

Existence, uniqueness and approximation of the jump-type stochastic Schrödinger equation for two-level systems

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Abstract

In quantum physics, recent investigations deal with the so-called “stochastic Schrödinger equations” theory. This concerns stochastic differential equations of non-usual-type describing random evolutions of open quantum systems. These equations are often justified with heuristic rules and pose tedious problems in terms of mathematical and physical justifications: notion of solution, existence, uniqueness, etc.

In this article, we concentrate on a particular case: the Poisson case. Random Measure theory is used in order to give rigorous sense to such equations. We prove the existence and uniqueness of a solution for the associated stochastic equation. Furthermore, the stochastic model is physically justified by proving that the solution can be obtained as a limit of a concrete discrete time physical model.

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0. Introduction

Many recent developments in quantum mechanics deal with “*Stochastic Schrödinger Equations*” [2,11,5,7,9,17,31,3]. These equations are classical stochastic differential equations (also called “*Belavkin equations*” [6,8,7,9,10]) which describe random phenomena in continuous measurement theory. The solutions of these equations are called “*quantum trajectories*”. They describe the time evolution of an open quantum system undergoing continuous measurement (see [20,12–14,33,34,30,17,11,19,22,18] for physical applications). Usually, in Quantum Optics or Quantum Communication [17,18,11,19,5], indirect measurement is performed in order

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to avoid phenomena like *Zeno effect* (see chapters 2 and 3 in [11]). The physical setup is the one of a small system (an open system) interacting with an environment and an observation is “indirectly” performed on the environment. In the literature, stochastic Schrödinger equations are expressed as perturbations of the Master Equation which describes normally the evolution of the small system without measurement.

Two characteristic types of Belavkin equations are described as follows.

- A diffusive equation (homodyne or heterodyne detection)

$$d\rho_t = L(\rho_t)dt + \left[\rho_t C^* + C\rho_t - \text{Tr}[\rho_t(C + C^*)]\rho_t \right] dW_t, \quad (1)$$

where W_t describes a one-dimensional Brownian motion.

- A jump equation (photon detection)

$$d\rho_t = L(\rho_t)dt + \left[\frac{\mathcal{J}(\rho_t)}{\text{Tr}[\mathcal{J}(\rho_t)]} - \rho_t \right] \left(d\tilde{N}_t - \text{Tr}[\mathcal{J}(\rho_t)]dt \right), \quad (2)$$

where \tilde{N}_t is a counting process with intensity $\int_0^t \text{Tr}[\mathcal{J}(\rho_s)]ds$.

The driving noise depends on the nature of the measurement. In this article, we shall focus on the jump equation (2) for a qubit, i.e. a two-level system (mathematically, the process (ρ_t) is valued in $M_2(\mathbb{C})$; see [11,22,20,33,34] for physical applications). The diffusive case (1) for similar models is treated in detail in [32]. Eq. (2) poses peculiar problems in terms of justifications: mathematical sense, existence and uniqueness of a solution. In particular, in the way of presenting Eq. (2), the driving process is not clearly and rigorously identified. This process depends on the solution of the equation that it is supposed to drive. In other terms, the notion of a solution of Eq. (2) is not immediate. Indeed, in order to determine the existence of (ρ_t) , we need to consider the existence of (\tilde{N}_t) but the existence of this latter is directly linked with that of (ρ_t) . In this article, we set up a probability framework, where all the processes are defined in a intrinsic way.

Regarding the physical justification of Belavkin equation models, heuristic rules are usually used to derive these equations (see [21,11]). Usually, rigorous and abstract approaches are based on von Neumann algebra, conditional expectation in operator algebra, Fock space, quantum filtering [6,7,10] or instrument process and notion of *a posteriori* states [4,3,4,2]. In this article, the model (2) is rigorously justified as a limit of a concrete discrete model of quantum repeated measurements. This approach, also developed for the model (1) in [32], is based on the model of quantum repeated interactions [1]. The setup is the one of a small system interacting with an infinite chain of quantum systems. Each piece of the chain interacts with the small system, one after the other, during a time h . After each interaction, a measurement is performed on the piece of the chain which has just interacted. According to the quantum measurement principle, the sequence of observations induces random perturbations of the small system. This is described by a Markov chain, called *Discrete Quantum Trajectory*, depending on the parameter h . In this article, by renormalizing the interaction, we show that particular quantum trajectories converge, when h goes to zero, to the solution of the jump-Belavkin equation (2).

As it is mentioned, similar results concerning existence, uniqueness and approximation for the diffusive equation are expressed in [32]. The particularity of the jump equation concerns the counting process (\tilde{N}_t) . In the diffusive case, the existence and uniqueness of solution concerns a notion of *strong* solution. In the jump case, we first use the notion of *Random Poisson Measure* to give a rigorous sense to the equation and secondly, we deal with existence and uniqueness of

a *weak* solution. Concerning the convergence result in the diffusive case, the approach is based on an abstract convergence result for stochastic integrals due to Kurtz and Protter [28,29]. In the jump case, such techniques cannot be applied and the arguments are totally different. In order to prove the convergence, we use a concrete comparison of the discrete and continuous processes. This approach, based upon a *random coupling method*, uses the explicit construction of a solution of Eq. (2) and a realization of discrete quantum trajectories in a same probability space. In order to prove the main convergence, we need also to prove the convergence of the Euler scheme of Eq. (2).

This article is structured as follows.

Section 1 is devoted to recall the discrete model of repeated quantum measurements. Next, we make precise the model of a qubit in contact with a spin chain (this model corresponds to the discrete model of Belavkin equations).

In Section 2, we investigate the continuous model of jump equation (2) for two-level systems. By using the theory of random Poisson measure, we define an appropriate probability space for studying this equation. Next, we solve the problems of existence and uniqueness by constructing an explicit solution.

In Section 3, we prove the convergence of the discrete model to the jump continuous model. We present the random coupling method, that is, the realization of discrete quantum trajectories in the probability space where the solution of (2) is constructed. In parallel, we show the convergence of the Euler scheme.

1. Quantum repeated measurements: A Markov chain

1.1. Quantum repeated measurements

In this section, we present the mathematical model describing the quantum repeated measurements setup (see [32] for complete details). Here, we just present the Markov property of discrete quantum trajectories.

The model is based on quantum repeated interactions model. We consider a small system \mathcal{H}_0 in contact with an infinite chain of quantum systems. All the pieces of the chain are identical and independent; they are denoted by \mathcal{H} . Ones after the others, the copies of \mathcal{H} interact with \mathcal{H}_0 during a time h . After each interaction, a measurement is performed on the last incoming copy \mathcal{H} . The “indirect” repeated measurements involve random perturbations of the state of \mathcal{H}_0 .

Let us start by describing a single interaction between \mathcal{H}_0 and \mathcal{H} and the indirect measurement on \mathcal{H} . Let \mathcal{H}_0 and \mathcal{H} be finite dimensional Hilbert spaces. Each Hilbert space is endowed with a positive operator of trace one, called *state*. Let ρ be the initial state on \mathcal{H}_0 and let β be the state of \mathcal{H} . The coupling system is described by $\mathcal{H}_0 \otimes \mathcal{H}$. The interaction is described by a total Hamiltonian H_{tot} , which is a self adjoint operator, defined by

$$H_{\text{tot}} = H_0 \otimes I + I \otimes H + H_{\text{int}}.$$

The operators H_0 and H are the Hamiltonians of \mathcal{H}_0 and \mathcal{H} which represent the free evolution of each system. The operator H_{int} is called the interaction Hamiltonian. The Hamiltonian H_{tot} gives rise to a unitary operator of evolution

$$U = e^{ih H_{\text{tot}}},$$

where h is the time of interaction. After the interaction, the initial state $\rho \otimes \beta$ on $\mathcal{H}_0 \otimes \mathcal{H}$ becomes $\mu = U(\rho \otimes \beta)U^*$ (this is usually called the *Schrödinger picture*).

Let us now describe the measurement of an observable A of \mathcal{H} . An observable is a self-adjoint operator and we consider its spectral decomposition

$$A = \sum_{i=0}^p \lambda_i P_i,$$

where the operators P_i are the spectral projectors associated with the eigenvalues λ_i . Naturally, on $\mathcal{H}_0 \otimes \mathcal{H}$, we consider the extension $I \otimes A$ as an observable of $\mathcal{H}_0 \otimes \mathcal{H}$. According to the law of quantum mechanics, the measurement of $I \otimes A$ gives a random result concerning the eigenvalues of $I \otimes A$. This obeys to the following probability law

$$P[\text{to observe } \lambda_i] = \text{Tr}[\mu I \otimes P_i].$$

Moreover, after the measurement, if we have observed the eigenvalue λ_i , the reference state μ becomes

$$\mu_1(i) = \frac{I \otimes P_i \mu I \otimes P_i}{\text{Tr}[\mu I \otimes P_i]}.$$

Such phenomena is called *Wave Packet Reduction Principle* and relies on the projection postulate of von Neumann (more general measurement procedures are described by instruments [4]). Conditionally to the result of the observation, the state $\mu_1(\cdot)$ is then the new reference state of $\mathcal{H}_0 \otimes \mathcal{H}$.

In general, one is only interested in the reduced evolution of the small system \mathcal{H}_0 . This evolution is obtained by using the *partial trace* operation (see the definition below) on the coupled system $\mathcal{H}_0 \otimes \mathcal{H}$.

Definition–Theorem 1. Let α be a state on the tensor product $\mathcal{H}_0 \otimes \mathcal{H}$. There exists a unique state on \mathcal{H}_0 , denoted by $\mathbf{E}_0[\alpha]$, which is characterized by the property

$$\forall X \in \mathcal{B}(\mathcal{H}_0) \quad \text{Tr}_{\mathcal{H}_0}[\mathbf{E}_0[\alpha] X] = \text{Tr}_{\mathcal{H}_0 \otimes \mathcal{H}}[\alpha(X \otimes I)].$$

Here $\text{Tr}_{\mathcal{H}_0}$ corresponds to the trace of operator on \mathcal{H}_0 and similar definition for $\text{Tr}_{\mathcal{H}_0 \otimes \mathcal{H}}$.

Now, let us define the state $\rho_1(\cdot)$ on \mathcal{H}_0 by

$$\rho_1(i) = \mathbf{E}_0[\mu_1(i)], \quad i = 0, \dots, p.$$

The state $\rho_1(\cdot)$ is then the new reference state of \mathcal{H}_0 . This is a random state: each possible state $\rho_1(i)$ appears with probability $p_i = P[\text{to observe } \lambda_i]$. This describes the transition from the state ρ to the possible states $\rho_1(i)$.

As a consequence, \mathcal{H}_0 is now endowed with the state ρ_1 and a second copy of \mathcal{H} can interact with \mathcal{H}_0 . In the same way, a measurement of $I \otimes A$ is then performed and by taking the partial trace, we get a new random variable ρ_2 (naturally, we have similar transition probabilities). In a recursive way, we get a random sequence (ρ_k) of states, which is called a *Discrete Quantum Trajectory*. This describes the evolution of the state of \mathcal{H}_0 undergoing quantum repeated interactions and quantum repeated measurements.

From the description of the transitions of one interaction and one measurement and by construction of the random sequence (ρ_k) , we can express the following proposition (see [32] for complete details).

