



Berry–Esseen and Edgeworth approximations for the normalized tail of an infinite sum of independent weighted gamma random variables

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Abstract

Consider the sum $Z = \sum_{n=1}^{\infty} \lambda_n (\eta_n - \mathbb{E}\eta_n)$, where η_n are independent gamma random variables with shape parameters $r_n > 0$, and the λ_n 's are predetermined weights. We study the asymptotic behavior of the tail $\sum_{n=M}^{\infty} \lambda_n (\eta_n - \mathbb{E}\eta_n)$, which is asymptotically normal under certain conditions. We derive a Berry–Esseen bound and Edgeworth expansions for its distribution function. We illustrate the effectiveness of these expansions on an infinite sum of weighted chi-squared distributions.

The results we obtain are directly applicable to the study of double Wiener–Itô integrals and to the “Rosenblatt distribution”.

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1. Introduction

We focus on the distribution of normalized tails of infinite weighted sums of *Gamma* random variables. Our results involve the interplay between the asymptotic behavior of the weights λ_n in the sum and the scales r_n of Gamma variables. We show that one cannot always expect a central limit theorem to hold. When the central limit theorem holds, we develop Berry–Esseen bounds and also Edgeworth expansions.

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The results we obtain are directly applicable to the study of double Wiener–Itô integrals since these can be represented as weighted sums of independent chi-squared variables which are special cases of Gamma variables.

The Rosenblatt distribution is a non-Gaussian distribution which often appears as a limit of normalized partial sums of random variables with long memory. The results obtained here play a key role in a major study of the “Rosenblatt distribution” in [21]. This is because a random variable with the Rosenblatt distribution can be represented as a double Wiener–Itô integral, and as we noted, these multiple integrals can be viewed as weighted sums of independent chi-squared random variables.

Consider then the random variable $\sum_{n=1}^{\infty} \lambda_n (\eta_n - \mathbb{E}\eta_n)$, where $\eta_n \sim \text{Gamma}(r_n, \theta_n)$ are independent gamma random variables with mean $\mathbb{E}\eta_n = r_n \theta_n$ and variance $\text{Var } \eta_n = r_n \theta_n^2$, where $r_n > 0$ and $\theta_n > 0$ are the shape and scale parameters, respectively. We may suppose without loss of generality that $\theta_n = r_n^{-1}$ for all n , by incorporating the extra parameters into the constants λ_n .¹

We thus consider

$$Z = \sum_{n=1}^{\infty} \lambda_n (\eta_n - 1), \tag{1}$$

where

$$\eta_n \sim \text{Gamma}(r_n, 1/r_n)$$

are independent gamma with pdf

$$f_{\eta_n}(x) = \frac{r_n^{r_n}}{\Gamma(r_n)} x^{r_n-1} e^{-r_n x}, \quad x > 0. \tag{2}$$

We suppose that $\{\lambda_n\}$ and $\{r_n\}$ are sequences of positive numbers such that

$$\sum_{n=1}^{\infty} \frac{\lambda_n^2}{r_n} < \infty. \tag{3}$$

With this setup, Z has mean zero and variance

$$\text{Var } Z = \sum_{n=1}^{\infty} \lambda_n^2 \text{Var} (\eta_n - 1) = \sum_{n=1}^{\infty} \frac{\lambda_n^2}{r_n}.$$

Of particular interest is the case where $r_n = r$ is constant and $\lambda_n \sim n^{-\alpha/2} \ell(n)$, where $\alpha > 1$ and ℓ is slowly varying as $n \rightarrow \infty$. The restriction $\alpha > 1$ ensures $\sum \lambda_n^2 / r_n = (1/r) \sum \lambda_n^2 < \infty$ but allows for cases when either $\sum \lambda_n = \infty$ or $\sum \lambda_n < \infty$.

¹ Since the characteristic function of $\eta \sim \text{Gamma}(r, \theta)$ is $\varphi_{\eta}(u) = (1 - iu\theta)^{-r}$ and that of $\hat{\eta} \sim \text{Gamma}(r, 1/r)$ is $\varphi_{\hat{\eta}}(u) = (1 - iu/r)^{-r}$, we have $\eta \stackrel{d}{=} \theta r \hat{\eta}$ and thus

$$\sum_{n=1}^{\infty} \lambda_n (\eta_n - \mathbb{E}\eta_n) = \sum_{n=1}^{\infty} \lambda_n (\eta_n - r_n \theta_n) \stackrel{d}{=} \sum_{n=1}^{\infty} \lambda_n r_n \theta_n (\hat{\eta}_n - 1) = \sum_{n=1}^{\infty} \hat{\lambda}_n (\hat{\eta}_n - 1)$$

by setting $\hat{\lambda}_n = \lambda_n r_n \theta_n$. In the text, we denote from now on $\hat{\lambda}_n$ and $\hat{\eta}_n$ by λ_n and η_n respectively.

Random variables of the form (1) make up a rich class of distributions. Indeed, consider the double Weiner–Itô integral

$$I = \int_{\mathbb{R}^2}'' H(x, y)Z(dx)Z(dy) \tag{4}$$

where Z is a complex-valued Gaussian random measure. The double prime on the integral indicates that one excludes the diagonals $\{x = \pm y\}$ from the integration (for more on integrals of this type, see [14]). In [7, Proposition 2], Dobrushin and Major show that the random variable I can be expressed in the form (1) with $r_n = 1/2$ for all n (chi-squared distributions). An important example in this case is the *Rosenblatt distribution*, discovered by Rosenblatt in [15], and later named after him in [19]. For an overview, see [20]. Properties of the Rosenblatt distribution are further developed in [21] using the results we obtain in the present paper.

A major difficulty that arises with distributions like (1) is that there is no closed form for their cumulative distribution function or density function. To make matters worse, even the characteristic function of Z is not easy to express or compute numerically (we shed some light on this issue below). An initial approach to this problem might be to truncate the sum (1) at a level $M \geq 1$, and write $Z = X_M + Y_M$ where

$$X_M = \sum_{n=1}^{M-1} \lambda_n(\eta_n - 1), \quad Y_M = \sum_{n=M}^{\infty} \lambda_n(\eta_n - 1) \tag{5}$$

and using X_M as an approximation of Z since it is a finite sum of weighted gamma random variables. An efficient method which computes the PDF/CDF of X_M is described in [22]. Alternatively, one could consider using infinite series expansions for the PDF/CDF of X_M which are given in [11,12].

How well does X_M approximate Z ? This question can be partially answered by looking at the variance of Y_M ,

$$\sigma_M^2 \equiv \text{Var } Y_M = \sum_{n=M}^{\infty} \frac{\lambda_n^2}{r_n}. \tag{6}$$

Depending on the behavior of the sequences λ_n and r_n , this can tend to 0 slowly. For instance, if $\lambda_n \sim Cn^{-\alpha/2}$ for some $\alpha > 1$ and $r_n = r$ is constant, then

$$\sigma_M^2 \sim \frac{C}{r} \int_M^{\infty} x^{-\alpha} dx \sim \frac{C}{r} M^{1-\alpha},$$

which tends to 0 slowly when α is close to 1, and thus in these cases M would have to be taken very large for X_M to be a reasonable approximation.

Instead of approximating Z by only X_M for M large, we will instead show that, under certain conditions, Y_M is asymptotically normal using a Berry–Essen estimate, and then we will give an Edgeworth expansion for the distribution function of Y_M . Combining this with the distribution of X_M will provide a method for computing more precisely the distribution function of Z . This can also be used for simulation of the random variable Z by simulating X_M exactly, and approximating the error with a $N(0, \sigma_M^2)$ random variable.

This paper is organized as follows. In Section 2, we give the characteristic function of Z and Y_M in the Lévy–Khintchine form. We then use this form of the characteristic function to show that Y_M is asymptotically normal under certain conditions in Section 3. We study the asymptotic

behavior of the cumulants of Y_M in Section 4 and prove an approximation lemma related to the characteristic function of Y_M in Section 5. In Section 6, we give an Edgeworth expansion of Y_M . Finally, we demonstrate the accuracy of these approximations in Section 7 on an example where the η_n are i.i.d. chi-squared, and the sequence λ_n is given. Some concluding remarks can be found in Section 8.

2. Lévy–Khintchine representation

Recall that a random variable X is infinitely divisible if for any positive integer n , one can find i.i.d. random variables $X_{1,n}, X_{2,n}, \dots, X_{n,n}$ such that

$$X \stackrel{d}{=} X_{1,n} + X_{2,n} + \dots + X_{n,n}.$$

The characteristic function of any real valued infinitely divisible random variable X with $\mathbb{E}X^2 < \infty$ can be expressed in the following form, known as the Lévy–Khintchine form.

$$\mathbb{E}e^{iuX} = \exp\left(iau - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{iux} - 1 - iux)\Pi(dx)\right) \tag{7}$$

where $a \in \mathbb{R}, \sigma^2 > 0$ and Π is a measure on $\mathbb{R} \setminus \{0\}$, known as the Lévy measure, which satisfies

$$\int_{\mathbb{R}\setminus\{0\}} \min(x^2, 1)\Pi(dx) < \infty. \tag{8}$$

For a background on such distributions, see [18,16,4], or [2].

The random variable η with PDF (2) with shape r is infinitely divisible and has the characteristic function

$$\mathbb{E}e^{iu\eta} = \exp\left(\int_0^\infty (e^{iux} - 1)\Pi(x)\right) \tag{9}$$

where the Lévy measure is given by $\Pi(dx) = rx^{-1}e^{-rx}dx$ for $x > 0$ [2, example 1.3.22]. Hence, if $\lambda > 0$, the random variable $\lambda(\eta - 1)$ is also infinitely divisible and its characteristic function is given by

$$\mathbb{E}\exp(iu\lambda(\eta - 1)) = \exp\left(\int_0^\infty (e^{iux} - 1 - iux)\left(\frac{r}{x} \exp\left(-\frac{rx}{\lambda}\right)\right) dx\right). \tag{10}$$

By taking an infinite sum of such distributions as in (1), it is not surprising that the resulting distribution is also infinitely divisible as indicated in the following proposition.

