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# Bootstrap of the offspring mean in the critical process with a non-stationary immigration

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**Abstract**

In applications of branching processes, usually it is hard to obtain samples of a large size. Therefore, a bootstrap procedure allowing inference based on a small sample size is very useful. Unfortunately, in the critical branching process with stationary immigration the standard parametric bootstrap is invalid. In this paper, we consider a process with non-stationary immigration, whose mean and variance vary regularly with nonnegative exponents $\alpha$ and $\beta$, respectively. We prove that $1 + 2\alpha$ is the threshold for the validity of the bootstrap in this model. If $\beta < 1 + 2\alpha$, the standard bootstrap is valid and if $\beta > 1 + 2\alpha$ it is invalid. In the case $\beta = 1 + 2\alpha$, the validity of the bootstrap depends on the slowly varying parts of the immigration mean and variance. These results allow us to develop statistical inferences about the parameters of the process in its early stages.


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*Keywords:* Branching process; Non-stationary immigration; Parametric bootstrap; Threshold; Martingale theorem; Skorokhod space

## 1. Introduction

We consider a discrete time branching process $Z(n), n \geq 0, Z(0) = 0$. It can be defined by two families of independent, nonnegative integer valued random variables $\{X_{ni}, n, i \geq 1\}$ and

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$\{\xi_n, n \geq 1\}$ recursively as

$$Z(n) = \sum_{i=1}^{Z(n-1)} X_{ni} + \xi_n, \quad n \geq 1. \tag{1.1}$$

Assume that the variables $X_{ni}$ have a common distribution for all $n$ and $i$, and families $\{X_{ni}\}$ and $\{\xi_n\}$ are independent. The variables $X_{ni}$ will be interpreted as the number of offspring of the $i$th individual in the $(n-1)$th generation and $\xi_n$ is the number of immigrating individuals in the $n$th generation. Then $Z(n)$ can be considered as the size of the $n$th generation of the population.

In this interpretation, $a = EX_{ni}$ is the mean number of offspring of a single individual. The process $Z(n)$ is called *subcritical, critical* or *supercritical* depending on $a < 1, a = 1$ or $a > 1$, respectively. The independence assumption of families $\{X_{ni}\}$ and $\{\xi_n\}$ means that reproduction and immigration processes are independent. However, unlike in the classical models, we do not assume that $\xi_n, n \geq 1$, are identically distributed. It is well known that asymptotic behavior of the process with immigration is very sensitive to any changes of the immigration process in time.

If a sample $\{Z(k), k = 1, \ldots, n\}$ is available, then the weighted conditional least squares estimator of the offspring mean is known to be (see [15])

$$\hat{a}_n = \frac{\sum_{k=1}^{n} (Z(k) - \alpha(k))}{\sum_{k=1}^{n} Z(k-1)}, \quad \alpha(k) = E\xi_k. \tag{1.2}$$

In the process with a stationary immigration the maximum likelihood estimators (MLE) for the offspring and immigration means, which were derived in [3] for the power series offspring and immigration distributions, are based on the sample of pairs $\{(Z(k), \xi_k), k = 1, \ldots, n\}$. The MLE for the offspring mean has the same form as $\hat{a}_n$ with $\xi_k$ in place of $\alpha(k)$, and the MLE for the immigration mean is just the average of the number of immigrating individuals.

Sriram [18] investigated the validity of the bootstrap estimator of the offspring mean based on MLE and demonstrated that in the critical case the asymptotic validity of the parametric bootstrap does not hold. Similar invalidity of the parametric bootstrap for the first-order autoregressive process with autoregressive parameter $\pm 1$ was earlier proved in [2]. The main cause of the failure is the fact that in the critical case the MLE does not have the desired rate of convergence (faster than $n^{-1}$).

Recently, it was shown [14] that in the process with non-stationary immigration when the immigration mean tends to infinity the conditional least squares estimator (CLSE) has a normal limit distribution and the rate of convergence of the CLSE is faster than $n^{-1}$. In connection with this the question on the validity of the bootstrap for the model under consideration is of interest. The present paper addresses this question. It turned out that the validity of the bootstrap depends on the relative rate of the immigration mean and variance. Assuming that the immigration mean and variance vary regularly with nonnegative exponents $\alpha$ and $\beta$, respectively, we prove that if $\beta < 1 + 2\alpha$, the bootstrap leads to a valid approximation for the CLSE. If $\beta > 1 + 2\alpha$, the conditional distribution of the bootstrap version of the CLSE has a random limit (in distribution). In the threshold case $\beta = 1 + 2\alpha$ the validity depends on slowly varying parts of the mean and variance of the immigration.

It follows from the above discussion that the question on criticality of the process is crucial for applications. To answer this question, one may test hypothesis $H_0 : a = 1$ against one of

$a \neq 1, a > 1$ or $a < 1$. Our results allow to develop rejection regions for these hypotheses based on bootstrap pivots. Thus, the bootstrap procedure is very useful in applications of branching processes, where it is hard to obtain large size samples. It makes possible to develop statistical inferences in early stages of the process, which is important, for example, in epidemic models.

Investigations of the problems related to the bootstrap methods and their applications can be seen in [9], in monographs [8,10] and in most recent review articles [7,13]. In [6] a modification of the standard bootstrap procedure was proposed, which eliminated the invalidity in the critical case. The second-order correctness of the bootstrap for a studentized version of MLE in subcritical case proved in [19].

In Section 2 of the paper, we describe the parametric bootstrap and state main results. Necessary limit theorems for CLSE will be derived in Section 3. The proofs of main theorems are given in Section 4. The Appendix contains proofs of some preliminary results.

## 2. Main results on the bootstrap

The process with time-dependent immigration is given by the offspring distribution of $X_{ki}, k, i \geq 1$, and by the family of distributions of the number of immigrating individuals $\xi_k, k \geq 1$. We assume that the offspring distribution has the probability mass function

$$p_j(\theta) = P\{X_{ki} = j\}, \quad j = 0, 1, \ldots, \tag{2.1}$$

depending on parameter $\theta$, where $\theta \in \Theta \subseteq \mathbb{R}$. Then $a = E_\theta X_{ki} = f(\theta)$ for some function $f$. We assume throughout the paper that $f$ is one-to-one mapping of $\Theta$ to $[0, \infty)$ and is a homeomorphism between its domain and range. It is known that these assumptions are satisfied, for example, by the distributions of the power series family [6]. We assume that for any $k \geq 1$the variable $\xi_k$ follows a known distribution with the probability mass function

$$q_j(k) = P\{\xi_k = j\}, \quad j = 0, 1, \ldots. \tag{2.2}$$

Throughout the paper "$\xrightarrow{D}$", "$\xrightarrow{d}$" and "$\xrightarrow{P}$" will denote convergence of random functions in the Skorokhod topology and convergence of random variables in distribution and in probability, respectively, and also $X \stackrel{d}{=} Y$ denotes equality of distributions. We assume that $b = Var X_{ni} < \infty$ and $\alpha(k) = E\xi_k, \beta(k) = Var\xi_k$ are finite for any $k \geq 1$ and are regularly varying sequences of nonnegative exponents $\alpha$ and $\beta$, respectively. Then $A(n) = EZ(n)$ and $B^2(n) = Var Z(n)$ are finite for each $n \geq 0$ and $a = 1$. To provide the asymptotic distribution of $\hat{a}_n$ defined in (1.2), we assume that there exists $c \in [0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\beta(n)}{n\alpha(n)} = c \tag{2.3}$$

and denote for any $\varepsilon > 0$

$$\delta_n(\varepsilon) = \frac{1}{B^2(n)} \sum_{k=1}^{n} E[(\xi_k - \alpha(k))^2; |\xi_k - \alpha(k)| > \varepsilon B(n)].$$

