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Systems of quasi-variational inequalities related to the switching problem

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Abstract

We prove the existence of weak solution for a system of quasi-variational inequalities related to a switching problem with dynamic driven by operator associated with a semi-Dirichlet form and with measure data. We give a stochastic representation of solutions in terms of solutions of a system of reflected BSDEs with oblique reflection. As a by-product, we prove the existence of an optimal strategy in the switching problem and show regularity of the payoff function.

Keywords: Switching problem; Reflected BSDEs; Semi-Dirichlet forms; Quasi-variational inequalities; Penalization scheme

MSC: 35J75; 60J45

1. Introduction

Let E be a locally compact separable metric space, m be a Radon measure on E with full support, and let $(L, D(L))$ be the generator of a regular semi-Dirichlet form $(\mathcal{E}, D[\mathcal{E}])$ on $L^2(E; m)$. The class of such operators is quite wide. The model example of local operator associated with semi-Dirichlet form is the second order uniformly elliptic divergence form

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operator with bounded drift, i.e. operator of the form

$$L = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\Big(a_{ij}(\cdot)\frac{\partial}{\partial x_i}\Big) + \sum_{i=1}^{d} b^i(\cdot)\frac{\partial}{\partial x_i}. \tag{1.1}$$

As the example of nonlocal operator of this class may serve

$$L = \Delta^{\alpha(\cdot)}, \tag{1.2}$$

i.e. fractional Laplacian with possibly varying exponent $\alpha : E \to (0,2)$ satisfying some regularity assumptions.

In the paper we consider the following problem: for given functions $f^j : E \times \mathbb{R}^N \to \mathbb{R}$, $h_{j,i} : E \times \mathbb{R} \to \mathbb{R}$, $i, j = 1, \dots, N$, smooth (with respect to the capacity associated with $(\mathcal{E}, D[\mathcal{E}])$) measures $\mu^1, \dots, \mu^N$ on E and sets $A_1, \dots, A_N$ such that $A_j \subset \{1, \dots, j-1, j+1, \dots, N\}$ find a pair (u, ν) consisting of a function $u = (u^1, \dots, u^N) : E \to \mathbb{R}^N$ and a vector $\nu = (\nu^1, \dots, \nu^N)$ of smooth measures on E such that

$$\begin{cases} -Lu^j = f^j(\cdot, u) + \mu^j + \nu^j, \\ \displaystyle\int_E (u^j - \max_{i\in A_j} h_{j,i}(\cdot, u^i))\, d\nu^j = 0, \\ u^j \ge \max_{i\in A_j} h_{j,i}(\cdot, u^i), \qquad j = 1, \dots, N. \end{cases} \tag{1.3}$$

Intuitively, we are looking for u satisfying the equations $-Lu^j = f^j(\cdot, u) + \mu^j$ on the sets $\{u^j > \max_{i\in A_j} h_{j,i}(\cdot, u^i)\}$. The measure ν^j represents the amount of energy we have to add to the system to keep u^j above the obstacle $H^j(\cdot, u) := \max_{i\in A_j} h_{j,i}(\cdot, u^i)$. The second equation in (1.3) says that ν^j is minimal in the sense that it acts only when $u^j = H^j(\cdot, u)$.

Systems of the form (1.3) arise when considering the so-called switching problem. They were studied by many authors (see, e.g., [5,6,8,9,12,10,11,23,24]) in case L is a diffusion operator or diffusion operator perturbed by nonlocal operator associated with a Poisson measure, and the data are L^2-integrable (hence, in particular, $\mu^i = 0$, $i = 1, \dots, N$). Note also that in all the papers cited above f is Lipschitz continuous with respect to u and viscosity solutions are considered.

In the present paper we generalize the existing results on (1.3) in the sense that we consider quite general class of operators and measure data. We also considerably weaken the assumptions on f, because we only assume that it is quasi-monotone with respect to u.

When $h_{j,i}$ do not depend on u, system (1.3) resembles the usual system of variational inequalities written in complementary form (see [14] and also [16,18,19] for the case of one equation). Such a form has proved useful in the study of variational inequalities with measure data (see [18,20,27]). One of the main reason is that it allows one to use known results on semilinear elliptic PDEs with measure data. On the other hand, the usual variational approach is applicable only to systems with L^2-data.

Our general approach to (1.3) (system of quasi-variational inequalities in complementary form) is similar to that in [18,20]. It can be briefly described as follows. Let $\mathbb{X} = (\{X_t, t \ge 0\}, \{P_x, x \in E\})$ be a Hunt process with life time ζ associated with $(\mathcal{E}, D[\mathcal{E}])$, and for smooth measure γ let A^γ denote the continuous additive functional of $\mathbb{X}$ in the Revuz correspondence with γ. By a solution of (1.3) we mean a pair (u, ν) satisfying the second and the third condition in (1.3), and such that for quasi-every $x \in E$ (with respect to the capacity associated with $\mathcal{E}$) the

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following generalized nonlinear Feynman–Kac formula is satisfied

$$u(x)=E_x\Big(\int_0^\zeta f(X_r,u(X_r))\,dr+\int_0^\zeta dA^\nu_r+\int_0^\zeta dA^\mu_r\Big). \tag{1.4}$$

Note that from (1.3) one can often deduce some regularity properties of u. For instance, if μ is a measure of bounded variation, (u,ν) satisfies (1.4) and we know that $f(\cdot,u)\in L^1(E;m)$ and ν has also bounded variation, then $T_k u\in D_e[\mathcal{E}]$ for every $k>0$, where $D_e[\mathcal{E}]$ is the extended Dirichlet space for $\mathcal{E}$ and $T_k u(x)=((-k)\vee u(x))\wedge k$. In fact, (u,ν) is then a renormalized solution of the first equation in (1.3) in the sense introduced in [21] (for the case where L is of the form (1.1) see also [2] and [27]).

Roughly speaking, to find a solution (u,ν) of (1.3) in the sense described above we find a solution of some system of Markov-type BSDEs with oblique reflection associated with (1.3), and we study various properties of these solutions. Then, using some ideas from the papers [19,22] devoted to PDEs with measure data, we translate the results on these systems of reflected BSDEs into results on (1.3).