Proposition 2.1. *The characteristic function of Z defined in (1) is given by*

$$\mathbb{E}e^{iuZ} = \exp\left(\int_0^\infty (e^{iux} - 1 - iux)v(x)dx\right) \tag{11}$$

where v is defined as

$$v(x) \equiv \sum_{n=1}^\infty \frac{r_n}{x} \exp\left(-\frac{r_n x}{\lambda_n}\right). \tag{12}$$

Proof. We have

$$\begin{aligned} \mathbb{E}e^{iuZ} &= \lim_{M \rightarrow \infty} \mathbb{E}e^{iuX_M} \\ &= \mathbb{E} \exp \left(\lim_{M \rightarrow \infty} \int_0^\infty (e^{iux} - 1 - iux) \left(\sum_{n=1}^{M-1} \frac{r_n}{x} \exp \left(-\frac{r_n x}{\lambda_n} \right) \right) dx \right). \end{aligned} \tag{13}$$

To pass the limit through the integral above, note that $|e^{iux} - 1 - iux| \leq \frac{1}{2}u^2x^2$ and thus it suffices to show (using the dominated convergence theorem) that

$$\frac{1}{2} \int_0^\infty x \sum_{n=1}^\infty r_n \exp \left(-\frac{r_n x}{\lambda_n} \right) dx < \infty. \tag{14}$$

This follows since

$$\int_0^\infty x \sum_{n=1}^\infty r_n \exp \left(-\frac{r_n x}{\lambda_n} \right) dx = \sum_{n=1}^\infty r_n \int_0^\infty x \exp \left(-\frac{r_n x}{\lambda_n} \right) dx = \sum_{n=1}^\infty \frac{\lambda_n^2}{r_n} < \infty.$$

Thus (14) holds and hence the Lévy measure is given by (12). \square

The form (11) of the characteristic function will be useful when we study the Edgeworth expansion of the tail Y_M defined in (5), whose Lévy measure is given by

$$v^{(M)}(x) = \sum_{n=M}^\infty \frac{r_n}{x} \exp \left(-\frac{r_n x}{\lambda_n} \right).$$

3. Berry–Esseen bound

In this section, we show that under certain conditions on the sequences λ_n and r_n , the distribution of the tail

$$Y_M = \sum_{n=M}^\infty \lambda_n (\eta_n - 1) \tag{15}$$

is asymptotically normal as $M \rightarrow \infty$. A Berry–Esseen type bound for infinitely divisible random variables was studied in [3], and we will apply a similar method to the random variable Y_M .

Consider the normalized distribution

$$\tilde{Y}_M = \sigma_M^{-1} Y_M \tag{16}$$

where σ_M is defined in (6) and let

$$\tilde{v}^{(M)}(x) = \sigma_M v^{(M)}(\sigma_M x) = \frac{1}{x} \sum_{n=M}^\infty r_n \exp \left(-\frac{r_n x \sigma_M}{\lambda_n} \right), \quad x > 0 \tag{17}$$

be the density of the Lévy measure of \tilde{Y}_M . As the remark below indicates, $\tilde{Y}_M = \sigma_M^{-1} Y_M$ does not always converge to a normal distribution. To determine whether \tilde{Y}_M is asymptotically normal, we consider the third cumulant of \tilde{Y}_M which we denote by

$$\kappa_{3,M} \equiv \int_0^\infty x^3 \tilde{v}^{(M)}(x) dx = \sum_{n=M}^\infty r_n \int_0^\infty x^2 e^{-r_n x \sigma_M / \lambda_n} dx = 2\sigma_M^{-3} \sum_{n=M}^\infty \frac{\lambda_n^3}{r_n^2}. \tag{18}$$

A priori, there is no reason for $\kappa_{3,M}$ to be finite, as this depends on the sequences r_n and λ_n . The following theorem uses a Berry–Esseen bound to show that \tilde{Y}_M is asymptotically normal as long as $\kappa_{3,M} \rightarrow 0$. The constant 0.7056 appearing in this bound is the smallest known to date, see [17].

Theorem 3.1. *Let Y_M be given by (15) and suppose the sequences λ_n and r_n satisfy (3) and are such that*

$$\frac{\sum_{n=M}^{\infty} \frac{\lambda_n^3}{r_n^2}}{\left(\sum_{n=M}^{\infty} \frac{\lambda_n^2}{r_n}\right)^{3/2}} \rightarrow 0 \quad \text{as } M \rightarrow \infty. \tag{19}$$

Then,

$$\tilde{Y}_M = \sigma_M^{-1} Y_M \rightarrow N(0, 1)$$

as $M \rightarrow \infty$, and, we have

$$\sup_{x \in \mathbb{R}} \left| P[\tilde{Y}_M \leq x] - \Phi(x) \right| \leq 0.7056 \kappa_{3,M} \tag{20}$$

where Φ is the standard normal CDF and $\kappa_{3,M}$ is defined in (18).

Remark. Condition (19) implies that the skewness of \tilde{Y}_M tends to 0 as $M \rightarrow \infty$. It can easily be checked that condition (19) is satisfied, for example, if $r_n = r$ is constant and λ_n decays as a power law, i.e. if $\lambda_n \sim Cn^{-\alpha/2}$ for some $\alpha > 1$. However, (19) is *not* satisfied if λ_n decays exponentially, and in this case convergence to $N(0, 1)$ will not always hold. For example, suppose $r_n = 1$ is constant and $\lambda_n = 2^{-n}$. Then

$$Y_M = \sum \lambda_n (\eta_n - 1) = \sum \lambda_n \eta_n - \sum \lambda_n \geq - \sum \lambda_n,$$

and so

$$\sigma_M^{-1} Y_M \geq \left(- \sum_{n=M}^{\infty} \lambda_n \right) / \sqrt{\sum_{n=M}^{\infty} \lambda_n^2} = - \left(\sum_{n=M}^{\infty} 2^{-n} \right) / \sqrt{\sum_{n=M}^{\infty} 2^{-2n}} = -\sqrt{3}$$

for all M . Since the normalized random variable $\sigma_M^{-1} Y_M$ is bounded below, it cannot converge in distribution to $N(0, 1)$.

Proof. Since Y_M is infinitely divisible, for each $m \geq 1$, we have

$$Y_M = Y_{M,1}^{(m)} + Y_{M,2}^{(m)} + \dots + Y_{M,m}^{(m)}, \tag{21}$$

where $Y_{M,i}^{(m)}$, $i = 1, 2, \dots, m$ are i.i.d. with mean 0 and variance σ_M^2/m . Applying the Berry–Esseen Theorem [8, Theorem 7.6.1] to the sum (21), we have for any $m \geq 1$,

$$\begin{aligned} \sup_{x \in \mathbb{R}} \left| P \left[\sigma_M^{-1} Y_M \leq x \right] - \Phi(x) \right| &= \sup_{x \in \mathbb{R}} \left| P \left[\frac{1}{\sigma_M \sqrt{m}} (\sqrt{m} Y_M) \leq x \right] - \Phi(x) \right| \\ &\leq 0.7056 \frac{\mathbb{E} \left[\left| \sqrt{m} Y_{M,1}^{(m)} \right|^3 \right]}{\sigma_M^3 \sqrt{m}} \end{aligned}$$

$$= 0.7056 \frac{m \mathbb{E}[|Y_{M,1}^{(m)}|]^3}{\sigma_M^3}. \tag{22}$$

Using Lemma 3.1 in [3],

$$\lim_{m \rightarrow \infty} m (\mathbb{E}[|\sigma_M^{-1} Y_{M,1}^{(m)}|])^3 = \int_0^\infty x^3 \tilde{v}_D^{(M)}(x) dx = \kappa_{3,M}.$$

Thus, we let $m \rightarrow \infty$ in (22), which gives (20).

To see that the right hand side of this bound tends to 0 as $M \rightarrow \infty$, note that by (18) and (6),

$$\kappa_{3,M} = 2\sigma_M^{-3} \sum_{m=M}^\infty \frac{\lambda_m^3}{r_m^2} = 2 \left(\sum_{m=M}^\infty \frac{\lambda_m^2}{r_m} \right)^{-3/2} \sum_{m=M}^\infty \frac{\lambda_m^3}{r_m^2}, \tag{23}$$

which tends to 0 by the assumption (19), so that (22) implies convergence to $N(0, 1)$. This finishes the proof. \square

Remarks. 1. A similar result applies to normalized weighted sums of arrays $Y_K = \sum_{n=1}^\infty \lambda_{n,K}(\eta_{n,K} - 1)$, where for any $K \geq 1$, $\{\eta_{n,K}\}_{n \geq 1}$ is a sequence of independent $\text{Gamma}(r_n, 1/r_n)$ random variables. The conditions are

$$\text{Var}Y_K = \sum_{n=1}^\infty \lambda_{n,K}^2 / r_{n,K} < \infty \quad \text{and} \quad \lim_{K \rightarrow \infty} \frac{\sum_{n=1}^\infty \lambda_{n,K}^3 / r_{n,K}^2}{\left(\sum_{n=1}^\infty \lambda_{n,K}^2 / r_{n,K} \right)^{3/2}} = 0.$$

2. A sufficient condition for (19) is $\liminf_{n \rightarrow \infty} r_n = \infty$. To see this, recall that for $0 < a < 1$ and for any $x, y \geq 0$, one has $(x + y)^a \leq x^a + y^a$, so that

$$\left[\sum_{n=M}^\infty \lambda_n^3 / r_n^2 \right]^{2/3} \leq \sum_{n=M}^\infty \lambda_n^2 / r_n^{4/3} \leq \left(\min_{n \geq M} r_n \right)^{-1/3} \sum_{n=M}^\infty \lambda_n^2 / r_n$$

and hence as $M \rightarrow \infty$,

$$\left[\frac{\sum_{n=M}^\infty \lambda_n^3 / r_n^2}{\left(\sum_{n=M}^\infty \lambda_n^2 / r_n \right)^{3/2}} \right]^{2/3} \leq \left(\min_{n \geq M} r_n \right)^{-1/3} \rightarrow 0.$$

However, note that it is necessary to consider λ_n as well to ensure that condition (3) holds. Corollary 4.1 below does this.

