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Martingale driven BSDEs, PDEs and other related deterministic problems

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Abstract

We focus on a class of BSDEs driven by a càdlàg martingale and the corresponding Markovian BSDEs which arise when the randomness of the driver appears through a Markov process. To those BSDEs we associate a deterministic equation which, when the Markov process is a Brownian diffusion, is nothing else but a parabolic semi-linear PDE. We prove existence and uniqueness of a *decoupled mild solution* of the deterministic problem, and give a probabilistic representation of this solution through the aforementioned BSDEs.

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1. Introduction

In the Brownian context, backward stochastic differential equations (BSDEs) were introduced by E. Pardoux and S. Peng in [24]. A subclass of BSDEs are said to be *Markovian*, if the randomness of the so called driver f depends on a Markovian diffusion X, and when the terminal condition depends on the terminal value X_T. Those are naturally linked to a parabolic PDE, which constitutes a particular deterministic problem. In particular, under

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reasonable conditions, which among other ensure well-posedness, the solutions of BSDEs produce *viscosity* type solutions for the mentioned PDE. In this paper we focus on *Pseudo-PDEs*, which are the corresponding deterministic problems associated to Markovian BSDEs driven by a càdlàg martingale, when the underlying forward process is a general Markov process. In this case, the concept of a viscosity solution (based on comparison theorems) is not completely appropriate. For this reason we propose an alternative notion called *decoupled mild solution*. This extends the usual formulation of a mild solution, expressed in terms of semigroups, which is well-known to the experts of PDEs. We establish an existence and uniqueness theorem among Borel functions having a certain growth condition.

Coming back to Brownian BSDEs, let s be an initial time and x an initial value. A Markovian BSDE appears as

$$\left\{\begin{array}{lll} X^{s,x}_t &=& x+\int_s^t \mu(r,X^{s,x}_r)dr+\int_s^t \sigma(r,X^{s,x}_r)dB_r, \quad t\in[s,T]\\ Y^{s,x}_t &=& g(X^{s,x}_T)+\int_t^T f\left(r,X^{s,x}_r,Y^{s,x}_r,Z^{s,x}_r\right)dr-\int_t^T Z^{s,x}_r dB_r, \quad t\in[s,T], \end{array}\right. \tag{1.1}$$

where B is a Brownian motion. In [26] and in [25] previous Markovian BSDE was linked to the semilinear PDE

$$\left\{\begin{array}{l} \partial_t u+\frac{1}{2}Tr\left(\sigma\sigma^\intercal\nabla^2_x u\right)+\mu\cdot\nabla^2_x u+f(\cdot,\cdot,u,\sigma\nabla_x u)=0 \quad \text{on } [0,T[\times\mathbb{R}^d\\ u(T,\cdot)=g. \end{array}\right. \tag{1.2}$$

The first link between (1.1) and (1.2) was established in [26], where the authors showed that when the PDE admits a $\mathcal{C}^{1,2}$ solution u, then the couple $(Y^{s,x},Z^{s,x})=(u(\cdot,X^{s,x}),\nabla u(\cdot,X^{s,x}))$ solves the BSDE. Conversely, if g is continuous (resp. f is continuous in (t,x) and is Lipschitz in the third and the fourth variable), [25] proved an important probability representation result of the (unique) viscosity solution u of the PDE, via the solutions of the Markovian BSDE for each (s,x). Indeed if $(Y^{s,x},Z^{s,x})$ is the solution of (1.1), then $u:(s,x)\longmapsto Y^{s,x}_s$ is a continuous viscosity solution of (1.2). In [5], it was shown that, whenever the coefficients belong to some Sobolev spaces, then the function u mentioned above is in fact a solution, in the sense of distributions, of the PDE. Later, [2] justified that, under certain conditions, u is a mild solution of the PDE.

An interesting fact is that, even without further regularity assumptions made on the coefficients of the BSDE, there exists another function v such that $(Y^{s,x},Z^{s,x})=(u(\cdot,X^{s,x}),$ $v(\cdot,X^{s,x}))$, see [16]. In [20] v was associated to u by use of the operator $\sigma\nabla$ suitably extended. However, when the viscosity solution u of the PDE has no additional regularity, it is a challenging question to specify the relation of the function v to u, or to the PDE (1.2). This is the so called *identification problem* and it will be a central theme in our investigation.

In [4] the authors introduced a new kind of BSDEs driven by a Brownian motion and a Poisson random measure. In the Markovian setup, the randomness of its coefficients comes from an underlying forward process X solving an SDE with jumps. They associated this new BSDE with a non-linear Integro-Partial Differential Equation (in short IPDE) and showed that, under some continuity and monotonicity conditions on the coefficients, the function $u:(s,x)\longmapsto Y^{s,x}_s$ constructed with the BSDEs, is again a viscosity solution of the IPDE. Remaining in the framework of Poisson random measures, but without any diffusion term, [13] considered BSDEs driven by marked point processes, see also [3].

From a different perspective, BSDEs driven by a general martingale and involving an orthogonal term were studied in [10,16], and [12]. In this paper, we consider a reformulation of such BSDEs, whose given data are a continuous increasing process $\hat{V}$, a square integrable martingale $\hat{M}$, a terminal condition ξ and a driver $\hat{f}$. A solution will be a couple (Y, M) satisfying

$$Y = \xi + \int_{\cdot}^{T} \hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M_T - M_{\cdot}), \tag{1.3}$$

where Y is càdlàg adapted and M is a square integrable martingale. We show the existence and the uniqueness of a solution for (1.3).

We will then be interested in a Markov process $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ taking values in some Polish space E and solving a martingale problem related to an operator $(\mathcal{D}(a), a)$ and a non-decreasing function V. By this we mean that, for any $\phi \in \mathcal{D}(a)$, and $(s, x) \in [0, T] \times E$, $M[\phi]^{s,x} := \phi(\cdot, X_{\cdot}) - \phi(s, x) - \int_s^{\cdot} a(\phi)(r, X_r) dV_r$ is a $\mathbb{P}^{s,x}$-martingale. We will fix some function $\psi := (\psi_1, \dots, \psi_d) \in \mathcal{D}(a)^d$ and at Notation 5.7 we will introduce some special BSDEs driven by a martingale which we will call again Markovian BSDEs.

Each BSDE will be indexed by a couple $(s, x) \in [0, T] \times E$, will hold under the probability $\mathbb{P}^{s,x}$ and will have the form

$$Y^{s,x} = g(X_T) + \int_{\cdot}^{T} f\left(r, X_r, Y_r^{s,x}, \frac{d\langle M^{s,x}, M[\psi]^{s,x}\rangle}{dV}(r)\right) dV_r - (M_T^{s,x} - M_{\cdot}^{s,x}), \tag{1.4}$$

where X is the canonical process, g is a Borel function with a growth condition and f is Borel, with a growth condition with respect to the second variable, and it is Lipschitz with respect to the third and fourth variables. In most of the examples, we will set ψ to be the identity, and $M[\psi]^{s,x}$ will be the martingale part of X under $\mathbb{P}^{s,x}$. We will however also include the case when X is not a semimartingale, and in particular $Id \notin \mathcal{D}(a)^d$.

Those Markovian BSDEs will be linked to the Pseudo-PDE

$$\begin{cases} a(u) + f\left(\cdot, \cdot, u, \Gamma^{\psi}(u)\right) &= 0 \quad \text{on } [0, T] \times E \\ u(T, \cdot) &= g, \end{cases} \tag{1.5}$$

where $\Gamma^{\psi}(u) := (a(u\psi_i) - ua(\psi_i) - \psi_i a(u))_{i\in[\![1;d]\!]}$, see Definition 5.3. A *classical* solution of the Pseudo-PDE will simply be an element of $\mathcal{D}(a)$ fulfilling (1.5). We call Γ^{ψ} the ψ-*generalized gradient*, due to the fact that when $E = \mathbb{R}^d$, $a = \partial_t + \frac{1}{2}\Delta$ and $\psi_i : (t, x) \longmapsto x_i$ for all $i \in [\![1, d]\!]$ then $\Gamma^{\psi}(u) = \nabla u$. In this particular setup, the forward Markov process is of course a Brownian motion and in this case, the space $\mathcal{D}(a) = C^{1,2}([0, T] \times \mathbb{R}^d)$.

We show the existence of a Borel function u in some extended domain $\mathcal{D}(\mathfrak{a})$ such that, for every $(s, x) \in [0, T] \times E$, $Y^{s,x}$ is a $\mathbb{P}^{s,x}$-modification of $u(\cdot, X_{\cdot})$ At Definition 5.9 we will introduce the notion of *martingale solution* for the Pseudo-PDE (1.5), where the operators $\mathfrak{a}$ and $\mathfrak{G}^{\psi}$ are respectively an extension of a and Γ^{ψ}. We also show that u is the unique *decoupled mild solution* of the same equation. We explain below that concept of solution, which will be introduced at Definition 5.13.

A Borel function u will be called decoupled mild solution if there exists an $\mathbb{R}^d$-valued Borel function $v := (v_1, \dots, v_d)$ such that, for every (s, x),

$$\left\{\begin{array}{rcl} u(s,x) & = & P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f(\cdot,\cdot,u,v)(r,\cdot)\right](x)dV_r \\ u\psi_1(s,x) & = & P_{s,T}[g\psi_1(T,\cdot)](x) - \int_s^T P_{s,r}\left[(v_1 + ua(\psi_1)\right. \\ & & \quad \left. -\psi_1 f(\cdot,\cdot,u,v))(r,\cdot)\right](x)dV_r \\ & \dots & \\ u\psi_d(s,x) & = & P_{s,T}[g\psi_d(T,\cdot)](x) - \int_s^T P_{s,r}\left[(v_d + ua(\psi_d)\right. \\ & & \quad \left. -\psi_d f(\cdot,\cdot,u,v))(r,\cdot)\right](x)dV_r, \end{array}\right. \tag{1.6}$$

where P is the time-dependent transition kernel associated to the Markov canonical class and to the operator a, see Notation 4.1. v coincides with $\mathfrak{G}^{\psi}(u)$ and the couple (u, v) will be called solution to the *identification problem*, see Definition 5.13. The intuition behind this notion of solution relies on the fact that the equation $a(u) = -f(\cdot,\cdot,u,\Gamma^{\psi}(u))$ can be decoupled into the system

$$\left\{\begin{array}{rcl} a(u) & = & -f(\cdot,\cdot,u,v) \\ v_i & = & \Gamma^{\psi_i}(u), \quad i \in [\![1;d]\!], \end{array}\right. \tag{1.7}$$

which can be rewritten

$$\left\{\begin{array}{rcl} a(u) & = & -f(\cdot,\cdot,u,v) \\ a(u\psi_i) & = & v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v), \quad i \in [\![1;d]\!]. \end{array}\right. \tag{1.8}$$

Martingale solutions were introduced in [6] and decoupled mild solutions in [8], but in relation to a specific type of Pseudo-PDE, for which v is one-dimensional and which does not include the usual parabolic PDE related to classical BSDEs. A first approach to classical solutions for a general deterministic problem, associated with forward BSDEs with applications to the so called *Föllmer–Schweizer decomposition*, was performed by [23].

The paper is organized as follows. In Section 3 we propose an alternative formulation (1.3) for BSDEs driven by càdlàg martingales discussed in [12]: in Theorem 3.3 (proved in Appendix A), we state existence and uniqueness for such equations. In Section 4, we refer to a canonical Markov class and its corresponding martingale problem. In Definition 4.13 we define the extended domain $\mathcal{D}(\mathfrak{a})$; in Definition 4.15 and Notation 4.18, appear the extended operators $\mathfrak{a}$ and $\mathfrak{G}^{\psi}$. In Section 5, we bring in the Pseudo-PDE (1.5) (see Definition 5.3) and the associated Markovian BSDEs (1.4), see Notation 5.7. We introduce the notion of martingale solution of the Pseudo-PDE in Definition 5.9 and the one of decoupled mild solution in Definition 5.13. Propositions 5.15 and 5.16 show the equivalence between martingale solutions and decoupled mild solutions. Proposition 5.17 states that any classical solution is a decoupled mild solution and conversely that any decoupled mild solution, belonging to $\mathcal{D}(\Gamma^{\psi})$, is a classical solution up to (what we call) a zero potential set. Let $(Y^{s,x}, M^{s,x})$ denote the unique solution of the associated BSDE (1.4), written as $BSDE^{s,x}(f, g)$. In Theorem 5.18 we show the existence of some $u \in \mathcal{D}(\mathfrak{a})$ such that for every $(s, x) \in [0, T] \times E$, $Y^{s,x}$ is a $\mathbb{P}^{s,x}$-modification of $u(\cdot, X_\cdot)$ on $[s, T]$. Theorem 5.20 states that the function $(s, x) \longmapsto Y_s^{s,x}$ is the unique decoupled mild solution of (1.5). Proposition 5.23 states that, if the couple (u, v) satisfies (1.6), then for any (s, x), the couple $\left(u(t, X_t), \quad u(t, X_t) - u(s, x) + \int_s^t f(\cdot,\cdot,u,v)(r, X_r)dV_r\right)_{t\in[s,T]}$ has a $\mathbb{P}^{s,x}$-version which solves $BSDE^{s,x}(f, g)$ on $[s, T]$. Finally, in Section 6, we study some application examples. In Section 6.1 we deal with parabolic semi-linear PDEs and in Section 6.2 with parabolic semi-linear PDEs with distributional drift.

2. Preliminary notions and basic notations

In this short section we introduce some basic notions, notations and vocabulary which will be used in this paper. $T \in \mathbb{R}_+$ will be a fixed horizon.

- For any topological spaces E and F, $\mathcal{B}(E)$ will denote the Borel σ-field of E. $\mathcal{C}(E, F)$ (resp. $\mathcal{C}_b(E, F)$, $\mathcal{B}(E, F)$, $\mathcal{B}_b(E, F)$) will denote linear the space of functions from E to F which are continuous (resp. bounded continuous, Borel, bounded Borel).
- A filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ will be called a **stochastic basis** and said to **fulfill the usual conditions** if the filtration is complete and right-continuous.
- Given a certain stochastic basis, $\mathcal{H}^2$ will denote the space of square integrable martingales, with the convention that indistinguishable elements are identified. $\mathcal{H}^2_0$ will denote the linear subspace constituted of elements vanishing at zero, and $\mathcal{H}^2_{loc}$ will be the space of locally square integrable martingales.
- For any $M, N \in \mathcal{H}^2_{loc}$, $[M, N]$ will denote the **quadratic covariation** and $\langle M, N\rangle$ their (predictable) **angle bracket**. If $M = N$ we will use the notations $[M]$ and $\langle M\rangle$.
- $\mathcal{P}ro$ will denote the progressive σ-field on $[0, T] \times \Omega$.
- If V is a non-decreasing process, $dV \otimes d\mathbb{P}$ will denote the positive measure on $(\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}([0, T]))$ defined for any $F \in \mathcal{F} \otimes \mathcal{B}([0, T])$ by $dV \otimes d\mathbb{P}(F) := \mathbb{E}\left[\int_0^T \mathbb{1}_F(\omega, r)dV_r(\omega)\right]$.
- If V is a non-decreasing predictable process and A is a predictable process which is absolutely continuous with respect to V, then $\frac{dA}{dV}$ will denote its Radon–Nikodym derivative. We recall that thanks to Proposition 3.2 in [6], this process can be chosen to be predictable.

3. An alternative formulation of BSDEs driven by a càdlàg martingale

We introduce now an alternative formulation for Backward Stochastic Differential Equations driven by a general càdlàg martingale investigated for instance by [12].

From now on, and until the end of this section, we are given a stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ fulfilling the usual conditions. We are also given some bounded continuous non-decreasing adapted process $\hat{V}$, we will indicate by $\mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$ the set of (up to indistinguishability) progressively measurable processes ϕ such that $\mathbb{E}[\int_0^T \phi_r^2 d\hat{V}_r] < \infty$. $\mathcal{L}^{2,cadlag}(d\hat{V} \otimes d\mathbb{P})$ will denote the subspace of càdlàg elements of $\mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$.

We will now fix an $\mathcal{F}_T$-measurable random variable ξ called the **final condition**, a square integrable **reference martingale** $\hat{M} := (\hat{M}^1, \dots, \hat{M}^d)$ taking values in $\mathbb{R}^d$ for some $d \in \mathbb{N}^*$, and a **driver** $\hat{f} : ([0, T] \times \Omega) \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$, measurable with respect to $\mathcal{P}ro \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$. We will assume that $(\xi, \hat{f}, \hat{M})$ satisfies the following.

Hypothesis 3.1.

1. $\xi \in L^2$;
2. $\hat{f}(\cdot, \cdot, 0, 0) \in \mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$;
3. There exist positive constants K^Y, K^Z such that, $\mathbb{P}$ a.s. for all t, y, y', z, z', we have
$$|\hat{f}(t, \cdot, y, z) - \hat{f}(t, \cdot, y', z')| \le K^Y|y - y'| + K^Z\|z - z'\|; \tag{3.1}$$
4. $\langle \hat{M}\rangle$ is absolutely continuous with respect to $\hat{V}$ and $\frac{d\langle \hat{M}\rangle}{d\hat{V}}$ is bounded.

We remark that, thanks to Kunita–Watanabe's inequality, the last assumption implies that for any $M \in \mathcal{H}^2_{loc}$, $\langle M, \hat{M}\rangle$ will also be absolutely continuous with respect to $\hat{V}$.

We will now formulate precisely our BSDE.

Definition 3.2. We say that a couple $(Y, M) \in \mathcal{L}^{2,cadlag}(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}^2_0$ is a solution of $BSDE(\xi, \hat{f}, V, \hat{M})$ if it satisfies

$$Y = \xi + \int_{\cdot}^{T} \hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M_T - M_{\cdot}) \tag{3.2}$$

in the sense of indistinguishability.

The proof of the theorem below is very similar to the one of Theorem 3.21 in [6]. For the convenience of the reader, it is therefore postponed to Appendix A.

Theorem 3.3. *If $(\xi, \hat{f}, \hat{V}, \hat{M})$ satisfies Hypothesis 3.1, then $BSDE(\xi, \hat{f}, \hat{V}, \hat{M})$ has a unique solution.*

Remark 3.4. Let $(\xi, \hat{f}, \hat{V}, \hat{M})$ satisfying Hypothesis 3.1. We can consider a BSDE on a restricted interval $[s, T]$ for some $s \in [0, T[$. Theorem 3.3 extend easily to this case. In particular there exists a unique couple of processes (Y^s, M^s), indexed by $[s, T]$ such that Y^s is adapted, càdlàg and satisfies $\mathbb{E}[\int_s^T (Y^s_r)^2 d\hat{V}_r] < \infty$, such that M^s is a martingale vanishing in s and such that $Y^s = \xi + \int_{\cdot}^{T} \hat{f}\left(r, \cdot, Y^s_r, \frac{d\langle M^s, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M^s_T - M^s_{\cdot})$ in the sense of indistinguishability on $[s, T]$.