Proposition 1. There exists a probability space (Ω, \mathcal{F}, P) , where the discrete quantum trajectory (ρ_k) is a Markov chain. More precisely, if $\rho_k = \chi_k$ then ρ_{k+1} takes one of the values

$$\mathbf{E}_0 \left[\frac{I \otimes P_i U(\chi_k \otimes \beta) U^* I \otimes P_i}{\text{Tr}[U(\chi_k \otimes \beta) U^* I \otimes P_i]} \right], \quad i = 0, \dots, p,$$

with probability $\text{Tr}[U(\chi_k \otimes \beta) U^* P_i]$.

From the next section onward, we concentrate on a special case of a qubit in contact with a spin chain.

1.2. A qubit interacting with a spin chain

The physical situation is modeled by $\mathcal{H}_0 = \mathcal{H} = \mathbb{C}^2$. Let us start by describing an evolution equation for the state (ρ_k) in this context. Let $A = \lambda_0 P_0 + \lambda_1 P_1$, be an observable which is repeatedly measured. Proposition 1 allows to describe the evolution of (ρ_k) by the discrete stochastic equation

$$\rho_{k+1} = \frac{\mathcal{L}_0(\rho_k)}{\text{Tr}[\mathcal{L}_0(\rho_k)]} \mathbf{1}_0^{k+1} + \frac{\mathcal{L}_1(\rho_k)}{\text{Tr}[\mathcal{L}_1(\rho_k)]} \mathbf{1}_1^{k+1}, \quad (3)$$

where for $i \in \{0, 1\}$ the terms $\mathcal{L}_i(\rho_k)$ corresponds to the “non-normalized” transition of ρ_{k+1} , that is $\mathcal{L}_i(\rho_k) = \mathbf{E}_0[I \otimes P_i U(\rho_k \otimes \beta) U^* I \otimes P_i]$. In Eq. (3), the term $\mathbf{1}_0^{k+1}$ corresponds to the random variable which takes the value 1 with probability $p_{k+1} = \text{Tr}[\mathcal{L}_0(\rho_k)]$ and 0 with probability $q_{k+1} = 1 - p_{k+1}$ (this corresponds to the observation of λ_0 at the $(k+1)$ th measurement). The analog holds for the random variable $\mathbf{1}_1^{k+1}$.

In order to detail precisely the evolution of (ρ_k) , we have to make explicit the terms $\mathcal{L}_i(\rho_k)$. To this end, we introduce an appropriate basis which makes easier the computation of the partial trace. On $\mathcal{H}_0 \otimes \mathcal{H}$, we consider the basis $(\Omega \otimes \Omega, X \otimes \Omega, \Omega \otimes X, X \otimes X)$, where (Ω, X) is an orthonormal basis of \mathbb{C}^2 . In this basis, the unitary-operator U can be written as

$$U = \begin{pmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{pmatrix},$$

where each L_{ij} are operators on \mathcal{H}_0 . For the state β , let choose the projector on $\mathbb{C}\Omega$, that is,

$$\beta = P_{\{\Omega\}}.$$

Now, by expressing $P_i = (p_{kl}^i)_{k,l=0,1}$ in the basis (Ω, X) , we get

$$\mathcal{L}_i(\rho_k) = p_{00}^i L_{00} \rho_k L_{00}^* + p_{01}^i L_{00} \rho_k L_{10}^* + p_{10}^i L_{10} \rho_k L_{00}^* + p_{11}^i L_{10} \rho_k L_{10}^*. \quad (4)$$

In order to compare the discrete evolution and the continuous evolution in Section 3, we introduce new random variables. For all $k \geq 0$, let put $v_{k+1} = \mathbf{1}_1^{k+1}$ and define the random variable

$$X_{k+1} = \frac{v_{k+1} - q_{k+1}}{\sqrt{q_{k+1}p_{k+1}}}.$$

These random variables are naturally associated with the filtration (\mathcal{F}_k) on $\Sigma^{\mathbb{N}}$ defined by

$$\mathcal{F}_k = \sigma(X_i, i \leq k)$$

and by construction we have $\mathbf{E}[X_{k+1}/\mathcal{F}_k] = 0$ and $\mathbf{E}[X_{k+1}^2/\mathcal{F}_k] = 1$. In terms of (X_k) , the discrete evolution equation for the discrete quantum trajectory becomes

$$\rho_{k+1} = \mathcal{L}_0(\rho_k) + \mathcal{L}_1(\rho_k) + \left[-\sqrt{\frac{q_{k+1}}{p_{k+1}}} \mathcal{L}_0(\rho_k) + \sqrt{\frac{p_{k+1}}{q_{k+1}}} \mathcal{L}_1(\rho_k) \right] X_{k+1}. \quad (5)$$

This way, Eq. (5) appears as a random perturbation of the deterministic equation $\rho_{k+1} = \mathcal{L}_0(\rho_k) + \mathcal{L}_1(\rho_k)$ which describes actually the evolution without measurement. In [1], it is shown that $\mathcal{L}_0 + \mathcal{L}_1$ is an approximation of the Master equation. Equations of type (5) are then perturbations of discrete Master equations (see Section 2).

In the following section, we concentrate on the continuous evolution.

2. The jump-Belavkin equation

In this article, we focus on the jump equation

$$d\rho_t = L(\rho_t)dt + \left[\frac{\mathcal{J}(\rho_t)}{\text{Tr}[\mathcal{J}(\rho_t)]} - \rho_t \right] (d\tilde{N}_t - \text{Tr}[\mathcal{J}(\rho_t)]dt), \quad (6)$$

where \tilde{N}_t is a counting process with intensity $\int_0^t \text{Tr}[\mathcal{J}(\rho_s)]ds$.

In this equation, the operator L is called the Lindbladian of the system. This is a classical generator of the dynamic of open quantum systems, and it gives rise to the continuous time Master equation

$$\frac{d}{dt}\rho_t = L(\rho_t) = -i[H, \rho_t] - \frac{1}{2} \{C^*C, \rho_t\} + C\rho_t C^*. \quad (7)$$

Here, the operator C is an arbitrary 2×2 matrix, the operator H_0 is the Hamiltonian of the qubit. In equation (6), the operator \mathcal{J} is defined as $\mathcal{J}(\rho) = C\rho C^*$.

The next section is devoted to construct an appropriate probability space for studying Eq. (6).

2.1. A probability framework of the jump equation

As it has been mentioned in Introduction, Eq. (6) and the process (\tilde{N}_t) are not correctly defined. Indeed, in the formulation (6), the definition of (\tilde{N}_t) relies on the existence of the solution of Eq. (6). However, in order to consider this solution, we need to consider first the definition of the driving process of (6).

Actually, the processes (ρ_t) and (\tilde{N}_t) cannot be dissociated. We adopt then the following definition of a solution for the jump equation (see [26,23] for complete references on similar topic).

Definition 1. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space. A process-solution of (6) is a càdlàg process (ρ_t) such that there exists a counting process (\tilde{N}_t) with stochastic intensity

$$t \rightarrow \int_0^t \text{Tr}[\mathcal{J}(\rho_{s-})]ds$$

and such that the couple (ρ_t, \tilde{N}_t) satisfies almost surely

$$\rho_t = \rho_0 + \int_0^t \left[L(\rho_{s-}) - \mathcal{J}(\rho_{s-}) + \text{Tr}[\mathcal{J}(\rho_{s-})]\rho_{s-} \right] ds + \int_0^t \left[\frac{\mathcal{J}(\rho_{s-})}{\text{Tr}[\mathcal{J}(\rho_{s-})]} - \rho_{s-} \right] d\tilde{N}_s.$$

Now, following Definition 1, we need to determine an appropriate probability space where the couple (ρ_t, \tilde{N}_t) is clearly defined. In this sense, the process (ρ_t) is called a *weak* solution.

To this end, we use the general theory of *Random Measure* (for all details, see [23] or [27]). Let us introduce this notion.

Definition 2. Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, a random measure is a family of measures $\mu = (\mu(\omega, \cdot), \omega \in \Omega)$ on $(\mathbb{R}_+ \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d))$.

- A random measure is said to be integer valued if
 1. For all $\omega \in \Omega$ $\mu(\omega, t \times \mathbb{R}^d) \leq 1$.
 2. For all $A \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d)$, the quantity $\mu(A)$ is valued in $\mathbb{N} \cup \{+\infty\}$.
 - A random Poisson measure on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a integer valued measure that verifies
 1. The measure $m(A) = \mathbf{E}(\mu(A))$ on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d)$ is non-atomic.
 2. $m(0 \times \mathbb{R}^d) = 0$.
 3. If $t \in \mathbb{R}_+$ and if $A_i \in \mathcal{B}(]t, +\infty[), i = 1, \dots, l$ are two by two disjoint with $m(A_i) < +\infty$, the random variables $\mu(A_i)$ are mutually independent and independent from \mathcal{F}_t .
- The measure m is called the intensity of the random Poisson measure μ .

Now, we shall write Eq. (6) in a rigorous form.

Theorem 1. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space which supports a random Poisson measure μ on $\mathbb{R} \times \mathbb{R}$ with intensity $dt \otimes dx$. Every process-solution of equation

$$\begin{aligned} \rho_t = \rho_0 &+ \int_0^t \left[L(\rho_{s-}) + \text{Tr}[\mathcal{J}(\rho_{s-})] \rho_{s-} - \mathcal{J}(\rho_{s-}) \right] ds \\ &+ \int_0^t \int_{\mathbb{R}_+} \left[\frac{\mathcal{J}(\rho_{s-})}{\text{Tr}[\mathcal{J}(\rho_{s-})]} - \rho_{s-} \right] \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\rho_{s-})]} \mu(ds, dx) \end{aligned} \quad (8)$$

is a process-solution of Eq. (6) satisfying Definition 1. For the process (\tilde{N}_t) , we put

$$\tilde{N}_t = \int_0^t \int_{\mathbb{R}} \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\rho_{s-})]} \mu(ds, dx). \quad (9)$$

We refer to [26] for general considerations regarding jump processes with stochastic intensity. Concerning Theorem 1, there are two parts to treat. Firstly, we must prove that the process given by (9) is well defined, that is, it is a non-explosive process. Secondly, we must prove that any solution of equation (8) satisfies Definition 1.

The non-explosive property of (\tilde{N}_t) is related to the bounded character of the stochastic intensity $t \rightarrow \text{Tr}[\mathcal{J}(\rho_{t-})]$. Here, a straightforward computation shows that there exists a constant K such that for all states ρ , we have $0 \leq \text{Tr}[\mathcal{J}(\rho)] \leq K$. It implies directly that $\text{Tr}[\mathcal{J}(\rho_{t-})] = \lim_{s < t, s \rightarrow t} \text{Tr}[\mathcal{J}(\rho_s)] \leq K$, for all t . The intensity is then bounded. Now, we shall prove Theorem 1.