3. As another example, if $\lambda_n \sim Cn^{-\alpha/2}$ for $\alpha > 1$ and r_n is constant, then $\kappa_{3,M} = O(M^{-1/2})$ (see (33) below), which describes the rate at which the right hand side of (20) tends to 0. While it is nice to have a practical bound on the error made when approximating the CDF of Y_M with that of a normal, this rate of convergence may be too slow. In the following sections, we improve this approximation by using Edgeworth expansions. These will do a better job of approximating the CDF of Y_M for small M , however it will no longer be easy to bound the error made in this approximation exactly.

4. Asymptotic behavior of the cumulants

In the following sections, we will suppose that the sequences λ_n and r_n satisfy

$$\lambda_n = \ell(n)n^{-\alpha/2} \tag{24}$$

$$r_n = \rho(n)n^\beta \tag{25}$$

where

$$\alpha + \beta > 1 \tag{26}$$

and ℓ are ρ are slowly varying functions at ∞ . With these assumptions, (3) is satisfied and

$$\sigma_M^2 \sim \int_M^\infty \frac{\ell(n)^2 n^{-\alpha}}{\rho(n)n^\beta} dn = \frac{\ell(M)^2}{\rho(M)(1 - \alpha - \beta)} M^{1-\alpha-\beta}. \tag{27}$$

Extending the definition of $\kappa_{3,M}$ in (18), we will denote all cumulants of $\tilde{Y}_M = \sigma_M^{-1}Y_M$ by (see [18, Theorem 7.4]),

$$\begin{aligned} \kappa_{k,M} &= \int_0^\infty x^k \tilde{\nu}^{(M)}(x) dx = \sum_{n=M}^\infty r_n \int_0^\infty x^{k-1} \exp\left(-\frac{r_n x \sigma_M}{\lambda_n}\right) dx \\ &= \frac{(k-1)!}{\sigma_M^k} \sum_{n=M}^\infty \frac{\lambda_n^k}{r_n^{k-1}}, \quad k \geq 2. \end{aligned} \tag{28}$$

Observe that $\kappa_{2,M} = 1$ and as $M \rightarrow \infty$, (24) and (27), (28) and properties of slowly varying functions imply

$$\kappa_{k,M} \sim (k-1)! \sigma_M^{-k} \int_M^\infty \frac{\ell(n)^k n^{-k\alpha/2}}{\rho(n)^{k-1} n^{(k-1)\beta}} dn \tag{29}$$

$$\sim C_k \left(\frac{\ell(M)^2}{\rho(M)}\right)^{-k/2} M^{-k(1-\alpha-\beta)/2} \left(\frac{\ell(M)^k}{\rho(M)^{k-1}}\right) M^{1-k\alpha/2-(k-1)\beta} \tag{30}$$

$$= \frac{C_k}{\rho(M)^{k/2-1}} M^{(1-k/2)(\beta+1)}, \quad k \geq 2 \tag{31}$$

for a constant C_k . Note that in particular, if $k = 3$, then $\kappa_{3,M}$ tends to 0 like $O(L(M)M^{-(\beta+1)/2})$, where L is a slowly varying function at ∞ . Observe that condition (19) is satisfied as long as $\beta > -1$. Therefore,

Corollary 4.1. *Suppose $\alpha + \beta > 1$ and $\beta > -1$. Then*

$$\tilde{Y}_M = \sigma_M^{-1}Y_M \xrightarrow{d} N(0, 1).$$

For $k \geq 3$, we also have that $\kappa_{k,M}$ tends to zero like a slowly varying function times $M^{-(1-k/2)(\beta+1)}$. In order to simplify notation, we will make the following definition.

Definition 4.2. *For sequences a_n and b_n , we write*

$$a_n = \tilde{O}(b_n) \tag{32}$$

to mean $a_n = O(L(n)b_n)$ for some slowly varying function $L(n)$ at infinity.

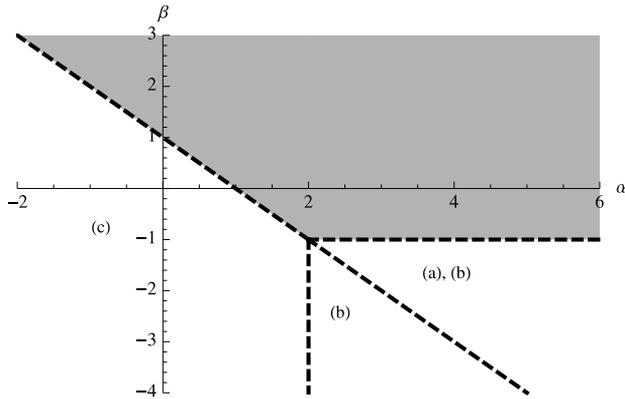


Fig. 1. The shaded region represents all α 's and β 's which satisfy the assumptions of Corollary 4.1 and Lemma 4.3. The regions labeled (a), (b) and (c) correspond to the regions described in Proposition 4.4.

Thus, (31) implies the following lemma.

Lemma 4.3. *If $\alpha + \beta > 1$ and $\beta > -1$, then*

$$\kappa_{k,M} = \tilde{O}(M^{(1-k/2)(1+\beta)}), \quad k \geq 2. \tag{33}$$

Moreover, if ρ is asymptotically constant, then \tilde{O} can be replaced with O .

The assumptions $\alpha + \beta > 1$ and $\beta > -1$ are necessary to ensure that the random variable Z is finite and that the third and higher cumulants of \tilde{Y}_M tend to 0. The following proposition shows what happens if $\beta < -1$ or $\alpha + \beta < 1$ under various renormalizations of $\sum_{n=M}^{\infty} \lambda_n (\eta_n - 1)$.

Proposition 4.4. (a) *Suppose $\alpha + \beta > 1$ and $\beta < -1$. Then*

$$\tilde{Y}_M = \sigma_M^{-1} Y_M \xrightarrow{d} 0.$$

(b) *Suppose $\beta < -1$ and $\alpha > 2$ and let $b_M = \sum_{n=M}^{\infty} \lambda_n$. Then*

$$b_M^{-1} Y_M = \frac{1}{b_M} \sum_{n=M}^{\infty} \lambda_n (\eta_n - 1) \xrightarrow{d} -1.$$

(c) *Suppose $\alpha + \beta < 1$ (in this case (3) fails) and $\alpha < 2$. Then*

$$\tilde{Y}_M = \sum_{n=M}^{\infty} \lambda_n (\eta_n - 1)$$

does not exist. (See Fig. 1)

Proof. For part (a), if $\beta < -1$, then for $\alpha + \beta > 1$ to hold, we must have $\alpha > 2$, and thus $\sum_{n=1}^{\infty} \lambda_n < \infty$. In this case, we have

$$\tilde{Y}_M = \frac{1}{\sigma_M} \sum_{n=M}^{\infty} \lambda_n (\eta_n - 1) = \frac{1}{\sigma_M} \sum_{n=M}^{\infty} \lambda_n \eta_n - \frac{1}{\sigma_M} \sum_{n=M}^{\infty} \lambda_n. \tag{34}$$

As $M \rightarrow \infty$, (27) implies that for some slowly varying function L_0 ,

$$\frac{1}{\sigma_M} \sum_{n=M}^{\infty} \lambda_n \sim L_0(M) M^{(\alpha+\beta-1)/2} M^{1-\alpha/2} = L_0(M) M^{1/2+\beta/2} \rightarrow 0 \tag{35}$$

since $\beta < -1$. Thus, it suffices to only consider the limit of the term $\frac{1}{\sigma_M} \sum_{n=M}^\infty \lambda_n \eta_n$, in (34). Because this random variable is not centered, the log of its characteristic function has the Lévy–Khintchine form

$$\log \mathbb{E} \exp \left(\frac{i\theta}{\sigma_M} \sum_{n=M}^\infty \lambda_n \eta_n \right) = \int_0^\infty (e^{i\theta x} - 1) \left(\frac{1}{x} \sum_{n=M}^\infty r_n e^{-r_n x \sigma_M / \lambda_n} \right) dx,$$

which is bounded by

$$\begin{aligned} \int_0^\infty \left| \frac{e^{i\theta x} - 1}{x} \right| \left(\sum_{n=M}^\infty r_n e^{-r_n x \sigma_M / \lambda_n} \right) dx &\leq |\theta| \sum_{n=M}^\infty r_n \int_0^\infty e^{-r_n x \sigma_M / \lambda_n} dx \\ &= |\theta| \sum_{n=M}^\infty \frac{\lambda_n}{\sigma_M} \rightarrow 0, \end{aligned} \tag{36}$$

which follows again from (35) and since $\beta < -1$. This verifies part (a).