Moreover, if (Y, M) denotes the solution of $BSDE(\xi, \hat{f}, \hat{V}, \hat{M})$ then $(Y, M_{\cdot} - M_s)$ and (Y^s, M^s) coincide on $[s, T]$. This follows by a uniqueness argument resulting by Theorem 3.3 on the time interval $[s, T]$.

Remark 3.5.

1. [12] considers a BSDE driven by a càdlàg martingale which corresponds to the BSDE (1.1), where the Brownian motion W is replaced with a martingale M with non-necessarily bounded angular bracket $\langle M\rangle$, with a remainder orthogonal martingale N. The solution is given by a triplet (Y, Z, N). The authors make use of weighted spaces of the type $\mathcal{H}^2_{T,\beta}$ and $\mathcal{L}^2_{\beta}$. For instance $\mathcal{H}^2_{T,\beta}$ is the space of all progressively measurable processes ϕ such that $\mathbb{E}(\int_0^T \phi_s^2 e^{\beta\langle M\rangle_s} d\langle M\rangle_s) < +\infty$. In particular they find a value for β such that existence and uniqueness holds within the class of triplets (Y, Z, N) such that $Y, Z \in \mathcal{H}^2_{T,\beta}$ and $N \in \mathcal{L}^2_{\beta}$.
2. Existence and uniqueness theorems for Brownian BSDEs can be also stated under more general assumptions than Lipschitz conditions. In [22], the author has obtained an existence result for possibly quadratic growth BSDEs, when the driver f is of the form $f(t, y, z) = f^1(t, z)y + f^2(t, y, z)$ where f^1 is bounded a.s., and for all t, y, z, $|f^2(t, y, z)| \leq K(1 + c(|y|)|z|^2)$ for some continuous function c. On the other hand the terminal condition ξ is supposed to be bounded.

We believe that several arguments developed in the two previous items can be adapted to our context. However, in this paper we have chosen not to explore the validity of Theorem 3.3 under more general assumptions along the line of items 1. and 2. It will be the object of future investigations.

4. Martingale problem and canonical Markov classes

We now introduce the Markov process which will be the forward underlying of our BSDE driven by a càdlàg martingale. That process will be defined as the solution of a martingale problem described below.

For details concerning the exact mathematical framework for our Markov process, we refer to our previous paper [7] about canonical Markov classes and additive functionals.

From now on, E is a Polish space and $\left(\Omega, \mathcal{F}, (X_t)_{t\in[0,T]}, (\mathcal{F}_t)_{t\in[0,T]}\right)$ denotes the canonical space defined in Notation 3.1 of [7]. We also fix a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ associated to a transition kernel $P = (P_{s,t})$ measurable in time as defined in Definitions 3.4, 3.5 and 3.7 in [7]. For any $(s,x) \in [0,T]\times E$, $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ will denote the stochastic basis in which $\mathbb{P}^{s,x}$-null sets are added to $\mathcal{F}$ and $\mathcal{F}_t$ for all t, and which fulfills the usual conditions. $\mathbb{E}^{s,x}$ will denote the corresponding expectation to $\mathbb{P}^{s,x}$. If $P_{s,t}$ only depends on $t-s$, P is called time-homogeneous and we will often use the notation P_t instead of $P_{0,t}$.

Notation 4.1. *In particular, for any $t \in [0,T]$ and $A \in \mathcal{B}(E)$*

$$\mathbb{P}^{s,x}(X_t \in A) = P_{s,t}(x, A), \tag{4.1}$$

and for any $s \le t \le u$

$$\mathbb{P}^{s,x}(X_u \in A|\mathcal{F}_t) = P_{t,u}(X_t, A) \quad \mathbb{P}^{s,x} \text{ a.s.} \tag{4.2}$$

Let s,t in $[0,T]$ with $s \le t$, $x \in E$ and $\phi \in \mathcal{B}(E,\mathbb{R})$. If ϕ is integrable with respect to $P_{s,t}(x,\cdot)$, then $P_{s,t}[\phi](x)$ will denote its integral.

We recall two important measurability properties, essentially stated in [8], even though with $V(t) \equiv t$.

Remark 4.2.

- Let $\phi \in \mathcal{B}(E,\mathbb{R})$ be such that for any (s,x,t), $\mathbb{E}^{s,x}[|\phi(X_t)|] < \infty$, then $(s,x,t) \longmapsto P_{s,t}[\phi](x)$ is Borel, see Proposition A.11 in [8].
- Let $\phi \in \mathcal{L}^1_X$, then $(s,x) \longmapsto \int_s^T P_{s,r}[\phi](x) dV_r$ is Borel, see Lemma A.10 in [8].

Definition 4.3. Let $V : [0,T] \to \mathbb{R}_+$ be a non-decreasing continuous function vanishing at 0. Let us consider a linear operator $a : \mathcal{D}(a) \subset \mathcal{B}([0,T]\times E,\mathbb{R}) \longrightarrow \mathcal{B}([0,T]\times E,\mathbb{R})$, where the domain $\mathcal{D}(a)$ is a linear space.

We say that $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ solves the **martingale problem associated to** $(\mathcal{D}(a), a, V)$ if, for any $(s,x) \in [0,T]\times E$, $\mathbb{P}^{s,x}$ satisfies the following.

(a) $\mathbb{P}^{s,x}(\forall t \in [0,s], X_t = x) = 1$;
(b) for every $\phi \in \mathcal{D}(a)$, $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r) dV_r$, $t \in [s,T]$, is a càdlàg $(\mathbb{P}^{s,x}, (\mathcal{F}_t)_{t\in[s,T]})$ square integrable martingale.

The Martingale Problem is said to be **well-posed** if for any $(s,x) \in [0,T]\times E$, $\mathbb{P}^{s,x}$ is the unique probability measure satisfying those two properties.

We anticipate that well-posedness for the martingale problem will not be a hypothesis in the sequel.

Notation 4.4. *For every $(s,x) \in [0,T]\times E$ and $\phi \in \mathcal{D}(a)$, the process* $t \mapsto \mathbb{1}_{[s,T]}(t)\left(\phi(t,X_t) - \phi(s,x) - \int_s^t a(\phi)(r,X_r) dV_r\right)$ *will be denoted $M[\phi]^{s,x}$.*

$M[\phi]^{s,x}$ is a càdlàg $(\mathbb{P}^{s,x},(\mathcal{F}^{s,x}_t)_{t\in[0,T]})$ square integrable martingale vanishing on $[0,s]$.

Notation 4.5. *Let $\phi\in\mathcal{D}(a)$. For $0\le t\le u\le T$, we set*

$$M[\phi]^t_u := \begin{cases} \phi(u,X_u)-\phi(t,X_t)-\int_t^u a(\phi)(r,X_r)dV_r \text{ if } \int_t^u |a(\phi)|(r,X_r)dV_r<\infty, \\ 0 \text{ otherwise}. \end{cases} \tag{4.3}$$

$M[\phi]$ is a square integrable Martingale Additive Functional (in short MAF), see Definition 4.1 in [7], whose càdlàg version under $\mathbb{P}^{s,x}$ for every $(s,x)\in[0,T]\times E$, is $M[\phi]^{s,x}$.

From now on we fix some $d\in\mathbb{N}^*$ and a vector $\psi=(\psi_1,\dots,\psi_d)\in\mathcal{D}(a)^d$. For any $(s,x)\in[0,T]\times E$, the $\mathbb{R}^d$-valued martingale $(M[\psi_1]^{s,x},\dots,M[\psi_d]^{s,x})$ will be denoted $M[\psi]^{s,x}$.

Definition 4.6. For any $\phi_1,\phi_2\in\mathcal{D}(a)$ such that $\phi_1\phi_2\in\mathcal{D}(a)$ we set $\Gamma(\phi_1,\phi_2):=a(\phi_1\phi_2)-\phi_1 a(\phi_2)-\phi_2 a(\phi_1)$. Γ will be called the **carré du champs operator**. We set $\mathcal{D}(\Gamma^\psi):=\left\{\phi\in\mathcal{D}(a):\forall i\in[\![1;d]\!],\phi\psi^i\in\mathcal{D}(a)\right\}$ and we define the linear operator $\Gamma^\psi:\mathcal{D}(\Gamma^\psi)\longrightarrow\mathcal{B}([0,T]\times E,\mathbb{R}^d)$ by

$$\Gamma^\psi(\phi):=\left(\Gamma^{\psi_i}(\phi)\right)_{i\in[\![1;d]\!]}:=(a(\phi\psi_i)-\phi a(\psi_i)-\psi_i a(\phi))_{i\in[\![1;d]\!]}. \tag{4.4}$$

Γ^ψ will be called the ψ-**generalized gradient operator**.

We emphasize that this terminology is justified by the considerations below (1.5). This operator appears in the expression of the angular bracket of the local martingales that we have defined.

Proposition 4.7. *If $\phi\in\mathcal{D}(\Gamma^\psi)$, then for any $(s,x)\in[0,T]\times E$ and $i\in[\![1;d]\!]$ we have*

$$\langle M[\phi]^{s,x},M[\psi_i]^{s,x}\rangle=\int_s^{\cdot\vee s}\Gamma^{\psi_i}(\phi)(r,X_r)dV_r, \tag{4.5}$$

in the stochastic basis $(\Omega,\mathcal{F}^{s,x},(\mathcal{F}^{s,x}_t)_{t\in[0,T]},\mathbb{P}^{s,x})$.

Proof. The result follows from a slight modification of the proof of Proposition 4.7 of [6] in which $\mathcal{D}(a)$ was assumed to be stable by multiplication and $M[\phi]^{s,x}$ could potentially be a local martingale which is not a martingale. □

We will later need the following assumption.

Hypothesis 4.8. For every $i\in[\![1;d]\!]$, the Additive Functional $\langle M[\psi_i]\rangle$ (which is well defined thanks to Corollary 4.9 in [7]) has càdlàg versions which are absolutely continuous with respect to dV.

Taking $\phi=\psi_i$ for some $i\in[\![1;d]\!]$ in Proposition 4.7, yields the following.

Corollary 4.9. *If $\psi_i^2\in\mathcal{D}(a)$ for all $i\in[\![1;d]\!]$, then Hypothesis 4.8 is fulfilled.*

We will now consider suitable extensions of the domain $\mathcal{D}(a)$.

For any $(s,x)\in[0,T]\times E$ we define the positive bounded **potential measure** $U(s,x,\cdot)$ on $([0,T]\times E,\mathcal{B}([0,T])\otimes\mathcal{B}(E))$ by

$$U(s,x,\cdot): \begin{array}{ccc} \mathcal{B}([0,T])\otimes\mathcal{B}(E) & \longrightarrow & [0,V_T] \\ A & \longmapsto & \mathbb{E}^{s,x}\left[\int_s^T \mathbb{1}_{\{(t,X_t)\in A\}} dV_t\right]. \end{array}$$

Definition 4.10. A Borel set $A \subset [0,T]\times E$ will be said to be **of zero potential** if, for any $(s,x)\in[0,T]\times E$ we have $U(s,x,A)=0$.

Notation 4.11. *Let $p>0$. We introduce*
$\mathcal{L}^p_{s,x} := \mathcal{L}^p(U(s,x,\cdot)) = \left\{ f\in\mathcal{B}([0,T]\times E,\mathbb{R}) : \mathbb{E}^{s,x}\left[\int_s^T |f|^p(r,X_r)dV_r\right] < \infty\right\}$.
For $p\geq 1$, that classical $\mathcal{L}^p$-space is equipped with the seminorm
$\|\cdot\|_{p,s,x} : f \mapsto \left(\mathbb{E}^{s,x}\left[\int_s^T |f(r,X_r)|^p dV_r\right]\right)^{\frac{1}{p}}$. *We also introduce*
$\mathcal{L}^0_{s,x} := \mathcal{L}^0(U(s,x,\cdot)) = \left\{ f\in\mathcal{B}([0,T]\times E,\mathbb{R}) : \int_s^T |f|(r,X_r)dV_r < \infty \quad \mathbb{P}^{s,x} \text{ a.s.}\right\}$.
For any $p\geq 0$ we set

$$\mathcal{L}^p_X = \bigcap_{(s,x)\in[0,T]\times E} \mathcal{L}^p_{s,x}. \tag{4.6}$$

Let $\mathcal{N}$ be the linear subspace of $\mathcal{B}([0,T]\times E,\mathbb{R})$ containing all functions which are equal to 0, $U(s,x,\cdot)$ a.e. for every (s,x). For any $p\geq 0$, we define the quotient space $L^p_X = \mathcal{L}^p_X/\mathcal{N}$. If $p\geq 1$, L^p_X can be equipped with the topology generated by the family of semi-norms $\left(\|\cdot\|_{p,s,x}\right)_{(s,x)\in[0,T]\times E}$ which makes it a separate locally convex topological vector space, see Theorem 5.76 in [1].

We recall that Proposition 4.13 in [6] states the following.

Proposition 4.12. *Let f and g be in $\mathcal{L}^0_X$. Then f and g are equal up to a set of zero potential if and only if for any $(s,x)\in[0,T]\times E$, the processes $\int_s^\cdot f(r,X_r)dV_r$ and $\int_s^\cdot g(r,X_r)dV_r$ are indistinguishable under $\mathbb{P}^{s,x}$. Of course in this case f and g correspond to the same element of L^0_X.*

We introduce now our notion of **extended generator** starting from its domain.

Definition 4.13. We first define the **extended domain** $\mathcal{D}(\mathfrak{a})$ as the set of functions $\phi\in\mathcal{B}([0,T]\times E,\mathbb{R})$ for which there exists
$\chi\in\mathcal{L}^0_X$ such that under any $\mathbb{P}^{s,x}$ the process

$$\mathbb{1}_{[s,T]}\left(\phi(\cdot,X_\cdot) - \phi(s,x) - \int_s^\cdot \chi(r,X_r)dV_r\right) \tag{4.7}$$

(which is not necessarily càdlàg) has a càdlàg modification in $\mathcal{H}^2_0$.

A direct consequence of Proposition 4.15 in [6] is the following.

Proposition 4.14. *Let $\phi\in\mathcal{B}([0,T]\times E,\mathbb{R})$. There is at most one (up to zero potential sets) $\chi\in\mathcal{L}^0_X$ such that under any $\mathbb{P}^{s,x}$, the process defined in (4.7) has a modification which belongs to $\mathcal{H}^2$.*

If moreover $\phi\in\mathcal{D}(a)$, then $a(\phi)=\chi$ up to zero potential sets. In this case, according to Notation 4.4, for every $(s,x)\in[0,T]\times E$, $M[\phi]^{s,x}$ is the $\mathbb{P}^{s,x}$ càdlàg modification in $\mathcal{H}^2_0$ of $\mathbb{1}_{[s,T]}\left(\phi(\cdot,X_\cdot)-\phi(s,x)-\int_s^\cdot \chi(r,X_r)dV_r\right)$.

Definition 4.15. Let $\phi \in \mathcal{D}(\mathfrak{a})$ as in Definition 4.13. We denote again by $M[\phi]^{s,x}$, the unique càdlàg version of the process (4.7) in $\mathcal{H}^2_0$. Taking Proposition 4.12 into account, this will not generate any ambiguity with respect to Notation 4.4. Proposition 4.12, also permits to define without ambiguity the operator

$$\mathfrak{a}: \begin{array}{ccc} \mathcal{D}(\mathfrak{a}) & \longrightarrow & L^0_X \\ \phi & \longmapsto & \chi. \end{array}$$

$\mathfrak{a}$ will be called the **extended generator**.

Remark 4.16. $\mathfrak{a}$ extends a in the sense that $\mathcal{D}(a) \subset \mathcal{D}(\mathfrak{a})$ (comparing Definitions 4.3 and 4.13) and if $\phi \in \mathcal{D}(a)$ then $a(\phi)$ is an element of the class $\mathfrak{a}(\phi)$, see Proposition 4.14.

We also introduce an extended ψ-generalized gradient.

Proposition 4.17. *Assume the validity of Hypothesis 4.8. Let $\phi \in \mathcal{D}(\mathfrak{a})$ and $i \in [\![1;d]\!]$. There exists a (unique up to zero-potential sets) function in $\mathcal{B}([0,T]\times E, \mathbb{R})$ which we will denote $\mathfrak{G}^{\psi_i}(\phi)$ such that under any $\mathbb{P}^{s,x}$, $\langle M[\phi]^{s,x}, M[\psi_i]^{s,x}\rangle = \int_s^{\cdot\vee s} \mathfrak{G}^{\psi_i}(\phi)(r,X_r)dV_r$ up to indistinguishability.*

Proof. We fix $i \in [\![1;d]\!]$. Let $M[\psi_i]$ be the square integrable MAF (see 4.1 in [7]) presented in Notation 4.5. We introduce the random field $M[\phi] = (M[\phi]^t_u)_{(0\leq t\leq u\leq T)}$ as follows. We fix some χ in the class $\mathfrak{a}(\phi)$ and set

$$M[\phi]^t_u := \begin{cases} \phi(u,X_u) - \phi(t,X_t) - \int_t^u \chi(r,X_r)dV_r \text{ if } \int_t^u |\chi|(r,X_r)dV_r < \infty, t \leq u, \\ 0 \text{ elsewhere,} \end{cases} \tag{4.8}$$

We emphasize that, a priori, the function χ is only in $\mathcal{L}^0_X$ implying that at fixed $t \leq u$, $\int_t^u |\chi|(r,X_r(\omega))dV_r$ is not finite for every $\omega \in \Omega$, but only on a set which is $\mathbb{P}^{s,x}$-negligible for all $(s,x) \in [0,t]\times E$.

According to Definition 4.1 in [7] $M[\phi]$ is an AF whose càdlàg version under $\mathbb{P}^{s,x}$ is $M[\phi]^{s,x}$. Of course $M[\psi_i]^{s,x}$ is the càdlàg version of $M[\psi_i]$ under $\mathbb{P}^{s,x}$.

