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Affine processes on positive semidefinite $d \times d$ matrices have jumps of finite variation in dimension $d > 1$

Eberhard Mayerhofer[1]

Deutsche Bundesbank, Wilhelm-Epstein-Straße 14, 60431 Frankfurt am Main, Germany

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Abstract

The theory of affine processes on the space of positive semidefinite $d \times d$ matrices has been established in a joint work with Cuchiero et al. (2011) [4]. We confirm the conjecture stated therein that in dimension $d > 1$ this process class does not exhibit jumps of infinite total variation. This constitutes a geometric phenomenon which is in contrast to the situation on the positive real line (Kawazu and Watanabe, 1971) [8]. As an application we prove that the exponentially affine property of the Laplace transform carries over to the Fourier–Laplace transform if the diffusion coefficient is zero or invertible.

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1. Introduction

Affine processes are a special class of stochastically continuous Markov processes with the following feature: some suitable integral transform (such as the characteristic function [5], Laplace transform [8,4], Fourier–Laplace transform, or even moment generating function [7]) of their transition function is exponentially affine in the state variable. It has become customary to describe affine processes in terms of a parameterization of their infinitesimal generator—quite similarly to the Lévy class case [14], where the so-called Lévy–Khintchine triplet $(a, c, m(d\xi))$[2]

E-mail address: eberhard.mayerhofer@gmail.com.

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[2] For simplicity of notation only the one-dimensional case is recalled here.

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relative to a truncation function $\chi(\xi)$ allows a parametric description of the generator

$$\mathcal{A}f(x) = af''(x) + bf'(x) + \int_{\mathbb{R}\setminus\{0\}} (f(x+\xi) - f(x) - f'(x)\chi(\xi))m(d\xi).$$

The affine property translates into affine drift, diffusive and jump behavior, and the coefficients of the affine functions involved determine the so-called "admissible parameter set" [5]. For instance, for the state space $\mathbb{R}_+ := [0, \infty)$, Kawazu and Watanabe [8] show that the infinitesimal generator of a conservative affine processes X takes the form[3]

$$\begin{aligned}\mathcal{A}f(x) &= \alpha x f''(x) + (b + \beta x)f'(x)\\ &\quad + \int_{\mathbb{R}_+\setminus\{0\}} (f(x+\xi) - f(x) - f'(x)\chi(\xi))(m(d\xi) + x\mu(d\xi)),\end{aligned}$$

with "parameters" ($\alpha \geq 0, b \geq 0, \beta \in \mathbb{R}, m(d\xi), \mu(d\xi)$), where the last two objects are sigma-finite measures on $\mathbb{R}_+ \setminus \{0\}$ such that

$$\int_{\mathbb{R}_+\setminus\{0\}} (\|\xi\| \wedge 1)m(d\xi) < \infty, \qquad \int_{\mathbb{R}_+\setminus\{0\}} (\|\xi\|^2 \wedge 1)\mu(d\xi) < \infty.$$

However, a full parametric characterization depends crucially on the geometry of the state space, and the probabilistic properties of affine processes may vary accordingly. Motivated by multivariate extensions in the affine term structure literature as well as in stochastic volatility, Duffie et al. [5] establish a unified theory on the so-called canonical state spaces $\mathbb{R}_+^m \times \mathbb{R}^n$ (for further insights, and certain simplifications, see [9,11]). The recent theory of Cuchiero et al. [4] for affine processes on positive semidefinite matrices S_d^+ is a response to suggestions in the finance literature concerning affine multi-asset models based on matrix factors. Those, in turn, have mostly used the class of Wishart processes as put forward in [3], or the OU-type processes driven by matrix-variate Lévy subordinators [2]. For a review on financial modeling issues with matrix factors, see the extensive introduction of [4], as well as the references given therein.

The aim of this paper is to show that affine processes on S_d^+, $d \geq 2$, do not exhibit jumps of infinite total variation (Theorem 3.2). This important result confirms a conjecture formulated in [4, Section 2.1.4]. In the conservative case, it allows us to simplify the semimartingale decomposition of [4, Theorem 2.6]; this is subject of Theorem 3.4. A crucial application of Theorem 3.2 concerns the affine character of the Fourier–Laplace transform of affine processes (Theorem 4.1). In particular, we show that in the presence of non-degenerate diffusion components, affine processes are affine in the sense of Duffie et al. (Theorem 4.2 and Corollary 4.3). This means that their characteristic function is exponentially affine in the state variable. A detailed introduction to this topic with technical remarks is given in Section 4.

The main result of this paper, Theorem 4.1, reveals a geometric phenomenon; indeed, in the much simpler case $d = 1$, where the state space simplifies to the positive real line $\mathbb{R}_+$, stochastic processes with jumps of infinite total variation actually exist. For instance, let $d\xi$ denote the Lebesgue measure on $[0, 1]$ and define a linear jump characteristic as

$$\mu(d\xi) := \xi^{-2} 1_{(0,1]}(\xi)d\xi.$$

[3] Note that in the case $\alpha = \beta = 0$, $\mu = 0$, X is a Lévy subordinator.

Clearly $1 = \int_0^1 \xi^2 \mu(d\xi) < \infty$; hence due to [8], an affine pure jump process X with infinitesimal generator

$$\mathcal{A}f(x) := x \int_{\mathbb{R}_+ \setminus \{0\}} \left(f(x+\xi) - f(x) - f'(x)\xi\right) \mu(d\xi)$$

exists. Let us start X at $x > 0$ and denote by X_t its càdlàg representation.[4] Then X_t is a special semimartingale with characteristics $(A = 0, B = 0, \nu(dt, d\xi))$, where the compensator of X_t equals

$$\nu(dt, d\xi) = X_t \mu(d\xi) dt.$$

The canonical decomposition of X_t is given in terms of the Poisson random measure associated with its jumps, $\mu^X(dt, d\xi)$:

$$X_t = x + \xi \star (\mu^X - \nu) = \lim_{\varepsilon \downarrow 0} \int_{\xi > \varepsilon} \int_0^t \xi (\mu^X(ds, d\xi) - \nu(ds, d\xi))$$

and clearly $X_t > 0$ a.s., because the jumps of X are positive throughout. Hence, almost surely it holds that[5]

$$\sum_{s \le t} |\Delta X_s| = \sum_{s \le t} \Delta X_s = \int_{\mathbb{R}_+ \setminus \{0\}} \xi \, \mu(d\xi) \times \left(\int_0^t X_s ds\right) = \infty.$$

