



Scaling limits of coupled continuous time random walks and residual order statistics through marked point processes

A. Barczyk, P. Kern*

Mathematical Institute, Heinrich-Heine-University Düsseldorf, D-40225 Düsseldorf, Germany

Received 27 August 2012; received in revised form 29 October 2012; accepted 30 October 2012

Available online 7 November 2012

Abstract

A continuous time random walk (CTRW) is a random walk in which both spatial changes represented by jumps and waiting times between the jumps are random. The CTRW is coupled if a jump and its preceding or following waiting time are dependent random variables (r.v.), respectively. The aim of this paper is to explain the occurrence of different limit processes for CTRWs with forward- or backward-coupling in Straka and Henry (2011) [37] using marked point processes. We also establish a series representation for the different limits. The methods used also allow us to solve an open problem concerning residual order statistics by LePage (1981) [20].

© 2012 Elsevier B.V. All rights reserved.

MSC: primary 60F17; 60G50; 60G55; secondary 60G18; 62G30

Keywords: Continuous time random walk; Operator stable law; Lévy process; Marked point process; Order statistics; Series representation

1. Introduction

Two i.i.d. sequences of \mathbb{R}^+ -valued waiting times $(J_n)_{n \in \mathbb{N}}$ and of \mathbb{R}^d -valued jumps $(\mathbf{X}_n)_{n \in \mathbb{N}}$ yield two versions of a CTRW by

$$S_{N_t} := \sum_{k=1}^{N_t} \mathbf{X}_k, \quad S_{N_t+1} := \sum_{k=1}^{N_t+1} \mathbf{X}_k,$$

* Corresponding author.

E-mail addresses: barczyk@math.uni-duesseldorf.de (A. Barczyk), kern@math.uni-duesseldorf.de (P. Kern).

where $N_t := \max\{n \in \mathbb{N}_0 : \sum_{k=1}^n J_k \leq t\}$ is the number of jumps up to time t . The CTRW is coupled if the sequences $(J_n)_{n \in \mathbb{N}}$ and $(\mathbf{X}_n)_{n \in \mathbb{N}}$ are dependent. Typically we assume, that the sequence $(J_n, \mathbf{X}_n)_{n \in \mathbb{N}}$ is i.i.d. with unknown dependence between the waiting time J_n and the jump \mathbf{X}_n for fixed $n \in \mathbb{N}$. Using this dependence structure S_{N_t} is called backward-coupled CTRW whereas S_{N_t+1} is called forward-coupled CTRW. Both processes represent the position of a jumper at time t , but in the backward-coupled case the particle first waits for a time J_1 before jumping to \mathbf{X}_1 , whereas in the forward-coupled case the particle jumps to \mathbf{X}_1 at time $t = 0$ and then waits for a time J_1 and so on. CTRW processes were introduced in [29] to study random walks on a lattice and have been studied intensively over the past few decades. Today there is a wide field of possible applications for CTRWs. They are used in physics to model phenomena of anomalous diffusion [35,28]. The jumps can also represent movements of an ensemble of particles being transported over the earth surface in geophysics [34] or represent log-returns in finance [32]. A comprehensive study of limit theorems for coupled CTRWs has been initiated in [3] covering previously known special models [35,17,18] from physics. Coupling methods different from forward-/backward-coupling can be found in [6,23], where certain correlations between the waiting times, respectively between the jumps were introduced. A similar approach in a more general setting appears in [13], where coupling is introduced through a Markov chain $(Y_n)_{n \in \mathbb{N}}$ and waiting times, respectively jumps are modeled by $(J_n = \tau(Y_n), X_n = V(Y_n))$ for some measurable functions τ, V with values in $(0, \infty)$, respectively \mathbb{R} . Note that these couplings in general do not fulfill our i.i.d. assumption on the sequence $(J_n, \mathbf{X}_n)_{n \in \mathbb{N}}$.

The limiting distributions of forward- and backward-coupled CTRWs have been investigated by Straka and Henry [37] using a continuous mapping approach on the space of their sample paths. Straka and Henry prove that the limiting processes of coupled CTRWs in general differ when waiting times precede or follow jumps, respectively. Also the differences between the properties of these processes are not marginal, cf. [14]. Unfortunately, neither the continuous mapping approach used in [37] nor the methods used in [14] are adequate to point out why different scaling limits occur from a mathematical point of view. So a new approach to fill this gap is made using marked point processes here. Another explanation for the different scaling limits of forward- and backward-coupled CTRW models can be found when introducing a random clustering procedure as in [39,15,36]. A subsequent clustering of M_n uncoupled i.i.d. waiting times and jumps (J_i, \mathbf{X}_i) , where $(M_n)_{n \in \mathbb{N}}$ is a sequence of positive integer valued i.i.d. random variables belonging to the domain of attraction of a γ -stable random variable with $0 < \gamma < 1$, leads to models with forward- and backward-coupling, regarding to the number of cluster jumps that have been occasionally or fully considered at the observation time. The corresponding scaling limits are described by over- and undershooting subordination in [39,15,36], where the over- and undershoot process is again the scaling limit of a tightly coupled CTRW with one-dimensional jumps $X_i = J_i$ equal to the corresponding waiting times. The special models of tightly coupled CTRW scaling limits have been extensively studied in [18,4,38] in connection with generalized arc-sine laws. Our approach using marked point processes will illuminate the occurrence of different scaling limits in more general situations than the case of a tightly coupled CTRW.

Defining the time of the n -th jump by $T_n := \sum_{k=1}^n J_k$ we first study the limit behavior of the point processes which arise by marking each jump time with its occurring jump, respectively, i.e. we analyze

$$\sum_{k=1}^n \mathcal{E}_{(T_k, \mathbf{X}_k)}, \quad \sum_{k=1}^n \mathcal{E}_{(T_{k-1}, \mathbf{X}_k)}. \tag{1}$$

It turns out that only the jumps with large norm contribute to the limit distributions of (1), as it is already known for real-valued partial sums which converge to an infinitely divisible r.v. without Gaussian part, cf. [1] and references therein. The methods used also solve an open problem concerning the convergence of residual order statistics by LePage, cf. [20,21,33]. The scaling limits of the CTRWs can be determined by summing up the marks of the points in (1) which have a jump time occurring before time t . Hence in the scaling limit of forward-coupled CTRWs an additional big jump occurs compared to its backward-coupled version, which illuminates the difference between the processes. This approach also provides a series representation for the different scaling limits which might be of interest for simulation purposes. Since the resulting limit processes are not Lévy processes, no efficient simulation algorithm is known yet.

2. Preliminaries

Let $(J_n, \mathbf{X}_n)_{n \in \mathbb{N}}$ be an i.i.d. sequence of $\mathbb{R}^+ \times \mathbb{R}^d$ -valued r.v. Assume that (J_1, \mathbf{X}_1) belongs to the generalized domain of attraction (GDOA) of a r.v. (D, \mathbf{A}) , where D is stable with index $\alpha \in (0, 1)$ and \mathbf{A} is full operator stable with index $E \in GL(\mathbb{R}^d)$ and without Gaussian part. Note that by Theorem 7.2.1 of [25] the real parts of the eigenvalues of E are greater than $1/2$. By classic results [16, Theorem 14.14] and [26, Theorem 4.1] this implies that there exist regularly varying sequences $(b_n)_{n \in \mathbb{N}} \in RV_{1/\alpha}$ and $(A_n)_{n \in \mathbb{N}} \in RV_{-E}$ such that the convergence

$$\left(b_n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} J_k, \sum_{k=1}^{\lfloor nt \rfloor} (A_n \mathbf{X}_k - \mathbb{E}(A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| \leq \tau}) \right)_{t \geq 0} \xrightarrow{\mathcal{D}} (D(t), \mathbf{A}(t))_{t \geq 0} \tag{2}$$

holds in $D([0, \infty), \mathbb{R}^+ \times \mathbb{R}^d)$ for any $\tau > 0$ such that for the Lévy measure η of \mathbf{A} and the sphere $\mathbb{S}_\tau^{d-1} = \{x \in \mathbb{R}^d : \|x\| = \tau\}$ we have $\eta(\mathbb{S}_\tau^{d-1}) = 0$, i.e. the sphere \mathbb{S}_τ^{d-1} is a continuity set for η . Throughout this paper, spaces of càdlàg paths $D([0, \infty), \mathcal{X})$, respectively $D([0, T], \mathcal{X})$, are always equipped with the corresponding Skorokhod J_1 -topology. Here the process $D(\cdot)$ denotes an α -stable subordinator and $\mathbf{A}(\cdot)$ denotes an operator Lévy motion. Note that the drift term of the Lévy process \mathbf{A} depends on τ . It is well known that we can choose $\tau = \infty$ if the real part of any eigenvalue of E belongs to $(1/2, 1)$, since then $\mathbb{E}(\mathbf{X}_1)$ exists. Moreover we can choose $\tau = 0$ if the real part of any eigenvalue of E exceeds 1. Due to the spectral decomposition in [25], centering by truncated expectations in (2) is only necessary if some eigenvalue of the exponent E has real part equal to 1.