As a matter of fact, in the first part of the paper we study general, nonMarkov-type BSDEs. First, in Section 2, we give an existence result for solutions of system of BSDEs of the type

$$Y^j_t=\xi^j+\int_t^T f^j(r,Y_r)\,dr+\int_t^T dV^j_r-\int_t^T dM^j_r,\quad t\in[0,T],$$

where V is a finite variation càdlàg process, with quasi-monotone right-hand side f, i.e. off-diagonal increasing and on-diagonal decreasing. This type of equation was not considered in the literature in such generality. Then, in Section 3, we prove the existence of a solution of the system of RBSDEs with oblique reflection of the form

$$\begin{cases} Y^j_t=\xi^j+\int_t^T f^j(r,Y_r)\,dr+\int_t^T dV^j_r+\int_t^T dK^j_r-\int_t^T dM^j_r,\quad t\in[0,T],\\ Y^j_t\ge \max_{i\in A_j} h_{ji}(t,Y^i_t),\quad t\in[0,T],\\ \int_0^T (Y^j_{t-}-\max_{i\in A_j}[h_{j,i}(\cdot,Y^i_\cdot)]_{t-})\,dK^j_t=0,\ j=1,\dots,N. \end{cases} \tag{1.5}$$

This result generalizes the existence results proved for L^2-data and Brownian filtration (see, e.g., [10]) or filtration generated by a Brownian motion and an independent Poisson measure (see [11,23]) to the case of general filtration and L^1-data. Moreover, as compared with [10,11,23], we impose less restrictive assumptions on the off-diagonal growth of the right-hand side. We also allow the terminal time T to be unbounded stopping time. In Section 3 we also show that solution of (1.5) may be approximated by solutions of the system of penalized BSDEs

$$Y^{n,j}_t=\xi^j+\int_t^T f^j(r,Y^n_r)\,dr+\int_t^T dV^j_r+\int_t^T n(Y^{n,j}_r-H^j(r,Y^n_r))^-\,dr-\int_t^T dM^{n,j}_r$$

with H^j of the form

$$H^j(t,y)=\max_{i\in A_j} h_{j,i}(t,y^i).$$

In Section 4 we study the switching problem (we describe it briefly below) and its connection with reflected BSDEs. Therefore we restrict our attention to $h_{j,i}$ of the form

$$h_{j,i}(t,y)=c_{j,i}(t)-y^i \tag{1.6}$$

for some adapted continuous processes $c_{j,i}$ (in applications $c_{j,i}(t)$ is the cost of switching the process of, say production, from mode j to mode i in time t). Our main result says that if f in (1.5) does not depend on y then the first component Y of the solution of (1.5) is the value function of the switching problem.

In Section 5, using the results of the probabilistic part of the paper, we first give an existence result for (1.3), and we show that u may be approximated by solutions of the following system of penalized PDEs

$$-Lu_n^j = f^j(\cdot, u_n) + n(u_n^j - H^j(\cdot, u_n))^- + \mu$$

with

$$H^j(x, y) = \max_{i \in A_j} h_{j,i}(x, y^i).$$

We also give conditions ensuring that $f(\cdot, u) \in L^1(E; m)$ and the measures ν^j have bounded variation. In particular, under these conditions, $T_k(u^j) \in D_e[\mathcal{E}]$ and u^j is a renormalized solution of the first equation in (1.3) (see comment following (1.4)). We next turn to the switching problem of Section 4, but in the Markovian setting, i.e. in case $f^j(t, y) = f^j(X_t)$, $c_{j,i}(t) = c_{j,i}(X_t)$ for some $f^j, c_{j,i} : E \to \mathbb{R}$. The problem can be described as follows. Consider a factory in which we can change a mode of production. Let $c_{j,i}(X)$ be the cost of the change from mode j to mode i, and let $\psi_i(X) + dA^{\mu^i}$ be the payoff rate in mode i. Then a management strategy $\mathcal{S} = (\{\tau_n\}, \{\xi_n\})$ consists of a pair of two sequences of random variables. The variable τ_n is the moment when we decide to switch the mode of production, and ξ_n is the mode to which we switch at time τ_n. If $\xi_0 = j$ then we start the production at mode j. Under strategy $\mathcal{S}$ the expected profit on the interval $[0, T]$ is given by the formula

$$J(x, \mathcal{S}, j) = E_x\Big(\int_0^T \psi_{w_r}(X_r)\,dr + \int_0^T dA_r^{\mu^{w_r}} - \sum_{n\geq 1} c_{w_{\tau_{n-1}}, w_{\tau_n}}(X_{\tau_n})\mathbf{1}_{\{\tau_n < \zeta\}} + \xi^{w_T}\Big),$$

where

$$w_t = \xi_0 \mathbf{1}_{[0,\tau_1)}(t) + \sum_{n\geq 1} \xi_n \mathbf{1}_{[\tau_n, \tau_{n+1})}(t).$$

A strategy $\mathcal{S}^*$ is called optimal (for fixed j) if $J(x, \mathcal{S}^*, j) = \sup_{\mathcal{S}} J(x, \mathcal{S}, j)$. In Section 5, we show that under some assumption on the data there exists an optimal strategy, and moreover, if $T = \zeta$, then u defined by the formula

$$u^j(x) = J(x, \mathcal{S}^*, j)$$

is a unique solution of (1.3) with $h_{j,i}$ defined by (1.6).

2. Systems of BSDEs with quasi-monotone generator

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ be a filtration satisfying the usual conditions, and let T be a stopping time. We denote by $\mathcal{T}$ the set of all $\mathbb{F}$-stopping times such that $\tau \leq T$.

In what follows $N \in \mathbb{N}$, $\xi = (\xi^1, \ldots, \xi^N)$ is an $\mathcal{F}_T$-measurable random vector, $V = (V^1, \ldots, V^N)$ is an $\mathbb{F}$-adapted process such that $V_0 = 0$ and each component V^j is a process of finite variation, $f : \Omega \times [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ is a measurable function such that for every $y \in \mathbb{R}^N$ the process $f(\cdot, y)$ is $\mathbb{F}$-progressively measurable. As usual, in the sequel, in our notation we will omit the dependence of f on $\omega \in \Omega$.

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We set $|V|_t = \sum_{j=1}^{N}|V^j|_t$, where $|V^j|_t$ stands for the variation of V^j on $[0, t]$, and we adopt the following notation:

$$f^j(t, y; a) = f^j(t, y_1, \dots, y_{j-1}, a, y_{j+1}, \dots, y_N), \quad y \in \mathbb{R}^N,\ a \in \mathbb{R}$$

and

$$f^j_{\inf}(t, a) = \inf_{y \in \mathbb{R}^N} f^j(t, y; a), \quad f^j_{\sup}(t, a) = \sup_{y \in \mathbb{R}^N} f^j(t, y; a), \quad a \in \mathbb{R}.$$

For $x = (x^1, \dots, x^N)$ we set $|x| = \sum_{j=1}^{N}|x^j|$, and for $x, y \in \mathbb{R}^N$ we write $x \le y$ if $x^j \le y^j$, $j = 1, \dots, N$. For processes X, Y we write $X \le Y$ if $X_t \le Y_t$, $t \in [0, T \wedge a]$ for all $a \ge 0$, and $X = Y$ if $X \le Y$ and $X \ge Y$. The abbreviation ucp means "uniformly on compacts in probability".

The following assumptions will be needed throughout the paper.

(A1) $E(|\xi| + \int_0^T d|V|_r) < \infty$,
(A2) for every $t \in [0, T]$ the function $f(t, \cdot)$ is on-diagonal decreasing, i.e. for $j = 1, \dots, N$ we have $f^j(t, y; a) \ge f^j(t, y; a')$ for all $a \le a'$, $a, a' \in \mathbb{R}$, $y \in \mathbb{R}^N$,
(A3) for every $t \in [0, T]$ the function $f(t, \cdot)$ is off-diagonal increasing, i.e. for $j = 1, \dots, N$ we have $f^j(t, y; a) \le f^j(t, y'; a)$ for all $a \in \mathbb{R}$ and $y, y' \in \mathbb{R}^N$ such that $y \le y'$ (i.e. $y^j \le y'^j$, $j = 1, \dots, N$),
(A4) $y \mapsto f(t, y)$ is continuous for every $t \in [0, T]$,
(A5) $\int_0^T |f(r, y)|\, dr < \infty$ for all $y \in \mathbb{R}^N$.

