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A new weak dependence condition and applications to moment inequalities

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Abstract

The purpose of this paper is to propose a unifying weak dependence condition. Mixing sequences, functions of associated or Gaussian sequences, Bernoulli shifts as well as models with a Markovian representation are examples of the models considered. We establish Marcinkiewicz–Zygmund, Rosenthal and exponential inequalities for general sequences of centered random variables. Inequalities are stated in terms of the decay rate for the covariance of products of the initial random variables subject to the condition that the gap of time between both products tends to infinity. As applications of those notions, we obtain a version of the functional CLT and an invariance principle for the empirical process

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1. Introduction

We propose a new weak dependence condition for time series. This definition makes explicit the asymptotic independence between 'past' and 'future'; this means that the 'past' is progressively forgotten. In terms of the initial time series, 'past' and 'future' are elementary events given through finite-dimensional marginal. Roughly speaking, for convenient functions h and k, we shall assume that

$$\mathrm{Cov}(h(\text{'past'}), k(\text{'future'}))$$

is small when the distance between the 'past' and the 'future' is sufficiently large. Asymptotic are expressed in terms of the distance between indices of the initial time series in the 'past' and 'future' terms; the convergence is not assumed to hold uniformly

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on the dimension of the marginal involved. As a special case of such a definition, Rosenblatt (1956) introduced strong mixing conditions but such conditions refer rather to σ-algebra than to random variables.

On the one hand, a main inconvenience of mixing assumptions is the difficulty of checking them; e.g. Doukhan (1994) provides, with evident difficulties, explicit bounds of the decay of mixing sequences. On the other hand, an important property of associated random variables is that zero correlation implies independence (see Newman, 1984). This means that one may hope that dependence will appear in this case only through the covariance structure, and also justifies the study of such processes: indeed a covariance is much easier to compute than a mixing coefficient.

The aim of this paper is to provide a unifying approach including mixing, association, Gaussian sequences and Bernoulli shifts. Sometimes we shall not obtain optimal results when they are particularized to some special subframe. In this frame we obtain moment inequalities and functional CLT for the partial sums and for the empirical distribution function. Those results apply to our classes of examples.

In the sequel, $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ denote respectively the set of nonnegative integers, integers, and the real line.

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables (r.v.s) centered at expectation. For stationary independent sequences of centered random variables, we recall the Marcinkiewicz–Zygmund inequality (cf. Petrov, 1995)

$$E|X_1 + \cdots + X_n|^q = \mathcal{O}(n^{q/2}) \tag{1.1}$$

and the Rosenthal inequality (cf. also Petrov, 1995)

$$E|X_1 + \cdots + X_n|^q \leqslant C_q \left(\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} E|X_i|^2 \right)^{q/2} \right). \tag{1.2}$$

Our dependence conditions on the process yielding the bound (1.1) or (1.2) only require a suitable non-correlation between the ‘past’ and the ‘future’ of the process. Indeed, we preceive that to bound $|E(X_1 + \cdots + X_n)^q|$ for integers q we only have to bound the covariance quantity $|\mathrm{Cov}(X_{t_1} \dots X_{t_m}, X_{t_{m+1}} \dots X_{t_q})|$ in terms of the gap $t_{m+1} - t_m = r$, between both groups of variables defined by $t_1 \leqslant \cdots \leqslant t_q$ and m, for $m \in \{1, \dots, q\}$ as in Doukhan and Portal (1983).

Under positive dependence, the bound (1.1) was developed in Birkel (1988a) under conditions on the $(q+\delta)$th-order moments of X_n and on the decay of the covariance function of the process (those conditions are optimal). For strongly mixing sequences, this bound was first obtained by Yokoyama (1980). Up to now, the best assumptions yielding the bound (1.1) under strongly mixing conditions were given in Rio (1994). His assumptions are given in terms of the quantile function of the sequence (X_n).

A more precise bound for (1.1) for dependent sequences is

$$E|X_1 + \cdots + X_n|^q \leqslant C_q n^{q/2} \max_i (E|X_i|^{q+\delta})^{q/(q+\delta)}. \tag{1.3}$$

Such inequalities are proved for associated sequences in Birkel (1988a): the constants C_q only involve correlations $\mathrm{Corr}(X_i, X_j)$. Our assumptions seem to be sharp. In particular, for associated sequences of bounded r.v.s, Birkel’s conditions yielding (1.3) specialize to our decay condition, and for strongly mixing sequences Rio's (1994) assumptions are reached. However, the use of combinatorics restricts us to even integer exponents.

Rosenthal type inequalities are also given in Doukhan and Portal (1983) and Doukhan (1994), and we also obtain analogues of the inequalities in Rio (1994) in our more general frame. For associated sequences, analogues of inequalities (1.2) are proved by Shao and Yu (1996). We refer also to Bahtin and Bulinski (1997) where moment inequalities are established for multiindexed sums of random variables in terms of covariances of some test functions.

Let $\mathscr{F}$ be a class of real-valued functions, such that for each $f \in \mathscr{F}$ there exists an integer $n \geqslant 1$ such that f is defined on $\mathbb{R}^n$. The integer n depends on the specific function considered. We now introduce

Definition 1. The sequence $(X_n)_{n\in\mathbb{N}}$ of r.v.s is called $(\theta, \mathscr{F}, \psi)$-weak dependent, if there exists a class $\mathscr{F}$ of real-valued functions, a sequence $\theta = (\theta_r)_{r\in\mathbb{N}}$ decreasing to zero at infinity, and a function ψ with arguments $(h,k,u,v) \in \mathscr{F}^2 \times \mathbb{N}^2$ such that for any u-tuple $(i_1,\ldots,i_u)$ and any v-tuple $(j_1,\ldots,j_v)$ with $i_1 \leqslant \cdots \leqslant i_u < i_u + r \leqslant j_1 \leqslant \cdots \leqslant j_v$, one has

$$|\mathrm{Cov}(h(X_{i_1},\ldots,X_{i_u}), k(X_{j_1},\ldots,X_{j_v}))| \leqslant \psi(h,k,u,v)\theta_r, \tag{1.4}$$

for all functions $h,k \in \mathscr{F}$ that are defined respectively on $\mathbb{R}^u$ and $\mathbb{R}^v$.

In the previous definition r always denotes the gap in time between 'past' and 'future'. Notice that the sequence θ depends both on the class $\mathscr{F}$ and on the function ψ. The function ψ may really depend on its arguments, contrarily e.g. to the case of mixing bounded sequences. An important point in the previous definition is its heredity through appropriate images as is the case for mixing conditions.

Doeblin and Fortet (1937), Rosenblatt (1956), Withers (1981), Tran (1990), Birkel (1992), among others, obtained limit theorems under some of the notions of weak dependence from Definition 1.

In order to justify this definition, we reformulate a general case of Theorem 18.4.1 in Ibragimov and Linnik (1971) in terms of our weak dependence conditions.

Define the class of "complex" exponential functions

$$\mathscr{E} = \{h_{s,u}; (s,u) \in \mathbb{R} \times \mathbb{N}\}, \tag{1.5}$$

by $h_s := h_{s,u}$ belongs to $\mathscr{E}$ if and only if there exist a real number s and an integer $u \geqslant 1$ such that $h_s(x_1,\ldots,x_u) = f(s(x_1 + \cdots + x_u))$, where f is the real-valued function defined on $\mathbb{R}$ by $f(x) = \cos x$ or $f(x) = \sin x$.

Corollary A. *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary sequence fulfilling $EX_0 = 0$ and $EX_0^2 < \infty$. Suppose that the sequence $(X_n)_{n\in\mathbb{N}}$ satisfies a $(\theta, \mathscr{E}, \psi)$-weak dependence condition with some bounded function ψ defined on $\mathscr{E}^2 \times \mathbb{N}^2$. Assume that*

1. $\lim_{n\to\infty} \theta_n = 0$.
2. $\lim_{n\to\infty} \mathrm{Var}\, S_n/n = \sigma^2 > 0$.
3. $E|S_n|^{2+\delta} = \mathscr{O}(n^{1+\delta/2})$, *for some* $\delta > 0$.

Then $S_n/\sqrt{n}$ converges in distribution to $\mathscr{N}(0,\sigma^2)$.

Withers (1981) proved the CLT for non-stationary triangular arrays of l-mixing sequences by using the blocking technique, his ideas yield us to the present definition. We also refer to Jakubowski (1993) (his condition B is analogous to the weak dependence condition of Corollary A: *covariance for "complex" exponentials*).

A paper devoted to investigate the basic properties of the functional estimation of a density in this framework is in preparation. Extensions of the present notions to random fields, continuous time processes or vector-valued sequences will be considered in forthcoming papers.

The paper is organized as follows. Section 6.2 is devoted to give our main results. Examples in Section 3 show that $(\theta, \mathscr{F}, \psi)$-weak dependence holds in many cases of interest. Our results are applied to important classes of modelling and the conditions that yield the bound (1.1) or (1.2) are discussed. Section 6.4 is aimed to give applications of the previous moment inequalities to a Donsker Corollary and to the convergence of the empirical process constructed from any stationary and weakly dependent sequences. The proof of the main results is given in Section 6.5. Section 6.6 is dedicated to the proofs related to Section 6.3, concerned with modelling.

2. Results

2.1. Weak dependence

We first introduce some classes of function. Set $\mathbb{L}^\infty$ for the set of real-valued and bounded functions on some space $\mathbb{R}^u$. Moreover $\mathrm{Lip}(h) = \sup_{x \neq y} |h(x) - h(y)|/\|x - y\|_1$ denotes the Lipschitz modulus of a function $h : \mathbb{R}^u \to \mathbb{R}$ where $\mathbb{R}^u$ is equipped with its l^1-norm. Define

$$\mathscr{L} = \{\text{set of bounded Lipschitz functions: } \mathbb{R}^u \to \mathbb{R}, \text{ for some } u \in \mathbb{N}^*\}. \tag{2.1}$$

The class $\mathscr{L}$ will be used together with functions ψ defined by

$$\psi(h, k, u, v) = c(u, v)\mu(\mathrm{Lip}(h), \mathrm{Lip}(k)), \tag{2.2}$$

where μ denotes some locally bounded function on $\mathbb{R}_+^2$ (here and in the sequel $\mathbb{R}_+$ is the set of non-negative real numbers). The functions h and k are defined respectively on $\mathbb{R}^u$ and $\mathbb{R}^v$, c is some function defined on $\mathbb{N}^{*2}$.

In the examples $\mu(x, y) = \text{const. } xy$ or $\mu(x, y) = \text{const. } \max\{x, y\}$ (cf. Section 3). In some cases, the class $\mathscr{L}$ will be replaced by the smaller class

$$\mathscr{L}_1 = \{h \in \mathscr{L}; \|h\|_\infty \leqslant 1\}. \tag{2.3}$$

The inclusions $\mathscr{E} \subset \mathscr{L}_1 \subset \mathscr{L}$ imply that Corollary A holds under $\mathscr{L}_1$-weak dependence. In fact we obtain

Proposition 1. *Let (X_n) be a stationary centered sequence. Suppose that $EX_0^2 < \infty$ and that conditions* (2) *and* (3) *of Corollary* A *are fulfilled. Suppose moreover that the sequence (X_n) is $(\theta, \mathscr{L}_1, \psi)$-weak dependent, where the function ψ is defined as in* (2.2) *by $c(u, v) = (u + v)^d$ and $\psi(h, k) = (\mathrm{Lip}(h) + \mathrm{Lip}(k))^c$, for some $d > 0$,*

$c \in [0,2]$. *If*

$$\theta_r = \mathcal{O}(r^{-D}), \quad \textit{for some } D > (d - c/2) \vee 0, \tag{2.4}$$

then the conclusion of Corollary A *holds.*

(We prove Proposition 1 in Section 4.)

Define now for some $x > 0$ the function $g_x : \mathbb{R} \to \mathbb{R}$ by

$$g_x(y) = \mathbb{1}_{x \leqslant y} - \mathbb{1}_{x \leqslant -y}, \tag{2.5}$$

where $\mathbb{1}_A$ denotes the indicator function of the event A. We shall also consider the class

$$\mathscr{I} = \left\{ \bigotimes_{i=1}^{u} g_{x_i};\ x_i \in \mathbb{R}_+^*,\ u \in \mathbb{N}^* \right\} \tag{2.6}$$

with $\psi(h,k,u,v) = c(u,v)$. In the examples (cf. Section 3), the function c is shown to be either $c(u,v) = \min\{u,v\}$ or $c(u,v) = (u+v)^2$.

The following lemma links $\mathscr{I}$-weak dependence with $\mathscr{L}$-weak dependence; indeed, examples are mainly proved to satisfy a weak dependence condition w.r.t. the class $\mathscr{L}$. It will show that the weaker $\mathscr{L}_0 \cap \mathscr{C}_b^1$-weak dependence condition defined by

$$\mathscr{L}_0 = \left\{ \bigotimes_{i=1}^{u} f_i;\ f_i \in \mathscr{L},\ f_i : \mathbb{R} \to \mathbb{R},\ i = 1,\ldots,u,\ u \in \mathbb{N}^* \right\},$$

$\mathscr{C}_b^1$ denotes the set of differentiable functions with continuous and bounded partial derivatives, and

$$\psi_{c,1}(h,k,u,v) = c(u,v)\max\{\mathrm{Lip}(h), \mathrm{Lip}(k)\} \tag{2.7}$$

or

$$\psi_{c,2}(h,k,u,v) = c(u,v)\mathrm{Lip}(h)\mathrm{Lip}(k), \tag{2.8}$$

(for a suitable function c) implies $\mathscr{I}$-weak dependence under concentration assumptions.