For (b), let $\beta < -1$ and $\alpha > 2$. Since $\sum_{n=M}^\infty \lambda_n < \infty$, we have

$$\frac{1}{b_M} Y_M = \frac{1}{b_M} \sum_{n=M}^\infty \lambda_n (\eta_n - 1). \tag{37}$$

Since b_M is decreasing in M , the sum on the right hand side of (37) is non-negative and is bounded above by $\sum_{n=M}^\infty \frac{\lambda_n}{b_n} \eta_n$. The log of the moment generating function of this bound is given by

$$\log \mathbb{E} \exp \left(\theta \sum_{n=M}^\infty \frac{\lambda_n}{b_n} \eta_n \right) = \sum_{n=M}^\infty r_n \log \left(1 + \frac{\theta \lambda_n}{b_n r_n} \right). \tag{38}$$

Since $\beta < -1$, $\sum_{n=M}^\infty r_n < \infty$. Also, $\lambda_n / (r_n b_n) = L_1(n) n^{-(1+\beta)} \rightarrow \infty$, for some slowly varying function L_1 . Thus, $L_2(n) = \rho(n) \log(1 + \theta \lambda_n / (r_n b_n))$ is slowly varying and the right hand side of (38) will tend to 0 like

$$\sum_{n=M}^\infty r_n \log \left(1 + \frac{\theta \lambda_n}{r_n b_n} \right) \sim \sum_{n=M}^\infty n^\beta L_2(n) \sim L_2(M) M^{1+\beta} \rightarrow 0 \quad \text{as } M \rightarrow \infty.$$

Hence, with this normalization, $\frac{1}{b_M} \sum_{n=M}^\infty \lambda_n \eta_n \xrightarrow{d} 0$, and (37) implies that $\frac{1}{b_M} Y_M \xrightarrow{d} -1$.

Finally, for part (c), assume $\alpha + \beta < 1$ and $\alpha < 2$. The log of the characteristic function of $\sum_{n=M}^N \lambda_n (\eta_n - 1)$ for $N > M$ is given by

$$\log \mathbb{E} \exp \left(i\theta \sum_{n=M}^N \lambda_n (\eta_n - 1) \right) = \sum_{n=M}^N \left[r_n \log \left(1 - \frac{i\theta \lambda_n}{r_n} \right) - i\theta \lambda_n \right]. \tag{39}$$

We claim that the limit of sum does not exist as $N \rightarrow \infty$. There are two cases to consider depending on the behavior of λ_n / r_n . If $\lambda_n / r_n \rightarrow 0$ (i.e. if $-\alpha/2 - \beta < 0$), then by taking a series expansion of log, the terms in the sum on the right hand side above are asymptotic to λ_n^2 / r_n , which is not summable if $\alpha + \beta < 1$.

On the other hand, if $\lambda_n / r_n \rightarrow \infty$, then (39) can be rewritten as

$$\sum_{n=M}^N \lambda_n \left[\frac{r_n}{\lambda_n} \log \left(1 - \frac{i\theta \lambda_n}{r_n} \right) - i\theta \right]. \tag{40}$$

Since r_n/λ_n tends to 0 in this case, the terms under the sum are asymptotic to $-i\theta\lambda_n$, which is not summable since $\alpha < 2$. Thus, the infinite sum $\sum_{n=M}^\infty \lambda_n(\eta_n - 1)$ does not exist in these cases. \square

Remark. 1. What happens on the boundaries $I_1 = \{\alpha > 2, \beta = -1\}$, $I_2 = \{\alpha = 2, \beta < -1\}$ and $I_3 = \{\alpha + \beta = 1, \beta > -1\}$ depends on the slowly varying functions ℓ and ρ .

For I_1 , note that in terms of the slowly varying functions ℓ and ρ , $L_0(M) = \frac{\sqrt{\rho(M)}}{\ell(M)} \cdot \ell(M) = \sqrt{\rho(M)}$. Thus, part (a) in Proposition 4.4 still holds if $\beta = -1$ and $\rho(M) \rightarrow 0$. On the other hand, if $\beta = -1$ and $\rho(M) \rightarrow \infty$, then (19) holds and the limit is normal.

For I_2 , note that the same result will hold in (b) as long as $b_M < \infty$, which happens if $\sum_{n=M}^\infty \lambda_n = \sum_{n=M}^\infty \ell(n)n^{-1} < \infty$. If this sum is infinite, then the result in (b) will not hold because b_M does not exist.

Finally, for I_3 , suppose that $\alpha + \beta = 1$, but σ_M is finite (which requires the sequence $\ell(n)^2\rho(n)^{-1}n^{-1}$ to be summable). To see if the limit of \tilde{Y}_M is normal, we can use condition (19) in the case where $\alpha + \beta = 1$. In this case, a computation shows

$$\frac{\sum_{n=M}^\infty \lambda_n^3/r_n^2}{\left(\sum_{n=M}^\infty \lambda_n^2/r_n\right)^{3/2}} \sim \frac{1}{\sqrt{\rho(M)}} M^{-\alpha/2-\beta} \tag{41}$$

which tends to 0 as $M \rightarrow \infty$ since $-\alpha/2 - \beta < 0$ if $\alpha + \beta = 1$ and $\alpha < 2$. Thus if σ_M is finite and $(\alpha, \beta) \in I_3$, then the limit of \tilde{Y}_M is normal by Theorem 3.1.

5. An approximation lemma

Sections 3 and 4 showed that if $\alpha + \beta > 1$ and $\beta > -1$, the tail Y_M can be approximated by a normal distribution for large M . We shall improve the approximation to the CDF of Y_M using an Edgeworth expansion. To establish the Edgeworth expansion, we will need a lemma involving an approximation of the characteristic function of \tilde{Y}_M by a polynomial involving the cumulants.

In view of Proposition 2.1, the difference between the log of the characteristic function of \tilde{Y}_M and that of a standard normal is given by the following function:

$$I_M(u) \equiv \int_0^\infty \left(e^{iux} - 1 - iux - \frac{(iux)^2}{2} \right) \tilde{v}^{(M)}(x) dx, \tag{42}$$

which can be rewritten as

$$I_M(u) = \int_0^\infty (e^{iux} - 1 - iux) \tilde{v}^{(M)}(x) dx - \left(-\frac{u^2}{2} \right), \tag{43}$$

since $\int_0^\infty x^2 \tilde{v}^{(M)}(x) dx = \kappa_{2,M} = 1$. A key step in developing an Edgeworth expansion is approximating the function $e^{I_M(u)}$ by a polynomial involving the cumulants, which is done in the following lemma.

Lemma 5.1. Suppose λ_n and r_n satisfy (24) and (25) with $\alpha + \beta > 1$ and $\beta > -1$. Then for $N \geq 3$ and $u > 0$, we have as $M \rightarrow \infty$,

$$\left| e^{I_M(u)} - \left[1 + \sum_{\eta(N)} \left(\prod_{m=3}^N \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M} \right)^{k_m} \right) \right] \right|$$

$$\leq Q_N(u) + \frac{u^{3N-3}}{(3!)^{N-1}(N-1)!} \kappa_{3,M}^{N-1} \exp\left(\frac{u^3}{6} \kappa_{3,M}\right), \tag{44}$$

where $\eta(N)$ denotes all non-negative indices k_3, k_4, \dots, k_N such that

$$1 \leq k_3 + 2k_4 + \dots + (N-2)k_N \leq N-2 \tag{45}$$

and $|Q_N(u)|$ is bounded by a polynomial in u whose coefficients are $\tilde{O}\left(M^{-\left(\frac{N-1}{2}\right)(1+\beta)}\right)$ as $M \rightarrow \infty$. If ρ is asymptotically constant, then \tilde{O} can be replaced by O .

Remark. This bound is a complicated function of u , but this will cause no problem because in the proof of Theorem 6.2 below, this bound is multiplied by $e^{-u^2/2}$ and integrated over $u \in [0, \kappa_{3,M}^{-1}]$.

Proof. By using Taylor’s Theorem on the function e^{iux} for $u \geq 0$, we have for each $N \geq 2$,

$$\begin{aligned} I_M(u) &= \int_0^\infty \left(e^{iux} - 1 - iux - \frac{(iux)^2}{2} \right) \tilde{v}^{(M)}(dx) \\ &= \int_0^\infty \left(\sum_{m=3}^N \frac{(iux)^m}{m!} + R_N(ux) \right) \tilde{v}^{(M)}(dx) \end{aligned} \tag{46}$$

where R_N is a remainder which satisfies

$$|R_N(ux)| \leq \frac{(ux)^{N+1}}{(N+1)!}.$$

Using the definition (28) of $\kappa_{k,M}$, I_M becomes

$$I_M(u) = \sum_{m=3}^N \frac{(iu)^m}{m!} \kappa_{m,M} + \tilde{R}_N(u) \tag{47}$$

where now,

$$|\tilde{R}_N(u)| \leq \int_0^\infty \frac{(ux)^{N+1}}{(N+1)!} \tilde{v}^{(M)}(x) dx = \frac{u^{N+1}}{(N+1)!} \kappa_{N+1,M}. \tag{48}$$

Note that $I_M(u) = \tilde{R}_2(u)$, which follows from (48) by setting $N = 2$.

Turning now to $\exp(I_M(u))$, we apply the classical inequality

$$\left| e^z - \sum_{n=0}^r \frac{z^n}{n!} \right| \leq \frac{z^{r+1}}{r!} e^{|z|}, \quad z \in \mathbb{R}, r \geq 0 \tag{49}$$

to $\exp(I_M(u))$ and using (48), we get

$$\begin{aligned} \left| \exp(I_M(u)) - \sum_{n=0}^{N-2} \frac{I_M(u)^n}{n!} \right| &\leq \frac{|I_M(u)|^{N-1}}{(N-1)!} \exp(|I_M(u)|) \\ &= \frac{|\tilde{R}_2(u)|^{N-1}}{(N-1)!} \exp(|\tilde{R}_2(u)|) \\ &\leq \frac{u^{3(N-1)}}{(3!)^{N-1}(N-1)!} \kappa_{3,M}^{N-1} \exp\left(\frac{u^3}{3!} \kappa_{3,M}\right). \end{aligned} \tag{50}$$

Thus, by adding and subtracting $\sum_{n=1}^{N-2} \frac{I_M(u)^n}{n!}$ on the left hand side of (44), we have

$$\begin{aligned} & \left| \exp(I_M(u)) - \left(1 + \left[\sum_{\eta(N)} \left(\prod_{m=3}^N \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M} \right)^{k_m} \right) \right] \right) \right| \\ & \leq \left| \exp(I_M(u)) - \sum_{n=0}^{N-2} \frac{I_M(u)^n}{n!} \right| \\ & \quad + \left| \sum_{n=0}^{N-2} \frac{I_M(u)^n}{n!} - \left(\left[\sum_{\eta(N)} \left(\prod_{m=3}^N \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M} \right)^{k_m} \right) \right] \right) \right|. \end{aligned} \tag{51}$$