Note that functions satisfying (A2) and (A3) are called quasi-monotone.

Recall that an adapted càdlàg process η is of class (D) if the collection $\{\eta_\tau : \tau$ is a finite valued stopping time$\}$ is uniformly integrable.

Definition 2.1. We say that a pair (Y, M) of N-dimensional $\mathbb{F}$-adapted processes is a solution of the system of backward stochastic differential equations on the interval $[0, T]$ with terminal condition ξ and right-hand side $f + dV$ (BSDE$^T(\xi, f + dV)$ for short) if

(i) Y^j is of class (D), M^j is a local martingale such that $M^j_0 = 0$, $j = 1, \dots, N$,
(ii) $\int_0^{T\wedge a}|f(r, Y_r)|\, dr < \infty$ for every $a \ge 0$,
(iii) for $j = 1, \dots, N$ and all $a \ge 0$,

$$Y^j_t = Y^j_{T\wedge a} + \int_t^{T\wedge a} f^j(r, Y_r)\, dr + \int_t^{T\wedge a} dV^j_r - \int_t^{T\wedge a} dM^j_r, \quad t \in [0, T \wedge a].$$

(iv) $Y_{T\wedge a} \to \xi$ P-a.s. as $a \to \infty$.

Remark 2.2. Let (Y, M) be a solution of BSDE$^T(\xi, f + dV)$. If

$$E\Big(|\xi| + \int_0^T |f(r, Y_r)|\, dr + \int_0^T d|V|_r\Big) < \infty, \tag{2.1}$$

then M is a uniformly integrable martingale and

$$Y^j_t = E(\xi^j + \int_t^T f^j(r, Y_r)\, dr + \int_t^T dV^j_r|\mathcal{F}_t), \quad t \le T, \quad j = 1, \dots, N. \tag{2.2}$$

To see this, we set $\tilde{M} = (\tilde{M}^1, \dots, \tilde{M}^N)$, where

$$\tilde{M}^j_t = E\Big(\xi^j + \int_0^T f^j(r, Y_r)\, dr + \int_0^T dV^j_r|\mathcal{F}_t\Big) - Y^j_0.$$

An elementary computation shows that $(Y, \tilde{M})$ is a solution of $\mathrm{BSDE}^T(\xi, f + dV)$. Hence $M = \tilde{M}$. Therefore we may pass to the limit as $a \to \infty$ in condition (iii) of the above definition. We then get

$$Y_t^j = \xi^j + \int_t^T f^j(r, Y_r)\,dr + \int_t^T dV_r^j - \int_t^T dM_r^j, \quad t \le T, \quad j = 1, \dots, N.$$

Since M is a uniformly integrable martingale, this yields (2.2).

2.1. One-dimensional equations

In this subsection we assume that $N = 1$.

Remark 2.3. Let $\eta_t = E(\xi|\mathcal{F}_t)$, $f_\eta(t, y) = f(t, y + \eta_t)$. If a pair $(\bar{Y}, \bar{M})$ is a solution of $\mathrm{BSDE}^T(0, f_\eta + dV)$, then the pair (Y, M) defined by

$$Y_t = \bar{Y}_t + \eta_t, \quad M_t = \bar{M}_t + \eta_t - \eta_0$$

is a solution of $\mathrm{BSDE}^T(\xi, f + dV)$.

Proposition 2.4. *Let (Y^i, M^i), $i = 1, 2$, be a solution to $\mathrm{BSDE}^T(\xi^i, f^i + dV^i)$ Assume that $\xi^1 \le \xi^2$, f^1 or f^2 satisfies (A2), $f^1(t, y) \le f^2(t, y)$ for all $y \in \mathbb{R}$ and a.e. $t \in [0, T]$, and that $dV^1 \le dV^2$. Then $Y_t^1 \le Y_t^2$, $t \in [0, T]$.*

Proof. By the Tanaka–Meyer formula, there exists an $\mathbb{F}$-adapted increasing process C with $C_0 = 0$ such that $((Y^1 - Y^2)^+, \int_0^\cdot \mathbf{1}_{\{Y_{r-}^1 > Y_{r-}^2\}}\,d(M_r^1 - M_r^2))$ is a solution to $\mathrm{BSDE}^T(0, \mathbf{1}_{\{Y^1 > Y^2\}}(f^1(\cdot, Y^1) - f^2(\cdot, Y^2)) + \mathbf{1}_{\{Y_{r-}^1 > Y_{r-}^2\}}\,d(V_r^1 - V_r^2) - dC)$. By the assumptions, the right-hand side of this BSDE is less than or equal to zero, so by [19, Proposition 3.1] we have $(Y^1 - Y^2)^+ = 0$. □

Proposition 2.5. *Let $\eta_t = E(\xi|\mathcal{F}_t)$, $t \ge 0$. If (A1), (A2), (A4), (A5) are satisfied, and moreover,*

$$E\int_0^T |f(r, \eta_r)|\,dr < \infty, \tag{2.3}$$

then there exists a solution of $\mathrm{BSDE}^T(\xi, f + dV)$.

Proof. Let $f_\eta(t, y) = f(t, y + \eta_t)$. Then by [19, Theorem 3.4] there exists a solution $(\bar{Y}, \bar{M})$ of $\mathrm{BSDE}^T(0, f_\eta + dV)$, and hence, by Remark 2.3, there exists a solution of $\mathrm{BSDE}^T(\xi, f + dV)$. □

Assumption (2.3) is quite natural in the theory of BSDEs with random terminal time (see, e.g., [1]). We would like, however, to weaken it and show that in fact assumptions (A1), (A2), (A4), (A5) together with (2.3) holding true with some semimartingale η of class (D) and integrable finite variation part are sufficient for the existence of a solution of $\mathrm{BSDE}^T(\xi, f + dV)$. That (2.3) can be weaken is quite easy to see in case T is finite. In the general case more work have to be done.

Remark 2.6. Condition (2.3) is too strong in many important application. To illustrate, let us consider the well known penalization scheme for reflected BSDE with terminal condition $\xi = 0$, coefficient equal to zero and lower barrier L, that is equation of the form

$$Y_t^n = \int_t^T n(Y_r^n - L_r)^-\,dr - \int_t^T dM_r^n. \tag{2.4}$$

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Of course, this is $\mathrm{BSDE}^T(0, f_n)$ with $f_n(t,y) = n(y - L_t)^-$. Suppose that $L_t = t^{-1}\mathbf{1}_{[1,\infty)}(t)$ and $T = \infty$. We then expect that there exists a solution (Y^n, M^n) of (2.4) and $\{Y^n\}$ converges to the Snell envelope of L, which exists since L is of class (D). Observe that in this example (2.3) does not hold with $\eta_t = E(\xi|\mathcal{F}_t) = 0$. However, (2.3) is satisfied with η replaced by the semimartingale L. The same phenomenon can happen for finite T. To see this, let us consider a finite stopping time τ such that $\tau \geq 1$ and $E \ln \tau = \infty$, and set $T = \tau + 1$. Let $L_t = t^{-1}\mathbf{1}_{[1,\tau)}(t)$. Then (2.3) is not satisfied with $\eta = 0$, but is satisfied with η replaced by the semimartingale L.