As it was proved in [15], if $a = 1, b \in (0, \infty), \alpha(n) \to \infty, \beta(n) = o(n\alpha^2(n))$, condition (2.3) is satisfied and $\delta_n(\varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$, then as $n \to \infty$

$$\frac{nA(n)}{B(n)}(\hat{a}_n - a) \xrightarrow{d} (2 + \alpha)\mathcal{N}(0, 1). \tag{2.4}$$

Furthermore, under the above conditions, $A(n)/B(n) \to \infty$ as $n \to \infty$ and when $c = 0$ the condition $\delta_n(\varepsilon) \to 0$ is satisfied automatically. More detailed discussion and examples can be seen in [15].

We now describe the bootstrap procedure to approximate the sampling distribution of the pivot

$$V_n = \frac{nA(n)}{B(n)}(\hat{a}_n - a).$$

Given a sample $\mathcal{X}_n = \{Z(k), k = 1, \dots, n\}$ of population sizes, we estimate the offspring mean $a$ by the weighted CLSE $\hat{a}_n$. Obtain the estimate of the parameter $\theta$ as $\hat{\theta}_n = f^{-1}(\hat{a}_n)$ from equation $a = f(\theta)$. Replace $\theta$ in the probability distribution (2.1) by its estimate. Given $\mathcal{X}_n$, let $\{X_{ki}^{*(n)}, k, i \geq 1\}$ be a family of i.i.d. random variables with the probability mass function $\{p_j(\hat{\theta}_n), j = 0, 1, \dots\}$. Now we obtain the bootstrap sample $\mathcal{X}_n^* = \{Z^{*(n)}(k), k = 1, \dots, n\}$ recursively from

$$Z^{*(n)}(k) = \sum_{i=1}^{Z^{*(n)}(k-1)} X_{ki}^{*(n)} + \xi_k, \quad n, k \geq 1, \tag{2.5}$$

with $Z^{*(n)}(0) = 0$, where $\xi_k$, $k \geq 1$, are independent random variables with the probability mass functions $\{q_j(k), j = 0, 1, \dots\}$. Then, we define the bootstrap analogue of the pivot $V_n$ by

$$V_n^* = \frac{nA(n)}{B(n)}(\hat{a}_n^* - \hat{a}_n), \tag{2.6}$$

where $\hat{a}_n^*$ is the weighted CLSE based on $\mathcal{X}_n^*$, i.e.

$$\hat{a}_n^* = \frac{\sum_{k=1}^{n} (Z^{*(n)}(k) - \alpha(k))}{\sum_{k=1}^{n} Z^{*(n)}(k-1)}. \tag{2.7}$$

To state our main result, we need the following conditions be satisfied.

A1. $a = 1$ and moments $E_\theta[(X_{ki})^2]$ and $E_\theta[(X_{ki})^{2+l}]$ are continuous functions of $\theta$ for some $l > 0$.
A2. $\delta_n(\varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$.
A3. $\alpha(n) \to \infty$, $\beta(n) = o(n\alpha^2(n))$ as $n \to \infty$.

**Theorem 2.1.** *If conditions* A1–A3*and* (2.3) *are satisfied, then*

$$\sup_x |P\{V_n^* \leq x | \mathcal{X}_n\} - \Phi(2 + \alpha, x)| \xrightarrow{P} 0 \tag{2.8}$$

*as $n \to \infty$, where $\Phi(\sigma, x)$ is the normal distribution with mean zero and variance $\sigma^2$.*

**Remarks 2.1.** Due to (2.4), the convergence (2.8) implies the validity of the standard bootstrap procedure to approximate the sampling distribution of $V_n$, i.e. as $n \to \infty$

$$\sup_x |P\{V_n^* \leq x | \mathcal{X}_n\} - P\{V_n \leq x\}| \xrightarrow{P} 0.$$

**2.2.** As it was mentioned before, in the case $c = 0$ condition A2 is automatically satisfied. In the case $c > 0$ the condition is equivalent to the Lindeberg condition for the family $\{\xi_n, n \geq 1\}$ of the number of immigrating individuals.

**Example 2.1.** Let $\xi_k, k \geq 1$, be Poisson with the mean $\alpha(k)$ such that $\alpha(k) \to \infty, k \to \infty$, and regularly varies with exponent $\alpha$. In this case $\beta(n) = o(n\alpha(n))$ as $n \to \infty$ and condition A3 is satisfied. Moreover, we realize that $c = 0$ in (2.3), which implies that condition A2 is also fulfilled. Thus we have the following result.

**Corollary 2.1.** *If $\xi_k, k \geq 1$, are Poisson with mean $\alpha(k) \to \infty, k \to \infty$, and $(\alpha(k))_{k=1}^{\infty}$ is regularly varying sequence with exponent $\alpha$ and condition* A1 *is satisfied, then* (2.8) *holds.*

Now we consider the case when the second relation in A3 is not satisfied. Instead, we assume that there exists $d \in [0, \infty)$ such that

$$\lim_{n\to\infty} \frac{n\alpha^2(n)}{\beta(n)} = d. \tag{2.9}$$

It is known that in this case the weighted CLSE is not asymptotically normal. More precisely, if $a = 1, b \in (0, \infty), \alpha(n) \to \infty$, condition (2.9) is satisfied and $\delta_n(\varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$, then as $n \to \infty$

$$n(\hat{a}_n - a) \xrightarrow{d} W_0 =: \frac{W(1)}{\int_0^1 W(t^{1+\beta})\mathrm{d}t + \gamma}, \tag{2.10}$$

where $W(t)$ is the standard Wiener process and $\gamma = ((\alpha + 1)(\alpha + 2))^{-1}\sqrt{d(1 + \beta)}$ (see [15], Theorem 3.2).

