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The Lamperti correspondence extended to Lévy processes and semi-stable Markov processes in locally compact groups

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Abstract

We establish Lamperti representations for semi-stable Markov processes in locally compact groups. We also study the particular cases of processes with values in $\mathbb{R}$ and $\mathbb{C}$ under the hypothesis that they do not visit 0. These Lamperti representations yield some properties of these semi-stable Markov processes.
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1. Introduction

Semi-stable Markov processes were introduced by Lamperti in [10], as particular processes taking their values in $\mathbb{R}_+$. Let $\alpha > 0$ be fixed. A Markov process X, with values in $\mathbb{R}_+$, is called an α-semi-stable Markov process, in the sense of Lamperti, if its transition function satisfies

$$P_{at}(x, A) = P_t(a^{-\alpha}x, a^{-\alpha}A) \tag{1}$$

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for any $a > 0$, $x > 0$, $A \in \mathcal{B}(\mathbb{R}_+)$. Then (see [10,1]) there exists a Lévy process ξ, such that the following Lamperti representation holds

$$\mathrm{e}^{\xi_t} = X_{A_t}, \quad A_t = \int_0^t \mathrm{e}^{\alpha^{-1}\xi_s}\,\mathrm{d}s.$$

The Lamperti representation obtained in [10] has proved to be very useful in the studies of the exponential functionals of Brownian motion, hyperbolic Brownian motion, windings of planar Brownian motion, Brownian motion in random media and in the studies of self-similar fragmentations.

In order to extend the above definitions to processes with values in $\mathbb{R}$ and more generally in $\mathbb{R}^d$ different ways have been proposed. Kiu [8,9] extended some results of [10] to $\mathbb{R}^d$, $d \geq 1$, by defining an α-semi-stable Markov process as a Markov process with a transition function such that (1) holds, but now $x \in \mathbb{R}^d$, and $A \in \mathcal{B}(\mathbb{R}^d)$. Graversen and Vuolle-Apiala in [4] considered the additional condition of isotropy for semi-stable Markov processes in $\mathbb{R}^d \setminus \{0\}$, $d \geq 2$. This means that

$$P_t(x, A) = P_t(\phi(x), \phi(A))$$

for any $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R})$ and $\phi \in O(d)$, where $O(d)$ denotes the group of orthogonal transformations on $\mathbb{R}^d$. Under this condition they obtained a skew product representation for this type of Markov processes. Yet another extension to $\mathbb{R}^d_+$ was proposed by Jacobsen and Yor in [7] which also leads to Lamperti representation for semi-stable Markov processes in $\mathbb{R}^d_+$.

In this paper we complete the line of research of [10,9] and [4] by obtaining Lamperti representations for semi-stable Markov processes in $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and more generally in a locally compact group with a countable basis for its topology. We give what seems to be a new definition of α-semi-stable Markov processes in $\mathbb{R}^*$ and I-semi-stable Markov processes, the latter being processes with a locally compact group G as state space with $I : G \to \mathbb{R}_+$ a homomorphism. A particular example of such defined semi-stable Markov processes in $\mathbb{R}^*$ was studied in [5] and lies at the source of inspiration for this work.

Here are some details about the organisation of the rest of the paper. First of all we study multiplicative Lévy processes in $\mathbb{R}^*$ and $\mathbb{C}^*$ using and extending the results from [13] and [11]. Then we exhibit an intimate relation between semi-stable Markov processes and multiplicative Lévy processes which extends an analogous result in [9]. Multiplicative Lévy processes in $\mathbb{R}^*$ and $\mathbb{C}^*$ are in fact semimartingales just as their additive counterparts—the Lévy processes. This fact permits us to use stochastic calculus techniques to prove our results. Our main result given by Theorem 16 generalises some known results in the literature and the examples studied in this paper. It also permits us to study semi-stable Markov processes in Lie groups using properties of corresponding Lévy processes in Lie groups. We hope to study a number of such examples in a subsequent paper.

2. Statement of results

Notation 1. *Let X be a Markov process with state space E. As in [7] $X^{(x)}$ denotes X starting from the given state $x \in E$. In general $X^{(x)}$ can be defined on different probability spaces. Only in special cases (3), (6), (10), (12) and (15) is it possible to define $X^{(x)}$ on the same probability space simultaneously for all x. Conversely $(X^{(x)})_{x \in E}$ will be called a Markov process if all $X^{(x)}$ (possibly defined on different probability spaces) enjoy the Markov property with the same transition function.*

2.1. Multiplicative Lévy processes and semi-stable Markov processes in $\mathbb{R}^*$ and $\mathbb{C}^*$

Definition 2. Let $\alpha > 0$ be fixed. A Markov process $(X^{(x)})_{x \in \mathbb{R}^*}$, respectively $(X^{(x)})_{x \in \mathbb{C}^*}$, with state space $\mathbb{R}^*$, respectively $\mathbb{C}^*$, is called a real α-semi-stable Markov process, respectively a complex α-semi-stable Markov process, if for any $c \in \mathbb{R}^*$, respectively $c \in \mathbb{C}^*$, and any initial state $x \in \mathbb{R}^*$, respectively $x \in \mathbb{C}^*$,

$$(cX_t^{(\frac{x}{c})})_{t \geq 0} \overset{(d)}{=} (X^{(x)}_{|c|^{\frac{1}{\alpha}} t})_{t \geq 0}. \tag{2}$$

Note that in the complex case our definition is equivalent to the definition of a semi-stable Markov process in $\mathbb{R}^2 \setminus \{0\}$ as a rotation-invariant strong Markov process with scaling property as in [4]. From now on, for simplicity, we fix $\alpha = \frac{1}{2}$. Obviously the results of this paper can be extended to any $\alpha > 0$. We now present a slightly modified definition of a multiplicative process as given in [13] and [11].

Definition 3. A process Z with state space $\mathbb{R}^*$, respectively $\mathbb{C}^*$, càdlàg in $\mathbb{R}^*$ (respectively in $\mathbb{C}^*$), $Z_0 = 1$, is called a multiplicative Lévy process if for any $t > 0$, $h > 0$ $Z_{t+h}Z_t^{-1}$ is independent from $\mathcal{F}_t = \sigma\{Z_u, u \leq t\}$ and the law of $Z_{t+h}Z_t^{-1}$ does not depend on t.

Note that X_t is càdlàg in $\mathbb{R}^*$ (respectively in $\mathbb{C}^*$) implies that $X_{t-} \in \mathbb{R}^*$ ($X_{t-} \in \mathbb{C}^*$) for any $t > 0$. Let Z be a multiplicative Lévy process in $\mathbb{R}^*$ (respectively in $\mathbb{C}^*$). Define

$$Z_t^{(x)} := xZ_t, \tag{3}$$

then from the results in ([11], Section 1.1) $(Z^{(x)})_{x \in \mathbb{R}^*}$ (respectively $(Z^{(x)})_{x \in \mathbb{C}^*}$) is a Feller process.

