SPA: 3152 Model 1 pp. 1–24 (col. fig: NIL)

ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications xx (xxxx) xxx–xxx

stochastic processes and their applications

www.elsevier.com/locate/spa

On the block counting process and the fixation line of the Bolthausen–Sznitman coalescent

Jonas Kukla, Martin Möhle*

Mathematisches Institut, Eberhard Karls Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany

Received 9 May 2016; received in revised form 28 November 2016; accepted 20 June 2017
Available online xxxx

Abstract

The block counting process and the fixation line of the Bolthausen–Sznitman coalescent are analyzed. It is shown that these processes, properly scaled, converge in the Skorohod topology to the Mittag-Leffler process and to Neveu's continuous-state branching process respectively as the initial state tends to infinity. Strong relations to Siegmund duality, Mehler semigroups and self-decomposability are pointed out. Furthermore, spectral decompositions for the generators and transition probabilities of the block counting process and the fixation line of the Bolthausen–Sznitman coalescent are provided leading to explicit expressions for functionals such as hitting probabilities and absorption times.

MSC: primary 60F05; 60J27; secondary 92D15; 97K60

Keywords: Absorption time; Block counting process; Bolthausen–Sznitman coalescent; Fixation line; Hitting probabilities; Mehler semigroup; Mittag-Leffler process; Neveu's continuous-state branching process; Self-decomposability; Siegmund duality; Spectral decomposition

1. Introduction

Exchangeable coalescents are Markovian processes $(\Pi_t)_{t\geq 0}$ with state space $\mathcal{P}$, the set of partitions of $\mathbb{N} := \{1, 2, \ldots\}$. These processes have attracted the interest of several researchers, mainly in biology, mathematics and physics, during the last decades. The full family of exchangeable coalescents (with simultaneous multiple collisions) is a class of partition valued Markovian processes with a rich probabilistic structure and hence important for mathematical

* Corresponding author.
E-mail addresses: jonas.kukla@uni-tuebingen.de (J. Kukla), martin.moehle@uni-tuebingen.de (M. Möhle).

http://dx.doi.org/10.1016/j.spa.2017.06.012
0304-4149/

studies. Moreover, coalescents are useful in mathematical population genetics to model the ancestry of a sample of individuals or genes and therefore important for biological applications.

Exchangeable coalescents with multiple collisions but without simultaneous multiple collisions are characterized by a measure Λ on the unit interval $[0, 1]$ and therefore called Λ-coalescents. For further information on these processes we refer the reader to the independent works of Pitman [32] and Sagitov [37]. The most important coalescent is probably the Kingman coalescent [20], which allows only for binary mergers of ancestral lineages. In this case the measure Λ is the Dirac measure at 0.

For $t \geq 0$ let N_t denote the number of blocks of Π_t and let $N_t^{(n)}$ denote the number of blocks of $\Pi_t^{(n)}$, where $\Pi_t^{(n)}$ denotes the partition of Π_t restricted to a sample of size $n \in \mathbb{N}$. The processes $(N_t)_{t\geq 0}$ and $(N_t^{(n)})_{t\geq 0}$ are called the block counting processes of $(\Pi_t)_{t\geq 0}$ and $(\Pi_t^{(n)})_{t\geq 0}$ respectively.

Hénard [17] introduced for $n \in \mathbb{N}$ the so-called fixation line $(L_t^{(n)})_{t\geq 0}$ of a Λ-coalescent. Recently [13] the fixation line was extended to arbitrary exchangeable coalescents. One possible definition of the fixation line is based on the lookdown construction going back to Donnelly and Kurtz [8,9]. A precise pathwise construction of the Markovian processes $(L_t^{(n)})_{t\geq 0}$, $n \in \mathbb{N}$, is provided in [17, p. 3010] for the Λ-coalescent and in [13, Section 1] for general exchangeable coalescents. By this construction, $(L_t^{(n)})_{t\geq 0}$ has state space $\{n, n+1, \ldots\} \cup \{\infty\}$, initial state $L_0^{(n)} = n$ and non-decreasing paths. Moreover, $L_t^{(n)} \leq L_t^{(n+1)}$ for all $n \in \mathbb{N}$ and $t \geq 0$. The infinitesimal rates of the process $(L_t)_{t\geq 0} := (L_t^{(1)})_{t\geq 0}$ are provided in [17, Lemma 2.3] for the Λ-coalescent and in [13, Proposition 2.2] for arbitrary exchangeable coalescents.

The fixation line can be traced back to Pfaffelhuber and Wakolbinger [31] for the Kingman coalescent. For the Λ-coalescent the fixation line appears in Labbé [22] and was further studied by Hénard [16,17].

Note that we omit the pathwise definition of the fixation line via the lookdown construction here because it is provided in detail in [13] and [17]. In fact, our proofs concerning the fixation line are mainly based on the infinitesimal rates and do not rely on the pathwise construction except for the fact that $L_t^{(n)}$ is non-decreasing in n.

The fact that the block counting process $(N_t)_{t\geq 0}$ of a coalescent with multiple collisions is Siegmund dual to the fixation line $(L_t)_{t\geq 0}$ is explicitly mentioned in [2, Remark 3.6] and already contained in Hénard [17, Lemma 2.4] even though the name Siegmund dual is not mentioned there. For the full class of coalescents with simultaneous multiple collisions this Siegmund duality is provided in [13, Theorem 2.9] and may also be derived from the pathwise relations $L_t^{(n)} = \sup\{k \in \mathbb{N} : N_t^{(k)} \leq n\}$ and $N_t^{(n)} = \inf\{k \in \mathbb{N} : L_t^{(k)} \geq n\}$, $t \geq 0, n \in \mathbb{N}$.

In this article we focus on the Bolthausen–Sznitman coalescent [6], which is the particular Λ-coalescent with Λ being the uniform distribution on the unit interval. The generator $Q = (q_{ij})_{i,j\in\mathbb{N}}$ of the block counting process and the generator $\Gamma = (\gamma_{ij})_{i,j\in\mathbb{N}}$ of the fixation line of the Bolthausen–Sznitman coalescent have entries (see, for example, [27, Eq. (1.1)] and [17, p. 3015, Eq. 2.8 with $\alpha = 1$])

$$q_{ij} = \begin{cases} \dfrac{i}{(i-j)(i-j+1)} & \text{for } j < i \\ 1-i & \text{for } j = i, \\ 0 & \text{for } j > i. \end{cases} \quad \text{and}$$

$$\gamma_{ij} = \begin{cases} \dfrac{i}{(j-i)(j-i+1)} & \text{for } j > i, \\ -i & \text{for } j = i, \\ 0 & \text{for } j < i. \end{cases}$$

The block counting process and the corresponding generator Q have been studied intensively in the literature. In this article we focus on both processes $(N_t)_{t\geq 0}$ and $(L_t)_{t\geq 0}$ with an emphasis on the fixation line $(L_t)_{t\geq 0}$, which has been studied less intensively so far. As already observed by Hénard [17], it follows from $\gamma_{i,i+j} = i/(j(j+1))$, $i, j \in \mathbb{N}$, that $(L_t)_{t\geq 0}$ is a continuous-time branching process with offspring law $p_k := 1/(k(k-1))$, $k \in \{2, 3, \ldots\}$ and probability generating function (pgf) $\mathbb{E}(s^{L_t^{(n)}}) = (1-(1-s)^{e^{-t}})^n$, $s \in [0, 1]$, $t \geq 0$, $n \in \mathbb{N}$. These properties of the fixation line turn out to be fundamental and simplify several calculations. We furthermore stress the duality relation between the block counting process and the fixation line.

Section 2 deals with the behavior of the block counting process $(N_t^{(n)})_{t\geq 0}$ and the fixation line $(L_t^{(n)})_{t\geq 0}$ as the initial state n tends to infinity. The main convergence result (Theorem 2.1) states that both processes, properly scaled, converge in the Skorohod sense as $n \to \infty$ to the Mittag-Leffler process and to Neveu's continuous-state branching process respectively.

In Section 3 the processes $(N_t)_{t\geq 0}$ and $(L_t)_{t\geq 0}$ are analyzed with an emphasis on spectral decompositions. These spectral decompositions lead to explicit expressions for several functionals of these processes such as hitting probabilities and absorption times.

The proofs provided in Section 4 rely on both analytic and probabilistic arguments which demonstrates the interplay between analysis and probability. The proofs concerning the asymptotic results in Section 2 do not depend on the proofs of the results concerning the spectral decomposition in Section 3 and vice versa. A short appendix collects some results of independent interest used in the proofs.

2. Asymptotics

We are interested in the behavior of the block counting process $(N_t^{(n)})_{t\geq 0}$ and the fixation line $(L_t^{(n)})_{t\geq 0}$ of the Bolthausen–Sznitman n-coalescent as the sample size n tends to infinity. In order to state the main convergence result (see Theorem 2.1) let us recall some properties of the Mittag-Leffler process $X = (X_t)_{t\geq 0}$ and Neveu's [28] continuous-state branching process $Y = (Y_t)_{t\geq 0}$.

The Mittag-Leffler process X is a Markovian process in continuous time with initial state $X_0 = 1$ and state space $E := [0, \infty)$. The name of this process comes from the fact that for every $t \geq 0$ the marginal random variable X_t is Mittag-Leffler distributed with parameter e^{-t}. Note that X_t has moments $\mathbb{E}(X_t^m) = \Gamma(1+m)/\Gamma(1+me^{-t})$, $m \in [0, \infty)$. The semigroup $(T_t^X)_{t\geq 0}$ of the Mittag-Leffler process X is given by

$$T_t^X f(x) = \mathbb{E}(f(x^{e^{-t}} X_t)), \qquad t, x \geq 0,\ f \in B(E), \tag{1}$$

where $B(E)$ denotes the set of all bounded measurable functions $f : E \to \mathbb{R}$. Conditional on $X_s = x$ the random variable $x^{-e^{-t}} X_{s+t}$ is Mittag-Leffler distributed with parameter e^{-t}. Some further information on the process X can be found in [3] and [26].

Neveu's [28] continuous-state branching process Y is as well a Markovian process in continuous time with initial state $Y_0 = 1$ and state space E. For every $t \geq 0$ the marginal random variable Y_t is α-stable with Laplace transform $\mathbb{E}(e^{-\lambda Y_t}) = e^{-\lambda^\alpha}$, $\lambda \geq 0$, where $\alpha := e^{-t}$. The semigroup $(T_t^Y)_{t\geq 0}$ of Neveu's continuous-state branching process Y is given by

$$T_t^Y g(y) = \mathbb{E}(g(y^{e^t} Y_t)), \qquad t, y \geq 0,\ g \in B(E). \tag{2}$$

Conditional on $Y_s = y$ the random variable $y^{-e^t} Y_{s+t}$ has the same distribution as Y_t. Note that (see, for example, [26]) the Mittag-Leffler process X is Siegmund dual to Neveu's continuous state branching process Y, i.e. $\mathbb{P}(X_t \leq y \,|\, X_0 = x) = \mathbb{P}(Y_t \geq x \,|\, Y_0 = y)$ for all $t, x, y \geq 0$.

Please cite this article in press as: J. Kukla, M. Möhle, On the block counting process and the fixation line of the Bolthausen–Sznitman coalescent, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.012.

Define the scaled block counting process $X^{(n)} = (X_t^{(n)})_{t\geq 0}$ and the scaled fixation line $Y^{(n)} := (Y_t^{(n)})_{t\geq 0}$ of the Bolthausen–Sznitman n-coalescent via

$$X_t^{(n)} := \frac{N_t^{(n)}}{n^{e^{-t}}} \quad \text{and} \quad Y_t^{(n)} := \frac{L_t^{(n)}}{n^{e^t}}, \qquad t \geq 0, n \in \mathbb{N}. \tag{3}$$

Note that, for $n \geq 2$, the processes $X^{(n)}$ and $Y^{(n)}$ are time-inhomogeneous because of the time-dependent scalings $n^{e^{-t}}$ and n^{e^t}. Let us consider the one-dimensional distributions of $X^{(n)}$ and $Y^{(n)}$ respectively. We first turn to $Y^{(n)}$. As already mentioned in the introduction, $(L_t)_{t\geq 0}$ is a continuous-time Galton–Watson branching process with offspring law $p_k := 1/(k(k-1))$, $k \in \{2, 3, \ldots\}$, and $\mathbb{E}(s^{L_t^{(n)}}) = (1-(1-s)^{e^{-t}})^n$, $s \in [0,1]$, $t \geq 0$, $n \in \mathbb{N}$. See also Eq. (7) in Corollary 3.2 in the following Section 3. Thus, for all $t, \lambda \geq 0$,

$$\mathbb{E}(e^{-\lambda Y_t^{(n)}}) = (1-(1-e^{-\lambda/n^{e^t}})^{e^{-t}})^n \;\to\; e^{-\lambda^{e^{-t}}} = \mathbb{E}(e^{-\lambda Y_t}), \qquad n \to \infty.$$

Hence, $Y_t^{(n)} \to Y_t$ in distribution as $n \to \infty$. The convergence $X_t^{(n)} \to X_t$ in distribution as $n \to \infty$ is now obtained via duality as follows. For $n \in \mathbb{N}$, $t \geq 0$ and $x > 0$ define $m := \lfloor x n^{e^{-t}} \rfloor$ for convenience. Since $(N_t)_{t\geq 0}$ is Siegmund dual to $(L_t)_{t\geq 0}$ we conclude that

$$\begin{aligned}
\mathbb{P}(X_t^{(n)} \leq x) \;&=\; \mathbb{P}(N_t^{(n)} \leq m) = \mathbb{P}(L_t^{(m)} \geq n) = \mathbb{P}(Y_t^{(m)} > (n-1)/m^{e^t}) \\
&\to \mathbb{P}(Y_t > x^{-e^t}) = \mathbb{P}(Y_t^{-e^{-t}} < x) = \mathbb{P}(Y_t^{-e^{-t}} \leq x), \qquad n \to \infty,
\end{aligned}$$

since $(n-1)/m^{e^t} \to x^{-e^t}$ as $n \to \infty$. It is well known that $Y_t^{-e^{-t}}$ is Mittag-Leffler distributed with parameter e^{-t}. Thus, $X_t^{(n)} \to X_t$ in distribution as $n \to \infty$. An alternative proof (avoiding duality) of the latter convergence based on moment calculations is provided in [26, p. 46, Step 1]. The convergence of the one-dimensional distributions motivates the following convergence result.