Lemma 1. *Let* (X_n) *be a sequence of r.v.s fulfilling for some* $\alpha > 0,\ C > 0$

$$C(\lambda) := \sup_{x \in \mathbb{R}} \sup_i P(x \leqslant X_i \leqslant x + \lambda) \leqslant C\lambda^\alpha. \tag{2.9}$$

If the sequence (X_n) *is* $(\theta_{\mathscr{L}_0}, \mathscr{L}_0 \cap \mathscr{L}_1 \cap \mathscr{C}_b^1, \psi_{c,1})$*-weak dependent, then* (X_n) *is* $(\theta_{\mathscr{I}}, \mathscr{I}, \psi)$*-weak dependent with*

$$\theta_{\mathscr{I},r} = \theta_{\mathscr{L}_0,r}^{\alpha/(1+\alpha)} \quad \textit{and} \quad \psi(h,k,u,v) = 2(8C)^{1/(1+\alpha)}(u+v)^{1/(1+\alpha)}[c(u,v)]^{\alpha/(1+\alpha)}.$$

If (X_n) *is* $(\theta_{\mathscr{L}_0}, \mathscr{L}_0 \cap \mathscr{L}_1 \cap \mathscr{C}_b^1, \psi_{c,2})$*-weak dependent, then it is* $(\theta_{\mathscr{I}}, \mathscr{I}, \psi)$*-weak dependent with*

$$\theta_{\mathscr{I},r} = \theta_{\mathscr{L}_0,r}^{\alpha/(2+\alpha)} \quad \textit{and} \quad \psi(h,k,u,v) = (8C)^{2/(2+\alpha)}(u+v)^{2/(2+\alpha)}[c(u,v)]^{\alpha/(2+\alpha)}.$$

The following lemma allows one to replace $\mathscr{L}$ by the wider class $\tilde{\mathscr{L}}$ of (perhaps unbounded) real-valued and Lipschitz functions defined on any $\mathbb{R}^u$-space.

Lemma 2. *If the sequence $(X_n)_{n\in\mathbb{N}}$ is $(\theta,\mathscr{L},\psi)$-weak dependent, with ψ associated to a coordinatewise non-decreasing function μ and if $\sup_{n\in\mathbb{N}} EX_n^2<\infty$, then the sequence $(X_n)_{n\in\mathbb{N}}$ is also $(\theta,\tilde{\mathscr{L}},\psi)$-weak dependent.*

Remark. Contrarily to the covariance inequalities for mixing sequences we do not need higher moments or tail assumptions (see Rio, 1993) to obtain bounds for a covariance in the case of $\mathscr{L}$-weak dependence. Thus, we rederive Rio's (1993) results without additional tail assumptions.

Proof of Lemma 2. First we note that for any function $f\in\tilde{\mathscr{L}}$ there holds $|f(x_1,\dots,x_u)| \leqslant |f(0,\dots,0)| + \mathrm{Lip}(f)\sum_{j=1}^u |x_j|$. Square integrability implies $Ef^2(X_{i_1},\dots,X_{i_u}) < \infty$ for any indices $i_1,\dots,i_u \in \mathbb{N}$. Consider now the continuous and piecewise linear function $i_M : \mathbb{R} \to [-M,M]$, which is the identity on $[-M,M]$ and is constant outside this interval. Then $\mathrm{Lip}(i_M)=1$ and the function $i_M o f$ is in $\mathscr{L}$ and satisfies $\mathrm{Lip}(i_M o f)\leqslant \mathrm{Lip}(f)$. We thus conclude by using the dominated convergence Corollary and μ's monotonicity. $\square$

2.2. Moment inequalities

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of centered r.v.s. Let $S_n=\sum_{i=1}^n X_i$. In this section, we obtain bounds for $|ES_n^q|$, when $q\in\mathbb{N}$ and $q\geqslant 2$.

Definition 2. Let (X_n) be a sequence of centered r.v.s. Define, for positive integer r, the coefficient of weak dependence as non-decreasing sequences $(C_{r,q})_{q\geqslant 2}$ such that

$$\sup|\mathrm{Cov}(X_{t_1}\dots X_{t_m}, X_{t_{m+1}}\dots X_{t_q})| =: C_{r,q},$$

where the supremum is taken over all $\{t_1,\dots,t_q\}$ such that $1\leqslant t_1\leqslant\cdots\leqslant t_q$ and m, r satisfy $t_{m+1}-t_m=r$.

In this paper, we provide explicit bounds of $C_{r,q}$ in order to obtain inequalities for moments of the partial sums S_n. We shall consider two types of assumptions, either there exist constants $c,\gamma>0$ such that for any convenient q-tuple $\{t_1,\dots,t_q\}$ (as in the definition):

$$|\mathrm{Cov}(X_{t_1}\dots X_{t_m}, X_{t_{m+1}}\dots X_{t_q})| \leqslant cq^\gamma M^{q-2}\theta_r, \tag{2.10}$$

or, denoting by Q_X the quantile function of $|X|$ (inverse of the tail function $t\to P(|X|>t)$),

$$|\mathrm{Cov}(X_{t_1}\dots X_{t_m}, X_{t_{m+1}}\dots X_{t_q})| \leqslant c\int_0^{\theta_r} Q_{X_{t_1}}(x)\dots Q_{X_{t_q}}(x)\,\mathrm{d}x. \tag{2.11}$$

The bound (2.10) holds mainly for bounded sequences such that $\|X_n\|_\infty\leqslant M$. E.g. $(\theta,\mathscr{L}_1,\psi)$-weak dependence yields the bounds

$$C_{r,q}\leqslant \max_{1\leqslant m<q} \psi(j^{\otimes m}, j^{\otimes(q-m)}, m, q-m)M^q\theta_r,$$

where $j(x)=x\mathbb{1}_{|x|\leqslant 1}+\mathbb{1}_{x>1}-\mathbb{1}_{x<-1}$. As in Lemma 2, we see that under $(\theta,\mathscr{L},\psi)$-weak dependence with $\psi(h,k,u,v)=c(u,v)\mathrm{Lip}(h)\mathrm{Lip}(k)$ a bound is

$$C_{r,q}\leqslant \max_{1\leqslant m<q} c(m,q-m)M^{q-2}\theta_r.$$

The bound (2.11) holds for more general r.v.s, using moment or tail assumptions.

Our first result is the following Marcinkiewicz–Zygmund inequality.

Theorem 1. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of centered r.v.s fulfilling for some fixed $q\in\mathbb{N}$, $q\geqslant 2$*

$$C_{r,q}=\mathrm{O}(r^{-q/2}) \quad \text{as } r\to\infty. \tag{2.12}$$

Then there exists a positive constant B not depending on n for which

$$|ES_n^q|\leqslant Bn^{q/2}. \tag{2.13}$$

2.3. Rosenthal-type inequalities

The following lemma gives moment inequalities of order $q\in\{2,4\}$ (its proof is essentially in Billingsley, 1968).

Lemma 3. *If $(X_n)_{n\in\mathbb{N}}$ is a sequence of centered r.v.s, then*

$$ES_n^2\leqslant 2n\sum_{r=0}^{n-1}C_{r,2},\quad ES_n^4\leqslant 4!\left\{\left(n\sum_{r=0}^{n-1}C_{r,2}\right)^2+n\sum_{r=0}^{n-1}(r+1)^2C_{r,4}\right\}. \tag{2.14}$$

Let us note that Lemma 3 in Bryc and Smoleński (1993) gives a Rosenthal-type inequality of order $q\in[2,4]$ (where q is not necessarily an integer); under a suitable decay of the so-called maximal correlation coefficients (recall that this mixing condition is more restrictive than ours).

The following theorems deal with higher order moments.

Theorem 2. *Let q be some fixed integer not less than 2. Suppose that the dependence coefficients $C_{r,p}$ associated to the sequence (X_n) satisfy, for every nonnegative integer p, $p\leqslant q$, and for some positive constants M,γ,C*

$$C_{r,p}\leqslant C\mathrm{e}^{\gamma p}M^{p-2}\theta_r. \tag{$\mathscr{H}$}$$

Then, for any integer $n\geqslant 2$

$$|ES_n^q|\leqslant\frac{(2q-2)!}{(q-1)!}\mathrm{e}^{q\gamma}\left\{\left(Cn\sum_{r=0}^{n-1}\theta_r\right)^{q/2}\vee\left(CM^{q-2}n\sum_{r=0}^{n-1}(r+1)^{q-2}\theta_r\right)\right\}. \tag{2.15}$$

Theorem 2 is adapted to work with bounded sequences. In order to consider the unbounded case, we shall consider $(\theta,\mathscr{I},\psi)$-weak dependence where ψ denotes $\psi(h,k,u,v)=c(u,v)$ and $\mathscr{I}$ is the class of functions defined by (2.6).

Theorem 3. *If $(X_n)_{n\in\mathbb{N}}$ is a centered and $(\theta,\mathscr{I},\psi)$-weak dependent sequence, then*

$$|ES_n^q| \leqslant \frac{(2q-2)!}{(q-1)!}\left\{C_q\sum_{i=1}^n\int_0^1[\min(\theta^{-1}(u),n)]^{q-1}Q_i^q(u)\,\mathrm{d}u \vee \left(C_2\sum_{i=1}^n\int_0^1[\min(\theta^{-1}(u),n)]Q_i^2(u)\,\mathrm{d}u\right)^{q/2}\right\},$$

where $C_q=(\max_{u+v\leqslant q}c(u,v))\vee 2$.

In the special case of strongly mixing and stationary sequences, this is Theorem 1 in Rio (1994) (cf. also Rio, 1997). The restriction of working with even integer exponents finds its compensation in the explicit form of the constants.

2.4. Exponential inequality

For any positive integers n and $q\geqslant 2$, define

$$M_{q,n}:=n\sum_{r=0}^{n-1}(r+1)^{q-2}C_{r,q}\leqslant A_n\frac{q!}{\beta^q}, \qquad (\mathscr{H})$$

where β is some positive constant and A_n is a sequence independent of q.

We shall prove as a consequence of Theorem 2 and Markov inequality that an exponential inequality holds.

Corollary 1. *Suppose that $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$ hold for some sequence $A_n\geqslant 1$ for any $n\geqslant 2$. Then for any positive real number x*

$$P(|S_n|\geqslant x\sqrt{A_n})\leqslant A\exp(-B\sqrt{\beta x}), \qquad (2.16)$$

for universal positive constants A and B.

Remark. (1) One may choose the explicit values $A=\mathrm{e}^{4+1/12}\sqrt{8\pi}$, and $B=\mathrm{e}^{5/2}$.

(2) Let us note that condition $(\mathscr{H}_2)$ holds if $C_{r,q}\leqslant C\sigma^2M^{q-2}\mathrm{e}^{\gamma q}\mathrm{e}^{-br}$ for positive constants C,σ,γ,b, as soon as $||X_n||_\infty\leqslant M$ and $||X_n||_2\leqslant\sigma$, for any integer $n\geqslant 0$. In such a case $A_n=n\sigma^2$.

E.g. this holds under $(\theta,\mathscr{L},\psi)$-weak dependence if $\theta_r=\mathcal{O}(\mathrm{e}^{-br})$ and $\psi(h,k,u,v)\leqslant \mathrm{e}^{\delta(u+v)}\mathrm{Lip}(h)\mathrm{Lip}(k)$ for some $\delta\geqslant 0$.

(3) The use of combinatorics in those inequalities makes them relatively weak. E.g. Bernstein inequality, valid for independent sequences allows to replace the term $\sqrt{x}$ in the previous inequality by x^2 under the same assumption $n\sigma^2\geqslant 1$; in the mixing cases analogue inequalities are also obtained by using coupling arguments (not available here), e.g. the case of absolute regularity is studied in Doukhan (1994).

3. Examples

In this section, we apply the preceding results to particular classes of sequences. In each case, we shall check condition $(\mathscr{H}_1)$, providing coefficients $C_{r,q}$. We will also make explicit the underlying weak dependence properties of the sequence.

3.1. Associated sequences

Definition 3 (Esary et al., 1967). The sequence $(X_n)_{n\in\mathbb{N}}$ is associated if for all coordinatewise non-decreasing real-valued functions h and k

$$\mathrm{Cov}(h(X_i, i\in A), k(X_i, i\in B))\geqslant 0,$$

holds for all finite subsets A and B of $\mathbb{N}$.

The lemmas below exhibit the weak dependence structure of the associated sequences under conditions involving only the covariance structure of the process. We note here a remarkable property of association: independence is equivalent to zero correlation (cf. Esary et al., 1967).