We already stated, that only points with large norm contribute to the limit of the point processes in (1). So we use a radial decomposition of the Lévy measure

$$\eta(A) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}_A(xv) \tilde{\eta}(dx, v) d\sigma(v)$$

where σ is a probability measure on the unit sphere \mathbb{S}^{d-1} of \mathbb{R}^d and $(\tilde{\eta}(\cdot, v))_{v \in \mathbb{S}^{d-1}}$ is a weakly measurable family of Lévy measures on $(0, \infty)$, cf. [31]. We also define the right-continuous inverse of $\tilde{\eta}(\cdot, v)$ by

$$\tilde{\eta}^{\leftarrow}(x, v) := \sup\{u > 0 : \tilde{\eta}([u, \infty), v) \geq x\}. \tag{3}$$

With this notation we are able to give a series representation for the process $\mathbf{A}(\cdot)$ in $D([0, T], \mathbb{R}^d)$ for fixed $T > 0$ by

$$\lim_{\varepsilon \downarrow 0} \left(\sum_{T \cdot \tau_k \leq t} \left(\tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \mathbf{V}_k \mathbf{1}_{\tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) > \varepsilon} \right) - t \int_{\varepsilon \leq \|x\| \leq \tau} x d\eta(x) \right), \tag{4}$$

cf. [20,31], where Γ_n is the n -th partial sum of i.i.d. standard exponential r.v., $(\tau_n)_{n \in \mathbb{N}}$ denotes an i.i.d. sequence of uniformly $\mathcal{U}(0, 1)$ -distributed r.v. and $(\mathbf{V}_n)_{n \in \mathbb{N}}$ denotes an i.i.d. sequence with distribution σ , with $(\Gamma_n)_{n \in \mathbb{N}}$, $(\tau_n)_{n \in \mathbb{N}}$ and $(\mathbf{V}_n)_{n \in \mathbb{N}}$ being independent.

Now it is well known that for a triangular array of infinitesimal row-wise independent \mathbb{R}^+ -valued r.v. $(Y_{k,n})_{1 \leq k \leq n}$, $n \in \mathbb{N}$, converging to an infinitely divisible r.v. Y with associated Lévy measure ϕ , only the extremes contribute to the limit distribution, cf. [1] and references therein. This result coincides with the convergence

$$\sum_{k=1}^n \mathcal{E}\left(\frac{k}{n}, Y_{k,n}\right) \xrightarrow{\mathcal{D}} \text{PRM}(\lambda \otimes \phi) \tag{5}$$

in $M_p([0, 1] \times (0, \infty])$, the set of all point measures on $[0, 1] \times (0, \infty]$, where $\text{PRM}(\lambda \otimes \phi)$ denotes a Poisson random measure with mean measure $\lambda \otimes \phi$. Furthermore, it is well known that $\sum_{k \in \mathbb{N}} \mathcal{E}(\tau_k, \phi^{\leftarrow}(\Gamma_k))$ is also a representation of $\text{PRM}(\lambda \otimes \phi)$ in $M_p([0, 1] \times (0, \infty])$, where ϕ^{\leftarrow} denotes the right-sided inverse of ϕ , cf. [31]. This fact can be understood by sorting the points on the left-hand side in (5)

$$\sum_{k=1}^n \mathcal{E}\left(\frac{k}{n}, Y_{k,n}\right) = \sum_{k=1}^n \mathcal{E}\left(\frac{d_k}{n}, Y_{n-k+1:n}\right), \tag{6}$$

where $(Y_{1:n}, \dots, Y_{n:n})$ denotes the order statistics of $(Y_{1,n}, \dots, Y_{n,n})$ with corresponding antirank vector (d_1, \dots, d_n) , i.e. the inverse permutation of the rank vector. Using Freedman’s Lemma, cf. [12], one can easily verify, that the convergence $(n^{-1}d_k)_{k \in \mathbb{N}} \xrightarrow{\mathcal{D}} (\tau_k)_{k \in \mathbb{N}}$ holds in $[0, 1]^{\mathbb{N}}$, where $d_k = 0$ for $k > n$. Moreover, (2.4) in [1] gives us $Y_{n-k+1:n} \xrightarrow{\mathcal{D}} \phi^{\leftarrow}(\Gamma_k)$. So the convergence in (5) can also be established by analyzing the convergence of the points $(n^{-1}d_k, Y_{n-k+1:n})_{1 \leq k \leq n}$. This approach can also be applied to the point processes in (1). As the r.v. $(\mathbf{X}_n)_{n \in \mathbb{N}}$ are \mathbb{R}^d -valued one cannot use traditional order statistics. LePage suggested to use a normwise sorting, cf. [20]. So for $x_1, \dots, x_n \in \mathbb{R}^d$ we introduce the residual order statistics $x_{1:n}, \dots, x_{n:n}$ by $\|x_{1:n}\| \leq \dots \leq \|x_{n:n}\|$.

3. Convergence of residual order statistics

The convergence of the normalized residual order statistics $A_n \mathbf{X}_{n-k+1:n}$ is still an open problem. LePage [20] conjectures, that a generalization of the one-dimensional case

$$(A_n \mathbf{X}_{n-k+1:n})_{k \in \mathbb{N}} \xrightarrow{\mathcal{D}} (\tilde{\eta}^{\leftarrow}(\Gamma_k, \mathbf{V}_k) \mathbf{V}_k)_{k \in \mathbb{N}} \tag{7}$$

holds. As usual one sets $\mathbf{X}_{k:n} = 0$, whenever $k \leq 0$ or $k > n$. This result has been proven in [21] for the case that the limit process $\mathbf{A}(\cdot)$ is multivariate α -stable. In this case the right-sided inverse $\tilde{\eta}^{\leftarrow}(x, v)$ defined in (3) is independent of $v \in \mathbb{S}^{d-1}$ as the projection of the Lévy measure η is the same for every direction, cf. Theorem 7.3.3 in [25]. Some years later a similar problem has been studied in [11] using a different norm $\|\cdot\|_H$ which respects the special structure of the operator E . In [33] the operator semistable case has been studied, but the result (7) also could only be established in the special case, that $\tilde{\eta}^{\leftarrow}(x, v)$ is independent of v , which coincides with

the multivariate α -stable case. The author also supposed that the convergence (7) holds only in this case. We will show that the limit on the right-hand side in (7) has to be modified. The proof is based on the following lemma.

Lemma 1. *Let $N_n = \sum_{k=1}^n \varepsilon_{\mathbf{X}_k^{(n)}}$, $n \in \mathbb{N}_0$, be a sequence of point processes in $M_p([-\infty, \infty]^d \setminus \mathbb{K}_\varepsilon^d)$, the set of all point measures on the space $[-\infty, \infty]^d \setminus \mathbb{K}_\varepsilon^d$, where $\mathbb{K}_\varepsilon^d := \{x \in \mathbb{R}^d : \|x\| \leq \varepsilon\}$ denotes the compact ε -ball in \mathbb{R}^d . Suppose $N_n \xrightarrow{\mathcal{D}} N_0$. If $\varepsilon < \|\mathbf{X}_i^{(0)}\| < \infty$ holds for every $i \in \mathbb{N}$ almost surely (a.s.) and $\|\mathbf{X}_i^{(0)}(\omega)\| > \|\mathbf{X}_j^{(0)}(\omega)\|$ for all $1 \leq i < j$ a.s. then the convergence*

$$(\mathbf{X}_{n-k+1:n})_{k \in \mathbb{N}} \xrightarrow{\mathcal{D}} (\mathbf{X}_k^{(0)})_{k \in \mathbb{N}}$$

holds in $(\mathbb{R}^d \setminus \mathbb{K}_\varepsilon^d)^{\mathbb{N}}$.

Proof. The proof is based on a continuous mapping approach and Lemma 7.1 in [30]. Define $M \subset M_p([-\infty, \infty]^d \setminus \mathbb{K}_\varepsilon^d)$ by

$$M := \left\{ m : m = \sum_{k=1}^P \varepsilon_{x_k}, \infty > \|x_1\| > \dots > \|x_P\| > \varepsilon \right\}.$$

Now we show that the mapping

$$\begin{aligned} \pi_k : M_p([-\infty, \infty]^d \setminus \mathbb{K}_\varepsilon^d) &\rightarrow [-\infty, \infty]^d \setminus \mathbb{K}_\varepsilon^d \\ \pi_k \left(\sum_{i=1}^P \varepsilon_{x_i} \right) &\mapsto x_{P-k+1:P}, \quad x_{P-k+1:P} = 0 \quad \text{for } k \leq 0 \text{ and } k > P \end{aligned}$$

is continuous in $m \in M$ for every $k \in \mathbb{N}$. Let $(m_n)_{n \in \mathbb{N}} \subset M_p([-\infty, \infty]^d \setminus \mathbb{K}_\varepsilon^d)$ be a sequence of point measures converging vaguely to a point measure $m_0 = \sum_{k=1}^P \varepsilon_{x_k^{(0)}} \in M_p([-\infty, \infty]^d \setminus \mathbb{K}_\varepsilon^d)$.