Lemma 2.7. *If (A2), (A4), (A5) are satisfied and*

$$E\Big(|\xi| + \int_0^T d|V|_r + \int_0^T |f(r,0)|\,dr\Big)^2 < \infty, \tag{2.5}$$

then there exists a solution of $\mathrm{BSDE}^T(\xi, f + dV)$.

Proof. Let g be a strictly positive function on $\mathbb{R}^+$ such that $\int_0^\infty g(r)\,dr < \infty$. Write

$$f_{n,m} = (f \wedge (n \cdot g)) \vee (-m \cdot g).$$

By Proposition 2.5, for all $n, m \in \mathbb{N}$ there is a solution $(Y^{n,m}, M^{n,m})$ of $\mathrm{BSDE}^T(\xi, f_{n,m} + dV)$. By Proposition 2.4, $Y^{n,m} \leq Y^{n+1,m}$. Set $Y^m_t = \sup_{n\geq 1} Y^{n,m}_t$. Applying the Tanaka–Meyer formula and (A2) we get

$$|Y^{n,m}_t| \leq E\Big(|\xi| + \int_0^T |f(r,0)|\,dr + \int_0^T d|V|_r|\mathcal{F}_t\Big) =: X_t, \quad t \leq T. \tag{2.6}$$

By (2.5), $E\sup_{t\geq 0}|X_t|^2 < \infty$, whereas by Remark 2.3 and [19, Lemma 2.3, Lemma 2.5],

$$\sup_{n,m\geq 1} E\Big(\int_0^T |f_{n,m}(r, Y^{n,m}_r)|\,dr\Big)^2 < \infty. \tag{2.7}$$

By Remark 2.2,

$$Y^{n,m}_t = E\Big(\xi + \int_t^T f_{n,m}(r, Y^{n,m}_r)\,dr + \int_t^T dV_r|\mathcal{F}_t\Big), \quad t \leq T.$$

Letting $n \to \infty$ in the above inequality and using (2.6), (A4), (A5), (2.7), we obtain

$$Y^m_t = E\Big(\xi + \int_t^T f(r, Y^m_r)\,dr + \int_t^T dV_r|\mathcal{F}_t\Big). \tag{2.8}$$

Set

$$M^m_t = E\Big(\xi + \int_0^T f(r, Y^m_r)\,dr + \int_0^T dV_r|\mathcal{F}_t\Big) - Y^m_0, \quad t \leq T.$$

Then the pair (Y^m, M^m) is a solution of $\mathrm{BSDE}^T(\xi, f_m + dV)$. Letting $m \to \infty$ in (2.8) and repeating the above argument, with obvious modification, shows the existence of a solution of $\mathrm{BSDE}^T(\xi, f + dV)$. □

Let us denote $T_k(y) = (y \wedge k) \vee (-k)$ for all $k \geq 0,\ y \in \mathbb{R}$.

Proposition 2.8. *Assume that (A1), (A2), (A4), (A5) are satisfied and*

$$E\int_0^T |f(r,0)|\,dr < \infty.$$

Then there exists a solution of $\mathrm{BSDE}^T(\xi, f + dV)$.

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Proof. Let $\xi_n = T_n(\xi)$, $V^n_t = \int_0^{t\wedge n} \mathbf{1}_{\{|V|_r\le n\}}\,dV_r$, and let

$$f_n(t,y) = f(t,y) - f(t,0) + T_n(f(t,0))\cdot g_n(t),$$

where $g_n(t) = 1/(1+t^2/n)$. Observe that the data ξ^n, V^n, f_n satisfy the assumptions of Lemma 2.7. Therefore, for every $n\ge 1$, there exists a solution (Y^n, M^n) of BSDE$^T(\xi_n, f_n + dV^n)$. By the Tanaka–Meyer formula and (A2), for $n<m$ we have

$$|Y^n_t - Y^m_t| \le E\Big(|\xi_n-\xi_m| + \int_n^T d|V|_r + \int_0^T \mathbf{1}_{\{n<|V|_r\le m\}}\,d|V|_r + \int_0^T |T_n(f(r,0))g_n(r) - T_m(f(r,0))g_m(r)|\,dr|\mathcal{F}_t\Big).$$

By [1, Lemma 6.1],

$$E\sup_{t\ge 0}|Y^n_t - Y^m_t|^q \to 0. \tag{2.9}$$

for every $q\in(0,1)$. It follows in particular that there is an adapted càdlàg process Y such that $Y^n\to Y$ in ucp. By the Tanaka–Meyer formula,

$$|Y^n_t| \le E\Big(|\xi| + \int_0^T |f(r,0)|\,dr + \int_0^T d|V|_r|\mathcal{F}_t\Big) =: X_t. \tag{2.10}$$

Furthermore, by Remark 2.3, [19, Lemma 2.3] and Fatou's lemma,

$$E\int_0^T |f(r,Y_r)|\,dr \le E\Big(|\xi| + \int_0^T |f(r,0)|\,dr + \int_0^T d|V|_r|\mathcal{F}_t\Big). \tag{2.11}$$

Set

$$\tau_k = \inf\{t\ge 0 : \int_0^t |f(r,X_r)|\,dr \ge k\}.$$

For every $a\ge 0$ we have

$$Y^n_t = E\Big(Y^n_{\tau_k\wedge a} + \int_t^{\tau_k\wedge a} f_n(r,Y^n_r)\,dr + \int_t^{\tau_k\wedge a} dV^n_r|\mathcal{F}_t\Big),\quad t\le \tau_k\wedge a.$$

Letting $n\to\infty$ in the above equality and using (A4), (A5) and (2.9), (2.10) we get

$$Y_t = E\Big(Y_{\tau_k\wedge a} + \int_t^{\tau_k\wedge a} f(r,Y_r)\,dr + \int_t^{\tau_k\wedge a} dV_r|\mathcal{F}_t\Big),\quad t\le \tau_k\wedge a.$$

Letting now $k, a\to\infty$ in the above equality and using (2.9), (2.11) we obtain

$$Y_t = E\Big(\xi + \int_t^T f(r,Y_r)\,dr + \int_t^T dV_r|\mathcal{F}_t\Big),\quad t\le T.$$

Set

$$M_t = E\Big(\xi + \int_0^T f(r,Y_r)\,dr + \int_0^T dV_r|\mathcal{F}_t\Big) - Y_0,\quad t\le T.$$

