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Superposition of COGARCH processes

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Abstract

We suggest three superpositions of COGARCH (sup-CO-GARCH) volatility processes driven by Lévy processes or Lévy bases. We investigate second-order properties, jump behaviour, and prove that they exhibit Pareto-like tails. Corresponding price processes are defined and studied. We find that the sup-CO-GARCH models allow for more flexible autocovariance structures than the COGARCH. Moreover, in contrast to most financial volatility models, the sup-CO-GARCH processes do not exhibit a deterministic relationship between price and volatility jumps. Furthermore, in one sup-CO-GARCH model not all volatility jumps entail a price jump, while in another sup-CO-GARCH model not all price jumps necessarily lead to volatility jumps.

MSC: primary 60G10; secondary 60G51; 60G57; 60H05

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1. Introduction

GARCH models have been used throughout the last decades to model returns sampled at regular intervals on stocks, currencies and other assets. They capture many of the stylized features of such data; e.g. heavy tails, volatility clustering and dependence without correlation. Also because of their interesting probabilistic properties as solutions to stochastic recurrence equations, they have attracted research by probabilists and statisticians; e.g. [16]. Various attempts have

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been made to capture the stylized features of financial time series using continuous-time models. The interest in continuous-time models originates in the current wide-spread availability of irregularly spaced and high-frequency data. There was a long debate whether price and volatility fluctuations are caused by jumps or not. This question was answered convincingly in previous years by Jacod and collaborators, who developed sophisticated statistical tools to extract jumps of price and volatility out of high-frequency data (cf. [1,19,21] and references therein).

A prominent continuous-time model is the stochastic volatility model of Barndorff-Nielsen and Shephard [4], in which the volatility process V and the martingale part of the logarithmic asset price G satisfy the equations

$$\begin{aligned} \mathrm{d}V_t &= -\lambda V_t\,\mathrm{d}t + \mathrm{d}L_{\lambda t}, \\ \mathrm{d}G_t &= \sqrt{V_t}\,\mathrm{d}W_t + \rho\,\mathrm{d}\widetilde{L}_{\lambda t}, \end{aligned} \tag{1.1}$$

where $\lambda > 0$, $\rho \leq 0$, $L = (L_t)_{t\geq 0}$ is a non-decreasing Lévy process with compensated version $\widetilde{L}$ and $W = (W_t)_{t\geq 0}$ is a standard Brownian motion independent of L. The volatility process V is taken to be the stationary solution of (1.1), in other words, a stationary Lévy-driven Ornstein–Uhlenbeck (OU) process. In this model, price jumps are modelled by (scaled) upwards jumps in the volatility.

It was noticed early on that the exponential autocovariance function of the OU process may be too restrictive. Two suggestions have been made to allow for more flexibility in the autocovariance function: Barndorff-Nielsen [2] suggested to replace V by a superposition of such processes (called supOU process), which yields more flexible monotone autocovariance functions. It is defined as

$$V_t = \int_{(-\infty,t]}\int_{(0,\infty)} \mathrm{e}^{-\lambda(t-s)}\,\Lambda(\mathrm{d}s,\mathrm{d}\lambda), \quad t \in \mathbb{R}, \tag{1.2}$$

where Λ is an independently scattered infinitely divisible random measure, also called Lévy basis. Superpositions of CARMA processes can be defined analogously; cf. [6,12]. As shown in e.g. [15, Proposition 2.6], supOU models can also model long-range dependence for specific superposition measures.

On the other hand, [10,31] suggested higher-order Lévy-driven CARMA models, which also allow for non-monotone autocovariance functions. The drawback of both model classes is their linearity and its consequences towards the stylized features of financial data. For instance, linear models inherit their distributions from that of the Lévy increments in a linear way. As a consequence, only when the driving Lévy process has heavy-tailed (regularly varying) increments, they model high-level volatility clusters; cf. [14, Proposition 5]. Moreover, in contrast to empirical findings (cf. [19]), these models allow only for negative price jumps coupled to the jumps in the volatility.

A continuous-time GARCH (COGARCH) model has been introduced in [23] with volatility process V and martingale part of the logarithmic asset price given by

$$\begin{aligned} \mathrm{d}V_t &= (\beta - \eta V_t)\,\mathrm{d}t + V_{t-}\varphi\,\mathrm{d}[L,L]_t, \\ \mathrm{d}G_t &= \sqrt{V_{t-}}\,\mathrm{d}L_t, \end{aligned} \tag{1.3}$$

where $\beta, \eta, \varphi > 0$ and L is an arbitrary mean-zero Lévy process. The volatility process V is taken to be the stationary solution of (1.3). This model satisfies all stylized features of financial prices, exactly as the GARCH model for low frequency data. The drawback of an exponentially decreasing covariance function has been taken care of by higher-order models; cf. [11], like generalizing from OU to CARMA.

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All models mentioned above have price jumps exactly at the times when the volatility jumps, since their prices are driven by the same Lévy process. Moreover, with the exception of the supOU/supCARMA process, all jump sizes in volatility and price exhibit a fixed deterministic relationship; cf. [19]. As this is not very realistic, multi-factor models are needed. In this paper we want to construct such a multi-factor model, based on the COGARCH.

In contrast to the OU or CARMA models, the COGARCH model is defined as a stochastic integral with stochastic integrand. But also in this framework there is a canonical way to construct a superposition.

Starting by the fact that the ratio of volatility jumps and squared price jumps is always equal to φ in the COGARCH model, we randomize this scale parameter φ. There are various ways how to do this in a meaningful way, and we present three different possibilities, all leading to multi-factor COGARCH models. Our three models have different qualitative behaviour. For instance, the first sup-CO-GARCH allows for jumps in the volatility, which do not necessarily lead to jumps in the price process. On the other hand, for certain choices of the distribution of the random parameter φ, the third sup-CO-GARCH model allows for jumps in the price without having a jump in the volatility. More properties will be reported.

An interesting feature is that some of the presented new sup-CO-GARCH volatility processes can be written in terms of a so-called *ambit process*, which has been introduced in [3] in the context of turbulence modelling. In our context the ambit process has a stochastic integrand, which is not independent of the integrator. This implies that we are no longer in the framework of [27]. Moreover, since COGARCH models are heavy-tailed, having possibly not even a second finite moment, the theory presented in [32] is also not applicable. Instead we need the concept presented in [12], which allows to integrate stochastic processes with respect to a Lévy basis in the generality needed for our sup-CO-GARCH models.

Our paper is organized as follows. In Section 2, we recall the COGARCH model and give a short summary of Lévy bases. In Section 3, we present three different superpositions of COGARCH volatility processes. For each of the three models we give necessary and sufficient conditions for strict stationarity and derive the second-order structure of the stationary process. The superpositions allow for more flexible autocorrelation structures than the COGARCH model (Propositions 3.4, 3.12 and 3.18). However, the stationary distributions of the sup-CO-GARCH processes preserve the Pareto-like tails of the COGARCH process (Propositions 3.5, 3.13 and 3.19). Section 4 is devoted to the corresponding price processes and the second-order properties of their stationary increments. Again, main characteristics of the COGARCH are preserved like the uncorrelated increments but positively correlated squared increments (Theorems 4.1–4.3). Nevertheless, each of the sup-CO-GARCH models has its specific characteristics as highlighted in Section 5. Furthermore, for all three models there is no longer a deterministic relationship between the jump sizes in volatility and price. Although in this paper we concentrate on the probabilistic properties of our new models, statistical issues are shortly addressed here. Finally, Section 6 contains the proofs of our results.

2. Notation and preliminaries

By the Lévy–Khintchine formula (e.g. [28, Theorem 8.1]) the *characteristic exponent* of a real-valued Lévy process $X = (X_t)_{t\geq 0}$ is given by

$$\psi_X(u) := \log \mathbb{E}\left[\mathrm{e}^{\mathrm{i}uX_1}\right] = \mathrm{i}\gamma_X u - \frac{1}{2}\sigma_X^2 u^2 + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}uy} - 1 - \mathrm{i}uy \mathbb{1}_{\{|y|\leq 1\}})\, \nu_X(\mathrm{d}y), \quad u \in \mathbb{R},$$

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where $(\gamma_X, \sigma_X^2, \nu_X)$ is the *characteristic triplet* of X with *Lévy measure* ν_X satisfying $\nu_X(\{0\}) = 0$ and $\int_{\mathbb{R}} 1 \wedge |y|^2\, \nu_X(\mathrm{d}y) < \infty$. If additionally $\int_{|y|\le 1} |y|\, \nu_X(\mathrm{d}y) < \infty$, we may also write the characteristic exponent in the form

$$\psi_X(u) = \mathrm{i}\gamma_X^0 u - \frac{1}{2}\sigma_X^2 u^2 + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}uy} - 1)\, \nu_X(\mathrm{d}y), \quad u \in \mathbb{R},$$

and call γ_X^0 the *drift* of X. This is in particular the case for subordinators, i.e. Lévy processes with increasing sample paths. We also recall that the *quadratic variation process* of the Lévy process X is given by

$$[X, X]_t := \sigma_X^2 t + [X, X]_t^{\mathrm{d}} := \sigma_X^2 t + \sum_{0<s\le t} (\Delta X_s)^2, \quad t \ge 0,$$

where $[X, X]^{\mathrm{d}}$ is called the *pure-jump part* of $[X, X]$.

Every Lévy process $(X_t)_{t\ge 0}$ can be extended to a *two-sided Lévy process* $(X_t)_{t\in\mathbb{R}}$ by setting $X_t = -X'_{-t-},\ t < 0$, for some i.i.d. copy X' of X. We say that $(X_t)_{t\in\mathbb{R}}$ has characteristic triplet $(\gamma_X, \sigma_X^2, \nu_X)$ if $(X_t)_{t\ge 0}$ has characteristic triplet $(\gamma_X, \sigma_X^2, \nu_X)$.

Throughout we use the notation $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_- = (-\infty, 0)$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2.1. The COGARCH model

Let $(L_t)_{t\ge 0}$ be a Lévy process with characteristic triplet $(\gamma_L, \sigma_L^2, \nu_L)$ and define

$$S_t := [L, L]_t^{\mathrm{d}} = \sum_{0<s\le t} (\Delta L_s)^2, \quad t \ge 0. \tag{2.1}$$

Then $(S_t)_{t\ge 0}$ is a subordinator without drift and its Lévy measure ν_S is the image measure of ν_L under the transformation $y \mapsto y^2$. For $\eta > 0$ and $\varphi \ge 0$ define another Lévy process by

$$X_t^{\varphi} = \eta t - \sum_{0<s\le t} \log(1 + \varphi \Delta S_s), \quad t \ge 0, \tag{2.2}$$

which is completely determined by S (and hence by L). Then X^{φ} has characteristic triplet $(\eta, 0, \nu_{X^{\varphi}})$, where $\nu_{X^{\varphi}}$ is the image measure of ν_S under the mapping $y \mapsto -\log(1 + \varphi y)$, and is therefore a spectrally negative Lévy process, i.e. it only has negative jumps. For $t \ge 0$ we have

$$\mathbb{E}[\mathrm{e}^{-uX_t^{\varphi}}] = \mathrm{e}^{t\,\Psi(u,\varphi)} \quad \text{with } \Psi(u, \varphi) = -\eta u + \int_{\mathbb{R}_+} ((1 + \varphi y)^u - 1)\, \nu_S(\mathrm{d}y), \tag{2.3}$$

where, whenever $\varphi > 0$, we have $\mathbb{E}[\mathrm{e}^{-uX_t^{\varphi}}] < \infty$ for $u > 0$ for some $t > 0$ or, equivalently, for all $t > 0$ if and only if $\mathbb{E}[S_1^u] < \infty$ [23, Lemma 4.1]. In particular, if $\mathbb{E}[S_1] < \infty$ or $\mathbb{E}[S_1^2] < \infty$, respectively, we have from [28, Example 25.12]

$$\Psi(1, \varphi) = \varphi \mathbb{E}[S_1] - \eta \quad \text{and} \quad \Psi(2, \varphi) = 2\varphi \mathbb{E}[S_1] + \varphi^2 \mathrm{Var}[S_1] - 2\eta. \tag{2.4}$$

Recall from [23] that the *COGARCH (volatility) process* driven by the Lévy process L (or the subordinator S) with parameter φ is given by

$$V_t^{\varphi} = \mathrm{e}^{-X_t^{\varphi}} \left(V_0^{\varphi} + \beta \int_{(0,t]} \mathrm{e}^{X_s^{\varphi}}\, \mathrm{d}s \right), \quad t \ge 0, \tag{2.5}$$

where $\beta > 0$ is a constant and V_0^{φ} is a nonnegative random variable, independent of $(S_t)_{t\ge 0}$.

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Moreover, the COGARCH volatility process V^φ is a special case of a generalized Ornstein–Uhlenbeck process (cf. [8,25]) and is the solution of the SDE

$$\mathrm{d}V_t^\varphi = (\beta - \eta V_t^\varphi)\,\mathrm{d}t + V_{t-}^\varphi \varphi\,\mathrm{d}S_t = V_{t-}^\varphi(\varphi \mathrm{d}S_t - \eta\,\mathrm{d}t) + \beta\,\mathrm{d}t, \quad t \geq 0. \tag{2.6}$$

It admits the integral representation

$$V_t^\varphi = V_0^\varphi + \beta t - \eta \int_{(0,t]} V_s^\varphi\,\mathrm{d}s + \sum_{0<s\leq t} V_{s-}^\varphi \varphi \Delta S_s, \quad t \geq 0. \tag{2.7}$$

The corresponding *price process* or *integrated COGARCH process* is then defined as

$$G_t = \int_{(0,t]} \sqrt{V_{s-}^\varphi}\,\mathrm{d}L_s, \quad t \geq 0. \tag{2.8}$$

2.2. *Stationary COGARCH processes*

By [23, Theorem 3.1], the process defined in (2.5) or equivalently in (2.7) has a strictly stationary distribution if and only if

$$\int_{\mathbb{R}_+} \log(1+\varphi y)\,\nu_S(\mathrm{d}y) = \int_{\mathbb{R}} \log(1+\varphi y^2)\,\nu_L(\mathrm{d}y) < \eta. \tag{2.9}$$

In this case, the stationary distribution of the COGARCH process is given by the distribution of $V_\infty^\varphi := \beta \int_{\mathbb{R}_+} \mathrm{e}^{-X_s^\varphi}\,\mathrm{d}s$. Note that for $\varphi = 0$, the stationary COGARCH reduces to $V_t^0 = \beta/\eta$ for all $t \geq 0$.

In the sequel we denote by the set Φ_L all $\varphi \geq 0$ where (2.9) is satisfied. By monotone convergence, the left-hand side of (2.9) is continuous in φ and converges to $+\infty$ as $\varphi \to \infty$, which means that $\varphi_{\max} := \sup \Phi_L$ is finite and hence $\Phi_L = [0, \varphi_{\max})$.

Let us recall the moment structure of V^φ in the stationary case. It follows by direct computation from [7, Theorem 3.1] that, if $\kappa > 0$ is a constant, then

$$\mathbb{E}[S_1^{\max\{\kappa,1\}}] < \infty \quad \text{and} \quad \log \mathbb{E}\left[\mathrm{e}^{-\kappa X_1^\varphi}\right] = \Psi(\kappa, \varphi) < 0 \tag{2.10}$$

imply $\mathbb{E}[(V_0^\varphi)^\kappa] < \infty$. If (2.10) holds for $\kappa = 1$ or $\kappa = 2$, respectively, for every $t \geq 0, h \geq 0$ the first two moments of the stationary process V^φ are given by [23, Corollary 4.1]

$$\mathbb{E}[V_t^\varphi] = -\frac{\beta}{\Psi(1,\varphi)} = \frac{\beta}{\eta - \varphi \mathbb{E}[S_1]}, \tag{2.11}$$

$$\mathbb{E}[(V_t^\varphi)^2] = \beta^2 \frac{2}{\Psi(1,\varphi)\,\Psi(2,\varphi)} \quad \text{and} \tag{2.12}$$

$$\begin{aligned} \mathrm{Cov}[V_t^\varphi, V_{t+h}^\varphi] &= \mathrm{e}^{h\,\Psi(1,\varphi)} \mathrm{Var}[V_0^\varphi] = \mathrm{e}^{h\,\Psi(1,\varphi)} \beta^2 \left(\frac{2}{\Psi(1,\varphi)\,\Psi(2,\varphi)} - \frac{1}{\Psi(1,\varphi)^2} \right) \\ &= \mathrm{e}^{h\,(\varphi\mathbb{E}[S_1]-\eta)} \frac{\beta^2\varphi^2 \mathrm{Var}[S_1]}{(\varphi\mathbb{E}[S_1]-\eta)^2(2\eta - 2\varphi\mathbb{E}[S_1] - \varphi^2\mathrm{Var}[S_1])}. \end{aligned} \tag{2.13}$$

From (2.10) we have the clear picture that, although a stationary V^φ exists for all $\varphi \in \Phi_L = [0, \varphi_{\max})$, moments only exist on some subinterval, which shrinks with the increasing order of the moment. Moreover, it is known that no COGARCH process has moments of all orders [23, Proposition 4.3]. For later reference we set

$$\Phi_L^{(\kappa)} := [0, \varphi_{\max}^{(\kappa)}) \quad \text{with } \varphi_{\max}^{(\kappa)} = \sup\{\varphi : \mathbb{E}[(V_0^\varphi)^\kappa] < \infty\}. \tag{2.14}$$

We have $0 < \varphi_{\max}^{(\kappa_2)} \le \varphi_{\max}^{(\kappa_1)} < \varphi_{\max} < \infty$ whenever $0 < \kappa_1 \le \kappa_2 < \infty$, i.e. $\Phi_L^{(\kappa_2)} \subset \Phi_L^{(\kappa_1)} \subset \Phi_L$.

In [24] the tail behaviour of the COGARCH process is studied. In particular, it is shown that under rather weak assumptions the distribution of V_0^φ has Pareto-like tails [24, Theorem 6].

Regarding the price process G^φ in the stationary case, it is known from [23, Proposition 5.1] that G^φ has stationary increments that are uncorrelated on disjoint intervals while the squared increments are, under some technical assumptions, positively correlated, an effect which is typical for financial time series.

For later reference we extend the stationary COGARCH volatility process (2.5) to a two-sided process in the following way. For a two-sided Lévy process $(L_t)_{t\in\mathbb{R}}$ we obtain a two-sided subordinator $(S_t)_{t\in\mathbb{R}}$ by setting

$$S_t := \sum_{0<s\le t} (\Delta L_s)^2, \quad t \ge 0 \quad \text{and} \quad S_t := -\sum_{t<s\le 0} (\Delta L_s)^2, \quad t \le 0. \tag{2.15}$$

Now we automatically obtain for every φ another two-sided Lévy process $(X_t^\varphi)_{t\in\mathbb{R}}$ given by

$$\begin{aligned} X_t^\varphi &= \eta t - \sum_{0<s\le t} \log(1+\varphi\Delta S_s), \quad t \ge 0, \\ X_t^\varphi &= \eta t + \sum_{t<s\le 0} \log(1+\varphi\Delta S_s), \quad t < 0. \end{aligned} \tag{2.16}$$

The two-sided COGARCH process $(V_t^\varphi)_{t\in\mathbb{R}}$ is then given by

$$V_t^\varphi := \beta \int_{(-\infty,t]} \mathrm{e}^{-(X_t^\varphi - X_s^\varphi)}\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{2.17}$$

and it is well-defined for every $\varphi \in \Phi_L$. Obviously, the restriction of this process to $t \ge 0$ equals the process given in (2.5) with starting random variable $V_0^\varphi := \beta \int_{(-\infty,0]} \mathrm{e}^{X_s^\varphi}\,\mathrm{d}s$. Hence the two-sided COGARCH is always stationary with the same finite-dimensional distributions as the one-sided stationary COGARCH.

2.3. *Lévy bases*

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$ be a filtered probability space satisfying the usual assumptions of completeness and right-continuity. Denote the space of all $\mathbb{P}$-a.s. finite random variables by L^0, the optional (resp. predictable) σ-field by $\mathcal{O}$ (resp. $\mathcal{P}$) and set $\tilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel-σ-field on $\mathbb{R}^d$. Now let $(E_k)_{k\in\mathbb{N}}$ be a sequence of measurable subsets increasing to $\mathbb{R}^d$ and define $\tilde{\mathcal{P}}_\mathrm{b}$ as the collection of all $\tilde{\mathcal{P}}$-measurable subsets of $\Omega \times (-k,k] \times E_k$ for $k \in \mathbb{N}$. Similarly, set $\mathcal{B}_\mathrm{b} := \bigcup_{k=1}^\infty \mathcal{B}((-k,k] \times E_k)$.

In this set-up, we use the term Lévy basis as follows:

Definition 2.1. A *Lévy basis* on $\mathbb{R} \times \mathbb{R}^d$ is a mapping $\Lambda : \tilde{\mathcal{P}}_\mathrm{b} \to L^0$ satisfying:

(a) $\Lambda(\emptyset) = 0$ a.s.

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(b) If $(A_n)_{n\in\mathbb{N}}$ are pairwise disjoint sets in $\tilde{\mathcal{P}}_{\mathrm{b}}$ whose union again lies in $\tilde{\mathcal{P}}_{\mathrm{b}}$, then

$$\Lambda\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \Lambda(A_n) \quad \text{a.s.}$$

(c) If $(B_n)_{n\in\mathbb{N}}$ are pairwise disjoint sets in $\mathcal{B}_{\mathrm{b}}$, then $(\Lambda(\Omega\times B_n))_{n\in\mathbb{N}}$ is a sequence of independent random variables with each of them having an infinitely divisible distribution.
(d) If $A \in \tilde{\mathcal{P}}_{\mathrm{b}}$ is a subset of $\Omega \times (-\infty, t] \times \mathbb{R}^d$ for some $t \in \mathbb{R}$, then $\Lambda(A)$ is $\mathcal{F}_t$-measurable.
(e) If $A \in \tilde{\mathcal{P}}_{\mathrm{b}}$, $t \in \mathbb{R}$ and $F \in \mathcal{F}_t$, then $\Lambda\big(A\cap(F\times(t,\infty)\times\mathbb{R}^d)\big) = \mathbb{1}_F \Lambda\big(A\cap(\Omega\times(t,\infty)\times\mathbb{R}^d)\big)$.
(f) For all $t \in \mathbb{R}$ and measurable $U \subset E_k$ for some $k \in \mathbb{N}$, we have $\Lambda(\Omega \times \{t\} \times U) = 0$ a.s.

In the following, we often write $\Lambda(B) = \Lambda(\Omega \times B)$ for a set $B \in \mathcal{B}_{\mathrm{b}}$. □

A natural choice for $\mathbb{F}$ is certainly the *augmented natural filtration* $\mathbb{G} = (\mathcal{G}_t)_{t\in\mathbb{R}}$ of the Lévy basis Λ, which means that for $t \in \mathbb{R}$, $\mathcal{G}_t$ is the completion of the σ-field generated by the collection of all $\Lambda(B)$ with $B \in \mathcal{B}_{\mathrm{b}}$, $B \subseteq (-\infty, t] \times \mathbb{R}^d$.

The first three points of Definition 2.1 are similar to the notion of infinitely divisible independently scattered random measures in [27]. Further we have added condition (f) because this ensures that Λ induces a jump measure μ^Λ by

$$\begin{aligned}&\mu^\Lambda(\omega, \mathrm{d}t, \mathrm{d}x, \mathrm{d}y)\\ &:= \sum_{s\in\mathbb{R}} \sum_{\xi\in\mathbb{R}^d} \mathbb{1}_{\{\Lambda(\{s\}\times\{\xi\})(\omega)\neq 0\}} \delta_{(s,\xi,\Lambda(\{s\}\times\{\xi\})(\omega))}(\mathrm{d}t, \mathrm{d}x, \mathrm{d}y), \quad \omega \in \Omega,\end{aligned} \tag{2.18}$$

where δ stands for the Dirac measure. We will follow the usual convention of suppressing ω in the sequel. Thanks to (d) and (e), μ^Λ is an optional $\tilde{\mathcal{P}}$-σ-finite random measure in the sense of [20, Theorem II.1.8]. Therefore, the predictable compensator Π of μ^Λ is well-defined.

In this paper, we will only consider Lévy bases Λ which are of the form

$$\Lambda(\mathrm{d}s, \mathrm{d}x) = \int_{\mathbb{R}} y\, \mu^\Lambda(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y). \tag{2.19}$$

In addition, the predictable compensator of μ^Λ in the augmented natural filtration $\mathbb{G}$ will always be given by $\Pi(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y) = \mathrm{d}s\,\pi(\mathrm{d}x)\,\nu(\mathrm{d}s)$, where π is some probability measure on $\mathbb{R}^d$ and ν the Lévy measure of a subordinator. In this particular case, if we write

$$W(s,x,y) * \mu_t^\Lambda := W * \mu_t^\Lambda := \begin{cases} \displaystyle\int_{(0,t]\times\mathbb{R}^d\times\mathbb{R}} W(s,x,y)\,\mu^\Lambda(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y), & \text{if } t \geq 0,\\ \displaystyle\int_{(t,0]\times\mathbb{R}^d\times\mathbb{R}} W(s,x,y)\,\mu^\Lambda(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y), & \text{if } t < 0, \end{cases}$$

for some $\mathcal{O} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$-measurable function W which is integrable w.r.t. μ^Λ (ω-wise as a Lebesgue integral), then we have

$$\mathbb{E}[W * \mu_t^\Lambda] = \mathbb{E}[W * \Pi_t] = \int_{(0,t]\times\mathbb{R}^d\times\mathbb{R}} \mathbb{E}[W(s,x,y)]\,\Pi(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y), \quad t \geq 0, \tag{2.20}$$

for all integrable functions W (and similarly for $t < 0$), see [20, Theorem II.1.8]. Moreover, when taking stochastic integrals with respect to Λ, these can be expressed in terms of μ^Λ:

$$\int_{(0,t]\times\mathbb{R}^d} H(s,x)\,\Lambda(\mathrm{d}s,\mathrm{d}x) = \int_{(0,t]\times\mathbb{R}^d\times\mathbb{R}} H(s,x)y\,\mu^\Lambda(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y), \quad t \geq 0,$$

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for all H which are integrable w.r.t. Λ on $(0,t]$ (similarly for $t<0$); see [12] for integrability conditions and further details on Lévy bases.