By Definition 4.15, since $\phi \in \mathcal{D}(\mathfrak{a})$, $M[\phi]^{s,x}$ is a square integrable martingale for every (s,x), so $M[\phi]$ is a square integrable MAF. Then by Corollary 4.9, the AF $\langle M[\psi_i]\rangle$ is absolutely continuous with respect to dV. The existence of $\mathfrak{G}^{\psi_i}(\phi)$ now follows from Proposition 4.14 in [7]. and the uniqueness follows by Proposition 4.12. $\square$

Notation 4.18. *If 4.8 holds, we can introduce the linear operator*

$$\mathfrak{G}^{\psi}: \begin{array}{ccc} \mathcal{D}(\mathfrak{a}) & \longrightarrow & (L^0_X)^d \\ \phi & \longmapsto & (\mathfrak{G}^{\psi_1}(\phi),\dots,\mathfrak{G}^{\psi_d}(\phi)), \end{array} \tag{4.9}$$

which will be called the ***extended ψ-generalized gradient****.*

Corollary 4.19. *Let $\phi \in \mathcal{D}(\Gamma^{\psi})$. Then $\Gamma^{\psi}(\phi) = \mathfrak{G}^{\psi}(\phi)$ up to zero potential sets.*

Proof. Comparing Propositions 4.7 and 4.17, for every $(s,x) \in [0,T]\times E$ and $i \in [\![1;d]\!]$, $\int_s^{\cdot\vee s} \Gamma^{\psi_i}(\phi)(r,X_r)dV_r$ and $\int_s^{\cdot\vee s} \mathfrak{G}^{\psi_i}(\phi)(r,X_r)dV_r$ are $\mathbb{P}^{s,x}$-indistinguishable. We can conclude by Proposition 4.12. $\square$

$\mathfrak{G}^{\psi}$ therefore extends Γ^{ψ} as well as $\mathfrak{a}$ extends a, see Remark 4.16.

5. Pseudo-PDEs and associated Markovian type BSDEs driven by a càdlàg martingale

5.1. The concepts

In this section, we keep working in the framework of the previous Section 4.

We now introduce a subclass of BSDEs driven by a càdlàg martingale which we will call **Markovian**. The process $\hat{V}$ will be the (deterministic) function V introduced in Definition 4.3, the terminal condition ξ will only depend on the final value of the canonical process X_T and the randomness of the driver $\hat{f}$ at time t will only depend on X_t. In other words, the driver will be of type $\hat{f}(t,\omega,y,z) = f(t, X_t(\omega), y, z)$ where $f : [0,T]\times E\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ is a measurable function.

Given d functions $\psi_1,\dots,\psi_d$ in $\mathcal{D}(a)$, we will set $\hat{M} := (M[\psi_1]^{s,x},\dots,M[\psi_d]^{s,x})$.

That BSDE will be connected with the deterministic problem in Definition 5.3.

We fix an integer $d\in\mathbb{N}^*$ and some functions $\psi_1,\dots,\psi_d\in\mathcal{D}(a)$ which in the sequel, will satisfy the following hypothesis.

Hypothesis 5.1. For any $i\in[\![1;d]\!]$ we have the following.

- Hypothesis 4.8 holds;
- $a(\psi_i)\in\mathcal{L}^2_X$;
- $\mathfrak{G}^{\psi_i}(\psi_i)$ is bounded.

Proposition 5.2. *Assume that Hypothesis 5.1 holds. Then for every $i\in[\![1;d]\!]$, we have the following.*

- *For any $(s,x)\in[0,T]\times E$, $\hat{M} := M[\psi]^{s,x}$ satisfies item 4. of Hypothesis 3.1 with respect to $\hat{V} := V$.*
- *for every $(s,x)\in[0,T]\times E$, $\sup\limits_{t\in[s,T]} |\psi_i(t,X_t)|^2$ belongs to L^1 under $\mathbb{P}^{s,x}$;*
- *$\psi_i\in\mathcal{L}^2_X$.*

Proof. The first item follows from the fact that, for any $(s,x)\in[0,T]\times E$, $\langle M[\psi_i]^{s,x}\rangle = \int_s^{\cdot\vee s}\mathfrak{G}^{\psi_i}(\psi_i)(r,X_r)dV_r$ (see Proposition 4.17), and the fact that $\mathfrak{G}^{\psi_i}(\psi_i)$ is bounded. Concerning the second item, for any $(s,x)\in[0,T]\times E$, the martingale problem gives $\psi_i(\cdot,X) = \psi_i(s,x)+\int_s^{\cdot} a(\psi_i)(r,X_r)dV_r + M[\psi_i]^{s,x}$, see Definition 4.3. By Jensen's inequality, we have $\sup\limits_{t\in[s,T]} |\psi_i(t,X_t)|^2 \leq C(\psi_i^2(s,x)+\int_s^T a^2(\psi_i)(r,X_r)dV_r + \sup\limits_{t\in[s,T]}(M[\psi_i]^{s,x}_t)^2)$ for some $C>0$. It is therefore L^1 since $a(\psi_i)\in\mathcal{L}^2_X$ and $M[\psi_i]^{s,x}\in\mathcal{H}^2$. The last item is a direct consequence of the second one. □

Definition 5.3. Let us consider some $g\in\mathcal{B}(E,\mathbb{R})$ and $f\in\mathcal{B}([0,T]\times E\times\mathbb{R}\times\mathbb{R},\mathbb{R}^d)$.

We will call **Pseudo-Partial Differential Equation** related to (f,g) (in short $Pseudo-PDE(f,g)$) the following equation with final condition:

$$\begin{cases} a(u)+f\left(\cdot,\cdot,u,\Gamma^{\psi}(u)\right) &= 0 \quad \text{on } [0,T]\times E \\ u(T,\cdot) &= g. \end{cases} \tag{5.1}$$

We will say that u is a **classical solution** of $Pseudo-PDE(f,g)$ if $u, u\psi_i, i\in[\![1;d]\!]$ belong to $\mathcal{D}(a)$ and if u satisfies (5.1).

The connection between a Markovian BSDE and a $Pseudo - PDE(f, g)$, will be possible under a hypothesis on some generalized moments on X, and some growth conditions on the functions (f, g). Those will be related to two fixed functions $\zeta, \eta \in \mathcal{B}(E, \mathbb{R}_+)$.

Hypothesis 5.4. The canonical Markov class will be said **to satisfy** $H^{mom}(\zeta, \eta)$ if

1. for any $(s, x) \in [0, T] \times E$, $\mathbb{E}^{s,x}[\zeta^2(X_T)]$ is finite;
2. for any $(s, x) \in [0, T] \times E$, $\mathbb{E}^{s,x}\left[\int_0^T \eta^2(X_r) dV_r\right]$ is finite.

Until the end of this section, we assume that some ζ, η are given and that the canonical Markov class satisfies $H^{mom}(\zeta, \eta)$.

Hypothesis 5.5. A couple (f, g) of functions $f \in \mathcal{B}([0, T]\times E\times\mathbb{R}\times\mathbb{R}^d, \mathbb{R})$ and $g \in \mathcal{B}(E, \mathbb{R})$ will be said **to satisfy** $H^{lip}(\zeta, \eta)$ if there exist positive constants K^Y, K^Z, C, C' such that

1. $\forall x: \quad |g(x)| \le C(1 + \zeta(x))$,
2. $\forall (t, x): \quad |f(t, x, 0, 0)| \le C'(1 + \eta(x))$,
3. $\forall (t, x, y, y', z, z'): \quad |f(t, x, y, z) - f(t, x, y', z')| \le K^Y|y - y'| + K^Z\|z - z'\|$.

(f, g) will be said **to satisfy** $H^{growth}(\zeta, \eta)$ if the following more general assumption holds. There exist positive constants C, C' such that

1. $\forall x: \quad |g(x)| \le C(1 + \zeta(x))$;
2. $\forall (t, x, y, z): \quad |f(t, x, y, z)| \le C'(1 + \eta(x) + |y| + \|z\|)$.

Remark 5.6. We fix for now a couple (f, g) satisfying $H^{lip}(\zeta, \eta)$. For any $(s, x) \in [0, T]\times E$, in the stochastic basis $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ and setting $\hat{V} := V$, the triplet $\xi := g(X_T)$, $\hat{f} : (t, \omega, y, z) \longmapsto f(t, X_t(\omega), y, z)$, $\hat{M} := M[\psi]^{s,x}$ satisfies Hypothesis 3.1.

With the equation $Pseudo - PDE(f, g)$, we will associate the following family of BSDEs indexed by $(s, x) \in [0, T] \times E$, driven by a càdlàg martingale.

Notation 5.7. *For any $(s, x) \in [0, T] \times E$, we consider in the stochastic basis $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ and on the interval $[0, T]$ the $BSDE(\xi, \hat{f}, V, \hat{M})$, where $\xi = g(X_T)$, $\hat{f} : (t, \omega, y, z) \longmapsto f(t, X_t(\omega), y, z)$, $\hat{M} = M[\psi]^{s,x}$.*

From now on that BSDE will be denoted $BSDE^{s,x}(f, g)$ and its unique solution (see Theorem 3.3 and Remark 5.6) will be denoted $(Y^{s,x}, M^{s,x})$.

If $H^{lip}(\zeta, \eta)$ is fulfilled by (f, g), then $(Y^{s,x}, M^{s,x})$ is therefore the unique couple in $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}_0^2$ satisfying

$$Y_\cdot^{s,x} = g(X_T) + \int_\cdot^T f\left(r, X_r, Y_r^{s,x}, \frac{d\langle M^{s,x}, M[\psi]^{s,x}\rangle}{dV}(r)\right) dV_r - (M_T^{s,x} - M_\cdot^{s,x}). \quad (5.2)$$

Remark 5.8. Even if the underlying process X admits no generalized moments, given a couple (f, g) such that $f(\cdot, \cdot, 0, 0)$ and g are bounded, the considerations of this section still apply. In particular the connections that we will establish between the $BSDE^{s,x}(f, g)$ and the corresponding $Pseudo - PDE(f, g)$ still take place.

Our main contribution consists in illustrating the precise link between the solutions of equations $BSDE^{s,x}(f, g)$ and those of $Pseudo - PDE(f, g)$. In particular we will emphasize that a solution of $BSDE^{s,x}(f, g)$ produces a solution of $Pseudo - PDE(f, g)$ and reciprocally.

We now introduce a probabilistic notion of solution for $Pseudo-PDE(f,g)$.

Definition 5.9. A Borel function $u:[0,T]\times E\rightarrow\mathbb{R}$ will be said to be a **martingale solution** of $Pseudo-PDE(f,g)$ if $u\in\mathcal{D}(\mathfrak{a})$ and

$$\left\{\begin{array}{rcl}\mathfrak{a}(u)&=&-f(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u))\\ u(T,\cdot)&=&g.\end{array}\right. \tag{5.3}$$

Remark 5.10. The first equation of (5.3) holds in L^0_X, hence up to a zero potential set. The second one is a pointwise equality.

The following lemma was the object of Lemma 5.13 in [6].

Lemma 5.11. *Let V be a non-decreasing function. If two measurable processes are $\mathbb{P}$-modifications of each other, then they are also equal $dV\otimes d\mathbb{P}$ a.e.*

Proposition 5.12. *Let (f,g) satisfy $H^{growth}(\zeta,\eta)$. Let u be a martingale solution of $Pseudo-PDE(f,g)$. Then for any $(s,x)\in[0,T]\times E$, the couple of processes*

$$\left(u(t,X_t),\quad u(t,X_t)-u(s,x)+\int_s^t f(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u))(r,X_r)dV_r\right)_{t\in[s,T]} \tag{5.4}$$

has a $\mathbb{P}^{s,x}$-version which is a solution on $[s,T]$ of $BSDE^{s,x}(f,g)$, see Remark 3.4.

Moreover, $u\in\mathcal{L}^2_X$.

Proof. Let $u\in\mathcal{D}(\mathfrak{a})$ be a solution of (5.3) and let $(s,x)\in[0,T]\times E$ be fixed. By Definition 4.13 and Remark 3.4, the process $u(\cdot,X_\cdot)$ under $\mathbb{P}^{s,x}$ admits a càdlàg modification $U^{s,x}$ on $[s,T]$, which is a special semimartingale with decomposition

$$\begin{array}{rcl}U^{s,x}&=&u(s,x)+\int_s^\cdot \mathfrak{a}(u)(r,X_r)dV_r+M[u]^{s,x}\\ &=&u(s,x)-\int_s^\cdot f\left(r,X_r,u(r,X_r),\mathfrak{G}^{\psi}(u)(r,X_r)\right)dV_r+M[u]^{s,x}\\ &=&u(s,x)-\int_s^\cdot f\left(r,X_r,U^{s,x}_r,\frac{d\langle M[u]^{s,x},M[\psi]^{s,x}\rangle}{dV}(r)\right)dV_r+M[u]^{s,x},\end{array} \tag{5.5}$$

where the third equality of (5.5) comes from Lemma 5.11 and Proposition 4.17. Moreover since $u(T,\cdot)=g$, then $U^{s,x}_T=u(T,X_T)=g(X_T)$ a.s. so the couple $(U^{s,x},M[u]^{s,x})$ satisfies the following equation on $[s,T]$ (with respect to $\mathbb{P}^{s,x}$):

$$U^{s,x}_\cdot=g(X_T)+\int_\cdot^T f\left(r,X_r,U^{s,x}_r,\frac{d\langle M[u]^{s,x},M[\psi]^{s,x}\rangle}{dV}(r)\right)dV_r-(M[u]^{s,x}_T-M[u]^{s,x}_\cdot). \tag{5.6}$$

$M[u]^{s,x}$ (introduced at Definition 4.15) belongs to $\mathcal{H}^2_0$ but we do not have a priori information on the square integrability of $U^{s,x}$. However we know that $M[u]^{s,x}$ is equal to zero at time s, and that $U^{s,x}_s$ is deterministic so square integrable. We can therefore apply Lemma A.12 which implies that $(U^{s,x},M[u]^{s,x})$ solves $BSDE^{s,x}(f,g)$ on $[s,T]$. In particular, $U^{s,x}$ belongs to $\mathcal{L}^2(dV\otimes d\mathbb{P}^{s,x})$ for every (s,x), so by Lemma 5.11 and Notation 4.11, $u\in\mathcal{L}^2_X$. $\square$

5.2. Decoupled mild solutions of Pseudo-PDEs

In this section we introduce an analytical notion of solution of our $Pseudo-PDE(f,g)$, that we will denominate *decoupled mild*, taking inspiration from the mild solutions of partial differential equation. That notion will be shown to be equivalent to the one of martingale solution introduced in Definition 5.9. Let $P = (P_{s,t})$ denote the transition kernel of the canonical Markov class, see Definition 3.4 in [7] and also Notation 4.1.

Our notion of decoupled mild solution relies on the fact that the equation $a(u) + f\left(\cdot, \cdot, u, \Gamma^{\psi}(u)\right) = 0$ can be naturally decoupled into

$$\left\{\begin{array}{lll} a(u) & = & -f(\cdot, \cdot, u, v) \\ v_i & = & \Gamma^{\psi_i}(u), \quad i \in [\![1; d]\!]. \end{array}\right. \tag{5.7}$$

Then, by definition of the carré du champ operator (see Definition 4.6), we formally have $a(u\psi_i) = \Gamma^{\psi_i}(u) + ua(\psi_i) + \psi_i a(u), i \in [\![1; d]\!]$. So the system of equations (5.7) can be rewritten as

$$\left\{\begin{array}{lll} a(u) & = & -f(\cdot, \cdot, u, v) \\ a(u\psi_i) & = & v_i + ua(\psi_i) - \psi_i f(\cdot, \cdot, u, v), \quad i \in [\![1; d]\!]. \end{array}\right. \tag{5.8}$$

Inspired by the usual notions of mild solution, this naturally brings us to the following definition of a (decoupled) mild solution.

Definition 5.13. Assume that (f, g) satisfies $H^{growth}(\zeta, \eta)$. Let $u \in \mathcal{B}([0, T] \times E, \mathbb{R})$ and $v \in \mathcal{B}([0, T] \times E, \mathbb{R}^d)$.

1. (u, v) is a **solution of the identification problem determined by** (f, g) or simply **solution of** $IP(f, g)$ if $u, v_1, \dots, v_d$ belong to $\mathcal{L}^2_X$ and if for every $(s, x) \in [0, T] \times E$,

$$\left\{\begin{array}{lll} u(s,x) & = & P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(\cdot, \cdot, u, v\right)(r, \cdot)\right](x) dV_r \\ u\psi_1(s,x) & = & P_{s,T}[g\psi_1(T,\cdot)](x) - \int_s^T P_{s,r}\left[\left(v_1 + ua(\psi_1)\right.\right. \\ & & \quad \left.\left. -\psi_1 f\left(\cdot, \cdot, u, v\right)\right)(r, \cdot)\right](x) dV_r \\ & \cdots & \\ u\psi_d(s,x) & = & P_{s,T}[g\psi_d(T,\cdot)](x) - \int_s^T P_{s,r}\left[\left(v_d + ua(\psi_d)\right.\right. \\ & & \quad \left.\left. -\psi_d f\left(\cdot, \cdot, u, v\right)\right)(r, \cdot)\right](x) dV_r. \end{array}\right. \tag{5.9}$$

2. u is a **decoupled mild solution** of $Pseudo - PDE(f, g)$ if there exists a function v such that (u, v) is a solution of $IP(f, g)$.

The following lemma is very close to Lemma 3.5 in [8] and the arguments for the proof are similar.

Lemma 5.14. *Let $u, v_1, \dots, v_d \in \mathcal{L}^2_X$, let (f, g) be a couple satisfying $H^{growth}(\zeta, \eta)$ and let $\psi_1, \dots, \psi_d$ satisfy Hypothesis 5.1. Then $f(\cdot, \cdot, u, v)$ belongs to $\mathcal{L}^2_X$ and for every $i \in [\![1; d]\!]$, $\psi_i f(\cdot, \cdot, u, v)$, and $ua(\psi_i)$, belong to $\mathcal{L}^1_X$. For any $(s, x) \in [0, T] \times E$, $i \in [\![1; d]\!]$, $g(X_T)\psi_i(T, X_T)$ belongs to L^1 under $\mathbb{P}^{s,x}$. In particular, all terms in (5.9) make sense.*

Proposition 5.15. *Let (f, g) satisfy $H^{growth}(\zeta, \eta)$. Let u be a martingale solution of $Pseudo- PDE(f, g)$, then $(u, \mathfrak{G}^{\psi}(u))$ is a solution of $IP(f, g)$ and in particular, u is a decoupled mild solution of $Pseudo - PDE(f, g)$.*

Proof. Let u be a martingale solution of $Pseudo- PDE(f, g)$. By Proposition 5.12, $u \in \mathcal{L}^2_X$. Taking into account Definition 4.15, for every (s, x), $M[u]^{s,x} \in \mathcal{H}^2_0$ under $\mathbb{P}^{s,x}$. So by

Lemma A.2, for any $i \in [\![1;d]\!]$, $\frac{d\langle M[u]^{s,x}, M[\psi^i]^{s,x}\rangle}{dV}$ belongs to $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x})$. By use of Proposition 4.17, this means that $\mathfrak{G}^{\psi_i}(u) \in \mathcal{L}^2_X$ for every i. By Lemma 5.14, it follows that $f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)$ belongs to $\mathcal{L}^2_X$ and so for any $i \in [\![1;d]\!]$, $\psi_i f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)$ and $ua(\psi_i)$, belong to $\mathcal{L}^1_X$.