3.1.1. Weak dependence and association

Lemma 4. *If $(X_n)_{n\in\mathbb{N}}$ is a sequence of associated and centered r.v.s, then $(X_n)_n$ is $(\theta,\mathscr{L},\psi)$-weak dependent with*

$$\theta_r=\sup_i \sum_{j:|i-j|\geqslant r}\mathrm{Cov}(X_i,X_j) \quad \textit{and} \quad \psi(h,k,u,v)=\min(u,v)\mathrm{Lip}(h)\mathrm{Lip}(k),$$

and it is $(\theta,\mathscr{C}_b^1,\psi)$-weak dependent with

$$\theta_r=\sup_{|i-j|\geqslant r}\mathrm{Cov}(X_i,X_j) \quad \textit{and} \quad \psi(h,k,u,v)=\sum_{i=1}^{u}\sum_{j=1}^{v}\left\|\frac{\partial h}{\partial x_i}\right\|_\infty\times\left\|\frac{\partial k}{\partial x_j}\right\|_\infty. \tag{3.1}$$

Remark. If the associated sequence is uniformly bounded, say by M, then inequality (3.1) yields

$$C_{r,q}\leqslant\frac{q^2}{4}M^{q-2}\sup_{|i-j|\geqslant r}\mathrm{Cov}(X_i,X_j). \tag{3.2}$$

For unbounded and associated sequences, we obtain the following.

Lemma 5. *If (X_n) is a sequence of associated r.v.s, then it is $(\gamma,\mathscr{I},\psi)$-weak dependent with*

$$\gamma_r=\sup_{|i-j|\geqslant r}\sup\{\mathrm{Cov}(\mathbb{1}_{X_i>x},\mathbb{1}_{X_j>y});x,y\in\mathbb{R}\} \quad \textit{and} \quad \psi(h,k,u,v)=(u+v)^2.$$

Hence, setting $Q=\max_{i\in\mathbb{N}}Q_i$, we obtain

$$C_{r,q}\leqslant q^2\int_0^{\gamma_r}Q^q(u)\,\mathrm{d}u. \tag{3.3}$$

Remark. If the associated sequence is bounded by M, then inequality (3.3) yields $C_{r,q}\leqslant q^2M^q\gamma_r$, which follows also from (3.2) since $\sup_{|i-j|\geqslant r}\mathrm{Cov}(X_i,X_j)\leqslant 4M^2\gamma_r$.

Conversely, γ_r may be bounded by means of $C_{r,2}$; indeed if the r.v.s X_i have a uniformly bounded density (w.r.t. i) then

$$\gamma_r\leqslant c\sup_{|i-j|\geqslant r}\mathrm{Cov}^{1/3}(X_i,X_j). \tag{3.4}$$

(see Yu (1993) for the proof of the last inequality).

3.1.2. Marcinkiewicz–Zygmund inequality under association

Theorem 2 together with Lemma 5, implies the Marcinkiewicz–Zygmund inequality (i.e. the bound (1.1) for an even integer q), under the condition

$$\int_0^{\gamma_r} Q^q(u)du = \mathcal{O}(r^{-q/2}). \tag{3.5}$$

Let us compare this result with Theorem 1 in Birkel (1988a). Suppose that

$$\sup_i P(|X_i| > t) = \mathcal{O}(t^{-q-\delta}) \quad \text{as } t \to \infty \text{ and for some } \delta > 0; \tag{3.6}$$

then $Q(t) = \mathcal{O}(t^{-1/(q+\delta)})$ and (3.5) holds whenever $\gamma_r = \mathcal{O}(r^{-q(q+\delta)/2\delta})$. Clearly, the tail condition (3.6) is weaker than the following one (condition (2.1) in Birkel, 1988a)

$$\sup_i E|X_i|^{q+\delta} < +\infty.$$

However, note that after simple calculations the preceeding decay on γ_r implies the condition (2.2) in Birkel (1988a). Moreover, the index of dependence, γ_r, that we investigate here is independent of any moment assumption while Birkel (1988a) uses explicitly the covariance structure of the process (and not the one of indicators); our inequality is thus also more intrinsic. Now, if the associated sequence is bounded then our condition in (2.12) is equivalent to the condition (2.4) in Birkel (1988a) that implies Marcinkiewicz–Zygmund inequality. This condition is shown to be optimal (cf. Birkel, 1988a).

3.1.3. Rosenthal-type inequality under association

Associated r.v.s are shown to be $(\gamma, \mathcal{I}, \psi)$-weak dependent (cf. Lemma 5). Hence, a new Rosenthal inequality for associated r.v.s follows from Theorem 3.

It is more important in practice to obtain moment inequalities for non-monotonic functions of associated r.v.s; indeed, this property fails to be hereditary under such transformations.

Theorem 4. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of associated r.v.s. Let f be a real-valued function with bounded first derivative. Suppose that $Ef(X_i) = 0$ and define $S_n(f) = \sum_{i=1}^n f(X_i)$. Then for every integer q, $q \geqslant 2$, there exists some positive constant C_q depending only on q, for which*

$$|ES_n(f)^q| \leqslant C_q \left\{ \left(n \sum_{r=0}^{n-1} (\|f\|_\infty \sup_{i\in\mathbb{N}} E|f(X_i)| \wedge \|f'\|_\infty^2 \theta_r) \right)^{q/2} + \|f\|_\infty^{q-2} n \sum_{r=0}^{n-1} (r+1)^{q-2} (\|f\|_\infty \sup_{i\in\mathbb{N}} E|f(X_i)| \wedge \|f'\|_\infty^2 \theta_r) \right\}, \tag{3.7}$$

where $\theta_r = \sup_{|i-j|\geqslant r} \operatorname{Cov}(X_i, X_j)$.

Remark. Let us note that Shao and Yu (1996) prove a general Rosenthal's inequality for non-monotonic functions of associated r.v.s. We compare their result with Theorem 4. The last term obtained on the right-hand side of (3.7) is slightly better than the corresponding one in Shao and Yu (1996) by a multiplicative factor n^ε; the first one is worse, however, it provides us with a sharp bound of Var $S_n(f)$. Hence, Theorem 4 is a good competitor with the result of Shao and Yu (1996) for the special case of even integer exponents.

3.2. Strongly mixing sequences

As a measure of dependence, Rosenblatt (1956) introduced the strong mixing coefficients. For any two σ-algebra $\mathscr{A}$ and $\mathscr{B}$, let

$$\alpha(\mathscr{A},\mathscr{B}) = \sup_{(A,B)\in\mathscr{A}\times\mathscr{B}} |\mathrm{Cov}(\mathbb{1}_A,\mathbb{1}_B)|.$$

The strong mixing coefficients of the sequence $(X_n)_{n\in\mathbb{Z}}$ are defined by

$$\alpha_n = \sup_{k\in\mathbb{Z}} \alpha(\mathscr{A}_k,\mathscr{B}_{k+n}) \quad \text{where } \mathscr{A}_k = \sigma(X_i, i\leqslant k) \text{ and } \mathscr{B}_k = \sigma(X_i, i\geqslant k).$$

For relevant literature on mixing, the reader is deferred to Doukhan (1994). The following lemma makes explicit the simple weak dependence structure of strongly mixing sequences.

Lemma 6 (Rosenblatt, 1956). *If the sequence $(X_n)_{n\in\mathbb{Z}}$ is strongly mixing, then it is $(\alpha,\mathbb{L}^\infty,\psi)$-weakly dependent with $\psi(h,k,u,v){=}4\|h\|_\infty\|k\|_\infty$. Moreover, if the strongly mixing sequence is centered at expectation and bounded by M, then*

$$C_{r,q}\leqslant 4M^q\alpha_r.$$

For unbounded strongly mixing sequences, we obtain

Lemma 7. *Every sequence of strongly mixing r.v.s is $(\alpha,\mathscr{I},4)$-weak dependent and satisfies*

$$C_{r,q}\leqslant 4\int_0^{\alpha_r} Q^q(u)\,\mathrm{d}u.$$

For strongly mixing sequences, Theorem 2 together with Lemma 7 yields a Marcinkiewicz–Zygmund inequality (for even integers q) under the optimal condition of Rio (1994):

$$\int_0^{\alpha_r} Q^q(u)\,\mathrm{d}u = \mathcal{O}(r^{-q/2}) \quad \text{as } r\to\infty.$$

3.3. Functions of Gaussian processes

Lemma 8. *If $(X_n)_{n\in\mathbb{N}}$ is a Gaussian process, centered at expectation, then it is $(\theta,\mathscr{C}_b^1\cap\mathbb{L}^\infty,\psi)$-weak dependent with, either $\theta_r=\sup_i\sum_{j:|i-j|\geqslant r}|\mathrm{Cov}(X_i,X_j)|$ and*

$$\psi(h,k,u,v) = \min(u,v)\max_i\left\|\frac{\partial h}{\partial x_i}\right\|_\infty \times \max_i\left\|\frac{\partial k}{\partial x_i}\right\|_\infty,$$

or

$$\theta_r = \sup_{|i-j|\geqslant r}|\mathrm{Cov}(X_i,X_j)| \quad \textit{and} \quad \psi(h,k,u,v) = \sum_{i=1}^{u}\sum_{j=1}^{v}\left\|\frac{\partial h}{\partial x_i}\right\|_\infty \times \left\|\frac{\partial k}{\partial x_j}\right\|_\infty.$$

Theorem 4 written for associated sequences holds true for functions of Gaussian sequences. Let us note the remarkable analogy between associated r.v.s and Gaussian processes (compare Lemma 4 with Lemma 8).

Other moment inequalities for Gaussian processes better than the one obtained here are given in Shao (1995). He obtains such Rosenthal inequalities under weak assumptions on the decay of the ρ-mixing coefficients (only logarithmic decay rates are needed). Now a stationary and Gaussian sequence is also ρ-mixing if the spectral density of the process is bounded below, and

$$\rho_n \leqslant \frac{1}{\inf_\lambda f(\lambda)} \sum_{k=n}^{\infty} |\mathrm{Cov}(X_0, X_k)| \to 0 \quad \text{as } n \to \infty$$

(cf. Doukhan, 1994 for a proof).

Remark. More general sequences satisfying the weak dependence condition defined as in (2.8) with $c(u,v) = (u+v)^2$, may be obtained by combinations: this is the case for the sum of processes $X_n = Y_n + Z_n$, where the Gaussian process (Y_n) is independent of the associated sequence (Z_n).

3.4. Bernoulli shifts

Now, we consider the weak dependence structure of the following class of Bernoulli shifts.

Definition 4. Let $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a sequence of independent real-valued r.v.s and F be a measurable function defined on $\mathbb{R}^{\mathbb{Z}}$. A Bernoulli shift is a sequence $(U_i)_{i\in\mathbb{Z}}$ defined by

$$U_i = F(\varepsilon_{i-j}, j \in \mathbb{Z}). \tag{3.8}$$

A main attraction of such sequences is that they provide examples of processes that are weakly dependent, but not mixing (see Rosenblatt, 1980). This way of constructing stationary sequences is very natural. Chaotic expansions of Gaussian functionals or, in the discrete time case, Volterra expansions are indeed a standard way of modelling stationary processes.

Definition 5. For any positive integer k, we set

$$\delta_k = \sup_{i\in\mathbb{Z}} E|F(\varepsilon_{i-j}, j \in \mathbb{Z}) - F(\varepsilon_{i-j}\mathbb{1}_{|j|<k}, j \in \mathbb{Z})|.$$

Such sequences $(\delta_k)_k$ are related to the modulus of uniform continuity of F. The sequence $(\delta_k)_k$ is evaluated under regularity conditions on the function F. In fact, if

$$|F(u_i\colon i \in \mathbb{Z}) - F(v_i\colon i \in \mathbb{Z})| \leqslant \sum_{i\in\mathbb{Z}} a_i |u_i - v_i|^b,$$

for some positive constants $(a_i)_{i\in\mathbb{Z}}$, $0 < b \leqslant 1$ and if the sequence $(\varepsilon_i)_{i\in\mathbb{Z}}$ has finite bth-order moment, then $\delta_k \leqslant \sum_{|i|\geqslant k} a_i E|\varepsilon_i|^b$.

The following lemma is aimed to prove a weak dependence property of such sequences.

Lemma 9. *The sequence* $(U_n - EU_n)_{n\in\mathbb{Z}}$ *is* $(\theta, \mathscr{L}, \psi)$*-weak dependent, with*

$$\psi(h,k,u,v) = 4(u\|k\|_\infty \mathrm{Lip}(h) + v\|h\|_\infty \mathrm{Lip}(k)) \quad \textit{and} \quad \theta_r = \delta_{r/2}.$$

Remark. More general processes have such weak dependence properties. E.g. instead of independence, assume that the sequence $(\varepsilon_n)_{n\in\mathbb{Z}}$ satisfies a $(\theta_\varepsilon, \mathscr{L}_1, \psi_\varepsilon)$-weak dependence condition; then the process $(U_n)_{n\in\mathbb{Z}}$ is $(\theta, \mathscr{L}_1, \psi)$-weak dependent with $\theta_r = \theta_{r,\varepsilon} + \delta_{r/2}$ and $\psi(h,k,u,v) = \psi_\varepsilon(h,k,u,v) + 4(u\mathrm{Lip}(h) + v\mathrm{Lip}(k))$. Such hereditry property of weak dependence is unknown under mixing.

Lemma 9, together with some elementary calculations, yields a bound for the coefficients $C_{r,q}$ associated to some bounded functions of the sequence $(U_i)_{i\in\mathbb{Z}}$.