It is easily seen that the pair (Y, M) is a solution of BSDE$^T(\xi, f+dV)$. □

Theorem 2.9. *Let (A1), (A2), (A4), (A5) be satisfied. Assume also that there exists a semimartingale S such that S is a difference of supermartingales of class* (D) *and*

$$E\int_0^T |f(r,S_r)|\,dr < \infty.$$

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Then there exists a solution (Y, M) *of* $\mathrm{BSDE}^T(\xi, f + dV)$*. Moreover,*

$$E\int_0^T |f(r, Y_r)|\,dr \le E\Big(|\xi| + |S_T| + \int_0^T |f(r, S_r)|\,dr + E\int_0^T d|V|_r + \int_0^T d|C|_r\Big),$$

where $S_t = S_0 + C_t + N_t$ *is the Doob–Meyer decomposition of* S*.*

Proof. Set

$$f_S(t, y) = f(t, S_t + y), \quad \tilde{\xi} = \xi - S_T, \quad \tilde{V}_t = V_t - C_t.$$

By Proposition 2.8 there exists a unique solution $(\tilde{Y}, \tilde{M})$ of $\mathrm{BSDE}^T(\tilde{\xi}, f_S + d\tilde{V})$. Set $(Y, M) = (\tilde{Y} + S, \tilde{M} + N)$. Then (Y, M) is a solution of $\mathrm{BSDE}^T(\xi, f + dV)$. By Remark 2.3, [19, Lemma 2.3],

$$E\int_0^T |f_S(r, \tilde{Y}_r)|\,dr \le E(|\tilde{\xi}| + \int_0^T |f_S(r, 0)|\,dr + \int_0^T d|\tilde{V}|_r),$$

which implies the desired inequality. □

2.2. Systems of equations

In the rest of this section we assume that $N \ge 1$.

Definition 2.10. We say that a pair (Y, M) is a subsolution (resp. supersolution) of $\mathrm{BSDE}^T(\xi, f + dV)$ if there exist $\underline{\xi}, \underline{V}, \underline{f}$ (resp. $\overline{\xi}, \overline{V}, \overline{f}$) satisfying (A1), (A2) such that $\underline{\xi} \le \xi$, $d\underline{V} \le dV$, $\underline{f}(t, y) \le f(t, y)$ for $y \in \mathbb{R}^N$, $t \in [0, T]$, $E\int_0^T |\underline{f}(r, Y_r)|\,dr < \infty$ (resp. $\overline{\xi} \ge \xi$, $d\overline{V} \ge dV$, $\overline{f}(t, y) \ge f(t, y)$ for $y \in \mathbb{R}^N$, $t \in [0, T]$, $E\int_0^T |\overline{f}(r, Y_r)\,dr| < \infty$), and (Y, M) is a solution of $\mathrm{BSDE}^T(\underline{\xi}, \underline{f} + d\underline{V})$ (resp. $\mathrm{BSDE}^T(\overline{\xi}, \overline{f} + d\overline{V})$).

We will make the following assumption:

(A6) there exist a subsolution $(\underline{Y}, \underline{M})$ and a supersolution $(\overline{Y}, \overline{M})$ of $\mathrm{BSDE}^T(\xi, f + dV)$ such that

$$\underline{Y} \le \overline{Y}, \qquad \sum_{j=1}^N E\Big(\int_0^T |f^j(r, \underline{Y}_r; S_r^j)|\,dr + \int_0^T |f^j(r, \overline{Y}_r; S_r^j)|\,dr\Big) < \infty$$

for some semimartingale S which is a difference of supermartingales of class (D).

Example 2.11. Let the assumptions (A1)–(A5) hold. If $f_{\sup}, f_{\inf}$ satisfy (A4), (A5) and

$$\sum_{j=1}^N E\Big(\int_0^T |f^j_{\sup}(r, S_r^j)|\,dr + \int_0^T |f^j_{\inf}(r, S_r^j)|\,dr\Big) < \infty, \tag{2.12}$$

for some semimartingale S which is a difference of supermartingales of class (D), then (A6) is satisfied with $(\underline{Y}^j, \underline{M}^j)$, $(\overline{Y}^j, \overline{M}^j)$ being solutions of $\mathrm{BSDE}^T(\xi^j, f^j_{\inf} + dV^j)$ and $\mathrm{BSDE}^T(\xi^j, f^j_{\sup} + dV^j)$, respectively.

Example 2.12. Assume that (A1), (A4), (A5) are satisfied, T is bounded and f is Lipschitz continuous in y uniformly in t, i.e. there exists $L > 0$ such that

$$|f(t, y) - f(t, y')| \le L|y - y'|, \quad y, y' \in \mathbb{R}^N. \tag{2.13}$$

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Then (A6) is satisfied by the pairs $(\overline{Y}, \overline{M})$, $(\underline{Y}, \underline{M})$ defined by

$$\overline{Y}^1 = \overline{Y}^2 = \cdots = \overline{Y}^N, \qquad \overline{M}^1 = \overline{M}^2 = \cdots = \overline{M}^N,$$
$$\underline{Y}^1 = \underline{Y}^2 = \cdots = \underline{Y}^N, \qquad \underline{M}^1 = \underline{M}^2 = \cdots = \underline{M}^N,$$

where $(\overline{Y}^1, \overline{M}^1)$, $(\underline{Y}^1, \underline{M}^1)$ are solutions of $\mathrm{BSDE}^T(\xi^1 \vee \cdots \vee \xi^N, f^1 \vee \cdots \vee f^N + dV^1 \vee \cdots \vee dV^N)$ and $\mathrm{BSDE}^T(\xi^1 \wedge \cdots \wedge \xi^N, f^1 \wedge \cdots \wedge f^N + dV^1 \wedge \cdots \wedge dV^N)$, respectively.

Theorem 2.13. *Let the assumptions (A1)–(A5) hold, and let (A6) be satisfied with some processes $\underline{Y}$, $\overline{Y}$. Then there exists a minimal solution (Y, M) of $\mathrm{BSDE}^T(\xi, f + dV)$ such that $\underline{Y} \le Y \le \overline{Y}$. Moreover,*

$$E \int_0^T |f(r, Y_r)|\, dr < \infty \tag{2.14}$$

and M is a uniformly integrable martingale.