For $d \geq 2$ the complex geometry of the boundary ∂S_d^+ of S_d^+ – it is no longer only the origin, nor it is a smooth manifold – leads to non-trivial restrictions concerning the linear jump behavior. One of these is (3.3), which expresses that transversal to ∂S_d^+ only finite variation jumps are allowed. In addition, there is a non-trivial tradeoff between linear drift and linear jumps; see Eq. (3.6). One of the nice consequences of Theorem 3.2 is that these two conditions may be disentangled from each other, into a simple condition that the drift must be inward pointing at the boundary (Eq. (3.4)) and the compensator of the affine processes satisfies a stronger integrability condition (see (3.3)). Furthermore, the admissible parameter set is now formulated independently of truncation functions, which is impossible in the setting of canonical state spaces [5], and in particular for $d = 1$.

It should perhaps be noted that the original characterization of affine processes [4, Theorem 2.4] and all consequences thereof are stated in a way which nests the one-dimensional one (cf. [8,5])—this is possible in view of the implicit nature of condition (3.6). Thus the preceding theory is perfectly valid in its original formulation; the contribution of the present work, however, is a technical simplification of the theory of affine processes on the cone of positive semidefinite matrices S_d^+ of arbitrary dimension $d \geq 2$, and the additional theoretical results concerning the Fourier–Laplace transform of this process class.

2. Notation and definition of the affine property

We try to keep notation and presentation of this paper as simple as possible. As a reference, both for applied and theoretic issues, see the quite extensive work [4]; this also concerns technically involved facts, which are here only recollected in prose.

4 Such a representation exists due to the Feller property of X; see [5].

5 Of course in the finite case, the two summands would differ in general.

1_A equals the indicator function corresponding to some set A. S_d denotes the linear space of real $d \times d$ symmetric matrices, and $\langle x, y\rangle := \mathrm{tr}(xy)$ is the standard scalar product thereon, given by the trace of the matrix product. Accordingly, $\| \|$ is the induced norm on S_d, and the pierced unit ball equals

$$B_1^\circ := \{z \in S_d^+ \mid 0 < \|z\| \le 1\}.$$

The natural order introduced by the closed convex cone S_d^+ is denoted by $\preceq$. The cone of positive definite matrices is denoted by S_d^{++} (and clearly is the interior of S_d^+). $\nabla f(x)$ denotes the Frechet derivative of a function f at $x \in S_d$. This coordinate free notation allows a much shorter and more elegant presentation; for an account of the details involved and the coordinatewise way to write what follows, the reader is referred to the nice demonstration of [1] as well as [4]. Only in the proof of the main theorem, Theorem 3.2, are coordinates used.

We consider a time-homogeneous Markov process X with state space S_d^+ and semigroup $(P_t)_{t\ge 0}$ acting on bounded Borel measurable functions f:

$$P_t f(x) = \int_{S_d^+} f(\xi) p_t(x, d\xi), \quad x \in S_d^+.$$

Here $p_t(x, d\xi)$ denotes the (possibly sub-)Markovian transition function of X.

Definition 2.1. The Markov process X is called *affine* if

(i) it is stochastically continuous, that is, $\lim_{s\to t} p_s(x,\cdot) = p_t(x,\cdot)$ weakly on S_d^+ for every t and $x \in S_d^+$, and
(ii) its Laplace transform has exponentially affine dependence on the initial state:

$$P_t e^{-\langle u,x\rangle} = \int_{S_d^+} e^{-\langle u,\xi\rangle} p_t(x, d\xi) = e^{-\phi(t,u)-\langle \psi(t,u),x\rangle}, \tag{2.1}$$

for all t and $u, x \in S_d^+$, for some functions $\phi : \mathbb{R}_+ \times S_d^+ \to \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \times S_d^+ \to S_d^+$.

3. The main result and proof

The so-called *admissible parameter set* is introduced in the following. Note that unlike in [4, Definition 2.3], truncation functions may be omitted, and the complicated admissibility condition ([4, (2.11)]; see also (3.6) in the proof below) is dropped:

Definition 3.1. Let $d \ge 2$. An *admissible parameter set* $(\alpha, b, B, c, \gamma, m(d\xi), \mu(d\xi))$ consists of

- a linear diffusion coefficient $\alpha \in S_d^+$,
- a constant drift term $b \in S_d^+$ which satisfies

$$b \succeq (d-1)\alpha, \tag{3.1}$$

- a constant killing rate term $c \in \mathbb{R}_+$,
- a linear killing rate coefficient $\gamma \in S_d^+$,
- a constant jump term: a Borel measure m on $S_d^+ \setminus \{0\}$ satisfying

$$\int_{S_d^+\setminus\{0\}} (\|\xi\| \wedge 1)\, m(d\xi) < \infty, \tag{3.2}$$

- a linear jump coefficient μ which is an S_d^+-valued,[6] sigma finite measure on $S_d^+ \setminus \{0\}$ and[7]

$$\int_{S_d^+ \setminus \{0\}} (\|\xi\| \wedge 1)\, \mu(d\xi) < \infty, \tag{3.3}$$

- and, finally, a linear drift B, which is a linear map from S_d to S_d and "inward pointing" at the boundary; that is,

$$\langle B(x), u\rangle \geq 0 \quad \text{for all } x, u \in S_d^+ \text{ with } \langle x, u\rangle = 0. \tag{3.4}$$

The main statement of this paper follows:

Theorem 3.2. *Let X be an affine processes on S_d^+ $(d \geq 2)$. Then its Markovian semigroup $(P_t)_t$ has an infinitesimal generator $\mathcal{A}$ acting on the space of rapidly decreasing functions[8] supported on S_d^+:*

$$\begin{aligned}\mathcal{A} f(x) &= 2\langle \nabla \alpha \nabla f(x), x\rangle + \langle \nabla f(x), b + B(x)\rangle - (c + \langle \gamma, x\rangle) f(x)\\ &\quad + \int_{S_d^+ \setminus \{0\}} (f(x+\xi) - f(x))\, (m(d\xi) + \langle \mu(d\xi), x\rangle)\end{aligned}$$

where $(\alpha, b, B, c, \gamma, m, \mu)$ are an admissible parameter set in the sense of Definition 3.1.