Proof. Let us show that the counting process (\tilde{N}_t) given by (9) is non-explosive. For all càdlàg matricial processes (X_t) , we define the explosion time

$$T^X := \inf \left(t : \tilde{N}_t^X = +\infty \right).$$

Let us show that, if (ρ_t) takes values in the set of states, the explosion time $T^\rho = \infty$ almost surely. To this aim, we introduce the following sequence of stopping times

$$T_n := \inf \left(t, \int_0^t \text{Tr}[\mathcal{J}(\rho_{s-})] ds \geq n \right), \quad n \geq 1.$$

According to the property of the intensity measure of the Poisson random measure, we get

$$\mathbf{E} \left[\tilde{N}_{T_n}^\rho \right] = \mathbf{E} \left[\int_0^{T_n} \text{Tr}[\mathcal{J}(\rho_{s-})] ds \right].$$

Hence, according to the fact that $\int_0^{T_n} \text{Tr}[\mathcal{J}(\rho_{s-})] ds \leq n$, we have $T_n \leq T^\rho$ almost surely. Now, if we prove that $\lim T_n = \infty$, the result holds. As $0 \leq \text{Tr}[\mathcal{J}(\rho_t)] \leq K$, we have $\int_0^t \text{Tr}[\mathcal{J}(\rho_{s-})] ds < \infty$, then we have $\lim_{n \rightarrow \infty} T_n = +\infty$. Finally, we have constructed a counting process without explosion for all càdlàg processes which take values in the set of states. Concerning the property that a process-solution of (8) satisfies Definition 1, this result follows from the construction of (\tilde{N}_t) . \square

Remark. This remark concerns a particular example of random Poisson measure with uniform intensity measure. Let (Ω, \mathcal{F}, P) be the probability space of a Poisson point process N on $\mathbb{R} \times \mathbb{R}$. The natural random Poisson measure attached with N is defined for all Borel subset $A \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ by

$$\mu(., A) = N(., A).$$

For all $A \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$, we have $\mathbf{E}[N(., A)] = \lambda(A)$ where λ denotes the Lebesgue measure. This particular random Poisson measure is used in Section 3 to realize continuous quantum trajectories and discrete quantum trajectories in a same probability space.

The next section is devoted to the problem of existence and uniqueness.

2.2. Existence and uniqueness

In this section, we consider a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ which supports a random Poisson measure μ . In order to treat the problems of existence and uniqueness, we first treat the existence of a solution for the ordinary differential part

$$\rho_t = \rho_0 + \int_0^t \left[L(\rho_{s-}) + \text{Tr}[\mathcal{J}(\rho_{s-})] \rho_{s-} - \mathcal{J}(\rho_{s-}) \right] ds, \quad (10)$$

and next we define the jump times. Let us stress that Eq. (10) is non-Lipschitz and the classical theorems cannot be applied. In our context, we show the following special result: if the initial condition is a state, the Eq. (10) admits a solution valued in the set of states. To this end, we use an auxiliary result which expresses that Eq. (10) preserves the property of being a *pure state* (a pure state is a particular state which is a one dimensional projector).

Proposition 2. Let x be a vector of norm one in \mathbb{C}^2 . If the Cauchy problem

$$\begin{cases} dx_t = \left[-iH_0 - \frac{1}{2}C^*C + \frac{1}{2}\eta_t \right] x_t dt \\ x_0 = x, \end{cases} \quad (11)$$

where $\eta_t = \langle x_t, C^*C x_t \rangle$ has a solution, then $\|x_t\| = 1$, for all $t > 0$.

Furthermore the process (ρ_t) of one-dimensional projector defined by $\rho_t = P_{\{x_t\}}$, for all $t > 0$, is a solution of the Cauchy problem

$$\begin{cases} d\rho_t = \left[L(\rho_t) + \text{Tr}[\mathcal{J}(\rho_t)]\rho_t - \mathcal{J}(\rho_t) \right] dt \\ \rho_0 = P_{\{x\}}. \end{cases} \quad (12)$$

Proof. Let (x_t) be the solution of (11). Since H_0 is self-adjoint and $\eta_t = \langle x_t, C^* C x_t \rangle = \langle C x_t, C x_t \rangle$, a straightforward computation gives

$$\frac{d}{dt} \langle x_t, x_t \rangle = -\eta_t + \eta_t \langle x_t, x_t \rangle.$$

As a consequence, if $\langle x_0, x_0 \rangle = 1$, then $\langle x_t, x_t \rangle = 1$, for all t . Now, we shall prove the second part. Let us put $\rho_t = P_{\{x_t\}}$, we have $\rho_t y = \langle x_t, y \rangle x_t$, for all y . The derivation of $\rho_t y$ gives then

$$\begin{aligned} \frac{d}{dt} \rho_t y &= \left\langle \frac{d}{dt} x_t, y \right\rangle x_t + \langle x_t, y \rangle \frac{d}{dt} x_t \\ &= \left\langle x_t, \left[-iH_0 - \frac{1}{2} C^* C + \frac{1}{2} \eta_t \right]^* y \right\rangle x_t + \left[-iH_0 - \frac{1}{2} C^* C + \frac{1}{2} \eta_t \right] \langle x_t, y \rangle x_t \\ &= \rho_t \left[-iH_0 - \frac{1}{2} C^* C + \frac{1}{2} \eta_t \right]^* y + \left[-iH_0 - \frac{1}{2} C^* C + \frac{1}{2} \eta_t \right] \rho_t y \\ &= \left[L(\rho_t) + \text{Tr}[\mathcal{J}(\rho_t)] \rho_t - \mathcal{J}(\rho_t) \right] y \end{aligned}$$

and the result follows. \square

In the proof of the existence of a solution for (10), we will use the following obvious characterization of pure states in \mathbb{C}^2 (actually this characterization is only true in dimension 2 and will be used in order to show that the solution takes values in the state space).

Lemma 1. *Let ρ be a state on \mathbb{C}^2 . If there exists a vector $x \in \mathbb{C}^2$ such that $\langle x; \rho x \rangle = 0$, the state ρ is a one dimensional projector.*

Now, we are in the condition to express the existence result of equation (10).

Proposition 3. *Let ρ be any state on \mathbb{C}^2 . The Cauchy problem*

$$\begin{cases} d\rho_t = \left[L(\rho_t) + \text{Tr}[\mathcal{J}(\rho_t)] \rho_t - \mathcal{J}(\rho_t) \right] dt \\ \rho_0 = \rho \end{cases} \quad (13)$$

has a unique solution defined for all time t .

Furthermore, if there exists t_0 such that ρ_{t_0} is a one dimensional projector, the solution of (13) after t_0 takes values in the set of pure states.

Proof. As the coefficients are not Lipschitz, the Theorem of Cauchy Lipschitz cannot be applied directly. However, the coefficients are C^∞ , so locally Lipschitz and we can use a truncation method. The ordinary equation is of the following form $d\rho_t = f(\rho_t) dt$, where f is C^∞ and $f(A) = L(A) + \text{Tr}[\mathcal{J}(A)]A - \mathcal{J}(A)$. We define the truncation function φ from \mathbb{R} to \mathbb{R} defined by

$$\varphi_k(x) = -k \mathbf{1}_{x \leq -k} + x \mathbf{1}_{-k \leq x \leq k} + k \mathbf{1}_{x \geq k}.$$

For a matrix $A = (a_{ij})$, we define by extension $\tilde{\varphi}_k(A) = \varphi_k(\text{Re}(a_{ij})) + i\varphi_k(\text{Im}(a_{ij}))$. Thus the function $f \circ \tilde{\varphi}_k$ is Lipschitz. Now, by applying Cauchy Lipschitz Theorem, the truncated equation

$$d\rho_{k,t} = f \circ \tilde{\varphi}_k(\rho_{k,t}) dt,$$

admits a unique solution $t \mapsto \rho_{k,t}$ defined for all t . In order to make the link between the equation (13) and the truncated equation, we define

$$T_k = \inf \left\{ t, \exists (ij) | |(\text{Re}(a_{ij}(\rho_{k,t})))| = k \text{ or } |(\text{Im}(a_{ij}(\rho_{k,t})))| = k \right\}.$$

As ρ_0 is a state, for $k > 1$, on $[0, T_k]$, we have $\tilde{\varphi}_k(\rho_{k,t}) = \rho_{k,t}$. Thus, the application $t \mapsto \rho_{k,t}$ is the unique solution of the ordinary equation (10) on $[0, T_k]$ (without truncation). Usually, in order to show that such a non-Lipschitz equation admits a solution, we prove that $T = \lim_k T_k = \infty$.

Here, the situation is simpler. Indeed, since $\|\rho\| \leq 1$ when ρ is a state, we have for example $\tilde{\varphi}_2(\rho) = \rho$. It is then sufficient to show that the solution, obtained by truncation, takes values in the set of states. This will then imply that $T_2 = \infty$. Let us show that the solution on $[0, T_2]$ is self adjoint, positive and of trace one.

On $[0, T_2]$, as the ordinary differential equation is Lipschitz, we can solve it by Picard method. This way, we define

$$\begin{cases} \rho_{n+1}(t) = \rho_n(0) + \int_0^t f \circ \tilde{\varphi}_k(\rho_n(s)) ds \\ \rho_0(t) = \rho. \end{cases} \quad (14)$$

Now, it is easy to see, with the right definition of f , that this sequence is made of self adjoint operators of trace one. These properties are then conserved at the limit and the matrices $\rho_{2,t}$ are self adjoint of trace one, for all $t \leq T_2$.

The positivity property poses more tedious problem and is not a direct consequence of the Picard method. We use the equivalence stated in Proposition 2 and Lemma 1 to prove it. We shall prove that $\langle y, \rho_{2,t} y \rangle \geq 0$, for all $y \in \mathbb{R}^2$ and for all $t \leq T_2$. To this end, we define

$$T^0 = \inf \left\{ t \leq T_2, \exists y \in \mathbb{R}^2 | \langle y, \rho_{2,t} y \rangle = 0 \right\}.$$

Now, if $T^0 = T_2$, an argument of continuity implies that $\langle y, \rho_{2,t} y \rangle \geq 0$, for all $t \leq T_2$ and all $y \in \mathbb{C}^2$ and the result holds.

Otherwise, if $T^0 < T_2$, by continuity, there exists $x \in \mathbb{R}^2$ such that $\langle x, \rho_{2,T^0} x \rangle = 0$ and $\langle y, \rho_{2,t} y \rangle \geq 0$, for all $t \leq T^0$ and for all $y \in \mathbb{R}^2$. Thus, on $[0, T^0]$, the solution $t \mapsto \rho_{2,t}$ takes values on the state space. Moreover, since $\langle x, \rho_{2,T^0} x \rangle = 0$, Lemma 1 states that the operator ρ_{2,T^0} is a one dimensional projector. Hence, we can now consider the ordinary differential equation with initial state $\rho_{T^0} = \rho_0$. We can then consider the Cauchy problem (11) which is equivalent to the problem (13) (Proposition 2). A truncation method allows to consider a truncated solution for (11), the fact that the solution is actually of norm one implies that the solution is defined for all t (the truncation has actually no effects). Proposition 2 implies then that the Cauchy problem (13) admits a solution, valued in the set of states, on the interval $[T^0, T_2]$.

At time T_2 , the solution is then a state. A local argument and the uniqueness in the Cauchy-Lipschitz theorem allows us to conclude that $T_2 = \infty$ and the result is proved. \square

This above proposition is essential in the proof of the following theorem.

Theorem 2. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space which supports a Poisson random measure μ whose intensity measure is $dx \otimes dt$ and let ρ_0 be any initial state. Then the jump-Belavkin equation

$$\begin{aligned} \rho_t = \rho_0 + \int_0^t & \left[L(\rho_{s-}) + \text{Tr}[\mathcal{J}(\rho_{s-})] \rho_{s-} - \mathcal{J}(\rho_{s-}) \right] ds \\ & + \int_0^t \int_{\mathbb{R}_+} \left[\frac{\mathcal{J}(\rho_{s-})}{\text{Tr}[\mathcal{J}(\rho_{s-})]} - \rho_{s-} \right] \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\rho_{s-})]} \mu(ds, dx) \end{aligned} \quad (15)$$

admits a unique solution. The process-solution (ρ_t) takes values in the set of states.

