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On the dynamic programming principle for uniformly nondegenerate stochastic differential games in domains

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Abstract

We prove the dynamic programming principle for uniformly nondegenerate stochastic differential games in the framework of time-homogeneous diffusion processes considered up to the first exit time from a domain. The zeroth-order "coefficient" and the "free" term are only assumed to be measurable. In contrast with previous results established for constant stopping times we allow arbitrary stopping times and randomized ones as well. The main assumption, which will be removed in a subsequent article, is that there exists a sufficiently regular solution of the Isaacs equation.

MSC: 35J60; 49N70; 91A15

Keywords: Dynamic programming principle; Stochastic games; Isaacs equation

1. Introduction

The dynamic programming principle is one of the basic tools in the theory of controlled diffusion processes. In the early 70s it allowed one to obtain results about the unique solvability in classes of differentiable functions of Bellman's equations, which, for about ten years, were the only known results for more or less general fully nonlinear second-order elliptic and parabolic equations.

In this paper we will be only dealing with the dynamic programming principle for stochastic differential games. Concerning all other aspects of the theory of stochastic differential games we refer the reader to [1,2,4,13,14] and the references therein.

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It seems to the author that Fleming and Souganidis in [2] were the first authors who proved the dynamic programming principle with nonrandom stopping times for stochastic differential games in the whole space on a finite time horizon. They used rather involved constructions to overcome some measure-theoretic difficulties, a technique somewhat resembling the one in Nisio [13], and the theory of viscosity solutions.

In [4] Kovats considers time-homogeneous stochastic differential games in a "weak" formulation in smooth *domains* and proves the dynamic programming principle again with nonrandom stopping times. He uses approximations of policies by piece-wise constant ones and proceeds similarly to [13].

Święch in [14] reverses the arguments in [2] and proves the dynamic programming principle for time-homogeneous stochastic differential games in the whole space with constant stopping times "directly" from knowing that the viscosity solutions exist. His method is quite similar to the so-called verification principle in the theory of controlled diffusion processes.

It is also worth mentioning the paper [1] by Buckdahn and Li where the dynamic programming principle for constant stopping times in the time-in homogeneous setting in the whole space is derived by using the theory of backward forward stochastic equations.

Basically, we adopt the strategy of Święch [14] which is based on using the fact that in many cases the Isaacs equation has a sufficiently regular solution. In [14] viscosity solutions are used and we rely on classical ones.

The main emphasis of [2,4,13,14] is on proving that (upper and lower) value functions for stochastic differential games are viscosity solutions of the corresponding Isaacs equations and the dynamic programming principle is used just as a tool to do that. In our setting the zeroth-order coefficient and the running payoff function can be just measurable and in this situation neither our methods nor the methods based on the notion of viscosity solution seem to be of much help while proving that the value function is a viscosity solution.

Our main future goal is to develop some tools which would allow us in a subsequent article to show that the value functions are of class $C^{0,1}$, provided that the data are there, for *possibly degenerate* stochastic differential games without assuming that the zeroth-order coefficient is large enough negative (see [12]). On the way to achieve this goal one of the main steps, apart from proving the dynamic programming principle, is to prove certain representation formulas which will be derived in a subsequent article (see [11]) from our Theorems 4.1 and 4.2, in the first of which the process is not assumed to be uniformly nondegenerate. Another important ingredient consists of approximations results allowing us to approximate stochastic differential games with the ones for which the corresponding Isaacs equations have sufficiently regular solutions. This issue will be addressed in a subsequent article (see [11]).

One of the main results of the present article, Theorem 2.2, is about the dynamic programming principle in a very general form including stopping and randomized stopping times. It is proved under the assumption that the corresponding Isaacs equations have sufficiently regular solutions.

In Theorem 2.3 we prove the Hölder continuity of the value function in our case where the zeroth-order coefficient and the running payoff function can be discontinuous.

Theorem 2.2 concerns time-homogeneous stochastic differential games unlike the time in homogeneous in [2] and generalizes the *corresponding* results of [14,4], where however degenerate case is not excluded.

The article is organized as follows. In Section 2 we state our main results to which actually, as we have pointed out implicitly above, belong Theorems 4.1 and 4.2 reminding the verification principle from the theory of controlled diffusion processes. The main technical tool for proving these theorems is laid out in a rather long Section 3 for processes which may be degenerate.

We prove there Theorems 3.1–3.3. In a short Section 4 we give their versions for uniformly nondegenerate case. These versions look stronger but Theorem 4.2 is proved only for uniformly nondegenerate case. In Section 5 we prove an auxiliary result which allows us to investigate the behavior of the value function near the boundary. In the final short Section 6 we combine previous results and prove Theorem 2.2.

2. Main results

Let $\mathbb{R}^d = \{x = (x_1, \dots, x_d)\}$ be a d-dimensional Euclidean space and let $d_1 \geq d$ and $k \geq 1$ be integers. Assume that we are given separable metric spaces A, B, and $\mathcal{P}$ and let, for each $\alpha \in A$, $\beta \in B$, and $p \in \mathcal{P}$ the following functions on $\mathbb{R}^d$ be given:

(i) $d \times d_1$ matrix-valued $\sigma^{\alpha\beta}(p, x) = \sigma(\alpha, \beta, p, x) = (\sigma^{\alpha\beta}_{ij}(p, x))$,

(ii) $\mathbb{R}^d$-valued $b^{\alpha\beta}(p, x) = b(\alpha, \beta, p, x) = (b^{\alpha\beta}_i(p, x))$, and

(iii) real-valued functions $c^{\alpha\beta}(p, x) = c(\alpha, \beta, p, x)$, $f^{\alpha\beta}(p, x) = f(\alpha, \beta, p, x)$, and $g(x)$.

Introduce

$$a^{\alpha\beta}(p, x) := (1/2)\sigma^{\alpha\beta}(p, x)(\sigma^{\alpha\beta}(p, x))^*,$$

fix a $\bar{p} \in \mathcal{P}$, and set

$$(\bar{\sigma}, \bar{a}, \bar{b}, \bar{c}, \bar{f})^{\alpha\beta}(x) = (\sigma, a, b, c, f)^{\alpha\beta}(\bar{p}, x).$$

Assumption 2.1. (i) All the above functions are continuous with respect to $\beta \in B$ for each (α, p, x) and continuous with respect to $\alpha \in A$ uniformly with respect to $\beta \in B$ for each (p, x). Furthermore, they are Borel measurable in (p, x) for each (α, β), the function $g(x)$ is bounded and uniformly continuous on $\mathbb{R}^d$, and $c^{\alpha\beta} \geq 0$.

(ii) The functions $\bar{\sigma}^{\alpha\beta}(x)$ and $\bar{b}^{\alpha\beta}(x)$ are uniformly continuous with respect to x uniformly with respect to $(\alpha, \beta) \in A \times B$ and for any $x \in \mathbb{R}^d$ and $(\alpha, \beta, p) \in A \times B \times \mathcal{P}$

$$\|\sigma^{\alpha\beta}(p, x)\|, |b^{\alpha\beta}(p, x)| \leq K_0,$$

where K_0 is a fixed constants and for a matrix σ we denote $\|\sigma\|^2 = \operatorname{tr} \sigma\sigma^*$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, let $\{\mathcal{F}_t, t \geq 0\}$ be an increasing filtration of σ-fields $\mathcal{F}_t \subset \mathcal{F}$ such that each $\mathcal{F}_t$ is complete with respect to $\mathcal{F}$, P, and let $w_t, t \geq 0$, be a standard d_1-dimensional Wiener process given on Ω such that w_t is a Wiener process relative to the filtration $\{\mathcal{F}_t, t \geq 0\}$.

The set of progressively measurable A-valued processes $\alpha_t = \alpha_t(\omega)$ is denoted by $\mathfrak{A}$. Similarly we define $\mathfrak{B}$ as the set of B-valued progressively measurable functions. By $\mathbb{B}$ we denote the set of $\mathfrak{B}$-valued functions $\beta(\alpha_\cdot)$ on $\mathfrak{A}$ such that, for any $T \in (0, \infty)$ and any $\alpha^1_\cdot, \alpha^2_\cdot \in \mathfrak{A}$ satisfying

$$P(\alpha^1_t = \alpha^2_t \text{ for almost all } t \leq T) = 1, \tag{2.1}$$

we have

$$P(\beta_t(\alpha^1_\cdot) = \beta_t(\alpha^2_\cdot) \text{ for almost all } t \leq T) = 1.$$

We closely follow the standard setup (but not the notation) from [2,4,14] allowing the control processes to depend on the past information contained in $\{\mathcal{F}_t\}$. By the way, in the situation of controlled diffusion processes (not stochastic differential games) these control processes were

first introduced in [6] and turned out to be extremely useful in developing the theory of Bellman's equations.

Definition 2.1. A function $p_t^{\alpha.\beta.} = p_t^{\alpha.\beta.}(\omega)$ given on $\mathfrak{A} \times \mathfrak{B} \times \Omega \times [0, \infty)$ is called *a control adapted process* if, for any $(\alpha., \beta.) \in \mathfrak{A} \times \mathfrak{B}$, it is progressively measurable in (ω, t) and, for any $T \in (0, \infty)$, we have

$$P(p_t^{\alpha^1.\beta^1.} = p_t^{\alpha^2.\beta^2.} \text{ for almost all } t \le T) = 1$$

as long as

$$P(\alpha_t^1 = \alpha_t^2, \beta_t^1 = \beta_t^2 \text{ for almost all } t \le T) = 1.$$

The set of control adapted $\mathcal{P}$-valued processes is denoted by $\mathfrak{P}$.

We fix a $p \in \mathfrak{P}$ (for the rest of the article) and for $\alpha. \in \mathfrak{A}$, $\beta. \in \mathfrak{B}$, and $x \in \mathbb{R}^d$ consider the following Itô equation

$$x_t = x + \int_0^t \sigma^{\alpha_s\beta_s}(p_s^{\alpha.\beta.}, x_s)\, dw_s + \int_0^t b^{\alpha_s\beta_s}(p_s^{\alpha.\beta.}, x_s)\, ds. \tag{2.2}$$

Assumption 2.2. Eq. (2.2) satisfies the usual hypothesis, that is for any $\alpha. \in \mathfrak{A}$, $\beta. \in \mathfrak{B}$, and $x \in \mathbb{R}^d$ it has a unique solution denoted by $x_t^{\alpha.\beta.x}$ and $x_t^{\alpha.\beta.x}$ is a control adapted process for each x.

Remark 2.1. As is well known, Eq. (2.2) satisfies the usual hypothesis if Assumption 2.1 is satisfied and for any $x, y \in \mathbb{R}^d$ and $(\alpha, \beta, p) \in A \times B \times \mathcal{P}$ the monotonicity condition

$$2\langle x - y, b^{\alpha\beta}(p, x) - b^{\alpha\beta}(p, y)\rangle + \|\sigma^{\alpha\beta}(p, x) - \sigma^{\alpha\beta}(p, y)\|^2 \le K_1|x - y|^2 \tag{2.3}$$

holds, where K_1 is a fixed constant. For instance, if $\sigma^{\alpha\beta}(p, x)$ and $b^{\alpha\beta}(p, x)$ are Lipschitz continuous in x with constant independent of α, β, p, then (2.3) holds. If $d = 1$, then (2.3) is satisfied if, for instance, $b^{\alpha\beta}(p, x)$ is a decreasing function and $\sigma^{\alpha\beta}(p, x)$ is Lipschitz continuous in x with constant independent of α, β, p. Even if σ and b are independent of p, this argument shows how control adapted processes may appear.

We discuss a different way in which control adapted processes appear naturally in Remark 2.4.