For multiplicative Lévy processes in $\mathbb{R}^*$ we obtain a representation given in the following theorem.

Theorem 4. *Let Z be a multiplicative Lévy process with state space $\mathbb{R}^*$. Then there exists a Lévy process ξ, $\xi_0 = 0$, and a compound Poisson process $N_t^U := \sum_{k \leq N_t} U_k$, obtained from a Poisson process N and an i.i.d. sequence $(U_k)_{k \geq 1}$, such that*

$$Z_t = \exp(\xi_t + N_t^U + i\pi N_t) \tag{4}$$

and $(\xi_t)_{t \geq 0}$, $(N_t)_{t \geq 0}$, $(U_k)_{k \geq 1}$ are independent.
The converse is true, i.e. let ξ, $\xi_0 = 0$ be a Lévy process, N_t^U be a compound Poisson process independent from ξ, then Z defined by (4) is a multiplicative Lévy process in $\mathbb{R}^*$.

In order to prove Theorem 4, we need the following lemma.

Lemma 5. *Let Z be a multiplicative Lévy process with state space $\mathbb{R}^*$. Then Z is a semimartingale and there is at most a finite number of sign switching jumps (i.e. $Z_s Z_{s-} < 0$) in any finite interval $[0, T]$, $T < \infty$.*

The correspondence between multiplicative Lévy processes and semi-stable Markov processes is given by

Theorem 6. (a) *Let Z with state space $\mathbb{R}^*$ be a multiplicative Lévy process and*

$$\int_0^\infty (Z_s)^2 \mathrm{d}s = +\infty \ a.s. \tag{5}$$

Define $X_t^{(x)}$ by

$$X_{\int_0^u (Z_v^{(x)})^2 \mathrm{d}v}^{(x)} = Z_u^{(x)}, \tag{6}$$

where $Z_t^{(x)}$ is defined by (3), then $(X^{(x)})_{x \in \mathbb{R}^*}$ is a $\frac{1}{2}$-semi-stable strong Markov process with state space $\mathbb{R}^*$, càdlàg in $\mathbb{R}^*$. Furthermore for any $x \in \mathbb{R}^*$

$$\int_0^\infty \frac{\mathrm{d}s}{(X_s^{(x)})^2} = \infty \ a.s.$$

(b) *Conversely let $(X^{(x)})_{x \in \mathbb{R}^*}$ be a $\frac{1}{2}$-semi-stable strong Markov process with state space $\mathbb{R}^*$ which is càdlàg in $\mathbb{R}^*$ and*

$$\int_0^{+\infty} \mathrm{d}s \, \frac{1}{(X_s^{(x)})^2} = +\infty \tag{7}$$

for all $x \in \mathbb{R}^*$. Define Z_t by

$$Z_{\int_0^t \mathrm{d}s \frac{1}{(X_s^{(x)})^2}} = \frac{1}{x} X_t^{(x)},$$

then Z is a multiplicative Lévy process with a distribution that does not depend on x.

Corollary 7. *Let Z be a multiplicative Lévy process in $\mathbb{R}^*$ verifying (5). Then the corresponding $\frac{1}{2}$-semi-stable Markov process $(X^{(x)})_{x \in \mathbb{R}^*}$ defined by (6) is a Feller process with state space $\mathbb{R}^*$.*

Similarly to the real case, we obtain the two following theorems and corollary.

Theorem 8. *Let Z be a multiplicative Lévy process with state space $\mathbb{C}^*$. Then there exists a two-dimensional Lévy process (ξ, η), $\xi_0 = 0$, $\eta_0 = 0$, such that*

$$Z_t = \exp(\xi_t + \mathrm{i}\eta_t). \tag{8}$$

The converse is true, i.e. let (ξ, η) be a two-dimensional Lévy process, $\xi_0 = 0$, $\eta_0 = 0$, then Z defined by (8) is a multiplicative Lévy process in $\mathbb{C}^*$.

Theorem 9. (a) *Let Z with state space $\mathbb{C}^*$ be a multiplicative Lévy process and*

$$\int_0^\infty |Z_s|^2 \mathrm{d}s = +\infty \ a.s. \tag{9}$$

Define $X_t^{(x)}$ by

$$X_{\int_0^u |Z_v^{(x)}|^2 \mathrm{d}v}^{(x)} = Z_u^{(x)}, \tag{10}$$

where $Z_t^{(x)}$ is defined by (3), then $(X^{(x)})_{x \in \mathbb{C}^*}$ is a $\frac{1}{2}$-semi-stable strong Markov process with state space $\mathbb{C}^*$, càdlàg in $\mathbb{C}^*$. Furthermore for any $x \in \mathbb{C}^*$

$$\int_0^\infty \frac{\mathrm{d}s}{(X_s^{(x)})^2} = \infty \ a.s.$$

(b) *Conversely let $(X^{(x)})_{x \in \mathbb{C}^*}$ be a $\frac{1}{2}$-semi-stable strong Markov process with state space $\mathbb{C}^*$ which is càdlàg in $\mathbb{C}^*$ and*

$$\int_0^{+\infty} \mathrm{d}s \, \frac{1}{|X_s^{(x)}|^2} = +\infty \tag{11}$$

for all $x \in \mathbb{C}^*$. Define Z_t by

$$Z_{\int_0^t \mathrm{d}s \frac{1}{|X_s^{(x)}|^2}} = \frac{1}{x} X_t^{(x)},$$

then Z is a multiplicative Lévy process with a distribution that does not depend on x.