Theorem 2.1 (*Asymptotics of the Block Counting Process and the Fixation Line*). *For the Bolthausen–Sznitman coalescent the following two assertions hold.*

(a) *As $n \to \infty$ the scaled block counting process $X^{(n)}$, defined in (3), converges in $D_E[0,\infty)$ to the Mittag-Leffler process $X = (X_t)_{t\geq 0}$.*
(b) *As $n \to \infty$ the scaled fixation line $Y^{(n)}$, defined in (3), converges in $D_E[0,\infty)$ to Neveu's continuous-state branching process $Y = (Y_t)_{t\geq 0}$.*

The proof of Theorem 2.1 provided in Section 4 indeed shows that it suffices to verify the convergence of the one-dimensional distributions. Theorem 2.1 demonstrates the intimate relation between the Bolthausen–Sznitman coalescent, the Mittag-Leffler process and Neveu's continuous state branching process. We refer the reader to Bertoin and Le Gall [4] for further insights concerning these relations.

Theorem 2.1(a) is known from the literature [26, Theorem 1.1] and provided here for completeness. Our proof of Theorem 2.1(a) is significantly shorter than the proof provided in [26] and gives further insights into the structure of the scaled block counting process $X^{(n)}$.

Part (b) of Theorem 2.1 is likely to be known from branching process theory, however the authors have not been able to trace this result in the literature. Note that the offspring distribution of the branching process $(L_t^{(n)})_{t\geq 0}$ has pgf $f(s) = s + (1-s)\log(1-s)$ and, hence, infinite mean. For related convergence results for the critical case when the offspring distribution has mean 1 we refer the reader to Sagitov [36] and the references therein. Note that in Theorem 2.1 of [36] the space-scaling is n and an additional time-scaling occurs. Theorem 2.1(b) may be viewed as

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Scaled block counting process}\\ (N_t^{(n)}/n^{e^{-t}})_{t\geq 0}\end{array}} \arrow[r, phantom, "\Rightarrow"] \arrow[d, phantom, "\updownarrow"] & \boxed{\begin{array}{c}\text{Mittag–Leffler process}\\ (X_t)_{t\geq 0}\end{array}} \arrow[d, phantom, "\updownarrow"] \\
\boxed{\begin{array}{c}\text{Scaled fixation line process}\\ (L_t^{(n)}/n^{e^{t}})_{t\geq 0}\end{array}} \arrow[r, phantom, "\Rightarrow"] & \boxed{\begin{array}{c}\text{Neveu's branching process}\\ (Y_t)_{t\geq 0}\end{array}}
\end{tikzcd}$$

Fig. 1. Commutative diagram for the block counting process $(N_t^{(n)})_{t\geq 0}$ and the fixation line $(L_t^{(n)})_{t\geq 0}$ of the Bolthausen–Sznitman coalescent. The right-arrows '$\Rightarrow$' stand for 'convergence in $D_E[0,\infty)$ as $n\to\infty$'. The vertical updown-arrows '$\updownarrow$' stand for 'duality', on the left hand side the duality of the block counting process $(N_t)_{t\geq 0}$ and the fixation line $(L_t)_{t\geq 0}$ with respect to the Siegmund duality kernel $H:\mathbb{N}^2\to\{0,1\}$ defined via $H(i,j):=1$ for $i\leq j$ and $H(i,j):=0$ otherwise, on the right hand side the duality of $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ with respect to the Siegmund duality kernel $H:[0,\infty)^2\to\{0,1\}$ defined via $H(x,y):=1$ for $x\leq y$ and $H(x,y):=0$ otherwise.

a kind of boundary case of Theorem 2.1 of [36] for $\alpha\to 1$. Similar convergence results for sequences of discrete-time branching processes can be traced back to Lamperti [23,24].

In summary the following commutative diagram holds (see Fig. 1).

Let us point out that Theorem 2.1 is strongly related to Mehler semigroups, to self-decomposability and to the Gumbel distribution. Clearly, Theorem 2.1 can be stated logarithmically as follows. The process $(\log N_t^{(n)} - e^{-t}\log n)_{t\geq 0}$ converges in $D_{\mathbb{R}}[0,\infty)$ to $\tilde{X}:=(\tilde{X}_t)_{t\geq 0}:=(\log X_t)_{t\geq 0}$ and the process $(\log L_t^{(n)} - e^{t}\log n)_{t\geq 0}$ converges in $D_{\mathbb{R}}[0,\infty)$ to $\tilde{Y}:=(\tilde{Y}_t)_{t\geq 0}:=(\log Y_t)_{t\geq 0}$ as $n\to\infty$. Note that the semigroup $(T_t^{\tilde{X}})_{t\geq 0}$ of $\tilde{X}$ is given by

$$T_t^{\tilde{X}}f(x)=\mathbb{E}(f(xe^{-t}+\tilde{X}_t)),\qquad t\geq 0,\ f\in B(\mathbb{R}),\ x\in\mathbb{R},\tag{4}$$

whereas the semigroup $(T_t^{\tilde{Y}})_{t\geq 0}$ of $\tilde{Y}$ is given by

$$T_t^{\tilde{Y}}g(y)=\mathbb{E}(g(ye^{t}+\tilde{Y}_t)),\qquad t\geq 0,\ g\in B(\mathbb{R}),\ y\in\mathbb{R}.\tag{5}$$

Semigroups of this form belong to the class of so called generalized Mehler semigroups [5] corresponding to generalized Ornstein–Uhlenbeck type processes. Note that (4) and (5) define the semigroups of $\tilde{X}$ and $\tilde{Y}$ completely, since for every $t\geq 0$ the distributions of the marginals $\tilde{X}_t=\log X_t$ and $\tilde{Y}_t=\log Y_t$ can be characterized as follows. Let E be standard exponentially distributed and independent of X and Y. Note that $G:=-\log E$ is standard Gumbel distributed. From $E\stackrel{d}{=}(E/Y_t)^{e^{-t}}$ (see, for example, [38]) we conclude by an application of the transformation $x\mapsto -\log x$ that the distribution of $\tilde{Y}_t$ is characterized via the self-decomposable distributional equation.

$$G\stackrel{d}{=}e^{-t}G+e^{-t}\tilde{Y}_t.$$

Thus, $\tilde{Y}_t$ has characteristic function $u\mapsto \Gamma(1-iue^t)/\Gamma(1-iu)$, $u\in\mathbb{R}$, and cumulants $\kappa_j(\tilde{Y}_t)=(e^{jt}-1)\kappa_j(G)$, $j\in\mathbb{N}$, $t\geq 0$, where $\kappa_j(G)$ are the cumulants of the Gumbel distribution, i.e. $\kappa_1(G)=\gamma$ (Euler–Mascheroni constant) and $\kappa_j(G)=(-1)^j\Psi^{(j-1)}(1)=(j-1)!\zeta(j)$ for $j\in\mathbb{N}\setminus\{1\}$, where Ψ and ζ denote the digamma function (logarithmic derivative of the gamma function) and the Riemann zeta function respectively.

Please cite this article in press as: J. Kukla, M. Möhle, On the block counting process and the fixation line of the Bolthausen–Sznitman coalescent, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.012.

Similarly, the distribution of $\tilde{X}_t$ is characterized via the self-decomposable distributional equation.

$$S \stackrel{d}{=} e^{-t}S + \tilde{X}_t,$$

where $S := -G$. Therefore, $\tilde{X}_t$ has characteristic function $u \mapsto \Gamma(1+iu)/\Gamma(1+iue^{-t})$, $u \in \mathbb{R}$, and cumulants $\kappa_j(\tilde{X}_t) = (-1)^j(1-e^{-jt})\kappa_j(G)$, $j \in \mathbb{N}, t \geq 0$.

3. Spectral decompositions and applications

Spectral decompositions are of fundamental interest since they lead to diagonal representations of the corresponding operators or matrices which simplify many mathematical calculations and numerical computations significantly. Explicit spectral decompositions for (the block counting process of) the Kingman coalescent and the Bolthausen–Sznitman coalescent are provided in [21] and [27]. We are interested in analog spectral decompositions for the fixation line. A spectral decomposition of the generator Γ of the fixation line of the Kingman coalescent is provided in the appendix (Lemma A.2) for completeness. Our first result in this section (Theorem 3.1) provides an explicit spectral decomposition for the generator Γ of the fixation line of the Bolthausen–Sznitman coalescent. This spectral decomposition is for example used afterwards to derive exact expressions and sharp approximations for the absorption time of the Bolthausen–Sznitman coalescent (see Corollaries 3.4 and 3.5). It turns out to be convenient to express the spectral decomposition in terms of the Stirling numbers $s(i, j)$ and $S(i, j)$ of the first and second kind respectively. Note that $(-1)^{i-j}s(i, j)$ is the number of permutations of i elements having j cycles whereas $S(i, j)$ counts the number of ways to partition a set of i elements into j nonempty subsets. For more insights why the Stirling numbers occur naturally in this context we refer the reader to [21], where a spectral decomposition of the generator of the full (partition valued) Bolthausen–Sznitman coalescent is provided.

Theorem 3.1 (*Spectral Decomposition of the Generator of the Fixation Line*)**.** *The generator $\Gamma = (\gamma_{ij})_{i,j\in\mathbb{N}}$ of the fixation line $(L_t)_{t\geq 0}$ of the Bolthausen–Sznitman coalescent has spectral decomposition $\Gamma = RDL$, where $D = (d_{ij})_{i,j\in\mathbb{N}}$ is the diagonal matrix with entries $d_{ij} = -i$ for $i = j$ and $d_{ij} = 0$ for $i \neq j$, and $R = (r_{ij})_{i,j\in\mathbb{N}}$ and $L = (l_{ij})_{i,j\in\mathbb{N}}$ are upper right triangular matrices with entries*

$$r_{ij} = \frac{i!}{j!}(-1)^{i+j}S(j,i) \quad \text{and} \quad l_{ij} = \frac{i!}{j!}(-1)^{i+j}s(j,i), \qquad i, j \in \mathbb{N}. \tag{6}$$

As already explained in the introduction, the fixation line $(L_t)_{t\geq 0}$ of the Bolthausen–Sznitman coalescent has the branching property. Alternatively, one may derive this branching property from the spectral decomposition of the generator (Theorem 3.1). We state the following corollary.

Corollary 3.2 (*Branching Property/Transition Probabilities of the Fixation Line*)**.** *For the Bolthausen–Sznitman coalescent, the random variable $L_t^{(i)}$ has probability generating function (pgf)*

$$\mathbb{E}(z^{L_t^{(i)}}) = (1-(1-z)^{e^{-t}})^i, \qquad |z| < 1, t \geq 0, i \in \mathbb{N}. \tag{7}$$

Thus, $(L_t)_{t\geq 0}$ is a Markovian continuous-time branching process with state space $\mathbb{N}$ and offspring distribution $p_k = 1/(k(k-1))$, $k \in \{2, 3, \ldots\}$ having infinite mean. Moreover, the transition *probabilities* $p_{ij}(t) := \mathbb{P}(L_t = j \mid L_0 = i) = \mathbb{P}(L_t^{(i)} = j)$ *are given by*

$$\begin{aligned} p_{ij}(t) &= (-1)^{i+j}\frac{i!}{j!}\sum_{k=i}^{j} S(k,i)e^{-tk}s(j,k) \\ &= (-1)^j \sum_{k=1}^{i}(-1)^k \binom{i}{k}\binom{e^{-t}k}{j}, \qquad i,j\in\mathbb{N}. \end{aligned} \tag{8}$$

Remarks.

1. For $i = 1$ it follows that $L_t = L_t^{(1)}$ has pgf $\mathbb{E}(z^{L_t}) = 1-(1-z)^\alpha = -\sum_{j=1}^\infty \binom{\alpha}{j}(-z)^j$ and Sibuya distribution

$$\mathbb{P}(L_t = j) = p_{1j}(t) = (-1)^{j+1}\binom{\alpha}{j} = \frac{\alpha\Gamma(j-\alpha)}{\Gamma(1-\alpha)\Gamma(j+1)}, \qquad j\in\mathbb{N}, \tag{9}$$

where $\alpha := e^{-t}$. Note that $\mathbb{P}(L_t = j) \sim \alpha/(\Gamma(1-\alpha)j^{\alpha+1})$ as $j\to\infty$ and that L_t has a Pareto like tail $\mathbb{P}(L_t \ge j) = \Gamma(j-\alpha)/(\Gamma(1-\alpha)\Gamma(j)) \sim 1/(\Gamma(1-\alpha)j^\alpha)$ as $j\to\infty$. Thus, $\mathbb{E}(L_t^q) = \sum_{j=1}^\infty j^q\mathbb{P}(L_t=j) < \infty$ if and only if $q<\alpha$. Particular reciprocal factorial moments of L_t are known explicitly. For example,

$$\begin{aligned} \mathbb{E}\left(\frac{1}{(L_t+1)(L_t+2)\cdots(L_t+k)}\right) &= \frac{\alpha}{\Gamma(1-\alpha)}\sum_{j=1}^\infty \frac{\Gamma(j-\alpha)}{\Gamma(j+k+1)} \\ &= \frac{\alpha}{k!(\alpha+k)}, \quad k\in\mathbb{N}. \end{aligned}$$

The Sibuya distribution (9) and similar distributions occur in [7, Eq. (2)], [18, p. 9], [19, p. 225] and [33, p. 70, Eq. 3.38].

2. The pgf $f(s) := \sum_{k=2}^\infty p_k s^k = s + (1-s)\log(1-s)$ of the offspring distribution satisfies

$$\int_{(1-\varepsilon,1)}\frac{\lambda(ds)}{f(s)-s} = \int_{(1-\varepsilon,1)}\frac{\lambda(ds)}{(1-s)\log(1-s)} = \int_{(0,\varepsilon)}\frac{\lambda(dx)}{x\log x} = -\infty$$

for all $\varepsilon\in(0,1)$, where λ denotes Lebesgue measure on $(0,1)$. This implies (Harris [15, p. 107]) that the fixation line $(L_t)_{t\ge 0}$ does not explode, in agreement (see [13]) with the fact that the Bolthausen–Sznitman coalescent stays infinite.

As a second application we study the probability $h(i,j) = \mathbb{P}(L_t^{(i)} = j \text{ for some} t \ge 0)$ that the fixation line hits state $j\in\mathbb{N}$ started from state $i\in\mathbb{N}$.