For later reference, we also introduce the pure-jump part of the quadratic variation measure of Λ defined as

$$[\Lambda,\Lambda]^{\mathrm{d}}(A) := \int_{\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}} \mathbb{1}_A(t,x)y^2\,\mu^\Lambda(\mathrm{d}t,\mathrm{d}x,\mathrm{d}y), \quad A\in\tilde{\mathcal{P}}_{\mathrm{b}}. \tag{2.21}$$

3. Superposition of COGARCH (sup-CO-GARCH) processes

In the following three subsections we propose different approaches to construct a superposition of COGARCH processes. As seen in Eq. (2.6), the parameters β and η only influence the continuous part of the COGARCH process, whereas φ scales its jump sizes. Since our goal is to find a model which shares the basic features of the COGARCH model but has a more flexible jump structure, we let β and η be fixed in the following three approaches and only allow the parameter φ to vary.

3.1. The sup-CO-GARCH 1 volatility process

The obvious idea of defining a sup-CO-GARCH process as a weighted integral of independent COGARCH processes with different parameters φ yields to consider

$$\bar{V}_t^{(1)} := \int_{[0,\infty)} V_t^\varphi\,\pi(\mathrm{d}\varphi), \quad t\geq 0, \tag{3.1}$$

for some probability measure π on $[0,\infty)$, where each COGARCH process V^φ is driven by $S^\varphi=[L^\varphi,L^\varphi]^{\mathrm{d}}$ and $(L^\varphi)_{\varphi\in[0,\infty)}$ are i.i.d. copies of a canonical Lévy process L, which, together with $S=[L,L]^{\mathrm{d}}$, we only use for notational convenience. As a consequence, $(V^\varphi)_{\varphi\in[0,\infty)}$ is a family of independent COGARCH processes such that the integral in (3.1) is only well-defined if π has countable support. This leads to the *sup-CO-GARCH* 1 *volatility process*

$$\bar{V}_t^{(1)} = \int_{[0,\infty)} V_t^\varphi\,\pi(\mathrm{d}\varphi) = \sum_{i=1}^\infty p_i V_t^{\varphi_i}, \quad t\geq 0, \tag{3.2}$$

where $\pi=\sum_{i=1}^\infty p_i\delta_{\varphi_i}$ for nonnegative weights $(p_i)_{i\in\mathbb{N}}$ with $\sum_{i=1}^\infty p_i=1$.

To avoid degenerate cases we will assume throughout that

$$\bar{V}_0^{(1)} = \sum_{i=1}^\infty p_i V_0^{\varphi_i} < \infty \quad \text{a.s.} \tag{3.3}$$

Note that this does not automatically imply finiteness of the sup-CO-GARCH process at all times unless we are in the stationary case (see below).

Remark 3.1. The sup-CO-GARCH 1 process can also be written in terms of a Lévy basis. First, define a Lévy basis on $\mathbb{R}_+\times[0,\infty)$ by

$$\Lambda^L((0,t]\times\{\varphi_i\}) := \sqrt{p_i}\,L_t^{\varphi_i}, \quad t\geq 0,\ i\in\mathbb{N},$$

and $\Lambda^L(\mathbb{R}\times([0,\infty)\setminus\bigcup_{i=1}^\infty\{\varphi_i\})) := 0$. Now with $\Lambda^S=[\Lambda^L,\Lambda^L]^{\mathrm{d}}$ being the pure-jump quadratic variation measure of Λ^L (in particular, $\Lambda^S((0,t]\times\{\varphi_i\}) = p_i S_t^{\varphi_i}$) and inserting (2.7) in (3.2),

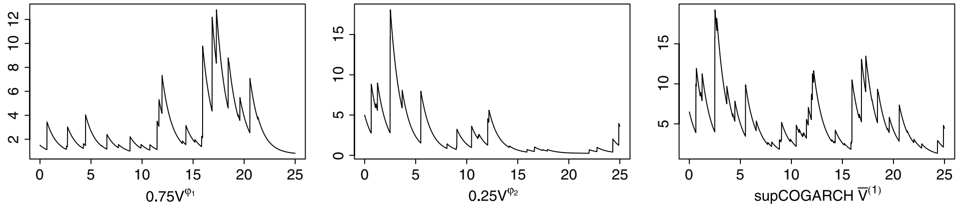

Fig. 1. Sample paths of two independent COGARCH processes with different values for φ, scaled with the corresponding p_i, and the resulting sup-CO-GARCH 1 process. The driving Lévy processes are independent compound Poisson processes with rate 1 and standard normal jumps. The parameters are: $\beta = 1, \eta = 1, \varphi_1 = 0.5, \varphi_2 = 0.95$ and $\pi = 0.75\delta_{\varphi_1} + 0.25\delta\varphi_2$, starting value was the respective mean.

we see that

$$\begin{aligned}\bar{V}_t^{(1)} &= \sum_{i=1}^{\infty} p_i V_0^{\varphi_i} + \beta t - \eta \sum_{i=1}^{\infty} p_i \int_{(0,t]} V_s^{\varphi_i}\,\mathrm{d}s + \sum_{i=1}^{\infty} \int_{(0,t]} p_i \varphi_i V_{s-}^{\varphi_i}\,\mathrm{d}S_s^{\varphi_i} \\ &= \bar{V}_0^{(1)} + \beta t - \eta \int_{(0,t]} \bar{V}_s^{(1)}\,\mathrm{d}s + \int_{(0,t]} \int_{[0,\infty)} \varphi V_{s-}^{\varphi}\, \Lambda^S(\mathrm{d}s, \mathrm{d}\varphi), \quad t \geq 0. \end{aligned} \tag{3.4}$$

Note that for each $i \in \mathbb{N}$, V^{φ_i} is driven by S^{φ_i}.

It follows directly from (3.4) that the jumps of the sup-CO-GARCH 1 process are given by

$$\Delta \bar{V}_t^{(1)} = \sum_{i=1}^{\infty} p_i \Delta V_t^{\varphi_i} = \sum_{i=1}^{\infty} p_i V_{t-}^{\varphi_i} \varphi_i \Delta S_t^{\varphi_i} = \int_{[0,\infty)} \varphi V_{t-}^{\varphi}\, \Lambda^S(\{t\} \times \mathrm{d}\varphi), \quad t \geq 0. \tag{3.5}$$

Since the independent subordinators a.s. jump at different times, a.s. only one summand in (3.5) is nonzero at each jump time.

The following example for a probability measure π with two-point support will be carried through the three different sup-CO-GARCH processes in this section to clarify their definitions.

Example 3.2. Let $\pi = p_1\delta_{\varphi_1} + p_2\delta_{\varphi_2}$ with $p_1 + p_2 = 1$ and $\varphi_1, \varphi_2 \in \mathbb{R}_+$. Then the sup-CO-GARCH 1 process is the weighted sum of two independent COGARCH processes. More precisely, we have $\bar{V}_t^{(1)} = p_1 V_t^{\varphi_1} + p_2 V_t^{\varphi_2}$ for $t \geq 0$, where V^{φ_1} and V^{φ_2} are driven by *independent* copies of the canonical Lévy process L. From Fig. 1, we clearly see that the sup-CO-GARCH 1 process inherits both the jumps of V^{φ_1} and V^{φ_2}, scaled with p_1 or p_2, respectively.

Stationarity and second-order properties of the sup-CO-GARCH 1 process are given in the following three results. Proofs are postponed to Section 6.1.

Theorem 3.3. *Let* $\pi = \sum_{i=1}^{\infty} p_i \delta_{\varphi_i}$ *be a probability measure on* $[0, \infty)$, $\{L^{\varphi_i} : i \in \mathbb{N}\}$ *a family of i.i.d. Lévy processes,* $\{S^{\varphi_i} : i \in \mathbb{N}\}$ *the corresponding family of subordinators and* $\{V^{\varphi_i} : i \in \mathbb{N}\}$ *the corresponding family of COGARCH processes. Assuming that (3.3) holds, a finite random variable* $\bar{V}_0^{(1)}$ *can be chosen such that* $\bar{V}^{(1)}$ *is strictly stationary if and only if*

$$\pi(\Phi_L) = 1. \tag{3.6}$$

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In the case that a stationary distribution exists, it is uniquely determined by the law of

$$\bar{V}_\infty^{(1)} := \int_{\Phi_L} V_\infty^\varphi \, \pi(\mathrm{d}\varphi) = \beta \int_{\Phi_L} \int_{\mathbb{R}_+} \mathrm{e}^{-X_t^\varphi} \, \mathrm{d}t \, \pi(\mathrm{d}\varphi) = \beta \sum_{i=1}^\infty p_i \int_{\mathbb{R}_+} \mathrm{e}^{-X_t^{\varphi_i}} \, \mathrm{d}t. \tag{3.7}$$

Proposition 3.4. *Assume we are in the setting of Theorem 3.3 and let $\bar{V}^{(1)}$ be a strictly stationary solution of (3.4). Recall the notation $\Phi_L^{(\kappa)}$ from Eq. (2.14).*

(a) *Suppose that $\pi(\Phi_L^{(1)}) = 1$. Then for every $t \geq 0$,*

$$\mathbb{E}[\bar{V}_t^{(1)}] = \int_{\Phi_L} \mathbb{E}[V_0^\varphi] \, \pi(\mathrm{d}\varphi) = \beta \sum_{i=1}^\infty \frac{p_i}{\eta - \varphi_i \mathbb{E}[S_1]}. \tag{3.8}$$

(b) *Suppose that $\pi(\Phi_L^{(2)}) = 1$. Then for every $t \geq 0$, $h \geq 0$ we have*

$$\mathrm{Var}[\bar{V}_t^{(1)}] = \sum_{i=1}^\infty p_i^2 \mathrm{Var}[V_0^{\varphi_i}] \quad \textit{and} \tag{3.9}$$

$$\mathrm{Cov}[\bar{V}_t^{(1)}, \bar{V}_{t+h}^{(1)}] = \sum_{i=1}^\infty p_i^2 \mathrm{Cov}[V_0^{\varphi_i}, V_h^{\varphi_i}], \tag{3.10}$$

with $\mathrm{Var}[V_0^{\varphi_i}]$ and $\mathrm{Cov}[V_0^{\varphi_i}, V_h^{\varphi_i}]$ as given in (2.12) and (2.13).

Note that the quantities in (3.8)–(3.10) may be infinite.

Proposition 3.5. *Assume we are in the setting of Theorem 3.3 and let $\bar{V}^{(1)}$ be a strictly stationary solution of (3.4). Set $\bar{\varphi} := \inf\{\varphi > 0 \colon \pi((\varphi, \infty)) = 0\} \leq \varphi_{\max} < \infty$ and assume that there exists $\bar{\kappa} > 0$ with*

$$\mathbb{E}[S_1^{\bar{\kappa}} \log^+(S_1)] < \infty \quad \textit{and} \quad \Psi(\bar{\kappa}, \bar{\varphi}) = 0. \tag{3.11}$$

Then we have for $\kappa > 0$

$$\lim_{x \to \infty} x^\kappa \mathbb{P}[\bar{V}_0^{(1)} > x] = \begin{cases} 0 & \textit{if } \kappa < \bar{\kappa}, \\ \infty & \textit{if } \kappa > \bar{\kappa}, \end{cases}$$

while for $\kappa = \bar{\kappa}$ there exists a constant $C > 0$ such that

$$\lim_{x \to \infty} x^{\bar{\kappa}} \mathbb{P}[\bar{V}_0^{(1)} > x] = \begin{cases} C & \textit{if } \pi(\{\bar{\varphi}\}) = \bar{p} > 0, \\ 0 & \textit{if } \pi(\{\bar{\varphi}\}) = 0. \end{cases}$$

Remark 3.6. Recall from [24, Theorem 5] that the stationary distribution of the COGARCH V^φ is self-decomposable, i.e. for all $b \in (0, 1)$ there exists a random variable Y_b such that $V_\infty^\varphi \stackrel{\mathrm{d}}{=} b(V_\infty^\varphi)' + Y_b$ where $(V_\infty^\varphi)'$ is an independent copy of V_∞^φ. Due to the fact that self-decomposability is preserved under scaling, convolution and taking limits, see e.g. [30, Proposition V.2.2], it follows directly from (3.7) that the stationary distribution of the sup-CO-GARCH 1 process $\bar{V}^{(1)}$ is self-decomposable, too.

Remark 3.7. Unless we are in the degenerate case $\pi = \delta_\varphi$ and the sup-CO-GARCH is in fact just the COGARCH with parameter φ, the sup-CO-GARCH process $\bar{V}^{(1)}$ is no longer a Markov

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process with respect to its augmented natural filtration, i.e. the smallest filtration such that $\bar{V}^{(1)}$ is adapted and which satisfies the usual hypotheses of right-continuity and completeness. But it follows directly from (3.4) that, letting $\mathbb{F}^{(1)} = (\mathcal{F}_t^{(1)})_{t\geq 0}$ be the augmented natural filtration of $((V_t^{\varphi_i})_{i\in\mathbb{N}})_{t\geq 0}$, we have for every measurable function $f: \mathbb{R}_+ \to \mathbb{R}$ and every $t \geq 0$

$$\mathbb{E}\Big[f\big(\bar{V}_t^{(1)}\big)\Big|\mathcal{F}_t^{(1)}\Big] = \mathbb{E}\Big[f\big(\bar{V}_t^{(1)}\big)\Big|(V_t^{\varphi_i})_{i\in\mathbb{N}}\Big].$$

Remark 3.8. In the representation $\bar{V}^{(1)} = \sum_{i=1}^{\infty} p_i V^{\varphi_i}$ a priori the φ_i do not have to be pairwise different and still the results of this section remain valid (apart from some obvious notational changes).

3.2. *The sup-CO-GARCH 2 volatility process*

In order to deal with uncountable superpositions, one possibility is to drop the assumption of independence, which led to the sup-CO-GARCH 1. Hence we fix a Lévy process L, define the subordinator $(S_t)_{t\geq 0}$ by (2.1) and define the superposition as a weighted integral of COGARCH processes V^{φ} as given in (2.7) with different parameters φ, but all driven by the single Lévy process L, i.e. we set

$$\bar{V}_t^{(2)} := \int_{\Phi_L} V_t^{\varphi}\,\pi(\mathrm{d}\varphi), \quad t \geq 0,$$

for some probability measure π on the parameter space Φ_L. To ensure that $\varphi \mapsto V_t^{\varphi}$ is measurable at all times and in particular at time $t = 0$, we will use two-sided COGARCH processes as in (2.17) and define the *sup-CO-GARCH* 2 *volatility process*

$$\bar{V}_t^{(2)} := \int_{\Phi_L} V_t^{\varphi}\,\pi(\mathrm{d}\varphi) = \beta \int_{\Phi_L}\int_{(-\infty,t]} \mathrm{e}^{-(X_t^{\varphi}-X_s^{\varphi})}\,\mathrm{d}s\,\pi(\mathrm{d}\varphi), \quad t \in \mathbb{R}, \tag{3.12}$$

for $(X_t^{\varphi})_{t\in\mathbb{R}}$ as given in (2.16). As a consequence, we have for $t \geq 0$

$$\begin{aligned}\bar{V}_t^{(2)} &= \int_{\Phi_L} V_0^{\varphi}\,\pi(\mathrm{d}\varphi) + \beta t - \eta \int_{\Phi_L}\int_{(0,t]} V_s^{\varphi}\,\mathrm{d}s\,\pi(\mathrm{d}\varphi) + \int_{\Phi_L}\int_{(0,t]} \varphi V_{s-}^{\varphi}\,\mathrm{d}S_s\,\pi(\mathrm{d}\varphi)\\ &= \bar{V}_0^{(2)} + \beta t - \eta \int_{(0,t]} \bar{V}_s^{(2)}\,\mathrm{d}s + \int_{(0,t]}\int_{\Phi_L} \varphi V_{s-}^{\varphi}\,\pi(\mathrm{d}\varphi)\,\mathrm{d}S_s. \end{aligned}\tag{3.13}$$

In order to ensure that (3.12) is finite, we always assume

$$\int_{\Phi_L} V_0^{\varphi}\,\pi(\mathrm{d}\varphi) < \infty. \tag{3.14}$$

If $\pi = \sum_{i=1}^{\infty} p_i \delta_{\varphi_i}$, we obviously have $\bar{V}^{(2)} = \sum_{i=1}^{\infty} p_i V^{\varphi_i}$ with dependent summands.

Observe that in this setting all single COGARCH processes jump at the same times and thus we have

$$\Delta \bar{V}_t^{(2)} = \int_{\Phi_L} \varphi V_{t-}^{\varphi}\,\pi(\mathrm{d}\varphi)\Delta S_t, \quad t \geq 0. \tag{3.15}$$

Example 3.9 (Example 3.2 *continued*)**.** Let $\pi = p_1\delta_{\varphi_1} + p_2\delta_{\varphi_2}$ with $p_1 + p_2 = 1$ and $\varphi_1, \varphi_2 \in \Phi_L$. Then the sup-CO-GARCH 2 process is the weighted sum of two COGARCH

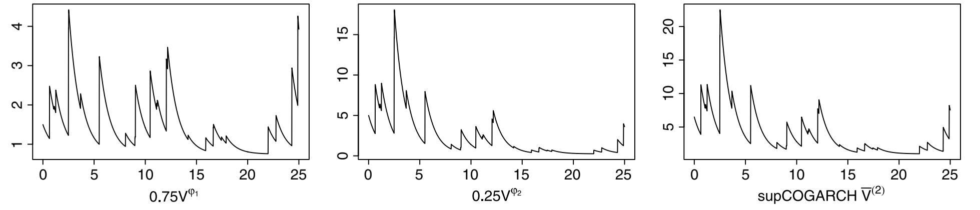

Fig. 2. Sample paths of two COGARCH processes V^{φ_1} and V^{φ_2} with different parameters, driven by the same Lévy process L, scaled with the corresponding p_i, and the resulting sup-CO-GARCH $\bar{V}^{(2)}$. The driving Lévy process L is a compound Poisson process with rate 1 and standard normal jumps. The parameters are the same as in Fig. 1.

processes with parameters φ_1 and φ_2, i.e. $\bar{V}_t^{(2)} = p_1 V_t^{\varphi_1} + p_2 V_t^{\varphi_2}$. In contrast to the sup-CO-GARCH 1 process in Example 3.2, V^{φ_1} and V^{φ_2} are driven by the same subordinator, say S, of the form (2.1). In Fig. 2 we illustrate the typical relationship between the original COGARCH processes and the resulting sup-CO-GARCH 2 process. We observe that V^{φ_1}, V^{φ_2} and $\bar{V}^{(2)}$ all jump at the same times, with the jump sizes of the sup-CO-GARCH being the weighted average jump sizes of the two COGARCH processes.

In the following we present stationarity and second-order properties of the sup-CO-GARCH process $\bar{V}^{(2)}$. Proofs are given in Section 6.2.

Theorem 3.10. *Assume that (3.14) holds. Then $(\bar{V}_t^{(2)})_{t\in\mathbb{R}}$ as defined in (3.12) is strictly stationary.*

Before we can calculate the moments of the stationary sup-CO-GARCH process $\bar{V}^{(2)}$ in Proposition 3.12 we need to establish covariances between single COGARCH processes with different parameters in the following proposition.

Proposition 3.11. *Let $(S_t)_{t\in\mathbb{R}}$ be a subordinator without drift, let $\varphi, \tilde{\varphi} \in \Phi_L$ be fixed and define the stationary two-sided COGARCH processes $(V_t^{\varphi})_{t\in\mathbb{R}}$, $(V_t^{\tilde{\varphi}})_{t\in\mathbb{R}}$ according to (2.17). If*

$$\mathbb{E}[S_1^2] < \infty, \quad \Psi(2,\varphi) < 0 \quad \text{and} \quad \Psi(2,\tilde{\varphi}) < 0,$$

then $\mathbb{E}[V_t^{\varphi} V_{t+h}^{\tilde{\varphi}}] < \infty$ for all $t \in \mathbb{R}$ and $h \geq 0$. In this case, we have for all $t \in \mathbb{R}$ that

$$\mathbb{E}[V_t^{\varphi} V_t^{\tilde{\varphi}}] = \frac{\beta^2((\varphi+\tilde{\varphi})\mathbb{E}[S_1] - 2\eta)}{(\varphi\mathbb{E}[S_1]-\eta)(\tilde{\varphi}\mathbb{E}[S_1]-\eta)((\varphi+\tilde{\varphi})\mathbb{E}[S_1] + \varphi\tilde{\varphi}\operatorname{Var}[S_1] - 2\eta)}, \tag{3.16}$$

$$\operatorname{Cov}[V_t^{\varphi}, V_t^{\tilde{\varphi}}] = \frac{\beta^2 \varphi\tilde{\varphi}\operatorname{Var}[S_1]}{(\varphi\mathbb{E}[S_1]-\eta)(\tilde{\varphi}\mathbb{E}[S_1]-\eta)(2\eta - (\varphi+\tilde{\varphi})\mathbb{E}[S_1] - \varphi\tilde{\varphi}\operatorname{Var}[S_1])}, \tag{3.17}$$

while for all $t \in \mathbb{R}$ and $h \geq 0$

$$\operatorname{Cov}[V_t^{\varphi}, V_{t+h}^{\tilde{\varphi}}] = \mathrm{e}^{h\,\Psi(1,\tilde{\varphi})}\operatorname{Cov}[V_0^{\varphi}, V_0^{\tilde{\varphi}}]. \tag{3.18}$$

Both covariances in (3.17) and (3.18) are nonnegative.

Now we can describe the covariance structure of the sup-CO-GARCH process $\bar{V}^{(2)}$.

Proposition 3.12. *Let $\bar{V}^{(2)}$ be the strictly stationary sup-CO-GARCH 2 process as defined in (3.12). Recall the notation $\Phi_L^{(\kappa)}$ from Eq. (2.14).*

(a) *Suppose that* $\pi(\Phi_L^{(1)}) = 1$. *Then we have for all* $t \geq 0$

$$\mathbb{E}[\bar{V}_t^{(2)}] = \int_{\Phi_L} \mathbb{E}[V_0^{\varphi}]\,\pi(\mathrm{d}\varphi) = \beta \int_{\Phi_L} \frac{1}{\eta - \varphi\mathbb{E}[S_1]}\,\pi(\mathrm{d}\varphi). \tag{3.19}$$

(b) *Suppose that* $\pi(\Phi_L^{(2)}) = 1$. *Then for* $t \in \mathbb{R}$ *and* $h \geq 0$ *we have*

$$\mathbb{E}[(\bar{V}_t^{(2)})^2] = \int_{\Phi_L}\int_{\Phi_L} \mathbb{E}[V_0^{\varphi} V_0^{\tilde{\varphi}}]\,\pi(\mathrm{d}\varphi)\,\pi(\mathrm{d}\tilde{\varphi}), \tag{3.20}$$

$$\mathrm{Var}[\bar{V}_t^{(2)}] = \int_{\Phi_L}\int_{\Phi_L} \mathrm{Cov}[V_0^{\varphi}, V_0^{\tilde{\varphi}}]\,\pi(\mathrm{d}\varphi)\,\pi(\mathrm{d}\tilde{\varphi}), \tag{3.21}$$

$$\mathrm{Cov}[\bar{V}_t^{(2)}, \bar{V}_{t+h}^{(2)}] = \int_{\Phi_L}\int_{\Phi_L} \mathrm{Cov}[V_0^{\varphi}, V_h^{\tilde{\varphi}}]\,\pi(\mathrm{d}\varphi)\,\pi(\mathrm{d}\tilde{\varphi}), \tag{3.22}$$

with $\mathbb{E}[V_0^{\varphi} V_0^{\tilde{\varphi}}]$ *and* $\mathrm{Cov}[V_0^{\varphi}, V_h^{\tilde{\varphi}}]$ *as given in Proposition* 3.11.

Note that the quantities in (3.19)–(3.22) *may be infinite.*

The tail behaviour of $\bar{V}^{(2)}$ is similar to the tail behaviour of the sup-CO-GARCH 1 process.