Proof. Let $(\overline{Y}, \overline{M})$, $(\underline{Y}, \underline{M})$ be as in (A6). Let $Y^0 := \underline{Y}$ and $(Y^{n,j}, M^{n,j})$, $j = 1, \ldots, N$, be a solution of $\mathrm{BSDE}^T(\xi^j, f^j(\cdot, Y^{n-1}; \cdot) + dV^j)$. Then

$$\begin{aligned} Y_t^{n,j} &= Y_{T\wedge a}^{n,j} + \int_t^{T\wedge a} f^j(r, Y_r^{n-1}; Y_r^{n,j})\, dr \\ &\quad + \int_t^{T\wedge a} dV_r^j - \int_t^{T\wedge a} dM_r^{n,j}, \qquad t \in [0, T \wedge a]. \end{aligned} \tag{2.15}$$

By Proposition 2.4,

$$Y^n \le Y^{n+1}, \qquad Y^n \le \overline{Y}. \tag{2.16}$$

Therefore, letting $n \to \infty$ in (2.15), we get

$$Y_t^j = Y_{T\wedge a}^j + \int_t^{T\wedge a} f^j(r, Y_r)\, dr + \int_t^{T\wedge a} dV_r^j - \int_t^{T\wedge a} dM_r^j, \qquad t \in [0, T \wedge a],$$

where $Y_t = \lim_{n\to\infty} Y_t^n$ and $M_t = \lim_{n\to\infty} M_t^n$, $t \in [0, T \wedge a]$. The process M is a local martingale, because by (2.16) the sequence $\{M^n\}$ is locally uniformly integrable as all the other terms in (2.15) are locally uniformly integrable with respect to n. To show that the pair (Y, M) is a solution of $\mathrm{BSDE}^T(\xi, f + dV)$ it remains to prove that $Y_{T\wedge a} \to \xi$ as $a \to \infty$. If T is finite, this follows immediately from the fact that $Y_t^n \nearrow Y_t$, $t \le T$. In general case an additional argument is required. By Theorem 2.9 there exist a solution $(\overline{X}^j, \overline{N}^j)$ of $\mathrm{BSDE}^T(\xi, f^j(\overline{Y}; \cdot) + dV^j)$ and a solution $(\underline{X}^j, \underline{N}^j)$ of $\mathrm{BSDE}^T(\xi, f^j(\underline{Y}; \cdot) + dV^j)$. Moreover, by Proposition 2.4, $\underline{X}_t \le Y_t^n \le \overline{X}_t$, $t \in [0, T \wedge a]$, $a \ge 0$, which implies the desired convergence. By (2.16), (A6) and Theorem 2.9,

$$\begin{aligned} E \int_0^T |f^j(r, Y_r)|\, dr &\le E\Big(|\xi^j| + |S_T^j| + \int_0^T |f^j(r, Y_r; S_r^j)|\, dr \\ &\qquad + \int_0^T d|V^j|_r + \int_0^T d|C^j|_r\Big) \\ &\le E\Big(|\xi^j| + |S_T^j| + \int_0^T d|V^j|_r + \int_0^T d|C^j|_r\Big) \\ &\qquad + E\Big(\int_0^T |f^j(r, \underline{Y}_r; S_r^j)|\, dr + \int_0^T |f^j(r, \overline{Y}_r; S_r^j)|\, dr\Big) < \infty. \end{aligned}$$

From this and the fact that Y is of class (D) we conclude that

$$M_t = E\big(\xi + \int_0^T f(r, Y_r)\,dr + \int_0^T dV_r|\mathcal{F}_t\big) - Y_0, \quad t \in [0, T].$$

It follows that M is a uniformly integrable martingale. Let (Y^*, M^*) be a solution of $\mathrm{BSDE}^T(\xi, f + dV)$ such that $\underline{Y} \le Y^* \le \overline{Y}$. Then by Proposition 2.4, $Y^n \le Y^*$, $n \ge 0$, which implies that $Y \le Y^*$. □

Corollary 2.14. *Assume that the data (ξ, f, V), (ξ', f', V') satisfy (A1)–(A5), and that (A6) is satisfied with the same processes $\underline{Y}$, $\overline{Y}$. Moreover, assume that*

$$\xi \le \xi', \quad f \le f', \quad dV \le dV',$$

and that (Y, M) (resp. (Y', M')) is the minimal solution of $\mathrm{BSDE}^T(\xi, f + dV)$ (resp. $\mathrm{BSDE}^T(\xi', f' + dV')$) such that $\underline{Y} \le Y \le \overline{Y}$ (resp. $\underline{Y} \le Y' \le \overline{Y}$). Then

$$Y_t \le Y'_t \quad t \in [0, T].$$

Proof. Follows from the construction of processes Y, Y' (see Theorem 2.13) and Proposition 2.4. □

3. Systems of BSDEs with oblique reflection

Consider a family $\{h_{j,i}; i, j = 1, \dots, N\}$ of measurable functions $h_{j,i} : \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ such that $h_{j,i}(\cdot, y^i)$ is progressively measurable for every $y^i \in \mathbb{R}$. For given sets $A_j \subset \{1, \dots, j-1, j+1, \dots, N\}$, $j = 1, \dots, N$, set

$$H^j(t, y) = \max_{i \in A_j} h_{j,i}(t, y^i), \quad H(t, y) = (H^1(t, y), \dots, H^N(t, y)), \quad t \in \mathbb{R}^+,\ y \in \mathbb{R}^N.$$

We adopt the convention that the maximum over the empty set equals $-\infty$. Consequently, if $A_j = \emptyset$ for some j, then $H^j(t, y) = -\infty$.

Apart from (A1)–(A5) we will also need the following assumptions:

(A7) There exist a subsolution $(\underline{Y}, \underline{M})$ and a supersolution $(\overline{Y}, \overline{M})$ of $\mathrm{BSDE}^T(\xi, f + dV)$ such that

$$H(\cdot, \overline{Y}) \le \overline{Y}, \quad \underline{Y} \le \overline{Y}, \quad \sum_{j=1}^N E\Big(\int_0^T |f^j(r, \underline{Y}_r; \overline{Y}^j_r)|\,dr + \int_0^T |f^j(r, \overline{Y}_r)|\,dr\Big) < \infty,$$

(A8) $(t, y) \mapsto H^j(t, y)$ is continuous, $y \mapsto H^j(t, y)$ is nondecreasing and

$$\limsup_{(t,y)\to(\infty,\xi)} H^j(t, y) \le \xi^j.$$

Example 3.1. Let the assumptions (A1)–(A5) hold. Moreover, assume that $f_{\sup}$, $f_{\inf}$ satisfy (A4), (A5), (2.12) (with $S^1 = \dots = S^N$), and $h_{j,i}(t, a) \le a$ for every $a \in \mathbb{R}$. Let

$$\overline{Y}^1 = \overline{Y}^2 = \dots = \overline{Y}^N, \qquad \overline{M}^1 = \overline{M}^2 = \dots = \overline{M}^N,$$

where $(\overline{Y}^1, \overline{M}^1)$ is a solution of $\mathrm{BSDE}^T(\sum_{j=1}^N \xi^{j,+}, \sum_{j=1}^N (f^{j,+}_{\sup} + dV^{j,+}))$. By $(\underline{Y}^j, \underline{M}^j)$ denote a solution of $\mathrm{BSDE}^T(\xi^j, f^j_{\inf} + dV^j)$. The solutions $(\overline{Y}^1, \overline{M}^1)$, $(\underline{Y}^j, \underline{M}^j)$ exist by Theorem 2.9. By Proposition 2.4, $\underline{Y} \le \overline{Y}$. It follows that the pair $(\underline{Y}, \overline{Y})$ satisfies (A7).

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Example 3.2. Let the assumptions of Example 2.12 hold, and let $h_{j,i}(t,a) \le a$ for every $a \in \mathbb{R}$. Then the processes $(\overline{Y}, \overline{M})$, $(\underline{Y}, \underline{M})$ defined in Example 2.12 satisfy (A7).