Take a $\zeta \in C_0^\infty(\mathbb{R}^d)$ with unit integral and for $\varepsilon > 0$ introduce $\zeta_\varepsilon(x) = \varepsilon^{-d}\zeta(x/\varepsilon)$. For locally summable functions $u = u(x)$ on $\mathbb{R}^d$ define

$$u^{(\varepsilon)}(x) = u * \zeta_\varepsilon(x).$$

Assumption 2.3. (i) For any $x \in \mathbb{R}^d$

$$\sup_{(\alpha,\beta)\in A\times B} (|\bar{c}^{\alpha\beta}| + |\bar{f}^{\alpha\beta}|)(x) < \infty. \tag{2.4}$$

(ii) There exist a constant $\delta_1 \in (0, 1]$ and a function $r^{\alpha\beta}(p, x)$ defined on $A \times B \times \mathcal{P} \times \mathbb{R}^d$ with values in $[\delta_1, \delta_1^{-1}]$ such that $r^{\alpha\beta}(\bar{p}, x) \equiv 1$ and on $A \times B \times \mathcal{P} \times \mathbb{R}^d$ we have

$$f^{\alpha\beta}(p, x) = r^{\alpha\beta}(p, x)\bar{f}^{\alpha\beta}(x).$$

(iii) For any bounded domain $D \subset \mathbb{R}^d$ we have

$$\|\sup_{(\alpha,\beta)\in A\times B} |\bar{f}^{\alpha\beta}|\|_{L_d(D)} + \|\sup_{(\alpha,\beta)\in A\times B} \bar{c}^{\alpha\beta}\|_{L_d(D)} < \infty,$$

$$\|\sup_{(\alpha,\beta)\in A\times B} |\bar{f}^{\alpha\beta} - (\bar{f}^{\alpha\beta})^{(\varepsilon)}|\|_{L_d(D)} \to 0,$$

$$\|\sup_{(\alpha,\beta)\in A\times B} |\bar{c}^{\alpha\beta} - (\bar{c}^{\alpha\beta})^{(\varepsilon)}|\|_{L_d(D)} \to 0,$$

as $\varepsilon \downarrow 0$.

(iv) There is a constant $\delta \in (0, 1]$ such that for $\alpha \in A$, $\beta \in B$, $p \in \mathcal{P}$, and $x, \lambda \in \mathbb{R}^d$ we have

$$\delta|\lambda|^2 \le a_{ij}^{\alpha\beta}(p, x)\lambda_i\lambda_j \le \delta^{-1}|\lambda|^2.$$

The reader understands, of course, that the summation convention is adopted throughout the article.

Set

$$\phi_t^{\alpha.\beta.x} = \int_0^t c^{\alpha_s\beta_s}(p_s^{\alpha.\beta.}, x_s^{\alpha.\beta.x})\, ds,$$

fix a bounded domain $D \subset \mathbb{R}^d$, define $\tau^{\alpha.\beta.x}$ as the first exit time of $x_t^{\alpha.\beta.x}$ from D, and introduce

$$v(x) = \inf_{\beta\in\mathbb{B}} \sup_{\alpha.\in\mathfrak{A}} E_x^{\alpha.\beta(\alpha.)}\left[\int_0^\tau f(p_t, x_t)e^{-\phi_t}\, dt + g(x_\tau)e^{-\phi_\tau}\right], \tag{2.5}$$

where the indices $\alpha.$, β, and x at the expectation sign are written to mean that they should be placed inside the expectation sign wherever and as appropriate, that is

$$E_x^{\alpha.\beta.}\left[\int_0^\tau f(p_t, x_t)e^{-\phi_t}\, dt + g(x_\tau)e^{-\phi_\tau}\right]$$
$$:= E\left[g(x_{\tau^{\alpha.\beta.x}}^{\alpha.\beta.x})e^{-\phi_{\tau^{\alpha.\beta.x}}^{\alpha.\beta.x}} + \int_0^{\tau^{\alpha.\beta.x}} f^{\alpha_t\beta_t}(p_t^{\alpha.\beta.}, x_t^{\alpha.\beta.x})e^{-\phi_t^{\alpha.\beta.x}}\, dt\right].$$

Observe that $v(x) = g(x)$ in $\mathbb{R}^d \setminus D$.

This definition makes perfect sense due to the following.

Lemma 2.1. *There is a constant N, depending only on K_0, δ, d, and the diameter of D, such that for any $\alpha. \in \mathfrak{A}$, $\beta. \in \mathfrak{B}$, $x \in D$, $n = 1, 2, \ldots$, $t \in [0, \infty)$, and $h \in L_d(D)$ we have (a.s.)*

$$I_{\tau^{\alpha.\beta.x}>t} E_x^{\alpha.\beta.}\left\{\left(\int_t^\tau |h(x_s)|\, ds\right)^n \mid \mathcal{F}_t\right\} \le n!N^n\|h\|_{L_d(D)}^n. \tag{2.6}$$

In particular, for any $n = 1, 2, \ldots$

$$E_x^{\alpha.\beta.}\tau^n \le n!N^n.$$

Proof. Estimate (2.6) with $n = 1$ is proved in Theorem 2.2.1 of [7]. If it is true for an n, then we have

$$\begin{aligned}
&I_{\tau^{\alpha.\beta.x}>t} E_x^{\alpha.\beta.}\left\{\left(\int_t^\tau |h(x_s)|\,ds\right)^{n+1} \mid \mathcal{F}_t\right\}\\
&\quad = (n+1) I_{\tau^{\alpha.\beta.x}>t} E_x^{\alpha.\beta.}\left\{\int_t^\tau |h(x_r)|\left(\int_r^\tau |h(x_s)|\,ds\right)^n dr \mid \mathcal{F}_t\right\}\\
&\quad = (n+1) I_{\tau^{\alpha.\beta.x}>t} E_x^{\alpha.\beta.}\left\{\int_t^\tau |h(x_r)| I_{\tau>r}\left[E_x^{\alpha.\beta.}\left(\int_r^\tau |h(x_s)|\,ds\right)^n \mid \mathcal{F}_r\right] dr \mid \mathcal{F}_t\right\}\\
&\quad \le N^n (n+1)!\|h\|^n_{L_d(D)} I_{\tau^{\alpha.\beta.x}>t} E_x^{\alpha.\beta.}\left\{\int_t^\tau |h(x_r)|\,dr \mid \mathcal{F}_t\right\}\\
&\quad \le N^{n+1}(n+1)!\|h\|^{n+1}_{L_d(D)}.
\end{aligned}$$

The lemma is proved. □

For a sufficiently smooth function $u = u(x)$ introduce

$$L^{\alpha\beta}u(p,x) = a_{ij}^{\alpha\beta}(p,x)D_{ij}u(x) + b_i^{\alpha\beta}(p,x)D_i u(x) - c^{\alpha\beta}(p,x)u(x),$$

where, naturally, $D_i = \partial/\partial x_i$, $D_{ij} = D_i D_j$. Recall that we fixed a $\bar p \in \mathcal{P}$ and denote

$$\begin{aligned}
&\bar L^{\alpha\beta}u(x) = L^{\alpha\beta}u(\bar p, x),\\
&H[u](x) = \sup_{\alpha\in A}\inf_{\beta\in B}[\bar L^{\alpha\beta}u(x) + \bar f^{\alpha\beta}(x)].
\end{aligned} \tag{2.7}$$

Definition 2.2. For a domain $U \subset \mathbb{R}^d$ we say that a $C^2_{loc}(U)$ function u is p-insensitive in U (relative to $(r^{\alpha\beta}, L^{\alpha\beta})$) if for any $x \in U$, $\alpha. \in \mathfrak{A}$, and $\beta. \in \mathfrak{B}$

$$d\big[u(x_t^{\alpha.\beta.x})e^{-\phi_t^{\alpha.\beta.x}}\big] = r^{\alpha_t\beta_t}(p_t^{\alpha.\beta.}, x_t^{\alpha.\beta.x})\bar L^{\alpha_t\beta_t}u(x_t^{\alpha.\beta.x})e^{-\phi_t^{\alpha.\beta.x}}\,dt + dm_t$$

for t less than the first exit time of $x_t^{\alpha.\beta.x}$ from U, where m_t is a local martingale starting at zero.

There are nontrivial cases when all sufficiently smooth functions are p-insensitive (see Example 2.1). On the other hand, any smooth function $u(x_1)$ will be p-insensitive if $(a_{11}, b_1)^{\alpha\beta}(p,x) = r^{\alpha\beta}(p,x)(\bar a_{11}, \bar b_1)^{\alpha\beta}(x)$ with *no* restrictions on other entries of a and b. A generalization of this particular example will play an extremely important role in one of subsequent articles (see [12]).

Definition 2.3. Let U be a domain in $\mathbb{R}^d$ for which the Sobolev embedding $W^2_d(U) \subset C(\bar U)$ is valid. We say that it is regular (for given g) if there exists a function $u \in W^2_d(U)$ such that $H[u] = 0$ in U (a.e.) and $u = g$ on ∂U and there exists a sequence $u_n \in C^2(\bar U)$ of p-insensitive in U functions such that $u_n \to u$ in $W^2_d(U)$ and in $C(\bar U)$.

In a subsequent article (see [11]) we will show that the following assumption can be dropped.

Assumption 2.4. There is a sequence of expanding regular subdomains D_n of D such that $D = \bigcup_{n\ge1} D_n$.

Finally we impose the following.

Assumption 2.5. There exists a bounded nonnegative $G \in C^2_{loc}(D)$ such that
(i) We have $G \in C(\bar{D})$ and $G = 0$ on ∂D;
(ii) For all $\alpha \in A$, $\beta \in B$, $p \in \mathcal{P}$, and $x \in D$

$$L^{\alpha\beta} G(p, x) \leq -1. \tag{2.8}$$

Here is our main result.

Theorem 2.2. *Under the above assumptions also suppose that there exists a sequence of functions g_n such that $\|g - g_n\|_{C(\bar{D})} \to 0$ as $n \to \infty$, for each $n \geq 1$, $\|g_n\|_{C^2(\bar{D})} < \infty$ and g_n is p-insensitive in D. Then*

(i) *The function $v(x)$ is independent of the chosen control adapted process $p \in \mathfrak{P}$, it is bounded and continuous in $\mathbb{R}^d$.*

(ii) *Let $\gamma^{\alpha.\beta.x}$ be an $\{\mathcal{F}_t\}$-stopping time defined for each $\alpha. \in \mathfrak{A}$, $\beta. \in \mathfrak{B}$, and $x \in \mathbb{R}^d$ and such that $\gamma^{\alpha.\beta.x} \leq \tau^{\alpha.\beta.x}$. Also let $\lambda_t^{\alpha.\beta.x} \geq 0$ be progressively measurable functions on $\Omega \times [0, \infty)$ defined for each $\alpha. \in \mathfrak{A}$, $\beta. \in \mathfrak{B}$, and $x \in \mathbb{R}^d$ and such that they have finite integrals over finite time intervals (for any ω). Then for any x*

$$v(x) = \inf_{\beta \in \mathbb{B}} \sup_{\alpha. \in \mathfrak{A}} E_x^{\alpha.\beta(\alpha.)} \left[v(x_\gamma) e^{-\phi_\gamma - \psi_\gamma} + \int_0^\gamma \{ f(p_t, x_t) + \lambda_t v(x_t) \} e^{-\phi_t - \psi_t} \, dt \right], \tag{2.9}$$

where inside the expectation sign $\gamma = \gamma^{\alpha.\beta(\alpha.)x}$ and

$$\psi_t^{\alpha.\beta.x} = \int_0^t \lambda_s^{\alpha.\beta.x} \, ds.$$

Remark 2.2. The function G is called a barrier in the theory of partial differential equations. Existence of such barriers is known for a very large class of domains, say such that there are $\rho_0 > 0$ and $\theta > 0$ such that for any point $x_0 \in \partial D$ and any $r \in (0, \rho_0]$ we have that the volume of the intersection of D^c with the ball of radius r centered at x_0 is greater than θr^d. The so-called uniform exterior cone condition will suffice.

Without Assumption 2.5 or similar ones one cannot assert that v is continuous in $\bar{D}$ even if no control parameters are involved.

Note that the possibility to vary λ in Theorem 2.2 might be useful while considering stochastic differential games with stopping in the spirit of [5].

Remark 2.3. Definition 2.2 is stated in the form which is easy to use and to check especially when (as in a subsequent article [12]) the state process consists of several components for each of which the corresponding equations have very different forms and u depends only on part of these components.

Still it is worth noting that, as follows immediately from Itô's formula, $u \in C^2_{loc}(U)$ will be p-insensitive if on $A \times B \times \mathcal{P} \times U$ we have

$$L^{\alpha\beta} u(p, x) = r^{\alpha\beta}(p, x) \bar{L}^{\alpha\beta} u(x). \tag{2.10}$$

Example 2.1. Let $\mathcal{O}$ be the set of $d_1 \times d_1$ orthogonal matrices and denote by $p = (p', p'')$ a generic point in $[-\delta_1, \delta_1^{-1}] \times \mathcal{O}$. Assume that the original σ, b are independent of p. Then introduce

$$\sigma^{\alpha\beta}(p, x) = (p')^{1/2} \sigma^{\alpha\beta}(x) p'', \qquad (b, c, f)^{\alpha\beta}(p, x) = p'(b, c, f)^{\alpha\beta}(x).$$

In this case

$$(1/2)\sigma^{\alpha\beta}(p,x)(\sigma^{\alpha\beta}(p,x))^* = p'a^{\alpha\beta}(x)$$

and (2.10) holds for any u with $r^{\alpha\beta}(p,x) = p'$.