Note that (50) gives a bound for the first term in (51). Thus, to finish the proof, it remains to bound the second term in (51). To do this, fix $1 \leq n \leq N - 2$ and observe that (47) implies

$$\frac{I_M(u)^n}{n!} = \frac{1}{n!} \left(\sum_{m=3}^N \frac{(iu)^m}{m!} \kappa_{m,M} + \tilde{R}_N(u) \right)^n. \tag{52}$$

Applying the multinomial theorem, this becomes

$$\begin{aligned} \frac{I_M(u)^n}{n!} &= \frac{1}{n!} \sum_{\{k_m\}_n} \binom{n}{k_3, k_4, \dots, k_N, k_{N+1}} \left[\prod_{m=3}^N \left(\frac{(iu)^m}{m!} \kappa_{m,M} \right)^{k_m} \right] \tilde{R}_N^{k_{N+1}} \\ &= \sum_{\{k_m\}_n} \left[\prod_{m=3}^N \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M} \right)^{k_m} \right] \frac{\tilde{R}_N^{k_{N+1}}}{k_{N+1}!} \end{aligned} \tag{53}$$

where $\{k_m\}_n$ denotes all sets of non-negative integers k_m , $3 \leq m \leq N + 1$ such that $k_3 + k_4 + \dots + k_N + k_{N+1} = n$. By (33), $\kappa_{m,M} = O(1)$. Moreover, by (48) and (33), $|\tilde{R}_N| \leq \frac{u^{N+1}}{(N+1)!} \kappa_{N+1,M} \sim \frac{u^{N+1}}{(N+1)!} \tilde{O}(M^{-(1+\beta)(N-1)/2})$, thus any term in (53) involving \tilde{R}_N (that is with $k_{N+1} \geq 1$) can be grouped into a function $Q_{n,N}^{(1)}(u)$ which is bounded by a polynomial with positive coefficients which are $\tilde{O}(M^{-(1+\beta)(N-1)/2})$. Doing this, (53) becomes

$$\frac{I_M(u)^n}{n!} = \sum_{\{k_m\}'_n} \left[\prod_{m=3}^N \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M} \right)^{k_m} \right] + Q_{n,N}^{(1)}(u) \tag{54}$$

where $\{k_m\}'_n$ denotes all k_m , $3 \leq m \leq N$ such that $k_3 + k_4 + \dots + k_N = n$. In the remaining sum, the coefficients are

$$\prod_{m=3}^N \frac{1}{k_m! (m!)^{k_m}} \kappa_{m,M}^{k_m}. \tag{55}$$

Using (33) again, these coefficients are of the order

$$\begin{aligned} \prod_{m=1}^N \kappa_{m,M}^{k_m} &= \tilde{O} \left(M^{- (1+\beta) \sum_{m=3}^N k_m \left(\frac{m-2}{2} \right)} \right) \\ &= \tilde{O} \left(M^{- (1+\beta) \frac{1}{2} \left[\sum_{m=3}^N m k_m - 2 \sum_{m=3}^N k_m \right]} \right) \end{aligned}$$

$$= \tilde{O} \left(M^{-(1+\beta)\frac{1}{2} \left[\sum_{m=3}^N mk_m - 2n \right]} \right). \tag{56}$$

We shall now isolate the terms in the sum (54) for which

$$\sum_{m=3}^N mk_m \geq N + 2n - 1. \tag{57}$$

They form a polynomial $Q_{n,N}^{(2)}(u)$ whose coefficients by (56) are of the order

$$\tilde{O} \left(M^{-(1+\beta)\frac{1}{2} \left[\sum_{m=3}^N mk_m - 2n \right]} \right) = \tilde{O} \left(M^{-(1+\beta)\frac{1}{2} [N+2n-1-2n]} \right) = \tilde{O} \left(M^{-(1+\beta)(N-1)/2} \right),$$

where we have used the fact that the k_m 's are chosen to satisfy (57). Thus,

$$\frac{I_M(u)^n}{n!} = \sum_{\{k_m\}'_n} \left[\prod_{m=3}^N \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M} \right)^{k_m} \right] + Q_{n,N}^{(1)}(u) + Q_{n,N}^{(2)}(u) \tag{58}$$

where $\{k_m\}'_n$ denotes all $k_m, 3 \leq m \leq N$ such that $\sum k_m = n$ and $\sum mk_m \leq N + 2n - 2$. Note that by combining these two inequalities, the k_m 's in this sum also satisfy

$$\sum_{m=3}^N (m - 2)k_m \leq (N + 2n - 2) - 2n = N - 2. \tag{59}$$

Now, returning to the second term in (51), in light of (58), we have

$$\begin{aligned} \sum_{n=1}^{N-2} \frac{I_M(u)^n}{n!} &= \left(\sum_{n=1}^{N-2} \left[\sum_{\{k_m\}'_n} \left(\prod_{m=3}^n \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M} \right)^{k_m} \right) \right] \right) \\ &\quad + \sum_{n=1}^{N-2} [Q_{n,N}^{(1)}(u) + Q_{n,N}^{(2)}(u)] \\ &= \left(\sum_{n=1}^{N-2} \left[\sum_{\{k_m\}'_n} \left(\prod_{m=3}^n \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M} \right)^{k_m} \right) \right] \right) + Q_N(u) \end{aligned} \tag{60}$$

where $Q_N(u) := \sum_{n=1}^{N-2} [Q_{n,N}^{(1)}(u) + Q_{n,N}^{(2)}(u)]$ is bounded by a polynomial in u whose coefficients are $\tilde{O}(M^{-(1+\beta)\left(\frac{N-1}{2}\right)})$. As for the double sum on the right hand side of (60), observe that from (59), this can be rewritten as the (single) sum over all $k_i, 3 \leq i \leq N$ such that

$$1 \leq k_3 + k_4 + \dots + k_N \leq N - 2 \quad \text{and} \quad k_3 + 2k_4 + \dots + (N - 2)k_N \leq N - 2.$$

Since $k_3 + k_4 + \dots + k_N \leq k_3 + 2k_4 + \dots + (N - 2)k_N$, these two conditions are satisfied if and only if

$$1 \leq k_3 + 2k_4 + \dots + (N - 2)k_N \leq N - 2,$$

which is the definition of $\eta(N)$ in (45). Thus,

$$\left| \sum_{n=1}^{N-2} \frac{I_M(u)^n}{n!} - \sum_{\eta(N)} \prod_{m=3}^N \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M} \right)^{k_m} \right| \leq |Q_N(u)|. \tag{61}$$

This bounds the second term in (51) and completes the proof. \square

6. Edgeworth expansions

We shall improve the approximation to the CDF of Y_M using an Edgeworth expansion. A two-term Edgeworth expansion of a general sequence of infinitely divisible distributions are studied in [10]. We apply a similar method to our case, but with an Edgewood expansion to any order.

Given a CDF F of a random variable X and a function G (not necessarily a CDF), we let d denote the supremum norm of the difference $F - G$:

$$d(F, G) = \sup_{x \in \mathbb{R}} |F(x) - G(x)|.$$

We can bound $d(F, G)$ using the characteristic function of X and the Fourier–Stieltjes transform of G . This is done in the following lemma which is proved in [5, Lemma 12.2].

Lemma 6.1. *Let ϕ be a characteristic function of a random variable X with CDF F . Let G be a function for which*

$$\lim_{x \rightarrow -\infty} G(x) = 0, \quad \lim_{x \rightarrow \infty} G(x) = 1, \quad \text{and} \quad \sup_{x \in \mathbb{R}} |G'(x)| < C,$$

for some constant C and let $g(u) = \int_{\mathbb{R}} e^{iux} dG(x)$ be the Fourier–Stieltjes transform of G . Furthermore, suppose that

$$\int_{\mathbb{R}} |x| dF(x) < \infty \quad \text{and} \quad \int_{\mathbb{R}} |x| dG(x) < \infty.$$

Then for every $U > 0$ and $t > t_0$,

$$d(F, G) \leq \frac{1}{4h(t) - \pi} \int_0^U |\phi(u) - g(u)| \frac{du}{u} + 4th(t) \frac{C}{U} \tag{62}$$

where h and t_0 are defined as

$$h(t) = \int_0^t \frac{\sin^2(x)}{x^2} dx, \quad t > 0, \quad \text{and} \quad h(t_0) = \frac{\pi}{4}.$$

This lemma involves two parameters t and U , which must balance each other (making U large decreases the second term on the right hand side of (62) and increases the first, and t has the opposite effect). In our application, U will tend to infinity and t will be an unspecified constant. This lemma will be used to study the convergence of an Edgewood expansion for \tilde{Y}_M .

We can now state a theorem detailing the convergence rate of an Edgeworth expansion for the CDF of \tilde{Y}_M as $M \rightarrow \infty$. Recall the *Hermite polynomials* which can be defined as $H_0(x) = 1$ and

$$H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}, \quad k \geq 1,$$

see [9, page 157]. A simple induction shows that H_k also satisfies the recursion formula

$$H_{k+1}(x) = -e^{x^2/2} \frac{d}{dx} \left(H_k(x)e^{-x^2/2} \right), \quad k \geq 0. \tag{63}$$

The first few H_k are given by $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$, $H_4(x) = x^4 - 6x^2 + 3$, $H_5(x) = x^5 - 10x^3 + 15x, \dots$

The following theorem provides an Edgeworth expansion of \tilde{Y}_M up to an arbitrary order $N \geq 2$.