We denote by $W_n = n(\hat{a}_n - a)$ the pivot corresponding to this result and by $W_n^* = n(\hat{a}_n^* - \hat{a}_n)$ the bootstrap pivot based on the bootstrap sample $\mathcal{X}_n^*$. The next theorem shows that in the case of fast immigration variance the parametric bootstrap is invalid. To formulate it, we introduce several new notations. Let $\nabla_\beta(t) = \nabla_\beta(c_0, t) = \mu_\beta(2c_0, t)$, where

$$\mu_\alpha(t) = \mu_\alpha(c_0, t) = \int_0^t u^\alpha \mathrm{e}^{(t-u)c_0}\mathrm{d}u, \qquad \gamma_0 = \gamma_0(c_0) = \left(\frac{d}{\nabla_\beta(1)}\right)^{1/2} \int_0^1 \mu_\alpha(u)\mathrm{d}u,$$

$$\psi(t) = \psi(c_0, t) = \frac{t^{1+\beta}}{(1 + \beta)\nabla_\beta(1)},$$

$$\Lambda(c_0, t) = W(\psi(t)) + c_0 \int_0^t \mathrm{e}^{c_0(t-u)} W(\psi(u))\mathrm{d}u.$$

Further, introduce ratio

$$\nu(c_0) = \frac{W(\psi(c_0, 1))}{\int_0^1 \Lambda(c_0, t)\mathrm{d}t + \gamma_0(c_0)} \tag{2.11}$$

and its cumulative distribution function $F(c_0, x) = P\{\nu(c_0) \leq x\}$.

**Theorem 2.2.** *If conditions* A1, A2 *and* (2.9) *are satisfied and $\alpha(n) \to \infty$ as $n \to \infty$, then*

$$P\{W_n^* \leq x | \mathcal{X}_n\} \xrightarrow{d} F(W_0, x) \tag{2.12}$$

*as $n \to \infty$ for each $x \in \mathbb{R}$.*

**Remark 2.3.** Theorem 2.2 shows that, if the immigration variance is large enough comparatively the mean, the conditional limit distribution of the bootstrap pivot $W_n^*$ does not coincide with the limit distribution of $W_n = n(\hat{a}_n - a)$ for $a = 1$. In other words, as in the case of stationary immigration, the standard bootstrap least squares estimate of the offspring mean is asymptotically invalid. It is not surprising, if we recall that even the bootstrap version of the sample mean is invalid in the infinite variance case [1]. In this case one may develop a modified version of the bootstrap as in [6] (see Section 5 for details).

In order to prove the main theorems, we obtained a series of results for the array of the branching processes in a more general set up, which are of independent interest [17]. The scheme of the proofs is as following. Since the bootstrap sample $\mathcal{X}_n^*$ is based on the sequence of branching processes (2.5), we first investigate the array of processes under suitable assumptions of the nearly criticality. In the second step, we derive limit distributions for the CLSE of the offspring mean in the sequence of nearly critical processes. In the third step, we show that conditions of the limit theorems for the CLSE are fulfilled by the bootstrap pivots $V_n^*$ and $W_n^*$.

## 3. Asymptotic distributions for the CLSE

Let $\{X_{ki}^{(n)}, k, i \geq 1\}$ and $\{\xi_k^{(n)}, k \geq 1\}$ be two families of independent, nonnegative and integer valued random variables for each $n \in \mathbb{N}$. The sequence of branching processes $(Z^{(n)}(k), k \geq 0)_{n\geq 1}$ with $Z^{(n)}(0) = 0, n \geq 1$, is defined recursively as

$$Z^{(n)}(k) = \sum_{i=1}^{Z^{(n)}(k-1)} X_{ki}^{(n)} + \xi_k^{(n)}, \quad n \geq 1. \tag{3.1}$$

Assume that $X_{ki}^{(n)}$ have a common distribution for all $k$ and $i$, and the families $\{X_{ki}^{(n)}\}$ and $\{\xi_k^{(n)}\}$ are independent. In this scheme $a(n) = EX_{ki}^{(n)}$ is the criticality parameter of the $n$th process. The sequence of branching processes (3.1) is said to be *nearly critical* if $a(n) \to 1$ as $n \to 1$. In this section we derive asymptotic distributions for the CLSE of the offspring mean in (3.1).

If a sequence $(f(n))_{n=1}^{\infty}$ is regularly varying with exponent $\rho$, we will write $(f(n))_{n=1}^{\infty} \in R_\rho$. We assume that $a(n) = EX_{ij}^{(n)}$ and $b(n) = VarX_{ij}^{(n)}$ are finite for each $n \geq 1$ and $\alpha(n,i) = E\xi_i^{(n)} < \infty, \beta(n,i) = Var\xi_i^{(n)} < \infty$ for all $n, i \geq 1$. Then $A_n(i) = EZ^{(n)}(i)$ and $B_n^2(i) = VarZ^{(n)}(i)$ are finite for each $n \geq 1, 1 \leq i \leq n$, and by a standard technique we find that

$$A_n(k) = \sum_{i=1}^{k} \alpha(n,i) a^{k-i}(n), \qquad B_n^2(k) = \Delta_n^2(k) + \sigma_n^2(k), \tag{3.2}$$

where

$$\Delta_n^2(k) = \sum_{i=1}^{k} \alpha(n,i) Var(X^{(n)}(k-i)), \qquad \sigma_n^2(k) = \sum_{i=1}^{k} \beta(n,i) a^{2(k-i)}(n),$$

$$Var(X^{(n)}(i)) = \frac{b(n)}{1-a(n)} a^{i-1}(n)(1-a^i(n)).$$

We also denote by $\Im^{(n)}(k)$ the $\sigma$-algebra containing all the history of the $n$th process up to $k$th generation, i.e. it is generated by $\{Z^{(n)}(0), Z^{(n)}(1), \ldots, Z^{(n)}(k)\}$ and put $M^{(n)}(k) =$

$Z^{(n)}(k) - E[Z^{(n)}(k)|\Im^{(n)}(k)]$. In our proofs we need approximation results for the processes

$$\mathcal{Z}_n(t) = \frac{Z^{(n)}([nt])}{A_n(n)}, \qquad \mathcal{Y}_n(t) = \frac{1}{B_n(n)} \sum_{k=1}^{[nt]} M^{(n)}(k), \quad t \in \mathbb{R}_+.$$

We assume that the following conditions are satisfied.

C1. There are sequences $(\alpha(i))_{i=1}^{\infty} \in R_\alpha$ and $(\beta(i))_{i=1}^{\infty} \in R_\beta$ with $\alpha, \beta \geq 0$, such that, as $n \to \infty$ for each $s \in \mathbb{R}_+$,

$$\max_{1 \leq k \leq ns} |\alpha(n,k) - \alpha(k)| = o(\alpha(n)), \qquad \max_{1 \leq k \leq ns} |\beta(n,k) - \beta(k)| = o(\beta(n)). \tag{3.3}$$

C2. $a(n) = 1 + n^{-1}c_0 + o(n^{-1})$ as $n \to \infty$ for some $c_0 \in \mathbb{R}$.
C3. $b(n) = o(\alpha(n))$ as $n \to \infty$.