Proposition 3.13. *Let* $\bar{V}^{(2)}$ *be the strictly stationary sup-CO-GARCH* 2 *process as defined in* (3.12). *Set* $\bar{\varphi} := \inf\{\varphi > 0 \colon \pi((\varphi, \infty)) = 0\} \leq \varphi_{\max} < \infty$ *and assume that there exists* $\bar{\kappa} > 0$ *such that* (3.11) *holds. Then we have for* $\kappa > 0$

$$\lim_{x\to\infty} x^{\kappa}\mathbb{P}[\bar{V}_0^{(2)} > x] = \begin{cases} 0 & \text{if } \kappa < \bar{\kappa}, \\ \infty & \text{if } \kappa > \bar{\kappa}, \end{cases}$$

while for $\kappa = \bar{\kappa}$ *there exists a constant* $C > 0$ *such that*

$$\lim_{x\to\infty} x^{\bar{\kappa}}\mathbb{P}[\bar{V}_0^{(2)} > x] = \begin{cases} C & \text{if } \pi(\{\bar{\varphi}\}) = \bar{p} > 0, \\ 0 & \text{if } \pi(\{\bar{\varphi}\}) = 0. \end{cases}$$

Remark 3.14. Similarly to $\bar{V}^{(1)}$, the process $\bar{V}^{(2)}$ is no Markov process with respect to its augmented natural filtration (unless in the degenerate case $\pi = \delta_{\varphi}$), but again we have a Markov property in a wide sense. More precisely, for $\mathbb{F}^{(2)} = (\mathcal{F}_t^{(2)})_{t\geq 0}$ being the augmented natural filtration of $((V_t^{\varphi})_{\varphi\in\Phi_L})_{t\geq 0}$, we obtain for every measurable function $f\colon \mathbb{R}_+ \to \mathbb{R}$ and every $t \geq 0$

$$\mathbb{E}\left[f\left(\bar{V}_t^{(2)}\right)\middle|\mathcal{F}_t^{(2)}\right] = \mathbb{E}\left[f\left(\bar{V}_t^{(2)}\right)\middle|(V_t^{\varphi})_{\varphi\in\Phi_L}\right].$$

3.3. *The sup-CO-GARCH 3 volatility process*

Our third superposition model invokes a Lévy basis Λ^L on $\mathbb{R} \times \Phi_L$ such that

$$L_t := \Lambda^L((0, t] \times \Phi_L), \quad t \geq 0, \qquad L_t := -\Lambda^L((-t, 0] \times \Phi_L), \quad t < 0,$$

exists for every $t \in \mathbb{R}$. With $\Lambda^S := [\Lambda^L, \Lambda^L]^{\mathrm{d}}$ in the sense of (2.21), Λ^S is of the form (2.19) and we assume that the predictable compensator of μ^{Λ^S} is $\Pi^S(\mathrm{d}t, \mathrm{d}y, \mathrm{d}\varphi) = \mathrm{d}t\,\nu_S(\mathrm{d}y)\pi(\mathrm{d}\varphi)$, where π is a probability measure on Φ_L and ν_S the Lévy measure of the following two-sided subordinator:

$$S_t := \Lambda^S((0, t] \times \Phi_L), \quad t \geq 0, \qquad S_t := -\Lambda^S((-t, 0] \times \Phi_L), \quad t < 0. \tag{3.23}$$

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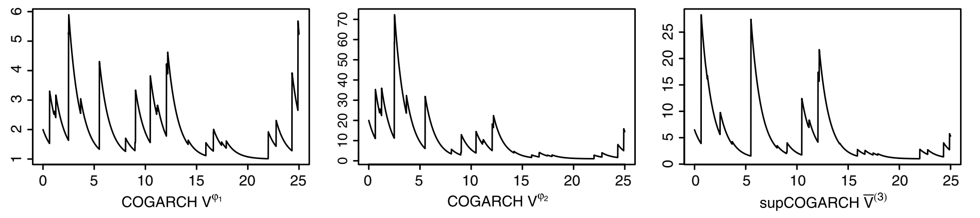

Fig. 3. Two COGARCH processes V^{φ_1} and V^{φ_2} driven by the same Lévy process L and the resulting sup-CO-GARCH $\bar{V}^{(3)}$. L is a compound Poisson process with rate 1 and standard normal jumps. The parameters are the same as in Fig. 1.

For every $\varphi \in \Phi_L$ we denote by V^φ the two-sided COGARCH process driven by S as in (2.17). The *sup-CO-GARCH* 3 *volatility process* $\bar{V}^{(3)}$ is then defined by the integral equation

$$\bar{V}_t^{(3)} = \bar{V}_0^{(3)} + \beta t - \eta \int_{(0,t]} \bar{V}_s^{(3)}\, \mathrm{d}s + \int_{(0,t]} \int_{\Phi_L} \varphi V_{s-}^\varphi\, \Lambda^S(\mathrm{d}s, \mathrm{d}\varphi), \quad t \geq 0, \tag{3.24}$$

where $\bar{V}_0^{(3)}$ is some starting random variable independent of the restriction of Λ^L to $\mathbb{R}_+ \times \Phi_L$ From (3.24) it follows directly that

$$\Delta \bar{V}_t^{(3)} = \int_{\mathbb{R}_+ \times \Phi_L} \varphi V_{t-}^\varphi\, y\, \mu^{\Lambda^S}(\{t\}, \mathrm{d}\varphi, \mathrm{d}y), \quad t \geq 0. \tag{3.25}$$

We present now conditions for stationarity and calculate the second-order properties. The proofs can be found in Section 6.3.

Proposition 3.15. *The stochastic integral equation* (3.24) *has a unique solution given by*

$$\bar{V}_t^{(3)} = \mathrm{e}^{-\eta t} \left(\bar{V}_0^{(3)} + \beta \int_{(0,t]} \mathrm{e}^{\eta s}\, \mathrm{d}s + \int_{(0,t]} \mathrm{e}^{\eta s}\, \mathrm{d}A_s \right), \quad t \geq 0, \tag{3.26}$$

where

$$A_t := \int_{(0,t]} \int_{\Phi_L} \varphi V_{s-}^\varphi\, \Lambda^S(\mathrm{d}s, \mathrm{d}\varphi), \quad t \geq 0, \tag{3.27}$$

is a semimartingale with increasing sample paths, finite at every fixed $t \geq 0$.

Example 3.16 (Examples 3.2 *and* 3.9 *continued*)**.** Let $\pi = p_1\delta_{\varphi_1} + p_2\delta_{\varphi_2}$ be a probability measure with $p_1 + p_2 = 1$ and $\varphi_1, \varphi_2 \in \Phi_L$. As opposed to the sup-CO-GARCH 1 process in Example 3.2 or the sup-CO-GARCH 2 process in Example 3.9, the sup-CO-GARCH 3 process is not the sum of two (independent or dependent) COGARCH processes. In fact, there is a subordinator S driving two COGARCH processes V^{φ_1} and V^{φ_2} and each time when S jumps, a value of φ is randomly chosen from $\{\varphi_1, \varphi_2\}$: φ takes the value φ_1 with probability p_1 and the value φ_2 with probability p_2. Now the jump size of the sup-CO-GARCH 3 at a particular jump time of S is exactly the jump size of the COGARCH with the chosen parameter φ. If $(T_i)_{i\in\mathbb{N}}$ denote the jump times of S, we have

$$\Delta \bar{V}_{T_i}^{(3)} = \Delta V_{T_i}^{\varphi_i} = \varphi_i V_{T_i-}^{\varphi_i} \Delta S_{T_i}, \quad i \in \mathbb{N},$$

and $(\varphi_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence with distribution π. Moreover, $(\varphi_i)_{i\in\mathbb{N}}$ is independent of S. This effect is illustrated in Fig. 3.

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The next theorem establishes necessary and sufficient conditions for the existence of a stationary distribution of the sup-CO-GARCH 3 process.

Theorem 3.17. *Define the sup-CO-GARCH* 3 *process* $(\bar V_t^{(3)})_{t\geq 0}$ *by* (3.26). *Then a finite random variable* $\bar V_0^{(3)}$ *can be chosen such that* $\bar V^{(3)}$ *is strictly stationary if and only if*

$$\int_{\mathbb{R}_+}\int_{\Phi_L}\int_{\mathbb{R}_+} 1\wedge (y\varphi V_s^{\varphi}\mathrm{e}^{-\eta s})\,\mathrm{d}s\,\pi(\mathrm{d}\varphi)\,\nu_S(\mathrm{d}y) < \infty \quad a.s. \tag{3.28}$$

In the case that a stationary distribution exists, it is uniquely determined by the law of $\frac{\beta}{\eta}+\int_{\mathbb{R}_+}\mathrm{e}^{-\eta s}\,\mathrm{d}A_s$. *In particular, setting* $\bar V_0^{(3)} := \frac{\beta}{\eta}+\int_{(-\infty,0]}\int_{\Phi_L}\mathrm{e}^{\eta s}\varphi V_{s-}^{\varphi}\,\Lambda^S(\mathrm{d}s,\mathrm{d}\varphi)$, *we obtain the two-sided stationary sup-CO-GARCH* 3 *process*

$$\begin{aligned}\bar V_t^{(3)} &= \mathrm{e}^{-\eta t}\left(\beta\int_{(-\infty,t]}\mathrm{e}^{\eta s}\,\mathrm{d}s+\int_{(-\infty,t]}\mathrm{e}^{\eta s}\,\mathrm{d}A_s\right)\\ &= \frac{\beta}{\eta}+\int_{(-\infty,t]}\int_{\Phi_L}\mathrm{e}^{-\eta(t-s)}\varphi V_{s-}^{\varphi}\,\Lambda^S(\mathrm{d}s,\mathrm{d}\varphi)\end{aligned} \tag{3.29}$$

for $t\in\mathbb{R}$. *Moreover,* (3.28) *holds in each of the following cases:*

(a) $\pi([0,\varphi_0]) = 1$ *with some* $\varphi_0 < \varphi_{\max}$.
(b) $\pi(\Phi_L^{(\kappa)}) = 1$ *for some* $\kappa > 0$.

The second-order properties of the strictly stationary sup-CO-GARCH 3 process are as follows.

Proposition 3.18. *Let* $\bar V^{(3)}$ *be the stationary sup-CO-GARCH* 3 *process given by* (3.29). *Recall the notation* $\Phi_L^{(\kappa)}$ *from Eq.* (2.14).

(a) *Assume that* $\pi(\Phi_L^{(1)}) = 1$. *Then for* $t\in\mathbb{R}$

$$\mathbb{E}[\bar V_t^{(3)}] = \int_{\Phi_L}\mathbb{E}[V_0^{\varphi}]\,\pi(\mathrm{d}\varphi) = \int_{\Phi_L}\frac{\beta}{\eta-\mathbb{E}[S_1]\varphi}\,\pi(\mathrm{d}\varphi). \tag{3.30}$$

(b) *Assume that* $\pi(\Phi_L^{(2)}) = 1$. *Then with* $\mathbb{E}[V_0^{\varphi}V_0^{\tilde\varphi}]$ *and* $\mathrm{Cov}[V_0^{\varphi},V_0^{\tilde\varphi}]$ *as given in Proposition* 3.11, *we have for* $t\in\mathbb{R}$ *and* $h\geq 0$

$$\mathbb{E}[(\bar V_t^{(3)})^2] = \int_{\Phi_L}\int_{\Phi_L}\left(\mathbb{E}[V_0^{\varphi}V_0^{\tilde\varphi}]+\frac{\beta}{\eta}\frac{\mathrm{Var}[V_0^{\varphi}]-\mathrm{Cov}[V_0^{\varphi},V_0^{\tilde\varphi}]}{\mathbb{E}[V_0^{\varphi}]}\right)\pi(\mathrm{d}\tilde\varphi)\,\pi(\mathrm{d}\varphi), \tag{3.31}$$

$$\begin{aligned}\mathrm{Cov}[\bar V_t^{(3)},\bar V_{t+h}^{(3)}] = \int_{\Phi_L}\int_{\Phi_L}\Bigg(&\mathrm{e}^{h\Psi(1,\varphi)}\mathrm{Cov}[V_0^{\varphi},V_0^{\tilde\varphi}]\\ &+\mathrm{e}^{-\eta h}\frac{\beta}{\eta}\frac{\mathrm{Var}[V_0^{\varphi}]-\mathrm{Cov}[V_0^{\varphi},V_0^{\tilde\varphi}]}{\mathbb{E}[V_0^{\varphi}]}\Bigg)\pi(\mathrm{d}\tilde\varphi)\,\pi(\mathrm{d}\varphi).\end{aligned} \tag{3.32}$$

Note that the quantities in (3.30)–(3.32) *may be infinite.*

The sup-CO-GARCH 3 process also exhibits Pareto-like tails.

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Proposition 3.19. *Let $\bar{V}^{(3)}$ be the stationary sup-CO-GARCH* 3 *process given by* (3.29)*. Set $\bar{\varphi} := \inf\{\varphi > 0 : \pi((\varphi,\infty)) = 0\} \le \varphi_{\max} < \infty$ and assume that there exists $\bar{\kappa} > 0$ such that* (3.11) *is fulfilled. Then for $\kappa > 0$*

$$\lim_{x\to\infty} x^{\kappa}\mathbb{P}[\bar{V}_0^{(3)} > x] = \begin{cases} 0 & \text{if } \kappa < \bar{\kappa}, \\ \infty & \text{if } \kappa > \bar{\kappa}, \end{cases}$$

and for $\kappa = \bar{\kappa}$ and $\pi(\{\bar{\varphi}\}) = 0$ we have

$$\lim_{x\to\infty} x^{\bar{\kappa}}\mathbb{P}[\bar{V}_0^{(3)} > x] = 0,$$

while for $\kappa = \bar{\kappa}$ and $\pi(\{\bar{\varphi}\}) = \bar{p} > 0$

$$0 < C_* := \liminf_{x\to\infty} x^{\bar{\kappa}}\mathbb{P}[\bar{V}_0^{(3)} > x] \le \limsup_{x\to\infty} x^{\bar{\kappa}}\mathbb{P}[\bar{V}_0^{(3)} > x] =: C^* < \infty.$$

Remark 3.20. Just like $\bar{V}^{(1)}$ and $\bar{V}^{(2)}$, the process $\bar{V}^{(3)}$ is not a Markov process with respect to its augmented natural filtration (unless in the case $\pi = \delta_{\varphi}$), but, denoting the augmented natural filtration of $((V_t^{\varphi})_{\varphi\in\Phi_L})_{t\ge 0}$ by $\mathbb{F}^{(3)} = (\mathcal{F}_t^{(3)})_{t\ge 0}$, we obtain for every measurable function $f:\mathbb{R}_+ \to \mathbb{R}$ and every $t \ge 0$

$$\mathbb{E}\left[f\left(\bar{V}_t^{(3)}\right)\middle|(\mathcal{F}_s^{(3)})_{s\le t}\right] = \mathbb{E}\left[f\left(\bar{V}_t^{(3)}\right)\middle|(V_t^{\varphi})_{\varphi\in\Phi_L}\right].$$

4. The price processes

Recall that in the COGARCH model, or its discrete-time analogue, the GARCH model the driving noises for volatility and price processes are the same, cf. [9] and (2.8). In this section, we suggest and investigate price processes corresponding to the sup-CO-GARCH volatility processes. All proofs can be found in Section 6.4.

4.1. The integrated sup-CO-GARCH 1 price process

For the sup-CO-GARCH 1 volatility process $\bar{V}^{(1)}$ as defined in Section 3.1, there is no canonical choice for a price process, since a whole sequence $(L^{\varphi_i})_{i\in\mathbb{N}}$ of Lévy processes is used in its definition. Hence a priori any function of this sequence is a reasonable candidate for the driver in the price process. As a simple example we take the Lévy process L^{φ_1} as integrator; i.e. we define

$$G_t^{(1)} := \int_{(0,t]} \sqrt{\bar{V}_{s-}^{(1)}}\, \mathrm{d}L_s^{\varphi_1}, \quad t \ge 0. \tag{4.1}$$

It is an interesting observation that this process not only allows for common jumps of volatility and price (as it is usual in the standard COGARCH model), but also for jumps only in the volatility and not in the price process. There is evidence that this happens in real data (cf. [21]).

It is obvious from the definition that, if $(\bar{V}_t^{(1)})_{t\ge 0}$ is strictly stationary, then $(G_t^{(1)})_{t\ge 0}$ has stationary increments. Furthermore, its second-order structure is comparable to that of the integrated COGARCH process [23, Proposition 5.1].

Theorem 4.1. *Let $\bar{V}^{(1)} = \sum_{i=1}^{\infty} p_i V^{\varphi_i}$, $\varphi_i \in \Phi_L$, be a stationary sup-CO-GARCH* 1 *process as defined in Section* 3.1*, where each V^{φ_i} is driven by $S^{\varphi_i} = [L^{\varphi_i}, L^{\varphi_i}]^{\mathrm{d}}$ and $(L^{\varphi_i})_{i\in\mathbb{N}}$ are* *i.i.d. copies of a Lévy processes L with zero mean. Define the price process $G^{(1)}$ by (4.1) and set*

$$\Delta^r G_t^{(1)} := G_{t+r}^{(1)} - G_t^{(1)} = \int_{(t,t+r]} \sqrt{\bar{V}_{s-}^{(1)}}\, \mathrm{d}L_s^{\varphi_1}, \quad t \geq 0,\ r > 0.$$

Recall the notation $\Phi_L^{(\kappa)}$ from Eq. (2.14) and that the support of π is countable in this case.

(a) *Assume that π has support in $\Phi_L^{(1/2)}$. Then*

$$\mathbb{E}[\Delta^r G_t^{(1)}] = 0, \quad t \geq 0,\ r > 0.$$

(b) *If further $\mathbb{E}[L_1^2] < \infty$ and π has support in $\Phi_L^{(1)}$, then for $t \in \mathbb{R}$ and $h \geq r > 0$*

$$\mathbb{E}[(\Delta^r G_t^{(1)})^2] = r\mathbb{E}[L_1^2]\mathbb{E}[\bar{V}_0^{(1)}] = r\mathbb{E}[L_1^2] \int_{\Phi_L^{(1)}} \frac{\beta}{\eta - \varphi(\mathbb{E}[L_1^2] - \sigma_L^2)}\, \pi(\mathrm{d}\varphi) \quad \text{and}$$

$$\mathrm{Cov}[\Delta^r G_t^{(1)}, \Delta^r G_{t+h}^{(1)}] = 0.$$

(c) *Assume further that $\mathbb{E}[L_1^4] < \infty$, $\int_{\mathbb{R}} y^3\, \nu_L(\mathrm{d}y) = 0$ and that $\pi \neq \delta_0$ has support in $\Phi_L^{(2)}$. Then for $t \in \mathbb{R}$ and $h \geq r > 0$*

$$\begin{aligned} &\mathrm{Cov}[(\Delta^r G_t^{(1)})^2, (\Delta^r G_{t+h}^{(1)})^2] \\ &\quad = \mathbb{E}[L_1^2] \int_{\Phi_L^{(2)}} \frac{\mathrm{e}^{h\Psi(1,\varphi)} - \mathrm{e}^{(h-r)\Psi(1,\varphi)}}{\Psi(1,\varphi)} \mathrm{Cov}[(\Delta^r G_0^{(1)})^2, V_r^{\varphi}]\, \pi(\mathrm{d}\varphi) \\ &\quad > 0. \end{aligned}$$

4.2. The integrated sup-CO-GARCH 2 price process

Let $(L_t)_{t\in\mathbb{R}}$ be a two-sided Lévy process, define the subordinator S by (2.15) and let $(\bar{V}_t^{(2)})_{t\in\mathbb{R}}$ be the sup-CO-GARCH 2 process driven by S as defined in Section 3.2. In view of the standard definition of the integrated COGARCH price process (2.8) it makes sense to define the *integrated sup-CO-GARCH 2 price process* by

$$\mathrm{d}G_t^{(2)} := \sqrt{\bar{V}_{t-}^{(2)}}\, \mathrm{d}L_t, \qquad G_0^{(2)} = 0, \quad t \in \mathbb{R}. \tag{4.2}$$

Hence, as in the standard COGARCH model, the process $G^{(2)}$ jumps exactly when the volatility $\bar{V}^{(2)}$ jumps. Also $(G_t^{(2)})_{t\in\mathbb{R}}$ has stationary increments if $(\bar{V}_t^{(2)})_{t\in\mathbb{R}}$ is strictly stationary. As shown in the following, the integrated sup-CO-GARCH 2 process has a similar second-order structure to the integrated sup-CO-GARCH 1 and the integrated COGARCH process.

Theorem 4.2. *Suppose that the two-sided Lévy process L has expectation* 0*, define S by (2.15), the sup-CO-GARCH volatility $\bar{V}^{(2)}$ as in Section 3.2 with $\pi(\Phi_L) = 1$ and the process $G^{(2)}$ by (4.2). Set*

$$\Delta^r G_t^{(2)} := G_{t+r}^{(2)} - G_t^{(2)} = \int_{(t,t+r]} \sqrt{\bar{V}_{s-}^{(2)}}\, \mathrm{d}L_s, \quad t \in \mathbb{R},\ r > 0.$$

(a) *Assume that π has support in $\Phi_L^{(1/2)}$. Then*

$$\mathbb{E}[\Delta^r G_t^{(2)}] = 0, \quad t \in \mathbb{R},\ r > 0.$$

(b) *If further $\mathbb{E}[L_1^2] < \infty$ and π has support in $\Phi_L^{(1)}$, then for $t \in \mathbb{R}$ and $h \geq r > 0$*

$$\mathbb{E}[(\Delta^r G_t^{(2)})^2] = r\mathbb{E}[L_1^2]\mathbb{E}[\bar{V}_0^{(2)}] = r\mathbb{E}[L_1^2] \int_{\Phi_L^{(1)}} \frac{\beta}{\eta - \varphi(\mathbb{E}[L_1^2] - \sigma_L^2)} \pi(\mathrm{d}\varphi),$$

$$\mathrm{Cov}[\Delta^r G_t^{(2)}, \Delta^r G_{t+h}^{(2)}] = 0.$$

(c) *Assume further that $\mathbb{E}[L_1^4] < \infty$, $\int_{\mathbb{R}} y^3\, \nu_L(\mathrm{d}y) = 0$ and $\pi \neq \delta_0$ has support in $\Phi_L^{(2)}$. Then for $t \in \mathbb{R}$ and $h \geq r > 0$*

$$\begin{aligned}&\mathrm{Cov}[(\Delta^r G_t^{(2)})^2, (\Delta^r G_{t+h}^{(2)})^2]\\ &\quad = \mathbb{E}[L_1^2] \int_{\Phi_L^{(2)}} \frac{\mathrm{e}^{h\Psi(1,\varphi)} - \mathrm{e}^{(h-r)\Psi(1,\varphi)}}{\Psi(1,\varphi)} \mathrm{Cov}[(\Delta^r G_0^{(2)})^2, V_r^{\varphi}]\, \pi(\mathrm{d}\varphi)\\ &\quad > 0.\end{aligned}$$

4.3. The integrated sup-CO-GARCH 3 price process

As in the case of the sup-CO-GARCH 2 there is a canonical choice for the driving noise in the price process of the sup-CO-GARCH 3. With L being a Lévy process and $V^{(3)}$ the stationary sup-CO-GARCH 3 as defined in (3.29), we define the *integrated sup-CO-GARCH* 3 *price process* by

$$G_t^{(3)} := \int_{(0,t]} \sqrt{\bar{V}_{t-}^{(3)}}\, \mathrm{d}L_t, \quad t \geq 0. \tag{4.3}$$

Evidently, $G^{(3)}$ has stationary increments and, if $\pi(\{0\}) = 0$, it jumps at exactly the times when $\bar{V}^{(3)}$ jumps. However, whenever $\pi(\{0\}) > 0$, the sup-CO-GARCH 3 model features price jumps without volatility jumps, a behaviour attested by the empirical findings of [21]. The second-order structure of $G^{(3)}$ is calculated in the following theorem.

Theorem 4.3. *Suppose that L is a Lévy process with expectation 0 and that $\pi(\Phi_L) = 1$. Define $V^{(3)}$ by (3.29) and set*

$$\Delta^r G_t^{(3)} := G_{t+r}^{(3)} - G_t^{(3)} = \int_{(t,t+r]} \sqrt{\bar{V}_{s-}^{(3)}}\, \mathrm{d}L_s, \quad t \geq 0,\ r > 0.$$

(a) *Assume that π has support in $\Phi_L^{(1/2)}$. Then*

$$\mathbb{E}[\Delta^r G_t^{(3)}] = 0, \quad t \geq 0,\ r > 0.$$

(b) *If further $\mathbb{E}[L_1^2] < \infty$ and π has support in $\Phi_L^{(1)}$, then for $t \geq 0$ and $h \geq r > 0$*

$$\mathbb{E}[(\Delta^r G_t^{(3)})^2] = r\mathbb{E}[L_1^2]\mathbb{E}[\bar{V}_0^{(3)}] = r\mathbb{E}[L_1^2] \int_{\Phi_L^{(1)}} \frac{\beta}{\eta - \varphi(\mathbb{E}[L_1^2] - \sigma_L^2)} \pi(\mathrm{d}\varphi),$$

$$\mathrm{Cov}[\Delta^r G_t^{(3)}, \Delta^r G_{t+h}^{(3)}] = 0.$$

(c) *Assume further that* $\mathbb{E}[L_1^4] < \infty$, $\int_{\mathbb{R}} y^3\, \nu_L(\mathrm{d}y) = 0$ *and* $\pi \neq \delta_0$ *has support in* $\Phi_L^{(2)}$. *Then for* $t \geq 0$ *and* $h \geq r > 0$

$$\begin{aligned}
\mathrm{Cov}[(\Delta^r G_t^{(3)})^2, (\Delta^r G_{t+h}^{(3)})^2] &= \mathbb{E}[L_1^2]\Bigg[\frac{\mathrm{e}^{-\eta(h-r)} - \mathrm{e}^{-\eta h}}{\eta}\mathrm{Cov}[(\Delta^r G_0^{(3)})^2, \bar{V}_r^{(3)}] \\
&\quad + \int_{\Phi_L^{(2)}}\left(\frac{\mathrm{e}^{h\Psi(1,\varphi)} - \mathrm{e}^{(h-r)\Psi(1,\varphi)}}{\Psi(1,\varphi)} + \frac{\mathrm{e}^{-\eta h} - \mathrm{e}^{-\eta(h-r)}}{\eta}\right) \\
&\quad \times \mathrm{Cov}[(\Delta^r G_0^{(3)})^2, V_r^{\varphi}]\,\pi(\mathrm{d}\varphi)\Bigg] \\
&> 0.
\end{aligned}$$

5. Comparison and conclusions

This section is devoted to highlight the analogies and differences between the three sup-CO-GARCH processes, and to compare them to the standard COGARCH process. First note that in all three models, setting $\pi = \delta_\varphi$ for $\varphi \in \Phi_L$ yields the standard COGARCH process $(V_t^{\varphi})_{t\geq 0}$ as defined in (2.5). Hence it seems natural that some features of the COGARCH process are preserved under superpositioning. The next remark summarizes the most important properties.

Remark 5.1. (a) Comparing the autocovariance functions of the sup-CO-GARCH volatility processes (cf. (3.10), (3.22) and (3.32)) to those of the COGARCH (cf. (2.13)), we find for large lags h exponential decay in all three sup-CO-GARCH models, but allowing for more flexibility than in the COGARCH model for small and medium lags.

(b) The important property of Pareto-like tails of the stationary distribution of a COGARCH process [24, Theorem 6] persists as shown in Propositions 3.5, 3.13 and 3.19.