Corollary 2. *Let* $(\varepsilon_i)_{i\in\mathbb{Z}}$ *be a sequence of independent r.v.s. Let* $(U_i)_{i\in\mathbb{Z}}$ *be the sequence defined as in* (3.8). *Let* $X_i = f_i(U_i) - Ef_i(U_i)$, *where the functions* $(f_i)_{i\in\mathbb{Z}}$ *satisfy* $\|f_i\|_\infty \leqslant 1/2$ *and* $\mathrm{Lip}(f_i) \leqslant K$. *Then the coefficients* $C_{r,q}$ *associated to the sequence* $(X_i)_{i\in\mathbb{N}}$ *satisfy*

$$C_{r,q} \leqslant 8qK\delta_{r/2}.$$

Remark. Note that condition $(\mathscr{H}_1)$ of Theorem 2 is satisfied, yielding new Rosenthal inequalities for such sequences.

Remark. Let $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a sequence of independent Bernoulli variables with parameter s. The AR(1) process (U_i) with innovation r.v.s (ε_i) and AR parameter $a \in]0, \frac{1}{2}]$ are defined by

$$U_i = aU_{i-1} + \varepsilon_i = \sum_{j\geqslant 0} a^j \varepsilon_{i-j}.$$

This sequence satisfies the requirement of Lemma 9, with $F(u_i,\ i \in \mathbb{Z}) = \sum_{i\geqslant 0} a^i u_i$, and $\delta_k = s\sum_{i>k} a^i$, but it is shown to be non-mixing (e.g. in Rosenblatt, 1980). Note that the process $((-1)^n U_n)$ is neither mixing nor associated, but concentration holds (e.g. U_n is uniform if $s = \frac{1}{2}$, and it has a Cantor marginal distribution if $s = \frac{1}{3}$).

3.5. Models with a Markovian representation

Let $(\varepsilon_i)_{i\in\mathbb{N}}$ be a sequence of independent r.v.s and F be a measurable function. Let $(X_i)_{i\in\mathbb{N}}$ be the Markov chain defined by

$$X_{n+1} = F(X_n, \varepsilon_{n+1}). \tag{3.9}$$

The initial distribution X_0 is supposed to be independent of the sequence $(\varepsilon_i)_{i\in\mathbb{N}}$. We suppose that F satisfies

$$E|F(0,\varepsilon_1)|^a < \infty \quad \text{and} \quad E|F(x,\varepsilon_1) - F(y,\varepsilon_1)|^a \leqslant \alpha^a |x-y|^a, \tag{$\mathscr{C}_1$}$$

for some $a \geqslant 1$ and $0 \leqslant \alpha < 1$. Duflo (1996) shows that under the condition $(\mathscr{C}_1)$, the Markov chain $(X_i)_{i\in\mathbb{N}}$ has a stationary law μ with finite moment of order a.

In the sequel, we suppose that X_0 has μ as distribution (i.e. the Markov chain is stationary).

Lemma 10. *If the Markov chain defined as in* (3.9) *satisfies* ($\mathscr{C}_1$), *then it is* $(\theta, \mathscr{L}, \psi)$-*weak dependent with*

$$\psi(h,k,u,v) = 2\min(u\|k\|_\infty \mathrm{Lip}(h), v\|h\|_\infty \mathrm{Lip}(k)) \quad \textit{and} \quad \theta_r = \alpha^r E|X_0|.$$

Corollary 3. *Let* $(X_i)_{i\in\mathbb{N}}$ *be the Markov chain defined as in* (3.9). *Suppose that* $(X_i)_{i\in\mathbb{N}}$ *satisfies the condition* ($\mathscr{C}_1$). *Let for* $i \in \mathbb{N}$, $Y_i = g_i(X_i) - Eg_i(X_i)$ *where the functions* $g_i:\mathbb{R} \to \mathbb{R}$ *satisfy* $\|g_i\|_\infty \leqslant 1$ *and* $\mathrm{Lip}(g_i) \leqslant K$. *Then the coefficients* $C_{r,q}$ *associated to the sequence* $(Y_i)_{i\in\mathbb{N}}$ *satisfy*

$$C_{r,q} \leqslant 2qK\alpha^r E|X_0|.$$

Remark. Let us note that condition ($\mathscr{H}_1$) of Theorem 2 is satisfied by this Markov chain $(X_i)_{i\in\mathbb{N}}$. Hence, we obtain Rosenthal inequalities which seem to be new.

4. Applications

Let $(X_n)_{n\in\mathbb{N}}$ be a stationary sequence. In this section, we investigate some properties of the Donsker line and of the empirical process constructed from the stationary sequence $(X_n)_{n\in\mathbb{N}}$. For associated sequences, such a result can be found in the papers of Newman (1984), Newman and Wright (1983) or in Bulinski and Keane (1996) for random fields.

4.1. Functional central limit theorem

Here we obtain a functional extension of the CLT in Proposition 1 under $(\theta, \mathscr{L}_1, \psi)$-weak dependence. Define

$$S_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} X_k,$$

where $0 \leqslant t \leqslant 1$ and $[x]$ denotes the integer part of the real number x. We suppose that the $(\theta, \mathscr{L}_1, \psi)$-weak dependence holds with

$$\psi(h,k,u,v) = (u+v)^d(\mathrm{Lip}(h) + \mathrm{Lip}(k))^c,$$

for some functions h, k defined respectively on $\mathbb{R}^u, \mathbb{R}^v$ and for some constants $d \geqslant 0$, and $0 \leqslant c \leqslant 2$.

Theorem 5. *Assume that the centered and stationary sequence* $(X_n)_{n\in\mathbb{N}}$ *fulfils the* $(\theta, \mathscr{L}_1, \psi)$-*weak dependence conditions where* ψ *is defined as before and* $E|X_0|^{4+\delta} < \infty$ *for some* $\delta > 0$. *Assume that* $\theta_r = \mathscr{O}(r^{-D})$ *with* $D > d$ *and* $D \geqslant 2 + 4(2-c)/\delta$. *Then* $\lim_{n\to\infty} ES_n^2 = \infty$ *implies that* $\sigma^2 = \lim_{n\to\infty} ES_n^2/n > 0$ *and that* $(S_n(t))_{t\in[0,1]}$ *converges to* σW *for some standard Brownian motion* W *in the Skohorod space* $\mathscr{D}([0,1])$.

Remark. Having in view the examples of Section 3 we can restrict ourselves to $c, d = 1$ which correspond to Bernoulli shifts; the assumption of the theorem becomes $D \geqslant 2 + \frac{4}{\delta}$ (for bounded r.v.s. this is only $D > 2$).

For $c = d = 2$, which is the case of both associated and Gaussian sequences, this is $D > 2$.

Proof of Theorem 5. Using Lemma 3 and the maximal inequality in Moricz et al. (1982), it is easy to see that assumption (3) of Corollary A holds (together with the tightness of the sequence of processes $(S_n(t)_{t\in[0,1]})$ if for any integers i, j, k such that $0 \leqslant i \leqslant j < k \leqslant l$

$$\sum_{m=0}^{\infty} m|EX_0X_m| < \infty \quad \text{and} \quad \mathrm{Cov}(X_iX_j, X_kX_l) = \mathcal{O}((k-j)^{-2}). \tag{4.1}$$

Condition (1) of Corollary A classically holds with $\sigma^2 \neq 0$ from (4.1) and from $\lim_{n\to\infty} ES_n^2 = \infty$. Now fidi convergence follows from Proposition 1. Hence it is enough to prove the relations (4.1).

We first control the covariance $|EX_0X_r|$.

For this write as in Lemma 2, $X_k = Y_k + (i_M(X_k) - Ei_M(X_k))$. Markov inequality yields for $0 < p < 4+\delta$, $E|Y_k|^p \leqslant 2^p EX_0^{4+\delta} M^{p-4-\delta}$. Now the covariance inequality implies $\mathrm{Cov}(i_M(X_0), i_M(X_r)) \leqslant M^2 2^d (2/M)^c \theta_r$. An optimization w.r.t. M yields $|EX_0X_r| = \mathcal{O}(\theta_r^{(2+\delta)/(4+\delta-c)})$.

We now prove the bound $|\mathrm{Cov}(X_iX_j, X_kX_l)| = \mathcal{O}(\theta_{k-j}^{\delta/(4+\delta-c)})$.

Indeed, the previous covariance is written as the sum of 2^4 covariance terms with the form $\mathrm{Cov}(U_1U_2, U_3U_4)$ for U_j's which are either in $\{Y_i, Y_j, Y_k, Y_l\}$ or are bounded by $2M$. Apply the weak dependence property in the case where each U is bounded. Thus, the Markov inequality implies that this covariance is $\mathcal{O}(M^{-\delta} + M^{4-2c}\theta_r)$. An optimization on M, yields the result. □

Remark. This theorem allows to consider examples of Bernoulli shifts or Lipschitz Markov models as shown by Lemmas 9 and 10 (since for $h, k \in \mathscr{L}_1, \psi(h,k,u,v) \leqslant 4(u+v)(\mathrm{Lip}(h) \vee \mathrm{Lip}(k))$, resp. $\psi(h,k,u,v) \leqslant 2(u+v)(\mathrm{Lip}(h) \wedge \mathrm{Lip}(k))$. Moreover, the case of strongly mixing sequences corresponds to $c = 0$. We may also consider the $\mathscr{L}$-weak dependence as in the examples of Gaussian or associated sequences. Using Lemma 2, the dependence condition yields $\theta_r = \mathcal{O}(r^{-D})$ with $D > d$ and $D \geqslant 2 + 2c/\delta$.

We conclude that Donsker theorem holds for each class of examples from Section 3.

4.2. Empirical processes

Here we prove the tightness of the empirical processes under weak dependence conditions. We assume without loss of generality that the marginal distribution of this sequence is the uniform law on $[0, 1]$. We denote by

$$F_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{X_i \leqslant t} \quad \text{and} \quad U_n(t) = \sqrt{n}[F_n(t) - t].$$

The sequence (X_n) is assumed to satisfy the following weak dependence condition:

$$\sup_{f\in\mathscr{F}} \left| \mathrm{Cov}\left(\prod_{i=1}^{2} f(X_{t_i}), \prod_{i=3}^{4} f(X_{t_i}) \right) \right| \leqslant \theta_r, \tag{4.2}$$

where $\mathscr{F} = \{x \mapsto \mathbb{1}_{s<x<t}$, for $s,t \in [0,1]\}, 0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant t_4$ and $r = t_3 - t_2$ (in this case a weak dependence condition holds for a class of functions $\mathbb{R}^u \to \mathbb{R}$ working only with the values $u = 1$ or 2).

Proposition 2. *Let* (X_n) *be a stationary sequence fulfilling* (4.2) *with*

$$\theta_r = \mathcal{O}(r^{-5/2-v}), \quad \textit{for some } v > 0. \tag{4.3}$$

Then the sequence of processes $(\{U_n(t); t \in [0,1]\})_{n>0}$ *is tight in the Skohorod space* $\mathscr{D}([0,1])$.

Comments. Stationary associated sequences (see Section 3.1) satisfy the requirement of Proposition 2 if $\gamma_r = \mathrm{O}(r^{-5/2-v)}$. Thus, the condition in Yu (1993) is obtained using the inequality (3.4). However, in this case, the recent paper by Louhichi (1998) ameliorates it in the sense that tightness is shown to hold if $\mathrm{Cov}(X_0, X_r) = \mathcal{O}(r^{-a})$, for $a > 4$.

In the same way, stationary mixing sequences satisfy the conditions of Proposition 2 if $\alpha_r = \mathcal{O}(r^{-5/2-v})$. This condition is slightly better than Yoshihara's condition $\alpha_r = \mathcal{O}(r^{-3-v})$ (cf. Yoshihara, 1973).

It is also slightly worse than the corresponding result in Rio (1997). Finally, we note that Theorem 2.3 in Shao and Yu (1996), concerned with ρ-mixing may also be compared with ours in the Gaussian case as in Section 3.3.

Proof of Proposition 2. Let $a = 5/2 + v$. The moment inequality (2.14), together with conditions (4.2) and (4.3), yields the existence of a positive constant C such that for any s, t in $[0,1]$

$$\begin{aligned} \|U_n(t) - U_n(s)\|_4 &\leqslant C\left\{\left(\sum_{r=0}^{n-1} r^{-a} \wedge |t-s|\right)^{1/2} + \left(\frac{1}{n}\sum_{r=0}^{n-1}(r+1)^2\theta_r\right)^{1/4}\right\} \\ &\leqslant C\left\{\left(\sum_{r \geqslant |t-s|^{-1/a}} r^{-a}\right)^{1/2} + \left(\sum_{r<|t-s|^{-1/a}} |t-s|\right)^{1/2} + n^{(2-a)/4}\right\} \\ &\leqslant C\{|t-s|^{(a-1)/2a} + n^{(2-a)/4}\}. \end{aligned}$$

Now it follows from Theorem 2.1 in Shao and Yu (1996) that the sequence $\{U_n(t), t \in [0,1]\}$ is tight. □

Next we prove a functional CLT using the previous proposition and the smoothing technique in Lemma 1. In the following, we assume that the sequence is $(\theta, \mathscr{L}_1, \psi)$-weak dependent with

$$\psi_1(h,k,u,v) = (\mathrm{Lip}(h) \vee \mathrm{Lip}(k))(u+v) \quad \text{or} \quad \psi_2(h,k,u,v) = \mathrm{Lip}(h)\mathrm{Lip}(k)\min(u,v).$$

Theorem 6. *Let* (X_n) *be a stationary sequence, with* X_n *uniformly distributed on* $[0,1]$. *Suppose that* (X_n) *is either* $(\theta, \mathscr{L}_1, \psi_1)$*-weak dependent, with*

$$\theta_r = \mathcal{O}(r^{-5-v}), \tag{4.4}$$

or $(\theta, \mathscr{L}_1, \psi_2)$*-weak dependent, with*

$$\theta_r = \mathcal{O}(r^{-15/2-\nu}). \tag{4.5}$$

Then the sequence of processes $(\{U_n(t);\ t \in [0,1]\})_{n>0}$ *converges in distribution (in* $\mathscr{D}([0,1])$*) to the centered Gaussian process indexed by* $[0,1]$ *with covariance defined by*

$$\Gamma(s,t) = \sum_{k=-\infty}^{+\infty} \operatorname{Cov}(\mathbb{1}_{X_0 \leqslant s}, \mathbb{1}_{X_{|k|} \leqslant t}).$$

Remark. (1) As a consequence of this theorem note that a $(\theta, \mathscr{L}_1, \psi_1)$ weakly dependent Bernoulli shift (i.e. such that $\delta_r \to 0$) with $\delta_r = \mathcal{O}(r^{-a})$ and with uniform marginal distributions has the following properties: its empirical process is a tight sequence in $\mathscr{D}([0,1])$ if $a > 5$ and it is convergent if $a > 11$. This general result seems to be new.