In this case the assertion that v is independent of the control adapted process $p_t^{\alpha.\beta.}$ is rather natural and is due to the fact that its effect on the state process is equivalent to that of a random time change and a random rotation of the increments of the original Wiener process.

The main advantage of introducing the above parameters, which by far are not the most general and important for the future, is that while estimating $v(x+\varepsilon\xi) - v(x)$ for small ε, where $\xi \in \mathbb{R}^d$, we can take $p \equiv (1, I)$, where I is the $d_1 \times d_1$ identity matrix, in the definition of $v(x)$ and a different p close to $(1, I)$ in the definition of $v(x+\varepsilon\xi)$. This may make the solutions of the corresponding stochastic equations to become closer than in the case where for both $v(x)$ and $v(x+\varepsilon\xi)$ we take $p \equiv (1, I)$.

Remark 2.4. One of ways to introduce control adapted processes can be explained in the situation of Example 2.1 when the original σ and b are Lipschitz continuous. Take a $[\delta_1, \delta_1^{-1}]$-valued function $r(x)$ and $\mathcal{O}$-valued function $Q(x)$ defined and Lipschitz continuous on $\mathbb{R}^d$. Fix an $x_0 \in \mathbb{R}^d$ and for $\alpha. \in \mathfrak{A}$ and $\beta. \in \mathfrak{B}$ define

$$p_t^{\alpha.\beta.} := (r(x_t), Q(x_t)),$$

where x_t is a unique solution of

$$x_t = x_0 + \int_0^t r^{1/2}(x_s)\sigma^{\alpha_s\beta_s}(x_s)Q(x_s)\,dw_s + \int_0^t r(x_s)b^{\alpha_s\beta_s}(x_s)\,ds.$$

Almost obviously $p \in \mathfrak{P}$ and the above solution is, actually, $x_t^{\alpha.\beta.x}$ for that p if $x = x_0$. In a subsequent article (see [12]) we will show a much more sophisticated use of control adapted processes defined by an auxiliary Itô equation.

As a simple byproduct of our proofs we obtain the following.

Theorem 2.3. *The function v is locally Hölder continuous in D with exponent $\theta \in (0,1)$ depending only on d and δ.*

The point is that v will be obtained as the limit of u_n which are solutions of class $W_d^2(D_n)$ of the equation $H[u_n] = 0$ in D_n (a.e.) with boundary data g. It is well known (see, for instance, Remark 1.3 in [10]) that such u_n satisfy linear uniformly elliptic equations with bounded coefficients and it is a classical result that such solutions admit uniform in n local Hölder estimates of some exponent $\theta \in (0,1)$ depending only on d and δ (see, for instance, [3] or [8]).

3. Proof of Theorem 2.2 in case that the Isaacs equation has a smooth solution

In this section Assumption 2.3(iii), (iv), the regularity Assumption 2.4 as well as Assumption 2.5 concerning G are not used and the domain D is not supposed to be bounded. Suppose that for each $\varepsilon > 0$ we are given real-valued functions $c_\varepsilon^{\alpha\beta}(x)$ and $f_\varepsilon^{\alpha\beta}(x)$ defined on $A \times B \times \mathbb{R}^d$.

Assumption 3.1. (i) Assumptions 2.1–2.3(i), (ii) are satisfied.

(ii) For a constant $\chi > 0$ we have $c^{\alpha\beta}(p, x) \geq \chi$ for all α, β, p, x.

(iii) For each $\varepsilon > 0$ the functions $(c, f)_\varepsilon^{\alpha\beta}(x)$ are bounded on $A \times B \times \bar{D}$ and uniformly continuous with respect to $x \in \bar{D}$ uniformly with respect to α, β.

(iv) For any x as $\varepsilon \downarrow 0$,

$$d_\varepsilon(x) := \sup_{(\alpha.,\beta.)\in\mathfrak{A}\times\mathfrak{B}} E_x^{\alpha.\beta.} \int_0^\tau (|\bar{c} - c_\varepsilon| + |\bar{f} - f_\varepsilon|)(x_t) e^{-\phi_t}\, dt \to 0.$$

(v) For any $x \in D$

$$\sup_{(\alpha.,\beta.)\in\mathfrak{A}\times\mathfrak{B}} E_x^{\alpha.\beta.} \int_0^\tau |f(p_t, x_t)| e^{-\phi_t}\, dt < \infty.$$

Observe that Assumption 3.1(v) implies that v is well defined.

In some applications we have in mind the following "degenerate" version of Theorem 2.2 plays an important role. We assume that we are given two p-insensitive in D functions $\hat{u}, \check{u} \in C^2(\bar{D})$ (with finite $C^2(\bar{D})$-norms) such that their second-order derivatives are uniformly continuous in $\bar{D}$ (in case D is unbounded).

Theorem 3.1. (i) *If $H[\hat{u}] \leq 0$ (everywhere) in D and $\hat{u} \geq g$ on ∂D (in case $\partial D \neq \emptyset$), then $v \leq \hat{u}$ in $\bar{D}$.*

(ii) *If $H[\check{u}] \geq 0$ (everywhere) in D and $\check{u} \leq g$ on ∂D (in case $\partial D \neq \emptyset$), then $v \geq \check{u}$ in $\bar{D}$.*

(iii) *If $\hat{u}$ and $\check{u}$ are as in* (i) *and* (ii) *and $\hat{u} = \check{u}$, then all assertions of Theorem 2.2 hold true. Moreover, $v = \hat{u}$.*

This theorem is an immediate consequence of the following two results in which we allow $\lambda_t^{\alpha.\beta.x}$ and $\gamma^{\alpha.\beta.x}$ to be as in Theorem 2.2.

Theorem 3.2. *Suppose that $H[\hat{u}] \leq 0$ (everywhere) in D. Then for all $x \in \bar{D}$ we have*

$$\hat{u}(x) \geq \inf_{\beta\in\mathbb{B}} \sup_{\alpha.\in\mathfrak{A}} E_x^{\alpha.\beta(\alpha.)} \left[\hat{u}(x_\gamma) e^{-\phi_\gamma - \psi_\gamma} + \int_0^\gamma [f(p_t, x_t) + \lambda_t \hat{u}(x_t)] e^{-\phi_t - \psi_t}\, dt \right]. \tag{3.1}$$

In particular, if $\hat{u} \geq g$ on ∂D, then for $\gamma \equiv \tau$ and $\lambda \equiv 0$ Eq. (3.1) yields that $\hat{u} \geq v$.

Theorem 3.3. *Suppose that $H[\check{u}] \geq 0$ (everywhere) in D. Then for all $x \in \bar{D}$ we have*

$$\check{u}(x) \leq \inf_{\beta\in\mathbb{B}} \sup_{\alpha.\in\mathfrak{A}} E_x^{\alpha.\beta(\alpha.)} \left[\check{u}(x_\gamma) e^{-\phi_\gamma - \psi_\gamma} + \int_0^\gamma [f(p_t, x_t) + \lambda_t \check{u}(x_t)] e^{-\phi_t - \psi_t}\, dt \right]. \tag{3.2}$$

In particular, if $\check{u} \leq g$ on ∂D, then for $\gamma \equiv \tau$ and $\lambda \equiv 0$ Eq. (3.2) yields that $\check{u} \leq v$.

Note that, formally, the value x_γ in (3.1) and (3.2) may not be defined if $\gamma = \infty$. In that case we set the corresponding terms to equal zero, which is natural because $\hat{u}$ and $\check{u}$ are bounded and $\phi_\infty^{\alpha.\beta.x} = \infty$.

To prove these theorems we need two lemmas. The reader can compare our arguments with the ones in [14] and see that they are very close.

For a stopping time γ we say that a process ξ_t is a submartingale on $[0, \gamma]$ if $\xi_{t\wedge\gamma}$ is a submartingale. Similar definition applies to supermartingales.

Lemma 3.4. *Let $H[\hat{u}] \leq 0$ (everywhere) in D. Then for any $x \in \mathbb{R}^d$, $\alpha. \in \mathfrak{A}$, and $\varepsilon > 0$, there exist a sequence $\beta^n_\cdot(\alpha.) = \beta^n_\cdot(\alpha., x, \varepsilon) \in \mathfrak{B}$, $n = 1, 2, \ldots$, and a sequence of increasing continuous $\{\mathcal{F}_t\}$-adapted processes $\eta^{n\varepsilon}_t(\alpha.) = \eta^{n\varepsilon}_t(\alpha., x)$ with $\eta^{n\varepsilon}_0(\alpha.) = 0$ such that*

$$\sup_n E\eta^{n\varepsilon}_\infty(\alpha.) < \infty, \tag{3.3}$$

the processes

$$\kappa^{n\varepsilon}_t(\alpha.) := \hat{u}(x^n_t)e^{-\phi^n_t} - \eta^{n\varepsilon}_t(\alpha.) + \int_0^t f^n_s(p^n_s, x^n_s)e^{-\phi^n_s}\,ds,$$

where

$$(x^n_t, \phi^n_t) = (x_t, \phi_t)^{\alpha.\beta^n_\cdot(\alpha.)x}, \qquad f^n_t(p, x) = f^{\alpha_t\beta^n_t(\alpha.)}(p, x), \qquad p^n_t = p_t^{\alpha.\beta^n_\cdot(\alpha.)} \tag{3.4}$$

are supermartingales on $[0, \tau^{\alpha.\beta^n_\cdot(\alpha.)x}]$, and

$$\varlimsup_{n\to\infty} \sup_{\alpha.\in\mathfrak{A}} E\eta^{n\varepsilon}_\tau(\alpha.) \leq \varepsilon/(\delta_1\chi) + Nd_\varepsilon(x), \tag{3.5}$$

where δ_1 is taken from Assumption 2.3(ii) and N is independent of x and ε. Furthermore, if for any n we are given a nonnegative, progressively measurable process $\lambda^n_t \geq 0$ having finite integrals over finite time intervals (for any ω), then the processes

$$\begin{aligned}\rho^{n\varepsilon}_t(\alpha.) := {} & \hat{u}(x^n_t)e^{-\phi^n_t-\psi^n_t} - \eta^{n\varepsilon}_t(\alpha.)e^{-\psi^n_t} \\ & + \int_0^t \left[f^n_s(p^n_s, x^n_s) + \lambda^n_s\hat{u}(x^n_s) - \lambda^n_s\eta^{n\varepsilon}_s(\alpha.)e^{\phi^n_s}\right]e^{-\phi^n_s-\psi^n_t}\,ds\end{aligned}$$

are supermartingales on $[0, \tau^{\alpha.\beta^n_\cdot(\alpha.)x}]$, where (we use notation (3.4) and)

$$\psi^n_t = \int_0^t \lambda^n_s\,ds. \tag{3.6}$$

Finally,

$$\sup_{\alpha.\in\mathfrak{A}}\sup_n E\sup_{t\geq 0}|\kappa^{n\varepsilon}_{t\wedge\tau}(\alpha.)| < \infty, \qquad \sup_{\alpha.\in\mathfrak{A}}\sup_n E\sup_{t\geq 0}|\rho^{n\varepsilon}_{t\wedge\tau}(\alpha.)| < \infty. \tag{3.7}$$

Proof. Since B is separable and $a^{\alpha\beta}$, $b^{\alpha\beta}$, $c^{\alpha\beta}$, and $f^{\alpha\beta}$ are continuous with respect to β one can replace B in (2.7) with an appropriate countable subset $B_0 = \{\beta_1, \beta_2, \ldots\}$. Then for each $\alpha \in A$ and $x \in D$ define $\beta(\alpha, x)$ as $\beta_i \in B_0$ with the least i such that

$$0 \geq \bar{L}^{\alpha\beta_i}\hat{u}(x) + \bar{f}^{\alpha\beta_i}(x) - \varepsilon. \tag{3.8}$$

For each i the right-hand side of (3.8) is Borel in x and continuous in α. Therefore, it is a Borel function of (α, x), implying that $\beta(\alpha, x)$ also is a Borel function of (α, x). For $x \notin D$ set $\beta(\alpha, x) = \beta^*$, where β^* is a fixed element of B. Then we have that in D

$$0 \geq \bar{L}^{\alpha\beta(\alpha,x)}\hat{u}(x) + \bar{f}^{\alpha\beta(\alpha,x)}(x) - \varepsilon. \tag{3.9}$$

After that fix x, define $\beta^{n0}_t(\alpha.) = \beta(\alpha_t, x)$, $t \geq 0$, and for $k \geq 1$ introduce $\beta^{nk}_t(\alpha.)$ recursively so that

$$\begin{aligned}&\beta^{nk}_t(\alpha.) = \beta^{n(k-1)}_t(\alpha.) \quad \text{for } t < k/n, \\ &\beta^{nk}_t(\alpha.) = \beta(\alpha_t, x^{nk}_{k/n}) \quad \text{for } t \geq k/n,\end{aligned} \tag{3.10}$$

where x_t^{nk} is a unique solution of

$$x_t = x + \int_0^t \sigma(\alpha_s, \beta_s^{n(k-1)}(\alpha.), p_s^{\alpha.\beta_\cdot^{n(k-1)}(\alpha.)}, x_s)\, dw_s + \int_0^t b(\alpha_s, \beta_s^{n(k-1)}(\alpha.), p_s^{\alpha.\beta_\cdot^{n(k-1)}(\alpha.)}, x_s)\, ds. \tag{3.11}$$

To show that the above definitions make sense, observe that, by Assumption 2.2, x_t^{n0} is well defined for all t. Therefore, $\beta_t^{n1}(\alpha.)$ is also well defined, and by induction we conclude that x_t^{nk} and $\beta_t^{nk}(\alpha.)$ are well defined for all k.