(c) Another similarity is given in the behaviour between jumps, where the COGARCH process exhibits exponential decay [24, Proposition 2]. More precisely, assuming that $\bar{V}^{(1)}$, $\bar{V}^{(2)}$ and $\bar{V}^{(3)}$ only have finitely many jumps on compact intervals, and fixing two consecutive jump times $T_j < T_{j+1}$, we obtain for $i \in \{1, 2, 3\}$ and $t \in (T_j, T_{j+1})$

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{V}_t^{(i)} = \beta - \eta\bar{V}_t^{(i)}, \quad \bar{V}_t^{(i)} = \frac{\beta}{\eta} + \left(\bar{V}_{T_j}^{(i)} - \frac{\beta}{\eta}\right)\mathrm{e}^{-\eta(t-T_j)}.$$

(d) An important difference between the sup-CO-GARCH processes and the COGARCH process is the jump behaviour. This is highlighted in Corollary 5.2 and Example 5.3.

(e) In general, all sup-CO-GARCH models have common jumps in volatility and price as it is characteristic for the COGARCH model. Additionally, the sup-CO-GARCH 1 model also features volatility jumps without price jumps and the sup-CO-GARCH 3, if $\pi(\{0\}) > 0$, also price jumps without volatility jumps. Moreover, if we replace L^{φ_1} in (4.1) by a (finite or infinite) linear combination of $(L^{\varphi_i})_{i\in\mathbb{N}}$, we can control the proportion of common volatility and price jumps to sole volatility jumps in the sup-CO-GARCH 1 model.

(f) Our three models have different degrees of randomness in the following sense. The sup-CO-GARCH 1 is defined via a sequence of independent Lévy processes. So by the adjustment of π there is an arbitrary degree of randomness in the model. The sup-CO-GARCH 2 model has only one single source of randomness, namely the driving Lévy process. Finally, the sup-CO-GARCH 3 incorporates two sources of randomness: one originating from the Lévy process $L = \Lambda^L((0, \cdot] \times \Phi_L)$ and one from the sequence $(\varphi_i)_{i\in\mathbb{N}}$ chosen at the jump times of L.

One of the motivations for this study was the observation made in [19] that for a COGARCH process (V^φ, G^φ) there is always a deterministic relationship between volatility jumps and price jumps given by

$$q_T^\varphi := \frac{\phi(V_{T-}^\varphi, V_T^\varphi)}{\psi(G_{T-}^\varphi, G_T^\varphi)} \equiv \varphi$$

for deterministic functions

$$\psi(x, y) = (y - x)^2, \qquad \phi(x, y) = y - x, \tag{5.1}$$

at every jump time T of the driving Lévy process.

From the following corollary, which is a direct consequence of the respective definitions, we see immediately that for all three sup-CO-GARCH models such a deterministic functional relationship between volatility and price jumps is no longer present.

Corollary 5.2. *Let T be a jump time of L^{φ_1} for the sup-CO-GARCH* 1*, and a jump time of L for the sup-CO-GARCH* 2 *and* 3*. Furthermore, define $\bar{\varphi} := \inf\{\varphi > 0 : \pi((\varphi, \infty)) = 0\}$ and $\underline{\varphi} := \sup\{\varphi > 0 : \pi((0, \varphi)) = 0\}$ (using the convention $\sup \emptyset := 0, \inf \emptyset := \infty$).*

(a) *We have*

$$\Delta \bar{V}_T^{(1)} = p_1 \varphi_1 V_{T-}^{\varphi_1} (\Delta L_T^{\varphi_1})^2, \qquad \Delta G_T^{(1)} = \sqrt{\sum_{i=1}^{\infty} p_i V_{T-}^{\varphi_i}} \, \Delta L_T^{\varphi_1}, \tag{5.2}$$

$$\Delta \bar{V}_T^{(2)} = \int_{\Phi_L} \varphi V_{T-}^\varphi \, \pi(\mathrm{d}\varphi) (\Delta L_T)^2, \qquad \Delta G_T^{(2)} = \sqrt{\int_{\Phi_L} V_{T-}^\varphi \, \pi(\mathrm{d}\varphi)} \, \Delta L_T, \tag{5.3}$$

$$\Delta \bar{V}_T^{(3)} = \varphi_T V_{T-}^{\varphi_T} (\Delta L_T)^2, \qquad \Delta G_T^{(3)} = \sqrt{\bar{V}_{T-}^{(3)}} \, \Delta L_T, \tag{5.4}$$

where in the last line φ_T is a random variable which has distribution π and is independent of L.

(b) *Define*

$$q_T^{(i)} := \frac{\phi(\bar{V}_{T-}^{(i)}, \bar{V}_T^{(i)})}{\psi(G_{T-}^{(i)}, G_T^{(i)})} \tag{5.5}$$

for $i = 1, 2, 3$ with ϕ and ψ given in (5.1). Then we have

$$q_T^{(1)} \le \bar{\varphi} \quad \text{and} \quad \underline{\varphi} \le q_T^{(2)} \le \bar{\varphi};$$

moreover, if $\varphi_T = \bar{\varphi}$ (resp. $\varphi_T = \underline{\varphi}$), we have

$$q_j^{(3)} \ge \bar{\varphi} \quad (\text{resp. } q_j^{(3)} \le \underline{\varphi}).$$

Example 5.3 (Examples 3.2, 3.9 *and* 3.16 *continued*)**.** Let us compare the jumps in the sup-CO-GARCH volatility processes for $\pi = p_1 \delta_{\varphi_1} + p_2 \delta_{\varphi_2}$ with $p_1 + p_2 = 1$ and $\varphi_1, \varphi_2 \in \Phi_L$: We see from (5.2) that in the sup-CO-GARCH 1 model a squared jump of L^{φ_i} is always scaled with $p_i \varphi_i V_{t-}^{\varphi_i}$ and, hence, the parameter φ_i as well as the weight p_i take part in the scaling. In contrast, defining $S^{\varphi_i} = \Lambda^S((0, \cdot] \times \{\varphi_i\})$ for $i = 1, 2$ in the case of the sup-CO-GARCH 3 process, each jump of $S = S^{\varphi_1} + S^{\varphi_2} = [L, L]^{\mathrm{d}}$ is scaled with $\varphi_1 V_{t-}^{\varphi_1}$ or $\varphi_2 V_{t-}^{\varphi_2}$, depending on whether S^1

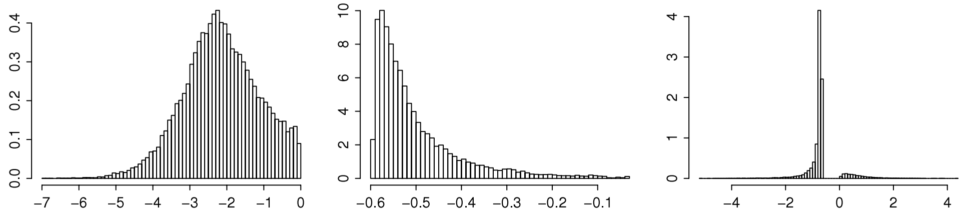

Fig. 4. The pictures (from left to right) show the histograms for $\log(q^{(1)})$, $\log(q^{(2)})$ and $\log(q^{(3)})$.

or S^2 actually jumps. Here the probabilities p_i do not influence the scaling of the jump, but the intensity of the driving processes S^{φ_i}, in other words, the p_i determine the probability for the value φ_i to be chosen at a specific jump time. Finally, for the sup-CO-GARCH 2 process, the jump size of the subordinator $S = [L, L]^{\mathrm{d}}$ is always scaled with $p_1\varphi_1 V_{t-}^{\varphi_1} + p_2\varphi_2 V_{t-}^{\varphi_2}$, so all weights and parameters are involved here.

Simulation results

To illustrate the theoretical findings above, we present simulations of the different sup-CO-GARCH volatility processes as well as the different price processes in Figs. 5 and 6. As Lévy process L we choose a variance gamma process arising through time changing a standard Brownian motion by an independent gamma process with mean and variance 1.

Note that we have chosen different parameters for the simulations presented in Figs. 5 and 6, respectively, in order to better visualize the differences between the three volatility and the three price processes.

To illustrate the profound difference between the COGARCH and the three sup-CO-GARCH models with reference to (5.1), we also compute $q^{(1)}$, $q^{(2)}$ and $q^{(3)}$ as defined in (5.5) for the jump times of the simulation in Fig. 6. The histograms of $\log q^{(i)}$ are given in Fig. 4. We see that both the sup-CO-GARCH 1 and 2 exhibit a certain interval of values for $\log q^{(1)}$ and $\log q^{(2)}$. As we would expect from Corollary 5.2, both $\log q^{(1)}$ and $\log q^{(2)}$ are bounded from above by $\log \varphi_2$, but only $\log q^{(2)}$ is bounded from below by $\log \varphi_1$ whereas the $\log q^{(1)}$ has a relatively long tail on the negative side. Also, in general, the values of $q^{(1)}$ tend to be smaller than those of $q^{(2)}$. This is due to the fact that at a common jump time of volatility and price, the volatility jump size is the sum of two terms for the sup-CO-GARCH 2 but only a single term for the sup-CO-GARCH 1 (see (5.3) and (5.2)). As a result, the nominator in (5.5) is usually smaller for the sup-CO-GARCH 1 than for the sup-CO-GARCH 2. Finally, again in coincidence with Corollary 5.2, the sup-CO-GARCH 3 shows two disjoint intervals for the values of $q^{(3)}$, corresponding to the two different values of φ chosen for the superposition.

Estimation

A thorough — statistical analysis of the sup-CO-GARCH processes via parameter estimation goes far beyond the scope of this paper. Nevertheless let us shortly comment on the main task, namely the estimation of the superposition measure π, for which no standard estimation procedure is available as it is typical for multifactor models.

In the case of the supOU stochastic volatility model, several attempts have been made to infer the underlying superposition measure. For example, assuming the form $\pi = \sum_{i=1}^{K} p_i \delta_{\varphi_i}$ for some known $K \in \mathbb{N}$, in [4,5] a least-square fit of the autocovariance function is employed.

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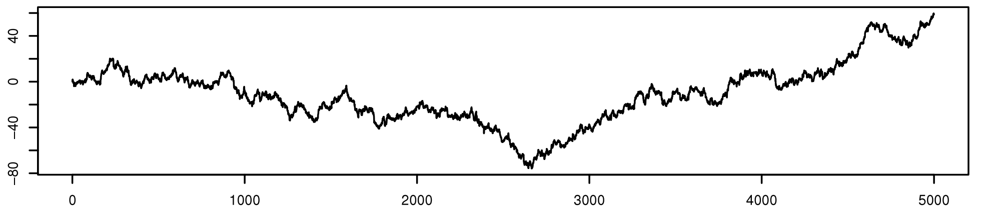
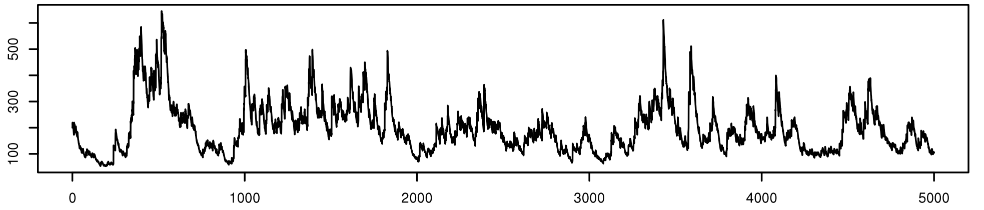
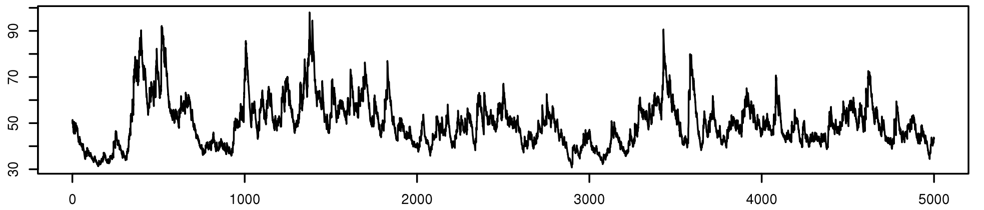


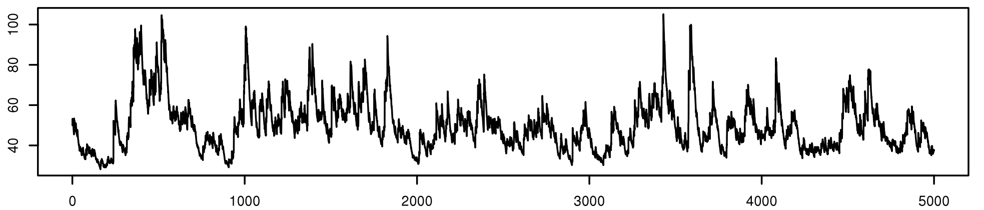
(a) L.



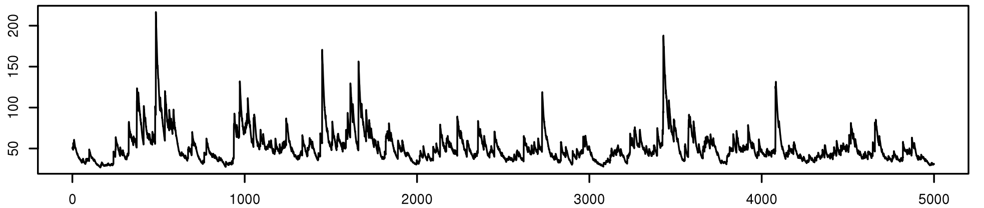
(b) V^{φ_2}.



(c) $\bar{V}^{(1)}$.



(d) $\bar{V}^{(2)}$.



(e) $\bar{V}^{(3)}$.

Fig. 5. The parameters are: $\beta = 1$, $\eta = 0.05$, $\varphi_1 = 0.02$, $\varphi_2 = 0.045$, $\pi = 0.9\delta_{\varphi_1} + 0.1\delta_{\varphi_2}$, starting value is the mean; (a) L is a variance gamma process with mean 0 and variance 1; (b) COGARCH process driven by L with parameter φ_2; (c) sup-CO-GARCH process $\bar{V}^{(1)}$ where V^{φ_2} is driven by L and V^{φ_1} is driven by an independent copy of L; (d) sup-CO-GARCH process $\bar{V}^{(2)}$ driven by L; (e) sup-CO-GARCH process $\bar{V}^{(3)}$ driven by L.

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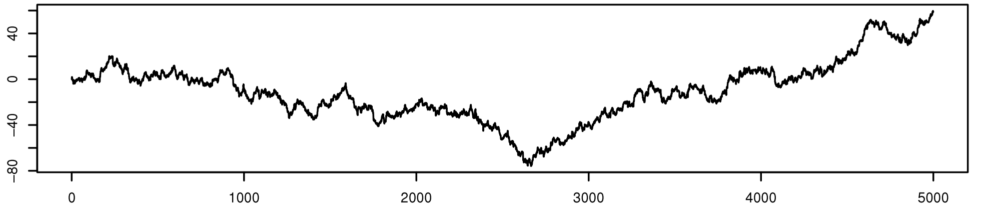
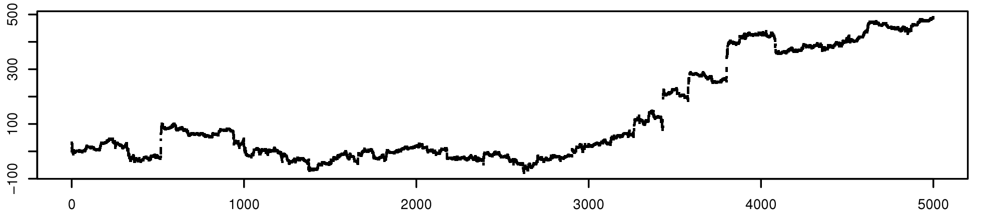
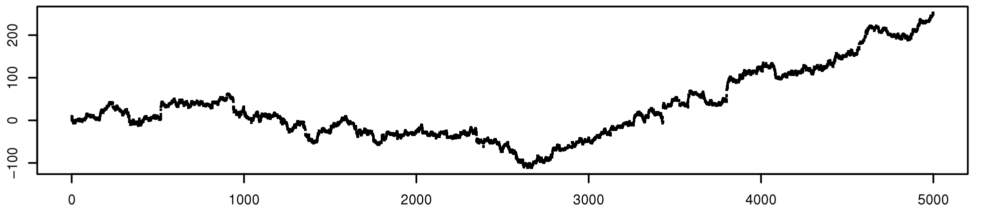


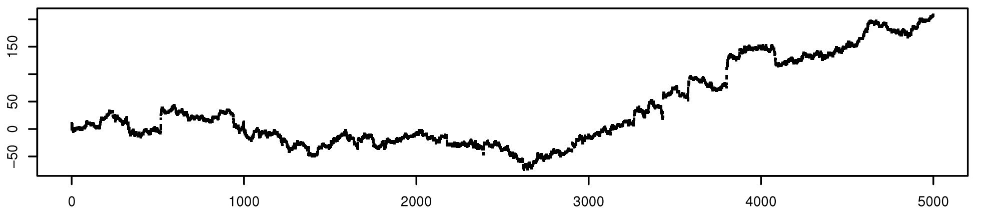
(a) L.



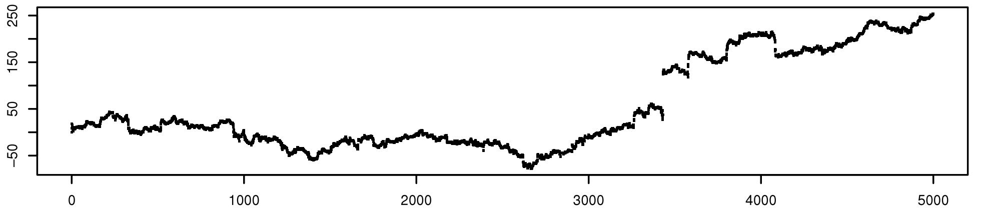
(b) G^{φ_2}.



(c) $G^{(1)}$.



(d) $G^{(2)}$.



(e) $G^{(3)}$.

Fig. 6. The parameters are: $\beta = 1, \eta = 1, \varphi_1 = 0.5, \varphi_2 = 0.995, \pi = 0.9\delta_{\varphi_1} + 0.1\delta_{\varphi_2}$; (a) L is the same Lévy process as in Fig. 5; (b) COGARCH price process driven by L with parameter φ_2; (c), (d) and (f) sup-CO-GARCH price processes $G^{(1)}$, $G^{(2)}$ and $G^{(3)}$ driven by L.

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In [29] a generalized method of moments is used to estimate the supOU model under the hypothesis of a gamma distribution for π. Further, in [17,18] a Bayesian nonparametric approach is proposed in the case that π is a discrete or continuous measure, respectively. Whether and how these approaches, or the estimation procedures for the COGARCH model mentioned in the Introduction can be adapted to the sup-CO-GARCH case, is open.

6. Proofs and auxiliary results

6.1. Proofs for Section 3.1

Proof of Theorem 3.3. First assume that (3.6) holds. Then we know that each COGARCH process V^{φ_i} in the representation $\bar{V}^{(1)} = \sum_{i=1}^{\infty} p_i V^{\varphi_i}$ admits a unique stationary distribution given by the law of $V_{\infty}^{\varphi_i} = \beta \int_{\mathbb{R}_+} \mathrm{e}^{-X_t^{\varphi_i}} \mathrm{d}t$ and that by choosing $V_0^{\varphi_i} \stackrel{\mathrm{d}}{=} V_{\infty}^{\varphi_i}$ independently of S^{φ_i}, the corresponding COGARCH process V^{φ_i} is strictly stationary. Thus, if we set, $\bar{V}_0^{(1)} := \sum_{i\in\mathbb{N}} p_i V_0^{\varphi_i}$, $\bar{V}^{(1)}$ becomes strictly stationary as shown in the following.

Assume for a moment that π has finite support. Then for every $0 \le t_1 < t_2 < \cdots < t_n$, $n \in \mathbb{N}$, $h > 0$ we can use the independence of $(V^{\varphi_i})_{i\in\mathbb{N}}$ to obtain

$$\begin{aligned}(\bar{V}_{t_1}^{(1)}, \ldots, \bar{V}_{t_n}^{(1)}) &= \left(\sum_{i=1}^{m} p_i V_{t_1}^{\varphi_i}, \ldots, \sum_{i=1}^{m} p_i V_{t_n}^{\varphi_i}\right) = \sum_{i=1}^{m} p_i (V_{t_1}^{\varphi_i}, \ldots, V_{t_n}^{\varphi_i}) \\ &\stackrel{\mathrm{d}}{=} \sum_{i=1}^{m} p_i (V_{t_1+h}^{\varphi_i}, \ldots, V_{t_n+h}^{\varphi_i}) = (\bar{V}_{t_1+h}^{(1)}, \ldots, \bar{V}_{t_n+h}^{(1)}).\end{aligned}$$

Due to the fact that $\sum_{i=1}^{m} p_i V_t^{\varphi_i}$ is strictly increasing in m, the case for π having countable support follows now by a standard monotonicity argument.

Conversely, assume that (3.6) is violated, i.e. there exists a φ_j with $\pi(\{\varphi_j\}) > 0$ such that V^{φ_j} has no stationary distribution. Then by [23, Theorem 3.1] $V_t^{\varphi_j}$ converges in probability to ∞ as $t \to \infty$. This yields that also $\bar{V}_t^{(1)} = p_j V_t^{\varphi_j} + \sum_{i=1, i\neq j}^{\infty} p_i V_t^{\varphi_i}$ converges in probability to ∞ as $t \to \infty$ since $\sum_{i=1, i\neq j}^{\infty} p_i V_t^{\varphi_i}$ is nonnegative. Hence $\bar{V}_t^{(1)}$ cannot be strictly stationary. □

Proof of Proposition 3.4. The moment conditions as well as the formulas for expectation and covariance follow directly from (3.7) together with the corresponding results for the COGARCH process (2.10), (2.11) and (2.13) observing that all appearing processes are strictly positive. □

Proof of Proposition 3.5. Throughout this proof we slightly change our notation as follows. Given i.i.d. subordinators $(S^i)_{i\in\mathbb{N}}$, we denote the COGARCH process driven by S^i with parameter $\varphi > 0$ by $V^{i,\varphi}$ such that we have $\bar{V}^{(1)} = \sum_{i=1}^{\infty} p_i V^{i,\varphi_i}$. If $\kappa < \bar{\kappa}$, then we know by the definition of Ψ in (2.3) and [23, Lemma 4.1(d)] that for every $\varphi \in (0, \bar{\varphi}]$ there exists a unique constant $\kappa(\varphi) > 0$ which satisfies (3.11) with $\bar{\kappa}$ replaced by $\kappa(\varphi)$ and such that $\kappa(\varphi)$ is strictly decreasing in φ. Moreover, as shown in [24, Theorem 6], for each $i \in \mathbb{N}$ the tail of $V^{i,\varphi}$ is asymptotically equivalent to $C(\varphi)x^{-\kappa(\varphi)}$ with some specific constant $C(\varphi) > 0$. So by [13, Lemma A3.26] we have

$$x^{\kappa}\mathbb{P}[\bar{V}_0^{(1)} > x] \le x^{\kappa-\bar{\kappa}} x^{\bar{\kappa}} \mathbb{P}\left[\sum_{i=1}^{\infty} p_i V_0^{i,\bar{\varphi}} > x\right] \to 0$$

as $x \to \infty$ for all $\kappa < \bar{\kappa}$. Conversely, if $\kappa > \kappa(\varphi_i)$ for some $i \in \mathbb{N}$, then

$$x^{\kappa}\mathbb{P}[\bar{V}_0^{(1)} > x] \geq x^{\kappa}\mathbb{P}[p_i V_0^{i,\varphi_i} > x] = x^{\kappa-\kappa(\varphi_i)} p_i^{\kappa(\varphi_i)} (x/p_i)^{\kappa(\varphi_i)} \mathbb{P}[V_0^{i,\varphi_i} > x/p_i] \to \infty.$$

Recalling that $\kappa(\varphi)$ is defined via the equation $\Psi(\kappa(\varphi), \varphi) = 0$, this result is still valid for $\kappa > \bar{\kappa}$ since we have $\inf_{i\in\mathbb{N}} \kappa(\varphi_i) = \bar{\kappa}$ by the implicit function theorem.

Finally, it remains to consider the case $\kappa = \bar{\kappa}$. If $\pi(\{\bar{\varphi}\}) = 0$, then using [24, Lemma 2] and again [13, Lemma A3.26], we obtain

$$x^{\bar{\kappa}}\mathbb{P}[\bar{V}_0^{(1)} > x] \leq x^{\bar{\kappa}}\mathbb{P}\left[\sum_{\varphi_i \leq \varphi} p_i V_0^{i,\varphi} + \sum_{\varphi_i > \varphi} p_i V_0^{i,\bar{\varphi}} > x\right] \sim x^{\bar{\kappa}}\mathbb{P}\left[\sum_{\varphi_i > \varphi} p_i V_0^{i,\bar{\varphi}} > x\right]$$
$$\to \sum_{\varphi_i > \varphi} p_i^{\bar{\kappa}} C(\bar{\varphi})$$

as $x \to \infty$ for every $\varphi \in (0, \bar{\varphi})$. Letting $\varphi \to \bar{\varphi}$, the assertion follows. The case $\pi(\{\bar{\varphi}\}) =: \bar{p} > 0$ now follows from the results above and ($\bar{i}$ is the index corresponding to $\bar{\varphi}$)

$$x^{\bar{\kappa}}\mathbb{P}[\bar{V}_0^{(1)} > x] = x^{\bar{\kappa}}\mathbb{P}\left[\sum_{\varphi_i < \bar{\varphi}} p_i V_0^{i,\varphi_i} + \bar{p} V_0^{\bar{i},\bar{\varphi}} > x\right]$$
$$\leq x^{\bar{\kappa}}\mathbb{P}[\bar{p} V_0^{\bar{i},\bar{\varphi}} > x(1-\varepsilon)] + x^{\bar{\kappa}}\mathbb{P}\left[\sum_{\varphi_i < \varphi} p_i V_0^{i,\varphi_i} > \varepsilon x\right] \to \left(\frac{\bar{p}}{1-\varepsilon}\right)^{\bar{\kappa}} C(\bar{\varphi}).$$

Letting $\varepsilon \to 0$, we may set $C := \bar{p}^{\bar{\kappa}} C(\bar{\varphi})$. □

6.2. *Proofs for Section 3.2*

For the proof of Theorem 3.10 we need the following lemma.