Proof. Such an equation is solved in path-wise manner. Since ρ_0 is a state, Proposition 3 ensures the existence and the uniqueness of the solution of the Cauchy problem (13). Now, we shall define the first time of jump. To this end, we put

$$\begin{cases} \rho(1)_t = \rho_0 + \int_0^t f(\rho(1)_s) ds \\ T_1 = \inf\{t, \tilde{N}_t^{\rho(1)} > 0\}. \end{cases} \quad (16)$$

The term T_1 is a random stopping time which defines the first jump. We have

$$\tilde{N}_{T_1}^{\rho(1)}(\omega) = \mu(\omega, G(\rho, T_1, 0)) = 1,$$

where $G(\rho, t, s) = \{(u, y) \in \mathbb{R}^2 | t < u < s, 0 < y < \text{Tr}[\mathcal{J}(\rho_u)]\}$. It is worth noting that the quantity $\mu(\omega, G(\rho, t, s))$ represents the number of points under the curve $t \rightarrow \text{Tr}[\mathcal{J}(\rho_t)]$.

Now, if $T_1 = \infty$, the solution of the jump equation is given by the solution of the ordinary differential equation, that is, there are no jumps.

If $T_1 < \infty$, at the jump time T_1 , the solution is defined by implementing the value of the jump, that is,

$$\rho_{T_1} = \rho_{T_1-} + \left[\frac{\mathcal{J}(\rho_{T_1-})}{\text{Tr}[\mathcal{J}(\rho_{T_1-})]} - \rho_{T_1-} \right] = \frac{\mathcal{J}(\rho_{T_1-})}{\text{Tr}[\mathcal{J}(\rho_{T_1-})]}.$$

Since $T_1 < \infty$, we have $\text{Tr}[\mathcal{J}(\rho_{T_1-})] > 0$ and this above matrix is well defined. Moreover, it is easy to see that ρ_{T_1} is a state. This gives rise to a new initial condition for the equation (10). We can solve equation (10) after T_1 and define a second jump time T_2 and so on. In a recursive way, we define a sequence of jump times (T_n) and a sequence of processes $(\rho_n(t))$ by the following expression

$$\begin{cases} \rho(n)_t = \rho(n-1)_t & \text{on } [0, T_{n-1}[\\ \rho_{T_{n-1}} = \frac{\mathcal{J}(\rho_{T_{n-1}-})}{\text{Tr}[\mathcal{J}(\rho_{T_{n-1}-})]} \\ \rho(n)_t = \rho_{T_{n-1}} + \int_{T_{n-1}}^t f(\rho(n-1)_s) ds \\ T_n = \inf\{t > T_{n-1}, \tilde{N}_t^{\rho(n)} > \tilde{N}_{T_1}^{\rho(n-1)}\}. \end{cases} \quad (17)$$

In other words, between the jump times, the solution is given by the solution of (10) and we implement the jump value at the jump times. Since the matrices, defined at each jump time, are states, the processes $(\rho_n(t))$ are well defined (the Cauchy problem can be solved).

Now, we shall define the solution of the jump equation. The sequence of random stopping times (T_n) satisfies $T_{n+1} > T_n$ on the set $\{T_n < \infty\}$. Hence, we can define the process-solution on $[0, T[$, where $T = \lim_{n \rightarrow \infty} T_n$. For all $t < T$, we put

$$\rho_t = \rho(n)_t \quad \text{on } [0, T_n[. \quad (18)$$

This process is clearly a solution of the jump-Belavkin equation (15) and it takes values in the state space. The uniqueness is implied by the uniqueness of the solutions of Cauchy problems. Moreover, any other solution is forced to have the same random jump times. This implies the uniqueness.

In order to conclude, we must show $T = \infty$ a.s. Since (ρ_t) takes values in the set of states, we have $\text{Tr}[\mathcal{J}(\rho_t)] < K$, for all t . This implies

$$\mathbf{E} \left[\tilde{N}_{T_p \wedge n}^{\rho} \right] \leq \mathbf{E} \left[\tilde{N}_n^{\rho} \right] \leq Kn.$$

Furthermore, as $\tilde{N}_{T_p \wedge n}^\rho = p$ on $\{T_p < n\}$, it follows that $pP[T_p < n] \leq Kn$, then we have $P[T \leq n] = 0$ for all n and the result is proved. \square

Remark. This theorem gives an explicit construction of the solution. Other approaches to treat stochastic Schrödinger equations are based on the use of a linear stochastic differential equation and a use of Girsanov transformation [3,31]. In this context, the solution is not explicit. Let us stress that our explicit construction will be used to prove the convergence result.

In order to prove the final result, we use a random coupling method. Hence, we need an explicit realization of the process (ρ_t) . To this end, we use the explicit Poisson random measure mentioned in Section 2.1. Let us consider the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ of a Poisson point process N on $\mathbb{R} \times \mathbb{R}$. Let $N(\omega, ds, dx)$ denote the differential increment of the random measure N . The continuous quantum trajectory (ρ_t) satisfies

$$\begin{aligned} \rho_t = \rho_0 &+ \int_0^t \left[L(\rho_{s-}) + \text{Tr}[\mathcal{J}(\rho_{s-})]\rho_{s-} - \mathcal{J}(\rho_{s-}) \right] ds \\ &+ \int_0^t \int_{[0, K]} \left[\frac{\mathcal{J}(\rho_{s-})}{\text{Tr}[\mathcal{J}(\rho_{s-})]} - \rho_{s-} \right] \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\rho_{s-})]} N(., ds, dx). \end{aligned} \quad (19)$$

It is worth noting that we can work on $[0, K]$ since $\text{Tr}[\mathcal{J}(\rho_t)] \leq K$ for all processes valued in the set of states. Based on this property, the following remark gives another equivalent way to define the solution of (8).

Remark. The function $t \rightarrow \text{card}(N(., [0, K] \times [0, t])) = \mathcal{N}_t$ defines a standard Poisson process with intensity K . Thus for the filtration \mathcal{F}_t , we can choose the natural filtration of this process. On $[0, T]$ (for a fixed T), the Poisson random measure and the previous process generate a sequence $\{(\tau_i, \xi_i), i \in \{1, \dots, \mathcal{N}_t\}\}$, where each τ_i represents one jump time of \mathcal{N} . Moreover, the random variables ξ_i are uniform random variables on $[0, K]$. Consequently, we can write our continuous quantum trajectory in the following way

$$\begin{aligned} \rho_t = \rho_0 &+ \int_0^t \left[L(\rho_{s-}) + \text{Tr}[\mathcal{J}(\rho_{s-})]\rho_{s-} - \mathcal{J}(\rho_{s-}) \right] ds \\ &+ \sum_{i=1}^{\mathcal{N}_t} \left[\frac{\mathcal{J}(\rho_{\tau_i-})}{\text{Tr}[\mathcal{J}(\rho_{\tau_i-})]} - \rho_{\tau_i-} \right] \mathbf{1}_{0 \leq \xi_i \leq \text{Tr}[\mathcal{J}(\rho_{\tau_i-})]}. \end{aligned}$$

The next section concerns the final convergence result.

3. Approximation and convergence theorems

This section is devoted to the convergence theorem. Particular discrete quantum trajectories, defined in Section 1, are shown to converge to the solution of the jump-Belavkin equation (19).

The procedure for showing this result is the following. First, we introduce the time interaction to be $h = 1/n$. This way, the discrete quantum trajectories depend on the parameter n . Next, we implement asymptotic assumptions on the unitary operator U . Next, for special form of observables, we show that the corresponding discrete quantum trajectories are “good candidates” to converge to the solution of the jump equation. The final result relies next on two steps.

- Random coupling method.
- Convergence and comparison to an Euler scheme.

3.1. Asymptotic form of discrete jump-Belavkin process

We start by investigating the asymptotic behavior of discrete quantum trajectories. This is based on proper asymptotic assumptions described as follows.

Let $h = 1/n$, the unitary evolution depends naturally on n and we put

$$U(n) = \begin{pmatrix} L_{00}(n) & L_{01}(n) \\ L_{10}(n) & L_{11}(n) \end{pmatrix}.$$

In the quantum repeated interactions setup [1], it is shown that the coefficients L_{ij} must obey precise asymptotic conditions to obtain a non-trivial limit, when n goes to infinity. More precisely, in [1], they have shown that quantum stochastic differential equations (also called “Hudson–Parthasarathy equations”), describing continuous time interaction models, can be obtained as continuous limit models of quantum repeated interactions by rescaling discrete interactions.

In our context the asymptotic conditions are the following

$$L_{00}(n) = I + \frac{1}{n} \left(-iH - \frac{1}{2}CC^* \right) + o\left(\frac{1}{n}\right), \quad (20)$$

$$L_{10}(n) = \frac{1}{\sqrt{n}}C + o\left(\frac{1}{n}\right). \quad (21)$$

The associated total Hamiltonian $H_{\text{tot}}(n)$ can be described as

$$H_{\text{tot}}(n) = H_0 \otimes I + I \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\sqrt{n}} \left[C \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + C^* \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] + o\left(\frac{1}{n}\right).$$

By introducing the parameter n , the discrete evolution equations are described as follows

$$\begin{aligned} \rho_{k+1}(n) &= \mathcal{L}_0(n)(\rho_k(n)) + \mathcal{L}_1(n)(\rho_k(n)) \\ &+ \left[-\sqrt{\frac{q_{k+1}(n)}{p_{k+1}(n)}} \mathcal{L}_0(n)(\rho_k(n)) + \sqrt{\frac{p_{k+1}(n)}{q_{k+1}(n)}} \mathcal{L}_1(n)(\rho_k(n)) \right] X_{k+1}(n). \end{aligned} \quad (22)$$

Remind that by definition, for (X_k) , we have

$$X_{k+1}(n)(i) = \begin{cases} -\sqrt{\frac{q_{k+1}(n)}{p_{k+1}(n)}} & \text{with probability } p_{k+1}(n) \text{ if } i = 0 \\ \sqrt{\frac{p_{k+1}(n)}{q_{k+1}(n)}} & \text{with probability } q_{k+1}(n) \text{ if } i = 1, \end{cases} \quad (23)$$

where $p_{k+1}(n) = \text{Tr} \left[I \otimes P_0 U(n)(\rho_k \otimes \beta) U(n)^* I \otimes P_0 \right] = 1 - q_{k+1}(n)$.

Now, the next step consists in applying the asymptotic assumptions in equation (22). Depending on the form of the observable, we get two different situations.