(2) The limiting process is the generalization of Brownian bridge for dependent sequences, the term corresponding to $k = 0$ in the covariance is the only one in the independent case and it corresponds to the Brownian bridge.

(3) The use of the space $\mathscr{L}_1$ allows to work with each of the class of models in the previous section (cf. the last remark at the end of the previous subsection). This yields really new results for the cases of Bernoulli shifts and also, apparently, for Markov sequences.

We now prove Theorem 6. We first propose as a lemma a version of the CLT under weak dependence conditions. A method to prove the CLT for the weakly dependent r.v.s $(\phi(X_i))_{i\in\mathbb{N}}$ is mentioned by Ibragimov et al. (1971) in their Theorems 18.4.1 and 18.4.2. The idea is to split S_n into Bernstein's blocks

$$S_n = \sum_{i=1}^{k} \varepsilon_i + \sum_{i=1}^{k+1} v_i := Z_k + Z'_{k+1},$$

$$\varepsilon_i = \sum_{(i-1)(p+q)+1}^{ip+(i-1)q} \phi(X_j), \quad v_i = \sum_{ip+(i-1)q+1}^{i(p+q)} \phi(X_j) \quad \text{for } 1 \leqslant i \leqslant k.$$

and $v_{k+1} = \sum_{k(p+q)+1}^{n} \phi(X_j)$, where $p = p(n)$, $q = q(n)$, $k = [n/p+q]$ are integer-valued functions satisfying

$$p \to \infty, \quad q \to \infty, \quad q = \mathrm{o}(p), \quad p = \mathrm{o}(n) \quad \text{as } n \to \infty. \tag{4.6}$$

Lemma 11. *Let* $S_n = \sum_{k=1}^{n} \phi(X_k)$ *be a sum of centered stationary r.v.s; set* $\sigma_n^2 = \operatorname{Var} S_n$. *Let g and h be one of the trigonometric functions* $x \to \cos x$, $x \to \sin x$. *Assume that* (4.6) *holds for some sequences* $p(n), q(n)$. *Suppose moreover that*

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} E Z'^2_{k+1} = 0, \tag{4.7}$$

$$\lim_{n\to\infty} \sum_{j=2}^{k} \left| \operatorname{Cov}\left(g\left(\frac{t}{\sigma_n} \sum_{i=1}^{j-1} \varepsilon_i \right), h\left(\frac{t}{\sigma_n} \varepsilon_j \right) \right) \right| = 0, \quad \textit{for all } t \in \mathbb{R}, \tag{4.8}$$

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{i=1}^{k} E|\varepsilon_i|^2 \mathbb{1}_{|\varepsilon_i| \geqslant \varepsilon\sigma_n} = 0, \quad \textit{for all } \varepsilon > 0, \tag{4.9}$$

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{i=1}^{k} E|\varepsilon_i|^2 = 1. \tag{4.10}$$

Then S_n/σ_n *converges in distribution to a standard Gaussian.*

Proof of Lemma 11. The proof is similar to the proof of Lemma 3.1 of Withers (1981) and will be omitted. □

Proof of Corollary A. By (1) and (2) of Corollary A, Condition B of Jakubowski (1993) holds. By (2) and (3), $\{S_n^2/\mathrm{Var}(S_n)\}$ is uniformly integrable. Following step by step the proof of Theorem 2 in Jakubowski and Szewczak (1990), we verify conditions (9.5)–(9.7) of Theorem 9.5 of Jakubowski (1993). Hence, $S_n/\sqrt{\mathrm{Var}(S_n)}$ converges in distribution to $\mathcal{N}(0,\sigma^2)$. Finally, the proof is complete using condition 2 of Corollary A. □

Proof of Proposition 1. We must find some positive integers p and q that fulfil all the requirement of Lemma 11 (with $\phi(x)=x$); more precisely we may have

$$\lim_{n\to\infty} p^{-2}qn = 0, \tag{4.11}$$

and

$$\lim_{n\to\infty} \sum_{j=2}^{k} \psi(f_{t/\sigma_n}, g_{t/\sigma_n}, p, p(j-1))\theta_q = 0, \quad \text{for any } t \in \mathbb{R}, \tag{4.12}$$

where k denotes the integer part of $n/(p+q)$, $\sigma_n^2 = \mathrm{Var}\, S_n$ and f_t, g_t are arbitrary functions in $\mathscr{E}$ defined by (1.5).

We deduce first, from the weak dependence considered, that condition (4.12) holds as soon as

$$\lim_{n\to\infty} k^{d+1} p^d \theta_q / n^{c/2} = 0. \tag{4.13}$$

Now let $p=[n^\alpha]$, $q=[n^\beta]$ with

$$(2(1+d)-c+2D)/2(2D+1) \vee 0 < \beta < \alpha < 1.$$

This choice of p and q is possible since $d - c/2 < D$. Now, p and q so chosen fulfil (4.13), (4.6) and (4.11). □

Proof of Theorem 6. Let $(t_i)_{1\leqslant i\leqslant m}$ be some fixed real numbers. The convergence in distribution of $(U_n(t_1),\ldots,U_n(t_m))$ follows if the sequence $(\phi(X_i))_{i\in\mathbb{N}}$, with ϕ defined by $\phi(x) = \sum_{j=1}^m \alpha_j(\mathbb{1}_{x\leqslant t_j} - t_j)$, satisfies the conditions of Lemma 11. Here, $(\alpha_i)_{1\leqslant i\leqslant m}$ are fixed real numbers such that $\sum_{i=1}^m \alpha_i \neq 0$.

First we note that $\mathrm{Var}\, S_n/n$ converges under $(\theta, \mathscr{L}_1, \psi_2)$ (resp. $(\theta, \mathscr{L}_1, \psi_1)$)-weak dependence if $\sum_{r=1}^{+\infty} \theta_r^{1/3} < \infty$ (resp. $\sum_{r=1}^{+\infty} \theta_r^{1/2} < \infty$) (the proof of this remark is along the same lines as the proof of Lemma 1 with $\alpha = 1$). We suppose now w.l.g that

$$\mathrm{Var}\ \phi(X_1) + 2\sum_{r=2}^{+\infty} \mathrm{Cov}(\phi(X_1), \phi(X_r)) > 0.$$

1. Condition (4.7) holds as soon as $\lim_{n\to\infty} nqp^{-2} = 0$ (recall that $\mathrm{Var}\, S_n/n$ converges).

2. Condition (4.9) holds if $E\varepsilon_1^4 = \mathcal{O}(p^2)$, which holds as soon as $\theta_r^{1/2} = \mathcal{O}(r^{-2})$ (resp. $\theta_r^{1/3} = \mathcal{O}(r^{-2})$) under the $(\theta, \mathcal{L}_1, \psi_1)$ (resp. $(\theta, \mathcal{L}_1, \psi_2)$)-weak dependence condition (use for this Theorem 1 with $q = 4$ and Lemma 1).

3. Let us now check condition (4.8). For this define

$$\phi_j(x) = \mathbb{1}_{x\leqslant t_j} + \left(\frac{-1}{\varepsilon}x + 1 + \frac{t_j}{\varepsilon}\right)\mathbb{1}_{t_j<x\leqslant t_j+\varepsilon} \quad \text{and} \quad \bar{\phi}(x) = \sum_{j=1}^m \alpha_j(\phi_j(x) - t_j).$$

We also define for $1 \leqslant i \leqslant k$: $\bar{\varepsilon}_i = \sum_{(i-1)(p+q)+1}^{ip+(i-1)q} \bar{\phi}(X_j)$.

The sequence $(X_n)_n$ is $(\theta, \mathcal{L}_1, \psi)$-weak dependent (where ψ is either ψ_1 or ψ_2), so that

$$\left|\mathrm{Cov}\left(g\left(\frac{t}{\sigma_n}\sum_{i=1}^{j-1}\bar{\varepsilon}_i\right), h\left(\frac{t}{\sigma_n}\bar{\varepsilon}_j\right)\right)\right| \leqslant \theta_q \mu\left(\frac{t}{\varepsilon\sigma_n}\sum_{j=1}^m \alpha_j, \frac{t}{\varepsilon\sigma_n}\sum_{j=1}^m \alpha_j\right) \times \max_{2\leqslant j\leqslant k} c(p, p(j-1)).$$

In the sequel, we denote by $A_{n,\varepsilon}$ the right-hand side of the last inequality. Hence

$$\sum_{j=2}^k \left|\mathrm{Cov}\left(g\left(\frac{t}{\sigma_n}\sum_{i=1}^{j-1}\bar{\varepsilon}_i\right), h\left(\frac{t}{\sigma_n}\bar{\varepsilon}_j\right)\right)\right| \leqslant kA_{n,\varepsilon}. \tag{4.14}$$

Using inequality (5.1) below, we obtain

$$k\max_{j\leqslant k}\left|\mathrm{Cov}\left(g\left(\frac{t}{\sigma_n}\sum_{i=1}^{j-1}\bar{\varepsilon}_i\right), h\left(\frac{t}{\sigma_n}\bar{\varepsilon}_j\right)\right) - \mathrm{Cov}\left(g\left(\frac{t}{\sigma_n}\sum_{i=1}^{j-1}\varepsilon_i\right), h\left(\frac{t}{\sigma_n}\varepsilon_j\right)\right)\right|$$
$$\leqslant 4\frac{tpk^2\varepsilon}{\sigma_n}\sum_{j=1}^m \alpha_j.$$

The last inequality, together with (4.14), yields

$$\sum_{j=2}^k \left|\mathrm{Cov}\left(g\left(\frac{t}{\sigma_n}\sum_{i=1}^{j-1}\varepsilon_i\right), h\left(\frac{t}{\sigma_n}\varepsilon_j\right)\right)\right| \leqslant 4\frac{tpk^2\varepsilon}{\sigma_n}\sum_{j=1}^m \alpha_j + kA_{n,\varepsilon}.$$

If the sequence is $(\theta, \mathcal{L}_1, \psi_2)$ (resp. $(\theta, \mathcal{L}_1, \psi_1)$)-weak dependent, then

$$\sum_{j=2}^k \left|\mathrm{Cov}\left(g\left(\frac{t}{\sigma_n}\sum_{i=1}^{j-1}\varepsilon_i\right), h\left(\frac{t}{\sigma_n}\varepsilon_j\right)\right)\right| \leqslant 4\frac{tpk^2\varepsilon}{\sigma_n}\sum_{j=1}^m \alpha_j + \frac{t^2k\theta_q p}{\varepsilon^2\sigma_n^2}\left(\sum_{j=1}^m \alpha_j\right)^2$$
$$\sim \text{const. } n\left(\frac{\theta_q}{p^2}\right)^{1/3}. \tag{4.15}$$

(The last inequality is obtained if $\varepsilon = (\theta_q/k\sigma_n)^{1/3}$).

Respectively, the left-hand side of inequality (4.15) is bounded, under $(\theta, \mathscr{L}_1, \psi_1)$)-weak dependence, by const. $(n^3\theta_q/p^2)^{1/2}$.

Now one can find some sequences p and q fulfilling (4.6), $\lim_{n\to+\infty} nqp^{-2} = 0$, and $\lim_{n\to\infty} n(p^{-2}\theta_q)^{1/3} = 0$, (resp. $\lim_{n\to\infty} n^3 p^{-2}\theta_q = 0$) as soon as $\theta_r = \mathcal{O}(r^{-15/2-v})$ (resp. $\theta_r = \mathcal{O}(r^{-5-v})$).

Hence the convergence of the fi-di distributions holds.

The tightness of $\{U_n(t),\ t \in [0,1]\}$ holds also since the requirements of Proposition 2 are fulfilled.

The theorem is thus proved. $\square$

5. Proofs of the main results

The purpose of this section is to prove the main results of Section 2. The inequality

$$|x_1 \ldots x_m - y_1 \ldots y_m| \leqslant \sum_{i=1}^{m} |x_i - y_i|, \tag{5.1}$$

valid for any real numbers $0 \leqslant x_i, y_i \leqslant 1$ is extensively used below.