Lemma 6.1. *Let $(S_t)_{t\in\mathbb{R}}$ be a subordinator without drift and define the double-indexed processes $(X_t^{\varphi})_{t\in\mathbb{R},\varphi\in\Phi_L}$ and $(V_t^{\varphi})_{t\in\mathbb{R},\varphi\in\Phi_L}$ according to (2.16) and (2.17). Then for all $n \in \mathbb{N}$, $-\infty < t_1 < t_2 < \cdots < t_n < \infty$, $h > 0$*

$$((V_{t_1}^{\varphi})_{\varphi\in\Phi_L}, (V_{t_2}^{\varphi})_{\varphi\in\Phi_L}, \ldots, (V_{t_n}^{\varphi})_{\varphi\in\Phi_L}) \stackrel{\mathrm{d}}{=} ((V_{t_1+h}^{\varphi})_{\varphi\in\Phi_L}, (V_{t_2+h}^{\varphi})_{\varphi\in\Phi_L}, \ldots, (V_{t_n+h}^{\varphi})_{\varphi\in\Phi_L}),$$

i.e. the $\mathbb{R}^{\Phi_L}$-valued stochastic process $((V_t^{\varphi})_{\varphi\in\Phi_L})_{t\in\mathbb{R}}$ is strictly stationary. In particular, every finite-dimensional process $(V_t^{\varphi_1}, \ldots, V_t^{\varphi_m})_{t\in\mathbb{R}}$, $m \in \mathbb{N}$, is strictly stationary.

Proof. Imitating the proof of [23, Theorem 3.2] for the finite-dimensional process $(V_t^{\varphi_1}, \ldots, V_t^{\varphi_m})_{t\in\mathbb{R}}$, $m \in \mathbb{N}$, one readily sees that

$$\left((V_{t_1}^{\varphi_1}, \ldots, V_{t_1}^{\varphi_m}), \ldots, (V_{t_n}^{\varphi_1}, \ldots, V_{t_n}^{\varphi_m})\right)$$
$$\stackrel{\mathrm{d}}{=} \left((V_{t_1+h}^{\varphi_1}, \ldots, V_{t_1+h}^{\varphi_m}), \ldots, (V_{t_n+h}^{\varphi_1}, \ldots, V_{t_n+h}^{\varphi_m})\right).$$

As stochastic processes with the same index space are equal in distribution, whenever their finite-dimensional distributions are equal (e.g. [22, Proposition 2.2]), this already yields the assertion. □

Proof of Theorem 3.10. The result follows from the definition of $\bar{V}^{(2)}$ and Lemma 6.1. □

To prove Proposition 3.11, another auxiliary lemma will be established.

Lemma 6.2. *Let $(S_t)_{t\in\mathbb{R}}$ be a subordinator without drift, let $\varphi, \tilde{\varphi} \geq 0$ be fixed and define the processes $(X_t^{\varphi})_{t\in\mathbb{R}}$ and $(X_t^{\tilde{\varphi}})_{t\in\mathbb{R}}$ according to (2.16). Set $X_t := X_t^{\varphi} + X_t^{\tilde{\varphi}},\ t \in \mathbb{R}$.*

(a) *The process $(X_t)_{t\in\mathbb{R}}$ is a Lévy process with characteristic triplet $(2\eta, 0, \nu_X)$ where $\nu_X = \nu_S \circ T^{-1}$ for $T\colon \mathbb{R}_+ \to \mathbb{R}_-,\ y \mapsto -\log(1 + (\varphi+\tilde{\varphi})y + \varphi\tilde{\varphi}y^2)$.*

(b) *Let $\varphi, \tilde{\varphi} > 0$. Then $\mathbb{E}[\mathrm{e}^{-\kappa X_t}]$ is finite at $\kappa > 0$ for some $t > 0$, or, equivalently, for all $t > 0$ if and only if $\mathbb{E}[S_1^{2\kappa}] < \infty$. In this case we have $\mathbb{E}[\mathrm{e}^{-\kappa X_t}] = \mathrm{e}^{t h_\kappa(\varphi,\tilde{\varphi})}$, where*

$$h_\kappa(\varphi,\tilde{\varphi}) = -2\eta\kappa + \int_{\mathbb{R}_+} \left(((1+\varphi y)(1+\tilde{\varphi}y))^\kappa - 1\right) \nu_S(\mathrm{d}y).$$

For $\kappa = 1$ we have

$$h(\varphi,\tilde{\varphi}) := h_1(\varphi,\tilde{\varphi}) = -2\eta + (\varphi+\tilde{\varphi})\mathbb{E}[S_1] + \varphi\tilde{\varphi}\,\mathrm{Var}[S_1]. \tag{6.1}$$

Proof. (a) Observe that by definition

$$\begin{aligned} X_t &= 2\eta t - \sum_{0<s\leq t} \log\left[(1+\varphi\Delta S_s)(1+\tilde{\varphi}\Delta S_s)\right] \\ &= 2\eta t - \sum_{0<s\leq t} \log(1 + (\varphi+\tilde{\varphi})\Delta S_s + \varphi\tilde{\varphi}(\Delta S_s)^2) \end{aligned}$$

for $t \geq 0$, which directly yields the assertion.

(b) By [28, Theorem 25.17] $\mathbb{E}[\mathrm{e}^{-\kappa X_t}]$ is finite for some, or, equivalently, for every $t > 0$ if and only if

$$\begin{aligned} \int_{|y|>1} \mathrm{e}^{-\kappa y}\, \nu_X(\mathrm{d}y) &= \int_{|y|>1} \mathrm{e}^{-\kappa y}\, \nu_S(T^{-1}(\mathrm{d}y)) \\ &= \int_{y\in D^c} (1 + (\varphi+\tilde{\varphi})y + \varphi\tilde{\varphi}y^2)^\kappa\, \nu_S(\mathrm{d}y) < \infty \end{aligned}$$

where $D = \left[\frac{-(\varphi+\tilde{\varphi})-\sqrt{(\varphi-\tilde{\varphi})^2+4\varphi\tilde{\varphi}\mathrm{e}}}{2\varphi\tilde{\varphi}}, \frac{-(\varphi+\tilde{\varphi})+\sqrt{(\varphi-\tilde{\varphi})^2+4\varphi\tilde{\varphi}\mathrm{e}}}{2\varphi\tilde{\varphi}}\right]$. This directly yields (b). □

Proof of Proposition 3.11. Due to Lemma 6.1 $(V_t^{\varphi}, V_t^{\tilde{\varphi}})_{t\in\mathbb{R}}$ is strictly stationary such that it suffices to consider $t > 0$. Assume w.l.o.g. that $0 < \varphi \leq \tilde{\varphi}$. Then it follows from the definition of the COGARCH process that $V^{\varphi} \leq V^{\tilde{\varphi}}$. Hence $\mathbb{E}[V_t^{\varphi} V_t^{\tilde{\varphi}}] \leq \mathbb{E}[V_t^{\tilde{\varphi}} V_t^{\tilde{\varphi}}]$ and similarly $\mathbb{E}[V_t^{\varphi} V_{t+h}^{\tilde{\varphi}}] \leq \mathbb{E}[V_t^{\tilde{\varphi}} V_{t+h}^{\tilde{\varphi}}]$, which are both finite as (2.10) is given for $\kappa = 2$. We start with the computation of $\mathbb{E}[V_t^{\varphi} V_t^{\tilde{\varphi}}]$ and use (2.5) to obtain

$$\begin{aligned} \mathbb{E}[V_t^{\varphi} V_t^{\tilde{\varphi}}] &= \beta^2 \mathbb{E}\left[\int_{(0,t]} \mathrm{e}^{X_s^{\varphi}-X_t^{\varphi}}\,\mathrm{d}s \int_{(0,t]} \mathrm{e}^{X_r^{\tilde{\varphi}}-X_t^{\tilde{\varphi}}}\,\mathrm{d}r\right] \\ &\quad + \beta\mathbb{E}[V_0^{\tilde{\varphi}}]\mathbb{E}\left[\int_{(0,t]} \mathrm{e}^{X_s^{\varphi}-X_t^{\varphi}-X_t^{\tilde{\varphi}}}\,\mathrm{d}s\right] \\ &\quad + \beta\mathbb{E}[V_0^{\varphi}]\mathbb{E}\left[\int_{(0,t]} \mathrm{e}^{X_r^{\tilde{\varphi}}-X_t^{\tilde{\varphi}}-X_t^{\varphi}}\,\mathrm{d}r\right] + \mathbb{E}[V_0^{\varphi}V_0^{\tilde{\varphi}}]\mathbb{E}[\mathrm{e}^{-X_t^{\varphi}-X_t^{\tilde{\varphi}}}] \\ &=: \beta^2 I_1 + \beta\mathbb{E}[V_0^{\tilde{\varphi}}]I_2 + \beta\mathbb{E}[V_0^{\varphi}]I_3 + \mathbb{E}[V_0^{\varphi}V_0^{\tilde{\varphi}}]I_4. \end{aligned} \tag{6.2}$$

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Recall the Lévy process X defined in Lemma 6.2 and observe that the increments of X and X^φ on disjoint intervals are mutually independent. Thus we have by (2.3) and Lemma 6.2(b)

$$\begin{aligned} I_1 &= \mathbb{E}\left[\int_{(0,t]}\int_{(0,r]} \mathrm{e}^{X_s^\varphi - X_r^\varphi + X_r^\varphi - X_t^\varphi + X_r^{\tilde\varphi} - X_t^{\tilde\varphi}}\,\mathrm{d}s\,\mathrm{d}r\right] \\ &\quad + \mathbb{E}\left[\int_{(0,t]}\int_{(r,t]} \mathrm{e}^{X_r^{\tilde\varphi} - X_s^{\tilde\varphi} + X_s^{\tilde\varphi} - X_t^{\tilde\varphi} + X_s^\varphi - X_t^\varphi}\,\mathrm{d}s\,\mathrm{d}r\right] \\ &= \int_{(0,t]}\int_{(0,r]} \mathrm{e}^{(r-s)\,\Psi(1,\varphi) + (t-r)h(\varphi,\tilde\varphi)}\,\mathrm{d}s\,\mathrm{d}r + \int_{(0,t]}\int_{(r,t]} \mathrm{e}^{(s-r)\,\Psi(1,\tilde\varphi) + (t-s)h(\varphi,\tilde\varphi)}\,\mathrm{d}s\,\mathrm{d}r \\ &= \frac{-a\mathrm{e}^{ct} + c\mathrm{e}^{at} + a - c}{a^2c - ac^2} + \frac{-b\mathrm{e}^{ct} + c\mathrm{e}^{bt} + b - c}{b^2c - bc^2}, \end{aligned}$$

where $a := \Psi(1,\varphi)$, $b := \Psi(1,\tilde\varphi)$ and $c := h(\varphi,\tilde\varphi)$. Very similar calculations lead to

$$I_2 = \frac{\mathrm{e}^{bt} - \mathrm{e}^{ct}}{b - c}, \qquad I_3 = \frac{\mathrm{e}^{at} - \mathrm{e}^{ct}}{a - c}$$

while we know from Lemma 6.2(b) that $I_4 = \mathrm{e}^{ct}$.

According to (2.11) we have $\mathbb{E}[V_0^\varphi] = -\beta/a$ and $\mathbb{E}[V_0^{\tilde\varphi}] = -\beta/b$. Furthermore, we have $\mathbb{E}[V_0^\varphi V_0^{\tilde\varphi}] = \mathbb{E}[V_t^\varphi V_t^{\tilde\varphi}]$ due to stationarity. Putting all this into (6.2), we obtain

$$(1 - \mathrm{e}^{ct})\mathbb{E}[V_t^\varphi V_t^{\tilde\varphi}] = \beta^2(1 - \mathrm{e}^{ct})\left(\frac{1}{ac} + \frac{1}{bc}\right).$$

Since $t > 0$ we have $1 - \mathrm{e}^{ct} \neq 0$, so dividing the last equation by this term yields

$$\mathbb{E}[V_t^\varphi V_t^{\tilde\varphi}] = \frac{\beta^2}{\Psi(1,\tilde\varphi)h(\varphi,\tilde\varphi)} + \frac{\beta^2}{\Psi(1,\varphi)h(\varphi,\tilde\varphi)},$$

from which (3.16) and (3.17) follow immediately.

To obtain the formula for $\mathrm{Cov}[V_t^\varphi, V_{t+h}^{\tilde\varphi}]$, observe first that

$$V_{t+h}^{\tilde\varphi} = A_{t,t+h}^{\tilde\varphi} V_t^{\tilde\varphi} + B_{t,t+h}^{\tilde\varphi}, \tag{6.3}$$

where

$$A_{t,t+h}^{\tilde\varphi} = \mathrm{e}^{-(X_{t+h}^{\tilde\varphi} - X_t^{\tilde\varphi})} \quad \text{and} \quad B_{t,t+h}^{\tilde\varphi} = \beta\int_{(t,t+h]} \mathrm{e}^{-(X_{t+h}^{\tilde\varphi} - X_s^{\tilde\varphi})}\,\mathrm{d}s.$$

In particular, we see that $A_{t,t+h}^{\tilde\varphi}$ and $B_{t,t+h}^{\tilde\varphi}$ are independent of $(V_t^\varphi, V_t^{\tilde\varphi})$ such that

$$\begin{aligned} \mathbb{E}[V_t^\varphi V_{t+h}^{\tilde\varphi}] &= \mathbb{E}[V_t^\varphi(A_{t,t+h}^{\tilde\varphi} V_t^{\tilde\varphi} + B_{t,t+h}^{\tilde\varphi})] \\ &= \mathbb{E}[A_{t,t+h}^{\tilde\varphi}]\mathbb{E}[V_t^\varphi V_t^{\tilde\varphi}] + \mathbb{E}[V_t^\varphi]\mathbb{E}[B_{t,t+h}^{\tilde\varphi}]. \end{aligned} \tag{6.4}$$

Now since

$$\mathbb{E}[A_{t,t+h}^{\tilde\varphi}] = \mathbb{E}[\mathrm{e}^{-(X_{t+h}^{\tilde\varphi} - X_t^{\tilde\varphi})}] = \mathbb{E}[\mathrm{e}^{-X_h^{\tilde\varphi}}] = \mathrm{e}^{h\,\Psi(1,\tilde\varphi)}$$

and

$$\begin{aligned} \mathbb{E}[B_{t,t+h}^{\tilde\varphi}] &= \beta\int_{(t,t+h]} \mathrm{e}^{(t+h-s)\,\Psi(1,\tilde\varphi)}\,\mathrm{d}s = \frac{\beta}{\Psi(1,\tilde\varphi)}\left(\mathrm{e}^{h\,\Psi(1,\tilde\varphi)} - 1\right) \\ &= \mathbb{E}[V_0^{\tilde\varphi}]\left(1 - \mathrm{e}^{h\,\Psi(1,\tilde\varphi)}\right), \end{aligned}$$

Eq. (6.4) directly yields

$$\mathbb{E}[V_t^{\varphi} V_{t+h}^{\tilde{\varphi}}] = \mathrm{e}^{h\Psi(1,\tilde{\varphi})}\mathbb{E}[V_0^{\varphi} V_0^{\tilde{\varphi}}] + \left(1 - \mathrm{e}^{h\Psi(1,\tilde{\varphi})}\right)\mathbb{E}[V_0^{\varphi}]\mathbb{E}[V_0^{\tilde{\varphi}}],$$

which gives (3.18). □

Proof of Proposition 3.12. Due to the fact that all appearing processes are nonnegative, we can use Tonelli's Theorem to determine the given formulas directly from the definition of $\bar{V}^{(2)}$. □

Proof of Proposition 3.13. The proof is mainly the same as the proof of Proposition 3.5, so we only indicate the differences. For $\kappa < \bar{\kappa}$ use the estimation $\mathbb{P}[\bar{V}_0^{(2)} > x] \leq \mathbb{P}[V_0^{\bar{\varphi}} > x]$. For $\kappa > \bar{\kappa}$ and $\pi(\{\bar{\varphi}\}) = 0$, it suffices to consider $\kappa > \kappa(\varphi_i)$ after having chosen sequences $(\varphi_i)_{i\in\mathbb{N}}$ and $(\varepsilon_i)_{i\in\mathbb{N}}$ with $\pi((\varphi_i - \varepsilon_i, \varphi_i]) > 0$ for each $i \in \mathbb{N}$. Using the fact $\mathbb{P}[\bar{V}_0^{(2)} > x] \geq \mathbb{P}\left[\pi((\varphi_i - \varepsilon_i, \varphi_i])V_0^{\varphi_i} > x\right]$ gives the result. Similarly, use $\pi(\{\bar{\varphi}\}) = 0$ and $\mathbb{P}[\bar{V}_0^{(2)} > x] \leq \mathbb{P}\left[\pi((0,\varphi])V_0^{\varphi} + \pi((\varphi,\bar{\varphi}])V_0^{\bar{\varphi}} > x\right]$ for $\kappa = \bar{\kappa}$. For $\kappa = \bar{\kappa}$ and $\pi(\{\bar{\varphi}\}) =: \bar{p} > 0$, we may use $\bar{V}_0^2 = \int_{(0,\bar{\varphi})} V_0^{\varphi}\,\pi(\mathrm{d}\varphi) + \bar{p}V_0^{\bar{\varphi}}$. Finally, the case $\kappa > \bar{\kappa}$ and $\pi(\{\bar{\varphi}\}) > 0$ follows from the preceding arguments. □

6.3. Proofs for Section 3.3

Proof of Proposition 3.15. By (2.16) and (2.17), the function $\varphi \mapsto V_s^{\varphi}$ is increasing in φ for every $s \in \mathbb{R}$. As a consequence, we have for all $t \geq 0$

$$A_t \leq \int_{(0,t]}\int_{\Phi_L} \varphi_{\max} V_s^{\varphi_{\max}}\, \Lambda^S(\mathrm{d}s, \mathrm{d}\varphi) = \varphi_{\max}\int_{(0,t]} V_s^{\varphi_{\max}}\,\mathrm{d}S_s < \infty.$$

Since A is by definition càdlàg, $\mathbb{G}^{(3)}$-adapted and increasing, A is a semimartingale [20, e.g. Definition I.4.21] such that uniqueness of the solution of (3.24) follows from [26, Theorem V.7]. It remains to show that (3.26) solves (3.24). Using integration by parts [20, Definition I.4.45] and [20, Proposition I.4.49d], we obtain

$$\begin{aligned}
\mathrm{d}\bar{V}_t^{(3)} &= \left(\bar{V}_0^{(3)} + \int_{(0,t]} \mathrm{e}^{\eta s}\,\mathrm{d}A_s + \beta\int_{(0,t]}\mathrm{e}^{\eta s}\,\mathrm{d}s\right)\mathrm{d}\left(\mathrm{e}^{-\eta t}\right) + \mathrm{e}^{-\eta t}(\mathrm{e}^{\eta t}\,\mathrm{d}A_t + \beta\mathrm{e}^{\eta t}\,\mathrm{d}t)\\
&= -\eta\bar{V}_t^{(3)}\,\mathrm{d}t + \mathrm{d}A_t + \beta\,\mathrm{d}t = (\beta - \eta\bar{V}_t^{(3)})\,\mathrm{d}t + \mathrm{d}A_t. \quad \square
\end{aligned}$$

In order to show that the sup-CO-GARCH 3 process $\bar{V}^{(3)}$ from (3.24) has a stationary solution we need a series of lemmata.

Lemma 6.3. *Let $n, m \in \mathbb{N}$. For $-\infty < t_1 < \cdots < t_{m+1} < \infty$, $0 < \varphi_1 < \cdots < \varphi_{n+1} < \varphi_{\max}$ and $h > 0$ we have*

$$\begin{aligned}
&(V_{t_j}^{\varphi_i}, \Lambda^S((t_j, t_{j+1}] \times (\varphi_i, \varphi_{i+1}]) : i \leq n,\ j \leq m)\\
&\overset{\mathrm{d}}{=} (V_{t_j+h}^{\varphi_i}, \Lambda^S((t_j + h, t_{j+1} + h] \times (\varphi_i, \varphi_{i+1}]) : i \leq n,\ j \leq m).
\end{aligned} \tag{6.5}$$

Proof. For $1 \leq i \leq n$ and $1 \leq j \leq m$ write $\Lambda_j^i := \Lambda^S((t_j, t_{j+1}] \times (\varphi_i, \varphi_{i+1}])$ and $\Lambda_{j,h}^i := \Lambda^S((t_j + h, t_{j+1} + h] \times (\varphi_i, \varphi_{i+1}])$, and let Z^m and Z_h^m denote the left- and right-hand side of (6.5), respectively. We first consider $m = 1$. On the one hand, we obtain from Lemma 6.1 that $(V_{t_1}^{\varphi_1}, \ldots, V_{t_1}^{\varphi_n}) \overset{\mathrm{d}}{=} (V_{t_1+h}^{\varphi_1}, \ldots, V_{t_1+h}^{\varphi_n})$. On the other hand, due to the independence of their

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single components, the vectors $(\Lambda_1^1, \ldots, \Lambda_1^n)$ and $(\Lambda_{1,h}^1, \ldots, \Lambda_{1,h}^n)$ have the same distribution. Since additionally the V-vector is independent of the Λ^S-vector, the assertion in the case $m = 1$ follows. For $m \geq 2$, using induction and the independence of Λ_m^i and Z^{m-1}, it suffices to show that the conditional distribution of $(V_{t_m}^{\varphi_i} : i = 1, \ldots, n)$ given Z^{m-1} does not change if shifted by h. By Markovianity (see [23, Theorem 3.2]) this distribution only depends on $(V_{t_{m-1}}^{\varphi_i}, \Lambda_{m-1}^i : i = 1, \ldots, n)$ such that by (6.3) and using the notation there, we only need to consider the distribution of $(A_{t_{m-1},t_m}^{\varphi_i}, B_{t_{m-1},t_m}^{\varphi_i} : i = 1, \ldots, n)$ given $(\Lambda_{m-1}^i : i = 1, \ldots, n)$. Since the former vector is a measurable transformation of $(\Delta S_s : t_{m-1} \leq s \leq t_m)$, it is evident that this distribution is invariant under a shift by h, which finishes the proof. □

In connection to (3.27), we show a further auxiliary result. To this end, define

$$\begin{aligned} A_t &:= \int_{(0,t]} \int_{\Phi_L} \varphi V_{s-}^{\varphi} \, \Lambda^S(\mathrm{d}s, \mathrm{d}\varphi), \quad t \geq 0, \\ A_t &:= -\int_{(t,0]} \int_{\Phi_L} \varphi V_{s-}^{\varphi} \, \Lambda^S(\mathrm{d}s, \mathrm{d}\varphi), \quad t < 0. \end{aligned} \tag{6.6}$$

Lemma 6.4. *The process $(A_t)_{t \in \mathbb{R}}$ defined in (6.6) has stationary increments, i.e. for every $n \in \mathbb{N}$, $-\infty < t_1 < \cdots < t_{n+1} < \infty$ and $h > 0$ we have*

$$(A_{t_2} - A_{t_1}, \ldots, A_{t_{n+1}} - A_{t_n}) \stackrel{\mathrm{d}}{=} (A_{t_2+h} - A_{t_1+h}, \ldots, A_{t_{n+1}+h} - A_{t_n+h}). \tag{6.7}$$

Proof. By an approximation via Riemann sums (note that $\varphi \mapsto V_s^{\varphi}$ is continuous in φ for all s), cf. [20, Proposition I.4.44], we may use Lemma 6.3 to obtain

$$\begin{aligned} &(A_{t_2} - A_{t_1}, \ldots, A_{t_{n+1}} - A_{t_n}) \\ &= \left(\int_{t_1}^{t_2} \int_{\Phi_L} \varphi V_{s-}^{\varphi} \, \Lambda^S(\mathrm{d}s, \mathrm{d}\varphi), \ldots, \int_{t_n}^{t_{n+1}} \int_{\Phi_L} \varphi V_{s-}^{\varphi} \, \Lambda^S(\mathrm{d}s, \mathrm{d}\varphi) \right) \\ &\stackrel{\mathrm{d}}{=} \left(\int_{t_1+h}^{t_2+h} \int_{\Phi_L} \varphi V_{s-}^{\varphi} \, \Lambda^S(\mathrm{d}s, \mathrm{d}\varphi), \ldots, \int_{t_n+h}^{t_{n+1}+h} \int_{\Phi_L} \varphi V_{s-}^{\varphi} \, \Lambda^S(\mathrm{d}s, \mathrm{d}\varphi) \right) \\ &= (A_{t_2+h} - A_{t_1+h}, \ldots, A_{t_{n+1}+h} - A_{t_n+h}). \quad \square \end{aligned}$$

Proof of Theorem 3.17. Since $\mathrm{e}^{-\eta t} \int_{(0,t]} \mathrm{e}^{\eta s} \, \mathrm{d}s \to \eta^{-1}$ as $t \to \infty$ the process $(V_t^{(3)})_{t \geq 0}$ converges in distribution to a finite random variable as $t \to \infty$ if and only if

$$\begin{aligned} \mathrm{e}^{-\eta t} \int_{(0,t]} \mathrm{e}^{\eta s} \, \mathrm{d}A_s &= \int_{(0,t]} \mathrm{e}^{\eta(s-t)} \, \mathrm{d}A_s \\ &= \int_{(-t,0]} \mathrm{e}^{\eta s} \, \mathrm{d}A_{s+t} \stackrel{\mathrm{d}}{=} \int_{(-t,0]} \mathrm{e}^{\eta s} \, \mathrm{d}A_s \stackrel{\mathrm{d}}{=} \int_{(0,t]} \mathrm{e}^{-\eta s} \, \mathrm{d}A_s \end{aligned}$$

converges to a finite random variable in distribution as $t \to \infty$, where we used Lemma 6.4 for the distributional equalities. By monotonicity this is equivalent to the existence of

$$\int_{\mathbb{R}_+} \mathrm{e}^{-\eta s} \, \mathrm{d}A_s = \int_{\mathbb{R}_+} \int_{\Phi_L} \mathrm{e}^{-\eta s} \varphi V_{s-}^{\varphi} \, \Lambda^S(\mathrm{d}s, \mathrm{d}\varphi)$$

in probability. As shown in [12, Theorem 3.1] and the following remark, this holds if and only if (3.28) is valid.