Definition 3.3. We say that a triple (Y, M, K) of adapted càdlàg processes is a solution of BSDE with oblique reflection (1.5) if Y is of class (D), M is a local martingale with $M_0 = 0$, K is an increasing process with $K_0 = 0$ and (1.5) is satisfied.

If $A_j = \emptyset$, then by convention, $H^j = -\infty$, and hence Y^j has no lower barrier. We then take $K^j = 0$ in the above definition.

3.1. One-dimensional reflected BSDEs

In the whole subsection we will assume that $N = 1$. Recall the following definition from [17].

Definition 3.4. Let L be a càdlàg process. We say that a triple (Y, M, K) of adapted càdlàg processes is a solution of reflected BSDE on the interval $[0, T]$ with terminal condition ξ, right-hand side $f + dV$ and lower barrier L ($\mathrm{RBSDE}^T(\xi, f + dV, L)$ for short) if

(i) Y is of class (D), M is a local martingale with $M_0 = 0$, K is an increasing process with $K_0 = 0$,
(ii) $Y_t \ge L_t$, $t \in [0, T \wedge a]$, $\int_0^{T\wedge a} (Y_{t-} - L_{t-})\,dK_t = 0$ for every $a \ge 0$,
(iii) $\int_0^{T\wedge a} |f(t, Y_t)|\,dt < \infty, \quad a \ge 0$,
(iv) for every $a \ge 0$,

$$Y_t = Y_{T\wedge a} + \int_t^{T\wedge a} f(r, Y_r)\,dr + \int_t^{T\wedge a} dV_r + \int_t^{T\wedge a} dK_r - \int_t^{T\wedge a} dM_r,\\ t \in [0, T \wedge a],$$

(v) $Y_{T\wedge a} \to \xi$ P-a.s. as $a \to \infty$.

Observe that a triple (Y, M, K) is a solution of (1.5) if and only if (Y^j, M^j, K^j) is a solution of $\mathrm{RBSDE}^T(\xi^j, f^j(\cdot, Y; \cdot) + dV^j, H^j(\cdot, Y))$ for every $j = 1, \dots, N$.

Remark 3.5. If (2.1) is satisfied then $EK_T < \infty$, M is a uniformly integrable martingale and

$$Y_t = \xi + \int_t^T f(r, Y_r)\,dr + \int_t^T dV_r + \int_t^T dK_r - \int_t^T dM_r, \qquad t \in [0, T].$$

Indeed, localizing the local martingale M we easily deduce that $EK_T < \infty$. The remaining two assertions then follow from Remark 2.2.

Remark 3.6. Let (Y, M, K) be a solution of $\mathrm{RBSDE}^T(\xi, f + dV, L)$. Under the assumptions of Remark 3.5,

$$Y_t = \operatorname*{ess\,sup}_{\tau \ge t} E\Big(\int_t^{T\wedge\tau} f(r, Y_r)\,dr + \int_t^{T\wedge\tau} dV_r + L_\tau \mathbf{1}_{\{\tau<T\}} + \xi \mathbf{1}_{\{T\wedge\tau=T\}} | \mathcal{F}_t\Big). \tag{3.1}$$

To see this, we first observe that by Remark 3.5, for every stopping time $\tau \ge t$,

$$\begin{aligned} Y_t &= E\Big(Y_{T\wedge\tau} + \int_t^{T\wedge\tau} f(r, Y_r)\,dr + \int_t^{T\wedge\tau} dK_r + \int_t^{T\wedge\tau} dV_r | \mathcal{F}_t\Big) \\ &\ge E\Big(\int_t^{T\wedge\tau} f(r, Y_r)\,dr + \int_t^{T\wedge\tau} dV_r + L_\tau \mathbf{1}_{\{\tau<T\}} + \xi \mathbf{1}_{\{T\wedge\tau=T\}} | \mathcal{F}_t\Big). \end{aligned}$$

This shows that Y_t is greater than or equal to the right-hand side of (3.1). To get the opposite inequality, we consider the stopping time

$$D_t^\varepsilon = \inf\{s \ge t,\ L_s + \varepsilon \ge Y_s\} \wedge T.$$

By the minimality property of K,

$$\begin{aligned} Y_t &= E(Y_{D_t^\varepsilon} + \int_t^{D_t^\varepsilon} f(r, Y_r)\, dr + \int_t^{D_t^\varepsilon} dV_r | \mathcal{F}_t) \\ &\le E(L_{D_t^\varepsilon} \mathbf{1}_{\{D_t^\varepsilon < T\}} + \xi \mathbf{1}_{\{D_t^\varepsilon = T\}} + \int_t^{D_t^\varepsilon} f(r, Y_r)\, dr + \int_t^{D_t^\varepsilon} dV_r | \mathcal{F}_t) + \varepsilon, \end{aligned}$$

from which it follows the result.

In [17] an existence and comparison result (see [17, Proposition 2.1, Theorem 2.13]) for RBSDE$^T(\xi, f + dV, L)$ are proved under the assumption that T is bounded. Proposition 2.1 in [17] holds also for arbitrary stopping time and its proof goes as the proof of [17, Proposition 2.1] with obvious changes. For the convenience of the reader we will formulate [17, Proposition 2.1] for arbitrary stopping time T. The proof of an existence result in the case of arbitrary T requires major modification of the proof given in [17]. For this we will prove Lemma 3.8 and next give an existence result in Theorem 3.9.

Proposition 3.7. *Let (Y^i, M^i, K^i) be a solution of RBSDE$^T(\xi^i, f^i + dV^i, L^i)$, $i = 1, 2$. Assume that $\xi^1 \le \xi^2$, $L^1 \le L^2$, $dV^1 \le dV^2$, and either f^1 satisfies (A2) and $f^1(t, Y_t^2) \le f^2(t, Y_t^2)$ for a.e. $t \in [0, T]$ or f^2 satisfies (A2) and $f^1(t, Y_t^1) \le f^2(t, Y_t^1)$ for a.e. $t \in [0, T]$. Then $Y_t^1 \le Y_t^2$, $t \in [0, T]$.*

Lemma 3.8. *Assume that L^+ is of class (D), $E|\xi| < \infty$ and $\limsup_{a\to\infty} L_{T\wedge a} \le \xi$. Then*

$$\limsup_{a\to\infty} Y_{T\wedge a} \le \xi, \tag{3.2}$$

where

$$Y_t = \operatorname*{ess\,sup}_{\tau \ge t} E(L_\tau \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{T\wedge\tau = T\}} | \mathcal{F}_t). \tag{3.3}$$

Proof. From the definition of Y it follows that $Y_t = Y_{T\wedge t}$. Therefore the assertion of the lemma is clear if $T < \infty$. Let $\varepsilon > 0$. By the assumptions of the lemma, for a.e. $\omega \in \Omega$ there exists t_ω such that

$$L_t(\omega) \le \xi(\omega) + \varepsilon, \quad t \ge t_\omega.$$

Let

$$\Lambda_n = \{\omega \in \Omega;\ t_\omega \ge n\}.$$

It is clear that $\Lambda_{n+1} \subset \Lambda_n$ and $P(\bigcap_{n\ge1} \Lambda_n) = 0$. Since L^+ is of class (D), there is $\delta > 0$ such that if $A \in \mathcal{F}$ and $P(A) < \delta$ then $\sup_\tau \int_A (L_\tau^+ \mathbf{1}_{\{T\wedge\tau < T\}} + |\xi|) \le \varepsilon$. Choose $N \in \mathbb{N}$ so that $P(\Lambda_N) \le \delta$.