Furthermore, owing to (3.10) it makes sense to define

$$\beta_t^n(\alpha.) = \beta_t^{nk}(\alpha.) \quad \text{for } t < k/n.$$

Notice that by definition $x_t^n = x_t^{\alpha.\beta_\cdot^n(\alpha.)x}$ satisfies the equation

$$x_t = x + \int_0^t \sigma(\alpha_s, \beta_s^n(\alpha.), p_s^{\alpha.\beta_\cdot^n(\alpha.)}, x_s)\, dw_s + \int_0^t b(\alpha_s, \beta_s^n(\alpha.), p_s^{\alpha.\beta_\cdot^n(\alpha.)}, x_s)\, ds. \tag{3.12}$$

For $t < k/n$ we have $\beta_t^n(\alpha.) = \beta_t^{n(k-1)}(\alpha.)$, so that for $t \le k/n$ Eq. (3.12) coincides with (3.11) owing to the fact that $p_t^{\alpha.\beta.}$ is control adapted. It follows that (a.s.)

$$x_t^n = x_t^{nk} \quad \text{for all } t \le k/n,$$

so that (a.s.)

$$\beta_t^{nk}(\alpha.) = \beta(\alpha_t, x_{k/n}^n)$$

for all $t \ge k/n$. Therefore, if $(k-1)/n \le t < k/n$, then

$$\beta_t^n(\alpha.) = \beta_t^{n(k-1)}(\alpha.) = \beta(\alpha_t, x_{(k-1)/n}^n)$$
$$\beta_s^n := \beta_s^n(\alpha.) = \beta(\alpha_s, x_{\kappa_n(s)}^n), \tag{3.13}$$

where $\kappa_n(t) = [nt]/n$, and x_t^n satisfies

$$x_t^n = x + \int_0^t \sigma(\alpha_s, \beta(\alpha_s, x_{\kappa_n(s)}^n), p_s^n, x_s^n)\, dw_s + \int_0^t b(\alpha_s, \beta(\alpha_s, x_{\kappa_n(s)}^n), p_s^n, x_s^n)\, ds, \tag{3.14}$$

with $p_s^n = p_s^{\alpha.\beta_\cdot^n}$.

Introduce τ^n as the first exit time of x_t^n from D and set

$$\phi_t^n = \phi_t^{\alpha.\beta_\cdot^n x}, \qquad r_s^n = r^{\alpha_s\beta_s^n}(p_s^n, x_s^n).$$

Observe that by Itô's formula

$$\hat{u}(x_{t\wedge\tau^n}^n)e^{-\phi_{t\wedge\tau^n}^n} = \hat{u}(x) + \int_0^{t\wedge\tau^n} e^{-\phi_s^n} L^{\alpha_s\beta_s^n}\hat{u}(p_s^n, x_s^n)\, ds + m_t^n,$$

where m_s^n is a martingale.

By Definition 2.2

$$\hat{u}(x^n_{t\wedge\tau^n})e^{-\phi^n_{t\wedge\tau^n}} + \int_0^{t\wedge\tau^n} f^{\alpha_s\beta^n_s}(p^n_s, x^n_s)e^{-\phi^n_s}\,ds$$
$$= \hat{u}(x) + \int_0^{t\wedge\tau^n} e^{-\phi^n_s} r^n_s \big[\bar{L}^{\alpha_s\beta^n_s}\hat{u}(x^n_s) + \bar{f}^{\alpha_s\beta^n_s}(x^n_s)\big]\,ds + m^n_t, \tag{3.15}$$

where, for $s < \tau^n$, (notice the change of $\bar{c}$ to c_ε)

$$\begin{aligned}\bar{L}^{\alpha_s\beta^n_s}\hat{u}(x^n_s) &= \bar{a}_{ij}(\alpha_s, \beta(\alpha_s, x^n_{\kappa_n(s)}), x^n_s)D_{ij}\hat{u}(x^n_s)\\ &\quad + \bar{b}_i(\alpha_s, \beta(\alpha_s, x^n_{\kappa_n(s)}), x^n_s)D_i\hat{u}(x^n_s) - \bar{c}(\alpha_s, \beta(\alpha_s, x^n_{\kappa_n(s)}), x^n_s)\hat{u}(x^n_s)\\ &= \bar{a}_{ij}(\alpha_s, \beta(\alpha_s, x^n_{\kappa_n(s)}), x^n_s)D_{ij}\hat{u}(x^n_s) + \bar{b}_i(\alpha_s, \beta(\alpha_s, x^n_{\kappa_n(s)}), x^n_s)D_i\hat{u}(x^n_s)\\ &\quad - c_\varepsilon(\alpha_s, \beta(\alpha_s, x^n_{\kappa_n(s)}), x^n_s)\hat{u}(x^n_s) + \xi^{n\varepsilon}_t,\end{aligned}$$

where $\xi^{n\varepsilon}_t$ (defined by the above equality) is a progressively measurable process such that by Assumption 3.1(iv)

$$\sup_n E\int_0^{\tau^n} |\xi^{n\varepsilon}_t|e^{-\phi^n_t}\,dt \le Nd_\varepsilon(x) \tag{3.16}$$

with N independent of α, ε, and x (equal to one). All such processes are denoted by $\xi^{n\varepsilon}_t$ below even if they may change from one occurrence to another.

According to our assumptions on the uniform continuity in x of the data and $D_{ij}u(x)$ we have that for $s < \tau^n$

$$\begin{aligned}\bar{L}^{\alpha_s\beta^n_s}\hat{u}(x^n_s) &\le \bar{a}_{ij}(\alpha_s, \beta(\alpha_s, x^n_{\kappa_n(s)}), x^n_{\kappa_n(s)})D_{ij}\hat{u}(x^n_{\kappa_n(s)})\\ &\quad + \bar{b}_i(\alpha_s, \beta(\alpha_s, x^n_{\kappa_n(s)}), x^n_{\kappa_n(s)})D_i\hat{u}(x^n_{\kappa_n(s)})\\ &\quad - \bar{c}(\alpha_s, \beta(\alpha_s, x^n_{\kappa_n(s)}), x^n_{\kappa_n(s)})\hat{u}(x^n_{\kappa_n(s)}) + \chi_\varepsilon(x^n_s - x^n_{\kappa_n(s)}) + |\xi^{n\varepsilon}_t|\end{aligned}$$

where, for each $\varepsilon > 0$, $\chi_\varepsilon(y)$ is a (nonrandom) bounded function on $\mathbb{R}^d$ such that $\chi_\varepsilon(y) \to 0$ as $y \to 0$. All such functions will be denoted by χ_ε even if they may change from one occurrence to another. Then (3.9) shows that, for $s < \tau^n$,

$$\begin{aligned}\bar{L}^{\alpha_s\beta^n_s}\hat{u}(x^n_s) &\le \varepsilon + \chi_\varepsilon(x^n_s - x^n_{\kappa_n(s)}) + |\xi^{n\varepsilon}_t| - \bar{f}(\alpha_s, \beta(\alpha_s, x^n_{\kappa_n(s)}), x^n_{\kappa_n(s)})\\ &\le \varepsilon + \chi_\varepsilon(x^n_s - x^n_{\kappa_n(s)}) + |\xi^{n\varepsilon}_t| - \bar{f}^{\alpha_s\beta^n_s}(x^n_s),\end{aligned}$$

which along with (3.15) implies that

$$\kappa^{n\varepsilon}_{t\wedge\tau^n} := \hat{u}(x^n_{t\wedge\tau^n})e^{-\phi^n_{t\wedge\tau^n}} + \int_0^{t\wedge\tau^n} f^{\alpha_s\beta^n_s}(p^n_s, x^n_s)e^{-\phi^n_s}\,ds - \eta^{n\varepsilon}_t = \zeta^{n\varepsilon}_t + m^n_t, \tag{3.17}$$

where $\zeta^{n\varepsilon}_t$ is a decreasing process and

$$\eta^{n\varepsilon}_t = \eta^{n\varepsilon}_t(\alpha.) = \varepsilon\delta_1^{-1}\int_0^{t\wedge\tau^n} e^{-\phi^n_s}\,ds + \int_0^{t\wedge\tau^n} e^{-\phi^n_s}[|\xi^{n\varepsilon}_s| + \chi_\varepsilon(x^n_s - x^n_{\kappa_n(s)})]\,ds.$$

Hence $\kappa^{n\varepsilon}_{t\wedge\tau^n}$ is at least a local supermartingale. Assumption 3.1 and (3.16) show that (3.3) and the first inequality in (3.7) hold. It follows that the local supermartingale $\kappa^{n\varepsilon}_{t\wedge\tau^n}$ is, actually, a supermartingale.

Furthermore, obviously

$$\int_0^\infty e^{-\phi_s^n}\,ds \le 1/\chi,$$

so that to prove the first assertion of the lemma, it only remains to show that

$$\sup_{\alpha_\cdot\in\mathfrak{A}} E\int_0^\infty e^{-\phi_s^n}\chi_\varepsilon(x_s^n - x_{\kappa_n(s)}^n)\,ds \to 0 \tag{3.18}$$

as $n\to\infty$. In light of the fact that $c^{\alpha\beta}\ge\chi$, this is done in exactly the same way as a similar fact is proved in [9].

That $\rho_{t\wedge\tau^n}^{n\varepsilon}(\alpha_\cdot)$ is a local supermartingale follows after computing its stochastic differential. Then the fact that it is a supermartingale follows from the second estimate in (3.7) which is proved by using

$$\int_0^\infty \lambda_t^n e^{-\psi_t^n}\,dt \le 1 \tag{3.19}$$

and the same argument as above. The lemma is proved. □

Proof of Theorem 3.2. First we fix $x\in\mathbb{R}^d$, $\alpha_\cdot\in\mathfrak{A}$, and $\varepsilon>0$, take $\beta_\cdot^n(\alpha_\cdot)$ form Lemma 3.4 and prove that the $\mathfrak{B}$-valued functions defined on $\mathfrak{A}$ by $\beta^n(\alpha_\cdot)=\beta_\cdot^n(\alpha_\cdot)$ belong to $\mathbb{B}$. To do that observe that if (2.1) holds and $T\le 1/n$, then (a.s.) $\beta_t^{n0}(\alpha_\cdot^1)=\beta_t^{n0}(\alpha_\cdot^2)$ for almost all $t\le T$. By definition also (a.s.)

$$p_s^{\alpha_\cdot^1\beta_\cdot^{n0}(\alpha_\cdot^1)} = p_s^{\alpha_\cdot^2\beta_\cdot^{n0}(\alpha_\cdot^2)} \quad \text{for almost all } s\le T.$$

By uniqueness of solutions of (2.2) (see Assumption 2.2), the processes x_t^{n1} found from (3.11) for $\alpha_\cdot=\alpha_\cdot^1$ and for $\alpha_\cdot=\alpha_\cdot^2$ coincide (a.s.) for all $t\le T$.

If (2.1) holds and $1/n<T\le 2/n$, then by the above solutions of (3.11) for $\alpha_\cdot=\alpha_\cdot^1$ and for $\alpha_\cdot=\alpha_\cdot^2$ coincide (a.s.) for $t=1/n$ and then (a.s.) $\beta_t^{n1}(\alpha_\cdot^1)=\beta_t^{n1}(\alpha_\cdot^2)$ not only for all $t<1/n$ but also for all $t\ge 1/n$, which implies that (a.s.)

$$p_s^{\alpha_\cdot^1\beta_\cdot^{n1}(\alpha_\cdot^1)} = p_s^{\alpha_\cdot^2\beta_\cdot^{n1}(\alpha_\cdot^2)} \quad \text{for almost all } s\le T$$

and again the processes x_t^n found from (3.11) for $\alpha_\cdot=\alpha_\cdot^1$ and for $\alpha_\cdot=\alpha_\cdot^2$ coincide (a.s.) for all $t\le T$.