Corollary 3.3 (*Hitting Probabilities*). *The hitting probabilities* $h(i,j)$ *have horizontal generating function*

$$\sum_{j=i}^\infty h(i,j)z^{j-1} = \frac{z^i}{(1-z)(-\log(1-z))}, \qquad i\in\mathbb{N}, |z|<1. \tag{10}$$

Moreover, for all $i\in\mathbb{N}$,

$$h(i,j) = \frac{1}{\log j} - \frac{\gamma}{\log^2 j} + O\left(\frac{1}{\log^3 j}\right), \qquad j\to\infty, \tag{11}$$

where $\gamma := -\Gamma'(1) \approx 0.577216$ *denotes the Euler–Mascheroni constant. The hitting probability* $h(i,j)$ *can be computed via*

$$h(i,j) = \sum_{k=1}^{j-i} \mathbb{P}(\eta_1 + \cdots + \eta_k = j-i), \qquad 1 \le i < j, \tag{12}$$

where $\eta_1, \eta_2, \ldots$ *are iid random variables with distribution* $\mathbb{P}(\eta_1 = n) := u_n := 1/(n(n+1))$, $n \in \mathbb{N}$. *The hitting probabilities can be also expressed in terms of the Stirling numbers* $s(.,.)$ *and* $S(.,.)$ *of the first and second kind as*

$$h(i,j) = (-1)^{i+j} \frac{i!}{(j-1)!} \sum_{k=i}^{j} \frac{s(j,k)S(k,i)}{k} \tag{13}$$

$$= (-1)^{j-i} \frac{1}{(j-i)!} \sum_{k=1}^{j-i+1} \frac{s(j-i+1,k)}{k}, \qquad 1 \le i \le j. \tag{14}$$

Moreover, $h(i,j)$ *has representations*

$$h(i,j) = \frac{1}{(j-i)!} \int_0^1 \frac{\Gamma(j-i+x)}{\Gamma(x)} \,\mathrm{d}x = \frac{1}{(j-i)!} \sum_{k=0}^{j-i} \frac{|s(j-i,k)|}{k+1}, \qquad 1 \le i \le j. \tag{15}$$

Remark. Concrete values of the hitting probabilities $h(i,j)$ for $i = 1$ and $j \in \{1, \ldots, 7\}$ are provided in the remark after the proof of Corollary 3.3.

We now turn to the block counting process $(N_t^{(n)})_{t \ge 0}$ of the Bolthausen–Sznitman n-coalescent. For $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$ let $\tau_{ni} := \inf\{t > 0 : N_t^{(n)} \le i\}$ denote the first time the block counting process $(N_t^{(n)})_{t \ge 0}$ jumps to a state smaller than or equal to i. Note that $\tau_n := \tau_{n1}$ is the absorption time of $N^{(n)}$.

Corollary 3.4 (*Distribution Function and Asymptotics of* τ_{ni}). *For all* $n \in \mathbb{N}$ *and* $i \in \{1, \ldots, n\}$, τ_{ni} *has distribution function*

$$\mathbb{P}(\tau_{ni} \le t) = \sum_{j=1}^{i} (-1)^{n+j} \binom{i}{j} \binom{je^{-t} - 1}{n-1}, \qquad t \in (0, \infty). \tag{16}$$

In particular, for every $i \in \mathbb{N}$, $\tau_{ni} - \log\log n \to \min(G_1, \ldots, G_i)$ *in distribution as* $n \to \infty$, *where* $G_1, G_2, \ldots$ *are independent standard Gumbel distributed random variables.*

Remark. Note that $\min(G_1, \ldots, G_i)$ has distribution function $F_i(x) := 1 - (1 - F(x))^i$, where $F(x) := e^{-e^{-x}}$, $x \in \mathbb{R}$. For $i = 1$ we recover the well known convergence result (see Goldschmidt and Martin [14, Proposition 3.4], Freund and Möhle [12, Corollary 1.2] or Hénard [17, Theorem 3.9]) that the scaled absorption time $\tau_n - \log\log n$ is asymptotically standard Gumbel distributed.

The fact that the distribution function (16) of τ_{ni} is known explicitly can be further exploited. For example, the following Edgeworth expansion holds.

Corollary 3.5 (*Edgeworth Expansion*). *For every* $i \in \mathbb{N}$ *and* $x \in \mathbb{R}$ *the following Edgeworth expansion of order* $K \in \mathbb{N}_0$ *holds.*

$$\mathbb{P}(\tau_{ni} - \log\log n \le x) = \sum_{k=0}^{K} c_k d_{ki}(x) \frac{e^{-kx}}{\log^k n} + O\left(\frac{1}{\log^{K+1} n}\right), \qquad n \to \infty, \tag{17}$$

where $c_0, c_1, \ldots$ *are the coefficients in the series expansion* $1/\Gamma(1-x) = \sum_{k=0}^{\infty} c_k x^k$, $|x| < 1$, *and*

$$d_{ki}(x) := \left(e^x \frac{\mathrm{d}}{\mathrm{d}x}\right)^k F_i(x) = \sum_{j=1}^{i} (F(x))^j (-1)^{j-1} \binom{i}{j} j^k, \qquad k \in \mathbb{N}_0, i \in \mathbb{N}, x \in \mathbb{R}, \tag{18}$$

with F_i *and* F *as defined in the previous remark. Alternatively,* $d_{0i}(x) = F_i(x)$ *and*

$$d_{ki}(x) = \sum_{j=1}^{k} S(k, j)(-1)^{j-1}(i)_j (F(x))^j (1 - F(x))^{i-j}, \qquad k, i \in \mathbb{N}, x \in \mathbb{R}, \tag{19}$$

where the $S(k, j)$ *are the Stirling numbers of the second kind and* $(i)_j := i(i-1)\cdots(i-j+1)$.

Remarks.

1. The coefficients c_k, $k \in \mathbb{N}_0$, are related to the moments of the Gumbel distribution (see Lemma 4.2). The concrete values c_k for $k \leq 3$ are provided in the remark after the proof of Lemma 4.2.
2. For $K = 1$ Corollary 3.5 reads $\mathbb{P}(\tau_{ni} - \log\log n \leq x) = F_i(x) - \gamma F_i'(x)/\log n + O(1/\log^2 n)$. In particular, for every $x \in \mathbb{R}$, $\mathbb{P}(\tau_{ni} - \log\log n \leq x) - F_i(x) \sim -\gamma F_i'(x)/\log n$ as $n \to \infty$. Thus, the speed of the convergence of $\tau_{ni} - \log\log n$ to G_i is of order $1/\log n$.

4. Proofs

Proof of Theorem 2.1(a). Let $Z^{(n)} := (X_t^{(n)}, t)_{t\geq 0}$ and $Z := (X_t, t)_{t\geq 0}$ denote the space–time processes of $X^{(n)} = (X_t^{(n)})_{t\geq 0}$ and $X = (X_t)_{t\geq 0}$ respectively. Note that $Z^{(n)}$ has state space $S_n := \{(j/n^{e^{-t}}, t) : j \in \{1, \ldots, n\}, t \geq 0\} = \bigcup_{t\geq 0}(E_{n,t} \times \{t\})$, where $E_{n,t} := \{j/n^{e^{-t}} : j \in \{1, \ldots, n\}\}$, and that Z has state space $S := E \times [0, \infty) = [0, \infty)^2$. The processes $Z^{(n)}$ and Z are time-homogeneous (see, for example, Revuz and Yor [35, p. 85, Exercise (1.10)]). In the following it is shown that $Z^{(n)}$ converges in $D_S[0, \infty)$ to Z as $n \to \infty$. Note that this convergence implies the desired convergence of $X^{(n)}$ in $D_E[0, \infty)$ to X as $n \to \infty$. Define $\pi_n : B(S) \to B(S_n)$ via $\pi_n f(x, s) := f(x, s)$ for all $f \in B(S)$ and $(x, s) \in S_n$. By Proposition A.4 it suffices to verify that, for every $t \geq 0$ and $\lambda, \mu > 0$,

$$\lim_{n\to\infty} \sup_{s\geq 0} \sup_{x\in E_{n,s}} |T_t^{(n)} \pi_n f_{\lambda,\mu}(x, s) - \pi_n T_t f_{\lambda,\mu}(x, s)| = 0,$$

where $(T_t^{(n)})_{t\geq 0}$ and $(T_t)_{t\geq 0}$ denote the semigroups of the space–time processes $Z^{(n)}$ and Z respectively and the test functions $f_{\lambda,\mu} : S \to \mathbb{R}$ are defined via $f_{\lambda,\mu}(x, s) := e^{-\lambda x - \mu s}$ for all $(x, s) \in S$. Fix $t \geq 0$ and $\lambda, \mu > 0$. For convenience, define $\alpha := e^{-t}$ and $\beta := e^{-s}$. We have

$$\begin{aligned} T_t^{(n)} \pi_n f_{\lambda,\mu}(x, s) &= \mathbb{E}(f_{\lambda,\mu}(X_{s+t}^{(n)}, s + t) \,|\, X_s^{(n)} = x) \\ &= (\alpha\beta)^\mu \mathbb{E}(\exp(-\lambda/n^{\alpha\beta} N_{s+t}^{(n)}) \,|\, N_s^{(n)} = xn^\beta) \\ &= (\alpha\beta)^\mu \mathbb{E}(\exp(-\lambda/n^{\alpha\beta} N_t^{(xn^\beta)})), \qquad (x, s) \in S_n, \end{aligned}$$

and

$$\begin{aligned} \pi_n T_t f_{\lambda,\mu}(x, s) &= \mathbb{E}(f_{\lambda,\mu}(X_{s+t}, s + t) \,|\, X_s = x) \\ &= (\alpha\beta)^\mu \mathbb{E}(\exp(-\lambda X_{s+t}) \,|\, X_s = x) \\ &= (\alpha\beta)^\mu \mathbb{E}(\exp(-\lambda x^\alpha X_t)), \qquad (x, s) \in S. \end{aligned}$$

Please cite this article in press as: J. Kukla, M. Möhle, On the block counting process and the fixation line of the Bolthausen–Sznitman coalescent, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.012.

Thus, we have to verify that

$$\lim_{n\to\infty}\sup_{s\geq 0}\sup_{x\in E_{n,s}}(\alpha\beta)^{\mu}|\mathbb{E}(\exp(-\lambda/n^{\alpha\beta}N_t^{(xn^{\beta})}))-\mathbb{E}(\exp(-\lambda x^{\alpha}X_t))|=0.$$

In the following it is shown that it suffices to verify the convergence of the one-dimensional distributions $X_t^{(k)}\to X_t$ in distribution as $k\to\infty$, $t\geq 0$. Since both expectations above are bounded between 0 and 1 and since $(\alpha\beta)^{\mu}=e^{-\mu(s+t)}$ tends to 0 as $s\to\infty$ it suffices to verify that, for every $s_0>0$,

$$\lim_{n\to\infty}\sup_{s\in[0,s_0]}\sup_{x\in E_{n,s}}|\mathbb{E}(\exp(-\lambda/n^{\alpha\beta}N_t^{(xn^{\beta})}))-\mathbb{E}(\exp(-\lambda x^{\alpha}X_t))|=0.$$

We will even verify that

$$\lim_{n\to\infty}\sup_{s\in[0,s_0]}\sup_{x>0}|\mathbb{E}(\exp(-\lambda/n^{\alpha\beta}N_t^{(\lfloor xn^{\beta}\rfloor)}))-\mathbb{E}(\exp(-\lambda x^{\alpha}X_t))|=0.$$

The difference of the two expectations depends on n and s only via $n^{\beta}=n^{e^{-s}}$. Since the map $s\mapsto n^{e^{-s}}$ is non-increasing it follows that the convergence for fixed $s\in[0,s_0]$ is slower as s is larger. So the slowest convergence holds at the right end point $s=s_0$. Thus, it suffices to verify that, for every $s\geq 0$,

$$\lim_{n\to\infty}\sup_{x>0}|\mathbb{E}(\exp(-\lambda/n^{\alpha\beta}N_t^{(\lfloor xn^{\beta}\rfloor)}))-\mathbb{E}(\exp(-\lambda x^{\alpha}X_t))|=0.$$

The map $x\mapsto\mathbb{E}(\exp(-\lambda x^{\alpha}X_t))$ is bounded, continuous, and non-increasing. Moreover, for every $n\in\mathbb{N}$ the map $x\mapsto\mathbb{E}(\exp(-\lambda/n^{\alpha\beta}N_t^{(\lfloor xn^{\beta}\rfloor)}))$ is bounded and non-increasing, since $N_t^{(1)}\leq N_t^{(2)}\leq\cdots$. Thus, by the theorem of Pólya [34, Satz I], it suffices to verify that, for every $s\geq 0$ and $x>0$,

$$\lim_{n\to\infty}\mathbb{E}(\exp(-\lambda/n^{\alpha\beta}N_t^{(\lfloor xn^{\beta}\rfloor)}))=\mathbb{E}(\exp(-\lambda x^{\alpha}X_t)).$$

Note that we have reduced the problem to verify the convergence uniformly for all $s\geq 0$ and $x\in E_{n,s}$ to the problem to verify the convergence pointwise for all points $(s,x)\in[0,\infty)\times(0,\infty)$.