Now choose n sufficiently large so that all points of m_n lie inside of $[-\infty, \infty]^d \setminus \mathbb{K}_\varepsilon^d$. By sorting the points of m_n in descending order of their norm

$$m_n = \sum_{i=1}^P \varepsilon_{x_i^{(n)}}, \quad \varepsilon < \|x_P^{(n)}\| \leq \dots \leq \|x_1^{(n)}\| < \infty$$

an application of Lemma 7.1 in [30] yields the convergence of the points

$$(x_1^{(n)}, \dots, x_P^{(n)}) \longrightarrow (x_1^{(0)}, \dots, x_P^{(0)}) \tag{8}$$

in $(\mathbb{R}^d \setminus \mathbb{K}_\varepsilon^d)^P$. Now by the definition of the mapping π_k

$$\begin{aligned} \pi_k(m_n) &= x_k^{(n)}, & \pi_k(m_0) &= x_k^{(0)} \quad \text{for } 1 \leq k \leq P \\ \pi_k(m_n) &= \pi_k(m_0) = 0 & & \text{for } k > P \end{aligned}$$

holds and π_k is continuous by (8) for every $k \in \mathbb{N}$. An easy application of the continuous mapping theorem yields the desired result. \square

Lemma 1 allows to identify the distribution of the limit of properly normalized residual order statistics.

Theorem 2. For $k \in \mathbb{N}$, $T > 0$ and $\omega \in \Omega$ define

$$\widehat{d}_k(\omega) = \arg \left(\max_{\substack{i \in \mathbb{N} \\ i \neq \widehat{d}_1(\omega), \dots, \widehat{d}_{k-1}(\omega)}} \widetilde{\eta}^{\leftarrow}(T^{-1}\Gamma_i(\omega), \mathbf{V}_i(\omega)) \right) \tag{9}$$

as the argument of the k -th largest element of the set $\{\widetilde{\eta}^{\leftarrow}(T^{-1}\Gamma_i(\omega), \mathbf{V}_i(\omega)), i \in \mathbb{N}\}$. Then convergence of the residual order statistics

$$(A_n \mathbf{X}_{[nT]-k+1:[nT]})_{k \in \mathbb{N}} \xrightarrow{\mathcal{D}} \left(\widetilde{\eta}^{\leftarrow}(T^{-1}\Gamma_{\widehat{d}_k}, \mathbf{V}_{\widehat{d}_k}) \mathbf{V}_{\widehat{d}_k} \right) \tag{10}$$

holds in $(\mathbb{R}^d)^{\mathbb{N}}$.

Remark 3. Note that \widehat{d}_k is well-defined for all $k \in \mathbb{N}$, since the number of elements in the set $\{i \in \mathbb{N} : \widetilde{\eta}^{\leftarrow}(T^{-1}\Gamma_i, \mathbf{V}_i) > \varepsilon\}$ is finite a.s. for all $\varepsilon > 0$. Moreover **Theorem 2** does not contradict any of the results proven in [11,21,33]. If $\mathbf{A}(\cdot)$ is a multivariate α -stable Lévy process, the monotonicity of the mapping $x \mapsto \widetilde{\eta}^{\leftarrow}(x, v)$ yields $\widehat{d}_k(\omega) = k$ a.s. for all $k \in \mathbb{N}$.

Proof of Theorem 2. First we have to determine the limit of the truncated point process

$$\sum_{k=1}^{\lfloor nT \rfloor} \varepsilon_{(A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| \geq \varepsilon})},$$

where $\varepsilon > 0$ has to be chosen such that $\eta(\mathbb{S}_\varepsilon^{d-1}) = 0$ holds. By **Theorem 3.2.2** in [25] assumption (2) yields the vague convergence

$$\lfloor nT \rfloor \mathbb{P}(A_n \mathbf{X}_k \in \cdot) \xrightarrow{v} T \cdot \eta(\cdot) \tag{11}$$

in $\mathbb{R}^d \setminus \{0\}$. Hence the convergence of the point processes

$$\sum_{k=1}^{\lfloor nT \rfloor} \varepsilon_{A_n \mathbf{X}_k} \xrightarrow{\mathcal{D}} \text{PRM}(T \cdot \eta)$$

holds in $M_p([-\infty, \infty]^d \setminus \{0\})$. Now by [31] $\sum_{k \in \mathbb{N}} \varepsilon_{\widetilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \mathbf{V}_k} = \text{PRM}(T \cdot \eta)$. Applying the a.s. continuous restriction functional

$$\pi' : M_p([-\infty, \infty]^d \setminus \{0\}) \rightarrow M_p([-\infty, \infty]^d \setminus \mathbb{K}_\varepsilon^d), \quad m \mapsto m|_{(\mathbb{K}_\varepsilon^d)^c}, \tag{12}$$

the continuous mapping theorem yields

$$\sum_{k=1}^{\lfloor nT \rfloor} \varepsilon_{A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| \geq \varepsilon}} \xrightarrow{\mathcal{D}} \sum_{k \in \mathbb{N}} \varepsilon_{\widetilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \mathbf{V}_k \mathbf{1}_{\widetilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \geq \varepsilon}}. \tag{13}$$

The continuity of π' is proven in [7] for instance. Now the points of the point process on the right-hand side of (13) have to be ordered in descending order of their norm

$$\sum_{k \in \mathbb{N}} \varepsilon_{\widetilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \mathbf{V}_k \mathbf{1}_{\widetilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \geq \varepsilon}} = \sum_{k \in \mathbb{N}} \varepsilon_{\widetilde{\eta}^{\leftarrow}(T^{-1}\Gamma_{\widehat{d}_k}, \mathbf{V}_{\widehat{d}_k}) \mathbf{V}_{\widehat{d}_k} \mathbf{1}_{\widetilde{\eta}^{\leftarrow}(T^{-1}\Gamma_{\widehat{d}_k}, \mathbf{V}_{\widehat{d}_k}) \geq \varepsilon}}.$$

An application of **Lemma 1** yields the convergence of the points

$$\begin{aligned} & (A_n \mathbf{X}_{[nT]-k+1:[nT]} \mathbf{1}_{\|A_n \mathbf{X}_{[nT]-k+1:[nT]}\| \geq \varepsilon})_{k \in \mathbb{N}} \\ & \xrightarrow{\mathcal{D}} \left(\widetilde{\eta}^{\leftarrow}(T^{-1}\Gamma_{\widehat{d}_k}, \mathbf{V}_{\widehat{d}_k}) \mathbf{V}_{\widehat{d}_k} \mathbf{1}_{\widetilde{\eta}^{\leftarrow}(T^{-1}\Gamma_{\widehat{d}_k}, \mathbf{V}_{\widehat{d}_k}) \geq \varepsilon} \right)_{k \in \mathbb{N}} \end{aligned}$$

in $(\mathbb{R}^d \setminus \mathbb{K}_\varepsilon^d)^\mathbb{N}$. The desired result follows by taking the limit as $\varepsilon \downarrow 0$ and an easy application of Theorem 4.2 in [5]. \square

4. Convergence of associated point processes

Now that the limit distribution of normalized residual order statistics is identified we can study the associated marked point processes

$$\sum_{k=1}^{\lfloor nT \rfloor} \varepsilon_{(b_n^{-1}T_k, A_n \mathbf{X}_k)}, \quad \sum_{k=1}^{\lfloor nT \rfloor} \varepsilon_{(b_n^{-1}T_{k-1}, A_n \mathbf{X}_k)}.$$

In the uncoupled case convergence results for this processes can be established with a continuous mapping approach using the time deformation defined in [30, (8.29)]. Since the continuity of this time deformation demands the processes $\mathbf{A}(\cdot)$ and $D(\cdot)$ to have a.s. no common jumps, which is not necessarily fulfilled in the coupled case, these standard methods cannot be applied in our case. So we use a sorting argument like in (6).

Lemma 4. *Let $(\tau_n)_{n \in \mathbb{N}}, (\Gamma_n)_{n \in \mathbb{N}}$ and $(\mathbf{V}_n)_{n \in \mathbb{N}}$ be as in (4). Then the convergence of the associated point processes*

$$\sum_{k=1}^{\lfloor nT \rfloor} \varepsilon_{(b_n^{-1}T_k, A_n \mathbf{X}_k)} \xrightarrow{\mathcal{D}} \sum_{k \in \mathbb{N}} \varepsilon_{(D(T \cdot \tau_k), \tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k)\mathbf{V}_k)} \tag{14}$$

$$\sum_{k=1}^{\lfloor nT \rfloor} \varepsilon_{(b_n^{-1}T_{k-1}, A_n \mathbf{X}_k)} \xrightarrow{\mathcal{D}} \sum_{k \in \mathbb{N}} \varepsilon_{(D(T \cdot \tau_k), \tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k)\mathbf{V}_k)} \tag{15}$$

holds in $M_p([0, \infty) \times [-\infty, \infty]^d \setminus \{0\})$ for every $T > 0$, where $D(x-)$ denotes the left-hand limit of the process $D(\cdot)$ in x .