By induction we get that if (2.1) holds for a $T\in(0,\infty)$ and we define k as the integer such that $k/n<T\le(k+1)/n$, then (a.s.)

$$\begin{aligned}&\beta_t^n(\alpha_\cdot^1)=\beta_t^{nk}(\alpha_\cdot^1)=\beta_t^{nk}(\alpha_\cdot^2)=\beta_t^n(\alpha_\cdot^2) \quad \text{for all } t<(k+1)/n,\\ &p_s^{\alpha_\cdot^1\beta_\cdot^{nk}(\alpha_\cdot^1)} = p_s^{\alpha_\cdot^2\beta_\cdot^{nk}(\alpha_\cdot^2)} \quad \text{for almost all } s\le T\end{aligned} \tag{3.20}$$

and the processes x_t^n found from (3.11) for $\alpha_\cdot=\alpha_\cdot^1$ and for $\alpha_\cdot=\alpha_\cdot^2$ coincide (a.s.) for all $t\le T$. This means that $\beta^n\in\mathbb{B}$ indeed.

Furthermore, by the supermartingale property of $\rho_t^{n\varepsilon}(\alpha)$, for any stopping times $\gamma^{\alpha_\cdot\beta_\cdot}\le\tau^{\alpha_\cdot\beta_\cdot x}$ defined for each $\alpha_\cdot\in\mathfrak{A}$ and $\beta_\cdot\in\mathfrak{B}$ we have

$$\begin{aligned}\hat u(x) \ge\ & E_x^{\alpha_\cdot\beta^n(\alpha_\cdot)}\hat u(x_\gamma)e^{-\phi_\gamma-\psi_\gamma} - E\eta_\gamma^{n\varepsilon}(\alpha_\cdot)e^{-\psi_\gamma}\\ &+ E_x^{\alpha_\cdot\beta^n(\alpha_\cdot)}\int_0^\gamma [f(p_t,x_t)+\lambda_t\hat u(x_t)-\lambda_t^n\eta_t^{n\varepsilon}(\alpha_\cdot)e^{\phi_t}]e^{-\phi_t-\psi_t}\,dt.\end{aligned}$$

Also observe that

$$E_x^{\alpha.\beta^n(\alpha.)}\left[\int_0^\gamma \lambda_t \eta_t^{n\varepsilon}(\alpha.)e^{-\psi_t}\,dt + \eta_\gamma^{n\varepsilon}(\alpha.)e^{-\psi_\gamma}\right] \le E\sup_{t\le\gamma}\eta_t^{n\varepsilon}(\alpha.) \le E\eta_\tau^{n\varepsilon}(\alpha.).$$

It follows that

$$\hat{u}(x) \ge E_x^{\alpha.\beta^n(\alpha.)}\left[\int_0^\gamma [f(p_t,x_t)+\lambda_t\hat{u}(x_t)]e^{-\phi_t-\psi_t}\,dt + \hat{u}(x_\gamma)e^{-\phi_\gamma-\psi_\gamma}\right] - E\eta_\tau^{n\varepsilon}(\alpha.),$$

which owing to (3.5) yields

$$\hat{u}(x) \ge \varliminf_{n\to\infty}\sup_{\alpha.\in\mathfrak{A}} E_x^{\alpha.\beta^n(\alpha.)}\left[\int_0^\gamma [f(p_t,x_t)+\lambda_t\hat{u}(x_t)]e^{-\phi_t-\psi_t}\,dt \right.$$
$$\left. + \hat{u}(x_\gamma)e^{-\phi_\gamma-\psi_\gamma}\right] - \varepsilon/(\delta_1\chi) - Nd_\varepsilon(x).$$

In light of the arbitrariness of ε we arrive at (3.1) and the theorem is proved. □

For treating $\check{u}$ we use the following result.

Lemma 3.5. *Let $H[\check{u}] \ge 0$ (everywhere) in D. Then for any $x \in \mathbb{R}^d$, $\beta \in \mathbb{B}$, and $\varepsilon > 0$, there exist a sequence $\alpha_\cdot^n \in \mathfrak{A}$, $n = 1, 2, \dots$, and a sequence of increasing continuous $\{\mathcal{F}_t\}$-adapted processes $\eta_t^{n\varepsilon}(\beta)$ with $\eta_0^{n\varepsilon}(\beta) = 0$ such that the processes*

$$\kappa_t^{n\varepsilon} := \check{u}(x_t^n)e^{-\phi_t^n} + \eta_t^{n\varepsilon}(\beta) + \int_0^t f_s^n(p_s^n, x_s^n)e^{-\phi_s^n}\,ds,$$

where

$$(x_t^n, \phi_t^n) = (x_t, \phi_t)^{\alpha_\cdot^n\beta(\alpha_\cdot^n)x}, \qquad f_t^n(p,x) = f^{\alpha_t^n\beta_t(\alpha_\cdot^n)}(p,x), \qquad p_t^n = p_t^{\alpha_\cdot^n\beta(\alpha_\cdot^n)}, \tag{3.21}$$

are submartingales on $[0, \tau^{\alpha_\cdot^n\beta(\alpha_\cdot^n)x}]$ and

$$\sup_n E\eta_\infty^{n\varepsilon}(\beta) < \infty, \tag{3.22}$$

$$\varlimsup_{n\to\infty} E\eta_\tau^{n\varepsilon}(\beta) \le \varepsilon/(\delta_1\chi) + Nd_\varepsilon(x), \tag{3.23}$$

where δ_1 is taken from Assumption 2.3(ii).

Furthermore, if for any n we are given a nonnegative, progressively measurable process $\lambda_t^n \ge 0$ having finite integrals over finite time intervals (for any ω), then the processes

$$\rho_t^{n\varepsilon} := \check{u}(x_t^n)e^{-\phi_t^n-\psi_t^n} - \eta_t^{n\varepsilon}(\beta)e^{-\psi_t^n}$$
$$+ \int_0^t \left[f_t^n(p_s^n, x_s^n) + \lambda_s^n\check{u}(x_s^n) - \lambda_s^n\eta_s^{n\varepsilon}(\beta)e^{\phi_s^n}\right]e^{-\phi_s^n-\psi_s^n}\,ds$$

are submartingales on $[0, \tau^{\alpha_\cdot^n\beta(\alpha_\cdot^n)x}]$, where we use notation (3.21) and ψ_t^n is taken from (3.6).

Finally,

$$\sup_n E\sup_{t\ge0}|\kappa_{t\wedge\tau}^{n\varepsilon}| < \infty, \qquad \sup_n E\sup_{t\ge0}|\rho_{t\wedge\tau}^{n\varepsilon}| < \infty.$$

Proof. Owing to Assumptions 2.1 and 2.3(i) the function

$$h(\alpha, x) := \inf_{\beta\in B}\left[\bar{L}^{\alpha\beta}\check{u}(x) + \bar{f}^{\alpha\beta}(x)\right]$$

is a finite Borel function of x and is continuous with respect to α. Its sup over A can be replaced with the sup over an appropriate countable subset of A and since

$$\sup_{\alpha\in A} h(\alpha, x) \geq 0,$$

similarly to how $\beta(\alpha, x)$ was defined in the proof of Lemma 3.4, one can find a Borel function $\bar{\alpha}(x)$ in such a way that

$$\inf_{\beta\in B}\big[\bar{L}^{\bar{\alpha}(x)\beta}\check{u}(x) + \bar{f}^{\bar{\alpha}(x)\beta}(x)\big] \geq -\varepsilon \tag{3.24}$$

in D. If $x \notin D$ we set $\bar{\alpha}(x) = \alpha^*$, where α^* is a fixed element of A.

After that we need some processes which we introduce recursively. Fix x and set $\alpha_t^{n0} \equiv \bar{\alpha}(x)$. Then define x_t^{n0} as a unique solution of the equation

$$x_t = x + \int_0^t \sigma(\alpha_s^{n0}, \beta_s(\alpha_\cdot^{n0}), p_s^{\alpha_\cdot^{n0}\beta(\alpha_\cdot^{n0})}, x_s)\, dw_s$$
$$+ \int_0^t b(\alpha_s^{n0}, \beta_s(\alpha_\cdot^{n0}), p_s^{\alpha_\cdot^{n0}\beta(\alpha_\cdot^{n0})}, x_s)\, ds.$$

For $k \geq 1$ introduce α_t^{nk} so that

$$\alpha_t^{nk} = \alpha_t^{n(k-1)} \quad \text{for } t < k/n,$$
$$\alpha_t^{nk} = \bar{\alpha}(x_{k/n}^{n(k-1)}) \quad \text{for } t \geq k/n,$$

where $x_t^{n(k-1)}$ is a unique solution of

$$x_t = x + \int_0^t \sigma(\alpha_s^{n(k-1)}, \beta_s(\alpha_\cdot^{n(k-1)}), p_s^{\alpha_\cdot^{n(k-1)}\beta(\alpha_\cdot^{n(k-1)})}, x_s)\, dw_s$$
$$+ \int_0^t b(\alpha_s^{n(k-1)}, \beta_s(\alpha_\cdot^{n(k-1)}), p_s^{\alpha_\cdot^{n(k-1)}\beta(\alpha_\cdot^{n(k-1)})}, x_s)\, ds. \tag{3.25}$$

As in the proof of Lemma 3.4 we show that the above definitions make sense as well as the definition

$$\alpha_t^n = \alpha_t^{n(k-1)} \quad \text{for } t < k/n. \tag{3.26}$$

Next, by definition $x_t^n = x_t^{\alpha_\cdot^n\beta(\alpha_\cdot^n)x}$ satisfies

$$x_t = x + \int_0^t \sigma(\alpha_s^n, \beta_s(\alpha_\cdot^n), p_s^{\alpha_\cdot^n\beta(\alpha_\cdot^n)}, x_s)\, dw_s + \int_0^t b(\alpha_s^n, \beta_s(\alpha_\cdot^n), p_s^{\alpha_\cdot^n\beta(\alpha_\cdot^n)}, x_s)\, ds.$$

Eq. (3.26) and the definitions of $\mathbb{B}$ and of control adapted processes show that x_t^n satisfies (3.25) for $t \leq k/n$. Hence, (a.s.) $x_t^n = x_t^{n(k-1)}$ for all $t \leq k/n$ and (a.s.) for all $t \geq 0$, $\alpha_t^n = \bar{\alpha}(x_{\kappa_n(t)}^n)$ and

$$x_t^n = x + \int_0^t \sigma(\bar{\alpha}(x_{\kappa_n(s)}^n), \beta_s(\alpha_\cdot^n), p_s^n, x_s^n)\, dw_s + \int_0^t b(\bar{\alpha}(x_{\kappa_n(s)}^n), \beta_s(\alpha_\cdot^n), p_s^n, x_s^n)\, ds,$$

where $p_s^n = p_s^{\alpha_\cdot^n\beta(\alpha_\cdot^n)}$.