Let $(s,x) \in [0,T] \times E$. Below we demonstrate that

$$\left\{\begin{array}{rcl} u(s,x) & = & P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)(r,\cdot)\right](x)dV_r \\ u\psi_1(s,x) & = & P_{s,T}[g\psi_1(T,\cdot)](x) - \int_s^T P_{s,r}\left[(\mathfrak{G}(u,\psi_1) + ua(\psi_1)\right. \\ & & \quad \left.-\psi_1 f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right))(r,\cdot)\right](x)dV_r \\ & \cdots & \\ u\psi_d(s,x) & = & P_{s,T}[g\psi_d(T,\cdot)](x) - \int_s^T P_{s,r}\left[(\mathfrak{G}(u,\psi_d) + ua(\psi_d)\right. \\ & & \quad \left.-\psi_d f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right))(r,\cdot)\right](x)dV_r. \end{array}\right. \tag{5.10}$$

We refer now to the probability $\mathbb{P}^{s,x}$: by Definitions 4.13, 4.15 and 5.9, the process $u(\cdot, X_\cdot)$ admits a modification $U^{s,x}$ being a special semimartingale with decomposition

$$U^{s,x} = u(s,x) - \int_s^{\cdot} f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)(r,X_r)dV_r + M[u]^{s,x}, \tag{5.11}$$

and $M[u]^{s,x} \in \mathcal{H}^2_0$.

Definition 5.9 also states that $u(T,\cdot) = g$, so

$$u(s,x) = g(X_T) + \int_s^T f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)(r,X_r)dV_r - M[u]^{s,x}_T \text{ a.s.} \tag{5.12}$$

By Fubini's theorem we deduce that

$$\begin{array}{rcl} u(s,x) & = & \mathbb{E}^{s,x}\left[g(X_T) + \int_s^T f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)(r,X_r)dV_r\right] \\ & = & P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(r,\cdot,u(r,\cdot),\mathfrak{G}^{\psi}(u)(r,\cdot)\right)\right](x)dV_r. \end{array} \tag{5.13}$$

We now fix $i \in [\![1;d]\!]$. By integration by parts, taking (5.11) and Definition 4.3 into account, we obtain

$$\begin{array}{rcl} d(U^{s,x}_t\psi_i(t,X_t)) & = & -\psi_i(t,X_t)f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)(t,X_t)dV_t + \psi_i(t^-,X_{t^-})dM[u]^{s,x}_t \\ & & +U^{s,x}_t a(\psi_i)(t,X_t)dV_t + U^{s,x}_{t^-}dM[\psi_i]^{s,x}_t \\ & & +d[M[u]^{s,x}, M[\psi_i]^{s,x}]_t, \end{array} \tag{5.14}$$

Integrating between s and T,

$$\begin{array}{rl} & u\psi_i(s,x) \\ = & g(X_T)\psi_i(T,X_T) + \int_s^T \psi_i(t,X_t)f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)(r,X_r)dV_r \\ & -\int_s^T \psi_i(r^-,X_{r^-})dM[u]^{s,x}_r \\ & -\int_s^T U^{s,x}_t a(\psi_i)(r,X_r)dV_r - \int_s^T U^{s,x}_{r^-}dM[\psi_i]^{s,x}_r - [M[u]^{s,x}, M[\psi_i]^{s,x}]_T \\ = & g(X_T)\psi_i(T,X_T) - \int_s^T \left(ua(\psi_i) - \psi_i f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)\right)(r,X_r)dV_r \\ & -\int_s^T \psi_i(r^-,X_{r^-})dM[u]^{s,x}_r \\ & -\int_s^T U^{s,x}_{r^-}dM[\psi_i]^{s,x}_r - [M[u]^{s,x}, M[\psi_i]^{s,x}]_T, \end{array} \tag{5.15}$$

thanks to Lemma 5.11.

By Proposition 4.17, $\langle M[\psi_i]^{s,x}\rangle = \int_s^{\cdot\vee s} \mathfrak{G}^{\psi_i}(\psi_i)(r, X_r)dV_r$. So the angular bracket of $\int_s^{\cdot} U^{s,x}_{r^-}dM[\psi_i]^{s,x}_r$ at time T is equal to $\int_s^T u^2\mathfrak{G}^{\psi_i}(\psi_i)(r, X_r)dV_r$ which is an integrable r.v. since $\mathfrak{G}^{\psi_i}(\psi_i)$ is bounded and $u \in \mathcal{L}^2_X$. Therefore $\int_s^{\cdot} U^{s,x}_{r^-}dM[\psi_i]^{s,x}_r$ is a square integrable martingale.

Then, by Hypothesis 5.1 and Proposition 5.2, $\sup\limits_{t\in[s,T]} |\psi_i(t, X_t)|^2 \in L^1$, and by Definition 4.15, $M[u]^{s,x} \in \mathcal{H}^2$ so by Lemma 3.17 in [6], $\int_s^{\cdot} \psi_i(r^-, X_{r^-})dM[u]^{s,x}_r$ is a martingale.

We can now perform the expectation in (5.15), to get

$$\begin{aligned} &u\psi_i(s, x)\\ =\ &\mathbb{E}^{s,x}\Big[g(X_T)\psi_i(T, X_T) - \textstyle\int_s^T \big(ua(\psi_i) - \psi_i f\left(\cdot, \cdot, u, \mathfrak{G}^{\psi}(u)\right)\big)(r, X_r)dV_r\\ &-[M[u]^{s,x}, M[\psi_i]^{s,x}]_T\Big]\\ =\ &\mathbb{E}^{s,x}\Big[g(X_T)\psi_i(T, X_T) - \textstyle\int_s^T \big(ua(\psi_i) + \mathfrak{G}^{\psi_i}(u)\\ &\quad -\psi_i f\left(\cdot, \cdot, u, \mathfrak{G}^{\psi}(u)\right)\big)(r, X_r)dV_r\Big], \end{aligned} \tag{5.16}$$

since u and ψ_i belong to $\mathcal{D}(\mathfrak{a})$. Indeed the second equality follows from the fact $[M[u]^{s,x}, M[\psi_i]^{s,x}] - \langle M[u]^{s,x}, M[\psi_i]^{s,x}\rangle$ is a martingale and Proposition 4.17.

Since we have assumed that $u \in \mathcal{L}^2_X$, Lemma 5.14 says that $f\left(\cdot, \cdot, u, \mathfrak{G}^{\psi}(u)\right) \in \mathcal{L}^2_X$, Hypothesis 5.1 implies that ψ_i and $a(\psi_i)$ are in $\mathcal{L}^2_X$, so all terms in the integral inside the expectation in the third line belong to $\mathcal{L}^1_X$. We can therefore apply Fubini's theorem to get

$$\begin{aligned} u\psi_i(s, x) = P_{s,T}[g\psi_i(T, \cdot)](x) - \int_s^T P_{s,r}\Big[\big(ua(\psi_i) + \mathfrak{G}^{\psi_i}(u)\\ -\psi_i f\left(\cdot, \cdot, u, \mathfrak{G}^{\psi}(u)\right)\big)(r, \cdot)\Big](x)dV_r. \end{aligned} \tag{5.17}$$

This concludes the proof. □

Proposition 5.15 admits a converse implication.

Proposition 5.16. *Let (f, g) satisfy $H^{growth}(\zeta, \eta)$, then every decoupled mild solution of $Pseudo-PDE(f, g)$ is a martingale solution. Moreover, if (u, v) solves $IP(f, g)$, then $v = \mathfrak{G}^{\psi}(u)$, up to zero potential sets.*

Proof. Let u and v_i, $i \in [\![1; d]\!]$ in $\mathcal{L}^2_X$ satisfy (5.9). We observe that the first line of (5.9) with $s = T$, implies that $u(T, \cdot) = g$.

Let $(s, x) \in [0, T] \times E$ be fixed. We will now work under the probability $\mathbb{P}^{s,x}$. On $[s, T]$, we set $U := u(\cdot, X)$ and $N := u(\cdot, X) - u(s, x) + \int_s^{\cdot} f(r, X_r, u(r, X_r), v(r, X_r))dV_r$.

For some $t \in [s, T]$, we combine the first line of (5.9) applied with $(s, x) = (t, X_t)$ and the Markov property, see e.g. (3.4) in [7]. Since $f(\cdot, \cdot, u, v)$ belongs to $\mathcal{L}^2_X$ (see Lemma 5.14) we a.s. have that

$$\begin{aligned} U_t &= u(t, X_t)\\ &= P_{t,T}[g](X_t) + \textstyle\int_t^T P_{t,r}\left[f(r, \cdot, u(r, \cdot), v(r, \cdot))\right](X_t)dV_r\\ &= \mathbb{E}^{t,X_t}\Big[g(X_T) + \textstyle\int_t^T f(r, X_r, u(r, X_r), v(r, X_r))dV_r\Big]\\ &= \mathbb{E}^{s,x}\Big[g(X_T) + \textstyle\int_t^T f(r, X_r, u(r, X_r), v(r, X_r))dV_r|\mathcal{F}_t\Big], \end{aligned} \tag{5.18}$$

so $N_t = \mathbb{E}^{s,x}\Big[g(X_T) + \int_s^T f(r, X_r, u(r, X_r), v(r, X_r))dV_r|\mathcal{F}_t\Big] - u(s, x)$ a.s. hence N is a martingale. Let $N^{s,x}$ denote its càdlàg version which we extend on $[0, s]$ with the value 0.

Then

$$U^{s,x} := u(s,x) - \int_s^{\cdot} f(r, X_r, u(r, X_r), v(r, X_r)) dV_r + N^{s,x}, \tag{5.19}$$

indexed on $[s,T]$ is a càdlàg version of U. Proceeding as in the proof of Proposition 3.8 in [8], we can show that $N^{s,x}$ is a square integrable martingale. The process $\left(u(\cdot, X_\cdot) - u(s,x) + \int_s^{\cdot} f(r, X_r, u(r, X_r), v(r, X_r)) dV_r\right) \mathbb{1}_{[s,T]}$ therefore admits for any (s,x) a $\mathbb{P}^{s,x}$-modification in $\mathcal{H}^2_0$. By Definitions 4.13, 4.15 this means that $u \in \mathcal{D}(\mathfrak{a})$, $\mathfrak{a}(u) = -f(\cdot,\cdot,u,v)$ and for any $(s,x) \in [0,T] \times E$, $M[u]^{s,x} = N^{s,x}$ $P^{s,x}$-a.s.

We are left to show $\mathfrak{G}^{\psi}(u) = v$, up to zero potential sets, hence that for every $(s,x) \in [0,T] \times E$ and $i \in [\![1; d]\!]$,

$$\langle M^{s,x}[u], M^{s,x}[\psi_i] \rangle = \int_s^{\cdot \vee s} v_i(r, X_r) dV_r, \quad \text{a.s.} \tag{5.20}$$

Let $(s,x) \in [0,T] \times E$, and $i \in [\![1; d]\!]$. Combining the $(i+1)$-th line of (5.9) applied to $(s,x) = (t, X_t)$ and the Markov property (see e.g. (3.4) in [7]), taking into account the fact that all terms belong to $\mathcal{L}^1_X$ (see Lemma 5.14, Hypothesis 5.1) we a.s. have

$$\begin{aligned} u\psi_i(t, X_t) &= P_{t,T}[g\psi_i(T,\cdot)](X_t) - \textstyle\int_t^T P_{t,r}\left[(v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v))(r,\cdot)\right](X_t) dV_r \\ &= \mathbb{E}^{t,X_t}\left[g(X_T)\psi_i(T,X_T) - \textstyle\int_t^T (v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v))(r, X_r) dV_r\right] \\ &= \mathbb{E}^{s,x}\left[g(X_T)\psi_i(T,X_T) - \textstyle\int_t^T (v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v))(r, X_r) dV_r | \mathcal{F}_t\right]. \end{aligned} \tag{5.21}$$

Setting, for $t \in [s,T]$, $N^i_t := u\psi_i(t, X_t) - \int_s^t (v_i + ua(\psi)^i - \psi_i f(\cdot,\cdot,u,v))(r, X_r) dV_r$, from (5.21) we deduce that, for any $t \in [s,T]$,

$$N^i_t = \mathbb{E}^{s,x}\left[g(X_T)\psi_i(T,X_T) - \int_s^T (v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v))(r, X_r) dV_r \middle| \mathcal{F}_t\right]$$

a.s. So N^i is a martingale. Let $N^{i,s,x}$ denote its càdlàg $\mathbb{P}^{s,x}$-modification. The process

$$\int_s^{\cdot} (v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v))(r, X_r) dV_r + N^{i,s,x}, \tag{5.22}$$

is a càdlàg $\mathbb{P}^{s,x}$-version of $u\psi_i(\cdot, X)$ on $[s,T]$. But we have by (5.19), that

$U^{s,x} = u(s,x) - \int_s^{\cdot} f(r, X_r, u(r, X_r), v(r, X_r)) dV_r + N^{s,x}$ is a version of $u(\cdot, X)$, hence by integration by parts on $U^{s,x}\psi_i(\cdot, X_\cdot)$ that

$$\begin{aligned} &u\psi_i(s,x) + \textstyle\int_s^{\cdot} U^{s,x}_r a(\psi_i)(r, X_r) dV_r + \int_s^{\cdot} U^{s,x}_{r^-} dM^{s,x}[\psi_i]_r \\ &- \textstyle\int_s^{\cdot} \psi_i f(\cdot,\cdot,u,v)(r, X_r) dV_r + \int_s^{\cdot} \psi_i(r^-, X_{r^-}) dM^{s,x}[u]_r + [M^{s,x}[u], M^{s,x}[\psi_i]] \end{aligned} \tag{5.23}$$

is another càdlàg semimartingale which is a $\mathbb{P}^{s,x}$-version of $u\psi_i(\cdot, X)$ on $[s,T]$. Now (5.23) equals

$$\mathcal{M}^i + \mathcal{V}^i, \tag{5.24}$$

where

$$\begin{aligned} \mathcal{M}^i_t = {} & u\psi_i(s,x) + \int_s^t U^{s,x}_{r^-} dM^{s,x}[\psi_i]_r + \int_s^t \psi_i(r^-, X_{r^-}) dM^{s,x}[u]_r \\ & + [M^{s,x}[u], M^{s,x}[\psi_i]]_t - \langle M^{s,x}[u], M^{s,x}[\psi_i] \rangle_t, \end{aligned}$$

is a local martingale and

$$\mathcal{V}^i_t = \langle M^{s,x}[u], M^{s,x}[\psi_i]\rangle_t + \int_s^t U^{s,x}_r a(\psi_i)(r, X_r)dV_r - \int_s^t \psi_i f(\cdot,\cdot,u,v)(r, X_r)dV_r,$$

is a predictable process with bounded variation vanishing at zero. Now (5.22) and (5.24) are two càdlàg versions of $u\psi_i(\cdot, X)$ on $[s, T]$.

By the uniqueness of the decomposition of a special semimartingale and using Lemma 5.11 we get

$$\begin{aligned}&\int_s^{\cdot} (v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v))(r, X_r)dV_r\\ &\qquad = \langle M^{s,x}[u], M^{s,x}[\psi_i]\rangle + \int_s^{\cdot} ua(\psi_i)(r, X_r)dV_r - \int_s^{\cdot} \psi_i f(\cdot,\cdot,u,v)(r, X_r)dV_r.\end{aligned}$$

This yields $\langle M^{s,x}[u], M^{s,x}[\psi_i]\rangle = \int_s^{\cdot\vee s} v_i(r, X_r)dV_r$, which implies (5.20). ☐

Proposition 5.17. *Let (f, g) satisfy $H^{growth}(\zeta,\eta)$. A classical solution of $Pseudo-PDE(f, g)$ is a decoupled mild solution.*

Conversely, a decoupled mild solution of $Pseudo-PDE(f, g)$ belonging to $\mathcal{D}(\Gamma^\psi)$ is a classical solution of $Pseudo-PDE(f, g)$ up to a zero-potential set, meaning that it satisfies the first equality of (5.1) up to a set of zero potential.

Proof. Let u be a classical solution of $Pseudo-PDE(f, g)$. Definition 5.3 and Corollary 4.19 imply that $u(T,\cdot) = g$, and the equalities up to zero potential set

$$\mathfrak{a}(u) = a(u) = -f(\cdot,\cdot,u,\Gamma^\psi(u)) = -f(\cdot,\cdot,u,\mathfrak{G}^\psi(u)), \tag{5.25}$$

which shows that u is a martingale solution and by Proposition 5.15 it is also a decoupled mild solution.

Similarly, the second statement follows by Proposition 5.16, Definition 5.9, and again Corollary 4.19. ☐

5.3. Existence and uniqueness of a decoupled mild solution

In this subsection, the positive functions ζ, η and the functions (f, g) appearing in $Pseudo-PDE(f, g)$ are fixed. We still assume that the canonical Markov class satisfies $H^{mom}(\zeta,\eta)$.

Theorem 5.18 can be proved using arguments which are very close to those developed in the proof of Theorem 5.15 in [6]. For the convenience of the reader, we postpone the adapted proof to Appendix B.

Let $(Y^{s,x}, M^{s,x})$ be for any $(s, x) \in [0, T]\times E$ the unique solution of (5.2), see Notation 5.7.

Theorem 5.18. *Let (f, g) satisfy $H^{lip}(\zeta,\eta)$. There exists $u \in \mathcal{D}(\mathfrak{a})$ such that for any $(s, x) \in [0, T] \times E$*

$$\begin{cases} \forall t \in [s, T] : Y^{s,x}_t &= u(t, X_t) \quad \mathbb{P}^{s,x} a.s.\\ M^{s,x} &= M[u]^{s,x},\end{cases}$$

and in particular $\frac{d\langle M^{s,x}, M[\psi]^{s,x}\rangle}{dV} = \mathfrak{G}^\psi(u)(\cdot, X_\cdot)dV \otimes d\mathbb{P}^{s,x}$ a.e. on $[s, T]$. Moreover, for every (s, x), $Y^{s,x}_s$ is $\mathbb{P}^{s,x}$ a.s. equal to a constant (which we shall still denote $Y^{s,x}_s$) and $u(s, x) = Y^{s,x}_s$ for every $(s, x) \in [0, T] \times E$.

Corollary 5.19. *Let (f, g) satisfy $H^{lip}(\zeta, \eta)$. For any $(s, x) \in [0, T] \times E$, the functions u obtained in Theorem 5.18 satisfy $\mathbb{P}^{s,x}$ a.s. on $[s, T]$*

$$u(t, X_t) = g(X_T) + \int_t^T f\left(r, X_r, u(r, X_r), \mathfrak{G}^{\psi}(u)(r, X_r)\right) dV_r - (M[u]_T^{s,x} - M[u]_t^{s,x}),$$

and in particular, $\mathfrak{a}(u) = -f(\cdot, \cdot, u, \mathfrak{G}^{\psi}(u))$.