Detailed discussion on conditions C1–C3 one can see in [17]. We provide different approximations for $\mathcal{Y}_n(t)$ under the following additional conditions.

C4. $\alpha(n) \to \infty$, $\beta(n) = o(n\alpha(n)b(n))$ as $n \to \infty$ and $\liminf_{n\to\infty} b(n) > 0$.
C5. $\alpha(n) \to \infty$ and $n\alpha(n)b(n) = o(\beta(n))$ as $n \to \infty$.
C6. $\alpha(n) \to \infty$ and $\beta(n) \sim cn\alpha(n)b(n)$ as $n \to \infty$, where $c \in (0, \infty)$.

The following functions appear in the time change of the approximating processes:

$$\mu_\alpha(t) = \int_0^t u^\alpha \mathrm{e}^{(t-u)c_0} \mathrm{d}u, \quad \nu_\alpha(t) = \int_0^t u^\alpha \mathrm{e}^{(t-u)c_0}(1 - \mathrm{e}^{(t-u)c_0})\mathrm{d}u. \tag{3.4}$$

In particular, it is useful to note that $\mu_\alpha(t) = t^{\alpha+1}/(\alpha+1)$ when $c_0 = 0$, and $\lim_{c_0 \to 0} \nu_\alpha(t)/c_0 = t^{\alpha+2}/(\alpha+1)(\alpha+2)$.
We denote $\delta_n^{(1)}(\varepsilon) = E[(X_{ki}^{(n)} - a(n))^2 \chi(|X_{ki}^{(n)} - a(n)| > \varepsilon B_n(n))]$, where $\chi(A)$ stands for the indicator of the event $A$. Let $\varphi(t) = c_0 \int_0^t \mu_\alpha(u)\mathrm{d}u/\nu_\alpha(1)$ for $c_0 \neq 0$ and $\varphi(t) = t^{2+\alpha}$, if $c_0 = 0$.

**Theorem 3.1.** *If conditions* C1–C4 *are satisfied and* $\delta_n^{(1)}(\varepsilon) \to 0$ *as* $n \to \infty$, *then* $\mathcal{Y}_n \xrightarrow{D} \mathcal{Y}^{(1)}$ *as* $n \to \infty$ *weakly in the Skorokhod space* $D(\mathbb{R}_+, \mathbb{R})$, *where* $\mathcal{Y}^{(1)}(t) = W(\varphi(t))$ *and* $(W(t), t \in \mathbb{R}_+)$ *is a standard Brownian motion.*

**Remark 3.1.** We note that the Lindeberg-type condition for the family $\{X_{ki}^{(n)}, k, i \geq 1\}$ is needed even in time-homogeneous models (see [11,18]). If $E(X_{ki}^{(n)})^{2+l} < \infty$ for all $n \in \mathbb{N}$ and some $l \in \mathbb{R}_+$, then

$$\delta_n^{(1)}(\varepsilon) \leq \frac{1}{\varepsilon^l B_n^l(n)} E|X_{ki}^{(n)} - a(n)|^{2+l}.$$

To state the next theorem, we denote $\bar{\xi}_k^{(n)} = \xi_k^{(n)} - \alpha(n,k)$ and

$$\delta_n^{(2)}(\varepsilon) = \frac{1}{\sigma_0^2(n)} \sum_{k=1}^n E[(\bar{\xi}_k^{(n)})^2 \chi(|\bar{\xi}_k^{(n)}| > \varepsilon\sigma_0(n))], \quad \sigma_0^2(n) = \sum_{k=1}^n \beta(n,k).$$

**Theorem 3.2.** *If conditions* C1–C3 *and* C5 *are satisfied and* $\delta_n^{(2)}(\varepsilon) \to 0$ *as* $n \to \infty$, *then* $\mathcal{Y}_n \xrightarrow{D} \mathcal{Y}^{(2)}$ *as* $n \to \infty$ *weakly in the Skorokhod space* $D(\mathbb{R}_+, \mathbb{R})$, *where* $\mathcal{Y}^{(2)}(t) = W(\psi(t))$, $\psi(t) = t^{1+\beta}/(1+\beta)\nabla_\beta(1)$ *and* $\nabla_\beta(t)$ *is defined right before Theorem 2.2.*

**Theorem 3.3.** *Let conditions* C1–C3 *and* C6 *be satisfied. If* $\delta_n^{(i)}(\varepsilon) \to 0$ *as* $n \to \infty$ *for each* $\varepsilon > 0$, $i = 1, 2$, *and* $\liminf_{n\to\infty} b(n) > 0$, *then* $\mathcal{Y}_n \xrightarrow{D} \mathcal{Y}^{(3)}$ *as* $n \to \infty$ *weakly in the Skorokhod space* $D(\mathbb{R}_+, \mathbb{R})$, *where* $\mathcal{Y}^{(3)}(t) = W(\omega(t))$ *and*

$$\omega(t) = \frac{1}{K(c_0, c)} \int_0^t (\mu_\alpha(u) + cu^\beta)\mathrm{d}u,$$

$K(c_0, c) = \nu_\alpha(1)/c_0 + c\nabla_\beta(1)$ *for* $c_0 \neq 0$ *and* $K(0, c) = 1/(1+\alpha)(2+\alpha) + c\mu_{\beta(1)}$.

**Remark 3.2.** It is not difficult to see that if $c_0 = 0$, then

$$\omega(t) = \left(\frac{1}{(1+\alpha)(2+\alpha)} + \frac{c}{1+\beta}\right)^{-1} \left[\frac{t^{2+\alpha}}{(1+\alpha)(2+\alpha)} + c\frac{t^{1+\beta}}{1+\beta}\right].$$

Furthermore, if in addition $b(n) \to b$ as $n \to \infty$ with $b \in (0, \infty)$, then $\omega(t) = t^{\alpha+2} = t^{\beta+1}$, i.e. the same time change as in the functional limit theorem for the critical process [14].

The proofs of Theorems 3.1–3.3 are based on a martingale limit result. (Theorem VIII, 3.33 from [12]). The details are provided in [17]. We also need the following theorem on a deterministic approximation of the process $\mathcal{Z}_n(t)$, which is proved in [16].

**Theorem 3.4.** *Let conditions* C1–C3 *be satisfied. If* $\alpha(n) \to \infty$ *and* $\beta(n) = o(n\alpha^2(n))$ *as* $n \to \infty$, *then* $\mathcal{Z}_n \xrightarrow{D} \pi_\alpha$ *as* $n \to \infty$ *weakly in the Skorokhod space* $D(\mathbb{R}_+, \mathbb{R})$, *where* $\pi_\alpha(t) = \mu_\alpha(t)/\mu_\alpha(1)$, $t \in \mathbb{R}_+$.