Theorem 6.2. *Suppose λ_n and r_n satisfy (24) and (25) with $\alpha + \beta > 1$ and $\beta > -1$. As $M \rightarrow \infty$, for each $N \geq 2$, the CDF of \tilde{Y}_M satisfies*

$$P[\tilde{Y}_M \leq x] = \Phi(x) - \phi(x) \left\{ \sum_{\eta(N)} \left[\prod_{m=1}^N \frac{1}{k_m!} \left(\frac{\kappa_{m,M}}{m!} \right)^{k_m} \right] H_{\zeta(k_3, \dots, k_N)}(x) \right\} + \tilde{O} \left(M^{-(1+\beta)\frac{N-1}{2}} \right), \tag{64}$$

where Φ and ϕ denote the standard normal CDF and PDF, $\eta(N)$ denotes all non-negative indices k_3, k_4, \dots, k_N such that

$$1 \leq k_3 + 2k_4 + \dots + (N - 2)k_N \leq N - 2 \tag{65}$$

and

$$\zeta(k_1, \dots, k_N) = 3k_3 + 4k_4 + \dots + Nk_N - 1. \tag{66}$$

The error $\tilde{O}(M^{-(1+\beta)(N-1)/2})$ is uniform for all $x \in \mathbb{R}$ and if ρ is a constant function, then \tilde{O} can be replaced with O .

For example, if $N = 2$, there is no solution to (65). If $N = 3$, the only solution to (65) is $k_3 = 1$. If $N = 4$, we have the additional solutions $k_3 = 2, k_4 = 0$ and $k_3 = 0, k_4 = 1$. Thus, for small values of N , the right hand side of (64) becomes

$$\begin{aligned} \mathbf{N} = 2: \quad & P[\tilde{Y}_M \leq x] = \Phi(x) + \tilde{O}(M^{-(1+\beta)/2}) \\ \mathbf{N} = 3: \quad & P[\tilde{Y}_M \leq x] = \Phi(x) - \phi(x) \frac{H_2(x)}{3!} \kappa_{3,M} + \tilde{O}(M^{-(1+\beta)}) \\ \mathbf{N} = 4: \quad & P[\tilde{Y}_M \leq x] = \Phi(x) - \phi(x) \left[\frac{H_2(x)}{3!} \kappa_{3,M} + \left(\frac{H_3(x)}{4!} \kappa_{4,M} + \frac{H_5(x)}{(2!)(3!)^2} \kappa_{3,M}^2 \right) \right] \\ & + \tilde{O}(M^{-3(1+\beta)/2}) \\ \mathbf{N} = 5: \quad & P[\tilde{Y}_M \leq x] = \Phi(x) - \phi(x) \left[\frac{H_2(x)}{3!} \kappa_{3,M} + \left(\frac{H_3(x)}{4!} \kappa_{4,M} + \frac{H_5(x)}{(2!)(3!)^2} \kappa_{3,M}^2 \right) \right] \\ & + \left(\frac{H_4(x)}{5!} \kappa_{5,M} + \frac{H_6(x)}{(3!)(4!)} \kappa_{4,M} \kappa_{3,M} + \frac{H_8(x)}{(3!)^4} \kappa_{3,M}^3 \right) \\ & + \tilde{O}(M^{-2(1+\beta)}). \end{aligned}$$

A more revealing (but slightly more complicated) statement of Theorem 6.2 is

$$P[\tilde{Y}_M \leq x] = \Phi(x) - \phi(x) \left\{ \sum_{n=3}^N \left(\sum_{\eta'(n)} \left[\prod_{m=3}^n \frac{1}{k_m!} \left(\frac{\kappa_{m,M}}{m!} \right)^{k_m} \right] H_{\zeta(k_3, \dots, k_n)}(x) \right) \right\}$$

$$+ \tilde{O}\left(M^{-(1+\beta)\frac{N-1}{2}}\right), \tag{67}$$

where $\eta'(n)$ denotes all k_3, k_4, \dots, k_N such that $k_3 + 2k_4 + \dots + (n - 2)k_N = n - 2$. In this form, it is clear that additional terms appear in the expansion as you increase n from 3 to N .

The following immediate corollary is used in [21] to study the Rosenblatt distribution.

Corollary 6.3. *Let $\tilde{Y}_M = \sum_{n=M}^\infty \lambda_n(\epsilon_n^2 - 1)$ where ϵ_n are i.i.d. $N(0, 1)$ random variables and $\lambda_n = \ell(n)n^{-\alpha/2}$ where $\alpha \geq 1$ and ℓ is slowly varying at ∞ . Then the statement of Theorem 6.2 holds with $\beta = 0$, \tilde{O} replaced by O , and $\kappa_{m,M} = O(M^{1-m/2})$.*

Proof. Since the ϵ_n^2 's are i.i.d. chi-squared with one degree of freedom, one has $r_n = 1/2$ for each $n \geq 1$. The conclusion follows from (33) and Theorem 6.2. \square

Proof of Theorem 6.2. Define $G(x)$ as

$$G(x) = \Phi(x) - \phi(x) \left\{ \sum_{\eta(N)} \left[\prod_{m=1}^N \frac{1}{k_m!} \left(\frac{\kappa_{m,M}}{m!}\right)^{k_m} \right] H_{\zeta(k_3, \dots, k_N)}(x) \right\}. \tag{68}$$

Then by (63), we also have

$$\frac{dG}{dx} = \phi(x) \left(1 + \sum_{\eta(N)} \left[\prod_{m=1}^N \frac{1}{k_m!} \left(\frac{\kappa_{m,M}}{m!}\right)^{k_m} \right] H_{\zeta(k_3, \dots, k_N)+1} \right).$$

Using this and the fact that

$$\int_{\mathbb{R}} H_k(x)\phi(x)e^{iux} dx = (-1)^k \int_{\mathbb{R}} \left(\frac{d^k}{dx^k}\phi(x)\right) e^{iux} dx = (iu)^k e^{-u^2/2},$$

we note that the Fourier–Stieltjes transform of G is given by

$$\begin{aligned} g(u) &= \int_{\mathbb{R}} e^{iux} dG(x) \\ &= \exp\left(-\frac{u^2}{2}\right) \left(1 + \sum_{\eta(N)} \left(\prod_{m=3}^N \frac{1}{k_m!} \left(\frac{\kappa_{m,M}}{m!}\right)^{k_m} (iu)^{\zeta(k_3, \dots, k_N)+1} \right) \right) \\ &= \exp\left(-\frac{u^2}{2}\right) \left(1 + \sum_{\eta(N)} \left(\prod_{m=3}^N \frac{1}{k_m!} \left(\frac{(iu)^m \kappa_{m,M}}{m!}\right)^{k_m} \right) \right), \end{aligned} \tag{69}$$

where we have used the definition of ζ in (66). Let $\varphi^{(M)}(u)$ be the characteristic function of \tilde{Y}_M :

$$\varphi^{(M)}(u) = \exp\left(\int_0^\infty (e^{iux} - 1 - iux)\tilde{\nu}^{(M)}(x)dx\right). \tag{70}$$

Since $N \geq 2$, choose $\epsilon > 0$ such that

$$\kappa_{3,M}^{-1} < \epsilon \kappa_{N+1,M}^{-1}$$

for all $M \geq 1$ (this exists by (33)). To show (64) using Lemma 6.1, it suffices to show that

$$J_M := \int_0^{U_M} |\varphi^{(M)}(u) - g(u)| \frac{du}{u} = \tilde{O}\left(M^{-(1+\beta)(N-1)/2}\right), \tag{71}$$

where

$$U_M := \epsilon \kappa_{N+1, M}^{-1} = \tilde{O}(M^{(1+\beta)(N-1)/2}) \tag{72}$$

from (33). Note that with this choice of U_M , the second term on the right hand side of (62) is already of order $O(U_M^{-1}) = \tilde{O}(M^{-(1+\beta)(N-1)/2})$ and thus we need to only bound J_M .

Using (43), note that

$$\varphi^{(M)}(u) = \exp\left(\int_0^\infty (e^{iux} - 1 - iux)\tilde{v}^{(M)}(x)dx\right) = \exp\left(I_M(u) - \frac{u^2}{2}\right).$$

Using this and the definition of g in (69), we can break up the integral J_M in (71) as

$$\begin{aligned} J_M &= \int_0^{U_M} \exp\left(-\frac{u^2}{2}\right) \left(\exp(I_M(u)) \right. \\ &\quad \left. - \left(1 + \sum_{\eta(N)} \left(\prod_{m=3}^N \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m, M}\right)^{k_m}\right)\right) \right) \frac{du}{u} \\ &:= J_{M,1} + J_{M,2} + J_{M,3}, \end{aligned}$$

where

$$\begin{aligned} J_{M,1} &= \int_0^{\kappa_{3, M}^{-1}} \exp\left(-\frac{u^2}{2}\right) \left(\exp(I_M(u)) \right. \\ &\quad \left. - \left(1 + \sum_{\eta(N)} \left(\prod_{m=3}^N \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m, M}\right)^{k_m}\right)\right) \right) \frac{du}{u} \\ J_{M,2} &= \int_{\kappa_{3, M}^{-1}}^{U_M} \exp\left(-\frac{u^2}{2} + I_M(u)\right) \frac{du}{u} \\ J_{M,3} &= - \int_{\kappa_{3, M}^{-1}}^{U_M} \exp\left(-\frac{u^2}{2}\right) \left(1 + \sum_{\eta(N)} \left(\prod_{m=3}^N \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m, M}\right)^{k_m}\right)\right) \frac{du}{u}. \end{aligned}$$

We will now show that $J_{M,i} = \tilde{O}(M^{-(1+\beta)(N-1)/2})$, $i = 1, 2, 3$, which with the help of Lemma 6.1, will imply the result.

