$ does not appear in any other remaining subset of the partition v; otherwise the partition would be decomposable. Without loss of generality (be relabeling the indices), let $j = 3$ and $l = 2$. Then

$$|B_1| \leq O(b_T/T)^{k/2-1} \int_{R^{k-2}} \left| \prod_{j=4}^{p} D_T\left(\sum_{l \in v_j} u_l \right) \phi_Z(u_l, l \in v_j) \right.$$
$$\left. \times \prod_{j=3}^{k} Q_T(\lambda_j - u_j) \right| du_3 \cdots du_k$$
$$\times \int_{-\infty}^{\infty} \left| D_T\left(u_2 + \sum_{l \in v_3'} u_l \right) \phi_Z(u_2, \{u_l, l \in v_3'\}) Q_T(\lambda_2 - u_2) \right| du_2. \tag{4.5}$$

The last integration in (4.5) is bounded by $O(\log T/b_T)$ by Lemma A.1 in the Appendix which states that

$$\int_{-\infty}^{\infty} \left| D_T(u) Q_T(\lambda - u) \right| du = O(\log T/b_T).$$

Hence,

$$|B_1| \leq O(b_T/T)^{k/2-1} O(\log T/b_T) \int_{R^{k-2}} \left| \prod_{j=4}^{p} D_T\left(\sum_{\ell \in v_j} u_l \right) \right.$$
$$\left. \times \phi_Z(u_l, l \in v_j) \prod_{j=3}^{k} Q_T(\lambda_j - u_j) \right| du_3 \cdots du_k.$$

If $p = 3$ then no more Dirichlet kernels D_T are left, and using the boundedness of ϕ_Z and the integrability of Q_T's we have $|B_1| = O[(1/T)^{k/2-1}(b_T)^{k/2-2} \log T] \to 0$ as $T \to \infty$ for all $k \geq 3$. Continuing in the same manner, using Lemma A.1 repeatedly, we will have $p - 4$ additional integrations involving $D_T Q_T$ and $k - 3 - (p - 4) = k - p + 1$ additional integrations involving the Q_T's only. Hence, the dominant term in (4.3) for a fixed $1 < p \leq k$ occurs when $p = k$,

$$\begin{aligned} |B_1| &= O\left[(b_T/T)^{k/2-1} \left(\frac{\log T}{b_T} \right)^{k-2} \right] \\ &= O((\log T)^{k-2}/(T b_T)^{k/2-1}) \to 0 \end{aligned} \tag{4.6}$$

as $T \to \infty$ for $k \geq 3$ since $b_T = O(T^{-\alpha})$, $0 < \alpha < 1$ by assumption. The same argument can be applied when at least one $O(1)$ term is present in the expansion of the product of the form $\prod_{i=1}^{p} [a_i + O(1)]$ in the integrand of (4.3). Since at least one D_T term will be replaced by $O(1)$, this will result in at least one factor $(\log T/b_T)$ less than that in the bound of the dominant term B_1 of (4.6). Hence, all other terms in the expansion are of smaller order. Thus (4.1) is bounded by (4.6). This completes the proof of the theorem. □

5. Discussion and examples

We first note that Theorems 3.1 and 3.2 imply that the spectral estimate $\hat{\phi}_X(\lambda)$ converges in quadratic mean to $\phi_X(\lambda)$ as $T \to \infty$. In order to obtain rates of convergence, we assume that Corollary 3.1 holds with $n = 2$ in which case the dominant term of the bias is

$$\text{bias}\,[\hat{\phi}_X(\lambda)] = -\frac{b_T^2 w^{(2)}(0)}{2}\,\phi_X^{(2)}(\lambda) \tag{5.1}$$

and by (3.15) the dominant term of the variance is

$$\mathrm{Var}\,[\hat{\phi}_X(\lambda)] = \frac{2\pi}{(Tb_T)\beta^4}\,\phi_Z^2(\lambda)(1+\delta_{0,\lambda})\int_{-\infty}^{\infty} W^2(u)\,\mathrm{d}u, \tag{5.2a}$$

where

$$\phi_Z(\lambda) = \beta^2\phi_X(\lambda) + \frac{\beta R_X(0)}{2\pi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}u\lambda} R_X(u)c_N(u)\,\mathrm{d}u. \tag{5.2b}$$