Define $\tau:=n^{\beta}$ and $k:=\lfloor x\tau\rfloor$. Using this notation it remains to verify that

$$\lim_{\tau\to\infty}\mathbb{E}(\exp(-\lambda\tau^{-\alpha}N_t^{(\lfloor x\tau\rfloor)}))=\mathbb{E}(\exp(-\lambda x^{\alpha}X_t))$$

or, equivalently, that

$$\lim_{k\to\infty}\mathbb{E}(\exp(-\lambda k^{-\alpha}N_t^{(k)}))=\mathbb{E}(\exp(-\lambda X_t)).\tag{20}$$

Thus, it suffices to verify the convergence of the one-dimensional distributions $X_t^{(k)}=k^{-\alpha}N_t^{(k)}\to X_t$ in distribution as $k\to\infty$. In the remaining part of the proof this convergence of the one-dimensional distributions is verified by the method of moments. We have

$$\mathbb{E}(\exp(-\lambda k^{-\alpha}N_t^{(k)}))=\sum_{m=0}^{\infty}\frac{(-\lambda)^m}{m!}\frac{\mathbb{E}((N_t^{(k)})^m)}{k^{\alpha m}}.$$

Note that the series on the right hand side is absolutely convergent, since $N_t^{(k)}\leq k$ and, hence, $\mathbb{E}((N_t^{(k)})^m)\leq k^m$. Applying the formula $z^m=\sum_{i=0}^{m}(-1)^{m-i}S(m,i)[z]_i$, $m\in\mathbb{N}_0$, $z>0$, where $[z]_i:=\Gamma(z+i)/\Gamma(z)$ for $z,i>0$, it follows that

$$\frac{\mathbb{E}((N_t^{(k)})^m)}{k^{\alpha m}}=\sum_{i=0}^{m}(-1)^{m-i}S(m,i)\frac{\mathbb{E}([N_t^{(k)}]_i)}{k^{\alpha m}}=\sum_{i=0}^{m}(-1)^{m-i}S(m,i)\mathbb{E}(X_t^i)\frac{[k]_{\alpha i}}{k^{\alpha m}}$$

Please cite this article in press as: J. Kukla, M. Möhle, On the block counting process and the fixation line of the Bolthausen–Sznitman coalescent, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.012.

by Lemma 3.1 of [26]. From $[k]_{\alpha i} \sim k^{\alpha i}$ as $k \to \infty$ we conclude that only the summand $i = m$ yields asymptotically a non-zero contribution and it follows that

$$\lim_{k\to\infty} \frac{\mathbb{E}((N_t^{(k)})^m)}{k^{\alpha m}} = \mathbb{E}(X_t^m).$$

Moreover,

$$\frac{\mathbb{E}((N_t^{(k)})^m)}{k^{\alpha m}} \leq \frac{\mathbb{E}([N_t^{(k)}]_m)}{k^{\alpha m}} = \mathbb{E}(X_t^m)\frac{[k]_{\alpha m}}{k^{\alpha m}}.$$

It is readily checked that the map $k \mapsto [k]_{\alpha m}/k^{\alpha m}$ is non-increasing in k. Thus, we obtain the upper bound

$$\frac{\mathbb{E}((N_t^{(k)})^m)}{k^{\alpha m}} \leq \mathbb{E}(X_t^m)\frac{[k_0]_{\alpha m}}{k_0^{\alpha m}} \quad \text{for all } k \geq k_0.$$

Note that

$$\frac{\lambda^m}{m!}\mathbb{E}(X_t^m)\frac{[k_0]_{\alpha m}}{k_0^{\alpha m}} = \frac{\lambda^m}{m!}\frac{m!}{\Gamma(1+\alpha m)}\frac{\Gamma(k_0+\alpha m)}{k_0^{\alpha m}\Gamma(k_0)} \sim \left(\frac{\lambda}{k_0^\alpha}\right)^m (\alpha m)^{k_0-1}$$

as $m \to \infty$. Thus, if we choose k_0 sufficiently large such that $\lambda/k_0^\alpha < 1$, for example $k_0 := (2\lambda)^{1/\alpha}$, then the dominating map $m \mapsto (\lambda^m/m!)\mathbb{E}(X_t^m)[k_0]_{\alpha m}/k_0^{\alpha m}$ is integrable with respect to the counting measure on $\mathbb{N}$. Thus, it is allowed to apply the dominated convergence theorem, which yields

$$\lim_{k\to\infty} \mathbb{E}(\exp(-\lambda k^{-\alpha} N_t^{(k)})) = \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!}\mathbb{E}(X_t^m) = \mathbb{E}(\exp(-\lambda X_t)).$$

Thus, (20) is established. The proof is complete. □

Remark. The proof of Theorem 2.1(a) shows (see (20)) that it suffices to verify the convergence of the one-dimensional distributions. The convergence of the one-dimensional distributions is then established by the method of moments. Alternatively, one may first prove Theorem 2.1(b) and then, as already explained before Theorem 2.1, use the fact that the block counting process is Siegmund dual to the fixation line in order to verify the convergence of the one-dimensional distributions $X_t^{(k)} \to X_t$ in distribution as $k \to \infty$, $t \geq 0$.

Before we come to the proof of Theorem 2.1(b), we provide a recursion for the Laplace transforms of the finite-dimensional distributions of Neveu's continuous-state branching process $Y = (Y_t)_{t\geq 0}$.

Lemma 4.1 (*Recursion for the Laplace Transforms of* **Y**)**.** *Let* $0 = t_0 \leq t_1 < t_2 < \cdots$*. For* $k \in \mathbb{N}$ *let* $\psi_k : [0,\infty)^k \to [0,1]$*, defined via* $\psi_k(\lambda_1,\ldots,\lambda_k) := \mathbb{E}(e^{-\lambda_1 Y_{t_1}} \cdots e^{-\lambda_k Y_{t_k}})$ *for all* $\lambda_1,\ldots,\lambda_k \geq 0$*, denote the Laplace transform of* $Y_{t_1},\ldots,Y_{t_k}$*. Then,* ψ_k *satisfies the recursion* $\psi_1(\lambda_1) = e^{-\lambda_1^{\alpha_1}}$ *for all* $\lambda_1 \geq 0$ *and*

$$\psi_k(\lambda_1,\ldots,\lambda_k) = \psi_{k-1}(\lambda_1,\ldots,\lambda_{k-2},\lambda_{k-1}+\lambda_k^{\alpha_k/\alpha_{k-1}}), \qquad k \in \mathbb{N}\setminus\{1\}, \lambda_1,\ldots,\lambda_k \geq 0,$$

where $\alpha_j := e^{-t_j}$, $1 \leq j \leq k$.

Please cite this article in press as: J. Kukla, M. Möhle, On the block counting process and the fixation line of the Bolthausen–Sznitman coalescent, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.012.

Proof of Lemma 4.1. Clearly, $\psi_1(\lambda_1) = \mathbb{E}(e^{-\lambda_1 Y_{t_1}}) = e^{-\lambda_1^{\alpha_1}}$ for all $\lambda_1 \geq 0$. Moreover, for all $\lambda_1, \ldots, \lambda_k \geq 0$,

$$\begin{aligned}\psi_k(\lambda_1, \ldots, \lambda_k) &= \mathbb{E}(e^{-\lambda_1 Y_{t_1}} \cdots e^{-\lambda_k Y_{t_k}})\\ &= \mathbb{E}(\mathbb{E}(e^{-\lambda_1 Y_{t_1}} \cdots e^{-\lambda_k Y_{t_k}} \mid Y_{t_1}, \ldots, Y_{t_{k-1}}))\\ &= \mathbb{E}(e^{-\lambda_1 Y_{t_1}} \cdots e^{-\lambda_{k-1} Y_{t_{k-1}}} \mathbb{E}(e^{-\lambda_k Y_{t_k}} | Y_{t_{k-1}})).\end{aligned}$$

Since $\mathbb{E}(e^{-\lambda_k Y_{t_k}} | Y_{t_{k-1}}) = e^{-\lambda_k^{\alpha_k/\alpha_{k-1}} Y_{t_{k-1}}}$ almost surely it follows that

$$\begin{aligned}\psi_k(\lambda_1, \ldots, \lambda_k) &= \mathbb{E}(e^{\lambda_1 Y_{t_1}} \cdots e^{-\lambda_{k-2} Y_{t_{k-2}}} e^{-(\lambda_{k-1} + \lambda_k^{\alpha_k/\alpha_{k-1}}) Y_{t_{k-1}}})\\ &= \psi_{k-1}(\lambda_1, \ldots, \lambda_{k-2}, \lambda_{k-1} + \lambda_k^{\alpha_k/\alpha_{k-1}}). \qquad \square\end{aligned}$$

We are now able to verify Theorem 2.1(b).

Proof of Theorem 2.1(b). The proof is divided into two parts. First the convergence of the finite-dimensional distributions is verified. Afterwards the convergence in $D_E[0, \infty)$ is considered. In fact Part 2 does not need results from Part 1, so one could omit Part 1. However, we think it is helpful for the reader to consider first the convergence of the finite-dimensional distributions.

Part 1. (Convergence of the finite-dimensional distributions) Fix $0 = t_0 \leq t_1 < t_2 < \cdots$. For $k, n \in \mathbb{N}$ let $\psi_k^{(n)} : [0, \infty)^k \to [0, 1]$ and $\psi_k : [0, \infty)^k \to [0, 1]$ denote the Laplace transforms of $(Y_{t_1}^{(n)}, \ldots, Y_{t_k}^{(n)})$ and $(Y_{t_1}, \ldots, Y_{t_k})$ respectively. In the following the pointwise convergence $\psi_k^{(n)} \to \psi_k$ as $n \to \infty$ is verified by induction on $k \in \mathbb{N}$.

Obviously, $L_{t_1}^{(n)}$ has generating function $\mathbb{E}(z_1^{L_{t_1}^{(n)}}) = (1 - (1 - z_1)^{\alpha_1})^n$, $z_1 \in [0, 1]$, where $\alpha_1 := e^{-t_1}$. Replacing z_1 by $e^{-\lambda_1/n^{1/\alpha_1}}$ with $\lambda_1 \geq 0$ it follows that

$$\psi_1^{(n)}(\lambda_1) = \mathbb{E}(e^{-\lambda_1 Y_{t_1}^{(n)}}) = (1 - (1 - e^{-\lambda_1/n^{1/\alpha_1}})^{\alpha_1})^n.$$

Clearly, $\psi_1(\lambda_1) = \mathbb{E}(e^{-\lambda_1 Y_{t_1}}) = e^{-\lambda_1^{\alpha_1}}$. Using the shorthand $x := \lambda_1/n^{1/\alpha_1}$ and the inequality $|a^n - b^n| \leq n|a - b|$, $|a|, |b| \leq 1$, it follows that

$$\begin{aligned}|\psi_1^{(n)}(\lambda_1) - \psi_1(\lambda_1)| &= |(1 - (1 - e^{-x})^{\alpha_1})^n - (e^{-x^{\alpha_1}})^n|\\ &\leq n|1 - (1 - e^{-x})^{\alpha_1} - e^{-x^{\alpha_1}}| = n(e^{-x^{\alpha_1}} - 1 + (1 - e^{-x})^{\alpha_1}),\end{aligned}$$

since $(1 - e^{-x})^{\alpha_1} \geq 1 - e^{-x^{\alpha_1}}$ by Lemma A.1. From $1 - e^{-x} \leq x$, $x \in \mathbb{R}$, and $e^{-t} - 1 + t \leq t^2/2$, $t \geq 0$, we conclude that

$$|\psi_1^{(n)}(\lambda_1) - \psi_1(\lambda_1)| \;\leq\; n(e^{-x^{\alpha_1}} - 1 + x^{\alpha_1}) \;\leq\; n\frac{(x^{\alpha_1})^2}{2} = \frac{\lambda_1^{2\alpha_1}}{2n} \;\to\; 0, \qquad n \to \infty.$$

Thus, the pointwise convergence $\psi_1^{(n)} \to \psi_1$ as $n \to \infty$ is established.

Now fix $k \in \mathbb{N} \setminus \{1\}$. The induction step from $k - 1$ to k works as follows. For convenience define $\alpha_j := e^{-t_j}$ for all $j \in \mathbb{N}$. For all $z_1, \ldots, z_k \in [0, 1]$,

$$\begin{aligned}\mathbb{E}(z_1^{L_{t_1}^{(n)}} \cdots z_k^{L_{t_k}^{(n)}}) &= \mathbb{E}(\mathbb{E}(z_1^{L_{t_1}^{(n)}} \cdots z_k^{L_{t_k}^{(n)}} \mid L_{t_1}^{(n)}, \ldots, L_{t_{k-1}}^{(n)}))\\ &= \mathbb{E}(z_1^{L_{t_1}^{(n)}} \cdots z_{k-1}^{L_{t_{k-1}}^{(n)}} \mathbb{E}(z_k^{L_{t_k}^{(n)}} \mid L_{t_{k-1}}^{(n)})).\end{aligned}$$

Since $\mathbb{E}(z_k^{L_{t_k}^{(n)}} \mid L_{t_{k-1}}^{(n)}) = (1 - (1 - z_k)^{\alpha_k/\alpha_{k-1}})^{L_{t_{k-1}}^{(n)}}$ almost surely it follows that

$$\mathbb{E}(z_1^{L_{t_1}^{(n)}} \cdots z_k^{L_{t_k}^{(n)}}) = \mathbb{E}(z_1^{L_{t_1}^{(n)}} \cdots z_{k-2}^{L_{t_{k-2}}^{(n)}} u_{k-1}^{L_{t_{k-1}}^{(n)}}),$$

Please cite this article in press as: J. Kukla, M. Möhle, On the block counting process and the fixation line of the Bolthausen–Sznitman coalescent, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.012.

where $u_{k-1} := z_{k-1}(1-(1-z_k)^{\alpha_k/\alpha_{k-1}})$. Replacing for each $j \in \{1,\ldots,k\}$ the variable z_j by $e^{-\lambda_j/n^{1/\alpha_j}}$ with $\lambda_j \geq 0$ it follows that

$$\begin{aligned}\psi_k^{(n)}(\lambda_1,\ldots,\lambda_k) &= \mathbb{E}(e^{-\lambda_1 Y_{t_1}^{(n)}}\cdots e^{-\lambda_k Y_{t_k}^{(n)}})\\ &= \mathbb{E}(e^{-\lambda_1 Y_{t_1}^{(n)}}\cdots e^{-\lambda_{k-2} Y_{t_{k-2}}^{(n)}} e^{-\mu_{k-1}(n) Y_{t_{k-1}}^{(n)}})\\ &= \psi_{k-1}^{(n)}(\lambda_1,\ldots,\lambda_{k-2},\mu_{k-1}(n)),\end{aligned} \tag{21}$$

where

$$\mu_{k-1}(n) := \lambda_{k-1} - n^{1/\alpha_{k-1}}\log(1-(1-e^{-\lambda_k/n^{1/\alpha_k}})^{\alpha_k/\alpha_{k-1}}).$$

A technical but straightforward calculation shows that $\mu_{k-1}(n) \to \lambda_{k-1} + \lambda_k^{\alpha_k/\alpha_{k-1}}$ as $n \to \infty$. Moreover, by induction, $\psi_{k-1}^{(n)}$ converges pointwise to ψ_{k-1} as $n \to \infty$. It is well known that the convergence $\psi_{k-1}^{(n)} \to \psi_{k-1}$ of Laplace transforms holds even uniformly on any compact subset of $[0,\infty)^{k-1}$. Taking these facts into account it follows from (21) that

$$\begin{aligned}\lim_{n\to\infty}\psi_k^{(n)}(\lambda_1,\ldots,\lambda_k) &= \lim_{n\to\infty}\psi_{k-1}^{(n)}(\lambda_1,\ldots,\lambda_{k-2},\mu_{k-1}(n))\\ &= \psi_{k-1}(\lambda_1,\ldots,\lambda_{k-2},\lambda_{k-1}+\lambda_k^{\alpha_k/\alpha_{k-1}}) = \psi_k(\lambda_1,\ldots,\lambda_k),\end{aligned}$$

where the last equality holds by Lemma 4.1. The induction is complete.

The pointwise convergence $\psi_k^{(n)} \to \psi_k$ of the Laplace transforms implies the convergence $(Y_{t_1}^{(n)},\ldots,Y_{t_k}^{(n)}) \to (Y_{t_1},\ldots,Y_{t_k})$ in distribution as $n \to \infty$.