Proof. The corollary follows from Theorem 5.18 and Lemma 5.11. □

Theorem 5.20. *Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be a canonical Markov class associated to a transition kernel measurable in time (see Definitions 3.4, 3.5 and 3.7 in [7]) which solves a martingale problem associated with the triplet $(\mathcal{D}(a), a, V)$. Moreover we suppose Hypothesis $H^{mom}(\zeta, \eta)$ for some positive ζ, η. Let (f, g) be a couple satisfying $H^{lip}(\zeta, \eta)$.*

Then $Pseudo - PDE(f, g)$ has a unique decoupled mild solution given by

$$u : \begin{array}{ccc} [0, T] \times E & \longrightarrow & \mathbb{R} \\ (s, x) & \longmapsto & Y_s^{s,x}, \end{array} \tag{5.26}$$

where $(Y^{s,x}, M^{s,x})$ denotes the (unique) solution of $BSDE^{s,x}(f, g)$ for fixed (s, x).

Proof. Let u be the function exhibited in Theorem 5.18. In order to show that u is a decoupled mild solution of $Pseudo - PDE(f, g)$, it is enough by Proposition 5.15 to show that it is a martingale solution.

In Corollary 5.19, we have already seen that $\mathfrak{a}(u) = -f(\cdot, \cdot, u, \mathfrak{G}^{\psi}(u))$. Concerning the second line of (5.3), for any $x \in E$, we have

$u(T, x) = u(T, X_T) = g(X_T) = g(x)\, \mathbb{P}^{T,x}$ a.s., so $u(T, \cdot) = g$, in the deterministic pointwise sense.

We now show uniqueness. By Proposition 5.16, it is enough to show that $Pseudo - PDE(f, g)$ admits at most one martingale solution. Let u, u' be two martingale solutions of $Pseudo - PDE(f, g)$. We fix $(s, x) \in [0, T] \times E$. By Proposition 5.12, both couples, indexed by $[s, T]$,

$\left(u(\cdot, X), \quad u(\cdot, X) - u(s, x) + \int_s^{\cdot} f(\cdot, \cdot, u, \mathfrak{G}^{\psi}(u))(r, X_r) dV_r\right)$ and

$\left(u'(\cdot, X), \quad u'(\cdot, X) - u'(s, x) + \int_s^{\cdot} f(\cdot, \cdot, u', \mathfrak{G}^{\psi}(u))(r, X_r) dV_r\right)$ admit a $\mathbb{P}^{s,x}$-version which solves $BSDE^{s,x}(f, g)$ on $[s, T]$. By Theorem 3.3 and Remark 3.4, $BSDE^{s,x}(f, g)$ admits a unique solution, so $u(\cdot, X_\cdot)$ and $u'(\cdot, X_\cdot)$ are $\mathbb{P}^{s,x}$-modifications one of the other on $[s, T]$. In particular, considering their values at time s, we have $u(s, x) = u'(s, x)$. We therefore have $u' = u$. □

Corollary 5.21. *Let (f, g) satisfy $H^{lip}(\zeta, \eta)$. There is at most one classical solution of $Pseudo - PDE(f, g)$ and this only possible classical solution is the unique decoupled mild solution $(s, x) \longmapsto Y_s^{s,x}$, where $(Y^{s,x}, M^{s,x})$ denotes the (unique) solution of $BSDE^{s,x}(f, g)$ for fixed (s, x).*

Proof. The proof follows from Proposition 5.17 and Theorem 5.20. □

Remark 5.22. Let (u, v) be the unique solution of the identification problem $IP(f, g)$, then v also admits a stochastic representation. Indeed, for every $(s, x) \in [0, T] \times E$, on $[s, T]$,

$\frac{d\langle M^{s,x}, M^{s,x}[\psi]\rangle}{dV} = v(\cdot, X_\cdot) dV \otimes d\mathbb{P}^{s,x}$ a.e. where $M^{s,x}$ is the second item of the solution of $BSDE^{s,x}(f, g)$.

The existence of a decoupled mild solution of $Pseudo - PDE(f, g)$ provides in fact an existence theorem for $BSDE^{s,x}(f, g)$ for any (s, x). The following constitutes the converse of Theorem 5.20.

Proposition 5.23. *Assume (f, g) satisfies $H^{mom}(\zeta, \eta)$. Let (u, v) be a solution of $IP(f, g)$, let $(s, x) \in [0, T] \times E$ and the associated probability $\mathbb{P}^{s,x}$ be fixed. The couple*

$$\left(u(t, X_t), \quad u(t, X_t) - u(s, x) + \int_s^t f(\cdot, \cdot, u, v)(r, X_r)dV_r\right)_{t\in[s,T]} \tag{5.27}$$

has a $\mathbb{P}^{s,x}$-version which solves $BSDE^{s,x}(f, g)$ on $[s, T]$.

In particular if (f, g) satisfies the stronger hypothesis $H^{lip}(\zeta, \eta)$ and (u, v) is the unique solution of $IP(f, g)$, then for any $(s, x) \in [0, T] \times E$,
$\left(u(t, X_t), \quad u(t, X_t) - u(s, x) + \int_s^t f(\cdot, \cdot, u, v)(r, X_r)dV_r\right)_{t\in[s,T]}$ *is a $\mathbb{P}^{s,x}$-modification of the unique solution of $BSDE^{s,x}(f, g)$ on $[s, T]$.*

Proof. It follows from Propositions 5.16 and 5.12. □

6. Examples of applications

We now develop some examples. In all the items below there will be a canonical Markov class with transition kernel being measurable in time which is solution of a martingale Problem associated to some triplet $(\mathcal{D}(a), a, V)$ as introduced in Definition 4.3. Therefore all the results of this paper will apply to all the examples below. In particular, Propositions 5.16, 5.17, Theorem 5.20, Corollary 5.21 and Proposition 5.23 will apply but we will mainly emphasize Theorem 5.20 and Corollary 5.21. In all the examples $T > 0$ will be fixed.

6.1. A new approach to Brownian BSDEs and associate semilinear PDEs

In this first application, the state space will be $E := \mathbb{R}^d$ for some $d \in \mathbb{N}^*$.

Notation 6.1. *A function $\phi \in \mathcal{B}([0, T] \times \mathbb{R}^d, \mathbb{R})$ will be said to have* ***polynomial growth*** *if there exists $p \in \mathbb{N}$ and $C > 0$ such that for every $(t, x) \in [0, T] \times \mathbb{R}^d$, $|\phi(t, x)| \leq C(1 + \|x\|^p)$. For any $k, p \in \mathbb{N}$, $\mathcal{C}^{k,p}([0, T] \times \mathbb{R}^d)$ (resp. $\mathcal{C}_b^{k,p}([0, T] \times \mathbb{R}^d)$, resp. $\mathcal{C}_{pol}^{k,p}([0, T] \times \mathbb{R}^d)$) will denote the sublinear algebra of $\mathcal{C}([0, T] \times \mathbb{R}^d, \mathbb{R})$ of functions admitting continuous (resp. bounded continuous, resp. continuous with polynomial growth) derivatives up to order k in the first variable and order p in the second.*

We consider bounded Borel functions $\mu \in \mathcal{B}_b([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$ and $\alpha \in \mathcal{B}_b([0, T] \times \mathbb{R}^d, S_d^+(\mathbb{R}))$ where $S_d^+(\mathbb{R})$ is the space of symmetric non-negative $d \times d$ real matrices. We define for $\phi \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^d)$ the operator a by

$$a(\phi) = \partial_t \phi + \frac{1}{2} \sum_{i,j\leq d} \alpha_{i,j} \partial^2_{x_i x_j} \phi + \sum_{i\leq d} \mu_i \partial_{x_i} \phi. \tag{6.1}$$

We will assume the following.

Hypothesis 6.2. There exists a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ which solves the Martingale Problem associated to $(\mathcal{C}_b^{1,2}([0, T] \times \mathbb{R}^d), a, V_t \equiv t)$ in the sense of Definition 4.3.

We now recall a non-exhaustive list of sets of conditions on μ, α under which Hypothesis 6.2 is satisfied.

1. α is continuous non-degenerate, in the sense that for any t, x, $\alpha(t, x)$ is invertible, see Theorem 4.2 in [30];
2. μ and α are continuous in the second variable, see Exercise 12.4.1 in [31];
3. $d = 1$ and α is uniformly positive on compact sets, see Exercise 7.3.3 in [31].

Remark 6.3.

- When the item 1. or 3. above is satisfied, the mentioned canonical Markov class is unique, but whenever only 2. holds, uniqueness may fail.
- We emphasize that given a fixed canonical Markov class, we obtain well-posedness results concerning the martingale solution (and so the decoupled mild solution) of an associated PDE.
- Nevertheless, for every canonical Markov class solving the martingale problem could correspond a different solution.

In this context, for ϕ, ψ in $\mathcal{D}(a)$, the carré du champs operator (see Definition 4.6) is given by $\Gamma(\phi, \psi) = \sum\limits_{i,j\leq d} \alpha_{i,j}\partial_{x_i}\phi\partial_{x_j}\psi$.

Remark 6.4. By a localization procedure, it is also clear that for every $(s, x) \in [0, T] \times \mathbb{R}^d$, for any $\phi \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^d)$, $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r)dr \in \mathcal{H}^2_{loc}$ with respect to $\mathbb{P}^{s,x}$. Consequently Proposition 4.7 extends to all $\phi \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^d)$.

We set now $\mathcal{D}(a) = \mathcal{C}^{1,2}_{pol}([0, T] \times \mathbb{R}^d)$.

For any $i \in [\![1; d]\!]$, the function Id_i denotes $(t, x) \longmapsto x_i$ which belongs to $\mathcal{D}(a)$ and $Id := (Id_1, \dots, Id_d)$. It is clear that for any i, $a(Id_i) = \mu_i$, and for any i, j, $Id_i Id_j \in \mathcal{D}(a)$ and $\Gamma(Id_i, Id_j) = \alpha_{i,j}$. In particular, by Corollary 4.9, $(Id_1, \dots, Id_d)$ satisfy Hypothesis 4.8 and, since μ, α are bounded, they satisfy Hypothesis 5.1.

For any i we can therefore consider the MAF $M[Id_i] : (t, u) \mapsto X^i_u - X^i_t - \int_t^u \mu_i(r, X_r)dr$ whose càdlàg version under $\mathbb{P}^{s,x}$ for every $(s, x) \in [0, T] \times \mathbb{R}^d$ is $M[Id_i]^{s,x}$ $= \mathbb{1}_{[s,T]}\left(X^i - x_i - \int_s^\cdot \mu_i(r, X_r)dr\right)$ and for any i, j we have $\langle M[Id_i]^{s,x}, M[Id_j]^{s,x}\rangle = \int_s^{\cdot\vee s} \alpha_{i,j}(r, X_r)dr$.

Lemma 6.5. *Let $(s, x) \in [0, T] \times \mathbb{R}^d$ and associated probability $\mathbb{P}^{s,x}$, $i \in [\![1; d]\!]$ and $p \in [1, +\infty[$ be fixed. Then $\sup\limits_{t\in[s,T]} |X^i_t|^p \in L^1$.*

Proof. We have $X^i = x_i + \int_s^\cdot \mu_i(r, X_r)dr + M[Id_i]^{s,x}$ where μ_i is bounded so it is enough to show that $\sup\limits_{t\in[s,T]} |M[Id_i]^{s,x}_t|^p \in L^1$. Since $\langle M[Id_i]^{s,x}\rangle = \int_s^{\cdot\vee s} \alpha_{i,i}(r, X_r)dr$, which is bounded, the result holds by Burkholder–Davis–Gundy inequality. □

Corollary 6.6. $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ *solves the Martingale Problem associated to $(\mathcal{C}^{1,2}_{pol}([0, T] \times \mathbb{R}^d), a, V_t \equiv t)$ in the sense of Definition 4.3.*

Proof. By Remark 6.4, for any $\phi \in \mathcal{C}^{1,2}_{pol}([0, T] \times \mathbb{R}^d)$ and $(s, x) \in [0, T] \times \mathbb{R}^d$, $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r)dr$ is a $\mathbb{P}^{s,x}$-local martingale. Since ϕ and $a(\phi)$ have polynomial growth, Lemma 6.5 and Jensen's inequality imply that it is also a square integrable martingale. □

We now consider a couple (f, g) satisfying $H^{lip}(\|\cdot\|^p, \|\cdot\|^p)$ for some $p \geq 1$. In this case Hypothesis 5.5 can be retranslated into what follows.

- g is Borel with polynomial growth;
- f is Borel with polynomial growth in x (uniformly in t), and Lipschitz in y, z.

We consider the PDE

$$\begin{cases} \partial_t u + \frac{1}{2}\sum\limits_{i,j\leq d}\alpha_{i,j}\partial^2_{x_i x_j}u + \sum\limits_{i\leq d}\mu_i\partial_{x_i}u + f(\cdot,\cdot,u,\alpha\nabla u) = 0 \\ u(T,\cdot) = g. \end{cases} \tag{6.2}$$

We emphasize that for $u \in \mathcal{C}^{1,2}_{pol}([0,T]\times\mathbb{R}^d)$, $\alpha\nabla u = \Gamma^{Id}(u)$. The associated decoupled mild equation is given by

$$\begin{cases} u(s,x) &=& P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f(\cdot,\cdot,u,v)(r,\cdot)\right](x)dr \\ u(s,x)x_i &=& P_{s,T}[gId_i](x) - \int_s^T P_{s,r}\left[(v_i + u\mu_i \right. \\ & & \left. - Id_i f(\cdot,\cdot,u,v))(r,\cdot)\right](x)dr, \, i \in [\![1;d]\!], \end{cases} \tag{6.3}$$

$(s,x) \in [0,T]\times\mathbb{R}^d$, where P is the transition kernel of the canonical Markov class.

Proposition 6.7. *Assume the validity of Hypothesis 6.2 and that (f,g) satisfies $H^{lip}(\|\cdot\|^p, \|\cdot\|^p)$ for some $p \geq 1$. Then equation (6.2) has a unique decoupled mild solution u.*

Moreover it has at most one classical solution which (when it exists) equals this function u.

Proof. $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ solves a martingale problem in the sense of Definition 4.3 and has a transition kernel which is measurable in time. Moreover $(Id_1,\ldots,Id_d)$ fulfills Hypothesis 5.1, $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ satisfies (by Lemma 6.5) $H^{mom}(\|\cdot\|^p, \|\cdot\|^p)$ for some $p \geq 1$ and (f,g) satisfies $H^{lip}(\|\cdot\|^p, \|\cdot\|^p)$. So Theorem 5.20 and Corollary 5.21 apply. ☐

Remark 6.8. The unique decoupled mild solution mentioned in the previous proposition admits the probabilistic representation given in Theorem 5.20.

Remark 6.9. In the classical literature, the Brownian BSDE (1.1) has been related to a slightly different type of parabolic PDE, i.e.

$$\begin{cases} \partial_t u + \frac{1}{2}\sum\limits_{i,j\leq d}(\sigma\sigma^\intercal)_{i,j}\partial^2_{x_i x_j}u + \sum\limits_{i\leq d}\mu_i\partial_{x_i}u + f(\cdot,\cdot,u,\sigma\nabla u) = 0 \\ u(T,\cdot) = g, \end{cases} \tag{6.4}$$

(where $\sigma = \sqrt{\alpha}$ in the sense of non-negative symmetric matrices) rather than (6.2). In fact, the only difference is that the term $\sigma\nabla u$ replaces $\alpha\nabla u$ in the fourth argument of the driver f. See the introduction of the present paper, or [25] for more details.

Our methodology also allows to represent (6.4). Under the probability $\mathbb{P}^{s,x}$ (for some fixed (s,x)), one can introduce the square integrable martingale $\tilde{M}[Id]^{s,x} := \int_s^{\cdot}(\sigma^\intercal)^+(r,X_r)\, dM[Id]^{s,x}_r$ where $A \mapsto A^+$ denotes the Moore–Penrose pseudo-inverse operator, see [9] chapter 1. The Brownian BSDE (1.1) can then be reexpressed here as

$$Y^{s,x}_t = g(X_T) + \int_t^T f\left(r, X_r, Y^{s,x}_r, \frac{d\langle M^{s,x}, \tilde{M}[Id]^{s,x}\rangle_r}{dr}\right)dr - (M^{s,x}_T - M^{s,x}_t). \tag{6.5}$$

Under the assumptions of Proposition 6.7 where $\alpha = \sigma\sigma^\intercal$, it is possible to show that (6.5) constitutes the probabilistic representation of (6.4) performing similar arguments as in our approach for (6.2). In particular we can show existence and uniqueness of a function $u \in \mathcal{L}^2_X$ for which there exists $v_1, \dots, v_d \in \mathcal{L}^2_X$ such that for all $(s, x) \in [0, T] \times \mathbb{R}^d$,

$$\left\{\begin{array}{rcl} u(s,x) &=& P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(\cdot,\cdot,u,(\sigma^\intercal)^+ v\right)(r,\cdot)\right](x)dr \\ u(s,x)x_i &=& P_{s,T}[gId_i](x) - \int_s^T P_{s,r}\left[\left(v_i + u\mu_i - Id_i f\left(\cdot,\cdot,u,(\sigma^\intercal)^+ v\right)\right)(r,\cdot)\right](x)dr, \\ i &\in& [\![1;d]\!]. \end{array}\right. \tag{6.6}$$

(6.6) constitutes indeed a suitable decoupled mild formulation corresponding to (6.4). Moreover, this function u, whenever it belongs to $\mathcal{C}^{1,2}_{pol}([0,T] \times \mathbb{R}^d)$, is the unique classical solution of (6.4).

This technique is however technically more complicated and for purpose of illustration we prefer to remain in our setup (which is by the way close to (6.4)) to keep our notion of decoupled-mild solution more comprehensible.

Remark 6.10. It is also possible to treat jump diffusions instead of continuous diffusions (see [30]), and under suitable conditions on the coefficients, it is also possible to prove existence and uniqueness of a decoupled mild solution for equations of type

$$\begin{cases} \partial_t u + \frac{1}{2}Tr(\alpha\nabla^2 u) + (\mu, \nabla u) + \int\left(u(\cdot, \cdot + y) - u - \frac{(y,\nabla u)}{1+\|y\|^2}\right)K(\cdot,\cdot,dy) \\ \quad + f(\cdot,\cdot,u,\Gamma^{Id}(u)) = 0 \\ u(T,\cdot) = g, \end{cases} \tag{6.7}$$

where K is a Lévy kernel: this means that for every $(t, x) \in [0, T] \times \mathbb{R}^d$, $K(t, x, \cdot)$ is a σ-finite measure on $\mathbb{R}^d \backslash \{0\}$, $\sup_{t,x} \int \frac{\|y\|^2}{1+\|y\|^2} K(t, x, dy) < \infty$ and for every Borel set $A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$, $(t, x) \longmapsto \int_A \frac{\|y\|^2}{1+\|y\|^2} K(t, x, dy)$ is Borel. In that framework we have

$$\Gamma^{Id} : \phi \longmapsto \alpha\nabla\phi + \left(\int y_i(\phi(\cdot, \cdot + y) - \phi(\cdot,\cdot))K(\cdot,\cdot,dy)\right)_{i\in[\![1;d]\!]}. \tag{6.8}$$

6.2. Parabolic semi-linear PDEs with distributional drift

The context of this subsection is the one introduced by Flandoli, Russo & Wolf in [18] and [19]), see also [14,28] for recent developments. We refer to Section 4.3 of [8] for a more detailed introduction. In particular [18,19] consider stochastic differential equations with distributional drift, whose solution are possibly non-semimartingales. These authors introduced a suitable framework of a martingale problem related to a PDE operator involving a distributional drift b' which is the derivative of a continuous function. [17] approached the n-dimensional setting for the first time and later developments were discussed by [11] studying singular SDEs involving paracontrolled distributions. Other Markov processes associated to diffusion operators which are not semimartingales were produced when the diffusion operator is in divergence form, see e.g. [27]. [29] linked second order ODEs with a distributional coefficient and BSDEs. In those BSDEs the final horizon was a stopping time. [21] and [17] have considered a class of BSDEs involving distributions in their setting.