Proof. Let $\chi(\xi) : S_d^+ \to S_d^+$ be a truncation function; that is $\chi(\xi) = \xi$ near zero, and χ is continuous, and (what may be assumed without loss of generality) bounded by 1. By [4, Theorem 2.4 and Definition 2.3] the semigroup $(P_t)_t$ has an infinitesimal generator $\mathcal{A}$ acting as

$$\begin{aligned}\mathcal{A} f(x) &= 2\langle \nabla \alpha \nabla f(x), x\rangle + \langle \nabla f(x), b + \widetilde{B}(x)\rangle - (c + \langle \gamma, x\rangle) f(x)\\ &\quad + \int_{S_d^+ \setminus \{0\}} (f(x+\xi) - f(x))\, m(d\xi)\\ &\quad + \int_{S_d^+ \setminus \{0\}} (f(x+\xi) - f(x) - \langle \chi(\xi), \nabla f(x)\rangle)\, \langle \mu(d\xi), x\rangle\end{aligned}$$

where $c \geq 0$, $\gamma \in S_d^+$ and the parameters $\alpha, b \in S_d^+$ satisfy (3.1) and $\mu(d\xi)$ is a sigma finite S_d^+-valued measure on $S_d^+ \setminus \{0\}$ which integrates $\|\xi\|^2 \wedge 1$. Furthermore, $m(d\xi)$ is a Borel measure on $S_d^+ \setminus \{0\}$ which satisfies (3.2). We also know from [4, Theorem 2.4 and Definition 2.3] that jumps of infinite total variation may only occur parallel to the boundary. In terms of admissibility conditions of the parameters, this is expressed in Eq. (3.2) as well as the following condition:

$$\int_{S_d^+ \setminus \{0\}} \langle \xi, u\rangle \langle \mu(d\xi), x\rangle < \infty, \qquad x, u \in S_d^+ \text{ with } \langle x, u\rangle = 0. \tag{3.5}$$

[6] We deviate here a little from the corresponding definition in [4], where μ is a finite measure on $S_d^+ \setminus \{0\}$ (later divided by $\|\xi\|^2 \wedge 1$). Here "S_d^+-valued" has to be understood as follows: for any Borel set E in S_d^+ such that its S_d^+-topological closure $\bar{E} \subset S_d^+ \setminus \{0\}$ we have $\mu(E) \in S_d^+$. This allows infinite activity jumps with state-dependent intensity. Indeed, there exists μ for which $\mu(S_d^+ \setminus \{0\}) = \infty$, which nevertheless satisfy Eq. (3.3). The latter simply means that $(\|\xi\| \wedge 1)\mu(d\xi)$ is a finite measure.

[7] The integral is of course matrix valued, and $<\infty$ expresses that it is finite.

[8] For further details, see [4, Appendix B].

Furthermore, the drift must be inward pointing at the boundary. That is expressed in the terms of the positivity of the constant drift $b \in S_d^+$ (as above), as well as the following constraint on the linear part $\widetilde{B}$ (a linear map from S_d to S_d):

$$\langle \widetilde{B}(x), u\rangle - \int_{S_d^+\setminus\{0\}} \langle \chi(\xi), u\rangle \, \langle \mu(d\xi), x\rangle \geq 0, \quad x, u \in S_d^+ \text{ with } \langle x, u\rangle = 0. \tag{3.6}$$

Note that $\langle x, u\rangle = 0$ is equivalent to $xu = ux = 0$, that is $x, u \in \partial S_d^+$; see also [4, Lemma 4.1(i)–(iii)].

Suppose for a moment that the validity of (3.3) has already been shown. Then μ integrates $\|\chi(\xi)\| \leq \|\xi\| \wedge 1$ and therefore a new drift may be introduced as

$$B(x) := \widetilde{B}(x) - \int_{S_d^+\setminus\{0\}} \chi(\xi)\langle \mu(d\xi), x\rangle \tag{3.7}$$

which, in view of (3.6), satisfies admissibility condition (3.4). Hence the proof of the theorem would be settled. So the essential point of the statement is (3.3). We use standard Euclidean coordinates on S_d for the remainder of the proof; all indices range between 1 and d, if not otherwise stated. Let $\{c^{ij}, i \leq j\}$ denote the canonical basis of S_d, that is, the klth coefficient of c^{ij} equals

$$c^{ij}_{kl} = \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}(1 - \delta_{ij}),$$

where δ_{ij} denotes the Kronecker delta. If $i = j$ then $c^{ii}_{kl} = \delta_{ik}\delta_{il}$ (the diagonal matrix with a 1 in the ith diagonal entry and zeros everywhere else). Otherwise c^{ij} is zero except in the ijth and jith entry, where it is equal to 1. We may evaluate μ coordinatewise as $\mu(A) = (\mu_{ij}(A))_{ij}$, $A \in S_d$ such that $0 \notin \bar{A}$, the latter denoting the topological closure of the set A. Let $c^*_i := 1 - c^{ii}$, where I is the unit matrix. Then clearly $\langle c^{ii}, c^*_i\rangle = 0$, and by Eq. (3.5) it holds that

$$\int_{B_1^\circ} \xi_{ii}\mu_{jj}(d\xi) < \infty, \quad 1 \leq i, j \leq d, \ i \neq j. \tag{3.8}$$

We show now that a similar integrability condition must also hold for $1 \leq i = j \leq d$. To circumvent integrability issues at the origin, the measure μ is pierced as follows near 0: For $\epsilon > 0$ we introduce the new (and by construction finite measures) $\mu^\varepsilon(d\xi) := \mu(d\xi)1_{\varepsilon<\|\xi\|\leq 1}(d\xi)$. In particular,

$$\mu^\varepsilon_{ij}(d\xi) = \mu_{ij}1_{\varepsilon<\|\xi\|\leq 1}(d\xi)$$

are all signed finite measures ($1 \leq i, j \leq d$) such that for all $\varepsilon > 0$ we have

$$-\infty < \int_{S_d^+} \xi_{kl}\mu^\varepsilon_{ij}(d\xi) < \infty, \quad 1 \leq i, j, k, l \leq d. \tag{3.9}$$