Part 2. (Convergence in $D_E[0,\infty)$) We proceed essentially in the same way as in the proof of Theorem 2.1(a), however the detail arguments differ slightly from those in the proof of part a). Recall that $E := [0,\infty)$ is the state space of the limiting process Y. For $n \in \mathbb{N}$ and $t \geq 0$ define $E_{n,t} := \{j/n^{e^t} : j = n, n+1, \ldots\}$. Note that the processes $Y^{(n)}$ are time-inhomogeneous. In order to obtain time-homogeneous processes let $Z^{(n)} := (Y_t^{(n)}, t)_{t\geq 0}$ and $Z := (Y_t, t)_{t\geq 0}$ denote the space–time processes of $(Y_t^{(n)})_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ respectively. Note that $Z^{(n)}$ has state space $S_n := \{(j/n^{e^t}, t) : j = n, n+1, \ldots, t \geq 0\} = \bigcup_{t\geq 0}(E_{n,t}\times\{t\})$ and that Z has state space $S := E \times [0,\infty) = [0,\infty)^2$. According to Revuz and Yor [35, p. 85, Exercise (1.10)] the processes $Z^{(n)}$ and Z are time-homogeneous. Define $\pi_n : B(S) \to B(S_n)$ via $\pi_n g(y,s) := g(y,s)$ for all $g \in B(S)$ and $(y,s) \in S_n$. In the following it is shown that $Z^{(n)}$ converges in $D_S[0,\infty)$ to Z as $n \to \infty$. Note that this convergence implies the desired convergence of $Y^{(n)}$ in $D_E[0,\infty)$ to Y as $n \to \infty$. For $\lambda,\mu > 0$ define the test function $g_{\lambda,\mu} \in \widehat{C}(S)$ via $g_{\lambda,\mu}(y,s) := e^{-\lambda y-\mu s}$, $(y,s) \in S$. By Proposition A.4 it suffices to verify that for every $t \geq 0$ and $\lambda,\mu > 0$,

$$\lim_{n\to\infty}\sup_{s\geq 0}\sup_{y\in E_{n,s}}|U_t^{(n)}\pi_n g_{\lambda,\mu}(y,s) - \pi_n U_t g_{\lambda,\mu}(y,s)| = 0, \tag{22}$$

where $U_t^{(n)} : B(S_n) \to B(S_n)$ is defined via $U_t^{(n)}g(y,s) := \mathbb{E}(g(Y_{s+t}^{(n)}, s+t)\,|\,Y_s^{(n)} = y)$, $g \in B(S_n)$, $s \geq 0$, $y \in E_{n,s}$. Note that $(U_t^{(n)})_{t\geq 0}$ is the semigroup of $Z^{(n)}$.

Fix $t \geq 0$ and $\lambda,\mu > 0$. As before define $\alpha := e^{-t}$. For all $n \in \mathbb{N}$, $s \geq 0$ and $y \in E_{n,s}$ we have (with the notation $\beta := e^{-s}$)

$$\begin{aligned}U_t^{(n)}\pi_n g_{\lambda,\mu}(y,s) &= \mathbb{E}(\pi_n g_{\lambda,\mu}(Y_{s+t}^{(n)}, s+t)\,|\,Y_s^{(n)} = y)\\ &= \mathbb{E}(\exp(-\lambda Y_{s+t}^{(n)} - \mu(s+t))\,|\,Y_s^{(n)} = y)\\ &= (\alpha\beta)^{\mu}\mathbb{E}(\exp(-\lambda/n^{1/(\alpha\beta)}L_{s+t}^{(n)})\,|\,L_s^{(n)} = yn^{1/\beta})\\ &= (\alpha\beta)^{\mu}\mathbb{E}(\exp(-\lambda/n^{1/(\alpha\beta)}L_t^{(yn^{1/\beta})}))\end{aligned}$$

Please cite this article in press as: J. Kukla, M. Möhle, On the block counting process and the fixation line of the Bolthausen–Sznitman coalescent, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.012.

and

$$\begin{aligned}\pi_n U_t g_{\lambda,\mu}(y,s) &= U_t g_{\lambda,\mu}(y,s) = \mathbb{E}(\exp(-\lambda Y_{s+t} - \mu(s+t)) \,|\, Y_s = y)\\ &= (\alpha\beta)^\mu \mathbb{E}(\exp(-\lambda Y_{s+t}) \,|\, Y_s = y) = (\alpha\beta)^\mu \mathbb{E}(\exp(-\lambda y^{1/\alpha} Y_t)).\end{aligned}$$

Thus, one has to verify that

$$\lim_{n\to\infty} \sup_{s\ge 0} \sup_{y\in E_{n,s}} (\alpha\beta)^\mu |\mathbb{E}(\exp(-\lambda/n^{1/(\alpha\beta)} L_t^{(yn^{1/\beta})})) - \mathbb{E}(\exp(-\lambda y^{1/\alpha} Y_t))| = 0. \tag{23}$$

We will even verify that

$$\lim_{n\to\infty} \sup_{s\ge 0} \sup_{y>0} |\mathbb{E}(\exp(-\lambda/n^{1/(\alpha\beta)} L_t^{(\lfloor yn^{1/\beta}\rfloor)})) - \mathbb{E}(\exp(-\lambda y^{1/\alpha} Y_t))| = 0.$$

The quantity inside the absolute values depends on n and s only via $n^{1/\beta} = n^{e^s}$. Since the map $s \mapsto n^{e^s}$ is non-decreasing it follows that the convergence for fixed $s \ge 0$ is slower as s is smaller. So the slowest convergence holds for $s = 0$ ($\Rightarrow \beta = 1$). Thus it suffices to verify that for every $t \ge 0$ and $\lambda > 0$

$$\lim_{n\to\infty} \sup_{y>0} |\mathbb{E}(\exp(-\lambda/n^{1/\alpha} L_t^{(\lfloor yn\rfloor)})) - \mathbb{E}(\exp(-\lambda y^{1/\alpha} Y_t))| = 0.$$

The map $y \mapsto \mathbb{E}(\exp(-\lambda y^{1/\alpha} Y_t))$ is bounded, continuous and non-increasing. Using that $L_t^{(1)} \le L_t^{(2)} \le \cdots$ it follows with the same argument as in the proof of Theorem 2.1(a) (Pólya's theorem [34, Satz I]) that it suffices to verify the above convergence pointwise for every $y > 0$. Defining $k := \lfloor yn \rfloor$ it is readily seen that this is equivalent to the convergence of the one-dimensional distributions $Y_t^{(k)} = k^{-1/\alpha} L_t^{(k)} \to Y_t$ in distribution as $k \to \infty$, $t \ge 0$. But the convergence of the one-dimensional distributions holds as already shown before Theorem 2.1 (or by Part 1). The proof of part (b) of Theorem 2.1 is complete.

The following calculations even provide an explicit upper bound for the difference

$$\begin{aligned} d &:= |\mathbb{E}(\exp(-\lambda/n^{1/(\alpha\beta)} L_t^{(yn^{1/\beta})})) - \mathbb{E}(\exp(-\lambda y^{1/\alpha} Y_t))|\\ &= |(1-(1-e^{-\lambda/n^{1/(\alpha\beta)}})^\alpha)^{yn^{1/\beta}} - e^{-y\lambda^\alpha}|\end{aligned}$$

occurring in (23) as well as an alternative proof of the convergence. Define $m := yn^{1/\beta} \in \{n, n+1, \ldots\}$ and $x := \lambda/n^{1/(\alpha\beta)}$. In the following it is assumed that $n \ge \lambda$ which implies that $x \le 1$. Using the inequality $|a^m - b^m| \le mr^{m-1}|a-b|$, $m \in \mathbb{N}$, where $r := \max(|a|, |b|)$, it follows that

$$\begin{aligned} d &= |(1-(1-e^{-\lambda/n^{1/(\alpha\beta)}})^\alpha)^{yn^{1/\beta}} - e^{-y\lambda^\alpha}|\\ &= |(1-(1-e^{-x})^\alpha)^m - (e^{-x^\alpha})^m|\\ &\le mr^{m-1}|1-(1-e^{-x})^\alpha - e^{-x^\alpha}|,\end{aligned}$$

where $r := \max(1-(1-e^{-x})^\alpha, e^{-x^\alpha}) = e^{-x^\alpha}$ by Lemma A.1. Note that $r \in (0, 1)$.

The map $z \mapsto zr^{z-1}$, $z \ge 0$ takes its maximum at the point $z = 1/(-\log r) = 1/x^\alpha$. Thus, $mr^{m-1} \le 1/x^\alpha r^{1/x^\alpha - 1} \le 1/x^\alpha$, since $r \le 1$ and $x \le 1$, i.e. $1/x^\alpha - 1 \ge 0$. Furthermore, $|1-(1-e^{-x})^\alpha - e^{-x^\alpha}| = e^{-x^\alpha} - 1 + (1-e^{-x})^\alpha \le e^{-x^\alpha} - 1 + x^\alpha \le (x^\alpha)^2/2$. Therefore, we obtain the upper bound

$$d \le \frac{1}{x^\alpha}\frac{(x^\alpha)^2}{2} = \frac{x^\alpha}{2} = \frac{\lambda^\alpha}{2n^{e^s}} \le \frac{\lambda^\alpha}{2n}.$$

Please cite this article in press as: J. Kukla, M. Möhle, On the block counting process and the fixation line of the Bolthausen–Sznitman coalescent, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.012.

Note that this upper bound does not depend on y and s. Thus, for all $t \geq 0$, $\lambda, \mu > 0$ and all $n \in \mathbb{N}$ with $n \geq \lambda$,

$$\begin{aligned}
&\sup_{s\geq 0}\sup_{y\in E_{n,s}} |U_t^{(n)}\pi_n g_{\lambda,\mu}(y,s) - \pi_n U_t g_{\lambda,\mu}(y,s)| \\
&\quad = \sup_{s\geq 0}\sup_{y\in E_{n,s}} \underbrace{(\alpha\beta)^\mu}_{\leq 1} |(1-(1-e^{-\lambda/n^{1/(\alpha\beta)}})^\alpha)^{yn^{1/\beta}} - e^{-y\lambda^\alpha}| \leq \frac{\lambda^\alpha}{2n}.
\end{aligned} \tag{24}$$

In particular, (22) holds for all $t \geq 0$ and all $\lambda, \mu > 0$. □

Remark. The presented proof of Theorem 2.1(b) does not use any duality argument and shows that it suffices to verify the convergence of the one-dimensional distributions. The proof gives some more information than stated in Theorem 2.1(b). Note that (24) provides the explicit upper bound $\lambda^\alpha/(2n)$, showing that the convergence of the semigroups is of order $1/n$, at least for test functions of the form $g_{\lambda,\mu}$, $\lambda, \mu > 0$.

To the best of the authors' knowledge the convergence result on the fixation line has no counterpart in the literature on branching processes and may hence trigger further research in the field of continuous-time branching processes (with infinite offspring mean).

We now turn to the proofs concerning the results in Section 3.

Proof of Theorem 3.1. Two proofs are provided. The first proof is self-contained and based on generating functions. The second proof uses duality and the spectral decomposition [27, Theorem 1.1] of the generator of the block counting process.

Proof 1 (*via Generating Functions*)**.** The proof is similar to that of Theorem 1.1 of [27]. Let $D = (d_{ij})_{i,j\in\mathbb{N}}$ be the diagonal matrix with entries $d_{ii} := -\gamma_i = \gamma_{ii}$, $i \in \mathbb{N}$. Furthermore, let $R = (r_{ij})_{i,j\in\mathbb{N}}$ be the upper right triangular matrix with entries defined for each $j \in \mathbb{N}$ recursively via $r_{jj} := 1$ and

$$r_{ij} := \frac{1}{\gamma_i - \gamma_j}\sum_{k=i+1}^{j} \gamma_{ik} r_{kj}, \qquad i \in \{j-1, j-2, \dots, 1\}. \tag{25}$$

Since $\gamma_{ii} = -\gamma_i$, $i \in \mathbb{N}$, we conclude that $r_{ij}\gamma_{jj} = \sum_{k=i}^{j} \gamma_{ik} r_{kj}$. Thus, the entries of R are defined such that $RD = \Gamma R$. Define $L := R^{-1}$. Then, the spectral decomposition $\Gamma = RDL$ holds. Moreover, $DL = L\Gamma$ and, hence, $\gamma_{ii} l_{ij} = \sum_{k=i}^{j} l_{ik}\gamma_{kj}$, $i, j \in \mathbb{N}$. Since $\gamma_{ii} = -\gamma_i$, $i \in \mathbb{N}$, we obtain for each $i \in \mathbb{N}$ the recursion $l_{ii} = 1$ and

$$l_{ij} = \frac{1}{\gamma_j - \gamma_i}\sum_{k=i}^{j-1} l_{ik}\gamma_{kj}, \qquad j \in \{i+1, i+2, \dots\}. \tag{26}$$

Let $U := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc. For $i \in \mathbb{N}$ define the generating function $l_i : U \to \mathbb{C}$ via $l_i(z) := \sum_{j=i}^{\infty} l_{ij} z^j$, $z \in U$, and consider the modified function $f_i : U \to \mathbb{C}$ defined via $f_i(z) := \sum_{j=i}^{\infty} (j-i) l_{ij} z^j$, $z \in U$. We have

$$f_i(z) = \sum_{j=i}^{\infty} j l_{ij} z^j - i\sum_{j=i}^{\infty} l_{ij} z^j = z l_i'(z) - i l_i(z).$$

Please cite this article in press as: J. Kukla, M. Möhle, On the block counting process and the fixation line of the Bolthausen–Sznitman coalescent, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.012.