Now, introduce τ^n as the first exit time of x_t^n from D, set

$$\beta_s^n = \beta_s(\alpha_\cdot^n), \qquad \phi_t^n = \phi_t^{\alpha_\cdot^n \beta_\cdot^n x}, \qquad r_s^n = r^{\alpha_\cdot^n \beta_\cdot^n}(p_s^n, x_s^n),$$

where $r^{\alpha\beta}(p, x)$ is taken from Assumption 2.3(ii), and observe that by Itô's formula

$$\check{u}(x_{t\wedge\tau^n}^n)e^{-\phi_{t\wedge\tau^n}^n} = \check{u}(x) + \int_0^{t\wedge\tau} e^{-\phi_s^n} r_s^n \bar{L}^{\alpha_s^n \beta_s^n} \check{u}(x_s^n)\, ds + m_t^n,$$

where m_s^n is a martingale and, for $s < \tau^n$,

$$\begin{aligned}\bar{L}^{\alpha_s^n \beta_s^n} \check{u}(x_s^n) &= \bar{a}_{ij}(\bar{\alpha}(x_{\kappa_n(s)}^n), \beta_s^n, x_s^n) D_{ij}\check{u}(x_s^n) \\ &\quad + \bar{b}_i(\bar{\alpha}(x_{\kappa_n(s)}^n), \beta_s^n, x_s^n) D_i \check{u}(x_s^n) - \bar{c}(\bar{\alpha}(x_{\kappa_n(s)}^n), \beta_s^n, x_s^n)\check{u}(x_s^n).\end{aligned}$$

Similarly to the proof of Lemma 3.4 we derive from (3.24) that, for $s < \tau^n$,

$$\begin{aligned}\bar{L}^{\alpha_s^n \beta_s^n} \check{u}(x_s^n) &\geq -\varepsilon - \chi(x_s^n - x_{\kappa_n(s)}^n) - \xi_t^{n\varepsilon} - \bar{f}(\bar{\alpha}(x_{\kappa_n(s)}^n), \beta_s^n, x_{\kappa_n(s)}^n) \\ &= -\varepsilon - \chi(x_s^n - x_{\kappa_n(s)}^n) - \xi_t^{n\varepsilon} - \bar{f}^{\alpha_s^n \beta_s^n}(x_s^n),\end{aligned}$$

where $\xi_t^{n\varepsilon}$ are nonnegative progressively measurable processes satisfying (3.16) and $\chi_\varepsilon(y)$ is a (nonrandom) bounded function on $\mathbb{R}^d$ such that $\chi_\varepsilon(y) \to 0$ as $y \to 0$. It follows that

$$\check{u}(x_{t\wedge\tau^n}^n)e^{-\phi_{t\wedge\tau^n}^n} + \int_0^{t\wedge\tau^n} f^{\alpha_s^n \beta_s^n}(p_s^n, x_s^n)e^{-\phi_s^n}\, ds + \eta_t^n = \zeta_t + m_t^n, \tag{3.27}$$

where ζ_t is an increasing process and

$$\eta_t^n = \eta_t^n(\beta) = \varepsilon\delta_1^{-1}\int_0^{t\wedge\tau^n} e^{-\phi_s^n}\, ds + \int_0^{t\wedge\tau^n} e^{-\phi_s^n}[\xi_s^{n\varepsilon} + \chi_\varepsilon(x_s^n - x_{\kappa_n(s)}^n)]\, ds.$$

Hence the left-hand side of (3.27) is a local submartingale and we finish the proof in the same way as the proof of Lemma 3.4. The lemma is proved. □

Proof of Theorem 3.3. Similarly to the proof of Theorem 3.2, for any $\beta \in \mathbb{B}$,

$$\begin{aligned}\check{u}(x) &\leq E_x^{\alpha_\cdot^n \beta(\alpha_\cdot^n)}\Bigg[\int_0^\gamma [f(p_t, x_t) + \lambda_t \check{u}(x_t) + \lambda_t \eta_t^{n\varepsilon}(\beta)e^{\phi_t}]e^{-\phi_t - \psi_t}\, dt \\ &\qquad + \check{u}(x_\gamma)e^{-\phi_\gamma - \psi_\gamma} + \eta_\gamma^{n\varepsilon}(\beta)e^{\psi_\gamma}\Bigg] \\ &\leq E_x^{\alpha_\cdot^n \beta(\alpha_\cdot^n)}\Bigg[\int_0^\gamma [f(p_t, x_t) + \lambda_t \check{u}(x_t)]e^{-\phi_t - \psi_t}\, dt + \check{u}(x_\gamma)e^{-\phi_\gamma - \psi_\gamma}\Bigg] + E\eta_\tau^{n\varepsilon}(\beta).\end{aligned}$$

It follows that

$$\begin{aligned}\check{u}(x) &\leq \sup_{\alpha_\cdot \in \mathfrak{A}} E_x^{\alpha_\cdot \beta(\alpha_\cdot)}\Bigg[\int_0^\gamma [f(p_t, x_t) + \lambda_t \check{u}(x_t)]e^{-\phi_t - \psi_t}\, dt \\ &\qquad + \check{u}(x_\gamma)e^{-\phi_\gamma - \psi_\gamma}\Bigg] + \varlimsup_{n\to\infty} E\eta_\tau^{n\varepsilon}(\beta),\end{aligned}$$

$$\begin{aligned}\check{u}(x) &\leq \sup_{\alpha_\cdot \in \mathfrak{A}} E_x^{\alpha_\cdot \beta(\alpha_\cdot)}\Bigg[\int_0^\gamma [f(p_t, x_t) + \lambda_t \check{u}(x_t)]e^{-\phi_t - \psi_t}\, dt \\ &\qquad + \check{u}(x_\gamma)e^{-\phi_\gamma - \psi_\gamma}\Bigg] + \varepsilon/(\delta_1\chi) + Nd_\varepsilon,\end{aligned}$$

which in light of the arbitrariness of ε and $\beta \in \mathbb{B}$ finally yields (3.2) and the theorem is proved. $\square$

4. Versions of Theorems 3.1–3.3 for "uniformly nondegenerate" case

In the first result of this section D is not assumed to be bounded.

Let $\hat{u}, \check{u} \in W^2_{1,loc}(D) \cap C(\bar{D})$ be given functions for which there exist sequences $\hat{u}_n, \check{u}_n \in C^2(\bar{D})$, $n \geq 1$, of p-insensitive in D functions which for each n have uniformly continuous second-order derivatives (if D is unbounded) and such that $\hat{u}_n, \check{u}_n$ converge to $\hat{u}$ and $\check{u}$, respectively, uniformly in $\bar{D}$. For a sufficiently regular function u we denote by Du its gradient and D^2u its Hessian. In case of $\hat{u}, \check{u}$ we take and fix any Borel measurable versions of their gradients and Hessians.

Theorem 4.1. *Suppose that Assumptions 2.1–2.3*(i), (ii), *Assumption 2.5*(ii) , *and Assumption 3.1*(iii), (v) *are satisfied. Also suppose that a stronger assumption than Assumption 3.1*(iv) *is satisfied, namely, for any x*

$$\sup_{(\alpha.,\beta.)\in\mathfrak{A}\times\mathfrak{B}} E^{\alpha.\beta.}_x \int_0^\tau \sup_{\alpha\in A,\beta\in B} (|\bar{c}^{\alpha\beta} - c^{\alpha\beta}_\varepsilon| + |\bar{f}^{\alpha\beta} - f^{\alpha\beta}_\varepsilon|)(x_t)e^{-\phi_t}\,dt \to 0 \tag{4.1}$$

as $\varepsilon \downarrow 0$. Finally, assume that for any $x \in D$

$$\sup_{(\alpha.,\beta.)\in\mathfrak{A}\times\mathfrak{B}} E^{\alpha.\beta.}_x \int_0^\tau \big(|D^2\hat{u} - D^2\hat{u}_n| + |D\hat{u} - D\hat{u}_n|\big)(x_t)e^{-\phi_t}\,dt \to 0 \tag{4.2}$$

as $n \to \infty$ and the same is true if we replace $\hat{u}$ with $\check{u}$.

Then the following holds true:

(i) *If $H[\hat{u}] \leq 0$ in D (a.e.) and $\hat{u} \geq g$ on ∂D (if $D \neq \mathbb{R}^d$), then $v \leq \hat{u}$ in $\bar{D}$ and (3.1) holds for any $\lambda^{\alpha.\beta.x}_t$ and $\gamma^{\alpha.\beta.x}$ as in Theorem 2.2.*

(ii) *If $H[\check{u}] \geq 0$ in D (a.e.) and $\check{u} \leq g$ on ∂D (if $D \neq \mathbb{R}^d$), then $v \geq \check{u}$ in $\bar{D}$ and (3.2) holds for any $\lambda^{\alpha.\beta.x}_t$ and $\gamma^{\alpha.\beta.x}$ as in Theorem 2.2.*

(iii) *If $\hat{u}$ and $\check{u}$ are as in* (i) *and* (ii) *and $\hat{u} = \check{u}$, then all assertions of Theorem 2.2 hold true. Moreover, $v = \hat{u}$.*

It is worth saying that, as in the case of Theorems 3.2 and 3.3, we set the terms containing x_γ in (3.1) and (3.2) to be zero on the event that $\gamma = \infty$.

Before we proceed with the proof we note the following.

Remark 4.1. For $x \in \mathbb{R}^d$ and $u = (u', u'')$, where $u' = (u'_0, u'_1, \ldots, u'_d) \in \mathbb{R}^{d+1}$ and u'' is in the set $\mathcal{S}$ of $d \times d$ symmetric matrices, introduce

$$H(u, x) = \sup_{\alpha\in A} \inf_{\beta\in B} \left(\bar{a}^{\alpha\beta}_{ij}(x)u''_{ij} + \sum_{i\geq 1} \bar{b}^{\alpha\beta}_i(x)u'_i - \bar{c}^{\alpha\beta}(x)u'_0 + \bar{f}^{\alpha\beta}(x) \right). \tag{4.3}$$

Owing to Assumption 2.1(i) one can replace A and B with their countable everywhere dense subsets. Then we see that $H(u, x)$ is a Borel function of x.

Also note that

$$
\begin{aligned}
|H(u,x)| &\le N\biggl(\sum_{i,j=1}^d |u''_{ij}| + \sum_{i=1}^d |u'_i|\biggr) + |u'_0| \sup_{\alpha,\beta} \bar{c}^{\alpha\beta}(x) \\
&\quad + \sup_{\alpha,\beta} |\bar{f}^{\alpha\beta}(x)|, |H(u,x) - H(v,x)| \\
&\le |u'_0 - v'_0| \sup_{\alpha,\beta} \bar{c}^{\alpha\beta}(x) + N\biggl(\sum_{i,j=1}^d |u''_{ij} - v''_{ij}| + \sum_{i=1}^d |u'_i - v'_i|\biggr), \qquad (4.4)
\end{aligned}
$$

where N is independent of u, v, x. In light of (2.4) the right-hand sides are finite, which, in particular, implies that $H(u,x)$ is a finite Borel function of (u,x).

If, in addition, $\bar{c}^{\alpha\beta}(x)$ and $\bar{f}^{\alpha\beta}(x)$ are bounded and continuous with respect to x uniformly with respect to (α,β), then the inequality

$$
\begin{aligned}
|H(u,x) - H(u,y)| &\le N|x-y|\biggl(\sum_{i,j=1}^d |u''_{ij}| + \sum_{i=1}^d |u'_i|\biggr) \\
&\quad + |u'_0| \sup_{\alpha,\beta} |\bar{c}^{\alpha\beta}(x) - \bar{c}^{\alpha\beta}(y)| + \sup_{\alpha,\beta} |\bar{f}^{\alpha\beta}(x) - \bar{f}^{\alpha\beta}(y)|
\end{aligned}
$$

shows that $H(u,x)$ is a continuous function of x, which along with (4.4) guarantees that $H(u,x)$ is a continuous function of (u,x).

Proof of Theorem 4.1. (i) Introduce $\hat{h}_n = H[\hat{u}_n]$,

$$
\begin{aligned}
c_n^{\alpha\beta}(p,x) &= c^{\alpha\beta}(p,x) + n^{-1} r^{\alpha\beta}(p,x), \\
f_n^{\alpha\beta}(p,x) &= f^{\alpha\beta}(p,x) - r^{\alpha\beta}(p,x)\hat{h}_n(x) + n^{-1} r^{\alpha\beta}(p,x)\hat{u}_n(x) \\
&= r^{\alpha\beta}(p,x)[\bar{f}^{\alpha\beta}(x) - \hat{h}_n(x) + n^{-1}\hat{u}_n(x)], \\
L_n^{\alpha\beta} u(p,x) &= L^{\alpha\beta} u(p,x) - n^{-1} r^{\alpha\beta}(p,x) u(x).
\end{aligned}
$$

Observe that $\hat{u}_n$ is p-insensitive in D with respect to $L_n^{\alpha\beta}$. Owing to Definition 2.2, this follows from the fact that (dropping the superscripts $\alpha., \beta., x$) for any $x \in D$ and $t < \tau$, we find that the coefficient of dt in the stochastic differential of

$$
\hat{u}_n(x_t) e^{-\phi_t^n}, \quad \text{where } \phi_t^n = \int_0^t c_n^{\alpha_s \beta_s}(p_s, x_s)\, ds,
$$

equals $e^{-\phi_t^n}$ times

$$
\begin{aligned}
L^{\alpha_t\beta_t}\hat{u}_n(p_t,x_t) - n^{-1} r^{\alpha_t\beta_t}(p_t,x_t)\hat{u}_n(x_t) &= r^{\alpha_t\beta_t}(p_t,x_t)\bigl[\bar{L}^{\alpha_t\beta_t}\hat{u}_n(x_t) - n^{-1}\hat{u}_n(x_t)\bigr] \\
&= r^{\alpha_t\beta_t}(p_t,x_t) L_n^{\alpha_t\beta_t}\hat{u}_n(\bar{p}_t, x_t).
\end{aligned}
$$

Furthermore,

$$
\sup_{\alpha\in A} \inf_{\beta\in B} \bigl[L_n^{\alpha\beta}\hat{u}_n(\bar{p},x) + f_n^{\alpha\beta}(\bar{p},x)\bigr] = 0,
$$

which makes us try to apply Theorem 3.2 for each n.