By (3.8), there exists a positive constant M such that for all $\varepsilon > 0$

$$0 \leq \int_{S_d^+} \xi_{ii}\mu^\varepsilon_{jj}(d\xi) < M, \quad i \neq j. \tag{3.10}$$

We introduce now the following boundary points of S_d^+:

$$e^{ij}_{\pm} := c^{ii} \pm c^{ij} + c^{jj}, \quad i \neq j.$$

By construction $\langle e_+^{ij}, e_-^{ij}\rangle = 0$. Setting $x = e_+^{ij}$ and $u = e_-^{ij}$ and applying (3.5), we must have

$$0 \le \int_{B_1^\circ} (\xi_{ii} - 2\xi_{ij} + \xi_{jj})(\mu_{ii}(d\xi) + 2\mu_{ij}(d\xi) + \mu_{jj}(d\xi)) < \infty, \quad i \ne j.$$

Similarly, we obtain by using $x = e_-^{ij}$ and $u = e_+^{ij}$ that

$$0 \le \int_{B_1^\circ} (\xi_{ii} + 2\xi_{ij} + \xi_{jj})(\mu_{ii}(d\xi) - 2\mu_{ij}(d\xi) + \mu_{jj}(d\xi)) < \infty, \quad i \ne j.$$

Accordingly, there exists a constant positive M_1 for which we have, for all $\varepsilon > 0$,

$$0 \le \int_{B_1^\circ} (\xi_{ii} - 2\xi_{ij} + \xi_{jj})(\mu_{ii}^\varepsilon(d\xi) + 2\mu_{ij}^\varepsilon(d\xi) + \mu_{jj}^\varepsilon(d\xi)) < M_1, \quad i \ne j \tag{3.11}$$

and

$$0 \le \int_{B_1^\circ} (\xi_{ii} + 2\xi_{ij} + \xi_{jj})(\mu_{ii}^\varepsilon(d\xi) - 2\mu_{ij}^\varepsilon(d\xi) + \mu_{jj}^\varepsilon(d\xi)) < M_1, \quad i \ne j. \tag{3.12}$$

Summing (3.11) and (3.12) we therefore obtain

$$\begin{aligned} 0 \le &\int_{B_1^\circ} \left(\xi_{ii}\mu_{ii}^\varepsilon(d\xi) - 2\xi_{ij}\mu_{ij}^\varepsilon(d\xi) + \xi_{jj}\mu_{jj}^\varepsilon(d\xi)\right) \\ &+ \int_{B_{\le 1}^\circ} \left(\xi_{ii}\mu_{jj}^\varepsilon(d\xi) - 2\xi_{ij}\mu_{ij}^\varepsilon(d\xi) + \xi_{jj}\mu_{ii}^\varepsilon(d\xi)\right) < 2M_1, \end{aligned}$$

for all $i \ne j$. The two integrals are non-negative, because μ^ε is an S_d^+-valued measure. We therefore conclude that both of them are finite:

$$0 \le \int_{B_1^\circ} \left(\xi_{ii}\mu_{ii}^\varepsilon(d\xi) - 2\xi_{ij}\mu_{ij}^\varepsilon(d\xi) + \xi_{jj}\mu_{jj}^\varepsilon(d\xi)\right) < 2M_1, \quad i \ne j \tag{3.13}$$

$$0 \le \int_{B_{\le 1}^\circ} \left(\xi_{ii}\mu_{jj}^\varepsilon(d\xi) - 2\xi_{ij}\mu_{ij}^\varepsilon(d\xi) + \xi_{jj}\mu_{ii}^\varepsilon(d\xi)\right) < 2M_1, \quad i \ne j. \tag{3.14}$$

By subtracting (3.10) from (3.14) twice, once for i, j and then for j, i, we have

$$-M_1 < \int_{B_1^\circ} \xi_{ij}\mu_{ij}^\varepsilon(d\xi) < M, \quad i \ne j$$

for all $\varepsilon > 0$. Plugging this information back into (3.13) and using the fact that $\xi_{ii} \ge 0$, and μ^ε is positive semidefinite, we obtain

$$0 \le \int_{B_1^\circ} \left(\xi_{ii}\mu_{ii}^\varepsilon(d\xi) + \xi_{jj}\mu_{jj}^\varepsilon(d\xi)\right) < 2(M_1 + M).$$

The choice of i was arbitrary. Taking into account (3.8) and the preceding uniform estimate in ε, we finally conclude that

$$0 \le \int_{B_1^\circ} \xi_{ii}\mu_{jj}(d\xi) < \infty, \quad 1 \le i, j \le d. \tag{3.15}$$

Define the positive measure $\mathrm{tr}(\mu)(d\xi)$ on Borel sets A with $0 \notin \bar{A}$ by $\mathrm{tr}(\mu(A))$. Eq. (3.15) implies immediately that

$$\int_{B_1^\circ} \mathrm{tr}(\xi)\mathrm{tr}(\mu)(d\xi) < \infty. \tag{3.16}$$

We finally show the admissibility condition (3.3): Let ξ be a positive semidefinite matrix with diagonalization $\xi = UDU^\top$, where U is orthogonal and $D = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$. By using this diagonalization and the cyclic property of the trace, we obtain

$$\begin{aligned} \|\xi\|^2 &= \mathrm{tr}(UD^2U) = \mathrm{tr}(D^2) = \sum_{i=1}^d \lambda_i^2 \le \left(\sum_{i=1}^d \lambda_i\right)^2 \\ &= (\mathrm{tr}\, D)^2 = (\mathrm{tr}\, UDU)^2 = \mathrm{tr}(\xi)^2, \end{aligned} \tag{3.17}$$

where $\le$ follows from the non-negativity of the eigenvalues λ_i. Using this technical detail, we infer from (3.16) the following estimate:

$$\int_{S_d^+ \setminus \{0\}} (\|\xi\| \wedge 1)\mathrm{tr}(\mu)(d\xi) < \infty. \tag{3.18}$$