The asymptotically optimal rate of mean-squares convergence is $T^{-4/5}$ obtained with $b_T \sim T^{-1/5}$ and then

$$T^{4/5}E[\hat{\phi}_X(\lambda) - \phi_X(\lambda)]^2 \to \frac{[w^{(2)}(0)\phi_X^2(\lambda)]^2}{4} + \frac{2\pi}{\beta^4}\phi_Z^2(\lambda)(1+\delta_{0,\lambda})\int_{-\infty}^{\infty} W^2(u)\,\mathrm{d}u. \tag{5.3}$$

We first compare the quadratic-mean performance of the above discrete-time estimate $\hat{\phi}_X(\lambda)$ with the classical continuous-time estimate $\hat{\phi}_{X,C}(\lambda)$, based on the observations $\{X(t), 0 \le t \le T\}$. The latter has the same bias as $\hat{\phi}_X(\lambda)$ and its variance is (Parzen, 1967)

$$\mathrm{Var}[\hat{\phi}_{X,C}(\lambda)] = \frac{2\pi}{Tb_T}\,\phi_X^2(\lambda)(1+\delta_{0,\lambda})\int_{-\infty}^{\infty} W^2(u)\,\mathrm{d}u. \tag{5.4}$$

It suffices therefore to compare the asymptotic constants of the two variance expressions since they have the same rate. Ignoring the common factor $(2\pi/Tb_T)(1+\delta_{0,\lambda})\int_{-\infty}^{\infty} W^2(u)\,\mathrm{d}u$ we write from (5.2)

$$V(\lambda) = [\phi_X(\lambda) + \frac{R_X(0)}{2\pi\beta} + \frac{1}{2\pi\beta^2}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}u\lambda} R_X(u)c_N(u)\,\mathrm{d}u]^2 \tag{5.5}$$

and

$$V_C(\lambda) = [\phi_X(\lambda)]^2. \tag{5.6}$$

Noting that

$$\frac{R_X(0)}{2\pi\beta} + \frac{1}{2\pi\beta^2}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}u\lambda} R_X(u)c_N(u)\,\mathrm{d}u = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}u\lambda} R_X(u)\,\mathrm{d}C_N(u) \geq 0,$$

since $\mathrm{d}C_N(u)$ is a covariance measure, we conclude that $V(\lambda) \geq V_C(\lambda)$ for all λ and thus there is no alias-free point process $\{\tau_k\}$ which asymptotically provides a smaller variance for $\hat{\phi}_X(\lambda)$ than $\hat{\phi}_{X,C}(\lambda)$ for any λ.

Among alias-free point processes $\{\tau_k\}$, the Poisson point process is the simplest and provides the simplest estimator for $\phi_X(\lambda)$: Here $c_N(u) \equiv 0$ and hence $\gamma(u) \equiv 0$ and $\Gamma(\lambda) \equiv 0$ and hence $\hat{\phi}_X(\lambda)$ of (2.14) becomes much simpler. In the Poisson case we have by (5.5),

$$V_{\mathrm{P}}(\lambda) = \left[\phi_X(\lambda) + \frac{R_X(0)}{2\pi\beta}\right]^2. \tag{5.7}$$

However, we show that there exist non-Poisson point processes for which $V(\lambda) < V_{\mathrm{P}}(\lambda)$ for *all* λ for *all* spectral densities ϕ_X and that the improvement factor can be substantial for certain frequency ranges.