Corollary 10. *Let Z be a multiplicative Lévy process in $\mathbb{C}^*$ verifying (9). Then the corresponding $\frac{1}{2}$-semi-stable Markov process $(X^{(x)})_{x \in \mathbb{C}^*}$ defined by (10) is a Feller process with state space $\mathbb{C}^*$.*

The condition: (X_t and $X_{t-} \neq 0$) can be weakened by considering a killed version of X: start with a Markov process defined on $\mathbb{R}$ which has the semi-stability property (2). Denote

$$\zeta = \inf\{t > 0 \mid X_t = 0 \text{ or } X_{t-} = 0\}.$$

It is clear that if X is a $\frac{1}{2}$-semi-stable Markov process in $\mathbb{R}$ that may hit 0, then $X^2 = ((X_t^{(x)})^2)_{t \geq 0, x \in \mathbb{R}^*}$ verifies (2) with $\alpha = 1$. Denote by $P_t(x, \mathrm{d}y)$ the semigroup of X. Then from (2)

$$P_t(x, A \cup -A) = P_t(-x, A \cup -A)$$

for any $A \in \mathcal{B}(\mathbb{R})$. Hence X^2 is a Markov process and $\zeta = \zeta'$, where

$$\zeta' = \inf\{t > 0 \mid X_t^2 = 0 \text{ or } X_{t-}^2 = 0\}.$$

Now one can use some of the results obtained by Lamperti in [10] for positive semi-stable Markov processes, in particular:

Lemma 11. (i) *Either $P_x(\zeta < \infty) = 1$ for all $x \in \mathbb{R}^*$, or else $P_x(\zeta < \infty) = 0$ for all $x \in \mathbb{R}^*$.*
(ii) *Denote $A_t := \int_0^t \mathrm{d}s \frac{1}{X_s^2}$. In the case $P_x(\zeta < \infty) = 0$ for all $x > 0$*

$$P_x(\lim_{t \to \infty} A_t = \infty) = 1, \quad x \in \mathbb{R}^*.$$

Hence there are only two cases for semi-stable Markov processes in $\mathbb{R}$: either $P_x(\zeta < \infty) = 0$ for all $x \in \mathbb{R}^*$, or $P_x(\zeta < \infty) = 1$ for all $x \in \mathbb{R}^*$. Due to (ii) in Lemma 11 in the first case the condition (7) is true and Theorem 6 can be applied. If $P_x(\zeta < \infty) = 1$ in order to apply our results one should work with the process X killed at time $T_0 = \zeta$. The complex case can be treated similarly.

2.2. Multiplicative Lévy processes and semi-stable Markov processes in general settings

Let $(G, *)$ be a locally compact group whose topology admits a countable basis and e denotes the neutral element of the group G. For convenience, we will write ab as short for $a * b$.

Definition 12. (a) A process Z with state space G, càdlàg in G, $Z_0 = e$, is called a left Lévy process if for any $t > 0$, $h > 0$ $Z_t^{-1} Z_{t+h}$ is independent from $\mathcal{F}_t = \sigma\{Z_u, u \leq t\}$ and the law of $Z_t^{-1} Z_{t+h}$ does not depend on t.
(b) A process Z with state space G, càdlàg in G, $Z_0 = e$, is called a right Lévy process if for any $t > 0$, $h > 0$ $Z_{t+h} Z_t^{-1}$ is independent from $\mathcal{F}_t = \sigma\{Z_u, u \leq t\}$ and the law of $Z_{t+h} Z_t^{-1}$ does not depend on t.

The discussion of left and right Lévy processes is given in ([11], Section 1.1). The results in ([11], Section 1.1) imply the following lemma which we will need in the proof of Theorem 16.

Lemma 13. *Let Z be a left Lévy process in G. Define*

$$Z_t^{(x)} := x Z_t, \tag{12}$$

then $(Z^{(x)})_{x \in G}$ is a Feller process with state space G. Furthermore for any finite stopping time τ the process $((Z_\tau^{(x)})^{-1} Z_{\tau+t}^{(x)})_{t \geq 0}$ is independent from $\mathcal{F}_\tau$ and has the same law as $(Z_t^{(1)})_{t \geq 0}$.

Remark 14. In Definition 12(a) one might require only that Z is continuous in probability in G, and not necessarily that Z is càdlàg in G. Then P_t, defined by

$$P_t f(x) = \mathbb{E} f(x Z_t),$$

for $x \in G$ and $f \in C_0(G)$, is a Feller semigroup due to the dominated convergence theorem. Therefore there is a càdlàg modification of Z and $Z_0^{-1} Z$ is a left Lévy process. The same is true for Definition 12(b).

Now let us fix a homomorphism of groups $(G, *)$ and $(\mathbb{R}_+, \cdot) I : (G, *) \rightarrow (\mathbb{R}_+, \cdot)$ i.e.

- $I(e) = 1$,
- $I(ab) = I(a)I(b)$.

We suppose that I is continuous on G.

Definition 15. (a) A Markov process $(X^{(x)})_{x \in G}$ with state space G is called a left I-semi-stable Markov process if for any $c \in G$ and any initial state $x \in G$

$$(c X_t^{(c^{-1}x)})_{t \geq 0} \overset{(d)}{=} (X_{I(c)t}^{(x)})_{t \geq 0}. \tag{13}$$

(b) A Markov process $(X^{(x)})_{x \in G}$ with state space G is called a right I-semi-stable Markov process if for any $c \in G$ and any initial state $x \in G$

$$(X_t^{(xc^{-1})} c)_{t \geq 0} \overset{(d)}{=} (X_{I(c)t}^{(x)})_{t \geq 0}.$$

For $x \in G$, $A \in \mathcal{B}(G)$ ($\mathcal{B}(G)$-Borel σ-algebra in G) let $P_t(x, A)$ be a transition function of a left I-semi-stable Markov process then (13) is equivalent to

$$P_{I(c)t}(x, A) = P_t(c^{-1}x, c^{-1}A).$$

Note that if Z is a left Lévy process, then Z^{-1} is a right Lévy process, and vice versa. Obviously the same is true for I-semi-stable Markov processes, i.e. if X is a left I-semi-stable Markov process, then X^{-1} is a right I-semi-stable Markov process, and vice versa. If G is Abelian then left and right I-semi-stable Markov processes (Lévy processes) coincide. From now on we consider only left Lévy processes and left I-semi-stable Markov processes.

Theorem 16. (a) *Let Z with state space G be a left Lévy process and*

$$\int_0^\infty I(Z_s) ds = +\infty \ a.s. \tag{14}$$

Define $X_t^{(x)}$ by

$$X_{\int_0^u I(Z_v^{(x)}) dv}^{(x)} = Z_u^{(x)}, \tag{15}$$

where $Z_t^{(x)} := x Z_t$, *then* $(X^{(x)})_{x \in G}$ *is a left I-semi-stable strong Markov process with state space G and càdlàg in G. Furthermore for any $x \in G$*

$$\int_0^\infty \frac{ds}{I(X_s^{(x)})} = \infty \ a.s. \tag{16}$$

(b) *Conversely let* $(X^{(x)})_{x \in G}$ *be a left I-semi-stable strong Markov process with state space G which is càdlàg in G and*

$$\int_0^{+\infty} ds \frac{1}{I(X_s^{(x)})} = +\infty \tag{17}$$

for all $x \in G$. Define Z_t by

$$Z_{\int_0^t ds \frac{1}{I(X_s^{(x)})}} = x^{-1} X_t^{(x)},$$

then Z is a left Lévy process with a distribution that does not depend on x.