By Lemma 3.3 we may conclude the validity of condition (3.3). Hence the definition of B by Eq. (3.7) is legitimate. $\square$

The following technical statement has just been used and will be used again in the proof of Theorem 4.1:

Lemma 3.3. *For any non-negative Borel-measurable function g we have*

$$\left\| \int_{S_d^+ \setminus \{0\}} g(\xi)\mu(d\xi) \right\| \le \int_{S_d^+ \setminus \{0\}} g(\xi)\mathrm{tr}(\mu)(d\xi) \le \infty.$$

Proof. Since μ is a positive semidefinite measure, we have by Eq. (3.17) for any Borel set $B \subset S_d^+ \setminus \{0\}$ the estimate $\|\mu(B)\| \le \mathrm{tr}(\mu(B))$. Hence approximating the function g by non-negative simple functions, the assertion follows. In particular, the integral (3.3) must be finite whenever (3.18) is finite. $\square$

3.1. The semimartingale decomposition

Suppose X is a conservative[9] affine process on S_d^+, $d \ge 2$, with admissible parameter set $(\alpha, b, B, 0, 0, m(d\xi), \mu(d\xi))$. In view of the Feller property [4, Theorem 2.4] of X, for each initial state $x \in S_d^+$, there exists a modification of $X_t := (X^x)_{t \ge 0}$ on the canonical path space, which is a càdlàg semimartingale. Since we know that the jumps of X are of total finite variation, we have as an immediate consequence of [4, Theorem 2.6]:

Theorem 3.4. *Let Σ be a $d \times d$ matrix such that $\Sigma^\top \Sigma = \alpha$. Then there exists, possibly on an enlargement of the probability space, a $d \times d$-matrix W of standard Brownian motions such that*

[9] For sufficient and necessary conditions for conservativeness, see [4, Remark 2.5] and [12, Section 3].

X admits the following representation:

$$X_t = x + bt + \int_0^t B(X_s)ds + \int_0^t \left(\sqrt{X_s}dW_s\Sigma + \Sigma^\top dW_s\sqrt{X_s}\right)$$
$$+ \int_0^t \int_{S_d^+} \xi\, \mu^X(ds, d\xi),$$

where $\mu^X(d\xi, ds)$ is the random measure associated with the jumps of X, having compensator

$$\nu(dt, d\xi) = (m(d\xi) + \langle X_t, \mu(d\xi)\rangle)dt.$$

Note that if the drift is of the particular form $B(x) = \beta x + x\beta^\top$, where β is a real $d \times d$ matrix, if $b = \delta\alpha$ ($\delta \geq 0$) and in the absence of jumps, X is a Wishart process [3,1,13].

4. The Fourier–Laplace transform of affine processes

Affine processes on positive semidefinite matrices are defined in terms of the Laplace transform of their transition probabilities, Eq. (2.1). In general, the Laplace transform is a natural choice for the integral transform for generalized functions on proper cones such as S_d^+. However, Duffie et al. [5] have defined affine processes on $\mathbb{R}_+^m \times \mathbb{R}^n$ in terms of the exponentially affine form of their characteristic function. Only in the one-dimensional case $\mathbb{R}_+$ do the two state spaces coincide and therefore also the two definitions of the affine property, either via the Laplace transform [8] or via the characteristic function.

Therefore, the question of whether the characteristic function of a positive semidefinite affine process is indeed exponentially affine in the state is of considerable interest. We will denote this property as being "affine in the sense of Duffie et al.".

Unless the diffusion coefficient α vanishes, X need not be infinitely divisible, or equivalently, infinitely decomposable (for the definition and characterization of these properties in the affine Markov setting, see [4, Definition 2.7, Example 2.8 and Theorem 2.9]). This complicates the problem of extending the affine formula Eq. (2.1) to the complex domain, because it is no longer guaranteed that the Fourier–Laplace transform of X exhibits no zeros, as in the infinite divisibility case [14, Theorem 25.17] (which is a necessary condition for writing it in an exponentially affine way). From the ODE perspective, there is a related technical problem, namely that of showing that the real part of $\psi(t, u)$ as a solution of the system of generalized Riccati equations (Eqs. (4.1)–(4.2)) with imaginary initial data does not explode in finite time. Indeed, we have the estimate

$$|e^{-\phi(t,u)-\langle\psi(t,u),x\rangle}| \leq e^{-\mathrm{Re}(\phi(t,u))-\langle\mathrm{Re}(\psi(t,u)),x\rangle}$$

and if the real part of ψ explodes in finite time, then the characteristic function must have a zero.[10] In this section we extend the affine transform formula to the full Fourier–Laplace transform, under the premise that the diffusion component must be non-degenerate or equal to zero. For technical difficulties in the degenerate case, see Remark 4.4.

We denote by $\mathcal{S}(S_d^+)$ the complex tube $S_d^+ + iS_d$, and similarly $\mathcal{S}(S_d^{++}) = S_d^{++} + iS_d$ and $\mathcal{S}(\mathbb{R}_+) = \mathbb{R}_+ + i\mathbb{R}$.

[10] We note that the ϕ coefficient does not matter here: $\mathrm{Re}(\phi(t, u)) \geq 0$ can be inferred from the specific form of the generalized Riccati differential equations.

Theorem 4.1. *Let X be an affine process on S_d^+ ($d \geq 2$), with a diffusion coefficient α which is either invertible or zero. Then the affine property (2.1) holds for all $t \geq 0$, $x \in S_d^+$, and for all $u \in \mathcal{S}(S_d^{++})$, with exponents $\phi : \mathbb{R}_+ \times \mathcal{S}(S_d^{++}) \to \mathcal{S}(\mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \times \mathcal{S}(S_d^{++}) \to \mathcal{S}(S_d^{++})$ which are the unique global solutions of the generalized Riccati differential equations*

$$\partial_t \phi(t,u) = \langle b, \psi(t,u)\rangle + c - \int_{S_d^+\setminus\{0\}} \left(e^{-\langle \psi(t,u),\xi\rangle} - 1\right) m(d\xi), \tag{4.1}$$

$$\begin{aligned}\partial_t \psi(t,u) &= -2\psi(t,u)\alpha\psi(t,u) + B^\top(\psi(t,u)) + \gamma \\ &\quad - \int_{S_d^+\setminus\{0\}} \left(e^{-\langle \psi(t,u),\xi\rangle} - 1\right) \mu(d\xi)\end{aligned} \tag{4.2}$$

given initial data $\phi(0,u) = 0$, $\psi(0,u) = u$.