We now discuss a specific class of alias-free point processes. The structure of the class of delayed renewal point processes is examined in detail in Masry (1978b). Here we only mention that it is generated by a *single* probability density function $g(x)$ on $[0, \infty)$ with mean $1/\beta$ (the "inter-arrival times" density). We assume that $g(x) > 0$ a.e. on $[0, \infty)$. Then the delayed renewal point process $\{\tau_k\}$ is alias-free relative to all spectral densities $\phi_X(\lambda)$ (Masry, 1978b). We can express the covariance density function $c_N(u)$ of $\{\tau_k\}$ in terms of g as follows. Let $g^{(n)}(x)$ be the nth fold convolution of g with itself and define the renewal density function h by

$$h(u) = \sum_{n=1}^{\infty} g^{(n)}(u). \tag{5.8}$$

We have $h(u) \to \beta$ as $u \to \infty$. Then

$$c_N(u) = \beta[h(|u|) - \beta]; \qquad \gamma(u) = 1 - \frac{\beta}{h(|u|)}. \tag{5.9}$$

This class of stationary point processes is admittable if condition (2.9) is satisfied. This requirement imposes further restrictions on $g(u)$: If $g(u) > 0$ for all $u \in [0, \infty)$ and $g(u)$ decays exponentially fast as $u \to \infty$ then we expect $\gamma(u)$ of (5.9) to be in L_1; a trivial example is $g(u) = \beta\mathrm{e}^{-\beta u}\, 1_{[0,\infty)}(u)$ which generates the Poisson point process. On the other hand, if g is Gamma type k, $k \geq 2$

$$g_k(u) = \frac{(\beta k)^k}{(k-1)!} u^{k-1} \mathrm{e}^{-k\beta u}\, 1_{[0,\infty)}(u), \quad k \geq 2, \tag{5.10}$$

then it is seen that $g_k(0) = 0$ and thus $g_k^{(n)}(0) = 0$ for $n \geq 1$, so that $h(0) = 0$ and by (5.9), $\gamma \notin L_1$ so that the Gamma class (5.10) is not admittable. A simple remedy is to modify (5.10) by making $g_k(0) > 0$ in a continuous manner, e.g., by introducing mixtures

$$\tilde{g}_k(u) = (1-\alpha)g_1(u) + \alpha g_k(u), \quad 0 < \alpha < 1, \quad k \geq 2. \tag{5.11}$$

We prove that all delayed renewal point processes $\{\tau_k\}$ generated by the mixture densities $\tilde{g}_k$ of (5.11) are admittable. The Laplace transform $\tilde{G}_k(s) = \int_0^\infty e^{-su}\tilde{g}_k(u)\,du$ of $\tilde{g}_k(u)$ is given by

$$\tilde{G}_k(s) = \frac{(1-\alpha)\beta}{\beta+s} + \frac{\alpha(\beta k)^k}{(\beta k+s)^k}, \quad k \geq 2$$

and the Laplace transform $H(s)$ of the corresponding renewal density $h(u)$ is equal to

$$H(s) = \frac{\tilde{G}_k(s)}{1-\tilde{G}_k(s)} = \frac{\beta(1-\alpha)\left(1+\dfrac{s}{\beta k}\right)^k + \alpha(\beta+s)}{\left(1+\dfrac{s}{\beta k}\right)^k(\beta\alpha+s) - \alpha(\beta+s)}, \quad k \geq 2. \tag{5.12}$$

We first note that by partial fraction expansion we have

$$h(u) = \beta + \sum_{j=1}^{k} A_j u^{n_j} e^{-a_j u}, \quad u \geq 0,$$

where the integers $n_j \geq 0$ and A_j and a_j are, possibly, complex number (appearing in complex conjugate pairs). Thus by (5.9)

$$c_N(u) = \beta \sum_{j=1}^{k} A_j |u|^{n_j} e^{-a_j|u|}. \tag{5.13}$$

By (2.9) the point process $\{\tau_k\}$ is admittable if $\gamma \in L_1$ and $\Gamma \in L_1$. By (5.13), $c_N(u) \to 0$ as $|u| \to \infty$. Hence, given $\varepsilon > 0$ there exists $N = N_\varepsilon$ such that $|c_N(u)| \leq \varepsilon$ for $|u| > N$. Thus, $\beta^2 + c_N(u) > \beta^2/2$ for $|u| > N$ with $\varepsilon = \frac{1}{2}\beta^2$. By (5.11) $\tilde{g}_k(u) \geq (1-\alpha)g_1(u)$ so that $\tilde{g}_k^{(n)}(u) \geq (1-\alpha)^n g_1^{(n)}(u)$ and by (5.8)

$$h(u) \geq \sum_{n=1}^{\infty} (1-\alpha)^n g_1^{(n)}(u) = \beta(1-\alpha)e^{-\alpha\beta u}.$$