Let $b, \sigma \in \mathcal{C}^0(\mathbb{R})$ such that $\sigma > 0$. In [18], the authors introduce a (generalized) notion for the equation $Lf = \ell$, for $f \in C^1(\mathbb{R})$. They suppose the existence of a function $\Sigma : \mathbb{R} \to \mathbb{R}$ which formally equals $2\int_0^{\cdot} \frac{b'}{\sigma^2}(y)dy$ and it is defined via mollification. A typical situation when Σ exists arises when either b or σ^2 have locally bounded variation. If Σ exists then the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(0) = 0$ and $h' = e^{-\Sigma}$ is *L-harmonic* function, in the sense that it fulfills $Lh = 0$, see Proposition 2.3 of [18]. $\mathcal{D}_L$ is defined as the set of $f \in \mathcal{C}^1(\mathbb{R})$ such that there exists some $\ell \in \mathcal{C}^0(\mathbb{R})$ with $Lf = \ell$ and it is a linear algebra.

Let v be the unique solution to $Lv = 1$, $v(0) = v'(0) = 0$, see Remark 2.4 in [18]; we will assume

$$v(-\infty) = v(+\infty) = +\infty, \tag{6.9}$$

which represents a non-explosion condition. In this case, Proposition 3.13 in [18] states that a certain martingale problem associated to $(\mathcal{D}_L, L)$ is well-posed. Its solution will be denoted $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$.

The canonical process X is a $\mathbb{P}^{s,x}$-Dirichlet process for every (s, x), i.e. the sum of a local martingale and a zero quadratic variation process and it is a semimartingale if and only if Σ is locally of bounded variation, see Corollary 5.11 in [19]. $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ defines a canonical Markov class and Proposition B.2 in [8] implies that its transition kernel is measurable in time.

We introduce below the domain that we will indeed use.

Definition 6.11. We set

$$\mathcal{D}^{max}(a) = \left\{\phi \in \mathcal{C}^{1,1}([0, T] \times \mathbb{R}) : \frac{\partial_x \phi}{h'} \in \mathcal{C}^{1,1}([0, T] \times \mathbb{R})\right\}. \tag{6.10}$$

On $\mathcal{D}^{max}(a)$, we set $L\phi := \frac{\sigma^2 h'}{2}\partial_x(\frac{\partial_x\phi}{h'})$ and $a(\phi) := \partial_t\phi + L\phi$. We then define the smaller domain

$$\mathcal{D}(a) = \left\{\phi \in \mathcal{D}^{max}(a) : \sigma\partial_x\phi \in \mathcal{C}^{0,0}_{pol}([0, T] \times \mathbb{R})\right\}. \tag{6.11}$$

We formulate here some supplementary assumptions that we will make, the first one being called (TA) in [18].

Hypothesis 6.12.

- There exist $c_1, C_1 > 0$ such that $c_1 \le \sigma h' \le C_1$;
- σ has linear growth.

The first item states in particular that $\sigma h'$ is bounded so $h \in \mathcal{D}(a)$. Proposition 3.2 in [18] states that for every (s, x), $\langle M[h]^{s,x}\rangle = \int_s^{\cdot\vee s}(\sigma h')^2(X_r)dr$. Moreover the AF defined by $\langle M[h]\rangle_u^t = \int_t^u(\sigma h')^2(X_r)dr$, is absolutely continuous with respect to $\hat{V}_t \equiv t$. Therefore Hypothesis 4.8 is satisfied (for $\psi = h$) and $\mathfrak{G}^h(h) = (\sigma h')^2$. Since this function is bounded and clearly $a(h) = 0$ then h satisfies Hypothesis 5.1.

We will therefore consider the h-generalized gradient Γ^h associated to a; Proposition 4.23 in [8] implies the following.

Proposition 6.13. *Let $\phi \in \mathcal{D}(\Gamma^h)$, then $\Gamma^h(\phi) = \sigma^2 h'\partial_x\phi$.*

The deterministic equation considered in this section is a semilinear PDE with singular (or distributional) drift b' given by

$$\begin{cases} \partial_t u + \frac{1}{2}\sigma^2\partial_x^2 u + b'\partial_x u + f(\cdot,\cdot,u,\sigma^2 h'\partial_x u) = 0 & \text{on } [0,T]\times\mathbb{R} \\ u(T,\cdot) = g. \end{cases} \tag{6.12}$$

The associated PDE in the decoupled mild sense is given by

$$\begin{cases} u(s,x) &= P_{T-s}[g](x) + \int_s^T P_{r-s}\left[f(\cdot,\cdot,u,v)(r,\cdot)\right](x)dr \\ u(s,x)h(x) &= P_{T-s}[gh](x) - \int_s^T P_{r-s}\left[(v - hf(\cdot,\cdot,u,v))(r,\cdot)\right](x)dr, \end{cases} \tag{6.13}$$

$(s,x) \in [0,T]\times\mathbb{R}$, where P is the (time-homogeneous) transition kernel of the canonical Markov class.

In order to consider the $BSDE^{s,x}(f,g)$ for functions (f,g) having polynomial growth in x, we had shown in [8] the following result, stated as Proposition 4.24.

Proposition 6.14. *We suppose that Hypothesis 6.12 is fulfilled. Then, for any $p\in\mathbb{N}$ and $(s,x)\in[0,T]\times\mathbb{R}$, $\mathbb{E}^{s,x}[|X_T|^p] < \infty$ and $\mathbb{E}^{s,x}[\int_s^T |X_r|^p dr] < \infty$. In other words, for any $p\geq 1$, the canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ satisfies $H^{mom}(|\cdot|^p, |\cdot|^p)$, see Hypothesis 5.4.*

Next we have the following.

Proposition 6.15. *We suppose that Hypothesis 6.12 is fulfilled. Then $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ solves the Martingale Problem associated to $(a, \mathcal{D}(a), V_t\equiv t)$ in the sense of Definition 4.3.*

Proof. Let $(s,x)\in[0,T]\times\mathbb{R}$ be fixed. Proposition 4.23 in [8] implies that for any $\phi\in\mathcal{D}(a)$, $\phi(\cdot,X_\cdot) - \int_s^\cdot a(\phi)(r,X_r)dr$ is a (continuous) $\mathbb{P}^{s,x}$-local martingale, so taking Definition 4.3 into account, it is enough to show that this local martingale is a square integrable martingale. Considering Definition 4.21, Proposition 4.22 and Proposition 2.6 in [8], we know that the angular bracket of this local martingale is given by $\int_s^\cdot (\sigma\partial_x\phi)^2(r,X_r)dr$. Since $\phi\in\mathcal{D}(a)$ then $\sigma\partial_x\phi$ has polynomial growth, so by Proposition 6.14, $\int_s^T(\sigma\partial_x\phi)^2(r,X_r)dr\in L^1$ and this implies that the aforementioned local martingale is a square integrable martingale. □

We can now state the main result of this section.

Proposition 6.16. *Assume the non-explosion condition (6.9), Hypothesis 6.12 and that (f,g) satisfies $H^{lip}(|\cdot|^p,|\cdot|^p)$ for some $p\geq 1$, see Hypothesis 5.5. Then, Eq. (6.12) has a unique decoupled mild solution u. Moreover, there is at most one classical solution which can only be equal to u.*

Proof. The assertions come from Theorem 5.20 and Corollary 5.21 which applies thanks to Propositions 6.15, 6.14, and the fact that h satisfies Hypothesis 5.1. □

Remark 6.17. The unique decoupled mild solution u can be of course represented by (5.26), Theorem 5.20.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proof of Theorem 3.3 and related technicalities

We adopt here the same notations as at the beginning of Section 3. We will denote $L^2(d\hat{V}\otimes d\mathbb{P})$ the quotient space of $\mathcal{L}^2(d\hat{V}\otimes d\mathbb{P})$ with respect to the subspace of processes equal to zero $d\hat{V}\otimes d\mathbb{P}$ a.e.

$L^2(d\hat{V}\otimes d\mathbb{P})$ is a Hilbert space equipped with its usual norm. $L^{2,cadlag}(d\hat{V}\otimes d\mathbb{P}))$ will stand for the subspace of $L^2(d\hat{V}\otimes d\mathbb{P}))$ of elements having a càdlàg representative. We emphasize that $L^{2,cadlag}(d\hat{V}\otimes d\mathbb{P})$ is not a closed subspace of $L^2(d\hat{V}\otimes d\mathbb{P})$. The application which to a process associate its class will be denoted $\phi\mapsto\dot{\phi}$.

Proposition A.1. *If (Y,M) solves $BSDE(\xi,\hat{f},V,\hat{M})$, and if we denote $\hat{f}\left(r,\cdot,Y_r,\frac{d\langle M,\hat{M}\rangle}{d\hat{V}}(r)\right)$ by $\hat{f}_r$, then for any $t\in[0,T]$, a.s. we have*

$$\left\{\begin{array}{lcl} Y_t &=& \mathbb{E}\left[\xi+\int_t^T\hat{f}_r d\hat{V}_r\middle|\mathcal{F}_t\right]\\ M_t &=& \mathbb{E}\left[\xi+\int_0^T\hat{f}_r d\hat{V}_r\middle|\mathcal{F}_t\right]-\mathbb{E}\left[\xi+\int_0^T\hat{f}_r d\hat{V}_r\middle|\mathcal{F}_0\right].\end{array}\right. \tag{A.1}$$

Proof. Since $Y_t=\xi+\int_t^T\hat{f}_r d\hat{V}_r-(M_T-M_t)$ a.s., Y being an adapted process and M a martingale, taking the expectation in (3.2) at time t, we directly get $Y_t=\mathbb{E}\left[\xi+\int_t^T\hat{f}_r d\hat{V}_r\middle|\mathcal{F}_t\right]$ and in particular that $Y_0=\mathbb{E}\left[\xi+\int_0^T\hat{f}_r d\hat{V}_r\middle|\mathcal{F}_0\right]$. Since $M_0=0$, looking at the BSDE at time 0 we get

$$M_T=\xi+\int_0^T\hat{f}_r d\hat{V}_r-\mathbb{E}\left[\xi+\int_0^T\hat{f}_r d\hat{V}_r\middle|\mathcal{F}_0\right].$$

Taking the expectation with respect to $\mathcal{F}_t$ in the above inequality, gives the second line of (A.1). □

Lemma A.2. *Let $M\in\mathcal{H}^2$ and ϕ be a bounded positive process. Then there exists a constant $C>0$ such that for any $i\in[\![1;d]\!]$,*
$\int_0^T\phi_r\left(\frac{d\langle M,\hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r\leq C\int_0^T\phi_r d\langle M\rangle_r$. In particular, $\frac{d\langle M,\hat{M}^i\rangle}{d\hat{V}}$ belongs to $L^2(d\hat{V}\otimes d\mathbb{P})$.

Proof. We fix $i\in[\![1;d]\!]$. By Hypothesis 3.1 $\frac{d\langle\hat{M}^i\rangle}{d\hat{V}}$ is bounded; using Proposition B.1 and Remark 3.3 in [6], we show the existence of $C>0$ such that

$$\begin{array}{rcl}\int_0^T\phi_r\left(\frac{d\langle M,\hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r &\leq& \int_0^T\phi_r\frac{d\langle\hat{M}^i\rangle}{d\hat{V}}(r)\frac{d\langle M\rangle}{d\hat{V}}(r)d\hat{V}_r\\ &\leq& C\int_0^T\phi_r\frac{d\langle M\rangle}{d\hat{V}}(r)d\hat{V}_r\\ &\leq& C\int_0^T\phi_r d\langle M\rangle_r.\end{array} \tag{A.2}$$

In particular, setting $\phi = 1$, we have $\int_0^T \left(\frac{d\langle M,\hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r \leq C\langle M\rangle_T$ which belongs to L^1 since $M \in \mathcal{H}^2_0$. $\square$

We fix for now a couple $(\dot{U}, N) \in L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}^2_0$ and we consider a representative U of $\dot{U}$. Until Proposition A.6 included, we will use the notation $\hat{f}_r := \hat{f}\left(r, \cdot, U_r, \frac{d\langle N,\hat{M}\rangle}{d\hat{V}}(r)\right)$.

Proposition A.3. *For any $t \in [0, T]$, $\int_t^T \hat{f}_r^2 d\hat{V}_r$ belongs to L^1 and $\left(\int_t^T \hat{f}_r d\hat{V}_r\right)$ is in L^2.*

Proof. By Jensen's inequality and by Lemma A.2, taking into account the Lipschitz conditions on $\hat{f}$ in Hypothesis 3.1, there exist positive constants C, C', C" such that, for any $t \in [0, T]$, we have

$$\begin{array}{rcl}\left(\int_t^T \hat{f}_r d\hat{V}_r\right)^2 & \leq & C\int_t^T \hat{f}_r^2 d\hat{V}_r \\ & \leq & C'\left(\int_t^T \hat{f}^2\left(r,\cdot,0,0\right)d\hat{V}_r + \int_t^T U_r^2 d\hat{V}_r + \sum_{i=1}^d \int_t^T \left(\frac{d\langle N,\hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r\right) \\ & \leq & C"\left(\int_t^T \hat{f}^2\left(r,\cdot,0,0\right)d\hat{V}_r + \int_t^T U_r^2 d\hat{V}_r + (\langle N\rangle_T - \langle N\rangle_t)\right).\end{array} \tag{A.3}$$

All terms on the right-hand side are in L^1. Indeed, N is taken in $\mathcal{H}^2$, $\dot{U}$ in $L^2(d\hat{V} \otimes d\mathbb{P})$ and by Hypothesis 3.1, $f(\cdot, \cdot, 0, 0)$ is in $\mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$. This concludes the proof. $\square$

We can therefore state the following definition.

Definition A.4. The random function

$$t \mapsto \mathbb{E}\left[\xi + \int_0^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_0\right], \tag{A.4}$$

is a square integrable martingale by Proposition A.3. Since the stochastic basis fulfills the usual conditions, by Theorem 4 in Chapter IV of [15], (A.4) admits a càdlàg version, that we denote M. We denote by Y the càdlàg process defined by $Y_t = \xi + \int_t^T \hat{f}_r d\hat{V}_r - (M_T - M_t)$. This will be called the **càdlàg reference process** and we will omit its dependence to $(\dot{U}, N)$.

Proposition A.5. *Y and M are square integrable processes.*

Proof. We already know that M is a square integrable martingale. As we have seen in Proposition A.3, $\int_t^T \hat{f}_r d\hat{V}_r$ belongs to L^2 for any $t \in [0, T]$ and by Hypothesis 3.1, $\xi \in L^2$. So by (A.1) and Jensen's inequality for conditional expectation we have

$$\begin{array}{rcl}\mathbb{E}\left[Y_t^2\right] & = & \mathbb{E}\left[\mathbb{E}\left[\xi + \int_t^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_t\right]^2\right] \\ & \leq & \mathbb{E}\left[\mathbb{E}\left[\left(\xi + \int_t^T \hat{f}_r d\hat{V}_r\right)^2 \middle| \mathcal{F}_t\right]\right] \\ & \leq & \mathbb{E}\left[2\xi^2 + 2\int_t^T \hat{f}_r^2 d\hat{V}_r\right],\end{array}$$

which is finite. $\square$

Proposition A.6. $\sup_{t\in[0,T]} |Y_t| \in L^2$ *and in particular,* $Y \in \mathcal{L}^{2,cadlag}(d\hat{V}\otimes\mathbb{P})$.

Proof. Since $dY_r = -\hat{f}_r d\hat{V}_r + dM_r$, by integration by parts formula we get

$$d(Y_r^2 e^{-\hat{V}_r}) = -2e^{-\hat{V}_r} Y_r \hat{f}_r d\hat{V}_r + 2e^{-\hat{V}_r} Y_{r-} dM_r + e^{-\hat{V}_r} d[M]_r - e^{-\hat{V}_r} Y_r^2 d\hat{V}_r.$$

So integrating from 0 to some $t\in[0,T]$, yields

$$\begin{array}{rcl} Y_t^2 e^{-\hat{V}_t} & = & Y_0^2 - 2\int_0^t e^{-\hat{V}_r} Y_r \hat{f}_r d\hat{V}_r + 2\int_0^t e^{-\hat{V}_r} Y_{r-} dM_r \\ & & + \int_0^t e^{-\hat{V}_r} d[M]_r - \int_0^t e^{-\hat{V}_r} Y_r^2 d\hat{V}_r \\ & \leq & Y_0^2 + \int_0^t e^{-\hat{V}_r} Y_r^2 d\hat{V}_r + \int_0^t e^{-\hat{V}_r} \hat{f}_r^2 d\hat{V}_r \\ & & +2\left|\int_0^t e^{-\hat{V}_r} Y_{r-} dM_r\right| + \int_0^t e^{-\hat{V}_r} d[M]_r - \int_0^t e^{-\hat{V}_r} Y_r^2 d\hat{V}_r \\ & \leq & Z + 2\left|\int_0^t e^{-\hat{V}_r} Y_{r-} dM_r\right|, \end{array}$$

where $Z = Y_0^2 + \int_0^T e^{-\hat{V}_r} \hat{f}_r^2 d\hat{V}_r + \int_0^T e^{-\hat{V}_r} d[M]_r$. Therefore, for any $t\in[0,T]$ we have $(Y_t e^{-\hat{V}_t})^2 \leq Y_t^2 e^{-\hat{V}_t} \leq Z + 2\left|\int_0^t e^{-\hat{V}_r} Y_{r-} dM_r\right|$. Thanks to Propositions A.3 and A.5, Z is integrable, so we can conclude by Lemma 3.18 in [6] applied to the process $Ye^{-\hat{V}}$, and the fact that $\hat{V}$ is bounded.