Proof. Choose $T > 0$ arbitrary. We start by sorting the points of the associated point process

$$\left(\sum_{k=1}^{\lfloor nT \rfloor} \varepsilon_{(b_n^{-1}T_k, A_n \mathbf{X}_k)} \right) = \left(\sum_{k=1}^{\lfloor nT \rfloor} \varepsilon_{(b_n^{-1}T_{d_{\lfloor nT \rfloor - k + 1}}, A_n \mathbf{X}_{\lfloor nT \rfloor - k + 1 : \lfloor nT \rfloor})} \right).$$

Again $(d_1, \dots, d_{\lfloor nT \rfloor})$ denotes the antirank vector of the r.v. $(\mathbf{X}_1, \dots, \mathbf{X}_{\lfloor nT \rfloor})$. The normalized residual order statistics $A_n \mathbf{X}_{n-k+1 : \lfloor nT \rfloor}$ have already been studied in Theorem 2. It remains to determine the limit distribution of

$$b_n^{-1}T_{d_{\lfloor nT \rfloor - k + 1}} = b_n^{-1} \sum_{l=1}^{d_{\lfloor nT \rfloor - k + 1}} J_l = b_n^{-1} \sum_{l=1}^{\lfloor nT \rfloor \cdot \frac{d_{\lfloor nT \rfloor - k + 1}}{\lfloor nT \rfloor}} J_l.$$

Since $\mathbf{X}_1, \dots, \mathbf{X}_{\lfloor nT \rfloor}$ are i.i.d., $(n^{-1}d_k)_{k \in \mathbb{N}} \xrightarrow{\mathcal{D}} (T \cdot \tau_k)_{k \in \mathbb{N}}$ suggests that the convergence

$$(b_n^{-1}T_{d_{\lfloor nT \rfloor - k + 1}})_{k \in \mathbb{N}} \xrightarrow{\mathcal{D}} (D(T \cdot \tau_k))_{k \in \mathbb{N}} \tag{16}$$

holds in $[0, \infty)^\mathbb{N}$. But this result cannot be established with a traditional continuous mapping approach. The mapping $\pi_t : D([0, \infty), \mathbb{R}^d) \times \mathbb{R}^+ \rightarrow \mathbb{R}, (x, t) \mapsto x(t)$ is only a.s. continuous if x is a.s. continuous in t . Also classical transfer theorems, cf. [9,10], are not helpful because

they require independence of the summands and their quantity or a stochastic convergence of the normalized antirank vector, cf. [2]. Since none of these conditions is fulfilled another approach is used.

Let $(\widehat{d}_1, \dots, \widehat{d}_{[nT]})$ denote the associated antirank vector of the waiting times $(J_1, \dots, J_{[nT]})$. Since the joint convergence of the well-centered and normalized sequential partial sums to the process $(D(\cdot), \mathbf{A}(\cdot))$ holds, one can easily prove convergence of the normalized antirank vector

$$(n^{-1}(d_{[nT]-k+1})_{k \in \mathbb{N}}, n^{-1}(\widehat{d}_{[nT]-k+1})_{k \in \mathbb{N}}) \xrightarrow{\mathcal{D}} ((T \cdot \tau_i)_{i \in \mathbb{N}}, (T \cdot \widehat{\tau}_i)_{i \in \mathbb{N}})$$

where $(\widetilde{\tau}_n)_{n \in \mathbb{N}}, (\widehat{\tau}_n)_{n \in \mathbb{N}}$ denote two i.i.d. sequence of $\mathcal{U}(0, 1)$ -distributed r.v. Note that the sequences $(\widetilde{\tau}_n)_{n \in \mathbb{N}}$ and $(\widehat{\tau}_n)_{n \in \mathbb{N}}$ are not independent in the coupled case. As a consequence of the convergence of the antirank vector, the convergence of indicator functions

$$\left(\mathbf{1}_{n^{-1}\widehat{d}_{[nT]-j+1} \leq n^{-1}d_{[nT]-i+1}} \right)_{j \in \mathbb{N}} \xrightarrow{\mathcal{D}} (\mathbf{1}_{\widehat{\tau}_j \leq \widetilde{\tau}_i})_{j \in \mathbb{N}}$$

holds. An application of Basu’s lemma, cf. [19, Theorem 5.1.2], yields the independence

$$((d_1, \dots, d_{[nT]}), (\widehat{d}_1, \dots, \widehat{d}_{[nT]})) \perp (J_1: [nT], \dots, J_{[nT]: [nT]}) \tag{17}$$

for every fixed $n \in \mathbb{N}$, which proves that the joint convergence

$$\left(b_n^{-1} J_{[nT]-j+1: [nT]} \mathbf{1}_{\widehat{d}_{[nT]-j+1} \leq d_{[nT]-i+1}} \right)_{j \in \mathbb{N}} \xrightarrow{\mathcal{D}} \left((\eta^\alpha)^{-1} (T^{-1} \widehat{\Gamma}_j) \mathbf{1}_{\widehat{\tau}_j \leq \widetilde{\tau}_i} \right)_{j \in \mathbb{N}}$$

holds, where $(\eta^\alpha)^{-1}$ is the right-sided inverse of the Lévy measure associated with $D(1)$ and $(\widehat{\Gamma}_n)_{n \in \mathbb{N}}$ denotes a distributional copy of the sequence $(\Gamma_n)_{n \in \mathbb{N}}$. Note that the sequences $(\Gamma_n)_{n \in \mathbb{N}}$ and $(\widehat{\Gamma}_n)_{n \in \mathbb{N}}$ are also not independent in the coupled case. Now summation verifies

$$\sum_{l \in \mathbb{N}} b_n^{-1} J_{[nT]-l+1: [nT]} \mathbf{1}_{n^{-1}\widehat{d}_{[nT]-l+1} \leq n^{-1}d_{[nT]-i+1}} \tag{18}$$

$$\begin{aligned} &= \sum_{l=1}^{[nT]} b_n^{-1} J_{[nT]-l+1: [nT]} \mathbf{1}_{n^{-1}\widehat{d}_{[nT]-l+1} \leq n^{-1}d_{[nT]-i+1}} \\ &= b_n^{-1} \sum_{l=1}^{[nT]} J_l \mathbf{1}_{l \leq d_{[nT]-i+1}} = b_n^{-1} \sum_{l=1}^{d_{[nT]-i+1}} J_l \xrightarrow{\mathcal{D}} \sum_{l \in \mathbb{N}} (\eta^\alpha)^{-1} (T^{-1} \widehat{\Gamma}_l) \mathbf{1}_{T \cdot \widehat{\tau}_l \leq T \cdot \widetilde{\tau}_i}. \end{aligned} \tag{19}$$

Using the Ferguson–Klass series representation of the process $D(\cdot)$, cf. [8], one identifies the right-hand side in (19) as series representation of $D(T \cdot \widetilde{\tau}_i)$. Since the convergence of antiranks holds simultaneously, we have proven

$$\left(b_n^{-1} \left[\sum_{l=1}^{\lfloor n \cdot \frac{d_{[nT]-i+1}}{n} \rfloor} J_l \right] \right)_{i \in \mathbb{N}} \xrightarrow{\mathcal{D}} (D(T \cdot \widetilde{\tau}_i))_{i \in \mathbb{N}} \tag{20}$$

in $(\mathbb{R}^+)^{\mathbb{N}}$. Again by the joint convergence of the properly normalized and scaled sequential partial sums to the process $(D(\cdot), \mathbf{A}(\cdot))$, the independence (17) and Theorem 2, convergence of the points

$$\left(b_n^{-1} T_{d_{[nT]-i+1}}, A_n \mathbf{X}_{[nT]-i+1: [nT]} \right)_{i \in \mathbb{N}} \xrightarrow{\mathcal{D}} \left(D(T \cdot \widetilde{\tau}_i), \widetilde{\eta}^{\leftarrow} (T^{-1} \Gamma_{\widehat{d}_i}, \mathbf{V}_{\widehat{d}_i}) \mathbf{V}_{\widehat{d}_i} \right)_{i \in \mathbb{N}}$$

holds. Since the mapping $x \mapsto \varepsilon_x$ is continuous, the convergence of points yields the convergence of point processes

$$\left(\varepsilon \left(b_n^{-1} T_{d_{[nT]_{-i+1}}, A_n \mathbf{X}_{[nT]_{-i+1}: [nT]}} \right) \right)_{i \in \mathbb{N}} \xrightarrow{\mathcal{D}} \left(\varepsilon \left(D(T \cdot \tilde{\tau}_i), \tilde{\eta}^{\leftarrow} (T^{-1} \tilde{\Gamma}_{\hat{d}_i}, \mathbf{V}_{\hat{d}_i}) \mathbf{V}_{\hat{d}_i} \right) \right)_{i \in \mathbb{N}}$$

in $M_p([0, \infty) \times [-\infty, \infty]^d \setminus \{0\})$. Now summation and an easy application of Theorem 4.2 in [5] yields

$$\sum_{k=1}^{[nT]} \varepsilon \left(b_n^{-1} T_{d_{[nT]_{-k+1}}, A_n \mathbf{X}_{[nT]_{-k+1}: [nT]}} \right) \xrightarrow{\mathcal{D}} \sum_{k \in \mathbb{N}} \varepsilon \left(D(T \cdot \tilde{\tau}_k), \tilde{\eta}^{\leftarrow} (T^{-1} \tilde{\Gamma}_{\hat{d}_k}, \mathbf{V}_{\hat{d}_k}) \mathbf{V}_{\hat{d}_k} \right)$$