Hence,

$$\beta^2 + c_N(u) = \beta h(|u|) \geq \beta^2(1-\alpha)e^{-\alpha\beta|u|} \geq \beta^2(1-\alpha)e^{-\alpha\beta N},$$

for $|u| \leq N$. Thus,

$$\beta^2 + c_N(u) \geq \min\left(\tfrac{1}{2}\beta^2, \beta^2(1-\alpha)e^{-\alpha\beta N}\right) > 0, \quad \forall u. \tag{5.14}$$

It follows that

$$\gamma(u) = \frac{c_N(u)}{\beta^2 + c_N(u)} \in L_1, \tag{5.15}$$

since $c_N(u) \in L_1$. We next show that $\Gamma \in L_1$. It is seen by (5.13)–(5.15) that on $[0, \infty)$ $\gamma(u)$ is infinitely differentiable. Now

$$\Gamma(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}u\lambda}\gamma(u)\,\mathrm{d}u = \frac{1}{\pi}\int_0^{\infty} (\cos u\lambda)\gamma(u)\,\mathrm{d}u. \tag{5.16}$$

Note that $\gamma^{(j)}(u) \in L_1$ on $[0, \infty)$ for $j = 1, 2$ and $\gamma^{(j)}(u) \to 0$ as $|u| \to \infty$ for $j = 0, 1$. It then follows by integration by parts twice that $\Gamma(\lambda) = \mathrm{O}(1/\lambda^2)$ as $|\lambda| \to \infty$ and thus $\Gamma \in L_1$. It follows that the delayed renewal points processes $\{\tau_k\}$ generated by the mixture densities $\tilde{g}_k(u)$ of (5.11) are indeed admittable for all $k \geq 2$.

We now examine in detail the case of the mixture density (5.11) with $k = 2$ and show that the spectral estimate $\hat{\phi}_X(\lambda)$, with this sampling process, has a smaller asymptotic variance than that of the Poisson sampling, for *all* spectral densities $\phi_X(\lambda)$. We have, after some algebra,

$$h(u) = \beta\{1 + B_1 \mathrm{e}^{-a_1 u} + B_2 \mathrm{e}^{-a_2 u}\}, \tag{5.17}$$

$$c_N(u) = \beta^2\{B_1 \mathrm{e}^{-a_1|u|} + B_2 \mathrm{e}^{-a_2|u|}\}, \tag{5.18a}$$

where

$$a_1 = \frac{\beta}{2}[4 + \alpha + \sqrt{\alpha(8 + \alpha)}]; \qquad a_2 = \frac{\beta}{2}[4 + \alpha - \sqrt{\alpha(8 + \alpha)}] \tag{5.18b}$$

$$B_1 = -\alpha\left[\frac{1}{2} + \frac{1 + \alpha/2}{\sqrt{\alpha(8 + \alpha)}}\right]; \qquad B_2 = \alpha\left[-\frac{1}{2} + \frac{1 + \alpha/2}{\sqrt{\alpha(8 + \alpha)}}\right]. \tag{5.18c}$$

The Fourier transform $\psi(\lambda)$ of $c_N(u)$ is given by

$$\psi(\lambda) = -\frac{\alpha\beta^3}{\pi}\left\{\frac{4\beta^2 + (3 + \alpha)\lambda^2}{(a_1^2 + \lambda^2)(a_2^2 + \lambda^2)}\right\}, \tag{5.19}$$

which is negative for all λ. Hence, by (5.2b)

$$\phi_Z(\lambda) = \beta^2\phi_X(\lambda) + \frac{\beta R_X(0)}{2\pi} + \int_{-\infty}^{\infty} \psi(\lambda - u)\phi_X(u)\,\mathrm{d}u < \beta^2\phi_X(\lambda) + \frac{\beta R_X(0)}{2\pi}$$

and thus $V(\lambda) < V_{\mathrm{P}}(\lambda)$.