*By a standard technique we obtain that the CLSE of the offspring mean* $a(n)$ *of nth process in* (3.1) *is*

$$\hat{A}_n = \frac{\sum_{k=1}^{n} (Z^{(n)}(k) - \alpha(n, k))}{\sum_{k=1}^{n} Z^{(n)}(k-1)}. \tag{3.5}$$

*We denote*

$$W_n^{(1)} = \frac{nA_n(n)}{B_n(n)}(\hat{A}_n - a(n)), \qquad W_n^{(2)} = n(\hat{A}_n - a(n)). \tag{3.6}$$

**Theorem 3.5.** *Let conditions* C1–C3 *be satisfied and* $\alpha(n) \to \infty$, $\beta(n) = o(n\alpha^2(n))$ *as* $n \to \infty$ *and* $\lambda_\alpha = \int_0^1 \pi_\alpha(u)\mathrm{d}u$.

(a) *If* C4 *holds and* $\delta_n^{(1)}(\varepsilon) \to 0$ *as* $n \to \infty$, *then* $W_n^{(1)} \xrightarrow{d} W(\varphi(1))/\lambda_\alpha$.
(b) *If* C5 *holds and* $\delta_n^{(2)}(\varepsilon) \to 0$ *as* $n \to \infty$, *then* $W_n^{(1)} \xrightarrow{d} W(\psi(1))/\lambda_\alpha$.
(c) *If* C6 *holds,* $\delta_n^{(i)}(\varepsilon) \to 0$, $i = 1, 2$, *as* $n \to \infty$ *for any* $\varepsilon > 0$ *and* $\liminf_{n\to\infty} b(n) > 0$, *then* $W_n^{(1)} \xrightarrow{d} W(\omega(1))/\lambda_\alpha$.

**Proof.** We consider the following equality:

$$\hat{A}_n - a(n) = \frac{\sum_{k=1}^{n} M^{(n)}(k)}{\sum_{k=1}^{n} Z^{(n)}(k-1)} =: \frac{D(n)}{Q(n)}. \tag{3.7}$$

First we obtain the asymptotic behavior of $Q(n)$. We use the following representation:

$$\frac{1}{nA_n(n)}\sum_{k=0}^{n} Z^{(n)}(k) = \sum_{k=1}^{n}\int_{(k-1)/n}^{k/n} \mathcal{Z}_n(t)\mathrm{d}t. \tag{3.8}$$

Now we consider functionals $\Psi_n : D(\mathbb{R}_+, \mathbb{R}) \mapsto \mathbb{R}$, defined for any $x \in D(\mathbb{R}_+, \mathbb{R})$ and $n \geq 1$ as

$$\Psi_n(x) = \sum_{k=1}^{n}\int_{(k-1)/n}^{k/n} x(t)\mathrm{d}t. \tag{3.9}$$

It is obvious that for all $x, x_n \in D(\mathbb{R}_+, \mathbb{R})$ such that $\|x_n - x\|_\infty \to 0$, we have $|\Psi_n(x_n) - \Psi(x)| \to 0$ as $n \to \infty$, where $\Psi(x) = \int_0^1 x(t)\mathrm{d}t$.

Due to Theorem 3.4, we have $\mathcal{Z}_n \overset{D}{\to} \pi_\alpha$ as $n \to \infty$ weakly in the Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$. Since $\pi_\alpha(t)$ is continuous, this implies that $\|\mathcal{Z}_n - \pi_\alpha\|_\infty \to 0$ as $n \to \infty$. Therefore, due to the extended continuous mapping theorem ([4], Theorem 5.5), we have $\Psi_n(\mathcal{Z}_n) \overset{d}{\to} \Psi(\pi_\alpha)$ as $n \to \infty$, where $\Psi(\pi_\alpha) = \int_0^1 \pi_\alpha(t)\mathrm{d}t$. Thus, when conditions C1–C3 are satisfied and $\alpha(n) \to \infty$, $\beta(n) = o(n\alpha^2(n))$ as $n \to \infty$, we have

$$\frac{Q(n)}{nA_n(n)} \overset{P}{\to} \int_0^1 \pi_\alpha(u)\mathrm{d}u. \tag{3.10}$$

Now we consider $D(n)$. It follows from Theorem 3.1 that when conditions C1–C4 are satisfied and $\delta_n^{(1)}(\varepsilon) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$ as $n \to \infty$

$$\frac{D(n)}{B_n(n)} \overset{d}{\to} W(\varphi(1)). \tag{3.11}$$

The proof of Part (a) of the theorem follows from (3.7), (3.10) and (3.11) due to Slutsky's theorem.

Using Theorems 3.2 and 3.3, by similar arguments we obtain parts (b) and (c) of Theorem 3.5. □

**Corollary 3.1.** *If the conditions of Theorem 3.5 are satisfied and $c_0 = 0$, then $W_n^{(1)} \overset{d}{\to} (2+\alpha)\mathcal{N}(0,1)$ as $n \to \infty$ in all cases* (a), (b) *and* (c).

The corollary follows from the fact that $\mu_\alpha(u) = u^{1+\alpha}/(1+\alpha)$ when $c_0 = 0$.

**Theorem 3.6.** *Let conditions* C1, C2 *and* (2.9) *be satisfied. If $\alpha(n) \to \infty$, $b(n) \to b \in (0, \infty)$ and $\delta_n^{(2)}(\varepsilon) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$, then*

$$W_n^{(2)} \overset{d}{\to} \nu(c_0), \tag{3.12}$$

*where $\nu(c_0)$ is defined in* (2.11).

**Proof.** We consider relation (3.7) in the proof of Theorem 3.5. It is obvious that conditions C3 and C5 of Theorem 3.2 are satisfied when (2.9) holds and $b(n) \to b \in (0, \infty)$ as $n \to \infty$. Therefore, taking into account that $D(n)/B(n) = \mathcal{Y}_n(1)$, we obtain that as $n \to \infty$

$$\frac{D(n)}{B_n(n)} \overset{d}{=\!\!\to} W(\psi(1)). \tag{3.13}$$

Now we evaluate $Q(n)$. When the condition $\beta(n) = o(n\alpha^2(n))$ as $n \to \infty$ is not valid we cannot use the deterministic approximation for $\mathcal{Z}_n(t)$ given in Theorem 3.4. Therefore, first we express the denominator in terms of the process $Y_n(t) = (Z^{(n)}([nt]) - A^{(n)}([nt]))/B_n(n)$. Namely, we consider the following equality

$$\frac{Q(n)}{nB_n(n)} = S_1(n) + S_2(n), \tag{3.14}$$

where

$$S_1(n) = \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} Y_n(t)\mathrm{d}t, \qquad S_2(n) = \frac{1}{nB_n(n)} \sum_{k=1}^{n} A_n(k-1).$$