Estimate for $J_{M,1}$: From Lemma 5.1, we have that

$$|J_{M,1}| \leq \int_0^{\kappa_{3, M}^{-1}} \exp\left(-\frac{u^2}{2}\right) \left| Q_N(u) + \frac{u^{3N-3}}{(3!)^{N-1}(N-1)!} \kappa_{3, M}^{N-1} \exp\left(\frac{u^3}{6} \kappa_{3, M}\right) \right| \frac{du}{u}$$

where Q_N is bounded by a polynomial in u whose coefficients are $\tilde{O}(M^{-(1+\beta)(N-1)/2})$. Thus,

$$\begin{aligned} |J_{M,1}| &\leq \int_0^{\kappa_{3, M}^{-1}} \exp\left(-\frac{u^2}{2}\right) |Q_N(u)| \frac{du}{u} \\ &\quad + \int_0^{\kappa_{3, M}^{-1}} \exp\left(-\frac{u^2}{3}\right) \frac{u^{3N-4}}{(3!)^{N-1}(N-1)!} \kappa_{3, m}^{N-1} du, \end{aligned} \tag{73}$$

since on the interval $0 < u < \kappa_{3, M}^{-1}$, we have $u\kappa_{3, M} \leq 1$ and

$$\exp\left(-\frac{u^2}{2}\right) \exp\left(\frac{u^3}{6} \kappa_{3, M}\right) \leq \exp\left(-\frac{u^2}{3}\right).$$

The first term in (73) is $\tilde{O}(M^{-(1+\beta)(N-1)/2})$ since all coefficients of Q_N are of this order. The second term in (73) is also of order $\tilde{O}(M^{-(1+\beta)(N-1)/2})$ by (33). Thus, $J_{M,1} = \tilde{O}(M^{-(1+\beta)(N-1)/2})$.

Estimate for $J_{M,2}$: We will in fact show that $J_{2,M} = o(M^{-(1+\beta)(N-1)/2})$. First, observe that

$$|J_{2,M}| \leq \int_{\kappa_{3,M}^{-1}}^{U_M} \exp\left(-\frac{u^2}{2} + \operatorname{Re}[I_M(u)]\right) \frac{du}{u}. \tag{74}$$

Thus, we must show that the integrand tends to zero fast enough. Note, using (43), that

$$-\frac{u^2}{2} + \operatorname{Re}[I_M(u)] = -\int_0^\infty (1 - \cos(ux)) \tilde{v}^{(M)}(x) dx.$$

Using (17), we compute this integral:

$$\begin{aligned} A_M(u) &:= \int_0^\infty (1 - \cos(ux)) \tilde{v}^{(M)}(x) dx \\ &= \int_0^\infty (1 - \cos(ux)) \left(\sum_{n=M}^\infty r_n \frac{e^{-xr_n\sigma_M/\lambda_n}}{x} \right) dx \\ &= \sum_{n=M}^\infty r_n \int_0^\infty \frac{1 - \cos(ux)}{x} e^{-xr_n\sigma_M/\lambda_n} dx \\ &= f \sum_{n=M}^\infty r_n \int_0^\infty \int_0^u \sin(tx) e^{-xr_n\sigma_M/\lambda_n} dt dx \\ &= \sum_{n=M}^\infty r_n \int_0^u \left(\int_0^\infty \sin(tx) e^{-xr_n\sigma_M/\lambda_n} dx \right) dt \\ &= \sum_{n=M}^\infty r_n \int_0^u \left(\frac{t}{t^2 + \frac{r_n^2\sigma_M^2}{\lambda_n^2}} \right) dt \\ &= \frac{1}{2} \sum_{n=M}^\infty r_n \log \left(1 + \frac{u^2\lambda_n^2}{r_n^2\sigma_M^2} \right), \end{aligned} \tag{75}$$

where we have used the integral identity $\int_0^\infty \sin(tx) e^{-zx} dx = t/(t^2 + z^2)$ in the fourth line, which can be shown by integration by parts.

Using the properties of slowly varying functions, we can find $\alpha' \geq \alpha \geq \alpha''$, $\beta' \geq \beta \geq \beta''$ and constants C_1, C_2, D_1, D_2 such that $\alpha'' + \beta'' > 1$, $\beta'' > -1$ and

$$\begin{aligned} C_1 n^{-\alpha'/2} &\leq \lambda_n \leq C_2 n^{-\alpha''/2} \\ D_1 n^{\beta''} &\leq r_n \leq D_2 n^{\beta'}. \end{aligned}$$

To simplify things, we will use “C” and “D” below to denote generic constants. Since (75) is increasing in u , on the interval $\kappa_{3,M}^{-1} < u < U_M$, as $M \rightarrow \infty$,

$$A_M(u) \geq A_M(\kappa_{3,M}^{-1}) = \frac{1}{2} \sum_{n=M}^\infty r_n \log \left(1 + \frac{\lambda_n^2}{r_n^2 \kappa_{3,M}^2 \sigma_M^2} \right)$$

$$\gtrsim D \int_M^\infty y^{\beta''} \log \left(1 + \frac{C y^{-\alpha' - 2\beta'}}{\kappa_{3,M}^2 \sigma_M^2} \right) dy \tag{76}$$

$$= D \int_M^\infty y^{\beta''} \log \left(1 + (B_M y)^{-\alpha' - 2\beta'} \right) dy \tag{77}$$

where

$$B_M = \left(\frac{C}{\kappa_{3,M}^2 \sigma_M^2} \right)^{-1/(\alpha' + 2\beta')}$$

Making the change of variables $w = B_M y$, the integral (77) becomes

$$\frac{D}{B_M^{1+\beta''}} \int_{MB_M}^\infty w^{\beta''} \log \left(1 + w^{-\alpha' - 2\beta'} \right) dw.$$

Eqs. (27) and (33) imply

$$\kappa_{3,M}^2 \sigma_M^2 \lesssim C M^{-(1+\beta'')} M^{1-\alpha''-\beta''} = C M^{-\alpha''-2\beta''}$$

for a constant $C > 0$, hence

$$B_M \lesssim C M^{-(\alpha''+2\beta'')/(\alpha'+2\beta')} = M^{-\gamma}$$

where $\gamma = (\alpha'' + 2\beta'')/(\alpha' + 2\beta')$. It follows that $0 < \gamma < 1$ and as $M \rightarrow \infty$,

$$A_M(\kappa_{3,M}^{-1}) \gtrsim D M^{\gamma(1+\beta'')} \int_{C'M^{1-\gamma}}^\infty w^{\beta''} \log(1 + w^{-\alpha' - 2\beta'}) dw.$$

Since $\log(1 + x) \sim x$ as $x \rightarrow 0$, we have

$$\begin{aligned} A_M(\kappa_{3,M}^{-1}) &\gtrsim D M^{\gamma(1+\beta'')} \int_{C'M^{1-\gamma}}^\infty w^{-\alpha' - 2\beta' + \beta''} dw \\ &= D M^{\gamma(1+\beta'')} M^{(1-\gamma)(1-\alpha' - 2\beta' + \beta'')} \\ &= D M^{f(\alpha', \beta', \alpha'', \beta'')} \end{aligned} \tag{78}$$

where

$$f(\alpha', \beta', \alpha'', \beta'') = \gamma(1 + \beta'') + (1 - \gamma)(1 - \alpha' - 2\beta' + \beta'').$$

Note that $f(\alpha, \beta, \alpha, \beta) = (1 + \beta) > 0$, and since f is continuous, we can choose the constants $\alpha', \beta', \alpha'', \beta''$ such that $\delta := f(\alpha', \beta', \alpha'', \beta'') > 0$. Hence (78) implies that $A_M(\kappa_{3,M}^{-1})$ is bounded below by DM^δ for some constant D . Now, returning to $J_{M,2}$, (72), (74), (78) and (33) imply

$$\begin{aligned} |J_{M,2}| &\leq \int_{\kappa_{3,M}^{-1}}^{U_M} \exp(-A_M(u)) \frac{du}{u} \\ &\leq \exp(-A_M(\kappa_{3,M}^{-1})) \int_{\kappa_{3,M}^{-1}}^{U_M} \frac{du}{u} \\ &= \exp(-A_M(\kappa_{3,M}^{-1})) \log \left(\frac{\epsilon \kappa_{3,M}}{\kappa_{N+1,M}} \right) \end{aligned}$$

$$\begin{aligned} &\lesssim C \exp(-DM^\delta) \log M, \\ &= o(M^{-(1+\beta)(N-1)/2}). \end{aligned}$$

Estimate for $J_{M,3}$: For $J_{M,3}$, we have

$$|J_{M,3}| \leq \left| \int_{\kappa_{3,M}^{-1}}^{U_M} \exp\left(-\frac{u^2}{2}\right) \left(1 + \sum_{\eta(N)} \left(\prod_{m=3}^N \frac{1}{k_m!} \left(\frac{(iu)^m}{m!} \kappa_{m,M}\right)^{k_m}\right)\right) \frac{du}{u} \right|.$$