Corollary 17. *Let Z be a left Lévy process in G verifying (14). Then the corresponding left I-semi-stable Markov process $(X^{(x)})_{x \in G}$ defined by (15) is a Feller process with state space G.*

Example 18. • Let $G = \mathbb{R}_+$ with operation of product and $I(x) = x^{\frac{1}{\alpha}}$. We get the classical definition of α-semi-stable Markov process due to Lamperti.

• Let $G = \mathbb{R}^*$ with the product operation and $I(x) = |x|^{\frac{1}{\alpha}}$. We get Definition 2.

• Let $G = \mathbb{C}^*$ with the product operation of complex numbers and $I(x) = |x|^2$ i.e. for any $c \in \mathbb{C}^*, x \in \mathbb{C}^*$

$$(c X_t^{(xc^{-1})})_{t \geq 0} \overset{(d)}{=} (X_{|x|^2 t}^{(x)})_{t \geq 0}.$$

We get a definition of a semi-stable Markov process as a rotation-invariant strong Markov process with scaling property as in [4].

• Let $G = \mathbb{R}_+^n$ and define the operation $*$ for $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$

$$x * y = (x_1 y_1, \dots, x_n y_n)$$

and take $I(x) = x_1 x_2 \dots x_n$ then we get the definition of a semi-stable Markov process from [7].

• Let $G = \mathbb{R}^n \setminus \{0\}$ and define the operation $*$ for $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$

$$x * y = (x_1 y_1, \dots, x_n y_n)$$

and take $I(x) = |x_1 x_2 \dots x_n|$.

• Let G be the Lie group $GL(n, \mathbb{R})$ (for the study of left Lévy processes in $GL(n, \mathbb{R})$ we refer to [11], Section 1.5). For $x \in G$ let $I(x) = |\det x|$, then we get a left I-semi-stable Markov process in $GL(n, \mathbb{R})$.

3. Examples

In this section we give two examples of semi-stable Markov processes which were studied in the literature. The first example is the planar Brownian motion

$$B = B^1 + i B^2,$$

where B^1, B^2 are independent real valued Brownian motions (see [3,2,12]). Since $\{0\}$ is a polar set for this process the planar Brownian motion is a $\frac{1}{2}$-semi-stable Markov process in $\mathbb{C}^*$. One can easily get

$$B_t = \exp\left(C_{\int_0^t \frac{ds}{|B_s|^2}}\right),$$

where C is still a planar Brownian motion.

Another example is the Dunkl process X which is a real-valued Feller process with the infinitesimal generator of the form

$$L_k f(x) = \frac{1}{2} f''(x) + k\left(\frac{1}{x} f'(x) + \frac{f(-x) - f(x)}{2x^2}\right),$$

where $f \in C^2(\mathbb{R})$ (see [5]). Suppose that $\nu = k - \frac{1}{2} \geq 0$ then the Dunkl process is a $\frac{1}{2}$-semi-stable Markov process with the Lamperti representation

$$X_t^{(x)} = x Y_{A_t},$$

where $A_t = \int_0^t \frac{ds}{X_s^2}$, $Y_u = \exp(\beta_u + \nu u + i\pi N_u^{(\lambda)})$, $\lambda = \frac{k}{2}$, $x \neq 0$ and $(N_u^{(\lambda)})$ is a Poisson process with parameter λ, (β_u) is a Brownian motion, and $N^{(\lambda)}$ and β are independent.

4. Proofs

4.1. Proofs of Lemma 5, and Theorem 8

Proof of Lemma 5. Let Z be an $\mathbb{R}^*$-valued multiplicative Lévy process, then for any $t > 0$ $Z_{t-} \neq 0$. Define

$$U_t := \ln |Z_t|,$$

then U is a Lévy process on $\mathbb{R}$ (in particular, it is càdlàg, finite with all left-limits finite), hence it is (almost surely) bounded on any finite interval and consequently Z itself is bounded away from 0 on any finite time interval and $|Z|$ is a semimartingale. Therefore for any $T > 0$ there is a random variable $C_T > 0$ such that $|Z_t| > C_T$, $t \in [0, T]$. Let $\tau \in [0, T]$ be the time of a sign switching jump, i.e. $Z_\tau Z_{\tau-} < 0$, then $|\Delta Z_\tau| > 2C_T$. Hence there is only a finite number of sign switching jumps on any finite interval and the following process is correctly defined

$$Z_t^0 := \prod_{s \leq t, Z_s Z_{s-} < 0} \frac{Z_s}{Z_{s-}} = \prod_{s_k \leq t} \frac{Z_{s_k}}{Z_{s_k-}},$$

the s_k denoting the times at which there is a sign change for Z with $0 < s_1 < s_2 < \cdots$. Then Z^0 is a multiplicative Lévy process with piecewise constant trajectories. Define U_k by

$$\frac{Z_{s_k}}{Z_{s_k-}} = -e^{U_k} = e^{U_k + i\pi},$$

then

$$Z_t^0 = \exp \sum_{s_k \leq t} (U_k + i\pi)$$

and $\sum_{s_k \leq t} (U_k + i\pi)$ is a Lévy process with piecewise constant trajectories so it is a compound Poisson process. Hence Z^0 is a semimartingale. Recalling that $Z_0 \equiv Z_0^0 \equiv 1$ one has

$$Z_t = |Z_t| Z_t^0 \exp - \sum_{s_k \leq t} U_k.$$

$\sum_{s_k \leq t} U_k$ is also a compound Poisson process and we have seen that $|Z|$ is a semimartingale. Therefore Z is a semimartingale. $\square$

Proof of Theorem 8. In order to prove Theorem 8 we extend the proof given in ([13], p. 242). Obviously if (ξ, η) is a two-dimensional Lévy process, $\xi_0 = 0$, $\eta_0 = 0$, then Z, as defined by (8), is a multiplicative Lévy process, so it remains to show that if Z is a multiplicative Lévy process in $\mathbb{C}^*$, then the representation (8) holds.