To evaluate $S_2(n)$, we use Lemma A.2 and Part (c) of Lemma A.3, which are proved in the Appendix. Since due to condition (2.9) $B_n^2(n) \sim n\beta(n)\nabla_\beta(1)$ as $n \to \infty$, we easily obtain that

$$\lim_{n\to\infty} S_2(n) = \gamma_0, \tag{3.15}$$

where $\gamma_0 = \gamma_0(c_0)$ is defined just before Theorem 2.2. In order to use the continuous mapping theorem, we need to express $S_1(n)$ in terms of the process $\mathcal{Y}_n(t)$. We obtain using (3.1) that $Z^{(n)}(k) - EZ^{(n)}(k) = M^{(n)}(k) + a(n)(Z^{(n)}(k-1) - EZ^{(n)}(k-1))$ for any $n, k \geq 1$. Thus

$$Z^{(n)}(k) - EZ^{(n)}(k) = \sum_{j=1}^{k} a^{k-j}(n)M^{(n)}(j).$$

From the last equality we easily obtain

$$Y_n(t) = \mathcal{Y}_n(t) + \frac{1}{B_n(n)} \sum_{j=1}^{[nt]} (a^{[nt]-j}(n) - 1)M^{(n)}(j). \tag{3.16}$$

Rearranging the sum on the right side of (3.16), we have

$$\frac{1}{B_n(n)} \sum_{j=1}^{[nt]} (a^{[nt]-j}(n) - 1)M^{(n)}(j) = (a(n) - 1) \sum_{i=2}^{[nt]} a^{[nt]-i}(n)\mathcal{Y}_n\left(\frac{i-1}{n}\right).$$

Hence, we obtain the following representation

$$S_1(n) = \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} \left(\mathcal{Y}_n(t) + c_0(n)\frac{1}{n} \sum_{i=2}^{[nt]} a^{[nt]-i}(n)\mathcal{Y}_n\left(\frac{i-1}{n}\right)\right) \mathrm{d}t, \tag{3.17}$$

where $c_0(n) = n(a(n)-1)$. Now we consider sequence of functionals $\Phi_n : D(\mathbb{R}_+, \mathbb{R}) \mapsto \mathbb{R}$, $n \geq 1$, which are defined for any $x \in D(\mathbb{R}_+, \mathbb{R})$ as

$$\Phi_n(x) = \frac{x(1)}{\Omega_n(x) + \gamma_0},$$

where functionals $\Omega_n : D(\mathbb{R}_+, \mathbb{R}) \mapsto \mathbb{R}$, $n \geq 1$ are defined by

$$\Omega_n(x) = \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} \left(x(t) + c_0(n)\frac{1}{n} \sum_{i=2}^{[nt]} a^{[nt]-i}(n)x\left(\frac{i-1}{n}\right)\right) \mathrm{d}t. \tag{3.18}$$

It is not difficult to see that for any sequence $x_n \in D(\mathbb{R}_+, \mathbb{R})$, $n \geq 1$, such that $\|x_n - x\|_\infty \to 0$ as $n \to \infty$ with $x \in D(\mathbb{R}_+, \mathbb{R})$, we have $\Phi_n(x_n) \to \Phi(x)$ as $n \to \infty$, where

$$\Phi(x) = x(1)\left(\int_0^1 x(t)\mathrm{d}t + c_0 \int_0^1 \int_0^t \mathrm{e}^{(t-u)c_0} x(u)\mathrm{d}u\mathrm{d}t + \gamma_0\right)^{-1}. \tag{3.19}$$

It follows from Theorem 3.2 that $\mathcal{Y}_n \xrightarrow{D} \mathcal{Y}^{(2)}$ as $n \to \infty$ weakly in the Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$, where $\mathcal{Y}^{(2)}(t) = W(\psi(t))$. Since $\mathcal{Y}^{(2)}(t)$ is continuous, this implies that $\|\mathcal{Y}_n - \mathcal{Y}^{(2)}\|_\infty \to 0$ as $n \to \infty$. Appealing again to the extended continuous mapping theorem ([4], Theorem 5.5), we conclude that $\Phi_n(\mathcal{Y}_n) \xrightarrow{d} \Phi(\mathcal{Y}^{(2)})$ as $n \to \infty$. To obtain the proof of Theorem 3.6, we rewrite (3.7) as

$$n(\hat{A}_n - a(n)) = \frac{\Phi_n(\mathcal{Y}_n)}{1 + \Upsilon_n(\mathcal{Y}_n)},$$

where $\Upsilon_n(\mathcal{Y}_n) = (S_2(n) - \gamma_0)/(\Omega_n(\mathcal{Y}_n) + \gamma_0)$. Since $\Upsilon_n(\mathcal{Y}_n) \xrightarrow{P} 0$ as $n \to \infty$, the assertion of Theorem 3.6 now follows from Slutsky's theorem. □

**Corollary 3.2.** *If conditions of Theorem 3.6 are satisfied and $c_0 = 0$, then*

$$W_n^{(2)} \xrightarrow{d} \frac{W(1)}{\int_0^1 W(t^{1+\beta})\mathrm{d}t + \gamma} \tag{3.20}$$

*as $n \to \infty$, where $\gamma = ((\alpha + 1)(\alpha + 2))^{-1}\sqrt{d(1 + \beta)}$.*

## 4. Proofs of the main results

**Proof of Theorem 2.1.** It is easy to see that given the sample $\mathcal{X}_n = \{Z(k), k = 1, \ldots, n\}$, the bootstrap process $\{Z^{*(n)}(k), k \geq 0\}$ is the sequence of branching processes defined in (3.1), where

$$a(n) = \hat{a}_n, \qquad b(n) = Var(X_{ki}^{*(n)}),$$

and $\alpha(n, k) = \alpha(k)$ and $\beta(n, k) = \beta(k)$ are the mean and variance of the probability distribution $\{q_j(k), j = 0, 1, \ldots\}$. Therefore, if conditions of Theorem 3.5 are satisfied by the bootstrap process (in probability) and $c_0 = 0$, then

$$\sup_x |H_n(\hat{a}_n, x) - \Phi(2 + \alpha, x)| \xrightarrow{P} 0, \tag{4.1}$$

where $H_n(\hat{a}_n, x) = P\{V_n^* \leq x|\mathcal{X}_n\}$. Thus we need to show that the conditions of Theorem 3.5 are satisfied. Conditions C1 are fulfilled trivially. It follows from (2.4) that $n(\hat{a}_n - 1) \xrightarrow{P} 0$ as $n \to \infty$, i.e. the condition C2 is also satisfied in probability with $c_0 = 0$. Since $\hat{a}_n \xrightarrow{P} 1$ as $n \to \infty$, $\hat{\theta}_n = f^{-1}(\hat{a}_n) \xrightarrow{P} f^{-1}(1) = \theta$. Since $b(n) = \varphi_1(\theta)$, where $\varphi_1$ is a continuous function, we have $Var(X_{ki}^{*(n)}|\mathcal{X}_n) = \varphi_1(\hat{\theta}_n) \xrightarrow{P} \varphi_1(\theta) = b$ as $n \to \infty$, i.e. condition C3 is satisfied.