Since Y is càdlàg progressively measurable, $\sup_{t\in[0,T]} |Y_t| \in L^2$ and since $\hat{V}$ is bounded, it is clear that $Y \in \mathcal{L}^{2,cadlag}(d\hat{V}\otimes d\mathbb{P})$ and the corresponding class $\dot{Y}$ belongs to $L^{2,cadlag}(d\hat{V}\otimes d\mathbb{P})$. $\square$

Thanks to Propositions A.5 and A.6, we are allowed to introduce the following.

Notation A.7. *We denote by Φ the operator which associates to a couple $(\dot{U},N)$ the couple $(\dot{Y},M)$.*

$$\Phi : \begin{array}{rcl} L^2(d\hat{V}\otimes d\mathbb{P})\times\mathcal{H}_0^2 & \longrightarrow & L^{2,cadlag}(d\hat{V}\otimes d\mathbb{P})\times\mathcal{H}_0^2 \\ (\dot{U},N) & \longmapsto & (\dot{Y},M). \end{array}$$

Proposition A.8. *The mapping $(Y,M)\longmapsto(\dot{Y},M)$ induces a bijection between the set of solutions of $BSDE(\xi,\hat{f},\hat{V},\hat{M})$ and the set of fixed points of Φ.*

Proof. First, let (U,N) be a solution of $BSDE(\xi,\hat{f},V,\hat{M})$, let $(\dot{Y},M):=\Phi(\dot{U},N)$ and let Y be the reference càdlàg process associated to U as in Definition A.4. By this same definition, M is the càdlàg version of
$t\mapsto \mathbb{E}\left[\xi + \int_0^T \hat{f}\left(r,\cdot,U_r,\frac{d\langle N,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r\middle|\mathcal{F}_t\right]$
$-\mathbb{E}\left[\xi + \int_0^T \hat{f}\left(r,\cdot,U_r,\frac{d\langle N,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r\middle|\mathcal{F}_0\right]$, but by Proposition A.1, so is N, meaning $M=N$. Again by Definition A.4, $Y=\xi+\int_\cdot^T \hat{f}\left(r,\cdot,U_r,\frac{d\langle N,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r-(N_T-N_\cdot)$ which is equal to U thanks to (3.2), consequently $Y=U$ in the sense of indistinguishability. In particular, $\dot{U}=\dot{Y}$, implying $(\dot{U},N)=(\dot{Y},M)=\Phi(\dot{U},N)$. Therefore, the mapping $(Y,M)\longmapsto(\dot{Y},M)$ does indeed map the set of solutions of $BSDE(\xi,\hat{f},V,\hat{M})$ into the set of fixed points of Φ.

The map Φ is surjective. Indeed let $(\dot{U},N)$ be a fixed point of Φ, the couple (Y,M) of Definition A.4 satisfies $Y=\xi+\int_\cdot^T \hat{f}\left(r,\cdot,U_r,\frac{d\langle N,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r-(M_T-M_\cdot)$ in the sense of indistinguishability, and $(\dot{Y}, M) = \Phi(\dot{U}, N) = (\dot{U}, N)$, so by Lemma 3.9 in [6], $\int_\cdot^T \hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r$ and $\int_\cdot^T \hat{f}\left(r, \cdot, U_r, \frac{d\langle N, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r$ are indistinguishable and $Y = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M_T - M_\cdot)$, meaning that (Y, M) (which is a preimage of $(\dot{U}, N)$) solves $BSDE(\xi, \hat{f}, V, \hat{M})$.

We finally show that it is injective. Let us consider two solutions (Y, M) and (Y', M) of $BSDE(\xi, \hat{f}, V, \hat{M})$ with $\dot{Y} = \dot{Y}'$. By Lemma 3.9 in [6] the processes $\int_\cdot^T \hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r$ and $\int_\cdot^T \hat{f}\left(r, \cdot, Y'_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r$ are indistinguishable, so taking (3.2) into account, we have $Y = Y'$. □

Proposition A.9. *Let $\lambda \in \mathbb{R}$, let $(\dot{U}, N)$, $(\dot{U}', N') \in L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$, let $(\dot{Y}, M)$, $(\dot{Y}', M')$ be their images through Φ and let Y, Y' be the càdlàg representatives of $\dot{Y}$, $\dot{Y}'$ introduced in Definition A.4. Then $\int_0^\cdot e^{\lambda \hat{V}_r} Y_{r-} dM_r$, $\int_0^\cdot e^{\lambda \hat{V}_r} Y'_{r-} dM'_r$, $\int_0^\cdot e^{\lambda \hat{V}_r} Y_{r-} dM'_r$ and $\int_0^\cdot e^{\lambda \hat{V}_r} Y'_{r-} dM_r$ are martingales.*

Proof. $\hat{V}$ is bounded and thanks to Proposition A.6 we know that $\sup_{t\in[0,T]} |Y_t|$ and $\sup_{t\in[0,T]} |Y'_t|$ are L^2. Moreover, since M and M' are square integrable, the statement yields therefore as a consequence of Lemma 3.17 in [6]. □

Starting from now, if $(\dot{Y}, M)$ is the image by Φ of some $(\dot{U}, N) \in L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$, by default, we will always refer to the càdlàg reference process Y of $\dot{Y}$ defined in Definition A.4.

For any $\lambda \geq 0$, on $L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ we define the norm $\|(\dot{Y}, M)\|_\lambda^2 := \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} Y_r^2 d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} d\langle M\rangle_r\right]$. Since $\hat{V}$ is bounded, these norms are all equivalent.

Proposition A.10. *There exists $\lambda > 0$ such that for any $(\dot{U}, N) \in L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$, $\left\|\Phi(\dot{U}, N)\right\|_\lambda^2 \leq \frac{1}{2}\left\|(\dot{U}, N)\right\|_\lambda^2$. In particular, Φ is a contraction in $L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ for the norm $\|\cdot\|_\lambda$.*

Proof.

Let $(\dot{U}, N)$ and $(\dot{U}', N')$ be two couples belonging to $L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$, let $(\dot{Y}, M)$ and $(\dot{Y}', M')$ be their images via Φ and let Y, Y' be the càdlàg reference process of $\dot{Y}$, $\dot{Y}'$ introduced in Definition A.4. We will write $\bar{Y}$ for $Y - Y'$ and we adopt a similar notation for other processes. We will also write

$\bar{f}_t := \hat{f}\left(t, \cdot, U_t, \frac{d\langle N, \hat{M}\rangle}{d\hat{V}}(t)\right) - \hat{f}\left(t, \cdot, U'_t, \frac{d\langle N', \hat{M}\rangle}{d\hat{V}}(t)\right)$.

By additivity, we have $d\bar{Y}_t = -\bar{f}_t d\hat{V}_t + d\bar{M}_t$. Since $\bar{Y}_T = \xi - \xi = 0$, applying the integration by parts formula to $\bar{Y}_t^2 e^{\lambda \hat{V}_t}$ between 0 and T we get

$$\bar{Y}_0^2 - 2\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r \bar{f}_r d\hat{V}_r + 2\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_{r-} d\bar{M}_r + \int_0^T e^{\lambda \hat{V}_r} d[\bar{M}]_r + \lambda \int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r^2 d\hat{V}_r = 0.$$

Since, by Proposition A.9, the stochastic integral with respect to $\bar{M}$ is a real martingale, by taking the expectations we get

$$\mathbb{E}\left[\bar{Y}_0^2\right] - 2\mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r \bar{f}_r d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} d\langle \bar{M}\rangle_r\right] + \lambda \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r^2 d\hat{V}_r\right] = 0.$$

So by re-arranging previous expression, by the Lipschitz condition on $\hat{f}$ stated in Hypothesis 3.1, by the linearity of the Radon–Nikodym derivative and by Lemma A.2, we get

$$\begin{aligned}
& \lambda\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{Y}_r^2 d\hat{V}_r\right]+\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}d\langle\bar{M}\rangle_r\right]\\
\leq\ & 2\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}|\bar{Y}_r\|\bar{f}_r|d\hat{V}_r\right]\\
\leq\ & 2K^Y\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}|\bar{Y}_r\|\bar{U}_r|d\hat{V}_r\right]+2K^Z\sum_{i=1}^d\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}|\bar{Y}_r|\left|\frac{d\langle\bar{N},\hat{M}^i\rangle}{d\hat{V}}(r)\right|d\hat{V}_r\right]\\
\leq\ & (K^Y\alpha+dK^Z\beta)\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{Y}_r^2 d\hat{V}_r\right]+\frac{K^Y}{\alpha}\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{U}_r^2 d\hat{V}_r\right]\\
& +\frac{K^Z}{\beta}\sum_{i=1}^d\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\left(\frac{d\langle\bar{N},\hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r\right]\\
\leq\ & (K^Y\alpha+dK^Z\beta)\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{Y}_r^2 d\hat{V}_r\right]+\frac{K^Y}{\alpha}\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{U}_r^2 d\hat{V}_r\right]\\
& +\frac{CdK^Z}{\beta}\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}d\langle\bar{N}\rangle_r\right],
\end{aligned}$$

for some positive C and any positive α and β. The latter equality holds by Hypothesis 3.1 4. Then we pick $\alpha=2K^Y$ and $\beta=2CdK^Z$, which gives us

$$\begin{aligned}
& \lambda\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{Y}_r^2 d\hat{V}_r\right]+\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}d\langle\bar{M}\rangle_r\right]\\
\leq\ & 2((K^Y)^2+C(dK^Z)^2)\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{Y}_r^2 d\hat{V}_r\right]\\
+\ & \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{U}_r^2 d\hat{V}_r\right]+\frac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}d\langle\bar{N}\rangle_r\right].
\end{aligned}$$

We choose now $\lambda=1+2((K^Y)^2+C(dK^Z)^2)$ and we get

$$\begin{aligned}
& \mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{Y}_r^2 d\hat{V}_r\right]+\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}d\langle\bar{M}\rangle_r\right]\\
\leq\ & \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{U}_r^2 d\hat{V}_r\right]+\frac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}d\langle\bar{N}\rangle_r\right],
\end{aligned}\tag{A.5}$$

which proves the contraction for the norm $\|\cdot\|_\lambda$. □

Proof of Theorem 3.3. The space $L^2(d\hat{V}\otimes d\mathbb{P})\times\mathcal{H}_0^2$ is complete and Φ defines on it a contraction for the norm $\|(\cdot,\cdot)\|_\lambda$ for some $\lambda>0$, so Φ has a unique fixed point in $L^2(d\hat{V}\otimes d\mathbb{P})\times\mathcal{H}_0^2$. Then by Proposition A.8, $BSDE(\xi,\hat{f},V,\hat{M})$ has a unique solution. □

Remark A.11. Let (Y,M) be the solution of $BSDE(\xi,\hat{f},V,\hat{M})$ and $\dot{Y}$ the class of Y in $L^2(d\hat{V}\otimes d\mathbb{P})$. Thanks to Proposition A.8, we know that $(\dot{Y},M)=\Phi(\dot{Y},M)$ and therefore by Propositions A.6 and A.9 that $\sup_{t\in[0,T]}|Y_t|$ is L^2 and that $\int_0^\cdot Y_{r-}dM_r$ is a real martingale.

The lemma below shows that, in order to check if a couple (Y,M) is the solution of $BSDE(\xi,\hat{f},V,\hat{M})$, it is not necessary to verify the square integrability of Y since it will be automatically fulfilled.

Lemma A.12. *We consider $(\xi,\hat{f},V,\hat{M})$ such that $\xi,\hat{M}$ satisfy items 1., 2. of Hypothesis 3.1 but where item 3. is replaced by the weaker following hypothesis on $\hat{f}$. There exists $C>0$ such that $\mathbb{P}$ a.s., for all t,y,z,*

$$|\hat{f}(t,\omega,y,z)|\leq C(1+|y|+\|z\|).\tag{A.6}$$

Assume that there exists a càdlàg adapted process Y with $Y_0 \in L^2$, and $M \in \mathcal{H}^2_0$ such that

$$Y = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M_T - M_\cdot), \tag{A.7}$$

in the sense of indistinguishability. Then $\sup_{t\in[0,T]} |Y_t|$ is L^2. In particular, $Y \in \mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$ and if $(\xi, \hat{f}, V, \hat{M})$ satisfies Hypothesis 3.1, then (Y, M) is the unique solution of $BSDE(\xi, \hat{f}, V, \hat{M})$ in the sense of Definition 3.2.

On the other hand if (Y, M) satisfies (A.7) on $[s, T]$ with $s < T$, $Y_s \in L^2$ and $M_s = 0$ then $\sup_{t\in[s,T]} |Y_t|$ is L^2. Consequently if $(\xi, \hat{f}, V, \hat{M})$ satisfies Hypothesis 3.1 and if we denote (U, N) the unique solution of $BSDE(\xi, \hat{f}, V, \hat{M})$, then (Y, M) and $(U, N_\cdot - N_s)$ are indistinguishable on $[s, T]$.

Proof.

Let $\lambda > 0$ and $t \in [0, T]$. By integration by parts formula applied to $Y^2 e^{-\lambda\hat{V}}$ between 0 and t we get

$$\begin{aligned} Y_t^2 e^{-\lambda\hat{V}_t} - Y_0^2 \quad = \quad & -2\int_0^t e^{-\lambda\hat{V}_r} Y_r \hat{f}\left(r, \cdot, Y_r, \tfrac{d\langle M,\hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r + 2\int_0^t e^{-\lambda\hat{V}_r} Y_{r-} dM_r \\ & + \int_0^t e^{-\lambda\hat{V}_r} d[M]_r - \lambda\int_0^t e^{-\lambda\hat{V}_r} Y_r^2 d\hat{V}_r. \end{aligned}$$

By re-arranging the terms and using the Lipschitz conditions stated in item 3. of in Hypothesis 3.1, we get

$$\begin{aligned} & Y_t^2 e^{-\lambda\hat{V}_t} + \lambda\int_0^t e^{-\lambda\hat{V}_r} Y_r^2 d\hat{V}_r \\ \leq \quad & Y_0^2 + 2\int_0^t e^{-\lambda\hat{V}_r} |Y_r \parallel \hat{f}|\left(r, \cdot, Y_r, \tfrac{d\langle M,\hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r + 2\left|\int_0^t e^{-\lambda\hat{V}_r} Y_{r-} dM_r\right| \\ & + \int_0^t e^{-\lambda\hat{V}_r} d[M]_r \\ \leq \quad & Y_0^2 + \int_0^t e^{-\lambda\hat{V}_r} \hat{f}^2(r, \cdot, 0, 0) d\hat{V}_r + (2K^Y + 1 + K^Z)\int_0^t e^{-\lambda\hat{V}_r} Y_r^2 d\hat{V}_r \\ & + K^Z \sum_{i=1}^d \int_0^t e^{-\lambda\hat{V}_r} \left(\tfrac{d\langle M,\hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r + 2\left|\int_0^t e^{-\lambda\hat{V}_r} Y_{r-} dM_r\right| + \int_0^t e^{-\lambda\hat{V}_r} d[M]_r. \end{aligned}$$

Picking $\lambda = 2K^Y + 1 + K^Z$ and using Lemma A.2, this gives

$$\begin{aligned} Y_t^2 e^{-\lambda\hat{V}_t} \quad \leq \quad & Y_0^2 + \int_0^t e^{-\lambda\hat{V}_r} |\hat{f}|^2(r, \cdot, 0, 0) d\hat{V}_r + K^Z C\int_0^t e^{-\lambda\hat{V}_r} d\langle M\rangle_r \\ & + 2\left|\int_0^t e^{-\lambda\hat{V}_r} Y_{r-} dM_r\right| + \int_0^t e^{-\lambda\hat{V}_r} d[M]_r, \end{aligned}$$

for some $C > 0$. Since $\hat{V}$ is bounded, there is a constant $C' > 0$, such that for any $t \in [0, T]$

$$Y_t^2 \leq C'\left(Y_0^2 + \int_0^T |\hat{f}|^2(r, \cdot, 0, 0) d\hat{V}_r + \langle M\rangle_T + [M]_T + \left|\int_0^t Y_{r-} dM_r\right|\right).$$

By Hypothesis 3.1, $Y_0 \in L^2$ and $M \in \mathcal{H}^2$, the first four terms on the right-hand side are integrable so that we can conclude by Lemma 3.18 in [6].

An analogous proof also holds on the interval $[s, T]$ taking into account Remark 3.4. In particular, if (U, N) is the unique solution of $BSDE(\xi, \hat{f}, V, \hat{M})$ then $(U, N - N_s)$ is a solution on $[s, T]$. The final statement result follows by the uniqueness argument of Remark 3.4. □

Notation A.13. *Let $\Phi : L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}^2_0$ be the operator introduced in Notation A.7.*

In the sequel we will not distinguish between a couple $(\dot{Y}, M)$ *in* $L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}^2_0$ *and* (Y, M)*, where* Y *is the reference càdlàg process of* $\dot{Y}$*, according to Definition A.4. We then convene the following.*

1. $(Y^0, M^0) := (0, 0)$*;*
2. $\forall k \in \mathbb{N}^* : (Y^k, M^k) := \Phi(Y^{k-1}, M^{k-1})$*,*

meaning that for $k \in \mathbb{N}^*$*,* (Y^k, M^k) *is the solution of the BSDE*

$$Y^k = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, Y^{k-1}, \frac{d\langle M^{k-1}, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M^k_T - M^k_\cdot). \tag{A.8}$$

Definition A.14. The processes $(Y^k, M^k)_{k\in\mathbb{N}}$ will be called the Picard iterations associated to $BSDE(\xi, \hat{f}, \hat{V}, \hat{M})$.

We know that Φ is a contraction in $L^2(d\hat{V} \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}^2_0$ for a certain norm, so that (Y^k, M^k) tends to (Y, M) in this topology. The proposition below also shows an a.e. corresponding convergence, adapting the techniques of Corollary 2.1 in [16].