5.1. A basic tool

Lemma 12. *Let $(U_q)_{q>0}$ and $(V_q)_{q>0}$ be two sequences of real numbers satisfying for some $\gamma \geqslant 0$, and for all $q \in \mathbb{N}^*$*

$$U_q \leqslant \sum_{m=1}^{q-1} U_m U_{q-m} + e^{q\gamma} V_q, \tag{5.2}$$

with $U_1 = 0 \leqslant V_1$. Suppose that for every integers m, q fulfilling $2 \leqslant m \leqslant q-1$, there holds

$$(V_2^{m/2} \vee V_m)(V_2^{(q-m)/2} \vee V_{q-m}) \leqslant (V_2^{q/2} \vee V_q). \tag{5.3}$$

Then, for any integer $q \geqslant 2$

$$U_q \leqslant \frac{\mathrm{e}^{q\gamma}}{q} \binom{2q-2}{q-1} (V_2^{q/2} \vee V_q). \tag{5.4}$$

Proof of Lemma 12. We first prove the following lemma.

Lemma 13. *Let $(U_q)_{q>0}$ be a sequence fulfilling for every positive integer q*

$$U_q \leqslant \sum_{m=1}^{q-1} U_m U_{q-m} + 1, \tag{5.5}$$

with $U_1 = 0$. Then, for every integer $q \geqslant 2$,

$$U_q \leqslant \frac{1}{q} \binom{2q-2}{q-1}. \tag{5.6}$$

Proof of Lemma 13. The proof is done by induction on q. Clearly (5.6) is true for $q=2$. Suppose now that (5.6) is true for every integer m less than $q-1$. Define the qth number of Catalan, $d_q=(1/q)\binom{2q-2}{q-1}$, $d_1=1$. The inductive hypothesis (recall that $U_1=0$) yields

$$U_q \leqslant \sum_{m=2}^{q-2} d_m d_{q-m} + 1. \tag{5.7}$$

The last inequality, together with the identity $d_q=\sum_{m=1}^{q-1} d_m d_{q-m}$ (cf. Comtet, 1970, p. 64), implies $U_q \leqslant d_q$.

Now to prove Lemma 12 it suffices to apply Lemma 13 to the sequence $\tilde{U}_q = U_q/\mathrm{e}^{q\gamma}(V_2^{q/2} \vee V_q)$. □

5.2. Application to moment inequalities

For any integer $q \geqslant 2$, set

$$A_q(n) = \sum_{1 \leqslant t_1 \leqslant \cdots \leqslant t_q \leqslant n} |EX_{t_1} \ldots X_{t_q}|. \tag{5.8}$$

Hence, in order to bound $|ES_n^q|$, it remains to bound $A_q(n)$ because

$$|ES_n^q| \leqslant q! A_q(n). \tag{5.9}$$

5.2.1. A basic lemma

Lemma 14. *Let $(X_n)_{n\in\mathbb{N}}$ be a centered sequence of r.v.s. Then,*

$$A_q(n) \leqslant \sum_{m=1}^{q-1} A_m(n) A_{q-m}(n) + V_q(n) \tag{5.10}$$

with

$$V_q(n) = \sum |\mathrm{Cov}(X_{t_1} \ldots X_{t_m}, X_{t_{m+1}} \ldots X_{t_q})|, \tag{5.11}$$

the sum is considered over $\{t_1,\ldots,t_q\}$ fulfilling $1 \leqslant t_1 \leqslant \cdots \leqslant t_q \leqslant n$ with $r = t_{m+1} - t_m = \max_{1\leqslant i<q}(t_{i+1}-t_i)$.

If condition (5.3) holds for a sequence $(\tilde{V}_i(n))_i$ such that $(\tilde{V}_i(n) \geqslant V_i(n)$, then for any integer $n \geqslant 2$,

$$\left| E\frac{S_n^q}{q!} \right| \leqslant \frac{1}{q} \binom{2q-2}{q-1} (\tilde{V}_q(n) \vee \tilde{V}_2^{q/2}(n)).$$

Proof of Lemma 14. The proof of this lemma is essentially in Doukhan and Portal (1983). Clearly

$$\begin{aligned} A_q(n) \leqslant & \sum_{1 \leqslant t_1 \leqslant \cdots \leqslant t_q \leqslant n} |EX_{t_1} \ldots X_{t_m} EX_{t_{m+1}} \ldots X_{t_q}| \\ & + \sum_{1 \leqslant t_1 \leqslant \cdots \leqslant t_q \leqslant n} |\mathrm{Cov}(X_{t_1} \ldots X_{t_m}, X_{t_{m+1}} \ldots X_{t_q})|. \end{aligned}$$

The first term on the right-hand side of the last inequality is bounded by

$$\sum_{1\leqslant t_1\leqslant\cdots\leqslant t_q\leqslant n} |EX_{t_1}\ldots X_{t_m}EX_{t_{m+1}}\ldots X_{t_q}| \leqslant \sum_{m=1}^{q-1} A_m(n)A_{q-m}(n).$$

Hence relation (5.10) holds. Thus, we deduce that the sequence $(A_q(n))_q$ satisfies (5.2) with $v=0$. If moreover, the sequence $(\tilde{V}_i)_i$ satisfies (5.3) then the conclusion of Lemma 12 holds. The proof of Lemma 14 follows then using (5.9). □

5.2.2. Comments

In this subsection, we bound the expression $V_q(n)$ defined by (5.11).

Lemma 15. *Let t_1 be a fixed positve integer. Let $\{t_1,\ldots,t_q\}$ be a collection of integers fulfilling $1\leqslant t_1\leqslant\cdots\leqslant t_q\leqslant n$. Let $r=t_{m+1}-t_m=\max_{2\leqslant i\leqslant q-1}(t_{i+1}-t_i)$. If*

$$|\mathrm{Cov}(X_{t_1}\ldots X_{t_m},X_{t_{m+1}}\ldots X_{t_q})|\leqslant C_{r,q}(t_1), \tag{5.12}$$

then

$$V_q(n)\leqslant \sum_{t_1=1}^{n}\sum_{r=0}^{n-1}(r+1)^{q-2}C_{r,q}(t_1). \tag{5.13}$$

Proof of Lemma 15. Clearly $V_q(n)\leqslant\sum_{t_1=1}^{n}\sum^{*}|\mathrm{Cov}(X_{t_1}\ldots X_{t_m},X_{t_{m+1}}\ldots X_{t_q})|$ where $\sum^{*}$ denotes a sum over such a collection $1\leqslant t_1\leqslant\cdots\leqslant t_q\leqslant n$ with fixed t_1, and $r=t_{m+1}-t_m=\max_{1\leqslant i\leqslant q-1}(t_{i+1}-t_i)\in[0,n-1]$. Again

$$\sum^{*}|\mathrm{Cov}(X_{t_1}\ldots X_{t_m},X_{t_{m+1}}\ldots X_{t_q})|\leqslant\sum_{r=0}^{n-1}\sum^{**}|\mathrm{Cov}(X_{t_1}\ldots X_{t_m},X_{t_{m+1}}\ldots X_{t_q})|.$$

$\sum^{**}$ denotes the $(q-2)$-dimensional sums each over $\{t_i\colon t_{i-1}\leqslant t_i\leqslant t_{i-1}+r,\ i\neq 1,\ldots,m+1\}$, Hence $\sum^{**}1=(r+1)^{q-2}$. Lemma 15 is so proved. □

Now we bound $V_q(n)$ for $(\theta,\mathscr{I},\psi)$-weak dependent sequences. We suppose in this case that $\psi(h,k,u,v)=c(u,v)$.

Lemma 16. *If the sequence $(X_n)_{n\in\mathbb{N}}$ is $(\theta,\mathscr{I},c)$-weak dependent, then*

$$|\mathrm{Cov}(X_{t_1}\ldots X_{t_m},X_{t_{m+1}}\ldots X_{t_q})|\leqslant\max(c_q,2)\int_0^{\min(\theta_r,1)}Q_{t_1}(u)\ldots Q_{t_q}(u)\,\mathrm{d}u,$$

where $c_q=\max_{u+v\leqslant q}c(u,v)$, and Q_{t_i} denotes the inverse of the tail function $t\to\mathbb{P}(|X_{t_i}|>t)$. Hence

$$V_q(n)\leqslant\max(c_q,2)\sum_{i=1}^{n}\int_0^1(\min(\theta^{-1}(u),n))^{q-1}Q_i^q(u)\,\mathrm{d}u,$$

where $\theta(u)=\theta_{[u]}$.

Proof of Lemma 16. We shall bound $|\mathrm{Cov}(X_{t_1}\ldots X_{t_m},X_{t_{m+1}}\ldots X_{t_q})|$ in terms of $r=t_{m+1}-t_m$. Let $X^+=\max(0,X)$ and $X^-=\max(0,-X)$,

$$X^+=\int_0^{+\infty}\mathbb{1}_{x\leqslant X^+}\,\mathrm{d}x=\int_0^{+\infty}\mathbb{1}_{x\leqslant X}\,\mathrm{d}x$$

and

$$X^- = \int_0^{+\infty} \mathbb{1}_{x \leqslant X^-}\, \mathrm{d}x = \int_0^{+\infty} \mathbb{1}_{x \leqslant -X}\, \mathrm{d}x.$$

A classical calculation shows that

$$X_1 \dots X_n = \prod_{i=1}^{n} (X_i^+ - X_i^-) = \sum (-1)^{n-r} X_{i_1}^+ \dots X_{i_r}^+ X_{i_{r+1}}^- \dots X_{i_n}^-,$$

denoting by $\sum$ a summation over all the permutations $\{i_1, \dots, i_n\}$ of $\{1, \dots, n\}$. Using Fubini's theorem, the preceding integral representation yields

$$\begin{aligned} X_1 \dots X_n &= \sum (-1)^{n-r} \int_{\mathbb{R}_+^d} \mathbb{1}_{x_1 \leqslant X_{i_1}} \dots \mathbb{1}_{x_r \leqslant X_{i_r}} \mathbb{1}_{x_{r+1} \leqslant -X_{r+1}} \dots \mathbb{1}_{x_n \leqslant -X_{i_n}}\, \mathrm{d}x_1 \dots \mathrm{d}x_n \\ &= \int_{\mathbb{R}_+^d} \prod_{i=1}^{n} (\mathbb{1}_{x_i \leqslant X_i} - \mathbb{1}_{x_i \leqslant -X_i})\, \mathrm{d}x_1 \dots \mathrm{d}x_n. \end{aligned}$$

Again Fubini's theorem yields

$$EX_1 \dots X_n = \int_{\mathbb{R}_+^d} E \prod_{i=1}^{n} (\mathbb{1}_{x_i \leqslant X_i} - \mathbb{1}_{x_i \leqslant -X_i})\, \mathrm{d}x_1 \dots \mathrm{d}x_n. \tag{5.14}$$

Now, Eq. (5.14) together with Fubini's theorem implies

$$\mathrm{Cov}(X_{t_1} \dots X_{t_m}, X_{t_{m+1}} \dots X_{t_q}) = \int_{\mathbb{R}_+^d} \mathrm{Cov}\left(\prod_{i=1}^{m} f_i(X_{t_i}), \prod_{i=m+1}^{q} f_i(X_{t_i}) \right) \mathrm{d}x_1 \dots \mathrm{d}x_q,$$

where $f_i(y) = \mathbb{1}_{x_i \leqslant y} - \mathbb{1}_{x_i \leqslant -y}$. Define

$$B = \left| \mathrm{Cov}\left(\prod_{i=1}^{m} f_i(X_{t_i}), \prod_{i=m+1}^{q} f_i(X_{t_i}) \right) \right|. \tag{5.15}$$

In the sequel, we give two bounds of the quantity B.

- The first bound does not use the dependence structure, only that $|f_i(y)| = \mathbb{1}_{x_i \leqslant |y|}$. Thus

  $$B \leqslant 2 \inf(\Phi_{X_{t_1}}(x_1), \dots, \Phi_{X_{t_q}}(x_q)), \tag{5.16}$$

  with $\Phi_X(x) = P(x \leqslant |X|)$.
- From $(\theta, \mathscr{I}, \psi)$-weak dependence, we obtain (recall that $r = t_{m+1} - t_m$)

  $$B \leqslant c_q \theta_r. \tag{5.17}$$

The bound (5.17) together with (5.16) yields

$$B \leqslant \max(c_q, 2) \inf(\theta_r, \Phi_{X_{t_1}}(x_1), \dots, \Phi_{X_{t_q}}(x_q)).$$

Hence,

$$\begin{aligned} &|\mathrm{Cov}(X_{t_1} \dots X_{t_m}, X_{t_{m+1}} \dots X_{t_q})| \\ &\quad \leqslant \max(c_q, 2) \int_0^{+\infty} \cdots \int_0^{+\infty} \inf(\theta_r, \Phi_{X_{t_1}}(x_1), \dots, \Phi_{X_{t_q}}(x_q))\, \mathrm{d}x_1 \dots \mathrm{d}x_q. \end{aligned}$$

The proof of Theorem 1-1 in Rio (1993) can be completely implemented here. We give it for completeness. Let U be an uniform-$[0,1]$ r.v; then

$$\begin{aligned}\inf(\theta_r, \Phi_{X_{t_1}}(x_1),\dots,\Phi_{X_{t_q}}(x_q)) &= P(U\leqslant\theta_r, U\leqslant\Phi_{X_{t_1}}(x_1),\dots,U\leqslant\Phi_{X_{t_q}}(x_q))\\ &= P(U\leqslant\theta_r, x_1\leqslant Q_{X_{t_1}}(U),\dots,x_q\leqslant Q_{X_{t_q}}(U)).\end{aligned}$$

So, $|\mathrm{Cov}(X_{t_1}\dots X_{t_m}, X_{t_{m+1}}\dots X_{t_q})|\leqslant\max(c_q,2)EQ_{X_{t_1}}(U)\dots Q_{X_{t_q}}(U)\mathbb{1}_{U\leqslant\theta_r}$. Hence, the first part of Lemma 16 follows. Now we shall prove the second part.