We next evaluate the improvement factor in the asymptotic variance of $\hat{\phi}_X(\lambda)$ using the sampling process generated by the mixture density $\tilde{g}_2(u)$ relative to a Poisson sampling process. We compute

$$f(\lambda; \beta) \triangleq \frac{V_{\mathrm{P}}(\lambda) - V(\lambda)}{V_{\mathrm{P}}(\lambda)} > 0. \tag{5.20}$$

Here we assume

$$R_X(t) = \sigma^2 \mathrm{e}^{-\rho|t|} \tag{5.21}$$

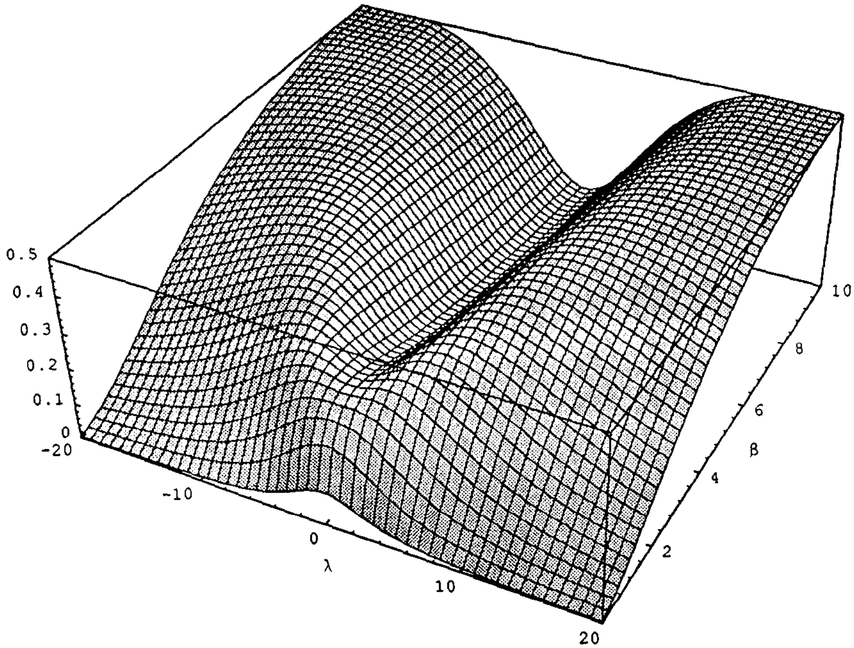

Fig. 1. Improvement factor $f(\lambda;\beta)$ as a function of λ and β.

and we set the mixture parameter α to $\alpha = 0.9$ (far away from the Poisson sampling which corresponds to $\alpha = 0$). By (5.2b) and (5.18) we find

$$V(\lambda) = \left\{\sigma^2\left\{\frac{\rho/\pi}{\rho^2+\lambda^2} + \frac{1}{2\pi\beta} + \frac{1}{\pi}\left[\frac{B_1(\rho+a_1)}{(\rho+a_1)^2+\lambda^2} + \frac{B_2(\rho+a_2)}{(\rho+a_2)^2+\lambda^2}\right]\right\}\right\}^2$$

and

$$V_{\mathrm{P}}(\lambda) = \left[\sigma^2\left\{\frac{\rho/\pi}{\rho^2+\lambda^2} + \frac{1}{2\pi\beta}\right\}\right]^2.$$

The half-power bandwidth of $\phi_X(\lambda)$ is $2B = 2\rho$ rad/(unit time) and the "nominal sampling rate" is $B/\pi = \rho/\pi$. We select $\rho = \pi$ in (5.21) so that the nominal sampling rate is normalized to one. Fig. 1 shows the improvement factor $f(\lambda;\beta)$ for $|\lambda| \le 20$ rad/(unit time) and $0 < \beta < 10$. It is clear that $f(\lambda;\beta) > 0$ throughout this region and that for large β the improvement factor can be as high as 50% for high frequencies.