Proposition A.15. $Y^k \underset{k\to\infty}{\longrightarrow} Y \quad d\hat{V} \otimes d\mathbb{P}$ *a.e. and for any* $i \in [\![1; d]\!]$*,* $\frac{d\langle M^k, \hat{M}^i\rangle}{d\hat{V}} \underset{k\to\infty}{\longrightarrow} \frac{d\langle M, \hat{M}^i\rangle}{d\hat{V}}$ $d\hat{V} \otimes d\mathbb{P}$ *a.e.*

Proof. For any $i \in [\![1; d]\!]$ and $k \in \mathbb{N}$ we set $Z^{i,k} := \frac{d\langle M^k, \hat{M}^i\rangle}{d\hat{V}}$ and $Z^i := \frac{d\langle M, \hat{M}^i\rangle}{d\hat{V}}$. By Proposition A.10, there exists $\lambda > 0$ such that for any $k \in \mathbb{N}^*$

$$\begin{aligned} &\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Y^{k+1}_r - Y^k_r|^2 d\hat{V}_r + \int_0^T e^{-\lambda\hat{V}_r} d\langle M^{k+1} - M^k\rangle_r\right] \\ \leq\ &\tfrac{1}{2}\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Y^k_r - Y^{k-1}_r|^2 d\hat{V}_r + \int_0^T e^{-\lambda\hat{V}_r} d\langle M^k - M^{k-1}\rangle_r\right], \end{aligned}$$

consequently

$$\begin{aligned} &\sum_{k\geq 0}\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Y^{k+1}_r - Y^k_r|^2 d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r} d\langle M^{k+1} - M^k\rangle_r\right] \\ \leq\ &\sum_{k\geq 0}\frac{1}{2^k}\left(\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Y^1_r|^2 d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r} d\langle M^1\rangle_r\right]\right) < \infty. \end{aligned} \tag{A.9}$$

For every fixed (i, k), we have $Z^{i,k+1}_r - Z^{i,k}_r = \frac{d\langle M^{k+1} - M^k, \hat{M}^i\rangle}{d\hat{V}}$. Therefore combining equation (A.9) and Lemma A.2, we get

$$\sum_{k\geq 0}\left(\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Y^{k+1}_r - Y^k_r|^2 d\hat{V}_r\right] + \sum_{i=1}^d \mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Z^{i,k+1}_r - Z^{i,k}_r|^2 d\hat{V}_r\right]\right) < \infty.$$

So by Fubini's theorem we have

$$\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}\left(\sum_{k\geq 0}\left(|Y^{k+1}_r - Y^k_r|^2 + \sum_{i=1}^d |Z^{i,k+1}_r - Z^{i,k}_r|^2\right)\right) d\hat{V}_r\right] < \infty.$$

Consequently the sum $\sum_{k\geq 0}\left(|Y^{k+1}_r(\omega) - Y^k_r(\omega)|^2 + \sum_{i=1}^d |Z^{i,k+1}_r(\omega) - Z^{i,k}_r(\omega)|^2\right)$ is finite on a set of full $d\hat{V} \otimes d\mathbb{P}$-measure. So on this set, the sequence $(Y^k_t(\omega), (Z^{i,k}_t(\omega))_{i\in[\![1;d]\!]})$ converges, and the limit is necessarily equal to $(Y_t(\omega), (Z^i_t(\omega))_{i\in[\![1;d]\!]})$ $\quad d\hat{V} \otimes d\mathbb{P}$ a.e. Indeed, as we have mentioned in the lines before the statement of the present Proposition A.15, we already know that Y^k converges to Y in $L^2(d\hat{V} \otimes d\mathbb{P})$. Since by Lemma A.2, $\mathbb{E}\left[\int_0^T e^{-\lambda \hat{V}_r} |Z_r^{i,k} - Z_r^i|^2 d\hat{V}_r\right] \leq C\mathbb{E}\left[\int_0^T e^{-\lambda \hat{V}_r} d\langle M^k - M\rangle_r\right]$, for every (i,k), where C is a positive constant which does not depend on (i,k), the convergence of M^k to M in $\mathcal{H}_0^2$ also implies the convergence of $Z^{i,k}$ to Z^i in $L^2(d\hat{V} \otimes d\mathbb{P})$. □

Appendix B. Proof of Theorem 5.18

Lemma B.1. *Let $\tilde{f} \in \mathcal{L}_X^2$. For every $(s,x) \in [0,T] \times E$, let $(\tilde{Y}^{s,x}, \tilde{M}^{s,x})$ be the unique (by Theorem 3.3 and Remark 3.4) solution of*

$$\tilde{Y}_\cdot^{s,x} = g(X_T) + \int_\cdot^T \mathbb{1}_{[s,T]}(r)\tilde{f}(r, X_r)\, dV_r - (\tilde{M}_T^{s,x} - \tilde{M}_\cdot^{s,x}) \tag{B.1}$$

in $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$. Then there exist $\tilde{u} \in \mathcal{D}(\mathfrak{a})$ such that for any $(s,x) \in [0,T] \times E$

$$\begin{cases} \forall t \in [s,T] : \tilde{Y}_t^{s,x} &= \tilde{u}(t, X_t) \quad \mathbb{P}^{s,x} a.s. \\ \tilde{M}^{s,x} &= M[\tilde{u}]^{s,x} \end{cases}$$

and in particular $\frac{d\langle \tilde{M}^{s,x}, M[\psi]^{s,x}\rangle}{dV} = \mathfrak{G}^\psi(\tilde{u})(\cdot, X_\cdot)\, dV \otimes d\mathbb{P}^{s,x}$ a.e. on $[s,T]$.

Proof. We set $\tilde{u} : (s,x) \mapsto \mathbb{E}^{s,x}\left[g(X_T) + \int_s^T \tilde{f}(r, X_r)\, dV_r\right]$ which is Borel by Proposition A.10 and Lemma A.11 in [8]. Therefore by the Markov property (see e.g. (3.4) in [7]), for every fixed $t \in [s,T]$ we have $\mathbb{P}^{s,x}$- a.s.

$$\begin{aligned} \tilde{u}(t, X_t) &= \mathbb{E}^{t,X_t}\left[g(X_T) + \int_t^T \tilde{f}(r, X_r)\, dV_r\right] = \mathbb{E}^{s,x}\left[g(X_T) + \int_t^T \tilde{f}(r, X_r)\, dV_r \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}^{s,x}\left[\tilde{Y}_t^{s,x} + (\tilde{M}_T^{s,x} - \tilde{M}_t^{s,x}) | \mathcal{F}_t\right] = \tilde{Y}_t^{s,x}. \end{aligned}$$

By (B.1) we have $d\tilde{Y}_t^{s,x} = -\tilde{f}(t, X_t)dV_t + d\tilde{M}_t^{s,x}$, so for every fixed $t \in [s,T]$, $\tilde{u}(t, X_t) = \tilde{u}(s,x) - \int_s^t \tilde{f}(r, X_r)dV_r - \tilde{M}_t^{s,x}$ $\mathbb{P}^{s,x}$- a.s. Since $\tilde{M}^{s,x}$ is square integrable and since previous relation holds for any (s,x) and t, Definition 4.15 implies that $\tilde{u} \in \mathcal{D}(\mathfrak{a})$, $\mathfrak{a}(\tilde{u}) = -\tilde{f}$ and $\tilde{M}^{s,x} = M[\tilde{u}]^{s,x}$ for every (s,x), hence the announced results. □

Notation B.2. *For a fixed $(s,x) \in [0,T] \times E$, we will denote by $(Y^{k,s,x}, M^{k,s,x})_{k\in\mathbb{N}}$ the Picard iterations associated to $BSDE^{s,x}(f,g)$.*

Proposition B.3. *For each $k \in \mathbb{N}$, there exists $u_k \in \mathcal{D}(\mathfrak{a})$, such that for every $(s,x) \in [0,T] \times E$*

$$\begin{cases} \forall t \in [s,T] : Y_t^{k,s,x} &= u_k(t, X_t) \quad \mathbb{P}^{s,x} a.s. \\ M^{k,s,x} &= M[u_k]^{s,x} \end{cases} \tag{B.2}$$

Remark B.4. In particular, (B.2) implies that $\frac{d\langle M^{k,s,x}, M[\psi]^{s,x}\rangle}{dV} = \mathfrak{G}^\psi(u_k)(\cdot, X_\cdot)\, dV \otimes d\mathbb{P}^{s,x}$ a.e. on $[s,T]$.

Proof. We proceed by induction on k. It is clear that $u_0 = 0$ satisfies the assertion for $k = 0$.

Now let us assume that the function u_{k-1} exists, for some integer $k \geq 1$, satisfying (B.2) and in particular Remark B.4, for k replaced with $k-1$.

We fix $(s,x) \in [0,T] \times E$. By Lemma 5.11, $(Y^{k-1,s,x}, \frac{d\langle M^{k-1,s,x}, M[\psi]^{s,x}\rangle}{dV}) = (u_{k-1}, \mathfrak{G}^{\psi}(u_{k-1}))(\cdot, X_\cdot)\, dV \otimes \mathbb{P}^{s,x}$ a.e. on $[s,T]$. Therefore by (A.8), on $[s,T]$ $Y^{k,s,x} = g(X_T) + \int_\cdot^T f\left(r, X_r, u_{k-1}(r,X_r), \mathfrak{G}^{\psi}(u_{k-1})(r,X_r)\right) dV_r - (M_T^{k,s,x} - M_\cdot^{k,s,x})$.

Since $\Phi^{s,x}$ maps $L^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}_0^2$ into itself (see Notation A.7), obviously all the Picard iterations belong to $L^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}_0^2$. In particular, by Lemma A.2 $Y^{k-1,s,x}$ and for every $i \in [\![1;d]\!]$, $\frac{d\langle M^{k-1,s,x}, M[\psi_i]^{s,x}\rangle}{dV}$ belong to $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x})$. So, by recurrence assumption on u_{k-1}, it follows that u_{k-1} and for any $i \in [\![1;d]\!]$, $\mathfrak{G}^{\psi_i}(u_{k-1})$ belong to $\mathcal{L}^2_X$.

Combining $H^{mom}(\zeta,\eta)$ and the growth condition of f (item 3.) in $H^{lip}(\zeta,\eta)$ (see Hypotheses 5.4 and 5.5), one shows that $f(\cdot,\cdot,0,0)$ also belongs to $\mathcal{L}^2_X$. Therefore thanks to the Lipschitz conditions on f assumed in $H^{lip}(\zeta,\eta)$, we have $f(\cdot,\cdot,u_{k-1},\mathfrak{G}^{\psi}(u_{k-1})) \in \mathcal{L}^2_X$.

The existence of u_k now comes from Lemma B.1 applied to $\tilde{f} := f(\cdot,\cdot,u_{k-1},\mathfrak{G}^{\psi}(u_{k-1}))$, which establishes the induction step for a general k and allows to conclude the proof. $\square$

Proof of Theorem 5.18. We set $\bar{u} := \limsup_{k\in\mathbb{N}} u_k$, in the sense that for any $(s,x) \in [0,T] \times E$, $\bar{u}(s,x) = \limsup_{k\in\mathbb{N}} u_k(s,x)$ and $v := \limsup_{k\in\mathbb{N}} v_k$. $\bar{u}$ and v are Borel functions. Let us fix now $(s,x) \in [0,T] \times E$. We know by Propositions B.3, A.15 and Lemma 5.11 that

$$\begin{cases} u_k(\cdot, X_\cdot) & \underset{k\to\infty}{\longrightarrow} & Y^{s,x} & dV \otimes d\mathbb{P}^{s,x} \text{ a.e. on } [s,T] \\ \mathfrak{G}^{\psi}(u_k)(\cdot, X_\cdot) & \underset{k\to\infty}{\longrightarrow} & Z^{s,x} & dV \otimes d\mathbb{P}^{s,x} \text{ a.e. on } [s,T], \end{cases}$$

where $Z^{s,x} := \frac{d\langle M^{s,x}, M[\psi]^{s,x}\rangle}{dV}$. Therefore, and on the subset of $[s,T] \times E$ of full $dV \otimes d\mathbb{P}^{s,x}$-measure on which these convergences hold, we have

$$\begin{cases} \bar{u}(t, X_t(\omega)) &= \limsup_{k\in\mathbb{N}} u_k(t, X_t(\omega)) &= \lim_{k\in\mathbb{N}} u_k(t, X_t(\omega)) &= Y_t^{s,x}(\omega) \\ v(t, X_t(\omega)) &= \limsup_{k\in\mathbb{N}} \mathfrak{G}^{\psi}(u_k)(t, X_t(\omega)) &= \lim_{k\in\mathbb{N}} \mathfrak{G}^{\psi}(u_k)(t, X_t(\omega)) &= Z_t^{s,x}(\omega). \end{cases} \tag{B.3}$$

Thanks to the $dV \otimes d\mathbb{P}^{s,x}$ equalities concerning v and $\bar{u}$ stated in (B.3), under $\mathbb{P}^{s,x}$ we actually have

$$Y^{s,x} = g(X_T) + \int_\cdot^T f\left(r, X_r, \bar{u}(r,X_r), v(r,X_r)\right) dV_r - (M_T^{s,x} - M_\cdot^{s,x}). \tag{B.4}$$

Now (B.4) can be considered as a BSDE where the driver does not depend on y and z. Since $Y^{s,x}$ and $Z^{s,x}$ belong to $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x})$ (see Lemma A.2), then by (B.3), so do $\bar{u}(\cdot, X_\cdot)\mathbb{1}_{[s,T]}$ and $v(\cdot, X_\cdot)\mathbb{1}_{[s,T]}$, meaning that $\bar{u}$ and v belong to $\mathcal{L}^2_X$. Combining $H^{mom}(\zeta,\eta)$ and the Lipschitz condition on f assumed in $H^{lip}(\zeta,\eta)$, $f(\cdot,\cdot,\bar{u},v)$ is also proved to belong to $\mathcal{L}^2_X$. We can therefore apply Lemma B.1 to $\tilde{f} := f(\cdot,\cdot,\bar{u},v)$, and conclude the proof of the first part of the theorem.

Concerning the last statement of Theorem 5.18, for any $(s,x) \in [0,T] \times E$, we have $Y_s^{s,x} = u(s,X_s) = u(s,x)\, \mathbb{P}^{s,x}$ a.s. so $Y_s^{s,x}$ is $\mathbb{P}^{s,x}$ a.s. equal to a constant and u is the mapping $(s,x) \longmapsto Y_s^{s,x}$. $\square$

References

[1] C.D. Aliprantis, K.C. Border, Infinite Dimensional Analysis. A Hitchhiker's Guide, second ed., Springer-Verlag, Berlin, 1999.

[2] V. Bally, E. Pardoux, L. Stoica, Backward stochastic differential equations associated to a symmetric Markov process, Potential Anal. 22 (1) (2005) 17–60.

[3] E. Bandini, Existence and uniqueness for backward stochastic differential equations driven by a random measure, Electron. Commun. Probab. 20 (71) (2015) 1–13.

[4] G. Barles, R. Buckdahn, E. Pardoux, Backward stochastic differential equations and integral-partial differential equations, Stochastics 60 (1–2) (1997) 57–83.

[5] G. Barles, E. Lesigne, SDE, BSDE and PDE, in: Backward Stochastic Differential Equations, Paris, 1995–1996, in: Pitman Res. Notes Math. Ser., vol. 364, Longman, Harlow, 1997, pp. 47–80.

[6] A. Barrasso, F. Russo, Backward stochastic differential equations with no driving martingale, Markov processes and associated pseudo partial differential equations, 2017, Preprint, hal-01431559.

[7] A. Barrasso, F. Russo, A note on time-dependent additive functionals, Commun. Stoch. Anal. 11 (3) (2017) 313–334, 9.

[8] A. Barrasso, F. Russo, BSDEs with no driving martingale, Markov processes and associated pseudo partial differential equations. Part II: Decoupled mild solutions and examples, 2020, Preprint hal-01505974.

[9] A. Ben-Israel, Th.N.E. Greville, Generalized Inverses, second ed., in: CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, vol. 15, Springer-Verlag, New York, 2003, Theory and applications.

[10] R. Buckdahn, Backward stochastic differential equations driven by a martingale, unpublished, 1993.

[11] G. Cannizzaro, K. Chouk, Multidimensional sdes with singular drift and universal construction of the polymer measure with white noise potential, Ann. Probab. 46 (3) (2018) 1710–1763.

[12] R. Carbone, B. Ferrario, M. Santacroce, Backward stochastic differential equations driven by càdlàg martingales, Teor. Veroyatn. Primen. 52 (2) (2007) 375–385.

[13] F. Confortola, M. Fuhrman, J. Jacod, Backward stochastic differential equation driven by a marked point process: an elementary approach with an application to optimal control, Ann. Appl. Probab. 26 (3) (2016) 1743–1773.

[14] F. Delarue, R. Diel, Rough paths and 1d SDE with a time dependent distributional drift: application to polymers, Probab. Theory Related Fields 165 (1–2) (2016) 1–63.

[15] C. Dellacherie, P.-A. Meyer, Probabilités et potentiel. Chapitres V à VIII, revised ed., in: Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], vol. 1385, Hermann, Paris, 1980, Théorie des martingales [Martingale theory].

[16] N. El Karoui, S. Peng, M.C. Quenez, Backward stochastic differential equations in finance, Math. Finance 7 (1) (1997) 1–71.

[17] F. Flandoli, E. Issoglio, F. Russo, Multidimensional stochastic differential equations with distributional drift, Trans. Amer. Math. Soc. 369 (3) (2017) 1665–1688.

[18] F. Flandoli, F. Russo, J. Wolf, Some SDEs with distributional drift. I. General calculus, Osaka J. Math. 40 (2) (2003) 493–542.

[19] F. Flandoli, F. Russo, J. Wolf, Some SDEs with distributional drift. II. Lyons-Zheng structure, Itô's formula and semimartingale characterization, Random Oper. Stoch. Equ. 12 (2) (2004) 145–184.

[20] M. Fuhrman, G. Tessitore, Generalized directional gradients, backward stochastic differential equations and mild solutions of semilinear parabolic equations, Appl. Math. Optim. 51 (3) (2005) 279–332.

[21] E. Issoglio, S. Jing, Forward-backward SDEs with distributional coefficients, Stochastic Process. Appl. 130 (1) (2020) 47–78.

[22] M. Kobylanski, Backward stochastic differential equations and partial differential equations with quadratic growth, Ann. Probab. 28 (2) (2000) 558–602.

[23] I. Laachir, F. Russo, BSDEs, càdlàg martingale problems, and orthogonalization under basis risk, SIAM J. Financial Math. 7 (1) (2016) 308–356.

[24] É. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation, Systems Control Lett. 14 (1) (1990) 55–61.

[25] É. Pardoux, S. Peng, Backward stochastic differential equations and quasilinear parabolic partial differential equations, in: Stochastic Partial Differential Equations and Their Applications, Charlotte, NC, 1991, in: Lecture Notes in Control and Inform. Sci., vol. 176, Springer, Berlin, 1992, pp. 200–217.

[26] S. Peng, Probabilistic interpretation for systems of quasilinear parabolic partial differential equations, Stoch. Stoch. Rep. 37 (1–2) (1991) 61–74.

[27] A. Rozkosz, Weak convergence of diffusions corresponding to divergence form operators, Stoch. Stoch. Rep. 57 (1–2) (1996) 129–157.
[28] F. Russo, G. Trutnau, Some parabolic PDEs whose drift is an irregular random noise in space, Ann. Probab. 35 (6) (2007) 2213–2262.
[29] F. Russo, L. Wurzer, Elliptic PDEs with distributional drift and backward SDEs driven by a càdlàg martingale with random terminal time, Stoch. Dyn. 17 (4) (2017) 1750030, 36.
[30] D.W. Stroock, Diffusion processes associated with Lévy generators, Z. Wahrscheinlichkeitstheor. Verwandte Geb. 32 (3) (1975) 209–244.
[31] D.W. Stroock, S.R.S. Varadhan, Multidimensional Diffusion Processes, in: Classics in Mathematics, Springer-Verlag, Berlin, 2006, Reprint of the 1997 edition.