in $M_p([0, \infty) \times [-\infty, \infty]^d \setminus \{0\})$. Now we need to reverse the order of the points again. We introduce

$$r_k(\omega) := 1 + \#\{i \in \mathbb{N} : \tilde{\eta}^{\leftarrow} (T^{-1} \Gamma_i(\omega), \mathbf{V}_i(\omega)) > \tilde{\eta}^{\leftarrow} (T^{-1} \Gamma_k(\omega), \mathbf{V}_k(\omega))\}$$

as inverse of \hat{d}_k . Note that r_k is well-defined for all $k \in \mathbb{N}$ by Remark 3. Moreover, since $(\tilde{\tau}_n)_{n \in \mathbb{N}}$ and $(r_n)_{n \in \mathbb{N}}$ are independent, an easy application of the disintegration formula shows, that $(\tilde{\tau}_n)_{n \in \mathbb{N}}$ is also i.i.d. and $\mathcal{U}(0, 1)$ distributed. Since the convergence in (5) towards (4) can be proven with the same technique, the sequences $(\tilde{\tau}_n)_{n \in \mathbb{N}}$, $(\Gamma_n)_{n \in \mathbb{N}}$ and $(\mathbf{V}_n)_{n \in \mathbb{N}}$ are also independent. So we define $\tau_n := \tilde{\tau}_{r_n}$ for all $n \in \mathbb{N}$ and obtain

$$\begin{aligned} \sum_{k=1}^{[nT]} \varepsilon \left(b_n^{-1} T_k, A_n \mathbf{X}_k \right) &= \sum_{k=1}^{[nT]} \varepsilon \left(b_n^{-1} T_{d_{[nT]_{-k+1}: [nT]}}, A_n \mathbf{X}_{[nT]_{-k+1}: [nT]} \right) \\ &\xrightarrow{\mathcal{D}} \sum_{k \in \mathbb{N}} \varepsilon \left(D(T \cdot \tau_k), \tilde{\eta}^{\leftarrow} (T^{-1} \Gamma_k, \mathbf{V}_k) \mathbf{V}_k \right). \end{aligned}$$

Hence we have proven (14). In order to prove (15) the limit distribution of

$$b_n^{-1} T_{d_{[nT]_{-i+1}-1}} = b_n^{-1} \sum_{k=1}^{d_{[nT]_{-i+1}-1}} J_k = b_n^{-1} \sum_{k=1}^{\left[n \cdot \frac{d_{[nT]_{-i+1}-1}}{n} \right]} J_k$$

has to be analyzed. Denoting $G([0, \infty), \mathbb{R}^d)$ the space of all left-continuous functions with right-hand limits from $[0, \infty)$ to \mathbb{R}^d one easily proves that the mapping $\mathcal{T} : D[0, \infty) \rightarrow G[0, \infty), x(t) \mapsto x(t-)$ is Lipschitz-continuous with Lipschitz-constant one and hence continuous. Since

$$\mathcal{T} \left(\sum_{k=1}^{[nt]} J_k \right) = \sum_{k=1}^{[nt]-1} J_k$$

holds, the continuity of \mathcal{T} suggests, that the convergence

$$\left(b_n^{-1} \sum_{k=1}^{\left[n \cdot \frac{d_{[nT]_{-i+1}-1}}{n} \right]} J_k \right)_{i \in \mathbb{N}} \xrightarrow{\mathcal{D}} (D(T \cdot \tilde{\tau}_i -))_{i \in \mathbb{N}} \tag{21}$$

holds. But since (20) could not be proven with the continuous mapping theorem, we use the above arguments again to obtain (21). We note that

$$\begin{aligned}
 b_n^{-1} \sum_{l=1}^{d_{[nT]-i+1}-1} J_l &= \sum_{l=1}^{[nT]} b_n^{-1} J_{[nT]-l+1:[nT]} \mathbf{1}_{\widehat{d}_{[nT]-l+1} \leq d_{[nT]-i+1}-1} \\
 &= \sum_{l=1}^{[nT]} b_n^{-1} J_{[nT]-l+1:[nT]} \mathbf{1}_{n^{-1} \widehat{d}_{[nT]-l+1} < n^{-1} d_{[nT]-i+1}} \\
 &\xrightarrow{\mathcal{D}} \sum_{l \in \mathbb{N}} \widehat{\eta}^{-1}(T^{-1} \widehat{\Gamma}_l) \mathbf{1}_{T \cdot \widehat{\tau}_l < T \cdot \widetilde{\tau}_i}
 \end{aligned} \tag{22}$$

holds. Again we identify the limit on the right-hand side in (22) as a series representation of $D(T \cdot \widetilde{\tau}_i -)$. As already stated this yields the desired result (15), which completes the proof. \square

5. Scaling limits of coupled CTRWs

In this section we are now able to identify the scaling limits of coupled CTRWs using the limit theorems for their associated point processes stated in Lemma 4. We need to introduce the set

$$\mathcal{S} := \{T \in \mathbb{R}^+ : \mathbb{P}(D(T \cdot \tau_i) = T) = 0 \text{ for all } i \in \mathbb{N}\}$$

for technical reasons. Due to self-similarity, one can easily show that the equality

$$D(xt \cdot \tau_i) \stackrel{\mathcal{D}}{=} x^{1/\alpha} D(t \cdot \tau_i)$$

holds for all $i \in \mathbb{N}$ and $x, t \in \mathbb{R}^+$. Hence the set \mathcal{S} is dense in \mathbb{R}^+ .

Theorem 5. *Let $E(t) := \inf\{x > 0 : D(x) > t\}$ denote the hitting-time process associated with $D(\cdot)$. Then convergence of the backward- and forward-coupled CTRW*

$$\begin{aligned}
 &\sum_{k=1}^{N_t b_n} (A_n \mathbf{X}_k - \mathbb{E}(A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| \leq \tau}) \\
 &\xrightarrow{\mathcal{D}} \lim_{\varepsilon \downarrow 0} \left(\sum_{D(T \cdot \tau_k) \leq t} \left(\widetilde{\eta}^{\leftarrow}(T^{-1} \Gamma_k, \mathbf{V}_k) \mathbf{V}_k \mathbf{1}_{\widetilde{\eta}^{\leftarrow}(T^{-1} \Gamma_k, \mathbf{V}_k) \geq \varepsilon} \right) \right. \\
 &\quad \left. - E(t) \int_{\varepsilon \leq \|x\| \leq \tau} x \, d\eta(x) \right)
 \end{aligned} \tag{23}$$

and

$$\begin{aligned}
 &\sum_{k=1}^{N_t b_n + 1} (A_n \mathbf{X}_k - \mathbb{E}(A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| \leq \tau}) \\
 &\xrightarrow{\mathcal{D}} \lim_{\varepsilon \downarrow 0} \left(\sum_{D(T \cdot \tau_k^-) \leq t} \left(\widetilde{\eta}^{\leftarrow}(T^{-1} \Gamma_k, \mathbf{V}_k) \mathbf{V}_k \mathbf{1}_{\widetilde{\eta}^{\leftarrow}(T^{-1} \Gamma_k, \mathbf{V}_k) \geq \varepsilon} \right) \right. \\
 &\quad \left. - E(t) \int_{\varepsilon \leq \|x\| \leq \tau} x \, d\eta(x) \right)
 \end{aligned} \tag{24}$$

holds in $D([0, T], \mathbb{R}^d)$ for every $T \in \mathcal{S}$ and every $\tau > 0$ such that $\eta(\mathbb{S}_\tau^{d-1}) = 0$.

Proof. Choose $T \in \mathcal{S}$ arbitrary. Similar to (12) we define another a.s. continuous restriction functional

$$\begin{aligned} \tilde{\pi}' &: M_p([0, \infty) \times [-\infty, \infty]^d \setminus \{0\}) \rightarrow M_p([0, \infty) \times [-\infty, \infty]^d \setminus \mathbb{K}_\varepsilon^d), \\ \tilde{\pi}'(m) &:= m_{|[0, \infty) \times [-\infty, \infty]^d \setminus \mathbb{K}_\varepsilon^d} \end{aligned}$$

for $\varepsilon > 0$ such that $\eta(S_\varepsilon^{d-1}) = 0$. Moreover we define the summation functional

$$\begin{aligned} \chi &: M_p([0, \infty) \times [-\infty, \infty]^d \setminus \mathbb{K}_\varepsilon^d) \rightarrow \overline{D}([0, T], \mathbb{R}^d) \\ \chi \left(\sum_{k \in \mathbb{N}} \varepsilon_{(t_k, x_k)} \right) (t) &= \left(\sum_{t_k \leq t} x_k \right)_{t \in [0, T]} \end{aligned}$$

which is a.s. continuous in the point

$$\sum_{k \in \mathbb{N}} \varepsilon_{(D(T \cdot \tau_k), \tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \mathbf{V}_k) \mathbf{1}_{\tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) > \varepsilon}}$$

for every $T \in \mathcal{S}$. A proof of the continuity of χ is given in [30, Section 7.2.3] for the case $d = 1$ and can easily be modified to hold for $d \geq 1$. So we apply the a.s. continuous mapping $\chi \circ \pi$ to the associated point processes in Lemma 4. Considering the equality $\{T_n \leq t\} = \{N_t \geq n\}$ we receive