For the following two results we assume, as in Theorem 4.1, that $d \geq 2$, and the diffusion coefficient α of X is either invertible or zero.

Theorem 4.2. *The affine property (2.1) also holds for $u \in \mathcal{S}(S_d^+)$, with exponents $\phi : \mathbb{R}_+ \times \mathcal{S}(S_d^+) \to \mathcal{S}(\mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \times \mathcal{S}(S_d^+) \to \mathcal{S}(S_d^+)$, being (not necessarily unique) solutions of the generalized Riccati differential equations (4.1)–(4.2).*

Applying the above to $u \in iS_d$, we finally obtain:

Corollary 4.3. *X is affine in the sense of Duffie et al.* [5].

4.1. Proof of Theorem 4.1

Proof. Let $u = v + iw \in \mathcal{S}(S_d^{++})$. We denote by $\psi(t,v)$ the unique global solution of Eq. (4.2) on S_d^{++}, which exists due to [4, Proposition 5.3]. $\psi(t,u)$ is defined as maximal solution of (4.2) on the open domain $\mathcal{S}(S_d^{++})$. Note that the right side $R(\psi)$ of Eq. (4.2) is an analytic function thereon, and hence it is in particular locally Lipschitz. Accordingly, the maximal lifetime of $\psi(t,u)$ equals

$$t_+(u) := \lim_{n\to\infty} \inf\{t > 0 \mid \operatorname{Re} \psi(t,u) \in \partial S_d^+ \text{ or } \|\psi(t,u)\| \geq n\}$$

and we have $0 < t_+(u) \leq \infty$.

First, we show that $\psi(t,u)$ does not touch the boundary of $\mathcal{S}(S_d^{++})$ in finite time. To this end, we introduce the function $\chi(t,u) := \operatorname{Re}(\psi(t,u+iv))$, which is well defined for $t \in [0, t_+(u))$ and has values in S_d^{++}. Denote by $u \mapsto R(u)$ the function on the right side of (4.2). By straightforward inspection, one observes that for all $t < t_+(u)$

$$\partial_t \chi(t,u) - \operatorname{Re}(R(\chi(t,u))) \succeq 0 = \partial_t \psi(t,v) - R(\psi(t,v)).$$

Since R is an analytic and quasi-monotone increasing function on S_d^{++} with respect to the cone S_d^+ (see [4, Definition 4.7 and Lemma 5.1]), we may invoke Volkmann's comparison result in the fashion of [4, Theorem 4.8] and derive

$$\chi(t,u) \succeq \psi(t,v), \quad \text{for all } t < t_+(u).$$

But $\psi(t,v) \in S_d^{++}$, for all $t \geq 0$ [4, Proposition 5.3]. Hence we have shown that $\psi(t,u)$ does not touch the boundary of $\mathcal{S}(S_d^{++})$, which is $\partial S_d^+ \times iS_d$, in finite time, and therefore we have

$$t_+(u) := \lim_{n\to\infty} \inf\{t > 0 \mid \|\psi(t,u)\| \geq n\}. \tag{4.3}$$

Hence it remains to show that $\psi(t,u)$ does not explode in finite time. Since affine transformations of the state space do not effect the blow-up property, we may without loss of generality assume that the diffusion coefficient equals zero or equals the identity matrix. To obtain the necessary transformation, one can adapt [4, Propositions 4.13 and 4.14]. We introduce the shorthand notation $K := K_1 + K_2$, where

$$K_1(u) := \int_{0<\|\xi\|\le 1} \left(\int_0^1 \langle u, \xi \rangle e^{-s\langle u,\xi\rangle} ds \right) \mu(d\xi)$$

and

$$K_2(u) := \int_{\|\xi\|>1} \left(1 - e^{-\langle u,\xi\rangle}\right) \mu(d\xi).$$

Using this decomposition, we can write

$$R(u) = -2u\alpha u + B^\top u + \gamma + K(u).$$

Using the Cauchy–Schwarz inequality, Lemma 3.3 and condition (3.3), we infer the existence of a constant $C_1 \ge 0$ such that for all $u \in \mathcal{S}(S_d^+)$,

$$\|K_1(u)\| \le \|u\| \int_{S_d^+\setminus\{0\}} (\|\xi\| \wedge 1) \mathrm{tr}(\mu)(d\xi) = C_1 \|u\|. \tag{4.4}$$

The same condition allows to conclude the existence of a positive constant C_2 such that

$$\|K_2(u)\| \le \int_{\|\xi\|>1} 2\mathrm{tr}(\mu)(d\xi) = C_2 < \infty \tag{4.5}$$

where we have once again used Lemma 3.3.

By Lemma B.1 in Appendix B we have that

$$\mathrm{Re}\langle \overline{\psi}(t,u), \psi(t,u)\alpha\psi(t,u)\rangle \ge 0 \tag{4.6}$$

for all $t < t_+(u)$. Using estimates (4.4)–(4.6) and the Cauchy–Schwarz inequality, the existence of a positive constant C follows, such that for all $u \in \mathcal{S}(S_d^{++})$ and $t < t_+(u)$,

$$\begin{aligned} \partial_t \|\psi(t,u)\|^2 &= 2\mathrm{Re}\langle \overline{\psi}(t,u), R(\psi(t,u))\rangle \\ &\le 2\mathrm{Re}\langle \overline{\psi}(t,u), B^\top(\psi(t,u)) + \gamma + K(\psi(t,u))\rangle \\ &\le 2C(1 + \|\psi(t,u)\|^2). \end{aligned}$$

Hence, by Gronwall's Lemma (or, equivalently, by standard comparison for scalar-valued ODEs) we obtain for all $t < t_+(u)$,

$$\|\psi(t,u)\| \le e^{Ct}\sqrt{1 + \|u\|^2} \tag{4.7}$$

which in view of (4.3) proves that $t_+(u) = \infty$. So we have shown that $t \mapsto \psi(t,u)$ is the global solution of (4.2) for all $u \in \mathcal{S}(S_d^+)$. Moreover, $\mathrm{Re}(\psi(t,u)) \in S_d^{++}$ for all $t \ge 0$ and the right side of (4.1) is well defined for all $u \in \mathcal{S}(S_d^+)$. Therefore plugging $\psi(t,u)$ into (4.1) and integrating with respect to time yields $\phi(t,u)$.