A more detailed picture is depicted in Fig. 2 where $f(\lambda;\beta)$ is plotted as a function of λ for 4 values of $\beta = 0.4, 1, 2, 3$. It is seen that in the frequency range $[-\pi\beta, \pi\beta]$ the improvement factor is quite substantial: For $\beta = 0.4$, the improvement factor in this range is approximately 23%. For $\beta = 1$, the improvement factor in this range is 28%. For large β, the improvement factor approaches 50% for broad frequency bands away

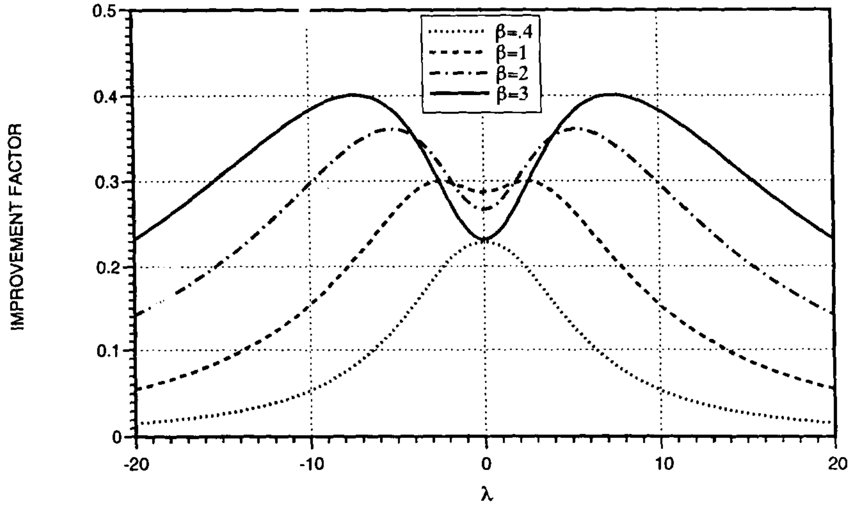

Fig. 2. Improvement factor $f(\lambda;\,\beta)$ as a function of λ for selected values of β.

from the origin (see Fig. 1). Thus, for example, Fig. 2 shows that for $\beta = 3$ we have an improvement factor of between 30% and 40% for $3 \leq |\lambda| \leq 16$ rad/(unit time).

The above example raises the issue of finding optimal sampling processes $\{\tau_k\}$ which minimize appropriate functions of the mean-squares error of $\hat{\phi}_X(\lambda)$. Since the bias of $\hat{\phi}_X(\lambda)$ does not depend on the statistics of $\{\tau_k\}$ it suffices to minimize functionals of the variance of $\hat{\phi}_X(\lambda)$, i.e., of $\phi_Z(\lambda)$ for a fixed mean sampling rate β. Given a real valued, even, nonnegative integrable weight function $H(\lambda)$ we may consider the functional

$$\mathscr{L}_1 = \int_{-\infty}^{\infty} H(\lambda)\phi_Z(\lambda)\,\mathrm{d}z \quad \text{or} \quad \mathscr{L}_2 = \int_{-\infty}^{\infty} H(\lambda)\phi_Z^2(\lambda)\,\mathrm{d}\lambda.$$

In view of expression (5.2b) for $\phi_Z(\lambda)$, one desires to minimize $\mathscr{L}_1$ or $\mathscr{L}_2$, for a given $\phi_X(\lambda)$, with respect to the covariance density $c_N(u)$ of the point process $\{\tau_k\}$ under certain constraints, i.e., the point process must be admittable,

$$\gamma(u) = \frac{c_N(u)}{\beta^2 + c_N(u)} \in L_1 \quad \text{and} \quad \Gamma(\lambda) \in L_1,$$

with a fixed $\beta > 0$. The problem appears to be difficult and even if the above optimization is solved, there is still the task of generating a stationary point process $\{\tau_k\}$ with the optimal $c_N(u)$. One possible approach is to restrict the optimization to a specific class of point processes whose structure is known (e.g., the delayed renewal point processes generated by a single "inter-arrival times" density function $g(x)$ with mean $1/\beta$). Practical considerations in the selection of point processes $\{\tau_k\}$ for the estimation of broadband and narrowband spectra are addressed in Lii and Masry (1992b).