$$\begin{aligned} \chi \circ \pi \left(\sum_{k=1}^{\lfloor nT \rfloor} \varepsilon_{(b_n^{-1}T_k, A_n \mathbf{X}_k)} \right) (t) &= \left(\sum_{b_n^{-1}T_k \leq t} A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| \geq \varepsilon} \right)_{t \in [0, T]} \\ &= \left(\sum_{k=1}^{N_{tb_n}} A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| \geq \varepsilon} \right)_{t \in [0, T]} \\ &\xrightarrow{\mathcal{D}} \chi \circ \pi \left(\sum_{k \in \mathbb{N}} \varepsilon_{(D(T \cdot \tau_k), \tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \mathbf{V}_k)} \right) \\ &= \sum_{D(T \cdot \tau_k) \leq t} \tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \mathbf{V}_k \mathbf{1}_{\tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \geq \varepsilon} \quad (25) \end{aligned}$$

in $D([0, T], \mathbb{R}^d)$. To study the centering constants we use the convergence

$$\left(\frac{N_{tb_n}}{n} \right)_{t \geq 0} \xrightarrow{\mathcal{D}} (E(t))_{t \geq 0}$$

in $D([0, \infty), \mathbb{R}^+)$, proved in Corollary 3.4 of [26]. Considering (11) this yields

$$\begin{aligned} \left(\sum_{k=1}^{N_{tb_n}} \mathbb{E} (A_n \mathbf{X}_k \mathbf{1}_{[\varepsilon, \tau]}(\|A_n \mathbf{X}_k\|)) \right)_{t \geq 0} &= \left(\frac{N_{tb_n}}{n} \int_{\varepsilon \leq \|x\| \leq \tau} x \, n d\mathbb{P}^{A_n \mathbf{X}_1}(x) \right)_{t \geq 0} \\ &\xrightarrow{\mathcal{D}} \left(E(t) \int_{\varepsilon \leq \|x\| \leq \tau} x \, d\eta(x) \right)_{t \geq 0} \end{aligned}$$

in $D([0, \infty), \mathbb{R}^+)$. Since the process $E(\cdot)$ has a.s. continuous sample paths, Theorem 4.1 in [40] allows us to put this and (25) together. We obtain

$$\begin{aligned} & \left(\sum_{k=1}^{N_{tb_n}} (A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| \geq \varepsilon} - \mathbb{E}(A_n \mathbf{X}_k \mathbf{1}_{\varepsilon \leq \|A_n \mathbf{X}_k\| \leq \tau}) \right)_{t \in [0, T]} \\ & \xrightarrow{\mathcal{D}} \left(\sum_{D(T \cdot \tau_k) \leq t} (\tilde{\eta}^{\leftarrow}(\Gamma_k, \mathbf{V}_k) \mathbf{V}_k \mathbf{1}_{\tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \geq \varepsilon}) - E(t) \int_{\varepsilon \leq \|x\| \leq \tau} x \, d\eta(x) \right)_{t \in [0, T]} \end{aligned}$$

in $D([0, T], \mathbb{R}^d)$. Taking limits as $\varepsilon \downarrow 0$ this yields the desired result (23). By Theorem 4.2 of [5] it remains to show

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{0 \leq t \leq T} \left\| \sum_{k=1}^{N_{tb_n}} (A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| \geq \varepsilon} - \mathbb{E}(A_n \mathbf{X}_k \mathbf{1}_{\varepsilon \leq \|A_n \mathbf{X}_k\| \leq \tau}) \right\| \right. \\ & \quad \left. - \sum_{k=1}^{N_{tb_n}} (A_n \mathbf{X}_k - \mathbb{E}(A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| \leq \tau}) \cdot \right\| \geq \delta \right) = 0 \end{aligned}$$

for all $\delta > 0$. Using a version of the Kolmogorov-inequality for integrable stopping times given in the Appendix and the norm-inequality $\|\cdot\| \leq \|\cdot\|_1$ we obtain

$$\begin{aligned} & \mathbb{P} \left(\sup_{0 \leq t \leq T} \left\| \sum_{k=1}^{N_{tb_n}} (A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| \geq \varepsilon} - \mathbb{E}(A_n \mathbf{X}_k \mathbf{1}_{\varepsilon \leq \|A_n \mathbf{X}_k\| \leq \tau}) \right\| \right. \\ & \quad \left. - \sum_{k=1}^{N_{tb_n}} (A_n \mathbf{X}_k - \mathbb{E}(A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| \leq \tau}) \right\| \geq \delta \right) \\ & \leq \mathbb{P} \left(\max_{1 \leq j \leq N_{Tb_n}} \left\| \sum_{k=1}^j A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| < \varepsilon} - \mathbb{E}(A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| < \varepsilon}) \right\|_1 \geq \delta \right) \\ & \leq \mathbb{P} \left(\sum_{i=1}^d \max_{1 \leq j \leq N_{Tb_n}} \left| \sum_{k=1}^j (A_n \mathbf{X}_k)^{(i)} \mathbf{1}_{\|A_n \mathbf{X}_k\| < \varepsilon} - \mathbb{E} \left((A_n \mathbf{X}_k)^{(i)} \mathbf{1}_{\|A_n \mathbf{X}_k\| < \varepsilon} \right) \right| \geq \delta \right) \\ & \leq \mathbb{P} \left(\bigcup_{i=1}^d \max_{1 \leq j \leq N_{Tb_n}} \left| \sum_{k=1}^j (A_n \mathbf{X}_k)^{(i)} \mathbf{1}_{\|A_n \mathbf{X}_k\| < \varepsilon} - \mathbb{E} \left((A_n \mathbf{X}_k)^{(i)} \mathbf{1}_{\|A_n \mathbf{X}_k\| < \varepsilon} \right) \right| \geq \frac{\delta}{d} \right) \\ & \leq \sum_{i=1}^d \mathbb{P} \left(\max_{1 \leq j \leq N_{Tb_n} + 1} \left| \sum_{k=1}^j (A_n \mathbf{X}_k)^{(i)} \mathbf{1}_{\|A_n \mathbf{X}_k\| < \varepsilon} - \mathbb{E} \left((A_n \mathbf{X}_k)^{(i)} \mathbf{1}_{\|A_n \mathbf{X}_k\| < \varepsilon} \right) \right| \geq \frac{\delta}{d} \right) \\ & \leq \left(\frac{\delta}{d} \right)^{-2} \mathbb{E} (N_{Tb_n} + 1) \sum_{i=1}^d \text{Var} \left((A_n \mathbf{X}_1)^{(i)} \mathbf{1}_{\|A_n \mathbf{X}_1\| < \varepsilon} \right) \\ & \leq \left(\frac{\delta}{d} \right)^{-2} \mathbb{E} (N_{Tb_n} + 1) \sum_{i=1}^d \mathbb{E} \left(\left((A_n \mathbf{X}_1)^{(i)} \mathbf{1}_{\|A_n \mathbf{X}_1\| < \varepsilon} \right)^2 \right), \end{aligned}$$

where $x^{(n)}$ denotes the n -th coordinate of the vector x . Note that we have to take $N_{tb_n} + 1$ since N_{tb_n} does not fulfill the conditions of Lemma 7. Now Theorem 9 in [24] states that $\mathbb{E}(N_t + 1)$

can asymptotically be expressed by the integrated tail of the distribution function of J_1

$$\mathbb{E}(N_t + 1) \Gamma(2 - \alpha) \Gamma(1 + \alpha) \sim \frac{t}{\int_0^t (1 - F_{J_1}(x)) dx},$$

where Γ denotes the gamma-function. By Karamata’s Theorem the limit behavior of the function $t \mapsto \int_0^t (1 - F_{J_1}(s)) ds$ can be expressed by F_{J_1}

$$\int_0^t (1 - F_{J_1}(s)) ds \sim \frac{t(1 - F_{J_1}(t))}{1 - \alpha}.$$

Putting this together we obtain

$$\mathbb{E}(N_t + 1) \sim (1 - F_{J_1}(t))^{-1} \cdot \Gamma(1 - \alpha)^{-1} \cdot \Gamma(1 + \alpha)^{-1}.$$

Defining $C := (\Gamma(1 - \alpha) \Gamma(1 + \alpha) \eta^\alpha((T, \infty)))^{-1}$ for abbreviation, the inequality $|x^{(n)}| \leq \|x\|$ yields

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \left(\frac{\delta}{d}\right)^{-2} \lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{E}(N_{Tb_n} + 1) \sum_{i=1}^d \mathbb{E} \left(\left((A_n \mathbf{X}_1)^{(i)} \mathbf{1}_{\|A_n \mathbf{X}_1\| < \varepsilon} \right)^2 \right) \\ & \leq \lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \left(\frac{\delta}{d}\right)^{-2} C \cdot \sum_{i=1}^d \lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} n \mathbb{E} \left(\left((A_n \mathbf{X}_1)^{(i)} \right)^2 \mathbf{1}_{|(A_n \mathbf{X}_1)^{(i)}| < \varepsilon} \right). \end{aligned}$$