It follows from Remark 3.1 that if for some $l > 0$

$$(1/B_n^l(n))E[|X_{ki}^{*(n)} - \hat{a}_n|^{2+l}|\mathcal{X}_n] \xrightarrow{P} 0 \tag{4.2}$$

as $n \to \infty$, then

$$\delta_n^{*(1)}(\varepsilon) =: \frac{1}{B_n^2(n)} E[(X_{ki}^{*(n)} - \hat{a}_n)^2 \chi(|X_{ki}^{*(n)} - \hat{a}_n| > \varepsilon B_n(n)) | \mathcal{X}_n] \xrightarrow{P} 0$$

as $n \to \infty$ for each $\varepsilon > 0$. We obtain from condition A1 that $E[(X_{ki}^{*(n)})^{2+l} | \mathcal{X}_n] \xrightarrow{P} E[(X_{ki})^{2+l}]$ as $n \to \infty$, which implies (4.2). Thus all conditions of Theorem 3.5 are satisfied and $c_0 = 0$. It follows from Lemmas A.1 and A.2 in the Appendix that $A_n(n) \sim A(n)$ and $B_n^2(n) \sim B^2(n)$ as $n \to \infty$ when $c_0 = 0$ and the assertion of (2.8) follows from Corollary 3.1. Theorem 2.1 is proved. $\square$

**Proof of Theorem 2.2.** As in [18], we use a quite standard technique based on the Skorokhod theorem (see [5], Theorem 29.6). We have from (2.10) that $n(\hat{a}_n - 1) \xrightarrow{d} W_0$ as $n \to \infty$. Therefore, due to the Skorokhod theorem, there exist a sequence $\{\hat{a}'_n, n \geq 1\}$ of random variables and a random variable $W'_0$ on a common probability space $(\Omega', \mathcal{F}, Q)$ such that $\hat{a}'_n \stackrel{d}{=} \hat{a}_n$ for all $n \geq 1$, $W'_0 \stackrel{d}{=} W_0$ and $n(\hat{a}'_n(\omega') - 1) \to W'_0$ as $n \to \infty$ for each $\omega' \in \Omega'$.

For any $\omega' \in \Omega'$ we estimate unknown $\theta$ by $\hat{\theta}'_n(\omega') = f^{-1}(\hat{a}'_n(\omega'))$. Then we obtain the bootstrap distribution $\{p_j(\hat{\theta}'_n), j \geq 0\}$ substituting $\theta$ by $\hat{\theta}'_n(\omega')$. Let now $\{X_{ki}^{\prime(n)}, k, i \geq 1\}$ be a family of i.i.d. random variables such that

$$P\{X_{ki}^{\prime(n)} = j\} = p_j(\hat{\theta}'_n)$$

for each $\omega' \in \Omega'$ and $n \geq 1$ and $\{\xi_k, k \geq 1\}$ be a sequence of random variables with the probability distributions $\{q_j(k), j \geq 0\}$. A new bootstrap sample $\mathcal{X}'_n = \{Z^{\prime(n)}(k), k = 1, \ldots, n\}$ will be obtained recursively from the relation

$$Z^{\prime(n)}(k) = \sum_{i=1}^{Z^{\prime(n)}(k-1)} X_{ki}^{\prime(n)} + \xi_k, \quad k = 1, 2, \ldots \tag{4.3}$$

for each $\omega' \in \Omega', n \geq 1$ with $Z^{\prime(n)}(0) = 0$. We define new pivot $W'_n = n(\tilde{a}_n - \hat{a}'_n(\omega'))$ for each $\omega' \in \Omega'$, where

$$\tilde{a}_n = \frac{\sum_{k=1}^{n} (Z^{\prime(n)}(k) - \alpha(k))}{\sum_{k=1}^{n} Z^{\prime(n)}(k-1)}. \tag{4.4}$$

If we denote $F_n(\theta, x) = P\{W_n \leq x\}$, we realize that

$$P\{W_n^* \leq x | \hat{a}_n\} = F_n(\hat{\theta}_n, x), \qquad P\{W'_n \leq x | \hat{a}'_n\} = F_n(\hat{\theta}'_n, x),$$

where $\hat{\theta}_n = f^{-1}(\hat{a}_n)$ and $\hat{\theta}'_n = f^{-1}(\hat{a}'_n)$. For each $\omega' \in \Omega'$ we apply Theorem 3.6 to $W'_n$. Under the assumptions of Theorem 2.2, conditions C1 and $\delta_n^{(2)}(\varepsilon) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$ are trivially satisfied. Condition C2 is also fulfilled for each $\omega' \in \Omega'$ with $c_0 = W'_0$. Due to our assumptions on the moments of the offspring distribution, we can write $b(n) = \varphi_1(\theta)$, where $\varphi_1$ is a continuous function, as in the proof of Theorem 2.1. Therefore, $Var(X_{ki}^{\prime(n)} | \hat{\theta}'_n) = \varphi_1(\hat{\theta}'_n) \to \varphi_1(\theta) = b$ as $n \to \infty$ for each $\omega' \in \Omega'$. Thus we obtain from Theorem 3.6 that $F_n(\hat{\theta}'_n, x) \to F(W'_0, x)$ as $n \to \infty$ for each $\omega' \in \Omega'$ and $x \in \mathbb{R}$. The assertion of Theorem 2.1 now follows from this, due to $F_n(\hat{\theta}_n, x) \stackrel{d}{=} F_n(\hat{\theta}'_n, x)$ and $F(W_0, x) \stackrel{d}{=} F(W'_0, x)$. The theorem is proved. □

## 5. Conclusions

According to Theorem 2.2, when $n\alpha^2(n) = o(\beta(n))$ as $n \to \infty$ the bootstrap version of CLSE is invalid. The cause of the failure is the same as in the case of stationary immigration [18], namely, in this case the estimator $\hat{a}_n$ does not have the desired rate of convergence to $a = 1$. As in [6], one may consider a modified version of the standard bootstrap procedure. The idea behind the modification is using in the initial estimator of the offspring mean an adaptive shrinkage towards $a = 1$.

If a sample of pairs $\{(Z(k), \xi_k), k = 1, \ldots, n\}$ is available, then a natural estimator of the offspring mean is

$$\tilde{a_n} = \frac{\sum_{k=1}^{n}(Z(k) - \xi_k)}{\sum_{k=1}^{n} Z(k-1)}.$$

The following questions related to this estimator are of interest. How much improvement in the sense of the rate of convergence we will get because of additional observations of the number of immigrating individuals? Will the standard parametric bootstrap procedure be valid for $\tilde{a_n}$ in the case of large immigration variance? Since

$$\tilde{a_n} - a = \frac{\sum_{k=1}^{n}\sum_{j=1}^{Z(k-1)}(X_{kj} - a)}{\sum_{k=1}^{n} Z(k-1)},$$

one can easily derive asymptotic distributions for the pivot, corresponding to $\tilde{a_n}$ from a martingale central limit theorem. By the arguments as in the proof of Proposition 4.1 in [15], it is possible to prove that $\tilde{a_n}$ is a strongly consistent estimator of $a$.