Now let $\mathbb{T}$ be the group of real numbers of $[0, 1[$ with operation $+ \pmod 1$, $S^1 := \{z \in \mathbb{C}^* \| z| = 1\}$. The distance on $\mathbb{T}$ is defined by $r(x, y) := \min(|x - y|, 1 - |x - y|)$. Then $\mathbb{T}$ is the quotient space $[0, 1]/ \sim$, where $\sim$ is defined by

$$x \sim y \Longleftrightarrow x - y = 0 \pmod 1.$$

Define $f : \mathbb{T} \to S^1$ by $f(x) := e^{i2\pi x}$, with inverse

$$g(z) = \frac{1}{2\pi} \arg z. \tag{18}$$

Then f and g establish a homeomorphism $\mathbb{T} \cong S^1$. Let us construct the following path transformation S from the set of càdlàg trajectories on $\mathbb{T}$ to the set of càdlàg trajectories on $\mathbb{R}$. Let x_t be a càdlàg trajectory on $\mathbb{T}$ such that $x_0 = 0$. Define the following times

$$\tau_0 = 0,$$
$$\tau_{k+1} = \inf \left\{ t > \tau_k \mid r(x_{\tau_k}, x_t) \geq \frac{1}{4} \right\},$$

then $\tau_k \to \infty$ as $k \to \infty$. Define a function from $\mathbb{T}$ to $\mathbb{R}$ by

$$\varkappa(x) := \begin{cases} x, & \text{if } x < 1/2, \\ x - 1, & \text{if } x \geq 1/2. \end{cases}$$

One can check that if $r(x, 0) < \frac{1}{4}$, $r(y, 0) \leq \frac{1}{4}$ then

$$\varkappa(x + y) = \varkappa(x) + \varkappa(y). \tag{19}$$

If $\tau_k \leq t < \tau_{k+1}$ define

$$Sx_t := \sum_{1 \leq i \leq k} [\varkappa(x_{\tau_i -} - x_{\tau_{i-1}}) + \varkappa(x_{\tau_i} - x_{\tau_i -})] + \varkappa(x_t - x_{\tau_k}). \tag{20}$$

Since $\varkappa(x) = x \pmod 1$ $Sx_t = x_t \pmod 1$. Sx_t is a càdlàg trajectory on $\mathbb{R}$ because $\varkappa$ is a continuous map from $\mathbb{T}$ (with distance r) to $\mathbb{R}$.

Let π_t^n be a partition of $[0, t]$, i.e. $\pi_t^n = \{t_0, t_1, \ldots, t_n\}$ and $0 = t_0 < t_1 < \cdots < t_n = t$. Denote $|\pi_t^n| := \sup_{0 < i \leq n} |t_i - t_{i-1}|$. Let us prove the following formula

$$Sx_t = \lim_{|\pi_t^n| \to 0} \sum_{i=1}^n \varkappa(x_{t_i} - x_{t_{i-1}}). \tag{21}$$

Suppose that $|\pi_t^n|$ is small enough so that there are at least two points t_j in any $]\tau_i, \tau_{i+1}[$ and $]\tau_k, t[$. For any τ_i let m_i, l_i be such that $t_{m_i} < \tau_{i-1} \le t_{m_i+1} < \cdots < t_{l_i} < \tau_i \le t_{l_i+1}$. Using (19) and the fact that x_t is càdlàg one obtains

$$\sum_{j=m_i+2}^{l_i} \varkappa(x_{t_j} - x_{t_{j-1}}) = \varkappa(x_{t_{l_i}} - x_{t_{m_i+1}}) \to \varkappa(x_{\tau_i-} - x_{\tau_{i-1}}), \quad \text{as } |\pi_t^n| \to 0.$$

Moreover, it holds:

$$\varkappa(x_{t_{l_i+1}} - x_{t_{l_i}}) \to \varkappa(x_{\tau_i} - x_{\tau_i-}), \quad \text{as } |\pi_t^n| \to 0.$$

Let m_k be such that $t_{m_k} < \tau_k \le t_{m_k+1} < \cdots < t_n = t$; then

$$\sum_{j=m_k+2}^{n} \varkappa(x_{t_j} - x_{t_{j-1}}) = \varkappa(x_t - x_{t_{m_k+1}}) \to \varkappa(x_t - x_{\tau_k}), \quad \text{as } |\pi_t^n| \to 0.$$

Finally one gets (21). By taking partitions $\bar{\pi}_t^n = \pi_t^n \cup \{s\}$, from (21) we get for $s < t$ that

$$Sx_t - Sx_s = \lim_{|\bar{\pi}_t^n| \to 0} \sum_{i=1, t_i > s}^{n} \varkappa(x_{t_i} - x_{t_{i-1}}). \tag{22}$$

Let Z_t be a multiplicative Lévy process in $\mathbb{C}^*$. Define

$$\xi_t := \ln|Z_t|, \quad \xi_0 = 0$$

and

$$\zeta_t := g\left(\frac{Z_t}{|Z_t|}\right), \quad \zeta_0 = 0,$$

where g is given by (18), then for any $s < t$ $\xi_t - \xi_s = \ln|\frac{Z_t}{Z_s}|$ and $\zeta_t - \zeta_s = \frac{1}{2\pi} \arg\frac{Z_t}{Z_s} (\text{mod } 1)$ are measurable with respect to $\mathcal{F}_t := \sigma(Z_u, u \le t)$ and independent from $\mathcal{F}_s$. Furthermore the joint distribution of $\xi_t - \xi_s$ and $\zeta_t - \zeta_s$ depends on t and s only through $t - s$. This implies by (21) that $S\zeta_t - S\zeta_s$ is measurable with respect to $\mathcal{F}_t := \sigma(Z_u, u \le t)$ and independent from $\mathcal{F}_s$. Let $j \in \bar{\pi}_t^n$ be such that $t_{j-1} = s$, then for any $\lambda, \mu \in \mathbb{C}$

$$\mathbb{E}e^{\mu(\xi_t - \xi_s) + \lambda(\varkappa(\zeta_{t_j} - \zeta_{t_{j-1}}) + \cdots + \varkappa(\zeta_{t_n} - \zeta_{t_{n-1}}))} = \mathbb{E}e^{\mu(\xi_{t_j} - \xi_{t_{j-1}}) + \lambda\varkappa(\zeta_{t_j} - \zeta_{t_{j-1}})}$$
$$\times \cdots \times \mathbb{E}e^{\mu(\xi_{t_n} - \xi_{t_{n-1}}) + \lambda\varkappa(\zeta_{t_n} - \zeta_{t_{n-1}})}$$

which is equal to

$$\mathbb{E}e^{\mu(\xi_{t_j-s} - \xi_{t_{j-1}-s}) + \lambda\varkappa(\zeta_{t_j-s} - \zeta_{t_{j-1}-s})} \times \cdots \times \mathbb{E}e^{\mu(\xi_{t_n-s} - \xi_{t_{n-1}-s}) + \lambda\varkappa(\zeta_{t_n-s} - \zeta_{t_{n-1}-s})}$$

and finally

$$\mathbb{E}e^{\mu(\xi_t - \xi_s) + \lambda(\varkappa(\zeta_{t_j} - \zeta_{t_{j-1}}) + \cdots + \varkappa(\zeta_{t_n} - \zeta_{t_{n-1}}))} = \mathbb{E}e^{\mu\xi_{t-s} + \lambda(\varkappa(\zeta_{t_j-s} - \zeta_{t_{j-1}-s}) + \cdots + \varkappa(\zeta_{t_n-s} - \zeta_{t_{n-1}-s}))}. \tag{23}$$