- If the observable A is diagonal in the orthonormal basis (Ω, X) of \mathcal{H} , that is $A = \lambda_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we obtain the asymptotic expression for the probabilities

$$p_{k+1} = 1 - \frac{1}{n} \text{Tr}[\mathcal{J}(\rho_k)] + o\left(\frac{1}{n}\right) \quad \text{and} \quad q_{k+1} = \frac{1}{n} \text{Tr}[\mathcal{J}(\rho_k)] + o\left(\frac{1}{n}\right).$$

In asymptotic form, the discrete equation becomes

$$\rho_{k+1} - \rho_k = \frac{1}{n}L(\rho_k) + o\left(\frac{1}{n}\right) + \left[\frac{\mathcal{J}(\rho_k)}{\text{Tr}(\mathcal{J}(\rho_k))} - \rho_k + o(1)\right]\sqrt{q_{k+1}p_{k+1}}X_{k+1}(n).$$

- If the observable is non-diagonal in the basis (Ω, X) , we put $P_0 = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$ and $P_1 = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix}$. Then we have

$$\begin{aligned} p_{k+1} = 1 - q_{k+1} &= p_{00} + \frac{1}{\sqrt{n}}\text{Tr}[\rho_k(p_{01}C + p_{10}C^*)] + \frac{1}{n}\text{Tr}[\rho_k(p_{00}(C + C^*))] \\ &+ o\left(\frac{1}{n}\right). \end{aligned}$$

The discrete equation becomes then

$$\begin{aligned} \rho_{k+1} - \rho_k &= \frac{1}{n}L(\rho_k) + o\left(\frac{1}{n}\right) + \left[e^{i\theta}C\rho_k + e^{-i\theta}\rho_kC^* \right. \\ &\quad \left. - \text{Tr}[\rho_k(e^{i\theta}C + e^{-i\theta}C^*)]\rho_k + o(1)\right]\frac{1}{\sqrt{n}}X_{k+1}. \end{aligned}$$

From these descriptions, we can define processes $\rho_{[nt]}(n)$ by

$$\begin{aligned} \rho_{[nt]}(n) &= \rho_0 + \sum_{k=1}^{[nt]-1} (\rho_{k+1} - \rho_k) \\ &= \rho_0 + \sum_{k=1}^{[nt]-1} \left(L(\rho_k) + o(1) \right) \frac{1}{n} + \sum_{k=1}^{[nt]-1} \mathcal{Q}_i(\rho_k) X_{k+1}(n), \end{aligned} \quad (24)$$

where the expression of \mathcal{Q}_i depends on the expression of the observable (diagonal or not).

Remark. In the non-diagonal case in [32], a first essential result is the proof of the convergence

$$W_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} X_k(n) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} W_t, \quad (25)$$

where \mathcal{D} denotes the convergence in distribution and (W_t) is a standard Brownian motion. Next, a result of Kurtz and Protter (cf [28,29]) implies that the discrete process converges to the solution of the diffusive equation. In other terms, the convergence of the discrete noise $(W_n(t))$ to the continuous noise implies the convergence of $(\rho_{[nt]})$.

In the diagonal case, we expect that the discrete quantum trajectory converges to the solution of jump equation. In this case, according to the asymptotic expressions of probabilities, the similar corresponding convergence of the discrete noise should be

$$N_n(t) = \sum_{k=1}^{[nt]} X_k(n) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \tilde{N}_t - \int_0^t \text{Tr}[\mathcal{J}(\rho_{s-})] ds. \quad (26)$$

It is important to see that the result (26) involves the process (ρ_t) . Moreover, the definition of $N_n(t)$ involves also the discrete process $(\rho_{[nt]}(t))$. For these reasons, the convergence (26) of the discrete noise $(N_n(t))$ requires the convergence of the discrete process to the continuous

process (which is the final expected result). As a conclusion, the approach based on the result of Kurtz and Protter cannot be used in the jump case. Here, the convergence follows from a explicit comparison between the discrete process $(\rho_{[nt]}(t))$ and the continuous process (ρ_t) .

Before presenting the main result, we need to establish the auxiliary convergence of the Euler scheme for the jump equation.

3.2. Euler-scheme of the jump-Belavkin equation

The literature abounds with references about Euler scheme approximation for stochastic differential equations (cf [16,24,25]). The non-usual case of jump-Belavkin equations is not really treated, that is why we detail this convergence.

An important property ensuring the convergence of an Euler scheme is the Lipschitz character of the functions defining the stochastic differential equation. In terms of the Poisson point process N , the jump equation can be written as

$$\mu_t = \mu_0 + \int_0^t f(\mu_{s-}) ds + \int_0^t \int_{[0,K]} [q(\mu_{s-})] \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\mu_{s-})]} N(., dx, ds), \quad (27)$$

where, the functions f and q are non-Lipschitz. Actually the problem concerns only the function q . Indeed, the function f is C^∞ and can be considered as Lipschitz by a truncation (as Section 2, such a truncation has no effects on the set of states). Let us show that we can consider a similar property for q . To this end, we must control the function defined on the states by

$$g : \rho \longrightarrow \left[\frac{\mathcal{J}(\rho)}{\text{Tr}[\mathcal{J}(\rho)]} - \rho \right] \mathbf{1}_{0 < \text{Tr}[\mathcal{J}(\rho)]}. \quad (28)$$

We shall prove that we can define a function q which is C^∞ and such that

$$g(\rho) = q(\rho) \mathbf{1}_{0 < \text{Tr}[\mathcal{J}(\rho)]}.$$

This depends on the invertible character of C .

If C is invertible, the function defined on the set of states by $\rho \rightarrow \text{Tr}[\mathcal{J}(\rho)]$ is continuous. Since C is invertible, we have $\text{Tr}[\mathcal{J}(\rho)] > 0$, for all states ρ . Since the set of states is compact, the function $\text{Tr}[\mathcal{J}(\cdot)]$ can be extended as a non-vanishing C^∞ function defined for all matrices.

If C is not invertible, there exists a unitary-operator V and two complex scalars α et β such that

$$VCV^* = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}. \quad (29)$$

Before going further, we have to show a unitary equivalence for jump equations. To this end, we define

$$\begin{aligned} \mathcal{J}_V(\rho) &= VCV^*(\rho)(VCV^*)^*, \\ f_V(\rho) &= -i[VHV^*, \rho] - \frac{1}{2} \{ VCV^*(VCV^*)^*, \rho \} + \text{Tr}[\mathcal{J}_V(\rho)]\rho, \\ g_V(\rho) &= \left[\frac{\mathcal{J}_V(\rho)}{\text{Tr}[\mathcal{J}_V(\rho)]} - \rho \right] \mathbf{1}_{0 < \text{Tr}[\mathcal{J}_V(\rho)]}, \end{aligned} \quad (30)$$

for all states ρ . We have the following proposition which expresses the unitary equivalence.

Proposition 4. Let V be any unitary operator and let (μ_t) be the solution of the jump-Belavkin equation. Then the process (γ_t) defined by

$$\gamma_t = V\mu_t V^*, \quad \forall t \geq 0$$

takes values in the state space. This process satisfies

$$\gamma_t = \gamma_0 + \int_0^t f_V(\gamma_{s-}) ds + \int_0^t \int_{[0, K]} [g_V(\gamma_{s-})] \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}_V(\gamma_{s-})]} N(., dx, ds), \quad (31)$$

which corresponds to a unitary transformation of the jump-Belavkin equation.

The proof is a straightforward computation. Such an unitary equivalence allows us to transform g without changing the Lipschitz property of f . Now, we are in the position to construct the function q in the case where C is not invertible. Let V be the unitary operator involved in expression (29), we get

$$g_V(\rho) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \rho \right] \mathbf{1}_{0 < \text{Tr}[\mathcal{J}_V(\rho)]}.$$

Hence, the expression of q is clear. This way, with a truncation method and the unitary equivalence, we can finally consider that the functions f and q are Lipschitz (without modifying their actions on the set of states). We denote by F and Q the corresponding Lipschitz constants.

All these technical precautions are necessary since we do not control the property of the process defined by the Euler scheme (namely this process cannot be valued in the set of states). Now, let us consider the Euler scheme

$$\theta_{k+1} = \theta_k + \frac{1}{n} f(\theta_k) + \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{[0, K]} [q(\theta_k)] \mathbf{1}_{0 \leq x \leq \text{Re}(\text{Tr}[\mathcal{J}(\theta_k)])} N(., dx, ds). \quad (32)$$

Let us fix an interval $[0, T]$ and for all $t < T$, we define $k_t = \max\{k \in \{0, 1, \dots\} | \frac{k}{n} \leq t\}$. For all t in $[\frac{k}{n}, \frac{k+1}{n}]$, we put

$$\tilde{\theta}_t = \theta_k + \int_{\frac{k}{n}}^t f(\theta_k) ds + \int_{\frac{k}{n}}^t \int_{[0, 1]} [q(\theta_k)] \mathbf{1}_{0 \leq x \leq \text{Re}(\text{Tr}[\mathcal{J}(\theta_k)])} N(., dx, ds). \quad (33)$$

It is worth noting that we have $\tilde{\theta}_{\frac{k}{n}} = \theta_k$, for all k . We then have for $t < T$

$$\begin{aligned} \tilde{\theta}_t(n) &= \sum_{k=0}^{k_t-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(\theta_k) ds + \sum_{k=0}^{k_t-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{[0, K]} [q(\theta_k)] \mathbf{1}_{0 \leq x \leq \text{Re}(\text{Tr}[\mathcal{J}(\theta_k)])} N(., dx, ds) \\ &\quad + \int_{k_t}^t f(\theta_{k_t}) ds + \int_{k_t}^t \int_{[0, K]} [q(\theta_{k_t})] \mathbf{1}_{0 \leq x \leq \text{Re}(\text{Tr}[\mathcal{J}(\theta_{k_t})])} N(., dx, ds). \end{aligned}$$

Let us stress that the solution (μ_t) of the Belavkin equation satisfies a similar decomposition.

Before expressing the convergence theorem, we need the following proposition.

Proposition 5. Let (μ_t) be the solution of the jump-Belavkin equation. Then there exists a constant M such that for all $(s, t) \in \mathbb{R}_+^2$

$$\mathbb{E}[\|\mu_t - \mu_s\|] \leq M|t - s|. \quad (34)$$

Proof. Let us first remark that $\|\mu_t\| \leq 1$ almost surely, for all $t > 0$ (since (μ_t) takes values in the state space). Now using the definition of the intensity measure, we have for $0 < s < t$

$$\begin{aligned} \mathbf{E}[\|\mu_t - \mu_s\|] &\leq \mathbf{E}\left[\left\|\int_s^t f(\mu_{u-})du\right\|\right] \\ &\quad + \mathbf{E}\left[\left\|\int_s^t \int_{[0,K]} [q(\mu_{u-})]\mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\mu_{u-})]} N(., dx, du)\right\|\right] \\ &\leq \int_s^t \mathbf{E}[\|f(\mu_{u-})\|] du + \mathbf{E}\left[\int_s^t \int_{[0,K]} \|q(\mu_{u-})\mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\mu_{u-})]}\| dx du\right] \\ &\leq \int_s^t \left(\sup_{\|R\| \leq 1} \|f(R)\| + 2\right) du \leq M(t-s), \end{aligned}$$

where M is a constant. The result is then proved. \square

The following theorem concerns the convergence of the Euler scheme.

Theorem 3. Let $T > 0$, let $(\tilde{\theta}_t)$ be the process (33) constructed by the Euler-scheme on $[0, T]$ and let (μ_t) be the unique solution of the jump-Belavkin equation (19).

There exists a constant Γ , independent of n , such that

$$Z_u(n) = \mathbf{E}\left[\sup_{0 \leq t \leq u} \|\tilde{\theta}_t(n) - \mu_t\|\right] \leq \frac{\Gamma}{n}, \quad (35)$$

for all $u < T$.

Let $\mathcal{D}([0, T])$ denote the space of càdlàg matrix processes endowed with the Skorohod topology. Hence, the Euler scheme approximation $(\tilde{\theta}_t)$ converges in distribution in $\mathcal{D}([0, T])$, for all T , to the process-solution (μ_t) of the jump-Belavkin equation.

Before giving the proof, it is interesting to compare this convergence with the classical results. Usually, in the literature, one considers L_2 norm result [16]. More precisely, one obtain estimations like

$$\mathbf{E}\left[\sup_{0 \leq t \leq u} \|\tilde{\theta}_t(n) - \mu_t\|^2\right] \leq \frac{\Gamma}{n^2}.$$

Next, one can obtain almost surely convergences. In the proof of Theorem 3, we will show that we cannot obtain similar results because we loose the homogeneity of the L_2 norm.