On the other hand, by the recursion (26), we obtain the factorization

$$\begin{aligned} f_i(z) &= \sum_{j=i+1}^{\infty}(j-i)l_{ij}z^j = \sum_{j=i+1}^{\infty}\sum_{k=i}^{j-1} l_{ik}\gamma_{kj}z^j \\ &= \sum_{k=i}^{\infty} l_{ik} \sum_{j=k+1}^{\infty}\gamma_{kj}z^j = \sum_{k=i}^{\infty} kl_{ik}z^k \sum_{j=k+1}^{\infty}\frac{z^{j-k}}{(j-k)(j-k+1)} \\ &= \sum_{k=i}^{\infty} kl_{ik}z^k \sum_{n=1}^{\infty}\frac{z^n}{n(n+1)} = zl_i'(z)a(z), \end{aligned}$$

where the auxiliary function $a : U \to \mathbb{C}$ is defined via $a(z) := \sum_{n=1}^{\infty} z^n/(n(n+1)) = 1-(1-z)(-\log(1-z))/z$, $z \in U$. Thus, l_i satisfies the differential equation $zl_i'(z)a(z) = zl_i'(z) - il_i(z)$ or, equivalently,

$$l_i'(z) = \frac{il_i(z)}{(1-a(z))z} = \frac{il_i(z)}{(1-z)(-\log(1-z))}.$$

The solution of this homogeneous differential equation with initial conditions $l_i(0) = \cdots = l_i^{(i-1)}(0) = 0$ and $l_i^{(i)}(0) = i!$ is $l_i(z) = (-\log(1-z))^i$, $i \in \mathbb{N}$, $z \in U$. Here $l_i^{(j)}$ denotes the jth derivative of l_i. For $f(z) = \sum_{j=0}^{\infty} a_j z^j$ let $[z^j]f(z) := a_j$ denote the coefficient in front of z^j in the series expansion of f. By [1, p. 824], $l_i(z) = (-\log(1-z))^i = i!\sum_{j=i}^{\infty}|s(j,i)|z^j/j!$ and, hence,

$$l_{ij} = [z^j]l_i(z) = \frac{i!}{j!}|s(j,i)| = \frac{i!}{j!}(-1)^{i+j}s(j,i),$$

which is the second formula in (6). Let us now turn to the inverse $R = L^{-1}$ of L. We have $L(z,z^2,\ldots)^\top = (l_1(z), l_2(z), \ldots)^\top$. Multiplying from the left with R it follows that $(z,z^2,\ldots)^\top = R(l_1(z), l_2(z), \ldots)^\top$. Thus, $z^i = \sum_{j=i}^{\infty} r_{ij}l_j(z) = \sum_{j=i}^{\infty} r_{ij}(-\log(1-z))^j$. Replacing z by $1-e^{-z}$ leads to $(1-e^{-z})^i = \sum_{j=i}^{\infty} r_{ij}z^j =: r_i(z)$, $i \in \mathbb{N}$, $z \in U$. The calculations between Eq. (2.9) and (2.10) in [27] show that r_i has expansion

$$r_i(z) = (1-e^{-z})^i = \sum_{j=0}^{\infty}(-1)^{i+j}\frac{i!}{j!}S(j,i)z^j,$$

which yields the formula in (6) for the coefficient $r_{ij} = [z^j]r_i(z)$ in front of z^j. □

Proof 2 (*via Duality*)**.** The duality kernel H can be interpreted as a non-singular matrix $H = (h_{ij})_{i,j\in\mathbb{N}}$ with entries $h_{ij} = 1$ for $j \ge i$ and $h_{ij} = 0$ for $j < i$. The entries of its inverse $H^{-1} =: (g_{ij})_{i,j\in\mathbb{N}}$ are given by $g_{ij} = \delta_{i,j} - \delta_{i+1,j}$. It is known [27] that the generator matrix Q of the block counting process has spectral decomposition $Q = \tilde{R}\tilde{D}\tilde{L}$, where the matrices $\tilde{R} = (\tilde{r}_{ij})_{i,j\in\mathbb{N}}$, $\tilde{D} = (\tilde{d}_{ij})_{i,j\in\mathbb{N}}$ and $\tilde{L} = (\tilde{l}_{ij})_{i,j\in\mathbb{N}}$ are given by $\tilde{r}_{ij} = ((j-1)!/(i-1)!)|s(i,j)|$, $\tilde{d}_{ij} = (i-1)\delta_{i,j}$ and $\tilde{l}_{ij} = (-1)^{i+j}((j-1)!/(i-1)!)S(i,j)$ respectively. The entries of $D = (d_{ij})_{i,j\in\mathbb{N}}$ can be read from the diagonal of Γ and are therefore given by $d_{ij} = i\delta_{i,j}$. Define the matrices $A = (a_{ij})_{i,j\in\mathbb{N}}$ and $B = (b_{ij})_{i,j\in\mathbb{N}}$ by $a_{ij} = \delta_{i+1,j}$ and $b_{ij} = \delta_{i-1,j}$. Clearly $\tilde{D} = BDA$. This together with the duality relation $H\Gamma^\top = QH$ and the spectral decomposition of the block counting process $Q = \tilde{R}\tilde{D}\tilde{L}$ yields

$$\Gamma^\top = H^{-1}\tilde{R}\tilde{D}\tilde{L}H = (-H^{-1}\tilde{R}B)D(-A\tilde{L}H).$$

Hence $\Gamma = RDL$ with $R := (-A\tilde{L}H)^\top$ and $L := (-H^{-1}\tilde{R}B)^\top$. It remains to calculate the entries of R and L. Using the recursion $S(i+1, j) = jS(i, j) + S(i, j-1)$ we obtain

$$\begin{aligned} r_{ji} &= (-A\tilde{L}H)_{ij} = -(\tilde{L}H)_{i+1,j} = -\sum_{k=1}^{j} \tilde{l}_{i+1,k} = \sum_{k=1}^{j} (-1)^{i+k} \frac{(k-1)!}{i!} S(i+1, k) \\ &= \sum_{k=1}^{j} (-1)^{i+k} \frac{k!}{i!} S(i, k) + \sum_{k=1}^{j} (-1)^{i+k} \frac{(k-1)!}{i!} S(i, k-1) \\ &= \sum_{k=1}^{j} (-1)^{i+k} \frac{k!}{i!} S(i, k) - \sum_{k=0}^{j-1} (-1)^{i+k} \frac{k!}{i!} S(i, k) = (-1)^{i+j} \frac{j!}{i!} S(i, j). \end{aligned}$$

Using the recursion $|s(i+1, j+1)| = |s(i, j)| + i|s(i, j+1)|$ we get

$$\begin{aligned} l_{ji} &= (-H^{-1}\tilde{R}B)_{ij} = -(H^{-1}\tilde{R})_{i,j+1} = \tilde{r}_{i+1,j+1} - \tilde{r}_{i,j+1} \\ &= \frac{j!}{i!} |s(i+1, j+1)| - \frac{j!}{(i-1)!} |s(i, j+1)| = \frac{j!}{i!} |s(i, j)|. \quad \square \end{aligned}$$

Proof of Corollary 3.2. By Theorem 3.1, $\Gamma = RDL$, where R and $L = R^{-1}$ have entries (6). Hence, the transition matrix $P(t) = e^{t\Gamma}$ has spectral decomposition $P(t) = e^{tRDL} = Re^{tD}L$. Thus, $p_{ij}(t) = \mathbb{P}(L_t = j \mid L_0 = i) = (Re^{tD}L)_{ij} = \sum_{k=i}^{j} r_{ik} e^{-\gamma_k t} l_{kj}$. The first formula in (8) for $p_{ij}(t)$ follows from $\gamma_k = k$ and from (6). Recall that $\alpha := e^{-t}$. Conditional on $L_0 = i$ the random variable L_t has probability generating function

$$\begin{aligned} \mathbb{E}(z^{L_t} \mid L_0 = i) &= \sum_{j=i}^{\infty} z^j p_{ij}(t) = \sum_{j=i}^{\infty} z^j (-1)^{i+j} \frac{i!}{j!} \sum_{k=i}^{j} S(k, i) \alpha^k s(j, k) \\ &= (-1)^i i! \sum_{k=i}^{\infty} S(k, i) \alpha^k \sum_{j=k}^{\infty} \frac{(-z)^j}{j!} s(j, k) \\ &= (-1)^i i! \sum_{k=i}^{\infty} S(k, i) \alpha^k \frac{(\log(1-z))^k}{k!} \\ &= (-1)^i (e^{\alpha \log(1-z)} - 1)^i = (1 - (1-z)^\alpha)^i, \quad |z| < 1, t \ge 0, i \in \mathbb{N}. \end{aligned}$$

Expansion leads to

$$\begin{aligned} \mathbb{E}(z^{L_t} \mid L_0 = i) &= \sum_{k=0}^{i} \binom{i}{k} (-1)^k (1-z)^{\alpha k} = \sum_{k=0}^{i} \binom{i}{k} (-1)^k \sum_{j=0}^{\infty} \binom{\alpha k}{j} (-z)^j \\ &= \sum_{j=0}^{\infty} (-z)^j \sum_{k=0}^{i} (-1)^k \binom{i}{k} \binom{\alpha k}{j}. \end{aligned}$$

The coefficient in front of z^j in this expansion yields the second formula for $p_{ij}(t)$. $\square$

Proof of Corollary 3.3. The hitting probability $h(i, j)$ is related to the entry $g(i, j) := \int_0^\infty \mathbb{P}(L_t^{(i)} = j)\,dt$ of the Green matrix via $h(i, j) = \gamma_j g(i, j) = j g(i, j)$ (see, for example, Norris [29, p. 146]). Thus, for all $i \in \mathbb{N}$ and $|z| < 1$,

$$h_i(z) := \sum_{j=i}^{\infty} h(i, j) z^{j-1} = \int_0^\infty \sum_{j=i}^{\infty} j \mathbb{P}(L_t^{(i)} = j) z^{j-1}\,dt = \int_0^\infty \frac{d}{dz} \sum_{j=i}^{\infty} \mathbb{P}(L_t^{(i)} = j) z^j\,dt.$$

Please cite this article in press as: J. Kukla, M. Möhle, On the block counting process and the fixation line of the Bolthausen–Sznitman coalescent, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.012.

Plugging in the formula (7) for the pgf of $L_t^{(i)}$ it follows that

$$h_i(z) = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}z}(1-(1-z)^{e^{-t}})^i \,\mathrm{d}t = \int_0^\infty i(1-(1-z)^{e^{-t}})^{i-1} e^{-t}(1-z)^{e^{-t}-1}\,\mathrm{d}t.$$

Substituting $x := e^{-t}$ and noting that $\mathrm{d}t/\mathrm{d}x = -1/x$ leads to $h_i(z) = (1-z)^{-1}\int_0^1 i(1-(1-z)^x)^{i-1}(1-z)^x\,\mathrm{d}x$. Substituting further $y := 1-(1-z)^x$ and noting that $\mathrm{d}x/\mathrm{d}y = 1/((1-y)(-\log(1-z)))$ we obtain

$$h_i(z) = \frac{1}{(1-z)(-\log(1-z))}\int_0^z iy^{i-1}\,\mathrm{d}y = \frac{z^i}{(1-z)(-\log(1-z))}, \qquad i \in \mathbb{N}, |z| < 1.$$

In particular, $h(i,j) = h(1, j-i+1)$. The asymptotic expansion (11) follows from Panholzer [30, Eq. (19)]. Formula (12) is obtained as follows. Let $(J_k)_{k\in\mathbb{N}_0}$ denote the jump chain of the fixation line $(L_t)_{t\geq 0}$. Given this chain is in state i it jumps to state $i+j$ with probability $\gamma_{i,i+j}/\gamma_i = 1/(j(j+1)) =: u_j$, $j \in \mathbb{N}$. From this property it is easily seen that the jump chain has independent increments, i.e. $J_0 = 1$, $J_1 = 1+\eta_1$, $J_2 = 1+\eta_1+\eta_2$ and so on, where $\eta_1, \eta_2, \ldots$ are iid random variables with distribution $\mathbb{P}(\eta_1 = j) = u_j$, $j \in \mathbb{N}$. For $1 \leq i < j$ it follows that $h(i,j) = h(1, j-i+1) = \sum_{k=1}^{j-i}\mathbb{P}(J_k = j-i+1) = \sum_{k=1}^{j-i}\mathbb{P}(\eta_1+\cdots+\eta_k = j-i)$. Formula (13) for $h(i,j)$ follows from $h(i,j) = jg(i,j) = j\int_0^\infty \mathbb{P}(L_t^{(i)} = j)\,\mathrm{d}t$ and

$$\begin{aligned}\int_0^\infty \mathbb{P}(L_t^{(i)} = j)\,\mathrm{d}t &= \int_0^\infty (-1)^{i+j}\frac{i!}{j!}\sum_{k=i}^{j} S(k,i)e^{-tk}s(j,k)\,\mathrm{d}t\\ &= (-1)^{i+j}\frac{i!}{j!}\sum_{k=i}^{j}\frac{S(k,i)s(j,k)}{k}.\end{aligned}$$

Eq. (14) follows from $h(i,j) = h(1,j-i+1)$ and $S(k,1) = 1$ for all $k \in \mathbb{N}$. Moreover, for $i = 1$ we have $\mathbb{P}(L_t = j) = \alpha\Gamma(j-\alpha)/(j!\Gamma(1-\alpha))$ with $\alpha := e^{-t}$. Thus,

$$g(1,j) = \int_0^\infty \mathbb{P}(L_t = j)\,\mathrm{d}t = \frac{1}{j!}\int_0^1 \frac{\Gamma(j-\alpha)}{\Gamma(1-\alpha)}\,\mathrm{d}\alpha = \frac{1}{j!}\int_0^1 \frac{\Gamma(j-1+x)}{\Gamma(x)}\,\mathrm{d}x$$

and, hence, we obtain the integral representation

$$\begin{aligned}h(i,j) &= h(1,j-i+1) = (j-i+1)g(1,j-i+1)\\ &= \frac{1}{(j-i)!}\int_0^1 \frac{\Gamma(j-i+x)}{\Gamma(x)}\,\mathrm{d}x, \quad 1 \leq i \leq j.\end{aligned}$$

The last formula for $h(i,j)$ in (15) follows from $\Gamma(n+x)/\Gamma(x) = \sum_{k=0}^n |s(n,k)|x^k$, $n \in \mathbb{N}_0$, $x \in \mathbb{R}$. The proof of Corollary 3.3 is complete. □

Remark. Note that $\mathbb{P}(\eta_1+\cdots+\eta_k = j-i) = \sum_{i_1,\ldots,i_k} u_{i_1}\cdots u_{i_k}$, where the sum extends over all $i_1,\ldots,i_k \in \mathbb{N}$ satisfying $i_1+\cdots+i_k = j-i$. Hence, concrete values of the hitting probabilities are $h(1,1) = 1$, $h(1,2) = \mathbb{P}(\eta_1 = 1) = u_1 = 1/2$, $h(1,3) = \mathbb{P}(\eta_1 = 2) + \mathbb{P}(\eta_1+\eta_2 = 2) = u_2 + u_1^2 = 1/6 + 1/4 = 5/12 \approx 0.41667$, $h(1,4) = \mathbb{P}(\eta_1 = 3) + \mathbb{P}(\eta_1+\eta_2 = 3) + \mathbb{P}(\eta_1+\eta_2+\eta_3 = 3) = u_3 + 2u_1u_2 + u_1^3 = 1/12 + 1/6 + 1/8 = 3/8 = 0.375$, $h(1,5) = u_4 + (2u_1u_3 + u_2^2) + 3u_1^2u_2 = 1/20 + 1/9 + 1/8 = 251/720 \approx 0.34861$, $h(1,6) = 95/288 \approx 0.32986$, $h(1,7) = 19087/60480 \approx 0.31559$ and so on.