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Hence in case that (3.28) is violated, no stationary distribution can exist. On the other hand, given (3.28), following the above computations, the process $(V_t^{(3)})_{t\geq 0}$ converges in distribution to $\bar{V}_\infty^{(3)} := \frac{\beta}{\eta} + \int_0^\infty \mathrm{e}^{-\eta s}\,\mathrm{d}A_s$, which is thus the only possible stationary distribution.

To show that $(V_t^{(3)})_{t\geq 0}$ is actually strictly stationary when started in a random variable $V_0^{(3)} \stackrel{\mathrm{d}}{=} \bar{V}_\infty^{(3)}$ which is independent of Λ^L on $\mathbb{R}_+ \times \Phi_L$, we set $\bar{V}_0^{(3)} := \frac{\beta}{\eta} + \int_{(-\infty,0]} \mathrm{e}^{\eta s}\,\mathrm{d}A_s$. Then using Lemma 6.4 we obtain for all $0 \leq t_1 < \cdots < t_n$ and $h > 0$

$$\begin{aligned}
&(\bar{V}_{t_1}^{(3)}, \ldots, \bar{V}_{t_n}^{(3)})\\
&\stackrel{\mathrm{d}}{=} \bigg(\int_{(-\infty,t_1]} \mathrm{e}^{-\eta(t_1-s)}\,\mathrm{d}A_s + \beta\int_{(-\infty,t_1]} \mathrm{e}^{-\eta(t_1-s)}\,\mathrm{d}s, \ldots, \int_{(-\infty,t_n]} \mathrm{e}^{-\eta(t_n-s)}\,\mathrm{d}A_s\\
&\qquad + \beta\int_{(-\infty,t_n]} \mathrm{e}^{-\eta(t_n-s)}\,\mathrm{d}s\bigg)\\
&= \bigg(\int_{(-\infty,0]} \mathrm{e}^{\eta s}\,\mathrm{d}A_{s+t_1} + \beta\int_{\mathbb{R}_+} \mathrm{e}^{-\eta s}\,\mathrm{d}s, \ldots, \int_{(-\infty,0]} \mathrm{e}^{\eta s}\,\mathrm{d}A_{s+t_n} + \beta\int_{\mathbb{R}_+} \mathrm{e}^{-\eta s}\,\mathrm{d}s\bigg)\\
&\stackrel{\mathrm{d}}{=} \bigg(\int_{(-\infty,0]} \mathrm{e}^{\eta s}\,\mathrm{d}A_{s+t_1+h} + \beta\int_{\mathbb{R}_+} \mathrm{e}^{-\eta s}\,\mathrm{d}s, \ldots, \int_{(-\infty,0]} \mathrm{e}^{\eta s}\,\mathrm{d}A_{s+t_n+h} + \beta\int_{\mathbb{R}_+} \mathrm{e}^{-\eta s}\mathrm{d}s\bigg)\\
&\stackrel{\mathrm{d}}{=} (\bar{V}_{t_1+h}^{(3)}, \ldots, \bar{V}_{t_n+h}^{(3)}),
\end{aligned}$$

and hence the process $(V_t^{(3)})_{t\geq 0}$ is strictly stationary.

It remains to show that (a) and (b) imply (3.28). First observe from (2.16) and (2.17) that for fixed s the function $\varphi \mapsto V_s^\varphi$ is increasing in φ. So if (a) holds, we have

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\Phi_L}\int_{\mathbb{R}_+} 1 \wedge (y\varphi V_s^\varphi \mathrm{e}^{-\eta s})\,\mathrm{d}s\,\pi(\mathrm{d}\varphi)\,\nu_S(\mathrm{d}y)\\
&\quad \leq \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} 1 \wedge (y\varphi_0 V_s^{\varphi_0}\mathrm{e}^{-\eta s})\,\mathrm{d}s\,\nu_S(\mathrm{d}y) < \infty
\end{aligned}$$

because (3.28) holds for $\pi = \delta_{\varphi_0}$ (in this case $\bar{V}^{(3)}$ is just the COGARCH process V^{φ_0}).

Finally, (b) follows from (a) together with the fact that $\varphi_{\max}^{(\kappa)} < \varphi_{\max}$. □

For the proof of Proposition 3.18 we need the following lemma.

Lemma 6.5. *Let $(A_t)_{t\in\mathbb{R}}$, V^φ and $\bar{V}^{(3)}$ be defined as in (6.6), (2.17) and (3.29), respectively. Then, under the assumptions of Proposition 3.18, we have for $t \geq 0$*

$$\begin{aligned}
[A, A]_t &= \int_{(0,t]}\int_{\Phi_L}\int_{\mathbb{R}_+} \varphi^2 (V_{s-}^\varphi)^2 y^2\,\mu^{\Lambda^S}(\mathrm{d}s, \mathrm{d}\varphi, \mathrm{d}y) \quad \textit{and}\\
[\bar{V}^{(3)}, V^\varphi]_t &= [A, V^\varphi]_t = \varphi\int_{(0,t]}\int_{\Phi_L}\int_{\mathbb{R}_+} \tilde{\varphi} V_{s-}^\varphi V_{s-}^{\tilde{\varphi}} y^2\,\mu^{\Lambda^S}(\mathrm{d}s, \mathrm{d}\tilde{\varphi}, \mathrm{d}y),
\end{aligned}$$

with μ^{Λ^S} as defined in (2.18). For $t < 0$, let the expressions on the left-hand side denote the respective quadratic (co-)variation on $(t, 0]$. Then the integrals have to be computed on $(-t, 0]$ instead of $(0, t]$.

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Proof. Obviously it suffices to consider $t \geq 0$. Since A is an increasing pure-jump process,

$$[A, A]_t = \sum_{0<s\leq t} (\Delta A_s)^2 = \sum_{0<s\leq t} \left(\Delta(\varphi V^{\varphi}_{\cdot -} y * \mu^{\Lambda^S})_s\right)^2$$
$$= \sum_{0<s\leq t} \left(\sum_{\varphi\in\Phi_L} \varphi V^{\varphi}_{s-} \Lambda^S(\{s\} \times \{\varphi\}) \right)^2.$$

Noting that for almost every ω there is at most one $\varphi \in \Phi_L$ at time s with $\Lambda^S(\{s\} \times \{\varphi\})(\omega) \neq 0$, we obtain

$$[A, A]_t = \sum_{0<s\leq t} \sum_{\varphi\in\Phi_L} \varphi^2 (V^{\varphi}_{s-})^2 \Lambda^S(\{s\} \times \{\varphi\})^2,$$

as desired. Similarly,

$$[A, V^{\varphi}]_t = \sum_{0<s\leq t} \Delta A_s \Delta V^{\varphi}_s = \sum_{0<s\leq t} \left(\sum_{\tilde\varphi\in\Phi_L} \tilde\varphi V^{\tilde\varphi}_{s-} \Lambda^S(\{s\} \times \{\tilde\varphi\}) \right) \varphi V^{\varphi}_{s-} \Delta S_s$$

according to (2.6). Now observe that for all $s \in \mathbb{R}$, $\Delta S_s = \Lambda^S(\{s\}\times\mathbb{R}_+) = \sum_{\varphi\in\Phi_L} \Lambda^S(\{s\}\times\{\varphi\})$ where again for almost every ω there is at most one $\varphi \in \Phi_L$ at time s with $\Lambda^S(\{s\} \times \{\varphi\})(\omega) \neq 0$. As a result,

$$[A, V^{\varphi}]_t = \sum_{0<s\leq t} \varphi V^{\varphi}_{s-} \sum_{\tilde\varphi\in\Phi_L} \tilde\varphi V^{\tilde\varphi}_{s-} \Lambda^S(\{s\} \times \{\tilde\varphi\})^2 = \varphi(\tilde\varphi V^{\varphi}_{\cdot -} V^{\tilde\varphi}_{\cdot -} y^2 * \mu^{\Lambda^S}_t).$$

Finally, we have $[\bar V^{(3)}, V^{\varphi}] = [A, V^{\varphi}]$ by (3.24). □

Proof of Proposition 3.18. First observe that Theorem 3.17(c) ensures the existence of the given stationary version of $\bar V^{(3)}$ under the assumptions of the present theorem.

We set $m_1 := \int_{\mathbb{R}_+} y\, \nu_S(\mathrm{d}y) = \mathbb{E}[S_1]$, $m_2 := \int_{\mathbb{R}_+} y^2\, \nu_S(\mathrm{d}y) = \mathrm{Var}[S_1]$ and assume w.l.o.g. $\pi(\{0\}) = 0$. For the mean we use (2.11) and obtain

$$\mathbb{E}[\bar V^{(3)}_t] = \mathbb{E}[\bar V^{(3)}_0] = \frac{\beta}{\eta} + \mathbb{E}\left[\int_{(-\infty,0]} \mathrm{e}^{\eta s}\, \mathrm{d}A_s\right]$$
$$= \frac{\beta}{\eta} + m_1 \int_{(-\infty,0]} \mathrm{e}^{\eta s}\, \mathrm{d}s \int_{\Phi_L} \varphi \mathbb{E}[V^{\varphi}_0]\, \pi(\mathrm{d}\varphi)$$
$$= \frac{\beta}{\eta} - \frac{\beta}{\eta} \int_{\Phi_L} \left(1 + \frac{\eta}{m_1\varphi - \eta}\right) \pi(\mathrm{d}\varphi)$$
$$= -\int_{\Phi_L} \frac{\beta}{m_1\varphi - \eta}\, \pi(\mathrm{d}\varphi) = \int_{\Phi_L} \mathbb{E}[V^{\varphi}_0]\, \pi(\mathrm{d}\varphi).$$

To compute the autocovariance function of $\bar V^{(3)}$, observe that for $t \geq 0$ and $h \geq 0$ we have from (3.29)

$$\mathrm{Cov}[\bar V^{(3)}_t, \bar V^{(3)}_{t+h}] = \mathrm{e}^{-2\eta t} \mathrm{e}^{-\eta h} \mathbb{E}\left[\int_{(-\infty,t]} \mathrm{e}^{\eta s}\, \mathrm{d}A_s \int_{(-\infty,t+h]} \mathrm{e}^{\eta s}\, \mathrm{d}A_s\right]$$
$$- \mathbb{E}\left[\int_{(-\infty,t]} \mathrm{e}^{-\eta(t-s)}\, \mathrm{d}A_s\right] \mathbb{E}\left[\int_{(-\infty,t+h]} \mathrm{e}^{-\eta(t+h-s)}\, \mathrm{d}A_s\right]$$

$$
\begin{aligned}
&= \mathrm{e}^{-2\eta t}\mathrm{e}^{-\eta h}\left(\mathbb{E}\left[\left(\int_{(-\infty,t]}\mathrm{e}^{\eta s}\,\mathrm{d}A_s\right)^2\right]+\mathbb{E}\left[\int_{(-\infty,t]}\mathrm{e}^{\eta s}\,\mathrm{d}A_s\int_t^{t+h}\mathrm{e}^{\eta s}\,\mathrm{d}A_s\right]\right)\\
&\quad -\frac{m_1^2}{\eta^2}\left(\int_{\Phi_L}\varphi\mathbb{E}[V_0^\varphi]\,\pi(\mathrm{d}\varphi)\right)^2\\
&=: \mathrm{e}^{-2\eta t}\mathrm{e}^{-\eta h}(E_1+E_2)-\frac{m_1^2}{\eta^2}\left(\int_{\Phi_L}\varphi\mathbb{E}[V_0^\varphi]\,\pi(\mathrm{d}\varphi)\right)^2. \qquad (6.8)
\end{aligned}
$$

For E_1 we can use integration by parts (see [20, Eq. I.4.45]) together with [26, Theorems II.19 and VI.29] and Lemma 6.5 to obtain

$$
\begin{aligned}
E_1 &= 2\mathbb{E}\left[\int_{(-\infty,t]}\left(\int_{(-\infty,s)}\mathrm{e}^{\eta r}\,\mathrm{d}A_r\right)\mathrm{e}^{\eta s}\,\mathrm{d}A_s\right]+\mathbb{E}\left[\int_{(-\infty,t]}\mathrm{e}^{2\eta s}\,\mathrm{d}[A,A]_s\right]\\
&= 2m_1\int_{\Phi_L}\int_{(-\infty,t]}\mathbb{E}\left[\left(\int_{(-\infty,s]}\mathrm{e}^{\eta r}\,\mathrm{d}A_r\right)V_s^\varphi\right]\mathrm{e}^{\eta s}\varphi\,\mathrm{d}s\,\pi(\mathrm{d}\varphi)\\
&\quad + m_2\int_{\Phi_L}\int_{(-\infty,t]}\mathrm{e}^{2\eta s}\varphi^2\mathbb{E}[(V_s^\varphi)^2]\,\mathrm{d}s\,\pi(\mathrm{d}\varphi)\\
&= 2m_1\int_{\Phi_L}\int_{(-\infty,t]}g(s,\varphi)\mathrm{e}^{\eta s}\varphi\,\mathrm{d}s\,\pi(\mathrm{d}\varphi)+\frac{m_2}{2\eta}\mathrm{e}^{2\eta t}\int_{\Phi_L}\varphi^2\mathbb{E}[(V_0^\varphi)^2]\,\pi(\mathrm{d}\varphi), \qquad (6.9)
\end{aligned}
$$

where $g(s,\varphi) := \mathbb{E}\left[V_s^\varphi \int_{(-\infty,s]}\mathrm{e}^{\eta r}\,\mathrm{d}A_r\right]$. Then again using integration by parts, Lemmas 6.3 and 6.5 and Eqs. (2.4), (2.6), (2.20) and (2.11), we obtain

$$
\begin{aligned}
g(s,\varphi) &= \mathbb{E}\left[\int_{(-\infty,s]}\int_{(-\infty,r)}\mathrm{e}^{\eta u}\,\mathrm{d}A_u\,\mathrm{d}V_r^\varphi\right]+\mathbb{E}\left[\int_{(-\infty,s]}V_{r-}^\varphi\mathrm{e}^{\eta r}\,\mathrm{d}A_r\right]\\
&\quad +\mathbb{E}\left[\int_{(-\infty,s]}\mathrm{e}^{\eta r}\,\mathrm{d}[A,V^\varphi]_r\right]\\
&= \mathbb{E}\left[\int_{(-\infty,s]}\left(\int_{(-\infty,r]}\mathrm{e}^{\eta u}\,\mathrm{d}A_u\right)(\beta-\eta V_r^\varphi)\,\mathrm{d}r\right]\\
&\quad +\mathbb{E}\left[\int_{(-\infty,s]}\left(\int_{(-\infty,r)}\mathrm{e}^{\eta u}\,\mathrm{d}A_u\right)\varphi V_{r-}^\varphi\,\mathrm{d}S_r\right]\\
&\quad +\mathbb{E}\left[\int_{(-\infty,s]}V_{r-}^\varphi\mathrm{e}^{\eta r}\,\mathrm{d}A_r\right]+\mathbb{E}\left[\int_{(-\infty,s]}\mathrm{e}^{\eta r}\,\mathrm{d}[A,V^\varphi]_r\right]\\
&= \beta m_1\int_{(-\infty,s]}\int_{(-\infty,r]}\mathrm{e}^{\eta u}\,\mathrm{d}u\,\mathrm{d}r\int_{\Phi_L}\tilde\varphi\mathbb{E}[V_0^{\tilde\varphi}]\,\pi(\mathrm{d}\tilde\varphi)\\
&\quad +(m_1\varphi-\eta)\int_{(-\infty,s]}g(r,\varphi)\,\mathrm{d}r+m_1\int_{(-\infty,s]}\mathrm{e}^{\eta r}\,\mathrm{d}r\int_{\Phi_L}\tilde\varphi\mathbb{E}[V_0^\varphi V_0^{\tilde\varphi}]\,\pi(\mathrm{d}\tilde\varphi)\\
&\quad +m_2\varphi\int_{(-\infty,s]}\mathrm{e}^{\eta r}\,\mathrm{d}r\int_{\Phi_L}\tilde\varphi\mathbb{E}[V_0^\varphi V_0^{\tilde\varphi}]\,\pi(\mathrm{d}\tilde\varphi)\\
&= \frac{\mathrm{e}^{\eta s}}{\eta}\left(\frac{m_1\beta}{\eta}\int_{\Phi_L}\tilde\varphi\mathbb{E}[V_0^{\tilde\varphi}]\,\pi(\mathrm{d}\tilde\varphi)+(m_1+m_2\varphi)\int_{\Phi_L}\tilde\varphi\mathbb{E}[V_0^\varphi V_0^{\tilde\varphi}]\,\pi(\mathrm{d}\tilde\varphi)\right)
\end{aligned}
$$

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$$+\,\Psi(1,\varphi)\int_{(-\infty,s]} g(r,\varphi)\,\mathrm{d}r$$
$$=\frac{\mathrm{e}^{\eta s}}{\eta}C(\varphi)+\Psi(1,\varphi)\int_{(-\infty,s]} g(r,\varphi)\,\mathrm{d}r,$$

with

$$C(\varphi):=\int_{\Phi_L} C(\varphi,\tilde\varphi)\,\pi(\mathrm{d}\tilde\varphi),$$
$$C(\varphi,\tilde\varphi):=-\frac{m_1}{\eta}\,\Psi(1,\varphi)\tilde\varphi\mathbb{E}[V_0^{\varphi}]\mathbb{E}[V_0^{\tilde\varphi}]+(m_1+m_2\varphi)\tilde\varphi\mathbb{E}[V_0^{\varphi}V_0^{\tilde\varphi}].$$

Solving this integral equation yields $g(s,\varphi)=\frac{C(\varphi)\mathrm{e}^{\eta s}}{\eta-\Psi(1,\varphi)}$. Inserting this result in (6.9) gives

$$E_1=\frac{m_1}{\eta}\mathrm{e}^{2\eta t}\int_{\Phi_L}\frac{\varphi C(\varphi)}{\eta-\Psi(1,\varphi)}\,\pi(\mathrm{d}\varphi)+\frac{m_2}{2\eta}\mathrm{e}^{2\eta t}\int_{\Phi_L}\varphi^2\mathbb{E}[(V_0^{\varphi})^2]\,\pi(\mathrm{d}\varphi).$$

Let us turn to E_2 and denote the augmented natural filtration of Λ^L by $\mathbb{G}^{(3)}=(\mathcal{G}_t^{(3)})_{t\in\mathbb{R}}$. Now taking conditional expectation w.r.t. $\mathcal{G}_t^{(3)}$ and observing that V^{φ}, $\bar V^{(3)}$ as well as A are all adapted to $\mathbb{G}^{(3)}$, we obtain

$$E_2=\mathbb{E}\left[\left(\int_{(-\infty,t]}\mathrm{e}^{\eta s}\,\mathrm{d}A_s\right)\mathbb{E}\left[\int_t^{t+h}\int_{\Phi_L}\mathrm{e}^{\eta s}\varphi V_{s-}^{\varphi}\,\Lambda^S(\mathrm{d}s,\mathrm{d}\varphi)\,\middle|\,\mathcal{G}_t^{(3)}\right]\right].$$

Observing that the restriction of Λ^S on $(t,t+h]$ is independent of $\mathcal{F}_t$, we have

$$E_2=\mathbb{E}\left[\left(\int_{(-\infty,t]}\mathrm{e}^{\eta s}\,\mathrm{d}A_s\right)m_1\int_{\Phi_L}\int_{(t,t+h]}\mathrm{e}^{\eta s}\varphi\mathbb{E}[V_{s-}^{\varphi}|\mathcal{G}_t^{(3)}]\,\mathrm{d}s\,\pi(\mathrm{d}\varphi)\right].$$

According to [23, Eq. (4.5)] we have $\mathbb{E}[V_{s-}^{\varphi}|\mathcal{G}_t^{(3)}]=(V_t^{\varphi}-\mathbb{E}[V_0^{\varphi}])\mathrm{e}^{(s-t)\,\Psi(1,\varphi)}+\mathbb{E}[V_0^{\varphi}]$ for $s>t$. So we get

$$\begin{aligned}
E_2&=m_1\mathbb{E}\Bigg[\left(\int_{(-\infty,t]}\mathrm{e}^{\eta s}\,\mathrm{d}A_s\right)\int_{\Phi_L}\int_{(t,t+h]}\mathrm{e}^{\eta s}\varphi\big((V_t^{\varphi}-\mathbb{E}[V_0^{\varphi}])\mathrm{e}^{(s-t)\,\Psi(1,\varphi)}\\
&\qquad+\mathbb{E}[V_0^{\varphi}]\big)\,\mathrm{d}s\,\pi(\mathrm{d}\varphi)\Bigg]\\
&=m_1\int_{\Phi_L}\varphi\mathbb{E}\left[V_t^{\varphi}\int_{(-\infty,t]}\mathrm{e}^{\eta s}\,\mathrm{d}A_s\right]\int_{(t,t+h]}\mathrm{e}^{\eta s}\mathrm{e}^{(s-t)\,\Psi(1,\varphi)}\,\mathrm{d}s\,\pi(\mathrm{d}\varphi)\\
&\qquad+m_1\mathbb{E}\left[\int_{(-\infty,t]}\mathrm{e}^{\eta s}\,\mathrm{d}A_s\right]\int_{\Phi_L}\varphi\mathbb{E}[V_0^{\varphi}]\int_{(t,t+h]}\mathrm{e}^{\eta s}(1-\mathrm{e}^{(s-t)\,\Psi(1,\varphi)})\,\mathrm{d}s\,\pi(\mathrm{d}\varphi)\\
&=\int_{\Phi_L}g(t,\varphi)\mathrm{e}^{\eta t}(\mathrm{e}^{m_1\varphi h}-1)\,\pi(\mathrm{d}\varphi)+m_1^2\int_{(-\infty,t]}\mathrm{e}^{\eta s}\,\mathrm{d}s\int_{\Phi_L}\varphi\mathbb{E}[V_0^{\varphi}]\,\pi(\mathrm{d}\varphi)\\
&\qquad\times\int_{\Phi_L}\varphi\mathbb{E}[V_0^{\varphi}]\mathrm{e}^{\eta t}\left(\frac{\mathrm{e}^{\eta h}-1}{\eta}-\frac{\mathrm{e}^{m_1\varphi h}-1}{m_1\varphi}\right)\pi(\mathrm{d}\varphi)\\
&=\mathrm{e}^{2\eta t}\left(\int_{\Phi_L}\frac{C(\varphi)}{\eta-\Psi(1,\varphi)}(\mathrm{e}^{m_1\varphi h}-1)\,\pi(\mathrm{d}\varphi)+\frac{m_1^2}{\eta^2}(\mathrm{e}^{\eta h}-1)\left(\int_{\Phi_L}\varphi\mathbb{E}[V_0^{\varphi}]\,\pi(\mathrm{d}\varphi)\right)^2\right.\\
&\qquad\left.-\frac{m_1}{\eta}\int_{\Phi_L}\varphi\mathbb{E}[V_0^{\varphi}]\,\pi(\mathrm{d}\varphi)\int_{\Phi_L}\mathbb{E}[V_0^{\varphi}](\mathrm{e}^{m_1\varphi h}-1)\,\pi(\mathrm{d}\varphi)\right).
\end{aligned}$$