Note that $\{t_{j-1} - s, \ldots, t_n - s\}$ is a partition of $[0, t-s]$. Passing to the limit in (23) as $|\bar{\pi}_t^n| \to 0$ one obtains that

$$\mathbb{E}e^{\mu(\xi_t - \xi_s) + \lambda(S\zeta_t - S\zeta_s)} = \mathbb{E}e^{\mu\xi_{t-s} + \lambda S\zeta_{t-s}}.$$

Define

$$\eta_t = 2\pi S \zeta_t = 2\pi S \left[g \left(\frac{Z_t}{|Z_t|} \right) \right].$$

$\xi_t := \ln |Z_t|$ is càdlàg because Z_t is càdlàg and $Z_{t-} \neq 0$. g is continuous and S maps càdlàg trajectories on $\mathbb{T}$ to càdlàg trajectories on $\mathbb{R}$. Therefore η_t is also càdlàg. Finally, one finds that $\xi_t + i\eta_t$ is a Lévy process on $\mathbb{C}$ and

$$Z_t = \exp(\xi_t + i\eta_t). \quad \square$$

Remark 19. Let y_t be a trajectory on the real line. The map $y_t \rightarrow e^{iy_t}$ "wraps" the trajectory round the circle S^1. For a given trajectory x_t on S^1 the map $x_t \rightarrow 2\pi S[g(x_t)]$ "unwraps" the trajectory x_t.

4.2. Proof of Theorems 4 and 8. A stochastic calculus approach

From Lemma 5 and Theorem 8 we have seen that the involved multiplicative Lévy processes are semimartingales. In this section, we shall assume a priori this semimartingale property. We will prove Theorem 4 and show that the proof of Theorem 8 can be simplified, thanks to stochastic calculus.

Lemma 20. *Suppose Z is a multiplicative Lévy process, with values in $\mathbb{C}^*$ (or in $\mathbb{R}^*$). Define*

$$Y_t := \int_0^t \frac{\mathrm{d}Z_s}{Z_{s-}},$$

then Y is a Lévy process.

Remark 21. Note that because all Z_s and Z_{s-} are $\neq 0$, Y_t is well defined and all $\Delta Y_t \neq -1$.

Proof. One has

$$Y_{t+h} - Y_t = \int_t^{t+h} \frac{\mathrm{d}Z_s/Z_t}{Z_{s-}/Z_t} = \int_0^h \frac{\mathrm{d}(Z_{t+s}/Z_t)}{Z_{(t+s)-}/Z_t} = \int_0^h \frac{\mathrm{d}\widehat{Y}_s}{\widehat{Y}_{s-}},$$

where $\widehat{Y}_s := \frac{Z_{t+s}}{Z_t}$ is independent from $\sigma(Z_u, u \leq t)$ and consequently is independent from $\sigma(Y_u, u \leq t)$. Since the law of $\widehat{Y}_s$ does not depend on t, the process Y_t is a Lévy process. $\quad \square$

One can write

$$Z_t = 1 + \int_0^t Z_{s-} \frac{\mathrm{d}Z_s}{Z_{s-}} = 1 + \int_0^t Z_{s-} \mathrm{d}Y_s.$$

We get the Doléans–Dade exponential in complex form as in [6]

$$Z_t = \exp \left(Y_t - \frac{1}{2} \langle Y^c, Y^c \rangle_t \right) \prod_{0 < s \leq t} (1 + \Delta Y_s) \exp(-\Delta Y_s).$$

Proof of Theorem 4. Obviously if ξ, $\xi_0 = 0$ is a Lévy process N_t^U is a compound Poisson process independent from ξ, then Z defined by (4) is a multiplicative Lévy process. Let us prove

the converse. From Lemma 5, Z is a semimartingale. Furthermore, the following process is correctly defined

$$
Z_t^0 := \prod_{0 < s \le t} (1 + \Delta Y_s)^{\mathbb{I}_{\{\Delta Y_s < -1\}}} = \prod_{s \le t} \left(\frac{Z_s}{Z_{s-}} \right)^{\mathbb{I}_{\{\frac{Z_s}{Z_{s-}} < 0\}}}
$$

$$
= \exp \left(\sum_{0 < s \le t, \, \Delta Y_s < -1} (\ln |1 + \Delta Y_s| + i\pi) \right).
$$

One gets

$$
Z_t^1 := \frac{Z_t}{Z_t^0} = \exp \left(Y_t - \sum_{0 < s \le t, \Delta Y_s < -1} \Delta Y_s - \frac{1}{2} \langle Y^c, Y^c \rangle_t \right.
$$

$$
\left. + \sum_{0 < s \le t, \Delta Y_s > -1} (\ln(1 + \Delta Y_s) - \Delta Y_s) \right)
$$

and $\langle Y^c, Y^c \rangle_t = \sigma^2 t$. Obviously the processes

$$
\xi_t = Y_t - \sum_{0 < s \le t, \Delta Y_s < -1} \Delta Y_s - \frac{1}{2} \langle Y^c, Y^c \rangle_t + \sum_{0 < s \le t, \Delta Y_s > -1} (\ln(1 + \Delta Y_s) - \Delta Y_s),
$$

$$
\eta_t = \sum_{0 < s \le t, \Delta Y_s < -1} (\ln |1 + \Delta Y_s| + i\pi)
$$

are Lévy processes that do not jump at the same times and η_t is a compound Poisson process. Hence these processes are independent. □