Arguing exactly as in Rio (1997), we obtain

$$\begin{aligned}V_q &\leqslant \frac{\max(c_q,2)}{q}\sum_{i=1}^{q}\sum_{r=0}^{n-1}\sum_{t_i=1}^{n}\int_0^{\theta_r}(r+1)^{q-2}Q_{t_i}^q(u)\,\mathrm{d}u\\ &\leqslant \max(c_q,2)\sum_{r=0}^{n-1}\sum_{i=1}^{n}\int_0^{\theta_r}(r+1)^{q-2}Q_i^q(u)\,\mathrm{d}u\\ &\leqslant \max(c_q,2)\sum_{i=1}^{n}\int_0^{1}(\min(\theta^{-1}(u),n))^{q-1}Q_i^q(u)\,\mathrm{d}u. \qquad\square\end{aligned}$$

Condition under which the hypothesis (5.3) *is satisfied*. We say that a sequence $(U_q)_{q\in\mathbb{N}}$ satisfies the convexity condition $(\mathscr{H}_0)$, if for every integers q and p such that $2\leqslant p\leqslant q-1$

$$\tilde{U}_p\leqslant\tilde{U}_q^{(p-2)/(q-2)}\tilde{U}_2^{(q-p)/(q-2)}. \qquad (\mathscr{H})$$

A Technical Lemma. *If the sequence $(\tilde{U}_q)_{q>0}$ satisfies $(\mathscr{H}_0)$, then it satisfies also condition* (5.3).

In fact, for any m: $2\leqslant m\leqslant q-1$, condition $(\mathscr{H}_0)$ together with some elementary calculations yields

$$\begin{aligned}(\tilde{U}_2^{m/2}\vee\tilde{U}_m)(\tilde{U}_2^{(q-m)/2}\vee\tilde{U}_{q-m}) &\leqslant \max(\tilde{U}_2^{q/2},\tilde{U}_2^{m/2}\tilde{U}_{q-m},\tilde{U}_m\tilde{U}_2^{(q-m)/2},\tilde{U}_m\tilde{U}_{q-m})\\ &\leqslant(\tilde{U}_2^{q/2}\vee\tilde{U}_q). \qquad\square\end{aligned}$$

Remark. If the coefficients $(C_{r,p}(t_1))_p$ defined as in Lemma 15 satisfy $(\mathscr{H}_0)$, then the sequence

$$\tilde{V}_q(n)=\sum_{t_1=1}^{n}\sum_{r=0}^{n-1}(r+1)^{q-2}C_{r,q}(t_1),$$

satisfies (5.3). Let us check this remark. Using $(\mathscr{H}_0)$ and some elementary estimations, we obtain

$$\tilde{V}_p(n)\leqslant\tilde{V}_q^{(p-2)/(q-2)}(n)\tilde{V}_2^{(q-p)/(q-2)}(n).$$

Thus condition $(\mathscr{H}_0)$ holds for $(\tilde{V}_p(n))_p$ which satisfies also (5.3). $\square$

Proof of Theorem 1. By induction on q, and using Lemmas 14, 15 and condition (2.12), it is easy to check that $A_q(n)\leqslant C_q n^{q/2}$. Hence Theorem 1 follows from (5.9). $\square$

Proof of Lemma 3. The proof of this lemma follows easily from the inequalities (5.10),(5.13) and (5.9) applied with $q=2$ and 4. □

Proof of Theorem 2. Condition $(\mathscr{H}_1)$ together with Lemmas 14 and 15 yields

$$A_q(n) \leqslant \sum_{m=1}^{q-1} A_m(n)A_{q-m}(n) + Ce^{q\gamma}M^{q-2}n\sum_{r=0}^{n-1}(r+1)^{q-2}\theta_r.$$

So that the sequence $(A_q(n))_q$ satisfies (5.2) with

$$\tilde{V}_q(n) = CM^{q-2}n\sum_{r=0}^{n-1}(r+1)^{q-2}\theta_r.$$

Hence it remains to check condition (5.3). This follows from the technical lemma and from the remark below (since the sequence $(CM^{q-2}\theta_r)_q$ satisfies $(\mathscr{H}_0)$). □

Proof of Theorem 3. As in the proof of Theorem 2, it suffices to note that the sequence $\tilde{V}_p(n){:=}\max(c_p,2)\sum_{i=1}^n\int_0^1 [\min(\theta^{-1}(u),n)]^{p-1}Q_i^p(u)\,\mathrm{d}u$ satisfies (5.3). □

Proof of Lemma 1. Let g, f be some fixed functions in $\mathscr{I}$; then there exist $u, v \in \mathbb{N}^*$ and $x_i,\ x'_j \geqslant 0$ such that

$$g(y_1,\ldots,y_u) = g_{x_1}(y_1)\ldots g_{x_u}(y_u) \quad \text{and} \quad f(y_1,\ldots,y_v) = g_{x'_1}(y_1)\ldots g_{x'_v}(y_v),$$

where the functions $(g_x)_{x\in\mathbb{R}^+}$ are defined as in (2.5). For fixed $x>0$ and $a>0$, let

$$f_x(y) = \mathbb{1}_{y>x} - \mathbb{1}_{y\leqslant -x} + \left(\frac{y}{a} - \frac{x}{a} + 1\right)\mathbb{1}_{x-a<y<x} + \left(\frac{y}{a} + \frac{x}{a} - 1\right)\mathbb{1}_{-x<y<-x+a}.$$

Clearly, $\mathrm{Lip}(f_x) = a^{-1}$ and $\|f_x\|_\infty = 1$. Hence $\mathrm{Lip}(h) \leqslant a^{-1}$, $\mathrm{Lip}(k) \leqslant a^{-1}$, where

$$h(y_1,\ldots,y_u) = f_{x_1}(y_1)\ldots f_{x_u}(y_u), \quad k(y_1,\ldots,y_v) = f_{x'_1}(y_1)\ldots f_{x'_v}(y_v).$$

Consider now, $i_1 \leqslant \cdots \leqslant i_u \leqslant i_u + r \leqslant j_1 \leqslant \cdots \leqslant j_v$ and set

$$\mathrm{Cov}(h,k){:=}\mathrm{Cov}(h(X_{i_1},\ldots,X_{i_u}), k(X_{j_1},\ldots,X_{j_v})).$$

On the one hand, the $(\theta_{\mathscr{L}_0}, \mathscr{L}_0 \cap \mathscr{L}_1 \cap \mathscr{C}_b^1, \psi_{c,1})$ (resp. $(\theta_{\mathscr{L}_0}, \mathscr{L}_0 \cap \mathscr{L}_1 \cap \mathscr{C}_b^1, \psi_{c,2})$)-weak dependence yields

$$|\mathrm{Cov}(h,k)| \leqslant \frac{1}{a}c(u,v)\theta_{\mathscr{L}_{0,r}} \quad \left(\text{resp. } |\mathrm{Cov}(h,k)| \leqslant \frac{1}{a^2}c(u,v)\theta_{\mathscr{L}_{0,r}}\right).$$

On the other hand, inequalities (2.9) and (5.1) yield

$$|\mathrm{Cov}(g,f) - \mathrm{Cov}(h,k)| \leqslant 8Ca^\alpha(u+v).$$

Hence the two last inequalities yield

$$|\mathrm{Cov}(g,f)| \leqslant 8Ca^\alpha(u+v) + \frac{1}{a}c(u,v)\theta_{\mathscr{L}_{0,r}}$$

$$(\text{resp. } |\mathrm{Cov}(g,f)| \leqslant 8Ca^\alpha(u+v) + \frac{1}{a^2}c(u,v)\theta_{\mathscr{L}_{0,r}}).$$

The proof of Lemma 1 follows then by setting, in the above inequality,

$$a = \left(\frac{c(u,v)\theta_{\mathscr{L}_{0,r}}}{8C(u+v)}\right)^{1/(1+\alpha)} \quad \left(\text{resp. } a = \left(\frac{c(u,v)\theta_{\mathscr{L}_{0,r}}}{8C(u+v)}\right)^{1/(2+\alpha)}\right). \quad □$$

Proof of Corollary 1. Theorem 2 written with $q=2p$ yields

$$ES_n^{2p} \leqslant \frac{(2p)!}{2p}\binom{4p-2}{2p-1} \mathrm{e}^{2p\gamma}[M_{2p,n} \vee M_{2,n}^p]. \tag{5.18}$$

Hence, inequality (5.18) together with condition $(\mathscr{H}_2)$ implies

$$\begin{aligned} ES_n^{2p} &\leqslant \frac{(4p-2)!}{(2p-1)!}\left(\left(\frac{2A_n}{\beta^2}\right)^p \vee A_n\frac{(2p)!}{\beta^{2p}}\right) \\ &\leqslant \frac{(4p-2)!}{(2p-1)!}\max(A_n, A_n^p)\frac{(2p)!}{\beta^{2p}} \\ &\leqslant \max(A_n, A_n^p)\frac{(4p)!}{\beta^{2p}}. \end{aligned}$$

From Stirling formula and the fact that $A_n \geqslant 1$, we obtain

$$\begin{aligned} P(|S_n|\geqslant x) &\leqslant \frac{ES_n^{2p}}{x^{2p}} \leqslant \frac{A_n^p}{x^{2p}\beta^{2p}}\mathrm{e}^{1/12-4p}\sqrt{8\pi p}(4p)^{4p} \\ &\leqslant \mathrm{e}^{1/12}\sqrt{8\pi}\left(\frac{16}{x\beta}\mathrm{e}^{-7/4}p^2\sqrt{A_n}\right)^{2p}. \end{aligned}$$

Now setting $h(y)=(C_n y)^{4y}$ with $C_n^2=(16/x\beta)\mathrm{e}^{-7/4}\sqrt{A_n}$, one obtains

$$P(|S_n|\geqslant x)\leqslant \mathrm{e}^{1/12}\sqrt{8\pi}h(p).$$

Define the convex function $g(y)=\log h(y)$. Clearly $\inf_{y\in\mathbb{R}^+} g(y)=g(1/\mathrm{e}C_n)$.

Suppose that $\mathrm{e}C_n\leqslant 1$ and let $p_0=[1/\mathrm{e}C_n]$, then

$$P(|S_n|\geqslant x)\leqslant \mathrm{e}^{1/12}\sqrt{8\pi}h(p_0)\leqslant \mathrm{e}^{4+1/12}\sqrt{8\pi}\exp\left(\frac{-4}{\mathrm{e}C_n}\right).$$

Suppose now that $\mathrm{e}C_n\geqslant 1$, then $1\leqslant \mathrm{e}^{4+1/12}\sqrt{8\pi}\exp(-4/\mathrm{e}C_n)$.

The above estimations then prove Corollary 1. □

6. Proofs for the examples

6.1. Associated sequences

Before adapting our moment inequalities to associated sequences we recall some basic lemmas.

Lemma 17 (Newman, 1984). *Let $(X_n)_{n\in\mathbb{N}}$ be an associated sequence. Let h, k, h_1, k_1 be some real-valued functions, such that the functions h_1-h, h_1+h, k_1-k, k_1+k are nondecreasing. Then*

$$||\mathrm{Cov}(h(X_i, i\in A), k(X_i, i\in B))|\leqslant \mathrm{Cov}(h_1(X_i, i\in A), k_1(X_i, i\in B)).$$

In particular, there holds

Lemma 18 (Birkel, 1988b). *If $(X_n)_{n\in\mathbb{N}}$ is a sequence of associated r.v.s, then for all real-valued functions $h,k\in C_b^1$, there holds*

$$|\mathrm{Cov}(h(X_i,i\in A),k(X_i,i\in B))|\leqslant\sum_{i\in A}\sum_{j\in B}\left\|\frac{\partial h}{\partial x_i}\right\|_\infty\left\|\frac{\partial k}{\partial x_j}\right\|_\infty \mathrm{Cov}(X_i,X_j).$$

Proof of Lemma 4. Lemma 4 is an immediate consequence of Lemmas 17 and 18. □

Proof of Lemma 5. Let B be the covariance quantity defined as in (5.15). Lemma 17 yields $B\leqslant q^2\gamma_r$. This implies the $(\gamma,\mathscr{I},\psi)$-weak dependence with $\psi(h,k,u,v)=(u+v)^2$. □

Proof of Theorem 4. Let $C_{r,q}(f)$ be the coefficient associated to the sequence $(f(X_i))_i$. On the one hand, it is obvious that $C_{r,q}(f)\leqslant 2||f||_\infty^{q-1}\sup_{i\in\mathbb{N}}E|f(X_i)|$. On the other, $C_{r,q}(f)\leqslant q^2||f||_\infty^{q-2}||f'||_\infty^2\sup_{|i-j|\geqslant r}\mathrm{Cov}(X_i,X_j)$ (cf. Lemma 18). Thus

$$C_{r,q}(f)\leqslant q^2||f||_\infty^{q-2}\min\left\{||f'||_\infty^2\sup_{|i-j|\geqslant r}\mathrm{Cov}(X_i,X_j),||f||_\infty\sup_{i\in\mathbb{N}}E|f(X_i)|\right\}.$$

Hence condition $(\mathscr{H}_1)$ of Theorem 2 holds and now Theorem 4 is proved. □

6.2. Gaussian sequences

Lemma 8 is a consequence of the following.