Proof of Corollary 3.4. By the definition of τ_{ni} and the duality of $(N_t)_{t\geq 0}$ and $(L_t)_{t\geq 0}$ we have $\mathbb{P}(\tau_{ni} \leq t) = \mathbb{P}(N_t^{(n)} \leq i) = \mathbb{P}(L_t^{(i)} \geq n) = \sum_{j=n}^\infty p_{ij}(t)$. Using the second formula for $p_{ij}(t)$ in

Please cite this article in press as: J. Kukla, M. Möhle, On the block counting process and the fixation line of the Bolthausen–Sznitman coalescent, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.012.

(8) yields

$$\begin{aligned}\mathbb{P}(\tau_{ni} \le t) &= \sum_{j=n}^{\infty}(-1)^j \sum_{k=1}^{i}(-1)^k \binom{i}{k}\binom{e^{-t}k}{j} \\ &= \sum_{k=1}^{i}(-1)^k \binom{i}{k}\sum_{j=n}^{\infty}(-1)^j \binom{e^{-t}k}{j} \\ &= \sum_{k=1}^{i}(-1)^k \binom{i}{k}(-1)^n \binom{e^{-t}k-1}{n-1},\end{aligned}$$

where the last equality holds since $\sum_{j=n}^{\infty}(-1)^j\binom{z}{j} = (-1)^n\binom{z-1}{n-1}$ for all $n \in \mathbb{N}$ and all $z \in \mathbb{R}$.

Fix $x \in \mathbb{R}$ and define $F(x) := e^{-e^{-x}}$ for convenience. Assume that n is sufficiently large such that $x + \log\log n > 0$. Choosing $t := x + \log\log n$ and noting that for all sufficiently large n

$$\begin{aligned}(-1)^{n-1}\binom{e^{-t}k-1}{n-1} &= \frac{\Gamma(n - ke^{-x}/\log n)}{\Gamma(n)\Gamma(1-ke^{-x}/\log n)} \\ &\sim \frac{\Gamma(n - ke^{-x}/\log n)}{\Gamma(n)} \to e^{-ke^{-x}} = (F(x))^k\end{aligned}$$

as $n \to \infty$ by an application of Stirling's formula $\Gamma(n+1) \sim (n/e)^n\sqrt{2\pi n}$ as $n \to \infty$, it follows that

$$\begin{aligned}\mathbb{P}(\tau_{ni} - \log\log n \le x) &= \mathbb{P}(\tau_{ni} \le x + \log\log n) \\ &\to \sum_{k=1}^{i}(-1)^{k-1}\binom{i}{k}(F(x))^k = 1-(1-F(x))^i, \qquad n \to \infty.\end{aligned}$$

It remains to note that $x \mapsto 1-(1-F(x))^i$, $x \in \mathbb{R}$, is the distribution function of the minimum of i standard Gumbel distributed random variables. $\square$

Before we will prove Corollary 3.5 we provide the Taylor expansion of the map $x \mapsto 1/\Gamma(1-x)$.

Lemma 4.2. *The map $x \mapsto 1/\Gamma(1-x)$ has Taylor expansion $1/\Gamma(1-x) = \sum_{k=0}^{\infty}c_k x^k$, $|x| < 1$, where the coefficients $c_0, c_1, \ldots$ are related to the moments $m_k = (-1)^k\Gamma^{(k)}(1)$, $k \in \mathbb{N}_0$, of the Gumbel distribution via $c_0 = m_0 = 1$ and*

$$c_k = \sum_{j=1}^{k}(-1)^j \sum_{\substack{k_1,\ldots,k_j \in \mathbb{N} \\ k_1+\cdots+k_j=k}} \frac{m_{k_1}\cdots m_{k_j}}{k_1!\cdots k_j!}, \qquad k \in \mathbb{N}. \tag{27}$$

Alternatively,

$$c_k = \frac{(-1)^k}{k!}\sum_{l=1}^{k}(-1)^l\binom{k+1}{l+1}(\Gamma^l)^{(k)}(1) \qquad k \in \mathbb{N}, \tag{28}$$

where $(\Gamma^l)^{(k)}$ denotes the kth derivative of the lth power of Γ.

Remark. Concrete values are $c_1 = -m_1 = -\gamma \approx -0.577216$, $c_2 = m_1^2 - m_2/2 = \gamma^2 - (\gamma^2 + \zeta(2))/2 = \gamma^2/2 - \pi^2/12 \approx -0.655878$, $c_3 = -m_3/6 + m_1m_2 - m_1^3 = \gamma\zeta(2)/2 - \zeta(3)/3 - \gamma^3/6 = \pi^2\gamma/12 - \zeta(3)/3 - \gamma^3/6 \approx 0.042003$ and so on.

Proof. A Gumbel distributed random variable τ has moment generating function $\mathbb{E}(e^{x\tau}) = \Gamma(1-x)$, $x < 1$. Thus, the map $x \mapsto \Gamma(1-x)$ has Taylor expansion $\Gamma(1-x) = \sum_{k=0}^{\infty} a_k x^k$, $|x| < 1$,where $a_k := m_k/k!$ and $m_k = \mathbb{E}(\tau^k)$, $k \in \mathbb{N}_0$, are the moments of the Gumbel distribution. For the reciprocal map $1/\Gamma(1-x)$ it follows that

$$\begin{aligned}\frac{1}{\Gamma(1-x)} &= \sum_{j=0}^{\infty}(1-\Gamma(1-x))^j = \sum_{j=0}^{\infty}\Big(\sum_{k=1}^{\infty} -a_k x^k\Big)^j \\ &= 1 + \sum_{j=1}^{\infty} \sum_{k_1,\ldots,k_j \in \mathbb{N}} (-a_{k_1})\cdots(-a_{k_j}) x^{k_1+\cdots+k_j} \\ &= 1 + \sum_{j=1}^{\infty}(-1)^j \sum_{k=1}^{\infty} x^k \sum_{\substack{k_1,\ldots,k_j \in \mathbb{N}\\ k_1+\cdots+k_j = k}} a_{k_1}\cdots a_{k_j} = \sum_{k=0}^{\infty} c_k x^k\end{aligned}$$

with $c_0 := 1$ and c_k, $k \in \mathbb{N}$, as given in (27), since $a_k = m_k/k!$, $k \in \mathbb{N}_0$. Since $m_k = (-1)^k \Gamma^{(k)}(1)$, (27) can be rewritten as

$$\begin{aligned}c_k &= \sum_{j=1}^{k}(-1)^{j+k} \sum_{\substack{k_1,\ldots,k_j \in \mathbb{N}\\ k_1+\cdots+k_j = k}} \frac{\Gamma^{(k_1)}(1)\cdots\Gamma^{(k_j)}(1)}{k_1!\cdots k_j!} \\ &= \sum_{j=1}^{k} \frac{(-1)^{j+k}}{k!} \sum_{l=1}^{j} (-1)^{j-l} \binom{j}{l} (\Gamma^l)^{(k)}(1), \qquad k \in \mathbb{N},\end{aligned}$$

where the last equality holds by Lemma 1 in the appendix of [25]. Interchanging the sums and noting that $\sum_{j=l}^{k}\binom{j}{l} = \binom{k+1}{l+1}$ yields (28). □

Proof of Corollary 3.5. Fix $x \in \mathbb{R}$ and define $F(x) := e^{-e^{-x}}$. By Corollary 3.4, for all sufficiently large n,

$$\mathbb{P}(\tau_{ni} - \log\log n \le x) = \sum_{j=1}^{i}(-1)^{j-1}\binom{i}{j}\frac{\Gamma(n - je^{-x}/\log n)}{\Gamma(n)\Gamma(1-je^{-x}/\log n)}. \tag{29}$$

For every $c \in \mathbb{R}$ it is easily checked that $\Gamma(n + c/\log n)/\Gamma(n) = e^c + O(1/(n\log n))$ as $n \to \infty$. For $c = -je^{-x}$ we obtain

$$\frac{\Gamma(n - je^{-x}/\log n)}{\Gamma(n)} = (F(x))^j + O\Big(\frac{1}{n\log n}\Big). \tag{30}$$

Moreover (see Lemma 4.2), from $1/\Gamma(1-x) = \sum_{k=0}^{\infty} c_k x^k$ we conclude that, for all $K \in \mathbb{N}_0$,

$$\frac{1}{\Gamma(1 - je^{-x}/\log n)} = \sum_{k=0}^{K} c_k \Big(\frac{je^{-x}}{\log n}\Big)^k + O\Big(\frac{1}{(\log n)^{K+1}}\Big). \tag{31}$$

Multiplying (30) with (31) yields

$$\frac{\Gamma(n - je^{-x}/\log n)}{\Gamma(n)\Gamma(1-je^{-x}/\log n)} = (F(x))^j \sum_{k=0}^{K} c_k \Big(\frac{je^{-x}}{\log n}\Big)^k + O\Big(\frac{1}{(\log n)^{K+1}}\Big).$$

Plugging this expansion into (29) and exchanging the sums yields

$$\mathbb{P}(\tau_{ni} - \log\log n \le x) = \sum_{k=0}^{K} c_k \left(\frac{e^{-x}}{\log n}\right)^k \sum_{j=1}^{i} (F(x))^j (-1)^{j-1} \binom{i}{j} j^k + O\left(\frac{1}{(\log n)^{K+1}}\right),$$

which is the desired Edgeworth expansion with coefficients $d_{ki}(x)$ as defined in (18). It remains to verify the alternative representation (19) of the coefficients $d_{ki}(x)$. It is readily checked by induction on $k \in \mathbb{N}_0$ that $(t\frac{\partial}{\partial t})^k f(t) = \sum_{j=0}^{k} S(k, j) t^j f^{(j)}(t)$ for every k-times differentiable function $f : \mathbb{R} \to \mathbb{R}$, where the $S(k, j)$ denote the Stirling numbers of the second kind. Applying this formula to $f(t) := 1 - (1-t)^i$ with $i \in \mathbb{N}$ it follows for all $k \in \mathbb{N}_0$ and $t \in \mathbb{R}$ that

$$\begin{aligned}
\sum_{j=1}^{i} (-1)^{j-1} \binom{i}{j} j^k t^j &= \left(t\frac{\partial}{\partial t}\right)^k \sum_{j=1}^{i} (-1)^{j-1} \binom{i}{j} t^j = \left(t\frac{\partial}{\partial t}\right)^k (1 - (1-t)^i) \\
&= \sum_{j=0}^{k} S(k, j) t^j \left(\frac{\partial}{\partial t}\right)^j (1 - (1-t)^i) \\
&= S(k, 0)(1 - (1-t)^i) + \sum_{j=1}^{k} S(k, j) t^j (-1)^{j-1} (i)_j (1-t)^{i-j},
\end{aligned}$$

where $(i)_j := i(i-1)\cdots(i-j+1)$. Replacing t by $F(x)$ and noting that $S(k, 0) = 0$ for $k \in \mathbb{N}$ shows that (18) coincides for $k \in \mathbb{N}$ with (19). □

Acknowledgments

The authors thank two anonymous referees for helpful comments and suggestions leading to a significant improvement of the structure of the article.

Appendix A

Lemma A.1. *For all $x \ge 0$ and all $\alpha \in [0, 1]$ we have $(1 - e^{-x})^\alpha \ge 1 - e^{-x^\alpha}$.*

Proof. Fix $\alpha \in [0, 1]$. If $x \ge 1$ then $x^\alpha \le x$ and, hence, $(1 - e^{-x})^\alpha \ge 1 - e^{-x} \ge 1 - e^{-x^\alpha}$. Assume now that $x \in [0, 1]$. Then $x^\alpha \ge x$. The function $f(x) := (1 - e^{-x})^\alpha - 1 + e^{-x^\alpha}$ satisfies $f(0) = 0$ and has derivative $f'(x) = \alpha e^{-x}(1 - e^{-x})^{\alpha-1} - \alpha x^{\alpha-1} e^{-x^\alpha}$, which is nonnegative on $[0, 1]$, since $e^{-x} \ge e^{-x^\alpha}$ and $(1 - e^{-x})^{\alpha-1} \ge x^{\alpha-1}$ for $x \in [0, 1]$. From $f(0) = 0$ and $f'(x) \ge 0$ for $x \in [0, 1]$ it follows that $f(x) \ge 0$ for $x \in [0, 1]$, which is the desired inequality. □

Lemma A.2 (*Spectral Decomposition of* $\mathbf{\Gamma}$ *for the Kingman Coalescent*)**.** *The generator $\Gamma = (\gamma_{ij})_{i,j\in\mathbb{N}}$ of the fixation line $(L_t)_{t\ge 0}$ of the Kingman coalescent has spectral decomposition $\Gamma = RDL$, where $D = (d_{ij})_{i,j\in\mathbb{N}}$ is the diagonal matrix with entries $d_{ij} = -i(i+1)/2$ for $i = j$ and $d_{ij} = 0$ for $i \ne j$, and $R = (r_{ij})_{i,j\in\mathbb{N}}$ and $L = (l_{ij})_{i,j\in\mathbb{N}}$ are upper right triangular matrices with entries*

$$r_{ij} = (-1)^{j-i} \frac{j!\,(j-1)!\,(i+j)!}{(j-i)!\,i!\,(i-1)!\,(2j)!}, \qquad i, j \in \mathbb{N}, i \le j, \tag{32}$$

and

$$l_{ij} = \frac{j!\,(j-1)!\,(2i+1)!}{i!\,(i-1)!\,(j-i)!\,(i+j+1)!}, \qquad i, j \in \mathbb{N}, i \le j. \tag{33}$$

Remark. Note that $l_i(z) := \sum_{j=i}^{\infty} l_{ij} z^{j+1}$ satisfies the differential equation $z^2(1-z)l_i''(z) = i(i+1)l_i(z)$, $i \in \mathbb{N}$, $|z| < 1$.