Now inserting the results for E_1 and E_2 in (6.8), we obtain

$$\begin{aligned}
\mathrm{Cov}[\bar V_t^{(3)}, \bar V_{t+h}^{(3)}] &= \mathrm{e}^{-\eta h}\left(\frac{m_1}{\eta}\int_{\Phi_L}\frac{\varphi C(\varphi)}{\eta-\Psi(1,\varphi)}\,\pi(\mathrm{d}\varphi)+\frac{m_2}{2\eta}\int_{\Phi_L}\varphi^2\mathbb{E}[(V_0^\varphi)^2]\,\pi(\mathrm{d}\varphi)\right)\\
&\quad+\int_{\Phi_L}\frac{C(\varphi)}{\eta-\Psi(1,\varphi)}(\mathrm{e}^{\Psi(1,\varphi)h}-\mathrm{e}^{-\eta h})\,\pi(\mathrm{d}\varphi)-\frac{m_1^2}{\eta^2}\mathrm{e}^{-\eta h}\left(\int_{\Phi_L}\varphi\mathbb{E}[V_0^\varphi]\,\pi(\mathrm{d}\varphi)\right)^2\\
&\quad-\frac{m_1}{\eta}\int_{\Phi_L}\tilde\varphi\mathbb{E}[V_0^{\tilde\varphi}]\,\pi(\mathrm{d}\tilde\varphi)\int_{\Phi_L}\mathbb{E}[V_0^\varphi](\mathrm{e}^{\Psi(1,\varphi)h}-\mathrm{e}^{-\eta h})\,\pi(\mathrm{d}\varphi)\\
&=\int_{\Phi_L}\int_{\Phi_L}\left(\frac{C(\varphi,\tilde\varphi)}{\eta-\Psi(1,\varphi)}-\frac{m_1}{\eta}\tilde\varphi\mathbb{E}[V_0^{\tilde\varphi}]\mathbb{E}[V_0^\varphi]\right)\mathrm{e}^{\Psi(1,\varphi)h}\,\pi(\mathrm{d}\varphi)\,\pi(\mathrm{d}\tilde\varphi)\\
&\quad+\mathrm{e}^{-\eta h}\int_{\Phi_L}\int_{\Phi_L}\left(\frac{m_1\varphi C(\varphi,\tilde\varphi)}{\eta(\eta-\Psi(1,\varphi))}+\frac{m_2\varphi^2\mathbb{E}[(V_0^\varphi)^2]}{2\eta}-\frac{C(\varphi,\tilde\varphi)}{\eta-\Psi(1,\varphi)}\right.\\
&\quad\left.-\frac{m_1^2}{\eta^2}\varphi\tilde\varphi\mathbb{E}[V_0^\varphi]\mathbb{E}[V_0^{\tilde\varphi}]+\frac{m_1}{\eta}\tilde\varphi\mathbb{E}[V_0^\varphi]\mathbb{E}[V_0^{\tilde\varphi}]\right)\pi(\mathrm{d}\varphi)\,\pi(\mathrm{d}\tilde\varphi), \qquad (6.10)
\end{aligned}$$

where using Proposition 3.11 together with Eqs. (2.4) and (6.1) gives

$$\begin{aligned}
&\frac{C(\varphi,\tilde\varphi)}{\eta-\Psi(1,\varphi)}-\frac{m_1}{\eta}\tilde\varphi\mathbb{E}[V_0^\varphi]\mathbb{E}[V_0^{\tilde\varphi}]\\
&\quad=-\frac{m_1\Psi(1,\varphi)\tilde\varphi\mathbb{E}[V_0^\varphi]\mathbb{E}[V_0^{\tilde\varphi}]}{\eta(\eta-\Psi(1,\varphi))}+\frac{(m_1+m_2\varphi)\tilde\varphi\mathbb{E}[V_0^\varphi V_0^{\tilde\varphi}]}{\eta-\Psi(1,\varphi)}-\frac{m_1}{\eta}\tilde\varphi\mathbb{E}[V_0^\varphi]\mathbb{E}[V_0^{\tilde\varphi}]\\
&\quad=\frac{(m_1+m_2\varphi)\tilde\varphi}{\eta-\Psi(1,\varphi)}\mathbb{E}[V_0^\varphi V_0^{\tilde\varphi}]-\frac{m_1\tilde\varphi}{\eta(\eta-\Psi(1,\varphi))}\mathbb{E}[V_0^\varphi]\mathbb{E}[V_0^{\tilde\varphi}](\Psi(1,\varphi)+\eta-\Psi(1,\varphi))\\
&\quad=\frac{\eta+\Psi(1,\tilde\varphi)+h(\varphi,\tilde\varphi)-\Psi(1,\varphi)-\Psi(1,\tilde\varphi)}{\eta-\Psi(1,\varphi)}\mathbb{E}[V_0^\varphi V_0^{\tilde\varphi}]\\
&\qquad-\frac{\eta+\Psi(1,\tilde\varphi)}{\eta-\Psi(1,\varphi)}\mathbb{E}[V_0^\varphi]\mathbb{E}[V_0^{\tilde\varphi}]\\
&\quad=\left(1+\frac{h(\varphi,\tilde\varphi)}{\eta-\Psi(1,\varphi)}\right)\mathbb{E}[V_0^\varphi V_0^{\tilde\varphi}]-\left(1+\frac{\Psi(1,\varphi)+\Psi(1,\tilde\varphi)}{\eta-\Psi(1,\varphi)}\right)\mathbb{E}[V_0^\varphi]\mathbb{E}[V_0^{\tilde\varphi}]\\
&\quad=\mathrm{Cov}[V_0^\varphi, V_0^{\tilde\varphi}], \qquad (6.11)
\end{aligned}$$

while for the second part of (6.10) by Eqs. (2.4), (2.11) and (2.13)

$$\begin{aligned}
&\frac{m_1\varphi C(\varphi,\tilde\varphi)}{\eta(\eta-\Psi(1,\varphi))}+\frac{m_2\varphi^2\mathbb{E}[(V_0^\varphi)^2]}{2\eta}-\frac{C(\varphi,\tilde\varphi)}{\eta-\Psi(1,\varphi)}\\
&\qquad-\frac{m_1^2}{\eta^2}\varphi\tilde\varphi\mathbb{E}[V_0^\varphi]\mathbb{E}[V_0^{\tilde\varphi}]+\frac{m_1}{\eta}\tilde\varphi\mathbb{E}[V_0^\varphi]\mathbb{E}[V_0^{\tilde\varphi}]\\
&\quad=\frac{\Psi(1,\varphi)C(\varphi,\tilde\varphi)}{\eta(\eta-\Psi(1,\varphi))}+\frac{\big(\Psi(2,\varphi)-2\Psi(1,\varphi)\big)}{2\eta}\mathbb{E}[(V_0^\varphi)^2]-\frac{m_1\tilde\varphi\Psi(1,\varphi)}{\eta^2}\mathbb{E}[V_0^\varphi]\mathbb{E}[V_0^{\tilde\varphi}]\\
&\quad=: F_1+F_2+F_3. \qquad (6.12)
\end{aligned}$$

Now observe that by (2.12) and (2.11)

$$F_2 = \frac{\Psi(2,\varphi)}{2\eta}\mathbb{E}[(V_0^\varphi)^2] - \frac{\Psi(1,\varphi)}{\eta}\mathbb{E}[(V_0^\varphi)^2] = -\frac{\beta}{\eta}\mathbb{E}[V_0^\varphi] + \frac{\beta}{\eta}\frac{\mathbb{E}[(V_0^\varphi)^2]}{\mathbb{E}[V_0^\varphi]},$$

while

$$\begin{aligned} F_3 &= -\frac{(\Psi(1,\tilde\varphi)+\eta)\Psi(1,\varphi)}{\eta^2}\mathbb{E}[V_0^\varphi]\mathbb{E}[V_0^{\tilde\varphi}] \\ &= -\frac{\beta^2}{\eta^2} + \frac{\Psi(1,\varphi)}{\eta}\mathrm{Cov}[V_0^\varphi, V_0^{\tilde\varphi}] - \frac{\Psi(1,\varphi)}{\eta}\mathbb{E}[V_0^\varphi V_0^{\tilde\varphi}] \\ &= -\frac{\beta^2}{\eta^2} - \frac{\beta}{\eta}\frac{\mathrm{Cov}[V_0^\varphi, V_0^{\tilde\varphi}]}{\mathbb{E}[V_0^\varphi]} + \frac{\beta}{\eta}\frac{\mathbb{E}[V_0^\varphi V_0^{\tilde\varphi}]}{\mathbb{E}[V_0^\varphi]}. \end{aligned}$$

On the other hand we obtain by similar means

$$\begin{aligned} \Psi(1,\varphi)C(\varphi,\tilde\varphi) &= \frac{(m_1+m_2\varphi)\tilde\varphi\beta^2(\Psi(1,\varphi)+\Psi(1,\tilde\varphi))}{h(\varphi,\tilde\varphi)\Psi(1,\tilde\varphi)} - \frac{m_1\beta^2\tilde\varphi\Psi(1,\varphi)}{\eta\Psi(1,\tilde\varphi)} \\ &= \frac{\beta^2}{\eta\Psi(1,\tilde\varphi)h(\varphi,\tilde\varphi)}\,(\eta(m_1+m_2\varphi)\tilde\varphi(\Psi(1,\varphi)+\Psi(1,\tilde\varphi)) \\ &\quad - m_1\tilde\varphi\Psi(1,\varphi)h(\varphi,\tilde\varphi)) \\ &= \frac{\beta^2(\eta-\Psi(1,\varphi))}{\eta\Psi(1,\tilde\varphi)h(\varphi,\tilde\varphi)}\big(h(\varphi,\tilde\varphi)\Psi(1,\tilde\varphi)+\eta(\Psi(1,\varphi)+\Psi(1,\tilde\varphi))\big) \end{aligned}$$

such that by Proposition 3.11

$$F_1 = \frac{\beta^2}{\eta^2} - \frac{\beta}{\eta}\frac{\mathbb{E}[V_0^\varphi V_0^{\tilde\varphi}]}{\mathbb{E}[V_0^\varphi]}.$$

Finally inserting (6.11) and (6.12) with the obtained formulas for F_1, F_2 and F_3 in (6.10) gives

$$\begin{aligned} \mathrm{Cov}[\bar V_t^{(3)}, \bar V_{t+h}^{(3)}] &= \int_{\Phi_L}\int_{\Phi_L}\Bigg(\mathrm{Cov}[V_0^\varphi, V_0^{\tilde\varphi}]\mathrm{e}^{\Psi(1,\varphi)h} \\ &\quad + \mathrm{e}^{-\eta h}\left(-\frac{\beta}{\eta}\mathbb{E}[V_0^\varphi] + \frac{\beta}{\eta}\frac{\mathbb{E}[(V_0^\varphi)^2]}{\mathbb{E}[V_0^\varphi]} - \frac{\beta}{\eta}\frac{\mathrm{Cov}[V_0^\varphi, V_0^{\tilde\varphi}]}{\mathbb{E}[V_0^\varphi]}\right)\Bigg)\,\pi(\mathrm{d}\varphi)\,\pi(\mathrm{d}\tilde\varphi), \end{aligned}$$

which yields the result. □

Proof of Proposition 3.19. To show the assertion for $\kappa < \bar\kappa$ we use $\mathbb{P}[\bar V_0^{(3)} > x] \le \mathbb{P}[V_0^{\bar\varphi} > x]$ and proceed as in the proof of Proposition 3.5. For the other cases, observe that

$$\bar V_0^{(3)} \stackrel{\mathrm{d}}{=} \frac{\beta}{\eta} + \int_{\mathbb{R}_+}\int_{\Phi_L}\mathrm{e}^{-\eta t}\varphi V_{t-}^\varphi\,\Lambda^S(\mathrm{d}t,\mathrm{d}\varphi) = \sum_{i=1}^\infty \mathrm{e}^{-\eta T_i}\varphi_i V_{T_i-}^{\varphi_i}\Delta S_{T_i},$$

where $(T_i)_{i\in\mathbb{N}}$ are the jump times of S and $(\varphi_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence with common distribution π which is also independent of S. We start by proving that, if I is a measurable subset of Φ_L with $\pi(I) =: p > 0$ and $\varphi \in \Phi_L$, then there are constants $0 < C_*(\varphi, p), C^*(\varphi, p) < \infty$, only dependent on I via p, with

$$C_*(\varphi, p) = \liminf_{x\to\infty} x^{\kappa(\varphi)} \mathbb{P}\Big[\sum_{\varphi_i \in I} \mathrm{e}^{-\eta T_i} \varphi V^{\varphi}_{T_i-} \Delta S_{T_i} > x\Big]$$

$$\le \limsup_{x\to\infty} x^{\kappa(\varphi)} \mathbb{P}\Big[\sum_{\varphi_i \in I} \mathrm{e}^{-\eta T_i} \varphi V^{\varphi}_{T_i-} \Delta S_{T_i} > x\Big] = C^*(\varphi, p), \tag{6.13}$$

and moreover, if $p \to 0$, then $C_*(\varphi, p), C^*(\varphi, p) \to 0$.

We abbreviate the sum in (6.13) by $V(\varphi, I)$ or $V(I)$. Since the sequence $(\varphi_i)_{i\in\mathbb{N}}$ is independent of everything else, the distribution of $V(I)$ only depends on p, which means that the constants $C_*(\varphi, p) =: C_*(p)$ and $C^*(\varphi, p) =: C^*(p)$ only depend on p. Also, they are obviously decreasing in p. Hence, for the claimed convergence to 0, it suffices to show $C^*(2^{-n}) \le ((1 + 2^{-\kappa(\varphi)})/2)^n C(\varphi)$ for all $n \in \mathbb{N}_0$, where $C(\varphi)$ is the tail constant of V^{φ}_0 as in the proof of Proposition 3.5. The case $n = 0$ corresponds to $V(I) \overset{\mathrm{d}}{=} V^{\varphi}_0$ and the statement is clear. For $n \ge 1$, find a set I' disjoint with I such that $\pi(I') = \pi(I) = 2^{-n}$ and therefore $\pi(J) = 2^{-(n-1)}$ for $J = I \cup I'$. Since

$$\mathbb{P}[V(J) > x] = \mathbb{P}[V(I) + V(I') > x] \ge 2\mathbb{P}[V(I) > x] - \mathbb{P}[V(I) > x, V(I') > x]$$
$$\ge 2\mathbb{P}[V(I) > x] - \mathbb{P}[V(J) > 2x],$$

we have by induction

$$C^*(2^{-n}) = \limsup_{x\to\infty} x^{\kappa(\varphi)} \mathbb{P}[V(I) > x] \le \frac{1 + 2^{-\kappa(\varphi)}}{2} C^*(2^{-(n-1)}).$$

It remains to show that $C^*(p) < \infty$ and $C_*(p) > 0$ for all $p > 0$. Again by monotonicity, the first inequality is obvious and in the second inequality we only need to consider $p = 1/n$. To this end, partition Φ_L into n disjoint sets $(I_k)_{k=1,\dots,n}$, each with $\pi(I_k) = 1/n$. Then observe that

$$\mathbb{P}[V^{\varphi}_0 > x] \le \mathbb{P}[V(I_1) > x/n \text{ or } \dots \text{ or } V(I_n) > x/n] \le n\mathbb{P}[V(I_1) > x/n],$$

which implies

$$C_*(1/n) = \liminf_{x\to\infty} x^{\kappa(\varphi)} \mathbb{P}[V(I_1) > x] \ge C n^{-(\kappa(\varphi)+1)} > 0.$$

Let us come back to the main line of the proof of Proposition 3.19. If $\varphi < \bar\varphi$, then we have by the above

$$\liminf_{x\to\infty} x^{\kappa} \mathbb{P}[\bar V^{(3)}_0 > x] \ge \liminf_{x\to\infty} x^{\kappa} \mathbb{P}\big[V(\varphi, [\varphi, \bar\varphi]) > x\big] \to \infty$$

for all $\kappa > \kappa(\varphi)$ and therefore, by the same argument as in the proof of Proposition 3.5, for all $\kappa > \bar\kappa$.

Next, consider the case $\kappa = \bar\kappa$ and $\bar p = 0$. Then, again by the above and the proof of [24, Lemma 2]

$$\limsup_{x\to\infty} x^{\bar\kappa} \mathbb{P}[\bar V^{(3)}_0 > x] \le \limsup_{x\to\infty} x^{\bar\kappa} \mathbb{P}\big[V(\varphi, (0, \varphi]) + V(\bar\varphi, (\varphi, \bar\varphi]) > x\big]$$
$$= \limsup_{x\to\infty} x^{\bar\kappa} \mathbb{P}\big[V(\bar\varphi, (\varphi, \bar\varphi]) > x\big] = C^*(\bar\varphi, \pi((\varphi, \bar\varphi])),$$

which converges to 0 as $\varphi \to \bar\varphi$. For the case $\bar p > 0$ first decompose

$$\bar V^{(3)}_0 = \frac{\beta}{\eta} + \sum_{\varphi_i \ne \bar\varphi} \mathrm{e}^{-T_i} \varphi_i V^{\varphi_i}_{T_i-} \Delta S_{T_i} + V(\bar\varphi, \{\bar\varphi\}) =: \frac{\beta}{\eta} + Z + V(\bar\varphi, \{\bar\varphi\})$$

and observe that $\limsup_{x\to\infty} x^{\bar{\kappa}}\mathbb{P}[Z > x] = 0$ by the results so far. Reading along the lines of the proof of [24, Lemma 2], we obtain

$$\liminf_{x\to\infty} x^{\bar{\kappa}}\mathbb{P}[\bar{V}_0^{(3)} > x] = \liminf_{x\to\infty} x^{\bar{\kappa}}\mathbb{P}[V(\bar{\varphi},\{\bar{\varphi}\}) > x] = C_*(\bar{\varphi},\bar{p}),$$
$$\limsup_{x\to\infty} x^{\bar{\kappa}}\mathbb{P}[\bar{V}_0^{(3)} > x] = \limsup_{x\to\infty} x^{\bar{\kappa}}\mathbb{P}[V(\bar{\varphi},\{\bar{\varphi}\}) > x] = C^*(\bar{\varphi},\bar{p}),$$

which finishes the proof. □

6.4. Proofs for Section 4

Proof of Theorem 4.1. First observe that the assumption that π has support in $\Phi_L^{(\kappa)}$ implies $\mathbb{E}\big[(\bar{V}_t^{(1)})^\kappa\big] < \infty$. Therefore, $\mathbb{E}[L_1^{\varphi_1}] = 0$ implies

$$\mathbb{E}[\Delta^r G_t^{(1)}] = \mathbb{E}\left[\int_{(t,t+r]} \sqrt{\bar{V}_s^{(1)}}\, \mathrm{d}L_s^{\varphi_1}\right] = 0.$$

Next assume $\mathbb{E}[L_1^2] < \infty$. Using integration by parts and the fact that $G^{(1)}$ has stationary increments, we have

$$\begin{aligned}\mathbb{E}[(\Delta^r G_t^{(1)})^2] &= \mathbb{E}[(G_r^{(1)})^2]\\ &= 2\mathbb{E}\left[\int_{(0,r]} G_{s-}^{(1)}\sqrt{\bar{V}_{s-}^{(1)}}\,\mathrm{d}L_s^{\varphi_1}\right] + \mathbb{E}\left[\int_{(0,r]} \bar{V}_{s-}^{(1)}\,\mathrm{d}[L^{\varphi_1},L^{\varphi_1}]_s\right]\\ &= 0 + \mathrm{Var}[L_1]\mathbb{E}[\bar{V}_0^{(1)}]r,\end{aligned}$$

which, together with Proposition 3.12 and the relation between S and L in (2.1), gives the stated formula. Furthermore, for $h \ge r > 0$ we have, in view of the above computations and again using integration by parts,

$$\begin{aligned}&\mathrm{Cov}[\Delta^r G_t^{(1)}, \Delta^r G_{t+h}^{(1)}] = \mathbb{E}\left[\Delta^r G_t^{(1)}\Delta^r G_{t+h}^{(1)}\right]\\ &= \mathbb{E}\left[\int_{(0,t+h+r]} \mathbb{1}_{(t,t+r]}(s)\sqrt{\bar{V}_{s-}^{(1)}}\,\mathrm{d}L_s^{\varphi_1}\int_{(0,t+h+r]} \mathbb{1}_{(t+h,t+h+r]}(u)\sqrt{\bar{V}_{u-}^{(1)}}\,\mathrm{d}L_u^{\varphi_1}\right]\\ &= \mathbb{E}\left[\int_{(0,t+h+r]} \mathbb{1}_{(t,t+r]}(s)\mathbb{1}_{(t+h,t+h+r]}(s)\bar{V}_{s-}^{(1)}\,\mathrm{d}[L^{\varphi_1},L^{\varphi_1}]_s\right]\\ &\quad + \mathbb{E}\left[\int_{(0,t+h+r]}\left(\int_{(0,u]} \mathbb{1}_{(t,t+r]}(s)\sqrt{\bar{V}_{s-}^{(1)}}\,\mathrm{d}L_s^{\varphi_1}\right)\mathbb{1}_{(t+h,t+h+r]}(u)\sqrt{\bar{V}_{u-}^{(1)}}\,\mathrm{d}L_u^{\varphi_1}\right]\\ &\quad + \mathbb{E}\left[\int_{(0,t+h+r]}\left(\int_{(0,u]} \mathbb{1}_{(t+h,t+h+r]}(s)\sqrt{\bar{V}_{s-}^{(1)}}\,\mathrm{d}L_s^{\varphi_1}\right)\mathbb{1}_{(t,t+r]}(u)\sqrt{\bar{V}_{u-}^{(1)}}\,\mathrm{d}L_u^{\varphi_1}\right]\\ &= 0.\end{aligned}$$

To compute the covariance of the squared increments, let $\mathbb{G}^{(1)} = (\mathcal{G}_t^{(1)})_{t\ge 0}$ denote the augmented natural filtration of $(L^{\varphi_i})_{i\in\mathbb{N}}$ and observe that

$$\begin{aligned}\mathbb{E}\left[(\Delta^r G_0^{(1)})^2(\Delta^r G_h^{(1)})^2\right] &= \mathbb{E}\left[\mathbb{E}\left[(\Delta^r G_0^{(1)})^2(\Delta^r G_h^{(1)})^2|\mathcal{G}_r^{(1)}\right]\right]\\ &= \mathbb{E}\left[(\Delta^r G_0^{(1)})^2\mathbb{E}\left[(\Delta^r G_h^{(1)})^2|\mathcal{G}_r^{(1)}\right]\right],\end{aligned}$$

where again by integration by parts

$$\begin{aligned}\mathbb{E}\left[(\Delta^r G_h^{(1)})^2|\mathcal{G}_r^{(1)}\right] &= \mathbb{E}\left[\left(\int_{(h,h+r]}\sqrt{\bar{V}_s^{(1)}}\,\mathrm{d}L_s^{\varphi_1}\right)^2\middle|\mathcal{G}_r^{(1)}\right]\\ &= 2\mathbb{E}\left[\int_{(h,h+r]}\left(\int_{(0,s]}\sqrt{\bar{V}_{u-}^{(1)}}\,\mathrm{d}L_u^{\varphi_1}\right)\sqrt{\bar{V}_{s-}^{(1)}}\mathrm{d}L_s^{\varphi_1}\middle|\mathcal{G}_r^{(1)}\right]\\ &\quad+\mathbb{E}\left[\int_{(h,h+r]}\bar{V}_{s-}^{(1)}\,\mathrm{d}[L^{\varphi_1},L^{\varphi_1}]_s\middle|\mathcal{G}_r^{(1)}\right]\\ &= 0+\mathbb{E}[L_1^2]\int_{(h,h+r]}\mathbb{E}[\bar{V}_{s-}^{(1)}|\mathcal{G}_r^{(1)}]\,\mathrm{d}s.\end{aligned}$$

Next, for $s > r$ we obtain, using the notation as in the proof of Proposition 3.11,

$$\begin{aligned}\mathbb{E}\left[\bar{V}_s^{(1)}|\mathcal{G}_r^{(1)}\right] &= \int_{\Phi_L^{(1)}}\mathbb{E}\left[V_s^{\varphi}|\mathcal{G}_r^{(1)}\right]\pi(\mathrm{d}\varphi) = \int_{\Phi_L^{(1)}}\mathbb{E}\left[(A_{r,s}^{\varphi}V_r^{\varphi}+B_{r,s}^{\varphi})|\mathcal{G}_r^{(1)}\right]\pi(\mathrm{d}\varphi)\\ &= \int_{\Phi_L^{(1)}}\left(\mathbb{E}[A_{r,s}^{\varphi}]V_r^{\varphi}+\mathbb{E}[B_{r,s}^{\varphi}]\right)\pi(\mathrm{d}\varphi)\\ &= \int_{\Phi_L^{(1)}}\left(\mathrm{e}^{(s-r)\Psi(1,\varphi)}V_r^{\varphi}+\mathbb{E}[V_0^{\varphi}]\left(1-\mathrm{e}^{(s-r)\Psi(1,\varphi)}\right)\right)\pi(\mathrm{d}\varphi).\end{aligned}$$

Together with the preceding computations, this yields

$$\begin{aligned}&\mathrm{Cov}[(\Delta^r G_0^{(1)})^2,(\Delta^r G_h^{(1)})^2]\\ &\quad= \mathbb{E}\left[(\Delta^r G_0^{(1)})^2\mathbb{E}[L_1^2]\int_{(h,h+r]}\mathbb{E}[\bar{V}_{s-}^{(1)}|\mathcal{G}_r^{(1)}]\,\mathrm{d}s\right]-\mathbb{E}\left[(\Delta^r G_0^{(1)})^2\right]\mathbb{E}\left[(\Delta^r G_h^{(1)})^2\right]\\ &\quad= \mathbb{E}[L_1^2]\mathbb{E}\Bigg[(\Delta^r G_0^{(1)})^2\int_{(h,h+r]}\int_{\Phi_L^{(1)}}\left(\mathrm{e}^{(s-r)\Psi(1,\varphi)}V_r^{\varphi}+\mathbb{E}[V_0^{\varphi}]\left(1-\mathrm{e}^{(s-r)\Psi(1,\varphi)}\right)\right)\\ &\qquad\times\pi(\mathrm{d}\varphi)\,\mathrm{d}s\Bigg]-\left(\mathbb{E}\left[(\Delta^r G_0^{(1)})^2\right]\right)^2\\ &\quad= \mathbb{E}[L_1^2]\mathbb{E}\Bigg[(\Delta^r G_0^{(1)})^2\int_{\Phi_L^{(1)}}\frac{1}{\Psi(1,\varphi)}\left(\mathrm{e}^{h\Psi(1,\varphi)}-\mathrm{e}^{(h-r)\Psi(1,\varphi)}\right)\\ &\qquad\times(V_r^{\varphi}-\mathbb{E}[V_0^{\varphi}])+r\mathbb{E}[V_0^{\varphi}]\,\pi(\mathrm{d}\varphi)\Bigg]-\left(\mathbb{E}\left[(\Delta^r G_0^{(1)})^2\right]\right)^2\\ &\quad= \mathbb{E}[L_1^2]\int_{\Phi_L^{(1)}}\frac{1}{\Psi(1,\varphi)}\left(\mathrm{e}^{h\Psi(1,\varphi)}-\mathrm{e}^{(h-r)\Psi(1,\varphi)}\right)\\ &\qquad\times\left(\mathbb{E}[(\Delta^r G_0^{(1)})^2V_r^{\varphi}]-\mathbb{E}[(\Delta^r G_0^{(1)})^2]\mathbb{E}[V_r^{\varphi}]\right)\pi(\mathrm{d}\varphi)\end{aligned}$$

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$$+\mathbb{E}[(\Delta^r G_0^{(1)})^2] r \mathbb{E}[L_1^2] \int_{\Phi_L^{(1)}} \mathbb{E}[V_0^\varphi]\,\pi(\mathrm{d}\varphi) - \left(\mathbb{E}\left[(\Delta^r G_0^{(1)})^2\right]\right)^2$$

$$= \mathbb{E}[L_1^2] \int_{\Phi_L^{(1)}} \frac{1}{\Psi(1,\varphi)} \left(\mathrm{e}^{h\,\Psi(1,\varphi)} - \mathrm{e}^{(h-r)\,\Psi(1,\varphi)}\right) \mathrm{Cov}[(\Delta^r G_0^{(1)})^2, V_r^\varphi]\,\pi(\mathrm{d}\varphi).$$