Proof. By comparing the Euler scheme and the solution of the jump-Belavkin equation, we get

$$\begin{aligned} \tilde{\theta}_t(n) - \mu_t &= \sum_{k=0}^{k_t-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} [f(\theta_k) - f(\mu_{s-})] ds + \int_{k_t}^t [f(\theta_{k_t}) - f(\mu_{s-})] ds \\ &\quad + \sum_{k=0}^{k_t-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{[0,K]} \left(q(\theta_k) \mathbf{1}_{0 \leq x \leq \text{Re}[\text{Tr}[\mathcal{J}(\theta_k)]]} - q(\mu_{s-}) \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\mu_{s-})]} \right) \\ &\quad \times N(., dx, ds) \\ &\quad + \int_{k_t}^t \int_{[0,K]} \left(q(\theta_{k_t}) \mathbf{1}_{0 \leq x \leq \text{Re}[\text{Tr}[\mathcal{J}(\theta_{k_t})]} - q(\mu_{s-}) \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\mu_{s-})]} \right) \\ &\quad \times N(., dx, ds). \end{aligned}$$

Let us consider separately the drift term and the term concerning the random measure. Let us start with the drift term. Since f is Lipschitz and by inserting terms like $\mu_{k/n}$, we have

$$\begin{aligned}
 & \mathbf{E} \left[\sup_{0 \leq t \leq u} \sum_{k=0}^{k_t-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \|f(\theta_k) - f(\mu_{s-})\| ds + \int_{k_t}^t \|f(\theta_{k_t}) - f(\mu_{s-})\| ds \right] \\
 & \leq \sum_{k=0}^{k_u-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \mathbf{E} \left[\|f(\tilde{\theta}_{\frac{k}{n}}) - f(\mu_{\frac{k}{n}})\| \right] ds + \int_{k_u}^u \mathbf{E} \left[\|f(\tilde{\theta}_{\frac{k_u}{n}}) - f(\mu_{\frac{k_u}{n}})\| \right] ds \\
 & \quad + \sum_{k=0}^{k_u-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \mathbf{E} \left[\|f(\mu_{s-}) - f(\mu_{\frac{k}{n}})\| \right] ds + \int_{k_u}^u \mathbf{E} \left[\|f(\mu_{s-}) - f(\mu_{\frac{k_u}{n}})\| \right] ds \\
 & \leq \sum_{k=0}^{k_u-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} F \mathbf{E} \left[\sup_{0 \leq t \leq s} \|\tilde{\theta}_t - \mu_t\| \right] ds + \int_{k_u}^u F \mathbf{E} \left[\sup_{0 \leq t \leq s} \|\tilde{\theta}_t - \mu_t\| \right] ds \\
 & \quad + \sum_{k=0}^{k_u-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} F M \left(s - \frac{k}{n} \right) ds + \int_{k_u}^u F M \left(s - \frac{k_u}{n} \right) ds \\
 & \leq A \left(\int_0^u Z_s ds + \frac{1}{n} \right) \quad (A \text{ is a suitable constant}).
 \end{aligned}$$

The analysis of the random measure terms is more complicated. For k being fixed, according to the properties of random measure and by introducing the term $[q(\mu_{s-})]\mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\theta_k)]}$, we have

$$\begin{aligned}
 & \mathbf{E} \left[\left\| \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{[0, K]} [q(\theta_k)] \mathbf{1}_{0 \leq x \leq \text{Re}(\text{Tr}[\mathcal{J}(\theta_k)])} - [q(\mu_{s-})] \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\mu_{s-})]} N(\cdot, dx, ds) \right\| \right] \\
 & \leq \mathbf{E} \left[\int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{[0, K]} \left\| [q(\theta_k)] \mathbf{1}_{0 \leq x \leq \text{Re}(\text{Tr}[\mathcal{J}(\theta_k)])} - [q(\mu_{s-})] \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\theta_k)]} \right\| \right. \\
 & \quad \times N(\cdot, dx, ds) \left. + \mathbf{E} \left[\int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{[0, K]} \| [q(\mu_{s-})] \| \right. \right. \\
 & \quad \times \left. \left| \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\mu_{s-})]} - \mathbf{1}_{0 \leq x \leq \text{Re}(\text{Tr}[\mathcal{J}(\theta_k)])} \right| N(\cdot, dx, ds) \right] \left. \right] \\
 & \leq \mathbf{E} \left[\int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{[0, K]} \left\| q(\tilde{\theta}_{\frac{k}{n}}) - q(\mu_{s-}) \right\| N(\cdot, dx, ds) \right] \\
 & \quad + 2 \mathbf{E} \left[\int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{[0, K]} \left| \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\mu_{s-})]} - \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\theta_k)]} \right| N(\cdot, dx, ds) \right] \\
 & \leq \mathbf{E} \left[\int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{[0, K]} Q \left\| \tilde{\theta}_{\frac{k}{n}} - \mu_{s-} \right\| N(\cdot, dx, ds) \right] \\
 & \quad + 2 \mathbf{E} \left[\int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{[0, K]} \left(\mathbf{1}_{0 \leq x \leq \max(\text{Tr}[\mathcal{J}(\mu_{s-})], \text{Re}(\text{Tr}[\mathcal{J}(\tilde{\theta}_{\frac{k}{n}})])} \right. \right. \\
 & \quad \left. \left. - \mathbf{1}_{0 \leq x \leq \min(\text{Tr}[\mathcal{J}(\mu_{s-})], \text{Re}(\text{Tr}[\mathcal{J}(\tilde{\theta}_{\frac{k}{n}})])} \right) N(\cdot, dx, ds) \right]. \tag{36}
 \end{aligned}$$

Furthermore, we have $\text{Re}(\text{Tr}[\mathcal{J}(\mu_{s-})]) = \text{Tr}[\mathcal{J}(\mu_{s-})]$, for all s . Hence, by linearity and continuity of the trace function, there exists a constant R such that

$$|\text{Re}(\text{Tr}[\mathcal{J}(A)]) - \text{Re}(\text{Tr}[\mathcal{J}(B)])| \leq R \|A - B\|,$$

for all matrices A and B . This implies

$$\begin{aligned} (36) &\leq \mathbf{E} \left[\int_{\frac{k}{n}}^{\frac{k+1}{n}} Q \left\| \tilde{\theta}_{\frac{k}{n}} - \mu_{s-} \right\| ds \right] + \mathbf{E} \left[\int_{\frac{k}{n}}^{\frac{k+1}{n}} \left| \text{Re}(\text{Tr}[\mathcal{J}(\tilde{\theta}_{\frac{k}{n}})]) - \text{Tr}[\mathcal{J}(\mu_{s-})] \right| ds \right] \\ &\leq \int_{\frac{k}{n}}^{\frac{k+1}{n}} (R + Q) \mathbf{E} \left[\left\| \tilde{\theta}_{\frac{k}{n}} - \mu_{s-} \right\| \right] ds \\ &\leq \int_{\frac{k}{n}}^{\frac{k+1}{n}} (R + Q) \mathbf{E} \left[\left\| \tilde{\theta}_{\frac{k}{n}} - \mu_{\frac{k}{n}} \right\| \right] ds + \int_{\frac{k}{n}}^{\frac{k+1}{n}} (R + Q) \mathbf{E} \left[\left\| \tilde{\mu}_{\frac{k}{n}} - \mu_{s-} \right\| \right] ds \\ &\leq (R + Q) \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} Z_s ds + \frac{1}{n^2} \right) \quad (\text{we do the same as the drift term}). \end{aligned} \quad (37)$$

The term between k_u and u can be treated in the same way. By summing, we obtain finally the same type of inequality for the term with the random measure. As a consequence there exist two constants F_1 and F_2 , which depend only on T , such that

$$Z_u \leq F_1 \int_0^u Z_s ds + \frac{F_2}{n}. \quad (38)$$

The Gronwall Lemma implies that there exists a constant Γ such that for all $u < T$

$$Z_u(n) \leq \frac{\Gamma}{n}, \quad (39)$$

where Γ is a constant independent of n . The convergence in $\mathcal{D}([0, T])$ is an easy consequence of the above inequality. The result is then proved.

Now, we can justify why the L_2 norm is not appropriate to treat such equations. In the last term of (36), we have a difference of two indicator functions. Such a difference satisfies

$$\left| \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\mu_{s-})]} - \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\theta_k)]} \right|^2 = \left| \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\mu_{s-})]} - \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\theta_k)]} \right|.$$

In the proof, we need to estimate the integral of this term and then we loose the homogeneity in terms of L_2 norms. As the final result relies on Gronwall Lemma, this homogeneity is actually necessary to obtain an appropriate estimation. \square

In the following section, we compare the discrete process with the Euler scheme.

3.3. Convergence of the discrete process

This section is devoted to the random coupling of the discrete quantum trajectory and the continuous quantum trajectory.

Consider the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where the solution (μ_t) of the jump-Belavkin equation (19) is defined. Let us construct the discrete quantum trajectory in this space. For n being fixed, we realize the random variables $\mathbf{1}_1^{k+1}$ by defining on the set of the states

$$\tilde{v}_{k+1}(\eta, \omega) = \mathbf{1}_{N(\omega, G_k(\eta)) > 0}, \quad (40)$$

where $G_k(\eta) = \left\{ (t, u) \mid \frac{k}{n} \leq t < \frac{k+1}{n}, 0 \leq u \leq -n \ln(\text{Tr}[\mathcal{L}_0(n)(\eta)]) \right\}$.

Let $\rho_0 = \rho$ be any state, we define the process $(\tilde{\rho}_k)$ by the recursive formula

$$\begin{aligned} \tilde{\rho}_{k+1} = & \mathcal{L}_0(\tilde{\rho}_k) + \mathcal{L}_1(\tilde{\rho}_k) + \left[-\frac{\mathcal{L}_0(\tilde{\rho}_k)}{\text{Tr}[\mathcal{L}_0(\tilde{\rho}_k)]} + \frac{\mathcal{L}_1(\tilde{\rho}_k)}{\text{Tr}[\mathcal{L}_1(\tilde{\rho}_k)]} \right] \\ & \times (\tilde{v}_{k+1}(\tilde{\rho}_k, \cdot) - \text{Tr}[\mathcal{L}_1(\tilde{\rho}_k)]), \end{aligned} \quad (41)$$

for $k < [nT]$. We get then the natural result.

Proposition 6. *Let $T > 0$ be fixed. The discrete process $(\tilde{\rho}_k)_{k < [nT]}$ defined by (41) has the same distribution as the discrete quantum trajectory $(\rho_k)_{k < [nT]}$ defined by the quantum repeated measurements principle.*

Remark. On $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, it is important to note that the process $(\tilde{\rho}_k)$ satisfies the same asymptotic behavior as the discrete quantum trajectory.