Define $\hat{h}_{n\varepsilon} = H_\varepsilon[\hat{u}_n]$, where H_ε is constructed in the same way as H with c_ε and f_ε in place of c and f, respectively, and observe that, for each n and $\varepsilon > 0$, $\hat{h}_n$ is a Borel function on $\bar{D}$

and $\hat{h}_{n\varepsilon}$ is bounded and uniformly continuous in $\bar{D}$ (cf. Remark 4.1). Also in D

$$|\hat{h}_{n\varepsilon} - \hat{h}_n| = |H_\varepsilon[\hat{u}_n] - H[\hat{u}_n]| \le (1 + \sup|\hat{u}_n|) \sup_{\alpha\in A,\beta\in B} (|\bar{c}^{\alpha\beta} - c_\varepsilon^{\alpha\beta}| + |\bar{f}^{\alpha\beta} - f_\varepsilon^{\alpha\beta}|).$$

Therefore, for any x

$$\lim_{\varepsilon\downarrow 0} \sup_{(\alpha.,\beta.)\in\mathfrak{A}\times\mathfrak{B}} E_x^{\alpha.\beta.} \int_0^\tau |\hat{h}_{n\varepsilon} - \hat{h}_n|(x_t)e^{-\phi_t}\,dt = 0.$$

All other assumptions of Theorem 3.2 are satisfied in light of the assumptions of the present theorem and the fact that we added n^{-1} to $\bar{c}$. By Theorem 3.2 after setting

$$\zeta_t^{\alpha.\beta.x} = \int_0^t r^{\alpha_s\beta_s}(p_s^{\alpha.\beta.}, x_s^{\alpha.\beta.x})\,ds$$

we obtain

$$\begin{aligned}\hat{u}_n(x) \ge \inf_{\beta\in\mathbb{B}} \sup_{\alpha.\in\mathfrak{A}} E_x^{\alpha.\beta(\alpha.)}\Big[&\hat{u}_n(x_\gamma)e^{-\phi_\gamma-\psi_\gamma-\zeta_\gamma/n} \\ &+ \int_0^\gamma \{f_n(p_t,x_t) + \lambda_t\hat{u}_n(x_t)\}e^{-\phi_t-\psi_t-\zeta_t/n}\,dt\Big].\end{aligned}$$

Furthermore,

$$\begin{aligned}&E_{t,x}^{\alpha.\beta.}\Big[|\hat{u}_n(x_\gamma) - \hat{u}(x_\gamma)|e^{-\phi_\gamma-\psi_\gamma-\zeta_\gamma/n} + \int_0^\gamma \lambda_t|\hat{u}_n(x_t) - \hat{u}(x_t)|e^{-\phi_t-\psi_t-\zeta_t/n}\,dt\Big] \\ &\quad\le E_{t,x}^{\alpha.\beta.} \sup_{t\le\gamma} |\hat{u}_n(x_t) - \hat{u}(x_t)|e^{-\phi_t-\zeta_t/n}.\end{aligned}$$

Therefore,

$$\begin{aligned}\hat{u}_n(x) \ge \inf_{\beta\in\mathbb{B}} \sup_{\alpha.\in\mathfrak{A}} E_x^{\alpha.\beta(\alpha.)}\Big[&\hat{u}(x_\gamma)e^{-\phi_\gamma-\psi_\gamma-\zeta_\gamma/n} \\ &+ \int_0^\gamma \{f(p_t,x_t) + \lambda_t\hat{u}(x_t)\}e^{-\phi_t-\psi_t-\zeta_t/n}\,dt\Big] \\ &- (n^{-1}\delta_1^{-1}\sup_D|\hat{u}_n| + \sup_D|\hat{u}_n - \hat{u}|) - \kappa_n,\end{aligned}$$

where

$$\kappa_n(x) = \delta_1^{-1} \sup_{\alpha.\in\mathfrak{A},\beta.\in\mathfrak{B}} E_x^{\alpha.\beta.} \int_0^\tau (\hat{h}_n(x_t))^+ e^{-\phi_t}\,dt.$$

Observe that

$$\begin{aligned}(\hat{h}_n)^+ = (H[\hat{u}_n])^+ \le (H[\hat{u}_n] - H[\hat{u}])^+ \le N(|D^2(\hat{u}_n - \hat{u})| + |D(\hat{u}_n - \hat{u})|) \\ + |\hat{u}_n - \hat{u}| \sup_{\alpha,\beta} \bar{c}^{\alpha\beta}.\end{aligned}$$

Here for any $\varepsilon > 0$

$$\sup_{\alpha,\beta} \bar{c}^{\alpha\beta} \le \sup_{\alpha,\beta} |\bar{c}^{\alpha\beta} - c_\varepsilon^{\alpha\beta}| + \sup_{\alpha,\beta} |c_\varepsilon^{\alpha\beta}|,$$

which along with our assumptions imply that $\kappa_n \to 0$ and leads to

$$\hat{u}(x) \geq \varlimsup_{n\to\infty} \inf_{\beta\in\mathbb{B}} \sup_{\alpha.\in\mathfrak{A}} E_x^{\alpha.\beta(\alpha.)}\Big[\hat{u}(x_\gamma)e^{-\phi_\gamma-\psi_\gamma-\zeta_\gamma/n} + \int_0^\gamma \{f(p_t,x_t)+\lambda_t\hat{u}(x_t)\}e^{-\phi_t-\psi_t-\zeta_t/n}\,dt\Big]. \tag{4.5}$$

Now we note that by considering $G+1$ in place of G we may assume that $G \geq 1$ on D. We set $G := 1$ outside D. Then, as follows easily from Itô's formula, the process

$$G(x_{t\wedge\tau})e^{\chi(t\wedge\tau)-\phi_{t\wedge\tau}} + (1/2)\int_0^{t\wedge\tau} e^{\chi s-\phi_s}\,ds,$$

is at least a local supermartingale, where $\chi = (2\sup_D G)^{-1}$,

$$(x_t,\phi_t,\tau) = (x_t,\phi_t,\tau)^{\alpha.\beta.x}, \quad \psi_t = \psi_t^{\alpha.\beta.}$$

and x, $\alpha.$, and $\beta.$ are arbitrary. Nonnegative local supermartingales are supermartingales. In particular, (note that the estimates below are consistent with our stipulation that functions of x_γ equal zero if $\gamma = \infty$)

$$E_x^{\alpha.\beta.}\sup_{t\leq\tau}\Big[G(x_t)e^{\chi t-\phi_t} + (1/2)\int_0^t e^{\chi s-\phi_s}\,ds\Big] \leq G(x),$$

$$\begin{aligned} &E_x^{\alpha.\beta.}\Big[|\hat{u}(x_\gamma)|e^{-\phi_\gamma-\psi_\gamma}(1-e^{-\zeta_\gamma/n}) + \int_0^\gamma \lambda_t|\hat{u}(x_t)|e^{-\phi_t-\psi_t}(1-e^{-\zeta_t/n})\,dt\Big] \\ &\quad\leq E_x^{\alpha.\beta.}\sup_{t\leq\gamma}|\hat{u}(x_t)|e^{-\phi_t}(1-e^{-\zeta_t/n}) \\ &\quad\leq \sup|\hat{u}|\delta_1^{-1}n^{-1}E_x^{\alpha.\beta.}\sup_{t\leq\gamma}[tG(x_t)e^{-\phi_t}] \leq N/n, \end{aligned}$$

where N depends only on G, $\hat{u}$, and δ_1.

As far as the term with f in (4.5) is concerned, we estimate it similarly to c and add that

$$E_x^{\alpha.\beta.}\int_0^\tau e^{-\phi_t}(1-e^{-\zeta_t/n})\,dt \leq \delta_1^{-1}n^{-1}E_x^{\alpha.\beta.}\int_0^\tau e^{-\phi_t}t\,dt.$$

After that by letting $n\to\infty$ in (4.5) we come to (3.1). Eq. (3.1) with $\gamma=\tau$ and $\lambda\equiv 0$ implies that $\hat{u}\geq v$.

(ii) Here the proof is very similar and yields (3.2), from which we conclude that $\check{u}\leq v$.

(iii) The combination of assertions in (i) and (ii) leads to $\hat{u}=\check{u}=v$, then (3.1) and (3.2) imply that v satisfies (2.9) and, since $\check{u}_n\to\check{u}=v$ uniformly by assumption, v is continuous in $\bar{D}$ and in $\mathbb{R}^d$. Finally, since $\hat{u}$ and $\check{u}$ have nothing to do with the fixed control adapted process $p_t^{\alpha.\beta.}$, the function v is independent of the choice of this process.

The theorem is proved. □

The assumptions of Theorem 4.1 admit an easy verification in the uniformly nondegenerate case.

Theorem 4.2. *Suppose that the domain D is bounded, all requirements of Assumptions 2.1–2.3 are satisfied, and $\hat{u}_n$ and $\check{u}_n$ not only converge uniformly in $\bar{D}$ but also converge in $W_d^2(D)$ to $\hat{u}$ and $\check{u}$, respectively. Then all assertions of Theorem 4.1 hold true.*

Indeed, the existence of a global barrier is well known for bounded domains and uniformly nondegenerate operators with bounded coefficients, so that Assumption 2.5(ii) is satisfied. Furthermore, in (4.1) we can take $(c_\varepsilon, f_\varepsilon) = (c^{(\varepsilon)}, f^{(\varepsilon)})$ owing to Assumption 2.3(iii) and Lemma 2.1. The same lemma guarantees that (4.2) holds and therefore Theorem 4.1 is applicable.

Remark 4.2. One may think that Theorem 4.2 is the only "reasonable" application of Theorem 4.1. However, in a subsequent articles (see [11,12]) we will see an application of Theorem 4.1 to a situation where $\check{u}$ depends only on part of the coordinates of a diffusion process, which does degenerate at each point, but the above mentioned part of it is uniformly nondegenerate.

5. An auxiliary result

In this section D is not assumed to be bounded. We assume that we are given a continuous $\mathcal{F}_t$-adapted process x_t in $\mathbb{R}^d$ and progressively measurable real-valued processes c_t and f_t. Suppose that $c_t \geq 0$.

Assumption 5.1. There exists a nonnegative bounded and continuous function Φ on $\bar{D}$ such that the process

$$\Phi(x_{t\wedge\tau})e^{-\phi_{t\wedge\tau}} + \int_0^{t\wedge\tau} |f_s|e^{-\phi_s}\,ds$$

is a supermartingale, where τ is the first exit time of x_t from D and

$$\phi_t = \int_0^t c_s\,ds.$$

Let $D(n)$, $n \geq 1$, be a sequence of subdomains of D. Introduce τ_n as the first exit time of x_t from $D(n)$.

Lemma 5.1. *We have*

$$E\int_0^\tau |f_t|e^{-\phi_t}\,dt \leq E\,\Phi(x_0)I_{x_0\in D}, \tag{5.1}$$

$$E\int_{\tau_n}^\tau |f_t|e^{-\phi_t}\,dt \leq \sup_{(\partial D_n)\setminus\partial D} \Phi \quad \left(\sup_{\emptyset} := 0\right). \tag{5.2}$$

Proof. By assumption, for any $t \in [0,\infty)$,

$$E\left[\Phi(x_{t\wedge\tau})e^{-\phi_{t\wedge\tau}} + \int_0^{t\wedge\tau} |f_s|e^{-\phi_s}\,ds\right] \leq E\,\Phi(x_0),$$

$$E\int_0^{t\wedge\tau} |f_s|e^{-\phi_s}\,ds \leq E[\Phi(x_0) - \Phi(x_{t\wedge\tau})e^{-\phi_{t\wedge\tau}}] \leq E\,\Phi(x_0)I_{\tau>0},$$

and sending $t \to \infty$ leads to (5.1).