Now for each $t > 0$, $x \in S_d^+$, the Fourier–Laplace transform

$$g(u) = \mathbb{E}[e^{-\langle u, X_t\rangle} \mid X_0 = x]$$

and the function

$$f(u) := e^{-\phi(t,u)-\langle x,\psi(t,u)\rangle}$$

are complex analytic functions on $\mathcal{S}(S_d^{++})$, and (in view of (2.1)) they coincide on the set of uniqueness, namely S_d^{++}. Hence $f \equiv g$ on $\mathcal{S}(S_d^{++})$, which proves the assertion. □

4.2. *Proof of Theorem 4.2*

We can write $u = v + iw$, where $v \in S_d^+$ and denote for each $n \geq 1$ the matrix $u_n := (v + \frac{1}{n}1) + iw$, where 1 is the unit $d \times d$ matrix. We further denote by $\psi_n(t)$ the solution of (4.2), subject to $\psi_n(0) = u_n$, which exists globally due to Theorem 4.1 because now $u_n \in \mathcal{S}(S_d^{++})$.

Let $\pi(x)$ be the projection of $x \in S_d$ onto S_d^+, which exists uniquely, because S_d^+ is a closed convex set. For $u = v + iw \in \mathcal{S}(S_d)$, we slightly abuse notation and write

$$\pi(u) := \pi(v) + iw \in \mathcal{S}(S_d^+).$$

Using the continuity of the right sides of (4.1)–(4.2) we may also consider $\phi_n(t)$ and $\psi_n(t)$ as solutions to the generalized Riccati differential equations

$$\partial_t \phi_n(t,u) = \langle b, \psi_n(t,u)\rangle + c - \int_{S_d^+\setminus\{0\}} \left(e^{-\langle \pi(\psi_n(t,u)),\xi\rangle} - 1\right) m(d\xi) \tag{4.8}$$

$$\begin{aligned}\partial_t \psi_n(t,u) \;&=\; -2\psi_n(t,u)\alpha\psi_n(t,u) + B^\top(\psi_n(t,u)) + \gamma \\ &\quad - \int_{S_d^+\setminus\{0\}} \left(e^{-\langle \pi(\psi_n(t,u)),\xi\rangle} - 1\right) \mu(d\xi)\end{aligned} \tag{4.9}$$

subject to $\psi_n(0) = u_n$, $\phi_n(0) = 0$ on the whole domain $\mathcal{S}(S_d)$.

Now by estimating (4.7) in the proof of Theorem 4.1, there exists a uniform constant C such that for each n,

$$\|\psi_n(t)\| \leq e^{Ct}\sqrt{1 + \|u_n\|^2}. \tag{4.10}$$

But this means that for any $T > 0$, the family of curves

$$\{\psi_n(t) \mid t \in [0,T]\}$$

lie in a single compact set K. Since T is arbitrary, an application of Lemma A.2 therefore yields that there exist functions $t \mapsto \phi(t,u)$, $t \mapsto \psi(t,u)$ on $[0,\infty)$ which are the pointwise limits of a subsequence $(\phi_{n_k}(t), \psi_{n_k}(t))$ $(k \to \infty)$ and they satisfy Eqs. (4.1)–(4.2). Furthermore, we have by dominated convergence,

$$\begin{aligned}e^{-\phi(t,u)-\langle\psi(t,u),x\rangle} &= \lim_{k\to\infty} e^{-\phi_{n_k}(t)-\langle\psi_{n_k}(t),x\rangle} = \lim_{k\to\infty} \mathbb{E}[e^{-\langle u_{n_k},X_t\rangle} \mid X_0 = x] \\ &= \mathbb{E}[e^{-\langle u,X_t\rangle} \mid X_0 = x].\end{aligned}$$

This ends the proof.

Remark 4.4. • It can easily be seen either by numerical experiments or explicit calculations that (an appropriate adaptation of) Lemma B.1 does not hold if α is not equal to a scalar multiple of the unit matrix. To be more precise, in general, the real part of

$$\mathrm{tr}(\bar{x}x\alpha x) = \mathrm{tr}(x\bar{x}x\alpha)$$

can be strictly negative. For instance, using

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad x = \begin{pmatrix} 1 & i \\ i & 4 \end{pmatrix}$$

we obtain $\mathrm{Re}(x) \in S_2^+$, but $\mathrm{Re}\ \mathrm{tr}(\bar{x}x\alpha x) = -1 < 0$. As a consequence, we cannot derive estimate (4.6), which in turn is a technical necessity for obtaining the a priori estimate (4.7) (resp., (4.10)). However, we conjecture that the problem concerning the degenerate, nonzero diffusion coefficient α admits the same answer as that of Theorem 4.2.
- It should be noted that the main technical complication concerning jumps in prior research had been the presence of a truncation function in the right side of Eq. (4.2). Only the finding of Theorem 3.2 allowed us to establish the general a priori estimate (4.7) (resp., (4.10)).

4.3. *Examples with degenerate, nonzero diffusion*

In the presence of a nonzero diffusion component α, Theorem 4.1 requires that α is invertible. It should, however, be reported that if X is "Wishart with state independent jump behavior", then not only is it evident that X is affine in the sense of [5], but also the affine character of the Laplace transform can be extended to the domain $\mathcal{S}(S_d^+)$. And in this case, we can solve the Riccati equations explicitly, with no non-degeneracy assumption on α.

Definition 4.5. A matrix-variate basic affine jump-diffusion X on S_d^+ (MBAJD for short) is an affine process with parameters $\gamma = 0,\ c = 0,\ \mu \equiv 0$, a constant drift

$$b = 2p\alpha, \quad p \geq \frac{d-1}{2},$$

and a linear drift B of the particular form

$$B(x) = \beta x + x\beta^\top,$$

where β is a real $d \times d$ matrix.