Appendix

Proof of Corollary 3.1. Expanding $w(t)$ in a Taylor series with integral remainder

$$w(t) = \sum_{j=0}^{n-1} \frac{w^{(j)}(0)t^j}{j!} + \frac{t^n}{(n-1)!}\int_0^1 x^{n-1} w^{(n)}[(1-x)t]\,dx$$

and using (3.7) we find

$$E[\hat{\phi}_X(\lambda)] = \frac{1}{2\pi}\sum_{j=0}^{n-1} \frac{b_T^j w^{(j)}(0)}{j!}\int_{-\infty}^{\infty} \exp[-iu\lambda]u^j R_X(u)\,du + e_T + O\left(\frac{1}{T}\right), \quad (A.1)$$

where

$$e_T = \frac{b_T^n}{2\pi(n-1)!}\int_{-\infty}^{\infty}\int_0^1 \exp[-iu\lambda]u^n R_X(u)x^{n-1}w^{(n)}[(1-x)b_T u]\,dx\,du.$$

The integrand $\to e^{-iu\lambda}u^n R_X(u)x^{n-1}w^{(n)}(0)$ as $T\to\infty$ and is bounded by const. $\times |u|^n |R_X(u)| x^{n-1} \in L_1(dx\times du)$. Thus by dominated convergence

$$\begin{aligned}(1/b_T^n)e_T &= \frac{w^{(n)}(0)}{(n-1)!}\left[\int_0^1 x^{n-1}\,dx\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\lambda}u^n R_X(u)\,du\right] + o(1)\\ &= \frac{w^{(n)}(0)}{n!}(i)^n \phi_X^{(n)}(\lambda) + o(1) \text{ as } T\to\infty.\end{aligned}$$

The result follows by (A.1). □

Lemma A.1. *If $W(\lambda)\in L_1\cap L_\infty$ and $\Gamma(\lambda)\in L_1$ then*

$$\int_{-\infty}^{\infty} |D_T(u)Q_T(\lambda-u)|\,du = O\left(\frac{\log T}{b_T}\right)$$

uniformly in λ.

Proof.

$$\begin{aligned}\int_{-\infty}^{\infty} Q_T(u)D_T(\lambda-u)\,du &= \int_{-\infty}^{\infty} W_T(u)D_T(\lambda-u)\,du\\ &\quad - \int_{R^2} \Gamma(u-v)W_T(v)D_T(\lambda-u)\,dv\,du\\ &= I_1 + I_2.\end{aligned}$$

For fixed $A>0$

$$\begin{aligned}I_1 &= \left[\int_{|\lambda-u|\le \frac{A}{T}} + \int_{\frac{A}{T}<|\lambda-u|\le A} + \int_{|\lambda-u|>A}\right] W_T(u)D_T(\lambda-u)\,du\\ &= J_1 + J_2 + J_3,\end{aligned}$$

$$|J_1| \le T \int_{|\lambda - u| \le \frac{A}{T}} |W_T(u)| \left| \frac{\sin T(\lambda - u)/2}{T(\lambda - u)/2} \right| \mathrm{d}u$$

$$\le \frac{T}{b_T} \int_{|\lambda - u| \le \frac{A}{T}} |W(u/b_T)| \, \mathrm{d}u \le \mathrm{O}\left(\frac{T}{b_T}\right) \int_{|\lambda - u| \le \frac{A}{T}} \mathrm{d}u = \mathrm{O}\left(\frac{1}{b_T}\right),$$

since $W(u)$ and $(\sin T(\lambda - u)/2)/(T(\lambda - u)/2)$ are bounded. Next,

$$|J_2| \le \int_{\frac{A}{T} < |\lambda - u| \le A} \left| W_T(u) \right| \frac{\mathrm{d}u}{|\lambda - u|} = \mathrm{O}\left(\frac{1}{b_T}\right) \int_{\frac{A}{T} < |\lambda - u| \le A} \frac{\mathrm{d}u}{|\lambda - u|}$$

$$= \mathrm{O}\left(\frac{\log T}{b_T}\right).$$

$$|J_3| \le \frac{2}{A} \int_{-\infty}^{\infty} |W_T(u)| \, \mathrm{d}u = \mathrm{O}(1).$$