Before comparing the discrete process (41) and the solution of the jump equation, we need to define an auxiliary process which is related to the approximation of the intensity. On $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, we define the sequence of random variables (\bar{v}_k) , defined on the set of states, by

$$\bar{v}_{k+1}(\eta, \omega) = \mathbf{1}_{N(\omega, H_k(\eta)) > 0}, \quad (42)$$

where $H_k(\eta) = \{(t, u) | \frac{k}{n} \leq t < \frac{k+1}{n}, 0 \leq u \leq \text{Tr}[\mathcal{J}(\eta)]\}$. Hence, we define recursively

$$\begin{aligned} \bar{\rho}_{k+1} = & \mathcal{L}_0(\bar{\rho}_k) + \mathcal{L}_1(\bar{\rho}_k) + \left[-\frac{\mathcal{L}_0(\bar{\rho}_k)}{\text{Tr}[\mathcal{L}_0(\bar{\rho}_k)]} + \frac{\mathcal{L}_1(\bar{\rho}_k)}{\text{Tr}[\mathcal{L}_1(\bar{\rho}_k)]} \right] \\ & \times (\bar{v}_{k+1}(\bar{\rho}_k, \cdot) - \text{Tr}[\mathcal{L}_1(\bar{\rho}_k)]), \end{aligned} \quad (43)$$

for $k < [nT]$. The process $(\bar{\rho}_k)$ satisfies also the same asymptotic behavior. Regarding the process $(\tilde{\rho}_k)_{0 \leq k \leq [nT]}$ and $(\bar{\rho}_k)_{0 \leq k \leq [nT]}$, we have the following proposition.

Proposition 7. *Let $(\tilde{\rho}_k)_{0 \leq k \leq [nT]}$ be the discrete quantum trajectory defined by the formula (41) and let $(\bar{\rho}_k)_{0 \leq k \leq [nT]}$ be the sequence defined by the formula (43). Let us assume that the two sequences are defined by the same initial state ρ .*

For all $k \leq [nT]$, we have

$$A_k(n) = \mathbf{E} \left[\sup_{0 < i \leq k} \|\tilde{\rho}_i(n) - \bar{\rho}_i(n)\| \right] \leq o\left(\frac{1}{n}\right),$$

where the little o is uniform in k .

Proof. Since the process $(\tilde{\rho}_k)$ is also valued in the set of states, in terms of f and q , we have

$$\tilde{\rho}_{k+1} - \tilde{\rho}_k = \frac{1}{n} [f(\tilde{\rho}_k) + o_{\tilde{\rho}_k}(1)] + [q(\tilde{\rho}_k) + o_{\tilde{\rho}_k}(1)] \tilde{v}_{k+1}(\tilde{\rho}_k, \cdot). \quad (44)$$

We can remark that all the rest $o_{\tilde{\rho}_k}(1)$ are uniform in k . Since we have a similar asymptotic form for the process (43), we can compare the two processes

$$\begin{aligned} \tilde{\rho}_i - \bar{\rho}_i = & \sum_{j=0}^{i-1} \left[\frac{1}{n} f(\tilde{\rho}_j) - f(\bar{\rho}_j) + o_{\tilde{\rho}_j}(1) - o_{\bar{\rho}_j}(1) \right] \\ & + \sum_{j=0}^{i-1} \left[(q(\tilde{\rho}_j) + o_{\tilde{\rho}_j}(1)) \tilde{v}_{j+1}(\tilde{\rho}_j, \cdot) - (q(\bar{\rho}_j) + o_{\bar{\rho}_j}(1)) \bar{v}_{j+1}(\bar{\rho}_j, \cdot) \right]. \end{aligned}$$

Hence, we have

$$\begin{aligned}
 \sup_{0 \leq i \leq k} \|\tilde{\rho}_i - \bar{\rho}_i\| &\leq \sum_{j=0}^{k-1} \frac{1}{n} \|f(\tilde{\rho}_j) - f(\bar{\rho}_j) + \circ_{\tilde{\rho}_j}(1)\| \\
 &\quad + \sum_{j=0}^{k-1} \left\| (q(\tilde{\rho}_j) + \circ_{\tilde{\rho}_j}(1))\tilde{v}_{j+1}(\tilde{\rho}_j, \cdot) - q(\bar{\rho}_j)\bar{v}_{j+1}(\bar{\rho}_j, \cdot) \right\| \\
 &\leq \sum_{j=0}^{k-1} \frac{1}{n} F \|\tilde{\rho}_j - \bar{\rho}_j\| + \sum_{j=0}^{k-1} \left\| (q(\tilde{\rho}_j) + \circ_{\tilde{\rho}_j}(1))(\tilde{v}_{j+1}(\tilde{\rho}_j, \cdot) - \bar{v}_{j+1}(\bar{\rho}_j, \cdot)) \right\| \\
 &\quad + \sum_{j=0}^{k-1} \left\| (q(\tilde{\rho}_j) + \circ_{\tilde{\rho}_j}(1)) - q(\bar{\rho}_j) - \circ_{\bar{\rho}_j}(1) \right\| \bar{v}_{j+1}(\bar{\rho}_j, \cdot) \right\|. \tag{45}
 \end{aligned}$$

By defining the filtration $\mathcal{G}_j = \sigma\{\tilde{v}_k(\tilde{\rho}_{k-1}, \cdot), \bar{v}_k(\bar{\rho}_{k-1}, \cdot), 0 < k \leq j\}$ for $j > 0$ and by the independence of the increments of a Poisson process, we have

$$\begin{aligned}
 &\mathbf{E} \left[\left\| (q(\tilde{\rho}^j) + \circ_{\tilde{\rho}_j}(1) - q(\bar{\rho}_j) - \circ_{\bar{\rho}_j}(1))\bar{v}_{j+1}(\bar{\rho}_j, \cdot) \right\| \right] \\
 &= \mathbf{E} \left[\left\| (q(\tilde{\rho}_j) + \circ_{\tilde{\rho}_j}(1) - q(\bar{\rho}_j) - \circ_{\bar{\rho}_j}(1)) \right\| \mathbf{E} [\bar{v}_{j+1}(\bar{\rho}_j, \cdot) / \mathcal{G}_j] \right] \\
 &= \mathbf{E} \left[\left\| q(\tilde{\rho}_j) + \circ_{\tilde{\rho}_j}(1) - q(\bar{\rho}_j) - \circ_{\bar{\rho}_j}(1) \right\| \left(1 - \exp \left(-\frac{1}{n} \text{Tr}[\mathcal{J}(\bar{\rho}_j)] \right) \right) \right] \\
 &\leq \mathbf{E} \left[Q \|\tilde{\rho}_j - \bar{\rho}_j\| \left(\frac{1}{n} \text{Tr}[\mathcal{J}(\bar{\rho}_j)] + \circ \left(\frac{1}{n} \right) \right) + \circ \left(\frac{1}{n} \right) \right], \tag{46}
 \end{aligned}$$

since all the \circ are uniform in j . In the same way, by using the filtration, we get

$$\begin{aligned}
 &\mathbf{E} \left[\left\| (q(\tilde{\rho}_j) + \circ_{\tilde{\rho}_j}(1))(\tilde{v}_{j+1}(\tilde{\rho}_j, \cdot) - \bar{v}_{j+1}(\bar{\rho}_j, \cdot)) \right\| \right] \\
 &= \mathbf{E} \left[\left\| q(\tilde{\rho}_j) + \circ_{\tilde{\rho}_j}(1) \right\| \mathbf{E} [|\tilde{v}_{j+1}(\tilde{\rho}_j, \cdot) - \bar{v}_{j+1}(\bar{\rho}_j, \cdot)| / \mathcal{G}_j] \right].
 \end{aligned}$$

For the second term, we have

$$\begin{aligned}
 &\mathbf{E} \left[|\tilde{v}_{j+1}(\tilde{\rho}_j, \cdot) - \bar{v}_{j+1}(\bar{\rho}_j, \cdot)| / \mathcal{G}_j \right] \\
 &= \mathbf{E} \left[\mathbf{1}_{\{N(\cdot, G_j(\tilde{\rho}_j)) > 0\} \Delta \{N(\cdot, H_j(\bar{\rho}_j)) > 0\}} / \mathcal{G}_j \right] \\
 &= P \left[\{N(\cdot, G_j(\tilde{\rho}_j)) > 0\} \Delta \{N(\cdot, H_j(\bar{\rho}_j)) > 0\} / \mathcal{G}_j \right].
 \end{aligned}$$

Now, we define $W_j = \left\{ (t, u) / \frac{j}{n} \leq t < \frac{j+1}{n}, \min(\text{Tr}[\mathcal{J}(\bar{\rho}_j)], -n \ln(\text{Tr}[\mathcal{L}_0(\rho_j)])) \leq u \leq \max(\text{Tr}[\mathcal{J}(\bar{\rho}_j)], -n \ln(\text{Tr}[\mathcal{L}_0(\rho_j)])) \right\}$. In terms of W_j , we have

$$\begin{aligned}
 &\mathbf{E} [|\tilde{v}_{j+1}(\tilde{\rho}_j, \cdot) - \bar{v}_{j+1}(\bar{\rho}_j, \cdot)| / \mathcal{G}_j] = P[\mathbf{1}_{W_j > 0} / \mathcal{G}_j] \\
 &= 1 - \exp \left(-\frac{1}{n} \left(\min(\text{Tr}[\mathcal{J}(\bar{\rho}_j)], -n \ln(\text{Tr}[\mathcal{L}_0(\rho_j)])) \right. \right. \\
 &\quad \left. \left. - \max(\text{Tr}[\mathcal{J}(\bar{\rho}_j)], -n \ln(\text{Tr}[\mathcal{L}_0(\rho_j)])) \right) \right)
 \end{aligned}$$

$$\begin{aligned}
&= 1 - \exp\left(-\frac{1}{n} \left| \text{Tr}[\mathcal{J}(\bar{\rho}_j)] + n \ln \text{Tr}[\mathcal{L}_0(\tilde{\rho}_j)] \right| \right) \\
&= \frac{1}{n} \left| \text{Tr}[\mathcal{J}(\bar{\rho}_j)] + n \ln (\text{Tr}[\mathcal{L}_0(\tilde{\rho}_j)]) \right| + o\left(\frac{1}{n}\right).
\end{aligned}$$

Now, since $\text{Tr}[\mathcal{L}_0(\tilde{\rho}_j)] = p_{j+1} = 1 - \frac{1}{n} \text{Tr}[\mathcal{J}(\rho_j)] + o(\frac{1}{n})$, we get

$$\mathbf{E} \left[\left| \tilde{v}_{j+1}(\tilde{\rho}_j, \cdot) - \bar{v}_{j+1}(\bar{\rho}_j, \cdot) \right| / \mathcal{G}_j \right] = \frac{1}{n} \left| \text{Tr}[\mathcal{J}(\bar{\rho}_j)] - \text{Tr}[\mathcal{J}(\tilde{\rho}_j)] \right| + o\left(\frac{1}{n}\right). \quad (47)$$

As $(\tilde{\rho}_k)$ is a process valued in the set of states, it is uniformly bounded, we then have

$$\mathbf{E} \left[\left\| (q(\tilde{\rho}_j) + o_{\tilde{\rho}_j}(1))(\tilde{v}_{j+1}(\tilde{\rho}_j, \cdot) - \bar{v}_{j+1}(\bar{\rho}_j, \cdot)) \right\| \right] \leq K \mathbf{E} \left[\left\| \bar{\rho}_j - \tilde{\rho}_j \right\| \right] + o\left(\frac{1}{n}\right). \quad (48)$$

By taking expectation in (45) and using the inequalities (46) and (47), we obtain finally

$$A_k \leq \sum_{j=0}^{k-1} \frac{L}{n} \mathbf{E} \left[\left\| \tilde{\rho}_j - \bar{\rho}_j \right\| \right] + o\left(\frac{1}{n}\right) \leq \sum_{j=0}^{k-1} \frac{L}{n} A_j + o\left(\frac{1}{n}\right). \quad (49)$$

The result follows with a discrete Gronwall Lemma. \square

Now, we can compare the process obtained by the Euler scheme and the process defined by the formula (43). This is the topic of the next proposition.