Proof of Theorem 8. Let us denote $Y_t = Y_t' + iY_t''$. One has

$$
Z_t = \exp \left(Y_t' + iY_t'' - \frac{1}{2} \langle Y'^c, Y'^c \rangle_t + \frac{1}{2} \langle Y''^c, Y''^c \rangle_t - i \langle Y'^c, Y''^c \rangle_t \right)
$$

$$
\times \prod_{0 < s \le t} (1 + \Delta Y_s) \exp(-\Delta Y_s),
$$

then

$$
Z_t = \exp \left(Y_t' + iY_t'' - \frac{1}{2} \langle Y'^c, Y'^c \rangle_t + \frac{1}{2} \langle Y''^c, Y''^c \rangle_t - i \langle Y'^c, Y''^c \rangle_t \right)
$$

$$
\times \exp \left(\sum_{s \le t} \ln |1 + \Delta Y_s| + i \mathrm{Arg}(1 + \Delta Y_s) - \Delta Y_s' - i \Delta Y_s'' \right),
$$

where for any $z \in \mathbb{C}^* \mathrm{Arg}(z) \in [0, 2\pi[$ is the principal argument of z. Finally one gets

$$
Z_t = \exp \left(Y_t' - \frac{1}{2} \langle Y'^c, Y'^c \rangle_t + \frac{1}{2} \langle Y''^c, Y''^c \rangle_t + \sum_{s \le t} [\ln |1 + \Delta Y_s| - \Delta Y_s'] \right)
$$

$$
\times \exp i \left(Y_t'' - \langle Y'^c, Y''^c \rangle_t + \sum_{s \le t} [\mathrm{Arg}(1 + \Delta Y_s) - \Delta Y_s''] \right)
$$

and $\langle Y'^c, Y'^c \rangle_t = \sigma_1^2 t$, $\langle Y''^c, Y''^c \rangle_t = \sigma_2^2 t$, $\langle Y'^c, Y''^c \rangle_t = \rho \sigma_1 \sigma_2 t$, where $|\rho| \le 1$. □

Remark 22. Note how different the situation of semi-stable Markov processes which can reach 0 is. They are not even necessarily semimartingales as the well-known example of $|B_t|^\alpha$, $\alpha < 1$, shows.

4.3. Proofs of Theorems 6, 9 and 16 and Corollaries 7, 10 and 17

We give the proofs only for Theorem 16 and Corollary 17. Theorems 6 and 9, Corollaries 7 and 10 follow as particular cases.

Proof of Theorem 16. We follow the proof of the similar result in [7].

(a) As soon as $\int_0^\infty I(Z_s)\mathrm{d}s = +\infty$ one gets

$$\int_0^\infty I(Z_s^{(x)})\mathrm{d}s = I(x)\int_0^\infty I(Z_s)\mathrm{d}s = \infty,$$

hence

$$H_t^{(x)} := \inf\left\{u \ge 0 \mid \int_0^u I(Z_s^{(x)})\mathrm{d}s = t\right\} \tag{24}$$

is a.s. finite and (15) determines $X^{(x)}$ uniquely through time substitution by the strictly increasing and continuous additive functional

$$A_u^{(x)} = \int_0^u I(Z_s^{(x)})\mathrm{d}s.$$

Therefore $X^{(x)}$ is càdlàg in G. Obviously one has

$$X_t^{(x)} = Z_{H_t^{(x)}}^{(x)}.$$

Since from Lemma 13 Z is a Feller process it is also a strong Markov process and the process $(X^{(x)})_{x \in G}$ is also strong Markov. Note that all the processes $X^{(x)}$ for arbitrary x are defined on the same probability space. Denote $\mathcal{F}_s := \sigma\{Z_u, u \le s\}$, then for any $t\, H_t^{(x)}$ are $\mathcal{F}_s$-stopping times. Denote $\mathcal{G}_t^{(x)} = \mathcal{F}_{H_t^{(x)}}$, then $X^{(x)}$ is $\mathcal{G}_t^{(x)}$ adapted. Since $A_{H_t^{(x)}}^{(x)} = t$ one has $\frac{\mathrm{d}}{\mathrm{d}t}H_t^{(x)} = 1/I(Z_{H_t^{(x)}}^{(x)})$ and

$$H_t^{(x)} = \int_0^t \mathrm{d}s\, \frac{1}{I(X_s^{(x)})}.$$

Since $A^{(x)}$ increases from 0 to ∞, so does the inverse $H_t^{(x)}$ and (16) follows. One has

$$H_t^{(x)} = \inf\left\{u \ge 0 \mid \int_0^u I(x)I(Z_s)\mathrm{d}s = t\right\}$$
$$= \inf\left\{u \ge 0 \mid \int_0^u I(Z_s)\mathrm{d}s = \frac{t}{I(x)}\right\}$$

and $X_t^{(x)} = x Z_{H_{t/I(x)}}$, where $H_t := H_t^{(e)}$. Replacing x by $c^{-1}x$ and t by $t/I(c)$ one gets

$$c X_{t/I(c)}^{(c^{-1}x)} = x Z_{H_{(t/I(c))/(I(x)/I(c))}} = X_t^{(x)},$$

which leads to (13). Let us show that all $X_t^{(x)}$'s share the same transition function, i.e. if

$$P_t(x, \cdot) = \mathbb{P}(X_t^{(x)} \in \cdot),$$

then

$$\mathbb{P}(X_{t+s}^{(x)} \in \cdot | \mathcal{G}_t^{(x)}, X_t^{(x)} = y) = P_s(y, \cdot).$$

One gets

$$H_{t+s}^{(x)} = H_t^{(x)} + \tilde{H}_s,$$

where

$$\tilde{H}_s = \inf \left\{ u \geq 0 \left| \int_{H_t^{(x)}}^{H_t^{(x)}+u} I(Z_v^{(x)}) \mathrm{d}v > s \right. \right\}$$

$$= \inf \left\{ u \geq 0 \left| \int_0^u I((Z_{H_t^{(x)}}^{(x)})^{-1} Z_{H_t^{(x)}+v}^{(x)}) \mathrm{d}v > \frac{s}{I(Z_{H_t^{(x)}}^{(x)})} \right. \right\}. \tag{25}$$

Since from Lemma 13 conditionally on $\mathcal{G}_t^{(x)}$ the law of

$$(Z_{H_t^{(x)}}^{(x)})^{-1} Z_{H_t^{(x)}+v}^{(x)}$$

is the same as the law of Z_v and using (25) one gets

$$\mathbb{P}(X_{t+s}^{(x)} \in \cdot | \mathcal{G}_t^{(x)}, X_t^{(x)} = y) = \mathbb{P}(Z_{H_t^{(x)}}^{(x)} (Z_{H_t^{(x)}}^{(x)})^{-1} Z_{H_t^{(x)}+\tilde{H}_s}^{(x)} \in \cdot | \mathcal{G}_t^{(x)}, X_t^{(x)} = y)$$

$$= \mathbb{P}(y (Z_{H_t^{(x)}}^{(x)})^{-1} Z_{H_t^{(x)}+\tilde{H}_s}^{(x)} \in \cdot | \mathcal{G}_t^{(x)}, X_t^{(x)} = y)$$

$$= \mathbb{P}(y Z_{H_{s/I(y)}} \in \cdot | \mathcal{G}_t^{(x)}, X_t^{(x)} = y) = \mathbb{P}(X_s^{(y)} \in \cdot).$$