Remark 4.6.
- If $d = 1$, and $m(d\xi)$ is a multiple of the density of an exponential distribution, then X is a BAJD as introduced by Duffie and Garleanu in [6].
- If $d \geq 2$ and $m \equiv 0$, then X is a Wishart process; see [1,3,13].

It is quite straightforward to check that any MBAJD is a conservative Markov process [4, Remark 2.5] and that Eqs. (4.1)–(4.2) take the particular form

$$\partial_t \phi(t,u) = 2p\langle \alpha, \psi(t,u)\rangle - \int_{S_d^+\setminus\{0\}} \left(e^{-\langle \psi(t,u),\xi\rangle} - 1\right) m(d\xi),$$

$$\partial_t \psi(t,u) = -2\psi(t,u)\alpha\psi(t,u) + \psi(t,u)\beta + \beta^\top \psi(t,u).$$

In the following we denote by ω_t^β the flow of the vector field $\beta x + x\beta^\top$, that is,

$$\omega^\beta : \mathbb{R} \times S_d^+ \to S_d^+, \qquad \omega_t^\beta(x) := e^{\beta t} x e^{\beta^\top t}.$$

Its twofold integral $\sigma_t^\beta : S_d^+ \to S_d^+$ for $t \geq 0$ is denoted by

$$\sigma^\beta : \mathbb{R}_+ \times S_d^+ \to S_d^+, \qquad \sigma_t^\beta(x) = 2\int_0^t \omega_s^\beta(x) ds.$$

By matrix analysis, we obtain the following semi-explicit solutions for initial data $u \in \mathcal{S}(S_d^+)$:

$$\phi(t,u) = p \log\det\left(I + u\sigma_t^\beta(\alpha)\right) - \int_{S_d^+\setminus\{0\}} \left(e^{-\langle \psi(t,u),\xi\rangle} - 1\right) m(d\xi),$$

$$\psi(t,u) = e^{\beta^\top t}\left(u^{-1} + \sigma_t^\beta(\alpha)\right)^{-1} e^{\beta t}.$$

Appendix A. Convergence of ordinary differential equations

The following results are consequences of standard ODE theory. The first one is clearly elaborated in [10, Lemma 8], and the second one is a variant of [10, Lemma 9] and can be proved like in [10] (the difference being that we drop the Lipschitz continuity of f, and hence one cannot show that every subsequence involved in A.2 converges, let alone to the same limit).

We recall them here for the convenience of the reader, and without any proof. We consider a system of ordinary differential equations on $\mathbb{R}^m$:

$$\partial_t \psi(t) = f(t, \psi(t)), \tag{A.1}$$

subject to an initial condition $\psi(0) = u \in \mathbb{R}^m$. Recall that Eq. (A.1) possesses a maximal solution on a half-open interval $[0, t_+(u))$ if the function $f : I \times U \to E$ is continuous. Note however that such a solution may be not unique if f is not locally Lipschitz continuous.

Lemma A.1. *Let $U \subset E$ be open. Let $f, f_1, f_2, \ldots$ be continuous maps from $I \times U$ to E. Suppose f is locally Lipschitz and the f_n converge to f uniformly on all compact subsets of $I \times U$. Let the $\psi_n \in C^1([0,\theta_n), U)$ be maximal solutions of*

$$\partial_t \psi_n(t) = f_n(t, \psi_n(t)) \tag{A.2}$$

such that the $\psi_n(0)$ converge to some $u \in U$ as $n \to \infty$. Then we have

$$t_+(u) \le \underline{\lim}\, \theta_n. \tag{A.3}$$

Let $0 \le a < t_+(u)$ and n_0 be such that $\theta_n > a$ for $n > n_0$. Then the sequence $\psi_{n_0+k}(t)$, $k = 1, 2, \ldots$, converges to $\psi(t)$ uniformly on $[0, a]$ as $k \to \infty$.

Lemma A.2. *Let $U \subset \mathbb{R}^m$ be open. Let $f, f_1, f_2, \ldots$ be continuous maps from $I \times U$ to $\mathbb{R}^m$. Suppose the f_n converge to f uniformly on compact subsets of $I \times U$. Let $0 < a < T$ and $\psi_n \in C^1([0,a], U)$ be solutions of (A.2) such that the $\psi_n(0)$ converge to some $u \in U$ as $n \to \infty$. If for some compact set $K \subset U$, $\psi_n(t) \in K$ for all $t \in [0,a]$, then there exists a (not necessarily unique) solution $\psi(t)$ of Eq. (A.1) on $[0,a]$, and a subsequence $\psi_{n_k}(t) \to \psi(t)$ for which $\partial_t \psi_{n_k}(t) \to \partial_t \psi(t)$ uniformly on $[0,a]$.*

Appendix B. A simple matrix inequality

Lemma B.1. *For any complex-valued $m \times n$ matrix a and for any $b \in \mathcal{S}(S_n^+)$ we have that*

$$\operatorname{Re}\operatorname{tr}(b\bar{a}^\top a) \ge 0.$$

Proof. Write $a = a_1 + ia_2$, and $b = b_1 + ib_2$. Then we have

$$\begin{aligned} \operatorname{Re}\operatorname{tr}(b\bar{a}^\top a) &= \operatorname{Re}\operatorname{tr}((b_1 + ib_2)(a_1^\top - ia_2^\top)(a_1 + ia_2)) \\ &= \operatorname{Re}\operatorname{tr}((b_1 + ib_2)(a_1^\top a_1 + ia_1^\top a_2 - ia_2^\top a_1 + a_2^\top a_2)) \\ &= \operatorname{tr}(b_1(a_1^\top a_1)) + \operatorname{tr}(b_1(a_2^\top a_2)) + \operatorname{tr}(b_2 a_2^\top a_1) - \operatorname{tr}(b_2 a_1^\top a_2) \\ &\geq 0 + 0 + \operatorname{tr}(b_2 a_2^\top a_1) - \operatorname{tr}(b_2^\top a_1^\top a_2) = 0. \end{aligned}$$

Here we have used that $a_1^\top a_1,\ b_1 \in S_n^+$, $b_2 = b_2^\top$ and the commutativity of the matrix trace. □

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