It remains to prove $\mathrm{Cov}[(\Delta^r G_0^{(1)})^2, V_r^\varphi] \geq 0$ with strict inequality if $\pi(\{\varphi\}) > 0$ in order to obtain the claimed positivity of the covariance of the squared increments. Again using integration by parts, we get

$$(\Delta^r G_0^{(1)})^2 = \left(\int_{(0,r]} \sqrt{\bar{V}_{s-}^{(1)}}\,\mathrm{d}L_s^{\varphi_1}\right)^2 = 2M_r + \int_{(0,r]} \bar{V}_{s-}^{(1)}\,\mathrm{d}[L^{\varphi_1}, L^{\varphi_1}]_s,$$

where

$$M_r := \int_{(0,r]} \sqrt{\bar{V}_{s-}^{(1)}} \left(\int_{(0,s)} \sqrt{\bar{V}_{u-}^{(1)}}\,\mathrm{d}L_u^{\varphi_1}\right) \mathrm{d}L_s^{\varphi_1}$$

satisfies $\mathbb{E}[M_r] = 0$ due to $\mathbb{E}[L_1] = 0$ and

$$\begin{aligned}
\mathbb{E}[M_r V_r^\varphi] &= \mathbb{E}\left[\int_{(0,r]} M_s(\beta - \eta V_s^\varphi)\,\mathrm{d}s\right] + \mathbb{E}\left[\int_{(0,r]} M_{s-}\varphi V_{s-}^\varphi\,\mathrm{d}S_s^\varphi\right] \\
&\quad + \mathbb{E}\left[\int_{(0,r]} V_{s-}^\varphi\,\mathrm{d}M_s\right] + \mathbb{E}\big[[V^\varphi, M]_r\big] \\
&= \Psi(1,\varphi) \int_{(0,r]} \mathbb{E}[M_s V_s^\varphi]\,\mathrm{d}s + \mathbb{E}\big[[V^\varphi, M]_r\big].
\end{aligned} \tag{6.14}$$

Applying $\int_{\mathbb{R}} y^3\,\nu_L(\mathrm{d}y) = 0$ and the independence of L^φ and L^{φ_1}, if $\varphi \neq \varphi_1$, we have

$$\begin{aligned}
\mathbb{E}\big[[V^\varphi, M]_r\big] &= \varphi \mathbb{E}\left[\int_{(0,r]} V_{s-}^\varphi \sqrt{\bar{V}_{s-}^{(1)}} \left(\int_{(0,s)} \sqrt{\bar{V}_{u-}^{(1)}}\,\mathrm{d}L_u^{\varphi_1}\right) \mathrm{d}[L^{\varphi_1}, S^\varphi]_s\right] \\
&= \begin{cases} 0 & \text{if } \varphi \neq \varphi_1, \\ \varphi \int_{\mathbb{R}} y^3\,\nu_L(\mathrm{d}y) \int_{(0,r]} \mathbb{E}\left[V_{s-}^\varphi \sqrt{\bar{V}_{s-}^{(1)}} \left(\int_{(0,s)} \sqrt{\bar{V}_{u-}^{(1)}}\,\mathrm{d}L_u^{\varphi_1}\right)\right] \mathrm{d}s & \text{if } \varphi = \varphi_1, \end{cases} \\
&= 0.
\end{aligned} \tag{6.15}$$

Therefore, (6.14) together with the fact that $\mathbb{E}[M_0 V_0^\varphi] = 0$ implies that $\mathbb{E}[M_r V_r^\varphi] = 0$ for all $r \geq 0$. As a consequence, we have

$$\begin{aligned}
\mathrm{Cov}[(\Delta^r G_0^{(1)})^2, V_r^\varphi] &= \mathrm{Cov}\left[2M_r + \int_{(0,r]} \bar{V}_{s-}^{(1)}\,\mathrm{d}[L^{\varphi_1}, L^{\varphi_1}]_s, V_r^\varphi\right] \\
&= \mathbb{E}\left[V_r^\varphi \int_{(0,r]} \bar{V}_{s-}^{(1)}\,\mathrm{d}[L^{\varphi_1}, L^{\varphi_1}]_s\right] - \mathbb{E}[V_1^\varphi]\mathbb{E}\left[\int_{(0,r]} \bar{V}_{s-}^{(1)}\,\mathrm{d}[L^{\varphi_1}, L^{\varphi_1}]_s\right] \\
&= \mathbb{E}\left[V_r^\varphi \int_{(0,r]} \bar{V}_{s-}^{(1)}\,\mathrm{d}[L^{\varphi_1}, L^{\varphi_1}]_s\right] - r\mathbb{E}[L_1^2]\mathbb{E}[\bar{V}_0^{(1)}]\mathbb{E}[V_0^\varphi],
\end{aligned}$$

where an application of the integration by parts formula yields

$$\begin{aligned}
f(r) &:= \mathbb{E}\left[V_r^{\varphi}\int_{(0,r]}\bar{V}_{s-}^{(1)}\,\mathrm{d}[L^{\varphi_1},L^{\varphi_1}]_s\right]\\
&= \mathbb{E}[L_1^2]\int_{(0,r]}\mathbb{E}[V_s^{\varphi}\bar{V}_s^{(1)}]\,\mathrm{d}s+\beta\int_{(0,r]}\mathbb{E}\left[\int_{(0,s]}\bar{V}_{u-}^{(1)}\,\mathrm{d}[L^{\varphi_1},L^{\varphi_1}]_u\right]\mathrm{d}s\\
&\quad+\Psi(1,\varphi)\int_{(0,r]}\mathbb{E}\left[V_s^{\varphi}\int_{(0,s]}\bar{V}_{u-}^{(1)}\,\mathrm{d}[L^{\varphi_1},L^{\varphi_1}]_u\right]\mathrm{d}s\\
&\quad+\mathbb{E}\left[\int_{(0,r]}\bar{V}_{s-}^{(1)}\,\mathrm{d}\big[S^{\varphi_1},V^{\varphi}\big]_s\right]\\
&= \mathbb{E}[L_1^2]\mathbb{E}[V_0^{\varphi}\bar{V}_0^{(1)}]r+\beta\mathbb{E}[L_1^2]\mathbb{E}[\bar{V}_0^{(1)}]\frac{r^2}{2}+\Psi(1,\varphi)\int_{(0,r]}f(s)\,\mathrm{d}s\\
&\quad+\mathbb{1}_{\{\varphi=\varphi_1\}}\varphi\int_{\mathbb{R}}y^2\,\nu_S(\mathrm{d}y)\mathbb{E}[V_0^{\varphi}\bar{V}_0^{(1)}]r,\\
f(0) &= 0.
\end{aligned}$$

Solving this integral equation yields ($m_2 := \int_{\mathbb{R}} y^2\,\nu_S(\mathrm{d}y)$)

$$\begin{aligned}
f(r) &= \frac{(\mathbb{E}[L_1^2]+\mathbb{1}_{\{\varphi=\varphi_1\}}\varphi m_2)\mathbb{E}[V_0^{\varphi}\bar{V}_0^{(1)}]\,\Psi(1,\varphi)(\mathrm{e}^{\Psi(1,\varphi)r}-1)}{\Psi(1,\varphi)^2}\\
&\quad+\frac{\beta\mathbb{E}[L_1^2]\mathbb{E}[\bar{V}_0^{(1)}](-\Psi(1,\varphi)r+\mathrm{e}^{\Psi(1,\varphi)r}-1)}{\Psi(1,\varphi)^2},
\end{aligned}$$

which by (2.11) yields the claimed positive correlation since

$$\begin{aligned}
&\mathrm{Cov}[(\Delta^r G_0^{(1)})^2, V_r^{\varphi}] = f(r)-\mathbb{E}[L_1^2]\mathbb{E}[V_0^{\varphi}]\mathbb{E}[\bar{V}_0^{(1)}]r\\
&\quad= \frac{(\mathbb{E}[L_1^2]+\mathbb{1}_{\{\varphi=\varphi_1\}}\varphi m_2)\mathbb{E}[V_0^{\varphi}\bar{V}_0^{(1)}]\,\Psi(1,\varphi)(\mathrm{e}^{\Psi(1,\varphi)r}-1)+\beta\mathbb{E}[L_1^2]\mathbb{E}[\bar{V}_0^{(1)}](\mathrm{e}^{\Psi(1,\varphi)r}-1)}{\Psi(1,\varphi)^2}\\
&\quad= \frac{\mathrm{e}^{\Psi(1,\varphi)r}-1}{\Psi(1,\varphi)}\left(\mathbb{E}[L_1^2]\mathrm{Cov}[V_0^{\varphi},\bar{V}_0^{(1)}]+\mathbb{1}_{\{\varphi=\varphi_1\}}\varphi\int_{\mathbb{R}}y^2\,\nu_S(\mathrm{d}y)\mathbb{E}[V_0^{\varphi}\bar{V}_0^{(1)}]\right)\geq 0 \qquad (6.16)
\end{aligned}$$

with $\mathrm{Cov}[V_0^{\varphi},\bar{V}_0^{(1)}] = \pi(\{\varphi\})\mathrm{Var}[V_1^{\varphi}]$. □

Proof of Theorem 4.2. The proof works similarly to the proof of Theorem 4.1 with obvious changes when independence of the single COGARCH processes was used (e.g. (6.15)). Also replace $\mathbb{G}^{(1)}$ by $\mathbb{G}^{(2)} = (\mathcal{G}_t^{(2)})_{t\in\mathbb{R}}$, the augmented natural filtration of L, and notice that $\mathrm{Cov}[V_0^{\varphi},\bar{V}_0^{(2)}] = \int_{\Phi_L^{(2)}}\mathrm{Cov}[V_0^{\varphi},V_0^{\tilde{\varphi}}]\,\pi(\mathrm{d}\tilde{\varphi}) > 0$ by Proposition 3.11. □

Proof of Theorem 4.3. Analogously to the proof of Theorem 4.1, one can show that (a) and (b) hold and that for (c) we have

$$\mathbb{E}\left[(\Delta^r G_0^{(3)})^2(\Delta^r G_h^{(3)})^2\right] = \mathbb{E}[L_1^2]\mathbb{E}\left[(\Delta^r G_0^{(3)})^2\int_{(h,h+r]}\mathbb{E}[\bar{V}_{s-}^{(3)}|\mathcal{G}_r^{(3)}]\,\mathrm{d}s\right], \qquad (6.17)$$

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where from (3.29) and [23, Eq. (4.5)] we have

$$\begin{aligned}
\mathbb{E}[\bar{V}_{s-}^{(3)}|\mathcal{G}_r^{(3)}] &= \mathrm{e}^{-\eta(s-r)}\bar{V}_r^{(3)} + \beta\mathrm{e}^{-\eta s}\int_{(r,s)}\mathrm{e}^{\eta u}\,\mathrm{d}u \\
&\quad + \mathbb{E}\left[\int_{(r,s)}\int_{\Phi_L^{(2)}}\mathrm{e}^{-\eta(s-u)}\varphi V_{u-}^{\varphi}\,\Lambda^S(\mathrm{d}u,\mathrm{d}\varphi)\middle|\mathcal{G}_r^{(3)}\right] \\
&= \mathrm{e}^{-\eta(s-r)}\bar{V}_r^{(3)} + \frac{\beta}{\eta}(1-\mathrm{e}^{-\eta(s-r)}) + \mathbb{E}[S_1]\int_{(r,s]}\int_{\Phi_L^{(2)}}\mathrm{e}^{-\eta(s-u)}\varphi\mathbb{E}[V_{u-}^{\varphi}|\mathcal{G}_r^{(3)}]\,\pi(\mathrm{d}\varphi)\,\mathrm{d}u \\
&= \mathrm{e}^{-\eta(s-r)}\bar{V}_r^{(3)} + \frac{\beta}{\eta}(1-\mathrm{e}^{-\eta(s-r)}) \\
&\quad + \mathbb{E}[S_1]\int_{(r,s]}\int_{\Phi_L^{(2)}}\mathrm{e}^{-\eta(s-u)}\varphi\big((V_r^{\varphi}-\mathbb{E}[V_0^{\varphi}])\mathrm{e}^{(u-r)\Psi(1,\varphi)} + \mathbb{E}[V_0^{\varphi}]\big)\,\pi(\mathrm{d}\varphi)\,\mathrm{d}u.
\end{aligned}$$

Applying (2.11) we obtain

$$\begin{aligned}
&\mathbb{E}[S_1]\int_{(r,s]}\int_{\Phi_L^{(2)}}\mathrm{e}^{-\eta(s-u)}\varphi\big((V_r^{\varphi}-\mathbb{E}[V_0^{\varphi}])\mathrm{e}^{(u-r)\Psi(1,\varphi)} + \mathbb{E}[V_0^{\varphi}]\big)\,\pi(\mathrm{d}\varphi)\,\mathrm{d}u \\
&= \mathbb{E}[S_1]\int_{\Phi_L^{(2)}}\left(\frac{\varphi(V_r^{\varphi}-\mathbb{E}[V_0^{\varphi}])}{\varphi\mathbb{E}[S_1]}\left(\mathrm{e}^{\Psi(1,\varphi)(s-r)} - \mathrm{e}^{-\eta(s-r)}\right)\right. \\
&\quad \left. + \frac{\varphi\mathbb{E}[V_0^{\varphi}]}{\eta}\left(1-\mathrm{e}^{-\eta(s-r)}\right)\right)\pi(\mathrm{d}\varphi) \\
&= \int_{\Phi_L^{(2)}}\mathrm{e}^{\Psi(1,\varphi)(s-r)}(V_r^{\varphi}-\mathbb{E}[V_0^{\varphi}])\,\pi(\mathrm{d}\varphi) - \mathrm{e}^{-\eta(s-r)}\left(\int_{\Phi_L^{(2)}}V_r^{\varphi}\,\pi(\mathrm{d}\varphi) - \mathbb{E}[\bar{V}_0^{(3)}]\right) \\
&\quad + \left(1-\mathrm{e}^{-\eta(s-r)}\right)\int_{\Phi_L^{(2)}}\frac{\mathbb{E}[S_1]\varphi}{\eta}\,\frac{-\beta}{\Psi(1,\varphi)}\,\pi(\mathrm{d}\varphi) \\
&= \int_{\Phi_L^{(2)}}\mathrm{e}^{\Psi(1,\varphi)(s-r)}(V_r^{\varphi}-\mathbb{E}[V_0^{\varphi}])\,\pi(\mathrm{d}\varphi) - \mathrm{e}^{-\eta(s-r)}\left(\int_{\Phi_L^{(2)}}V_r^{\varphi}\,\pi(\mathrm{d}\varphi) - \mathbb{E}[\bar{V}_0^{(3)}]\right) \\
&\quad - \frac{\beta}{\eta}\left(1-\mathrm{e}^{-\eta(s-r)}\right)\int_{\Phi_L^{(2)}}\left(1+\frac{\eta}{\Psi(1,\varphi)}\right)\pi(\mathrm{d}\varphi) \\
&= \int_{\Phi_L^{(2)}}\mathrm{e}^{\Psi(1,\varphi)(s-r)}(V_r^{\varphi}-\mathbb{E}[V_0^{\varphi}])\,\pi(\mathrm{d}\varphi) - \mathrm{e}^{-\eta(s-r)} \\
&\quad \times\int_{\Phi_L^{(2)}}V_r^{\varphi}\,\pi(\mathrm{d}\varphi) - \frac{\beta}{\eta}\left(1-\mathrm{e}^{-\eta(s-r)}\right) + \mathbb{E}[\bar{V}_0^{(3)}]
\end{aligned}$$

such that

$$\begin{aligned}
\mathbb{E}[\bar{V}_{s-}^{(3)}|\mathcal{G}_r^{(3)}] &= \mathrm{e}^{-\eta(s-r)}\left(\bar{V}_r^{(3)} - \int_{\Phi_L^{(2)}}V_r^{\varphi}\,\pi(\mathrm{d}\varphi)\right) \\
&\quad + \int_{\Phi_L^{(2)}}\mathrm{e}^{\Psi(1,\varphi)(s-r)}(V_r^{\varphi}-\mathbb{E}[V_0^{\varphi}])\,\pi(\mathrm{d}\varphi) + \mathbb{E}[\bar{V}_0^{(3)}].
\end{aligned}$$

Inserting this into (6.17) yields

$$
\begin{aligned}
&\mathrm{Cov}[(\Delta^r G_0^{(3)})^2, (\Delta^r G_h^{(3)})^2] \\
&\quad = \mathbb{E}[L_1^2]\mathbb{E}\left[(\Delta^r G_0^{(3)})^2 \int_{(h,h+r]} \mathbb{E}[\bar{V}_{s-}^{(3)}|\mathcal{G}_r^{(3)}]\,\mathrm{d}s\right] - \mathbb{E}[(\Delta^r G_0^{(3)})^2]^2 \\
&\quad = \mathbb{E}[L_1^2]\mathbb{E}\left[(\Delta^r G_0^{(3)})^2 \left(\frac{\mathrm{e}^{-\eta h} - \mathrm{e}^{-\eta(h-r)}}{-\eta}\left(\bar{V}_r^{(3)} - \int_{\Phi_L^{(2)}} V_r^{\varphi}\,\pi(\mathrm{d}\varphi)\right)\right)\right. \\
&\qquad \left. + \int_{\Phi_L^{(2)}} \frac{\mathrm{e}^{\Psi(1,\varphi)h} - \mathrm{e}^{\Psi(1,\varphi)(h-r)}}{\Psi(1,\varphi)}(V_r^{\varphi} - \mathbb{E}[V_0^{\varphi}])\,\pi(\mathrm{d}\varphi)\right] \\
&\quad = \mathbb{E}[L_1^2]\left[\frac{\mathrm{e}^{-\eta h} - \mathrm{e}^{-\eta(h-r)}}{-\eta}\mathrm{Cov}[(\Delta^r G_0^{(3)})^2, \bar{V}_r^{(3)}]\right. \\
&\qquad + \int_{\Phi_L^{(2)}} \left(\frac{\mathrm{e}^{\Psi(1,\varphi)h} - \mathrm{e}^{\Psi(1,\varphi)(h-r)}}{\Psi(1,\varphi)} - \frac{\mathrm{e}^{-\eta h} - \mathrm{e}^{-\eta(h-r)}}{-\eta}\right) \\
&\qquad \left. \times \mathrm{Cov}[(\Delta^r G_0^{(3)})^2, V_r^{\varphi}]\,\pi(\mathrm{d}\varphi)\right].
\end{aligned}
$$

Since $\Psi(1,\varphi) > -\eta$ and the function $x \mapsto (\mathrm{e}^{hx} - \mathrm{e}^{(h-r)x})/x$ is increasing in x for $x < 0$, it remains to prove $\mathrm{Cov}[(\Delta^r G_0^{(3)})^2, \bar{V}_r^{(3)}] > 0$ and $\mathrm{Cov}[(\Delta^r G_0^{(3)})^2, V_r^{\varphi}] > 0$. For the latter, proceed as in the proof of Theorem 4.1 and note that $\mathrm{Cov}[V_0^{\varphi}, \bar{V}_0^{(3)}] > 0$. Indeed, using integration by parts,

$$
\begin{aligned}
V_r^{\varphi}\bar{V}_r^{(3)} &= V_0^{\varphi}\bar{V}_0^{(3)} + \int_{(0,r]} \bar{V}_{s-}^{(3)}\,\mathrm{d}V_s^{\varphi} + \int_{(0,r]} V_{s-}^{\varphi}\,\mathrm{d}\bar{V}_s^{(3)} + [V^{\varphi}, \bar{V}^{(3)}]_r \\
&= V_0^{\varphi}\bar{V}_0^{(3)} + \int_{(0,r]} \bar{V}_s^{(3)}(\beta - \eta V_s^{\varphi})\,\mathrm{d}s + \int_{(0,r]} \bar{V}_{s-}^{(3)}\varphi V_{s-}^{\varphi}\,\mathrm{d}S_s \\
&\quad + \int_{(0,r]} V_{s-}^{\varphi}(\beta - \eta\bar{V}_s^{(3)})\,\mathrm{d}s + \int_{(0,r]}\int_{\Phi_L^{(2)}} V_{s-}^{\varphi}\tilde{\varphi}V_{s-}^{\tilde{\varphi}}\,\Lambda^S(\mathrm{d}s, \mathrm{d}\tilde{\varphi}) \\
&\quad + \varphi\int_{(0,r]}\int_{\Phi_L^{(2)}}\int_{\mathbb{R}_+} V_{s-}^{\varphi}\tilde{\varphi}V_{s-}^{\tilde{\varphi}}y^2\,\mu^{\Lambda^S}(\mathrm{d}s, \mathrm{d}\tilde{\varphi}, \mathrm{d}y),
\end{aligned}
$$

with $[V^{\varphi}, \bar{V}^{(3)}]_r$ as given in Lemma 6.5. Taking expectations, differentiating w.r.t. r and using the stationarity of $V^{\varphi}\bar{V}^{(3)}$, which is a consequence of Lemma 6.3, we find that ($m_1 := \int_{\mathbb{R}_+} y\,\nu_S(\mathrm{d}y)$ and $m_2 := \int_{\mathbb{R}_+} y^2\,\nu_S(\mathrm{d}y)$)

$$
\begin{aligned}
&\beta(\mathbb{E}[\bar{V}_0^{(3)}] + \mathbb{E}[V_0^{\varphi}]) + (\varphi m_1 - 2\eta)\mathbb{E}[V_0^{\varphi}\bar{V}_0^{(3)}] + (m_1 + \varphi m_2)\int_{\Phi_L^{(2)}} \tilde{\varphi}\mathbb{E}[V_0^{\varphi}V_0^{\tilde{\varphi}}]\,\pi(\mathrm{d}\tilde{\varphi}) \\
&\quad = 0,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\mathrm{Cov}[\bar{V}_0^{(3)}, V_0^{\varphi}] \\
&\quad = \frac{\beta(\mathbb{E}[\bar{V}_0^{(3)}] + \mathbb{E}[V_0^{\varphi}]) + (m_1 + \varphi m_2)\int_{\Phi_L^{(2)}} \tilde{\varphi}\mathbb{E}[V_0^{\varphi}V_0^{\tilde{\varphi}}]\,\pi(\mathrm{d}\tilde{\varphi}) - (\eta - \Psi(1,\varphi))\mathbb{E}[\bar{V}_0^{(3)}]\mathbb{E}[V_0^{\varphi}]}{\eta - \Psi(1,\varphi)}.
\end{aligned}
$$

To show the positivity of this term, we only have to consider the numerator, which by (2.11), (3.30) and (6.1) can be simplified to

$$\int_{\Phi_L^{(2)}}(m_1+\varphi m_2)\tilde{\varphi}\mathrm{Cov}[V_0^{\varphi},V_0^{\tilde{\varphi}}]\,\pi(\mathrm{d}\tilde{\varphi})+\beta(\mathbb{E}[\bar{V}_0^{(3)}]+\mathbb{E}[V_0^{\varphi}])+h(\varphi,\tilde{\varphi})\mathbb{E}[V_0^{\varphi}]\mathbb{E}[\bar{V}_0^{(3)}]$$
$$=\int_{\Phi_L^{(2)}}(m_1+\varphi m_2)\tilde{\varphi}\mathrm{Cov}[V_0^{\varphi},V_0^{\tilde{\varphi}}]\,\pi(\mathrm{d}\tilde{\varphi})+\beta^2 m_2\int_{\Phi_L^{(2)}}\frac{\varphi\tilde{\varphi}}{\Psi(1,\varphi)\Psi(1,\tilde{\varphi})}\,\pi(\mathrm{d}\tilde{\varphi})>0.$$

Finally, using the same methods as in the proof of Theorem 4.1, one can derive the following analogue of Eq. (6.16):

$$\mathrm{Cov}[(\Delta^r G_0^{(3)})^2,\bar{V}_0^{(3)}]=g(r)-\mathbb{E}[L_1^2]\mathbb{E}[\bar{V}_0]^2 r,$$

where

$$g(r)=\mathrm{e}^{-\eta r}\left(\int_{(0,r]}\mathrm{e}^{\eta s}\left(a+bs+\int_{\Phi_L^{(2)}}m_1\varphi f(\varphi,s)\,\pi(\mathrm{d}\varphi)\right)\mathrm{d}s\right),\qquad r\geq 0,$$
$$a=\mathbb{E}[L_1^2]\mathbb{E}[(\bar{V}_0^{(3)})^2]+\int_{\mathbb{R}_+}y^2\,\nu_S(\mathrm{d}y)\int_{\Phi_L^{(2)}}\varphi\mathbb{E}[V_0^{\varphi}\bar{V}_0]\,\pi(\mathrm{d}\varphi),$$
$$b=\beta\mathbb{E}[L_1^2]\mathbb{E}[\bar{V}_0^{(3)}]\quad\text{and}\quad f(\varphi,r)=\mathbb{E}\left[V_r^{\varphi}\int_{(0,r]}\bar{V}_{u-}^{(3)}\,\mathrm{d}[L,L]_u\right].$$

The positivity now follows from

$$\mathrm{Cov}[(\Delta^r G_0^{(3)})^2,\bar{V}_0^{(3)}]\geq \mathrm{e}^{\eta r}\int_{(0,r]}\mathrm{e}^{-\eta s}\,\mathrm{d}s\mathbb{E}[L_1^2]\mathbb{E}[(\bar{V}_0^{(3)})^2]-\mathbb{E}[L_1^2]\mathbb{E}[\bar{V}_0^{(3)}]^2 r$$

and the fact that $\mathrm{e}^{\eta r}\int_{(0,r]}\mathrm{e}^{-\eta s}\,\mathrm{d}s=(\mathrm{e}^{\eta r}-1)/\eta>r$ for all $r>0$. □

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