Proof. For a pure birth process the recursion (25) reduces to $r_{ij} = \gamma_i/(\gamma_i - \gamma_j) r_{i+1,j}$, $i \in \{j-1, j-2, \ldots, 1\}$, with solution $r_{ij} = \prod_{k=i}^{j-1} \gamma_k/(\gamma_k - \gamma_j)$, $i \le j$. Thus, for the Kingman coalescent, for all $i, j \in \mathbb{N}$ with $i \le j$,

$$\begin{aligned} r_{ij} &= \prod_{k=i}^{j-1} \frac{k(k+1)}{k(k+1) - j(j+1)} = \prod_{k=i}^{j-1} \frac{k(k+1)}{(k-j)(k+j+1)} \\ &= (-1)^{j-i} \frac{j!\,(j-1)!\,(i+j)!}{(j-i)!\,i!\,(i-1)!\,(2j)!}. \end{aligned}$$

Similarly, the recursion (26) reduces to $l_{ij} = \gamma_{j-1}/(\gamma_j - \gamma_i) l_{i,j-1}$, $j \in \{i+1, i+2, \ldots\}$, with solution $l_{ij} = \prod_{k=i+1}^{j} \gamma_{k-1}/(\gamma_k - \gamma_i)$, $i \le j$. Thus, for the Kingman coalescent, for all $i, j \in \mathbb{N}$ with $i \le j$,

$$\begin{aligned} l_{ij} &= \prod_{k=i+1}^{j} \frac{k(k-1)}{k(k+1) - i(i+1)} = \prod_{k=i+1}^{j} \frac{k(k-1)}{(k-i)(k+i+1)} \\ &= \frac{j!\,(j-1)!\,(2i+1)!}{i!\,(i-1)!\,(j-i)!\,(i+j+1)!}. \quad \square \end{aligned}$$

Let E be locally compact, i.e. every point $x \in E$ has a compact neighborhood. A function $f : E \to \mathbb{R}$ vanishes at infinity, if for every $\varepsilon > 0$ there exists a compact $K \subseteq E$ such that $|f(x)| < \varepsilon$ for all $x \in E \setminus K$. In other words $\{x \in E : |f(x)| \ge \varepsilon\}$ is compact. In the following $\widehat{C}(E)$ denotes the set of all real-valued continuous functions on E vanishing at infinity.

Lemma A.3. *Let $d \in \mathbb{N}$. The set D of all functions $g : [0,\infty)^d \to \mathbb{R}$ of the form $g(y) = \sum_{i_1,\ldots,i_d=1}^{m} a_{i_1,\ldots,i_d} e^{-(i_1 y_1 + \cdots + i_d y_d)}$ with $m \in \mathbb{N}$ and $a_{i_1,\ldots,i_d} \in \mathbb{R}$ is dense in $\widehat{C}([0,\infty)^d)$.*

Proof. Let $g \in \widehat{C}([0,\infty)^d)$. Define $f : [0,1]^d \to \mathbb{R}$ via $f(x) := g(-\log x_1, \ldots, -\log x_d)$ for $x \in (0,1]^d$ and $f(x) := 0$ if $x_j = 0$ for some $j \in \{1, \ldots, d\}$. Since g is continuous and vanishes at infinity it follows that f is continuous. For $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_d) \in [0,1]^d$ let

$$p_n(x) := \sum_{k_1,\ldots,k_d=1}^{n} f\Big(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\Big) \prod_{j=1}^{d} \binom{n}{k_j} x_j^{k_j} (1-x_j)^{n-k_j}.$$

denote the nth multivariate Bernstein polynomial of f. Note that the sum runs only over $k = (k_1, \ldots, k_d) \in \{1, \ldots, n\}^d$ (not as usual over $k \in \{0, \ldots, n\}^d$) since $f(x) = 0$ if $x_j = 0$ for some $j \in \{1, \ldots, d\}$. By a d-dimensional version of Bernstein's approximation theorem (see, for example, [10, Theorem 8]), $p_n \to f$ as $n \to \infty$ uniformly on $[0,1]^d$. Replacing x_j by e^{-y_j} it follows that $g_n \to g$ as $n \to \infty$ uniformly on $[0,\infty)^d$, where $g_n(y) := p_n(e^{-y_1}, \ldots, e^{-y_d})$. It remains to note that $g_n \in D$. $\square$

Proposition A.4 (*Convergence of Markov Processes*)**.** *Let $d \in \mathbb{N}$, $E := [0,\infty)^d$ and $X = (X_t)_{t\ge0}$ be an E-valued time-homogeneous Markov process. Furthermore, for every $n \in \mathbb{N}$ let $X^{(n)} = (X_t^{(n)})_{t\ge0}$ be an E_n-valued time-homogeneous Markov process with state space $E_n \subseteq E$. Let $(T_t)_{t\ge0}$ and $(T_t^{(n)})_{t\ge0}$ denote the corresponding semigroups. Define $\pi_n : B(E) \to B(E_n)$ via*

Please cite this article in press as: J. Kukla, M. Möhle, On the block counting process and the fixation line of the Bolthausen–Sznitman coalescent, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.012.

$\pi_n f(x) := f(x)$ for all $f \in B(E)$ and $x \in E_n$. If, for every $t \geq 0$ and $\lambda \in \mathbb{N}^d$,

$$\lim_{n\to\infty} \|T_t^{(n)}\pi_n f_\lambda - \pi_n T_t f_\lambda\| := \lim_{n\to\infty} \sup_{x\in E_n} |T_t^{(n)}\pi_n f_\lambda(x) - \pi_n T_t f_\lambda(x)| = 0,$$

where $f_\lambda(x) := e^{-\langle\lambda,x\rangle} := e^{-(\lambda_1 x_1+\cdots+\lambda_d x_d)}$ for all $\lambda \in \mathbb{N}^d$ and $x \in E$, then $X^{(n)}$ converges in $D_E[0,\infty)$ to X as $n \to \infty$.

Proof. By assumption, $\lim_{n\to\infty}\|T_t^{(n)}\pi_n f - \pi_n T_t f\| = 0$ for all $f \in D$, where $D := \{f : E \to \mathbb{R} \,:\, f(x) = \sum_{i=1}^m a_i e^{-\langle\lambda,x\rangle}, m \in \mathbb{N}, \lambda \in \mathbb{N}^d, a_i \in \mathbb{R}\}$ Let $f \in \widehat{C}(E)$ and fix $\varepsilon > 0$. Since D is dense in $\widehat{C}(E)$ by Lemma A.3 there exists $h \in D$ such that $\|f - h\| < \varepsilon$. It follows that

$$\begin{aligned}\|T_t^{(n)}\pi_n f - \pi_n T_t f\| &\leq \|T_t^{(n)}\pi_n(f-h)\| + \|T_t^{(n)}\pi_n h - \pi_n T_t h\| + \|\pi_n T_t(h-f)\|\\ &\leq \|T_t^{(n)}\|\,\|f-h\| + \|T_t^{(n)}\pi_n h - \pi_n T_t h\| + \|T_t\|\,\|h-f\|\\ &\leq 2\varepsilon + \|T_t^{(n)}\pi_n h - \pi_n T_t h\| \;\to\; 2\varepsilon, \qquad n\to\infty.\end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily we conclude that $\lim_{n\to\infty}\|T_t^{(n)}\pi_n f - \pi_n T_t f\| = 0$ for all $f \in \widehat{C}(E)$. The result follows from [11, p. 172, Theorem 2.11]. □

References

[1] Milton Abramowitz, Irene A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York; John Wiley & Sons, Inc., New York, 1984, p. xiv+1046 Reprint of the 1972 edition, Selected Government Publications, MR0757537.

[2] Ellen Baake, Ute Lenz, Anton Wakolbinger, The common ancestor type distribution of a Λ-Wright-Fisher process with selection and mutation, Electron. Commun. Probab. 21 (2016) Paper No. 59, 16, MR3548771.

[3] Erich Baur, Jean Bertoin, The fragmentation process of an infinite recursive tree and Ornstein-Uhlenbeck type processes, Electron. J. Probab. 20 (98) (2015) 20, MR3399834.

[4] Jean Bertoin, Jean-François Le Gall, The Bolthausen-Sznitman coalescent and the genealogy of continuous-state branching processes, Probab. Theory Related Fields 117 (2) (2000) 249–266, MR1771663.

[5] Vladimir I. Bogachev, Michael Röckner, Byron Schmuland, Generalized Mehler semigroups and applications, Probab. Theory Related Fields 105 (2) (1996) 193–225, MR1392452.

[6] E. Bolthausen, A.-S. Sznitman, On Ruelle's probability cascades and an abstract cavity method, Comm. Math. Phys. 197 (2) (1998) 247–276, MR1652734.

[7] Gerd Christoph, Karina Schreiber, Discrete stable random variables, Statist. Probab. Lett. 37 (3) (1998) 243–247, MR1614930.

[8] Peter Donnelly, Thomas G. Kurtz, A countable representation of the Fleming-Viot measure-valued diffusion, Ann. Probab. 24 (2) (1996) 698–742, MR1404525.

[9] Peter Donnelly, Thomas G. Kurtz, Particle representations for measure-valued population models, Ann. Probab. 27 (1) (1999) 166–205, MR1681126.

[10] Miloslav Duchoň, A generalized Bernstein approximation theorem, Tatra Mt. Math. Publ. 49 (2011) 99–109, MR2867253.

[11] Stewart N. Ethier, Thomas G. Kurtz, Markov Processes, in: Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York, 1986, p. x+534. Characterization and convergence, MR0838085.

[12] F. Freund, M. Möhle, On the time back to the most recent common ancestor and the external branch length of the Bolthausen-Sznitman coalescent, Markov Process. Related Fields 15 (3) (2009) 387–416, MR2554368.

[13] Florian Gaiser, Martin Möhle, On the block counting process and the fixation line of exchangeable coalescents, ALEA Lat. Am. J. Probab. Math. Stat. 13 (2) (2016) 809–833, MR3546382.

[14] Christina Goldschmidt, James B. Martin, Random recursive trees and the Bolthausen-Sznitman coalescent, Electron. J. Probab. 10 (21) (2005) 718–745, MR2164028.

[15] Theodore E. Harris, The Theory of Branching Processes, in: Die Grundlehren der Mathematischen Wissenschaften, Bd., vol. 119, Springer-Verlag, Berlin; Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963, p. xiv+230, MR0163361.

[16] Olivier Hénard, Change of measure in the lookdown particle system, Stochastic Process. Appl. 123 (6) (2013) 2054–2083, MR3038498.

Please cite this article in press as: J. Kukla, M. Möhle, On the block counting process and the fixation line of the Bolthausen–Sznitman coalescent, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.012.

[17] Olivier Hénard, The fixation line in the Λ-coalescent, Ann. Appl. Probab. 25 (5) (2015) 3007–3032, MR3375893.
[18] Thierry Huillet, Martin Möhle, On the extended Moran model and its relation to coalescents with multiple collisions, Theor. Popul. Biol. 87 (2013) 5–14, MR number not available.
[19] Alex Iksanov, Martin Möhle, On the number of jumps of random walks with a barrier, Adv. Appl. Probab. 40 (1) (2008) 206–228, MR2411821.
[20] J.F.C. Kingman, The coalescent, Stochastic Process. Appl. 13 (3) (1982) 235–248, MR0671034.
[21] Jonas Kukla, Helmut H. Pitters, A spectral decomposition for the Bolthausen-Sznitman coalescent and the Kingman coalescent, Electron. Commun. Probab. 20 (87) (2015) 13, MR3434204.
[22] Cyril Labbé, From flows of Λ-Fleming-Viot processes to lookdown processes via flows of partitions, Electron. J. Probab. 19 (55) (2014) 49, MR3227064.
[23] John Lamperti, Continuous state branching processes, Bull. Amer. Math. Soc. 73 (1967) 382–386, MR0208685.
[24] John Lamperti, The limit of a sequence of branching processes, Z. Wahrscheinlichkeitstheor. Verwandte Geb. 7 (1967) 271–288, MR0217893.
[25] Martin Möhle, Looking forwards and backwards in the multi-allelic neutral Cannings population model, J. Appl. Probab. 47 (3) (2010) 713–731, MR2731344.
[26] Martin Möhle, The Mittag-Leffler process and a scaling limit for the block counting process of the Bolthausen-Sznitman coalescent, ALEA Lat. Am. J. Probab. Math. Stat. 12 (1) (2015) 35–53, MR3333734.
[27] Martin Möhle, Helmut Pitters, A spectral decomposition for the block counting process of the bolthausen-Sznitman coalescent, Electron. Commun. Probab. 19 (47) (2014) 11, MR3246966.
[28] Jacques Neveu, A continuous-state branching process in relation with the GREM model of spin glass theory. Rapport interne no 267, École Polytechnique, 1992.
[29] James R. Norris, Markov Chains, in: Cambridge Series in Statistical and Probabilistic Mathematics, vol. 2, Cambridge University Press, Cambridge, 1998, p. xvi+237. Reprint of 1997 original, MR1600720.
[30] Alois Panholzer, Destruction of recursive trees, in: Mathematics and Computer Science. III, in: Trends Math., Birkhäuser, Basel, 2004, pp. 267–280, MR2090518.
[31] Peter Pfaffelhuber, Anton Wakolbinger, The process of most recent common ancestors in an evolving coalescent, Stochastic Process. Appl. 116 (12) (2006) 1836–1859, MR2307061.
[32] Jim Pitman, Coalescents with multiple collisions, Ann. Probab. 27 (4) (1999) 1870–1902, MR1742892.
[33] Jim Pitman, Combinatorial Stochastic Processes, in: Lecture Notes in Mathematics, vol. 1875, Springer-Verlag, Berlin, 2006, p. x+256 Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002, With a foreword by Jean Picard, MR2245368.
[34] Georg Pólya, Über den zentralen Grenzwertsatz der Wahrscheinlichkeitsrechnung und das Momentenproblem, Math. Z. 8 (3–4) (1920) 171–181, MR1544437.
[35] Daniel Revuz, Marc Yor, Continuous Martingales and Brownian Motion, third ed., in: Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 293, Springer-Verlag, Berlin, 1999, p. xiv+602, MR1725357.
[36] Serik Sagitov, A key limit theorem for critical branching processes, Stochastic Process. Appl. 56 (1) (1995) 87–100, MR1324323.
[37] Serik Sagitov, The general coalescent with asynchronous mergers of ancestral lines, J. Appl. Probab. 36 (4) (1999) 1116–1125, MR1742154.
[38] Damodar N. Shanbhag, Maddipatla Sreehari, On certain self-decomposable distributions, Z. Wahrscheinlichkeitstheor. Verwandte Geb. 38 (3) (1977) 217–222, MR0436267.

Please cite this article in press as: J. Kukla, M. Möhle, On the block counting process and the fixation line of the Bolthausen–Sznitman coalescent, Stochastic Processes and their Applications (2017), http://dx.doi.org/10.1016/j.spa.2017.06.012.