By bounding all the coefficients of the polynomial in u by their maximum value, we have

$$\begin{aligned} |J_{M,3}| &\leq \max_{\eta(N)} \left(1, \prod_{m=3}^N \kappa_{m,M}^{k_m}\right) \int_{\kappa_{3,M}^{-1}}^{\infty} \exp\left(-\frac{u^2}{2}\right) \left(1 + \sum_{\eta(N)} \left(\prod_{m=3}^N \frac{1}{k_m!} \left(\frac{u^m}{m!}\right)^{k_m}\right)\right) du \\ &\sim \max_{\eta(N)} \left(1, \prod_{m=3}^N \kappa_{m,M}^{k_m}\right) \int_{L(M)M^{(1+\beta)/2}}^{\infty} \exp\left(-\frac{u^2}{2}\right) p(u) du \end{aligned} \tag{79}$$

where $L(M)$ is slowly varying and $p(u)$ is a polynomial in u whose coefficients do not depend on M . Choosing a constant $C > 0$ large enough such that $p(u) \leq Ce^u$ for all $u > L(M)M^{1/2}$, we see

$$\begin{aligned} \int_{L(M)M^{(1+\beta)/2}}^{\infty} \exp\left(-\frac{u^2}{2}\right) p(u) du &\leq \int_{L(M)M^{(1+\beta)/2}}^{\infty} \exp\left(-\frac{u^2}{2}\right) Ce^u du \\ &= Ce^{1/2} \int_{L(M)M^{(1+\beta)/2-1}}^{\infty} e^{-w^2/2} dw \end{aligned} \tag{80}$$

$$= C \sqrt{\frac{e\pi}{2}} \operatorname{Erfc}\left(\frac{L(M)M^{(1+\beta)/2} - 1}{\sqrt{2}}\right) \tag{81}$$

where $\operatorname{Erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} e^{-w^2} dw$. Using the fact that $\operatorname{Erfc}(u) \sim e^{-u^2}/(\sqrt{\pi}u)$, [13, Eq. 40:9:2] (79) and (81) imply

$$\begin{aligned} |J_{M,3}| &\leq \max_{\eta(N)} \left(1, \prod_{m=3}^N \kappa_{m,M}^{k_m}\right) C \sqrt{\frac{e\pi}{2}} \operatorname{Erfc}\left(\frac{L(M)M^{(1+\beta)/2} - 1}{\sqrt{2}}\right) \\ &\sim \tilde{O}(1) \frac{\exp\left(-\frac{1}{2}L(M)^2 M^{1+\beta}\right)}{L(M)^2 M^{1+\beta}} \\ &= o(M^{-(1+\beta)(N-1)/2}). \end{aligned}$$

Combining the estimates for $J_{M,i}$, $i = 1, 2, 3$, together with Lemma 6.1 implies the desired result. Finally, Lemma 5.1 implies that the \tilde{O} in (64) can be replaced by O when the slowly varying function ρ is asymptotically constant. \square

7. A numerical example

In this section, we will demonstrate the utility of the Edgeworth expansion given in Theorem 6.2 for computing the CDF of $Z = \sum_{n=1}^{\infty} \lambda_n(\eta_n - 1)$. Consider the example where

$r_n = r = 1/2$ is constant, i.e. η_n are i.i.d. chi-squared with 1 degree of freedom, and the λ_n 's are given simply by

$$\lambda_n = Cn^{-3/4}$$

where the normalization constant C is such that $\text{Var } Z = 2 \sum \lambda_n^2 = 1$, and is given by $C = (2\zeta(6/4))^{-1/2} = (2 \sum_{n=1}^{\infty} n^{-6/4})^{-1/2} \approx 0.4375$, where ζ denotes the Riemann zeta function. To compute the CDF of

$$Z = \sum_{n=1}^{\infty} \lambda_n(\eta_n - 1) = X_M + Y_M,$$

where X_M and Y_M are defined in (5), we will proceed in three steps.

1. Choose a truncation level $M \geq 1$. We will see below that M does not need to be too large. Once an M is chosen, one must be able to compute the CDF $F_{X_M}(x)$ of X_M , which is a finite sum of weighted chi-squared random variables. There are multiple techniques for doing this, for instance methods using expansions of the CDF/PDF, [11,12], methods based on Laplace transform inversion, [22,6], or Fourier transform inversion, [1]. We found the method in [22] effective. One could also think of using the infinite series expansions for the distribution function of X_M . We noted, however, that these infinite series expansions are slow to converge if the scaling terms (λ_n) become small, as they do here. As an illustration, we found that over 1000 terms for the expansion in [12] are needed to obtain an adequate approximation for the distribution of $\eta_1 + 0.001\eta_2$, where η_1 and η_2 are i.i.d. exponential with mean 1.
2. Choose an $N \geq 3$ and compute the appropriate terms in the Edgeworth expansion for the CDF $F_{Y_M}(x)$ of Y_M in Theorem 6.2. This involves σ_M and $\kappa_{k,M}$ for $k = 3, \dots, N$. For this example, σ_M defined in (6), and the $\kappa_{k,M}$'s defined in (28), can be computed in terms of the Riemann Zeta function:

$$\sigma_M^2 = 2 \sum_{n=M}^{\infty} \lambda_n^2 = 2C^2 \left(\zeta(2\gamma) - \sum_{n=1}^{M-1} n^{-6/4} \right)$$

$$\kappa_{k,M} = 2^{k-1}(k-1)! \sigma_M^{-2} \sum_{n=M}^{\infty} \lambda_n^k = 2^{k-1}(k-1)! \sigma_M^{-k} C^k \left(\zeta(k\gamma) - \sum_{n=1}^{M-1} n^{-3k/4} \right).$$

3. The CDF of the sum $Z = X_M + Y_M$ is given by the convolution

$$F_Z(x) = \int_{-\infty}^{\infty} F_{X_M}(x-y) dF_{Y_M}(y). \tag{82}$$

We compute this integral numerically in MATLAB using standard techniques.

We have studied approximations of Y_M for various values of M and N . Figs. 2 and 3 give a sense of how good these approximations are. We look at the Edgeworth approximations to the density of Y_M and see how these behave as both M and N grow. Fig. 2 shows plots of the $N = 2, 3, 4, 5$ Edgeworth approximations to the density of \tilde{Y}_M for $M = 2, 5, 10$ and 20.

An Edgeworth expansion with values of $N \geq 2$ involves corrections to the normal distribution. Increasing N improves on this correction. If the improvement is already negligible if one goes from $N = 2$ to $N = 3$, then the distribution is close to normal. This appears to be the case in Fig. 2 for already small values of M ($M = 10$).

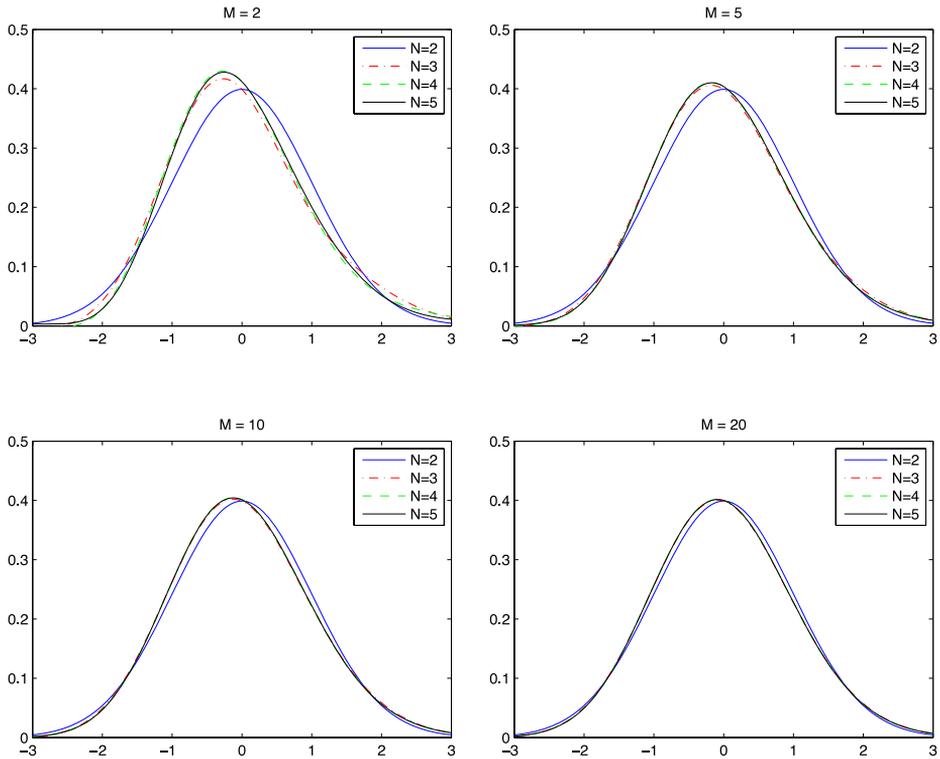


Fig. 2. Edgeworth approximations to the density of \tilde{Y}_M for various values of M . We can see that for this example, the converge of Y_M to a normal distribution is fast as M grows and increasing N beyond 5 causes a negligible change in the distribution function.

What happens at smaller values of M ? We note that in Fig. 2, that even for $M = 2$, there seems to be no change in the Edgeworth correction as N goes from 4 to 5. Hence it appears that for small values of M , a high level of accuracy is already reached by $N = 5$ as it is hard to distinguish the $N = 4$ and $N = 5$ curves.

For this reason, we will use $N = 5$ to approximate the CDF of the full distribution Z . Fig. 3 shows the CDF computed using an $N = 5$ Edgeworth expansion for Y_M for various values of M . Since the resulting approximation is nearly independent of $M = 2, 5, 10, 20$, it is clear that the convergence of the Edgeworth expansions is fast for this example. The techniques developed here are used in [21] to obtain the numerical evaluation of the CDF and PDF of the Rosenblatt distribution.

8. Conclusion

We considered weighted sums of centered independent Gamma random variables and studied how well the tail of these sums is approximated by a normal distribution. We improved the approximation by considering Berry–Esseen type results and Edgeworth expansions. To do so, we applied the classical theory of convergence of sums of independent random variables to an infinitely divisible setting. As noted by a referee, a number of our results can be extended to arrays.

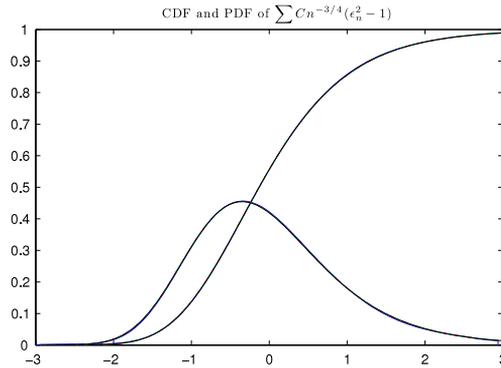


Fig. 3. Approximation to the CDF and PDF of Z in the case $\lambda_n = Cn^{-3/4}$ using the $N = 5$ Edgeworth expansion for the tail Y_M . There are 4 curves corresponding to $M = 2, 5, 10$ and 20 in both curves and are almost indistinguishable suggesting fast convergence of this method.

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