Again by Assumption 5.1 for any $T \in [0, \infty)$ we have

$$E\Big[\Phi(x_{\tau_n\wedge T})e^{-\phi_{\tau_n\wedge T}} + \int_0^{\tau_n\wedge T} |f_t|e^{-\phi_t}\,dt\Big] \geq E\Big[\Phi(x_{\tau\wedge T})e^{-\phi_{\tau\wedge T}} + \int_0^{\tau\wedge T} |f_t|e^{-\phi_t}\,dt\Big],$$

$$\begin{aligned} E\int_{\tau_n\wedge T}^{\tau\wedge T} |f_t|e^{-\phi_t}\,dt &\leq E\big[\Phi(x_{\tau_n\wedge T})e^{-\phi_{\tau_n\wedge T}} - \Phi(x_{\tau\wedge T})e^{-\phi_{\tau\wedge T}}\big] \\ &= E\big[\Phi(x_{\tau_n\wedge T})e^{-\phi_{\tau_n\wedge T}} - \Phi(x_{\tau\wedge T})e^{-\phi_{\tau\wedge T}}\big]I_{\tau_n<\tau} \\ &\leq E\,\Phi(x_{\tau_n\wedge T})I_{\tau_n<\tau} \qquad . \end{aligned}$$

By sending $T \to \infty$ and using the monotone and dominated convergence theorems we arrive at

$$E\int_{\tau_n}^{\tau} |f_t|e^{-\phi_t}\,dt \leq E\,\Phi(x_{\tau_n})I_{\tau_n<\tau}$$

and (5.2) follows. The lemma is proved. □

6. Proof of Theorem 2.2

In this section all assumptions of Section 2 are supposed to be satisfied.

So far we did not use Assumption 2.5(i) concerning the existence of G vanishing on ∂D, which we need now to deal with the case of general D. Take an expanding sequence of smooth domains $D_n \subset D$ from Assumption 2.4 and introduce

$$v_n(x) = \inf_{\beta\in\mathbb{B}}\sup_{\alpha\in\mathfrak{A}} E_x^{\alpha.\beta(\alpha.)}\Big[\int_0^{\tau_n} f(p_t, x_t)e^{-\phi_t}\,dt + g(x_{\tau_n})e^{-\phi_{\tau_n}}\Big],$$

where $\tau_n^{\alpha.\beta.x}$ is the first exit time of $x_t^{\alpha.\beta.x}$ from $D(n)$. By Theorem 4.2 we have that v_n are continuous in $\mathbb{R}^d$ and

$$\begin{aligned} v_n(x) = \inf_{\beta\in\mathbb{B}}\sup_{\alpha.\in\mathfrak{A}} E_x^{\alpha.\beta(\alpha.)}\Big[&v_n(x_{\gamma_n})e^{-\phi_{\gamma_n}-\psi_{\gamma_n}} \\ &+ \int_0^{\gamma_n}\{f(p_t, x_t) + \lambda_t v_n(x_t)\}e^{-\phi_t-\psi_t}\,dt\Big], \end{aligned} \tag{6.1}$$

where $\gamma_n^{\alpha.\beta(\alpha.)x} = \gamma^{\alpha.\beta(\alpha.)x} \wedge \tau_n^{\alpha.\beta(\alpha.)x}$.

We claim that, as $n \to \infty$,

$$\kappa_n := \sup_{\mathbb{R}^d}|v_n - v| = \sup_D |v_n - v| \to 0, \tag{6.2}$$

which, in particular, would imply the continuity of v and the fact that v is independent of the choice of $p_t^{\alpha.\beta.}$.

To prove (6.2) introduce v_m and v_{nm} by replacing g with g_m in the definitions of v and v_n, respectively, where g_m are taken from the statement of the theorem. Observe that, obviously, $\sup_n |v_n - v_{nm}| + |v - v_m| \to 0$ as $m \to \infty$ uniformly on $\mathbb{R}^d$. Therefore, *while proving* (6.2) we may assume that $\|g\|_{C^2(D)} < \infty$ and g is p-insensitive.

Then notice that in such a case we have

$$E_x^{\alpha.\beta.}\Big[\int_0^{\tau} f(p_t, x_t)e^{-\phi_t}\,dt + g(x_\tau)e^{-\phi_{\gamma\wedge\tau}}\Big] = g(x) + E_x^{\alpha.\beta.}\int_0^{\tau} \hat{f}(p_t, x_t)e^{-\phi_t}\,dt,$$

where

$$\hat{f}^{\alpha\beta}(p,x) := \left[f^{\alpha\beta}(p,x) + r^{\alpha\beta}(p,x)\bar{L}^{\alpha\beta}g(x)\right]I_D(x)$$

satisfies Assumption 2.3(i)–(iii). Hence,

$$u(x) := v(x) - g(x) = \inf_{\beta\in\mathbb{B}}\sup_{\alpha\in\mathfrak{A}} E_x^{\alpha.\beta(\alpha.)}\int_0^\tau \hat{f}(p_t,x_t)e^{-\phi_t}\,dt.$$

This argument shows that we may also assume in the remaining part of the proof of (6.2) that $g = 0$. Then

$$v_n(x) = \inf_{\beta\in\mathbb{B}}\sup_{\alpha\in\mathfrak{A}} E_x^{\alpha.\beta(\alpha.)}\int_0^{\tau_n} f(p_t,x_t)e^{-\phi_t}\,dt.$$

Next, by using Itô's formula, for any $\alpha. \in \mathfrak{A}$, $\beta. \in \mathfrak{B}$, and $x \in \mathbb{R}^d$, we find that the process

$$G(x_{t\wedge\tau})e^{-\phi_{t\wedge\tau}} + \int_0^{t\wedge\tau} e^{-\phi_s}\,ds \tag{6.3}$$

is at least a local supermartingale, where

$$(x_t,\phi_t,\tau) = (x_t,\phi_t,\tau)^{\alpha.\beta.x}.$$

Since G is nonnegative in D, the process (6.3) is a supermartingale (constant if $x \notin D$).

Now, for $\chi > 0$ introduce

$$N_\chi = \sup_{(\alpha,\beta,x)\in A\times B\times D} |(\bar{f}^{\alpha\beta})^{(\chi)}|$$

and observe that by Lemmas 5.1 and 2.1

$$|v_n(x) - v(x)| \le I_n(x),$$

where

$$\begin{aligned}
I_n(x) &:= \sup_{\alpha.\in\mathfrak{A},\beta.\in\mathfrak{B}} E_x^{\alpha.\beta.}\int_{\tau_n}^\tau |f(p_t,x_t)|e^{-\phi_t}\,dt \\
&\le \delta_1^{-1}\sup_{\alpha.\in\mathfrak{A},\beta.\in\mathfrak{B}} E_x^{\alpha.\beta.}\int_{\tau_n}^\tau |\bar{f}(x_t)|e^{-\phi_t}\,dt \\
&\le \delta_1^{-1}N_\chi \sup_{\alpha.\in\mathfrak{A},\beta.\in\mathfrak{B}} E_x^{\alpha.\beta.}\int_{\tau_n}^\tau e^{-\phi_t}\,dt \\
&\quad + \sup_{\alpha.\in\mathfrak{A},\beta.\in\mathfrak{B}} E_x^{\alpha.\beta.}\int_0^\tau |\bar{f} - \bar{f}^{(\chi)}|(x_t)e^{-\phi_t}\,dt \\
&\le \delta_1^{-1}N_\chi \sup_{(\partial D_n)\setminus\partial D} G + N\left\|\sup_{\alpha,\beta}|\bar{f}^{\alpha,\beta} - (\bar{f}^{\alpha,\beta})^{(\chi)}|\right\|_{L_d(D)},
\end{aligned}$$

where N is independent of χ, n, and x. This and the fact that $G = 0$ on ∂D certainly imply (6.2).

After that (6.1) (cf. (3.19)) yields

$$v(x) \geq \inf_{\beta\in\mathbb{B}} \sup_{\alpha.\in\mathfrak{A}} E_x^{\alpha.\beta(\alpha.)}\Big[v(x_{\gamma_n})e^{-\phi_{\gamma_n}-\psi_{\gamma_n}} + \int_0^{\gamma_n}\{f(p_t,x_t)+\lambda_t v(x_t)\}e^{-\phi_t-\psi_t}\,dt\Big] - \kappa_n.$$

We use estimates like

$$|v(x_{\gamma_n})e^{-\phi_{\gamma_n}-\psi_{\gamma_n}} - v(x_\gamma)e^{-\phi_\gamma-\psi_\gamma}| = I_{\tau_n<\gamma}|v(x_{\tau_n})e^{-\phi_{\tau_n}-\psi_{\tau_n}} - v(x_\gamma)e^{-\phi_\gamma-\psi_\gamma}| \leq 2I_{\tau_n<\gamma}\sup_{t\in[\tau_n,\tau]}|v(x_t)|e^{-\phi_t},$$

$$I_{\tau_n<\gamma}\int_{\gamma_n}^{\gamma}\lambda_t|v(x_t)|e^{-\psi_t}\,dt \leq I_{\tau_n<\gamma}\sup_{t\in[\tau_n,\tau]}|v(x_t)|e^{-\phi_t},$$

where and sometimes in the future we drop the superscripts $\alpha.$, $\beta.$, and x for simplicity.

Then we see that

$$v(x) \geq \inf_{\beta\in\mathbb{B}} \sup_{\alpha.\in\mathfrak{A}} E_x^{\alpha.\beta(\alpha.)}\Big[v(x_\gamma)e^{-\phi_\gamma-\psi_\gamma} + \int_0^{\gamma}\{f(p_t,x_t)+\lambda_t v(x_t)\}e^{-\phi_t-\psi_t}\,dt\Big] - \kappa_n - J_n(x), \tag{6.4}$$

where

$$J_n(x) = I_n(x) + 3R_n(x),$$
$$R_n(x) := \sup_{(\alpha.,\beta.)\in\mathfrak{A}\times\mathfrak{B}} E_x^{\alpha.\beta.} I_{\tau_n<\gamma}\sup_{t\in[\tau_n,\tau]}|v(x_t)|e^{-\phi_t}.$$

To estimate $R_n(x)$ we observe that, by Lemmas 5.1 and 2.1 for $x\in\bar{D}$

$$|v(x)| \leq \delta_1^{-1}N_\chi G(x) + \varepsilon(\chi),$$

where

$$\varepsilon(\chi) = N\|\sup_{\alpha,\beta}|\bar{f}^{\alpha,\beta} - (\bar{f}^{\alpha,\beta})^{(\chi)}|\,\|_{L_d(D)} \to 0,$$

as $\chi \downarrow 0$ (uniformly with respect to x). Furthermore, since $G(x_{t\wedge\tau})\exp(-\phi_{t\wedge\tau})$ is a supermartingale, we have

$$E_x^{\alpha.\beta.} I_{\tau_n<\tau}\sup_{t\in[\tau_n,\tau]}[G(x_t)e^{-\phi_t}]^{1/2} \leq N\big[E_x^{\alpha.\beta.} I_{\tau_n<\tau}G(x_{\tau_n})e^{-\phi_{\tau_n}}\big]^{1/2},$$

where N is an absolute constant, and since v is bounded,

$$E_x^{\alpha.\beta.} I_{\tau_n<\tau}\sup_{t\in[\tau_n,\tau]}|v(x_t)|e^{-\phi_t} \leq NE_x^{\alpha.\beta.} I_{\tau_n<\tau}\sup_{t\in[\tau_n,\tau]}[|v(x_t)|e^{-\phi_t}]^{1/2}$$
$$\leq NN_\chi\big[E_x^{\alpha.\beta.} I_{\tau_n<\tau}G(x_{\tau_n})\big]^{1/2} + N\varepsilon^{1/2}(\chi),$$

where the constants N are independent of χ, n, and x. By assumption $G = 0$ on ∂D and therefore we have that

$$\sup_{(\alpha.,\beta.)\in\mathfrak{A}\times\mathfrak{B}} E_x^{\alpha.\beta.} I_{\tau_n<\tau} G(x_{\tau_n}) \to 0$$

as $n \to \infty$ (uniformly with respect to x). It follows that

$$\overline{\lim_{n\to\infty}} R_n(x) \le N\varepsilon^{1/2}(\chi).$$

Above we have also proved that

$$\overline{\lim_{n\to\infty}} I_n(x) \le N\varepsilon^{1/2}(\chi).$$

It follows now from (6.4) that

$$v(x) \ge \inf_{\beta\in\mathbb{B}} \sup_{\alpha.\in\mathfrak{A}} E_x^{\alpha.\beta(\alpha.)}\Big[v(x_\gamma)e^{-\phi_\gamma-\psi_\gamma} + \int_0^\gamma \{f(p_t,x_t)+\lambda_t v(x_t)\}e^{-\phi_t-\psi_t}\,dt\Big] - N\varepsilon^{1/2}(\chi),$$

which after sending $\chi \downarrow 0$ finally shows that $v(x)$ is greater than the right-hand side of (2.9). The reader understands that the opposite inequality is proved similarly and this brings the proof of the theorem to an end.

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