Lemma 19. *If $(X_n)_{n\in\mathbb{N}}$ is a Gaussian centered process, then for all real-valued functions h,k in $C_b^1\cap\mathbb{L}^\infty$,*

$$|\mathrm{Cov}(h(X_i,i\in A),k(X_i,i\in B))|\leqslant\sum_{i\in A}\sum_{j\in B}\left\|\frac{\partial h}{\partial x_i}\right\|_\infty\left\|\frac{\partial k}{\partial x_j}\right\|_\infty |\mathrm{Cov}(X_i,X_j)|. \quad (6.1)$$

Define the process $(Y_n)_{n\in\mathbb{N}}$

$$Y_n=f(X_n)-Ef(X_n), \quad (6.2)$$

where f is a bounded function with bounded first derivative. Lemma 19 yields

Lemma 20. *Define the sequence (Y_n) as in (6.2). Then*

$$|\mathrm{Cov}(Y_{t_1}\dots Y_{t_m},Y_{t_{m+1}}\dots Y_{t_q})|\leqslant||f||_\infty^{q-2}||f'||_\infty^2\sum_{i=1}^m\sum_{j=m+1}^q|\mathrm{Cov}(X_{t_i},X_{t_j})|.$$

Even if we think that inequality (6.1) certainly exists in the literature (but we did not meet it) we prove Lemma 19, our guideline for its proof is Pitt (1982). Let h and k be two differentiable functions with bounded partial derivative, $X=(X_1,\dots,X_n)$ be a Gaussian centered vector with non-singular covariance matrix $\Sigma=(\sigma_{i,j})$. Pitt (1982) proves that

$$\mathrm{Cov}(h(X),k(X))=\int_0^1 F'(\lambda)\,\mathrm{d}\lambda, \tag{6.3}$$

with

$$F'(\lambda)=\frac{1}{\lambda}\int_{\mathbb{R}^n}\phi(x)\left\{\sum_{i,j}\sigma_{i,j}\frac{\partial h(x)}{\partial x_j}\frac{\partial k(\lambda,x)}{\partial x_j}\right\}\mathrm{d}x, \tag{6.4}$$

where ϕ denotes the density of X, $k(\lambda,x)=\int k(\lambda x-y)\phi_\lambda(y)\,\mathrm{d}y$, and $\phi_\lambda(x)=(1-\lambda^2)^{-n/2}\phi((1-\lambda^2)^{-1/2}x)$. Now, the partial derivative $\partial k(\lambda,x)/\partial x_j$ exists and

$$\left\|\frac{\partial k(\lambda,.)}{\partial x_j}\right\|_\infty \leqslant \lambda\left\|\frac{\partial k(\lambda,.)}{\partial x_j}\right\|_\infty\int\phi_\lambda(y)\,\mathrm{d}y\leqslant\lambda\left\|\frac{\partial k(\lambda,.)}{\partial x_j}\right\|_\infty. \tag{6.5}$$

Inequalities (6.3)–(6.5) yield

$$|\mathrm{Cov}(h(X),k(X))|\leqslant\sum_{i,j}\sigma_{i,j}\left\|\frac{\partial h}{\partial x_i}\right\|_\infty\left\|\frac{\partial k}{\partial x_j}\right\|_\infty. \tag{6.6}$$

Now, we may omit the restrictive condition that Σ is non-singular. Let $X=(X_1,\dots,X_n)$ be a Gaussian vector with covariance matrix Σ and $X^{(\varepsilon)}$ be the Gaussian vector with non-singular covariance matrix $\Sigma_\varepsilon=\Sigma+\varepsilon I$, where I is the identical matrix. $X^{(1/k)}$ converges in distribution to X as k tends to ∞. So that for bounded functions h and k,

$$\lim_{k\to+\infty}\mathrm{Cov}(h(X^{(1/k)}),k(X^{(1/k)}))=\mathrm{Cov}(h(X),k(X)).$$

Lemma 19 holds then taking the limit in the inequality (6.6) which is satisfied by $X^{(1/k)}$. □

6.3. Bernoulli shifts

Proof of Lemma 9. Define for $i_1\leqslant\cdots\leqslant i_u<i_u+r\leqslant j_1\leqslant\cdots\leqslant j_v$, $A=\{i_1,\dots,i_u\}$, $B=\{j_1,\dots,j_v\}$ and for $h,k\in\mathscr{L}$, $h_A(X)=h(\boldsymbol{X}_i,i\in A)$, $U_i^{(p)}=F(\varepsilon_{i-j}\mathbb{1}_{|j|<p}\colon j\in\mathbb{Z})$. Finally let $X_i^{(p)}=U_i^{(p)}-EU_i^{(p)}$, where p is an arbitrary positive interger p. Clearly,

$$\begin{aligned}\mathrm{Cov}(h_A(X),k_B(X))=&\,\mathrm{Cov}(h_A(X)-h_A(X^{(p)}),k_B(X))\\&+\mathrm{Cov}(h_A(X^{(p)}),k_B(X)-k_B(X^{(p)}))\\&+\mathrm{Cov}(h_A(X^{(p)}),k_B(X^{(p)})).\end{aligned} \tag{6.7}$$

We now consider those three terms.

- First, note that $h_A(X^{(p)})$ is a measurable function of $\{\varepsilon_i,i_1-p<i<i_u+p\}$ and that $k_B(X^{(p)})$ is a measurable function of $\{\varepsilon_i,j_1-p<i<j_v+p\}$. The sequence (ε_i) is independent, so that $\mathrm{Cov}(h_A(X^{(p)}),k_A(X^{(p)}))=0$, as soon as $i_u+p\leqslant i_u+r-p$, i.e. $p\leqslant r/2$.
- since h,k belong to the set $\mathscr{L}$, we deduce that

$$\begin{aligned}|\mathrm{Cov}(h_A(X)-h_A(X^{(p)}),k_B(X))|&\leqslant 2\|k\|_\infty E|h_A(X)-h_A(X^{(p)})|\\&\leqslant 4u\|k\|_\infty\delta_p\mathrm{Lip}(h),\end{aligned}$$

the last inequality is obtained from (5.1). The last term of Eq. (6.7) is analogously bounded.

Hence, the proof of Lemma 9 follows by taking $p = r/2$. □

6.4. Models with a Markovian representation

Proof of Lemma 10. Using notations from the previous section we also set $Eh_B(X^x)$ for the conditional expectation of $h(X_i, i \in B)$ given $X_{i_u} = x$. Markov's property yields

$$\mathrm{Cov}(h_A(X), k_B(X)) = \int .. \int h(x_{i_1}, \ldots, x_{i_u})(Ek_B(X^{x_u}) - Ek_B(X))\,\mathrm{d}P(x_{i_1}, \ldots, x_{i_u}).$$

Clearly,

$$\begin{aligned}|Ek_B(X^{x_u}) - Ek_B(X)| &\leqslant \int \mu(\mathrm{d}x)|k_B(X^{x_u}) - k_B(X^x)| \\ &\leqslant \mathrm{Lip}(k) \sum_{i \in B} \int \mu(\mathrm{d}x) E|X_i^{x_u} - X_i^x|.\end{aligned}$$

Hence we obtain, by induction and using Property ($\mathscr{C}_1$) (we recall that for all $i \in B$, $i - i_u \geqslant r$)

$$E|X_i^{x_u} - X_x^x| \leqslant \alpha^{i-i_u}|x_u - x| \leqslant \alpha^r |x_u - x|.$$

Hence

$$\int\int \mu(\mathrm{d}x_u)\mu(\mathrm{d}x) E|X_i^{x_u} - X_i^x| \leqslant \alpha^r \int\int \mu(\mathrm{d}x_u)\mu(\mathrm{d}x)|x_u - x| \leqslant 2\alpha^r E|X_0|.$$

The lemma is so proved. □

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References

Bahtin, Y., Bulinski, A.V., 1997. Fund. Appl. Math. 3, 1101–1108. (in Russian).
Billingsley, P., 1968. Probability and Measure. Wiley, New York.
Birkel, T., 1988aa. Moment bounds for associated sequences. Ann. Probab. 16, 1184–1193.
Birkel, T., 1988bb. The invariance principle for associated processes. Stochastic Process. Appl. 27, 57–71.
Birkel, T., 1992. A note on the strong law of large numbers of positively dependent random variables. Statist. Probab. Lett. 14, 355–362.
Bulinski, A., Keane, M.S., 1996. Invariance principle for associated random fields. J. Math. Sci. NY 81, 2905–2911.
Bryc, W., Smoleński, W., 1993. Moment conditions for almost sure convergence of weakly correlated random variables. Proc. Amer. Math. Soc. 119, 629–635.
Comtet, L., 1970. Analyse Combinatoire, Tome 1. Coll. S.U.P., Presses Universitaires de France.
Doeblin, W., Fortet, R., 1937. Sur des chaines à liaisons complètes. Bull. Soc. Math. France 65, 132–148.
Doukhan, P., 1994. Mixing: Properties and Examples, Lecture Notes in Statistics, Vol. 85. Springer, Berlin.
Doukhan, P., Portal, F., 1983. Moments de variables aléatoires mélangeantes. C.R. Acad. Sci. Paris. Sér. 1 297, 129–132.

Duflo, M., 1996. Algorithmes stochastiques, Math. Appl., Vol. 23, Springer Verlag, Berlin.
Esary, J., Proschan, F., Walkup, D., 1967. Association of random variables with applications. Ann. Math. Statist. 38, 1466–1476.
Ibragimov, I.A., Linnik, Y.V., 1971. Independent and Stationary Sequences of Random Variables. Walter Nordhoof pub., Groningen.
Jakubowski, A., 1993. Minimal conditions in p-stable limit theorems. Stochastic Process. Appl. 44, 291–327.
Jakubowski, A., Szewczak, Z., 1990. A normal convergence criterion for strongly mixing stationary sequences. In: Révész, P. (Ed.), Limit Theorems in Probability and Statistics, Pécs, 1989, Coll. Math. Soc. Janos Bolyai, Vol. 57, North-Holland, Amsterdam, pp. 281–292.
Louhichi, S., 1998. Weak convergence for empirical processes of associated sequences. Prépublication 98.36, Université de Paris-Sud.
Moricz, F.A., Serfling, R.J., Stout, W.F., 1982. Moment and probability bounds with quasi-superadditive structure for the maximum partial sum. Ann. Probab. 10, 1032–1040.
Newman, C.M., 1984. Asymptotic independence and limit theorems for positively and negatively dependent random variables. In: Tong, Y.L. (Ed.), Inequalities in Statistics and Probability, IMS Lecture Notes-Monograph Series, Vol. 5, pp. 127–140
Petrov, V.V., 1995. Limit Theorems of Probability Theory: Sequences of Independent Random Variables. Clarendon Press, Oxford.
Pitt, L., 1982. Positively Correlated normal variables are associated. Ann. Probab. 10, 496–499.
Rio, E., 1993. Covariance inequalities for strongly mixing processes. Ann. Inst. H. Poincaré 29, 587–597.
Rio, E., 1994. Inégalités de moments pour les suites stationnaires et fortement mélangeantes. C.R. Acad. Sci. Paris Sér. 1 318, 355–360.
Rio, E., 1997. Théorèmes limites pour les suites de variables aléatoires faiblement dépendantes. Prépublication 97.81, Université de Paris-Sud.
Rosenblatt, M., 1956. A central limit Corollary And a strong mixing condition. Proc. Natl. Acad. Sci. USA 42, 43–47.
Rosenblatt, M., 1980. Linear Processes and bispectra. J. Appl. Probab. 17, 265–270.
Shao, Q.M., 1995. Maximal inequalities for partial sums of ρ-mixing sequences. Ann. Probab. 23-2, 948–965.
Shao, Q.M., Yu, H., 1996. Weak convergence for weighted empirical processes of dependent sequences. Ann. Probab. 24 (4), 2052–2078.
Tran, L., 1990. Recursive kernel density estimators under a weak dependence condition. Ann. Inst. Statist. Math. 42, 305–329.
Withers, C.S., 1981. Central limit Theorems for dependent variables. Z. Wahrsch. Verw. Gebiete 57, 509–534 (Corrigendum (1983), 63, 555).
Yu, H., 1993. A Gilvenko-Cantelli lemma and weak convergence for empirical processes of associated sequences. Probab. Theory Related fields 95, 357–370.
Yokoyama, R., 1980. Moment bounds for stationary mixing sequences. Z. Wahrsch. Verw. Gebiete 52, 45–57.
Yoshihara, K., 1973. Billingsley's theorems on empirical processes of strong mixing sequences. Yokohama Math. J. 23, 1–7.