(b) Let $(X^{(x)})_{x \in G}$ be a left I-semi-stable strong Markov process. Consider $X^{(x)}$ for an arbitrary initial state x. Let $y = I(x)$. Define the process $Y^{(y)} := I(X^{(x)})$, then by (13)

$$\left(\frac{1}{y} Y_{yt}^{(y)} \right)_{t \geq 0} \overset{(d)}{=} (Y_t^{(1)})_{t \geq 0},$$

where $Y^{(1)} = I(X^{(e)})$. Hence the law of $Y^{(y)}$ depends on x only through $y = I(x)$ and $(Y^{(y)})_{t \geq 0}$ is a 1-semi-stable strong Markov process. By the result in [10] there exists a real valued Lévy process ξ such that

$$\exp \xi_u^{(a)} = Y_{A_u^{(a)}}^{(y)}, \tag{26}$$

where $A_u^{(a)} = \int_0^u \mathrm{d}v \exp \xi_v^{(a)}$, $a = \log y$, $\xi^{(a)} = \xi + a$. As in the proof of (a), one finds that the inverse of $A_u^{(a)}$

$$H_t^{(a)} := \inf\{u \geq 0 | A_u^{(a)} = t\}$$

satisfies

$$H_t^{(a)} = \int_0^t \mathrm{d}s \frac{1}{Y_s^{(y)}}. \tag{27}$$

Hence from (17) $\lim_{u \to \infty} A_u^{(a)} = \infty$ a.s.

Define the process $Z^{(x)}$ with values in G by

$$Z_u^{(x)} = X_{A_u^{(a)}}^{(x)}, \tag{28}$$

in particular from (26) $\xi_u^{(a)} = \log I(Z_u^{(x)})$. Let us define $\mathcal{G}_s := \sigma\{X_u^{(x)}, u \leq s\}$. Note that for each $uA_u^{(a)}$ is $\mathcal{G}_t$-stopping time and $Z_u^{(x)}$ is $\mathcal{F}_u$-adapted, where $\mathcal{F}_u = \mathcal{G}_{A_u^{(a)}}$. As in the proof of Theorem 1(b) in [7] in order to complete the proof let us show that for any $u \geq 0$, $h > 0$ $(Z_u^{(x)})^{-1} Z_{u+h}^{(x)}$ is independent of $\mathcal{F}_u$ with a law that depends on x, u, h through h only. One has that

$$A_{u+h}^{(a)} = A_u^{(a)} + \inf\left\{t \geq 0 \mid \int_0^t \mathrm{d}s / Y_{A_u^{(a)}+s}^{(y)} = h\right\}.$$

Therefore by the strong Markov property for $X^{(x)}$, for any $x_0 \in G$ the conditional law of

$$(Z_u^{(x)})^{-1} Z_{u+h}^{(x)} = (X_{A_u^{(a)}}^{(x)})^{-1} X_{A_{u+h}^{(a)}}^{(x)} \tag{29}$$

given $\mathcal{G}_{A_u^{(a)}}$, $X_{A_u^{(a)}}^{(x)} = x_0$ is that of

$$x_0^{-1} X_\tau^{(x_0)}, \tag{30}$$

where τ is the stopping time for $X^{(x_0)}$ given by

$$\tau = \inf\left\{t \geq 0 \left| \int_0^t \mathrm{d}s / Y_s^{(y_0)} = h\right.\right\}$$

and $Y_s^{(y_0)} = I(X^{(x_0)})$, $y_0 = I(x_0)$. Note that

$$\tau = \tau' y_0, \tag{31}$$

where

$$\tau' = \inf\left\{t \geq 0 \left| \int_0^t \mathrm{d}v \frac{1}{\frac{1}{y_0} Y_{y_0 v}^{(y_0)}} = h\right.\right\}.$$

By (13) for any $b \in G$

$$(b^{-1} X_{I(b)t}^{(b)})_{t\geq 0} \overset{(d)}{=} (X_t^{(e)})_{t\geq 0},$$

hence inserting (31) in (30) one gets that the conditional law from (29) is the marginal law of $X_{\tau_0}^{(e)}$ and τ_0 is the stopping time for $X^{(e)}$ defined by

$$\tau_0 = \inf\left\{t \geq 0 \left| \int_0^t \mathrm{d}v \frac{1}{Y_v^{(1)}} = h\right.\right\}.$$

Since this marginal law neither depends on $\mathcal{F}_u$ nor x nor u, the proof is complete. $\square$

Remark 23. Using Theorem 16 we are able to construct the canonical left I-semi-stable Markov process X via a given left Lévy process Z. We take

$$X_t^{(x)} = x Z_{H_{t/I(x)}},$$

where H is given by

$$H_t = \inf \left\{ u \geq 0 \,\middle|\, \int_0^u I(Z_s) \mathrm{d}s = t \right\}.$$

Proof of Corollary 17. From Lemma 13 $(Z^{(x)})_{x \in G}$ is a Feller process. Hence $Z^{(x)}$ is quasi-left-continuous. Let $t_n \uparrow t$, as $\to \infty$. $X_t^{(x)} = Z_{\tau_t}^{(x)}$, where $\tau_t = \inf\{u \geq 0| \int_0^u I(Z_s^{(x)})\mathrm{d}s > t\}$. Define $\mathcal{F}_s = \sigma\{Z_u^{(x)}, u \leq s\}$ and $\mathcal{G}_t = \mathcal{F}_{\tau_t}$. One has

$$\tau_{t_n} \uparrow \tau_t,$$

τ_{t_n}, τ_t are $\mathcal{F}.$-stopping times and $\tau_t < +\infty$ a.s. Since $Z_{\tau_{t_n}}^{(x)} \to Z_{\tau_t}^{(x)}$ a.s. one has $X_{t_n}^{(x)} \to X_t^{(x)}$ a.s. Then since the trajectories of X are càdlàg, the transition function of X: $P_t f(x)$ is continuous in t for any fixed $x \in G$ and $f \in C_0(G)$. Finally due to (13) one gets for any $x, y \in G$

$$P_t f(y) = \int_G f(z) P_t(y, \mathrm{d}z) = \int_G f(z) P_{tI(xy^{-1})}(x, xy^{-1}\mathrm{d}z)$$
$$= \int_G f(yx^{-1}z) P_{tI(xy^{-1})}(x, \mathrm{d}z) \underset{y \to x}{\to} P_t f(x)$$

due to the dominated convergence theorem and the fact that the maps $(x, y) \to xy$, $(x, y) \to xy^{-1}$, $x \to I(x)$ are continuous. Consequently, X is a Feller process. $\quad\square$

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