Proposition 8. *Let $(\bar{\rho}_k)_{0 \leq k \leq [nT]}$ be the process defined by the formula (43) and let $(\theta_k)_{0 \leq k \leq [nT]}$ be the process obtained by the Euler scheme of the jump-Belavkin equation. Let us assume that the two sequences are defined by the same initial state ρ .*

There exists a constant F , independent of n , such that

$$S_k(n) = \mathbf{E} \left[\sup_{0 \leq i \leq k} \|\theta_i(n) - \bar{\rho}_i(n)\| \right] \leq \frac{F}{n},$$

for all $k \leq [nT]$.

The proof is based on the Gronwall Lemma and a finer property of the random measure induced by the Poisson point process. This is a generalization of the Poisson approximation studied by Brown in [15].

Proof. According to the bounded property of the random sequence $(\bar{\rho}_j)$, we have $o_{\bar{\rho}_j} = o(1)$ (in the following we suppress the parameter n to ease the way of writing the different formulas). Next, for $S_k(n)$, we have

$$\begin{aligned}
S_k &\leq \mathbf{E} \left[\sum_{j=0}^{k-1} \frac{1}{n} \left\| [f(\theta_j) - f(\bar{\rho}_j) + o(1)] \right\| \right] \\
&\quad + \mathbf{E} \left[\sum_{j=0}^{k-1} \left\| \int_{\frac{j}{n}}^{\frac{j+1}{n}} \int_{[0,1]} q(\theta_j) \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\theta_j)]} N(\cdot, dx, ds) \right. \right. \\
&\quad \left. \left. - (q(\bar{\rho}_j) + o(1)) \bar{v}_{j+1}(\bar{\rho}_j, \cdot) \right\| \right].
\end{aligned}$$

As f is Lipschitz, we have

$$\mathbf{E} \left[\sum_{j=0}^{k-1} \frac{1}{n} \|f(\theta_j) - f(\bar{\rho}_j) + o(1)\| \right] \leq F \sum_{j=0}^{k-1} \frac{1}{n} S_j + o\left(\frac{1}{n}\right).$$

For the second term, we get

$$\begin{aligned} & \mathbf{E} \left[\sum_{j=0}^{k-1} \left\| \int_{\frac{j}{n}}^{\frac{j+1}{n}} \int_{[0,1]} q(\theta_j) \mathbf{1}_{0 \leq x \leq \text{Re}(\text{Tr}[\mathcal{J}(\theta_j)])} N(\cdot, dx, ds) - (q(\bar{\rho}_j) + o(1)) \bar{v}_{j+1}(\bar{\rho}_j, \cdot) \right\| \right] \\ &= \mathbf{E} \left[\sum_{j=0}^{k-1} \|q(\theta_j) N(\cdot, H_j(\theta_j)) - (q(\bar{\rho}_j) + o(1)) \bar{v}_{j+1}(\bar{\rho}_j, \cdot)\| \right] \\ &\leq \mathbf{E} \left[\sum_{j=0}^{k-1} \|q(\theta_j) \bar{v}_{j+1}(\bar{\rho}_j, \cdot) - (q(\bar{\rho}_j) + o(1)) \bar{v}_{j+1}(\bar{\rho}_j, \cdot)\| \right] \\ &\quad + \mathbf{E} \left[\sum_{j=0}^{k-1} \|q(\theta_j)\| \times |N(\cdot, H_j(\theta_j)) - \bar{v}_{j+1}(\bar{\rho}_j, \cdot)| \right] \\ &\leq \sum_{j=0}^{k-1} \mathbf{E} [(Q\|\theta_j - \bar{\rho}_j\| + o(1)) \times |\bar{v}_{j+1}(\bar{\rho}_j, \cdot)|] \\ &\quad + \sum_{j=0}^{k-1} \mathbf{E} [\|q(\theta_j)\| \times |N(\cdot, H_j(\theta_j)) - \bar{v}_{j+1}(\bar{\rho}_j, \cdot)|]. \end{aligned}$$

Now, we introduce the following discrete filtration $\mathcal{F}_j = \sigma \{\bar{v}_l(\bar{\rho}_{l-1}, \cdot), N(\cdot, H_l(\theta_l)) / l \leq j\}$. This allows to compute the previous terms. It is clear that the random variables $\bar{\rho}_j$ and θ_j are \mathcal{F}_j measurable, we then have

$$\begin{aligned} & \mathbf{E} \left[\sum_{j=0}^{k-1} \|q(\theta_j) N(\cdot, H_j(\theta_j)) - (q(\bar{\rho}_j) + o(1)) \bar{v}_{j+1}(\bar{\rho}_j, \cdot)\| \right] \\ &\leq \sum_{j=0}^{k-1} \mathbf{E} [(Q\|\theta_j - \bar{\rho}_j\| + o(1)) \times \mathbf{E} [|\bar{v}_{j+1}(\bar{\rho}_j, \cdot)| / \mathcal{F}_j]] \\ &\quad + \sum_{j=0}^{k-1} \mathbf{E} [\|q(\theta_j)\| \times \mathbf{E} [|N(\cdot, H_j(\theta_j)) - \bar{v}_{j+1}(\bar{\rho}_j, \cdot)| / \mathcal{F}_j]]. \end{aligned}$$

By conditioning with respect to \mathcal{F}_j , the random variable $\bar{v}_{j+1}(\bar{\rho}_j, \cdot)$ is of Bernoulli type. Hence, we have

$$\mathbf{E} [|\bar{v}_{j+1}(\bar{\rho}_j, \cdot)| / \mathcal{F}_j] = 1 - \exp \left(-\frac{1}{n} \text{Tr}[\mathcal{J}(\bar{\rho}_j)] \right) = \frac{1}{n} \text{Tr}[\mathcal{J}(\bar{\rho}_j)] + o\left(\frac{1}{n}\right).$$

For the second part, we have almost surely

$$\begin{aligned} & \mathbf{E} [|N(\cdot, H_j(\theta_j)) - \bar{v}_{j+1}(\bar{\rho}_j, \cdot)| / \mathcal{F}_j] \\ &\leq \mathbf{E} [|N(\cdot, H_j(\theta_j)) - N(\cdot, H_j(\bar{\rho}_j))| / \mathcal{F}_j] + \mathbf{E} [N(\cdot, H_j(\bar{\rho}_j)) - \bar{v}_{j+1}(\bar{\rho}_j, \cdot) / \mathcal{F}_j] \\ &\leq \frac{1}{n} |\text{Tr}[\mathcal{J}(\bar{\rho}_j)] - \text{Tr}[\mathcal{J}(\theta_j)]| + \mathbf{E} [N(\cdot, H_j(\bar{\rho}_j)) - \bar{v}_{j+1}(\bar{\rho}_j, \cdot) / \mathcal{F}_j] \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{n} |\text{Tr}[\mathcal{J}(\bar{\rho}_j)] - \text{Tr}[\mathcal{J}(\theta_j)]| + \left[\frac{1}{n} \text{Tr}[\mathcal{J}(\bar{\rho}_j)] - \left(1 - \exp\left(-\frac{1}{n} \text{Tr}[\mathcal{J}(\bar{\rho}_j)]\right) \right) \right] \\ &\leq \frac{A}{n} \|\bar{\rho}_j - \theta_j\| + \frac{A'}{n^2} + o\left(\frac{1}{n^2}\right). \end{aligned}$$

The $o\left(\frac{1}{n^2}\right)$ are uniform in j since $(\bar{\rho}_j)$ is uniformly bounded. For the second term, the above inequalities and the fact that the Euler scheme is bounded implies that there exist two constants K_1 and K_2 such that

$$\sum_{j=0}^{k-1} \mathbf{E} [\|q(\theta_j)\| \times \mathbf{E} [N(\cdot, H_j(\theta_j)) - \bar{v}_{j+1}(\bar{\rho}_j, \cdot) | \mathcal{F}_j]] \leq K_1 \sum_{j=0}^{k-1} \frac{1}{n} S_j + \frac{K_2}{n} + o\left(\frac{1}{n}\right).$$

For the first part, we have an equivalent inequality. Thus we can conclude that there exist two constants G_1 and G_2 such that

$$S_k \leq G_1 \sum_{j=0}^{k-1} \frac{1}{n} S_j + \frac{G_2}{n} + o\left(\frac{1}{n}\right). \quad (50)$$

The discrete Gronwall Lemma implies that there exists a constant F independent of n such that for all $k \leq [nT]$

$$S_k \leq \frac{F}{n}.$$

The proposition is then proved. \square

By using this two properties we can now express the final theorem.

Theorem 4. Let $T > 0$ be a fixed time and let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be the probability space of the Poisson point process N . Let n be an integer and let $(\tilde{\rho}_{[nt]})_{0 \leq t \leq T}$ be the discrete quantum trajectory defined for $k < [nT]$ by equation

$$\tilde{\rho}_{k+1} = \mathcal{L}_0(\tilde{\rho}_k) + \mathcal{L}_1(\tilde{\rho}_k) + \left[-\frac{\mathcal{L}_0(\tilde{\rho}_k)}{\text{Tr}[\mathcal{L}_0(\tilde{\rho}_k)]} + \frac{\mathcal{L}_1(\tilde{\rho}_k)}{\text{Tr}[\mathcal{L}_1(\tilde{\rho}_k)]} \right] (\tilde{v}_{k+1}(\tilde{\rho}_k, \cdot) - \text{Tr}[\mathcal{L}_1(\tilde{\rho}_k)]).$$

Let $(\mu_t)_{0 \leq t \leq T}$ be the quantum trajectory solution of the jump-Belavkin equation on $[0, T]$ which satisfies

$$\mu_t = \mu_0 + \int_0^t f(\mu_{s-}) ds + \int_0^t \int_{[0, K]} \left[\frac{\mathcal{J}(\mu_{s-})}{\text{Tr}[\mathcal{J}(\mu_{s-})]} - \mu_{s-} \right] \mathbf{1}_{0 \leq x \leq \text{Tr}[\mathcal{J}(\mu_{s-})]} N(\cdot, dx, ds).$$

If $\mu_0 = \tilde{\rho}_0$, then the discrete quantum trajectory $(\rho_{[nt]})_{0 \leq t \leq T}$ converges in distribution to the continuous quantum trajectory $(\mu_t)_{0 \leq t \leq T}$ in $\mathcal{D}([0, T])$ for all T .

Proof. Let n be large enough. According to Proposition 8 and Theorem 3 concerning the Euler scheme, there exists a constant R , independent of n , such that

$$B_k = \mathbf{E} \left[\sup_{0 \leq i \leq k} \|\tilde{\rho}_i - \tilde{\mu}_i\| \right] \leq \frac{R}{n}, \quad (51)$$

for all $k \leq [nT]$, where we have defined $\tilde{\mu}_k = \mu_{\frac{k}{n}}$.

It is worth noting that the process $(\tilde{\mu}_{[nt]})_{0 \leq t \leq T}$ converges in distribution to $(\mu_t)_{0 \leq t \leq T}$ for all T in $\mathcal{D}([0, T])$. According to this fact and the inequality (51), the convergence in distribution of $(\rho_{[nt]})_{0 \leq t \leq T}$ to $(\mu_t)_{0 \leq t \leq T}$ is proved. \square

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