Thus, $I_1 = \mathrm{O}(\log T/b_T)$. Similarly $I_2 = \mathrm{O}(\log T/b_T)$ since $\Gamma \in L_1$. $\square$

Proof of Lemma 3.1.

$$\int_{-\infty}^{\infty} Q_T(\lambda - u) \Delta_T(u) \, \mathrm{d}u = \int_{-\infty}^{\infty} [W_T(\lambda - u) - K_T(\lambda - u)] \Delta_T(u) \, \mathrm{d}u. \tag{A.2}$$

Now

$$\int_{-\infty}^{\infty} K_T(\lambda - u) \Delta_T(u) \, \mathrm{d}u = \int_{R^2} \Gamma(\lambda - u - x) W_T(x) \Delta_T(u) \, \mathrm{d}u \, \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \Gamma(v) \, \mathrm{d}v \int_{-\infty}^{\infty} W_T(\lambda - v - u) \Delta_T(u) \, \mathrm{d}u. \tag{A.3}$$

We prove below that

$$\int_{-\infty}^{\infty} W_T(\lambda - u) \Delta_T(u) \, \mathrm{d}u = 2\pi W_T(\lambda) + \mathrm{o}(1/b_T), \tag{A.4}$$

where $\mathrm{o}(1/b_T)$ is uniform in λ. It then follows by (A.3) that

$$\int_{-\infty}^{\infty} K_T(\lambda - u) \Delta_T(u) \, \mathrm{d}u = 2\pi \int_{-\infty}^{\infty} \Gamma(v) W_T(\lambda - v) \, \mathrm{d}v + \mathrm{o}(1/b_T), \tag{A.5}$$

since $\Gamma \in L_1$. By (A.2), (A.4), and (A.5) we then have

$$\int_{-\infty}^{\infty} Q_T(\lambda - u) \Delta_T(u) \, \mathrm{d}u = 2\pi \left\{ W_T(\lambda) - \int_{-\infty}^{\infty} \Gamma(v) W_T(\lambda - v) \, \mathrm{d}v \right\} + \mathrm{o}(1/b_T)$$

$$= 2\pi Q_T(\lambda) + \mathrm{o}(1/b_T).$$

We now proceed to prove (A.4). Let

$$J_T \triangleq \int_{-\infty}^{\infty} W_T(\lambda - u)\Delta_T(u)\,\mathrm{d}u - 2\pi W_T(\lambda).$$

Then

$$\begin{aligned} J_T &= \int_{-\infty}^{\infty} \left[W\left(\frac{\lambda}{b_T} - u\right) - W\left(\frac{\lambda}{b_T}\right)\right] \Delta_T(b_T u)\,\mathrm{d}u \\ &= b_T \int_{-\infty}^{\infty} \left[W\left(\frac{\lambda}{b_T} - \frac{u}{Tb_T}\right) - W\left(\frac{\lambda}{b_T}\right)\right] \Delta(u)\,\mathrm{d}u \\ &\le b_T \int_{-\infty}^{\infty} w_W\left(\frac{|u|}{Tb_T}\right)\Delta(u)\,\mathrm{d}u, \end{aligned} \tag{A.6}$$

where $w_W(x)$ is the modulus of continuity of W and $\Delta(u) = ((\sin u/2)/(u/2))^2$. The integrand in (A.6) $\to 0$ as $T \to \infty$ and is bounded by const. $\Delta(u) \in L_1$. Hence, by dominated convergence $(1/b_T)\,J_T \to 0$ as $T \to \infty$. □

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