The estimation problems and a justification of the validity of the bootstrap for subcritical and supercritical processes with non-stationary immigration are also open. In order to derive the asymptotic distributions for an estimator of the offspring mean, one needs to establish functional limit theorems in these cases. Further, as in the classical models, one may obtain results for the estimator without any assumption of the criticality.

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## Appendix

In the proofs of the main theorems we used several preliminary lemmas. Now we provide these results with proofs. We start with a simple but useful result related to regularly varying sequences.

**Lemma A.1.** *If* $(C(n))_{n=1}^{\infty} \in R_\rho$ *and* $a(n)$ *satisfies condition* C2*, then for any* $\rho \in [0, \infty)$ *and* $\theta \in \mathbb{R}$

$$\frac{1}{nC(n)} \sum_{k=1}^{[ns]} a^{k\theta}(n) C(k) \to \int_0^s t^\rho \mathrm{e}^{t\theta c_0} \mathrm{d}t \tag{A.1}$$

*as* $n \to \infty$ *uniformly in* $s \in [0, T]$ *for each fixed* $T > 0$.

**Proof.** It follows from condition C2 that

$$a^{ns}(n) \to \mathrm{e}^{c_0 s} \tag{A.2}$$

as $n \to \infty$ uniformly in $s \in [0, T]$. Since $(C(n))_{n=1}^{\infty} \in R_\rho$, (A.2) implies that

$$\frac{1}{n} \sum_{k=1}^{[ns]} \frac{C(k)}{C(n)} a^{k\theta}(n) - \frac{1}{n} \sum_{k=1}^{[ns]} \left(\frac{k}{n}\right)^\rho \mathrm{e}^{k/n\theta c_0} \to 0$$

as $n \to \infty$ uniformly in $s \in [0, T]$. Now the second sum tends as $n \to \infty$ to the integral

$$\int_0^s t^\rho \mathrm{e}^{t\theta c_0} \mathrm{d}t,$$

which is a continuous function of $s$. Hence, the convergence is uniform in $s \in [0, T]$. The lemma is proved. $\square$

The next result is related to the asymptotic behavior of the mean and the variance of the process.

**Lemma A.2.** *If conditions* C1 *and* C2 *are satisfied, then uniformly in* $s \in [0, T]$ *for each fixed* $T > 0$

$$\text{(a)} \quad \lim_{n\to\infty} \frac{A_n([ns])}{n\alpha(n)} = \mu_\alpha(s), \qquad \lim_{n\to\infty} \frac{\sigma_n^2([ns])}{n\beta(n)} = \nabla_\beta(s),$$

$$\text{(b)} \quad \lim_{n\to\infty} \frac{\Delta_n^2([ns])}{n^2\alpha(n)b(n)} = \begin{cases} (1/c_0)\nu_\alpha(s), & \text{if } c_0 \neq 0, \\ s^{\alpha+2}/(\alpha+1)(\alpha+2), & \text{if } c_0 = 0. \end{cases}$$

**Proof.** To prove the first relation in Part (a), we consider

$$A_n([ns]) = \sum_{k=1}^{[ns]} \alpha(k) a^{[ns]-k}(n) + \sum_{k=1}^{[ns]} (\alpha(n,k) - \alpha(k)) a^{[ns]-k}(n). \tag{A.3}$$

Applying Lemma A.1, we easily obtain that the first term on the right side of (A.3), divided by $n\alpha(n)$, as $n \to \infty$ tends to $\mu_\alpha(s)$ uniformly in $s \in [0, T]$. The second term is dominated by

$$\max_{1\le k\le nT} |\alpha(n,k) - \alpha(k)| \sum_{k=1}^{[ns]} a^{[ns]-k}(n).$$

Due to Lemma A.1, the sum in this expression, divided by $n$, tends to $\int_0^s e^{(1-u)c_0}du$ as $n \to \infty$ uniformly in $s \in [0, T]$. Therefore, taking into account C1, we obtain the assertion.

The proofs of the remaining claims are similar and, therefore, are omitted. □

**Lemma A.3.** *If conditions C1 and C2 are satisfied, then for any $\theta \in \mathbb{R}$, $s \in \mathbb{R}_+$*

$$\text{(a)} \lim_{n\to\infty} \frac{1}{n^3\alpha(n)b(n)} \sum_{i=1}^{[ns]} a^{\theta i}(n)\Delta_n^2(i) = \begin{cases} (1/c_0)\int_0^s e^{u\theta c_0}\nu_\alpha(u)du, & \textit{if } c_0 \neq 0, \\ s^{\alpha+3}/(\alpha+1)(\alpha+2)(\alpha+3), & \textit{if } c_0 = 0, \end{cases}$$

$$\text{(b)} \lim_{n\to\infty} \frac{1}{n^2\beta(n)} \sum_{i=1}^{[ns]} a^{\theta i}(n)\sigma_n^2(i) = \int_0^s e^{u\theta c_0}\nabla_\beta(u)du,$$

$$\text{(c)} \lim_{n\to\infty} \frac{1}{n^2\alpha(n)} \sum_{i=1}^{[ns]} a^{\theta i}(n)A_n(i) = \int_0^s e^{u\theta c_0}\mu_\alpha(u)du,$$

$$\text{(d)} \lim_{n\to\infty} \frac{1}{n^2\alpha(n)\beta(n)} \sum_{i=1}^{[ns]} a^{\theta i}(n)\beta(i)A_n(i) = \int_0^s e^{u\theta c_0}\mu_\alpha(u)u^\beta du.$$

**Proof.** We prove Part (a). Let $c_0 \neq 0$. In this case due to Part (b) of Lemma A.2, we have

$$\frac{1}{n}\sum_{i=1}^{[ns]} a^{\theta i}(n)\frac{\Delta_n^2(i)}{n^2\alpha(n)b(n)} - \frac{1}{nc_0}\sum_{i=1}^{[ns]} e^{(i/n)\theta c_0}\nu_\alpha\left(\frac{i}{n}\right) \to 0$$

as $n \to \infty$. Since $e^{u\theta c_0}\nu_\alpha(u)$ is bounded in $u \in [0, s]$ for each fixed $s$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{[ns]} e^{(i/n)\theta c_0}\nu_\alpha\left(\frac{i}{n}\right) = \int_0^s e^{u\theta c_0}\nu_\alpha(u)du. \tag{A.4}$$

In the case $c_0 = 0$, we have $\int_0^s (u^{\alpha+2}/(\alpha+1)(\alpha+2))du$ on the right side of (A.4). The proofs of Parts (b), (c) and (d) are similar. We just use the second relation of Part (a) and Part (b) of Lemma A.2. The lemma is proved. □

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