So it remains to show

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} n \mathbb{E} \left(\left((A_n \mathbf{X}_1)^{(i)} \right)^2 \mathbf{1}_{|(A_n \mathbf{X}_1)^{(i)}| < \varepsilon} \right) = 0$$

for all $1 \leq i \leq d$. Defining the tail and truncated second moment of \mathbf{X}_1 in direction v by

$$V(r, v) := \mathbb{P}(|\langle \mathbf{X}_1, v \rangle| > r), \quad U(r, v) := \mathbb{E}(\langle \mathbf{X}_1, v \rangle^2 \mathbf{1}_{|\langle \mathbf{X}_1, v \rangle| < r})$$

for every $v \in \mathbb{S}^{d-1}$ and every $r > 0$ an easy calculation shows that

$$\mathbb{E} \left(\left((A_n \mathbf{X}_1)^{(i)} \right)^2 \mathbf{1}_{|(A_n \mathbf{X}_1)^{(i)}| < \varepsilon} \right) = r_n^2 U(r_n^{-1} \varepsilon, v_n)$$

holds. Here $r_n > 0$ and $v_n \in \mathbb{S}^{d-1}$ are taken such that $A_n^* e_i = r_n v_n$ holds for every $n \in \mathbb{N}$, where A^* is the adjoint of A and e_1, \dots, e_d denotes the standard basis of \mathbb{R}^d . With this notation we have to analyze

$$n \cdot r_n^2 U(r_n^{-1} \varepsilon, v_n) = \varepsilon^2 \frac{U(r_n^{-1} \varepsilon, v_n)}{\varepsilon^2 r_n^{-2} V(r_n^{-1} \varepsilon, v_n)} \cdot \frac{V(r_n^{-1} \varepsilon, v_n)}{V(\varepsilon^{-1} r_n^{-1} \varepsilon, v_n)} \cdot n \cdot V(r_n^{-1}, v_n). \tag{26}$$

One easily verifies that the third factor in (26) is bounded by $\eta(\{x \in \mathbb{R}^d : |x^{(i)}| > 1\})$. Since the real parts $a_1 \leq \dots \leq a_d$ of all eigenvalues of the operator E are greater than $1/2$ we can find $\tilde{\varepsilon} > 0$ such that $2 - \tilde{\varepsilon} - a_1^{-1} > 0$ holds. So an application of Theorem 6.3.4 of [25] yields the existence of a constant C_1 , such that for $n \in \mathbb{N}$ large enough the second factor in (26) is bounded by $C_1 \varepsilon^{2 - \tilde{\varepsilon} - a_1^{-1}}$. Finally, Corollary 6.3.9 in [25] yields that the first factor in (26) is bounded by a constant C_2 . Putting things together this completes the proof.

The proof of the convergence (24) works the same way. \square

Theorem 5 provides a series representation for the limit distribution. This representation might be useful for simulation purposes. Now it is of considerable interest to identify the scaling limits with the ones stated in [37]. Let $\mathbf{A}(t-)^+$ denote the right-continuous version

of the process $\mathbf{A}(t-)$, i.e. $\mathbf{A}(t-)^+$ is an element of $D([0, \infty), \mathbb{R}^d)$. Using a continuous mapping approach on $D([0, \infty), \mathbb{R}^d)$, Straka and Henry [37] show that the limit laws in (23) and (24) coincide with the distribution of $\mathbf{A}(E(t)-)^+$, respectively $\mathbf{A}(E(t))$. More precisely, Straka and Henry follow a more general approach and consider triangular arrays instead of i.i.d. sequences following [27] and they determine the joint scaling limit of the time process together with the CTRW. Note that for an uncoupled CTRW, where waiting times and jumps are independent, the processes $(\mathbf{A}(t))_{t \geq 0}$ and $(E(t))_{t \geq 0}$ are independent and thus the two limiting processes $(\mathbf{A}(E(t)-)^+)_{t \geq 0}$ and $(\mathbf{A}(E(t)))_{t \geq 0}$ coincide in distribution. This equality fails for a coupled CTRW and differences in the two limiting processes are illustrated in [14], where the limiting distributions, their Fourier–Laplace transforms and the corresponding governing pseudo differential equations for their densities are given in terms of the joint distribution of $(D(t), \mathbf{A}(t))_{t \geq 0}$ and its Lévy exponent.

Our arguments for the identification of the scaling limits are based on the following equalities:

$$\{D(x) < t\} = \{x < E(t)\}, \quad \{D(x-) \leq t\} = \{x \leq E(t)\}. \tag{27}$$

The left-hand side of (27) is already proven in (3.2) of [26]. For the proof of the right-hand side assume $D(x-) \leq t$ holds. So we have $D(y) \leq t$ for all $y < x$. Hence $x \leq E(t)$. Otherwise if $D(x-) > t$ holds, there exists an $\varepsilon > 0$ such that $D(y) > t$ holds for all $y \geq x - \varepsilon$. Hence $E(t) \leq x - \varepsilon < x$.

Corollary 6. *The convergence*

$$\sum_{k=1}^{N_{tbn}} (A_n \mathbf{X}_k - \mathbb{E}(A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| \leq \tau}) \xrightarrow{\mathcal{D}} \mathbf{A}(E(t)-)^+ \tag{28}$$

and

$$\sum_{k=1}^{N_{tbn}+1} (A_n \mathbf{X}_k - \mathbb{E}(A_n \mathbf{X}_k \mathbf{1}_{\|A_n \mathbf{X}_k\| \leq \tau}) \xrightarrow{\mathcal{D}} \mathbf{A}(E(t)) \tag{29}$$

holds in $D([0, T], \mathbb{R}^d)$ for every $T \in \mathcal{S}$ and every $\tau > 0$ such that $\eta(\mathbb{S}_\tau^{d-1}) = 0$.

Proof. The convergence (29) can easily be verified applying the right-hand side of (27) to the series representation (4). To prove (28) we apply the left-hand side of (27) to (23) and obtain

$$\begin{aligned} & \sum_{k=1}^{N_{tbn}} (A_n \mathbf{X}_k - \mathbb{E}_\tau(A_n \mathbf{X}_k)) \\ & \xrightarrow{\mathcal{D}} \lim_{\varepsilon \downarrow 0} \left(\sum_{D(T \cdot \tau_k) \leq t} \left(\tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \mathbf{V}_k \mathbf{1}_{\tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \geq \varepsilon} \right) \right. \\ & \quad \left. - E(t) \int_{\varepsilon \leq \|x\| \leq \tau} x \, d\eta(x) \right) \\ & = \lim_{\varepsilon \downarrow 0} \left(\sum_{T \cdot \tau_k < E(t)} \left(\tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \mathbf{V}_k \mathbf{1}_{\tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \geq \varepsilon} \right) \right. \\ & \quad \left. + \sum_{D(T \cdot \tau_k) = t} \left(\tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \mathbf{V}_k \mathbf{1}_{\tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \geq \varepsilon} \right) - E(t) \int_{\varepsilon \leq \|x\| \leq \tau} x \, d\eta(x) \right) \tag{30} \end{aligned}$$

in $D([0, T], \mathbb{R}^d)$. As we already stated

$$\lim_{\varepsilon \downarrow 0} \left(\sum_{T \cdot \tau_k < E(t)} \left(\tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \mathbf{V}_k \mathbf{1}_{\tilde{\eta}^{\leftarrow}(T^{-1}\Gamma_k, \mathbf{V}_k) \geq \varepsilon} \right) - E(t) \int_{\varepsilon \leq \|x\| \leq \tau} x \, d\eta(x) \right)$$

is a series representation of $\mathbf{A}(E(t)-)$. The extra summands only have to be considered if a jump occurs at a time t with $D(T \cdot \tau_k) = t$. This yields the right-continuity of the limit and we have proven (28). \square

Note that, since \mathcal{S} is dense in \mathbb{R}^+ , due to Theorem 2 in [22] together with Theorem 14.5 in [5] the scaling limits in Theorem 5 exist as elements of $D([0, \infty), \mathbb{R}^d)$ and coincide in distribution with the corresponding limits in Corollary 6.

Acknowledgment

The authors are grateful to an anonymous referee for several helpful comments leading to an improvement of the presentation.

Appendix. A generalization of Kolmogorov’s inequality

The following generalization of Kolmogorov’s inequality can be shown by standard techniques. However, we were not able to find a suitable proof in the literature and will only give a sketch of proof.

Lemma 7. *Let $(Y_n)_{n \in \mathbb{N}}$ be i.i.d. with $\mathbb{E}(Y_1) = 0$ and T be an \mathbb{N}_0 -valued integrable stopping time with respect to the filtration $\mathcal{F}_n := \sigma(Y_1, \dots, Y_n)$. Then*

$$\mathbb{P} \left(\max_{1 \leq k \leq T} \left| \sum_{j=1}^k Y_j \right| \geq \delta \right) \leq \delta^{-2} \cdot \mathbb{E}(T) \cdot \text{Var}(Y_1)$$

holds for all $\delta > 0$.

Proof. First we restrict our attention to the truncated stopping time $T \wedge n$. An easy calculation shows that

$$\mathbb{E} \left(\sum_{k=1}^{T \wedge n} (S_k - S_{k-1})^2 \right) = \mathbb{E}(S_{T \wedge n}^2)$$

holds, where $S_n := \sum_{k=1}^n Y_k$ denotes the n -th partial sum. Defining

$$M_0 := 0, \quad M_{k+1} := \begin{cases} S_{k+1}, & \text{if } \max_{1 \leq j \leq k} |S_j| < \delta \\ M_k, & \text{else} \end{cases}$$

the same calculation shows

$$\mathbb{E} \left(\sum_{k=1}^{T \wedge n} (M_k - M_{k-1})^2 \right) = \mathbb{E}(M_{T \wedge n}^2) - 2 \cdot \mathbb{E} \left(\sum_{k \in \mathbb{N}} (M_k - M_{k-1}) S_{k-1} \mathbf{1}_{k \leq T \wedge n} \right).$$

Using the definition of M_n one can show

$$\mathbb{E} \left(\sum_{k=1}^{T \wedge n} (M_k - M_{k-1})^2 \right) = \mathbb{E}(M_{T \wedge n}^2).$$

Now $|M_k - M_{k-1}| \leq |S_k - S_{k-1}|$ holds for all $k \in \mathbb{N}$. Hence the Markov-inequality yields

$$\begin{aligned} \mathbb{P} \left(\max_{1 \leq k \leq T \wedge n} |S_k| \geq \delta \right) &= \mathbb{P} (|M_{T \wedge n}| \geq \delta) \leq \delta^{-2} \mathbb{E} \left(M_{T \wedge n}^2 \right) \\ &= \delta^{-2} \mathbb{E} \left(\sum_{k=1}^{T \wedge n} (M_k - M_{k-1})^2 \right) \\ &\leq \delta^{-2} \mathbb{E} \left(\sum_{k=1}^{T \wedge n} (S_k - S_{k-1})^2 \right) = \delta^{-2} \mathbb{E} \left(S_{T \wedge n}^2 \right). \end{aligned}$$

An application of Wald's inequality yields the desired result for $T \wedge n$. The generalization for the stopping time T follows by the martingale convergence theorem. \square

Remark 8. Let $(Z_n)_{n \in \mathbb{N}}$ be a sequence such that $(Y_n, Z_n)_{n \in \mathbb{N}}$ is i.i.d. Then Lemma 7 also holds replacing \mathcal{F}_n by $\mathcal{F}_n := \sigma((Y_1, Z_1), \dots, (Y_n, Z_n))$.

References

- [1] A. Barczyk, A. Janssen, M. Pauly, The asymptotics of L -statistics for non i.i.d. variables with heavy tails, *Probab. Math. Statist.* 31 (2011) 285–299.
- [2] P. Becker-Kern, Random sums of independent random vectors attracted by (semi)-stable hemigroups, *J. Appl. Anal.* 10 (2004) 83–104.
- [3] P. Becker-Kern, M.M. Meerschaert, H.P. Scheffler, Limit theorems for coupled continuous time random walks, *Ann. Probab.* 32 (2004) 730–756. Correction: *Ann. Probab.* 40 (2012) 890–891.
- [4] J. Bertoin, *Lévy Processes*, Cambridge University Press, Cambridge, 1996.
- [5] P. Billingsley, *Convergence of Probability Measures*, in: *Wiley Series in Probability and Statistics*, New York, 1968.
- [6] A.V. Chechkin, M. Hofmann, I.M. Sokolov, Continuous-time random walk with correlated waiting times, *Phys. Rev. E* 80 (2009) 031112.
- [7] P. Feigin, M.F. Krantz, S.I. Resnick, Parameter estimation for moving averages with positive innovations, *Ann. Appl. Probab.* 6 (1996) 1157–1190.
- [8] T.S. Ferguson, M.J. Klass, A representation of independent increment processes without Gaussian components, *Ann. Math. Stat.* 43 (1972) 1634–1643.
- [9] M. Finkelstein, H.G. Tucker, A necessary and sufficient condition for convergence in law of random sums of random variables under nonrandom centering, *Proc. Amer. Math. Soc.* 107 (1989) 1061–1070.
- [10] B. Gnedenko, A.V. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley, Reading, MA, 1968.
- [11] M.G. Hahn, W.N. Hudson, J.A. Veeh, Operator stable laws: series representation and domains of normal attraction, *J. Theoret. Probab.* 2 (1989) 3–35.
- [12] A. Janssen, D.M. Mason, Non-Standard Rank Tests, in: *Lect. Notes Stat.*, vol. 65, Springer, New York, 1990.
- [13] M. Jara, T. Komorowski, Limit theorems for some continuous-time random walks, *Adv. Appl. Probab.* 43 (2011) 782–813.
- [14] A. Jurliewicz, P. Kern, M.M. Meerschaert, H.P. Scheffler, Fractional governing equations for coupled random walks, *Comput. Math. Appl.* 64 (10) (2012) 3021–3036.
- [15] A. Jurliewicz, M.M. Meerschaert, H.P. Scheffler, Cluster continuous time random walks, *Studia Math.* 205 (2011) 13–30.
- [16] O. Kallenberg, *Random Measures*, Akad.-Verl., Berlin, 1986.
- [17] J. Klafter, A. Blumen, M.F. Shlesinger, Stochastic pathways to anomalous diffusion, *Phys. Rev. A* 35 (1987) 3081–3085.
- [18] M. Kotulski, Asymptotic distributions of the continuous time random walks: a probabilistic approach, *J. Stat. Phys.* 81 (1995) 777–792.
- [19] E.L. Lehmann, J.P. Romano, *Testing Statistical Hypotheses*, Springer, New York, 2005.
- [20] R. LePage, Multidimensional infinitely divisible variables and processes part II, *Lecture Notes in Math.* 860 (1981) 279–284.

- [21] R. LePage, M. Woodroffe, J. Zinn, Convergence to a stable distribution via order statistics, *Ann. Probab.* 9 (1981) 624–632.
- [22] T. Lindvall, Weak convergence of probability measures and random functions in the function space $D[0, \infty)$, *J. Appl. Probab.* 10 (1973) 109–121.
- [23] M. Magdziarz, R. Metzler, W. Szczotka, P. Zebrowski, Correlated continuous-time random walks—scaling limits and Langevin picture, *J. Stat. Mech.* (2012) P04010.
- [24] F. Mallor, E.A.M. Omeý, *Univariate and Multivariate Weighted Renewal Theory*, Public University of Navarra, Pamplona, 2006.
- [25] M.M. Meerschaert, H.P. Scheffler, *Limit Distributions for Sums of Independent Random Vectors*, in: *Wiley Series in Probability and Statistics*, Wiley, New York, 2001.
- [26] M.M. Meerschaert, H.P. Scheffler, Limit theorems for continuous-time random walks with infinite mean waiting times, *J. Appl. Probab.* 41 (2004) 623–638.
- [27] M.M. Meerschaert, H.P. Scheffler, Triangular array limits for continuous time random walks, *Stochastic Process. Appl.* 118 (2008) 1606–1633; *Stochastic Process. Appl.* 120 (2010) 2520–2521 (erratum).
- [28] R. Metzler, J. Klafter, The random walk’s guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.* 339 (2000) 1–77.
- [29] E. Montroll, G.H. Weiss, Random walks on lattices. II, *J. Math. Phys.* 6 (1965) 167–181.
- [30] S.I. Resnick, *Heavy-Tail Phenomena*, Springer, New York, 2007.
- [31] J. Rosiński, Series representations of Lévy processes from the perspective of point processes, in: O.E. Barndorff-Nielsen, et al. (Eds.), *Lévy Processes—Theory and Applications*, Birkhäuser, Boston, 2001, pp. 401–415.
- [32] E. Scalas, Five years of continuous-time random walks in econophysics, in: *Proc. WEHIA*, vol. 55, 2004, pp. 223–230.
- [33] H.P. Scheffler, Series representation for operator semistable laws and domains of normal attraction, *J. Math. Sci.* 92 (1998) 4062–4084.
- [34] R. Schumer, M.M. Meerschaert, B. Baeumer, Fractional advection–dispersion equations for modeling transport at the Earth surface, *J. Geophys. Res.* 114 (2009) F00A07.
- [35] M. Shlesinger, J. Klafter, Y.M. Wong, Random walks with infinite spatial and temporal moments, *J. Stat. Phys.* 27 (1982) 499–512.
- [36] A. Stanislavsky, K. Weron, Anomalous diffusion with under- and overshooting subordination: a competition between the very large jumps in physical and operational times, *Phys. Rev. E* 82 (2010) 051120.
- [37] P. Straka, B.I. Henry, Lagging and leading coupled continuous time random walks, renewal times and their joint limits, *Stochastic Process. Appl.* 121 (2011) 324–336.
- [38] W. Szczotka, P. Żebrowski, On fully coupled continuous time random walks, *Appl. Math.* 39 (2012) 87–102. (Warsaw).
- [39] K. Weron, A. Jurlewicz, M. Magdziarz, A. Weron, J. Trzmiel, Overshooting and undershooting subordination scenario for fractional two-power-law relaxation responses, *Phys. Rev. E* 81 (2010) 041123.
- [40] W. Whitt, Some useful functions for functional limit theorems, *Math. Oper. Res.* 5 (1980) 67–85.