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Then for $t \ge N$,

$$\begin{aligned}Y_t &= \operatorname*{ess\,sup}_{\tau\ge t} E((L_\tau \mathbf{1}_{\{\tau<T\}} + \xi\mathbf{1}_{\{T\wedge\tau=T\}})\mathbf{1}_{\Lambda_N^c}|\mathcal{F}_t)\\ &\quad + \operatorname*{ess\,sup}_{\tau\ge t} E((L_\tau \mathbf{1}_{\{\tau<T\}} + \xi\mathbf{1}_{\{T\wedge\tau=T\}})\mathbf{1}_{\Lambda_N}|\mathcal{F}_t)\\ &\le \varepsilon + E(\xi\mathbf{1}_{\Lambda_N^c}|\mathcal{F}_t) + \operatorname*{ess\,sup}_{\tau\ge t} E((L^+_\tau \mathbf{1}_{\{\tau<T\}} + |\xi|\mathbf{1}_{\{T\wedge\tau=T\}})\mathbf{1}_{\Lambda_N}|\mathcal{F}_t)\\ &\le 2\varepsilon + E(\xi\mathbf{1}_{\Lambda_N^c}|\mathcal{F}_t).\end{aligned}$$

Letting $t \to \infty$ and then $N \to \infty$ yields $\limsup_{t\to\infty} Y_t \le 2\varepsilon + \xi$, which implies (3.2). □

Theorem 3.9. *Assume that (A1), (A2), (A4), (A5) are satisfied and L is a càdlàg adapted process such that* $\limsup_{a\to\infty} L_{T\wedge a} \le \xi$ *and* $L \le X$ *for some semimartingale* X *such that* X *is a difference of supermartingales of class (D) and*

$$E\int_0^T |f(r, X_r)|\,dr < \infty.$$

Then there exists a solution (Y, M, K) *of* $\mathrm{RBSDE}^T(\xi, f + dV, L)$*. Moreover,*

$$E\int_0^T |f(r, Y_r)|\,dr + EK_T < \infty$$

and M *is a uniformly integrable martingale.*

Proof. The proof runs as the proof of [17, Theorem 2.13], with small modifications. By Theorem 2.9 there exists a solution (Y^n, M^n) of $\mathrm{BSDE}^T(\xi, f_n + dV)$ with

$$f_n(t, y) = f(t, y) + n(y - L_t)^-.$$

By Proposition 2.4, $Y^n \le Y^{n+1}$. As in [17] we construct a supersolution $(\overline{X}, \overline{N})$ of $\mathrm{BSDE}^T(\xi, f + dV)$ such that $\overline{X} \ge L$ and

$$Y^1 \le Y^n \le \overline{X}, \quad n \ge 1. \tag{3.4}$$

By Theorem 2.9,

$$E\int_0^T |f(r, Y^1_r)|\,dr + E\int_0^T |f(r, \overline{X}_r)|\,dr < \infty. \tag{3.5}$$

Therefore by (A2), (3.4) and the Lebesgue dominated convergence theorem,

$$E\int_0^T |f(r, Y^n_r) - f(r, Y_r)|\,dr \to \infty, \tag{3.6}$$

where $Y_t = \sup_{n\ge1} Y^n_t$, $t \ge 0$. Repeating now, on each interval $[0, T\wedge a]$, the reasoning following (2.22) in the proof of [17, Theorem 2.13] we show that Y is càdlàg and there exists a predictable càdlàg increasing process K with $K_0 = 0$ and a local martingale M with $M_0 = 0$ such that for every $a \ge 0$,

$$Y_t = Y_{T\wedge a} + \int_t^{T\wedge a} f(r, Y_r)\,dr + \int_t^{T\wedge a} dV_r + \int_t^{T\wedge a} dK_r - \int_t^{T\wedge a} dM_r,$$
$$t \in [0, T\wedge a]$$

and

$$Y \geq L, \quad \int_0^{T\wedge a} (Y_{r-} - L_{r-})\,dK_r = 0.$$

By (3.4), Y is of class (D), which combined with (3.5) yields $E\int_0^T |f(r, Y_r)|\,dr + EK_T < \infty$. This inequality implies that M is a uniformly integrable martingale (see Remark 3.5). What is left is to show that $Y_{T\wedge a} \to \xi$, $a \to \infty$. By (3.4) $\xi \leq \liminf_{a\to\infty} Y_{T\wedge a}$, so it suffices to show that

$$\limsup_{a\to\infty} Y_{T\wedge a} \leq \xi. \tag{3.7}$$

Observe that the triple (Y^n, M^n, K^n), where $K^n_t = \int_0^t (Y^n_r - L_r)^-\,dr$, is a solution of $\mathrm{RBSDE}^T(\xi, f + dV, L^n)$ with $L^n_t = L_t - (Y^n_t - L_t)^-$. Therefore by Remark 3.6 and the definition of L^n,

$$Y^n_t \leq \operatorname*{ess\,sup}_{\tau\geq t} E\Big(\int_t^{T\wedge\tau} f(r, Y^n_r)\,dr + \int_t^{T\wedge\tau} dV_r + L_\tau \mathbf{1}_{\{T\wedge\tau<T\}} + \xi\mathbf{1}_{\{T\wedge\tau=T\}}|\mathcal{F}_t\Big).$$

Letting $n \to$ and using (3.6) we get

$$Y_t \leq \operatorname*{ess\,sup}_{\tau\geq t} E\Big(\int_t^{T\wedge\tau} f(r, Y_r)\,dr + \int_t^{T\wedge\tau} dV_r + L_\tau \mathbf{1}_{\{T\wedge\tau<T\}} + \xi\mathbf{1}_{\{T\wedge\tau=T\}}|\mathcal{F}_t\Big).$$

From this and Lemma 3.8 we conclude that (3.7) is satisfied. □

To prove the existence result for (1.5) we will need the monotone convergence theorem for BSDEs stated below. In the case of Brownian filtration this result was proved in [15,26]. In the case of general filtration it follows from [17].

Proposition 3.10. *Let (A1) be satisfied. Assume that (Y^n, M^n) is a solution of $\mathrm{BSDE}^T(\xi, dV^n + dK^n)$, where K^n is an increasing predictable càdlàg process such that $K^n_0 = 0$, and V^n is a finite variation càdlàg process with $V^n_0 = 0$. Moreover, assume that $Y^n \leq Y^{n+1}$, there exists a càdlàg process $\overline{Y}$ of class D such that $Y^n \leq \overline{Y}$, and that $\{|V^n|\}$ is locally bounded in L^2 and $V^n \to V$ in ucp for some finite variation càdlàg process V. Then there exists a local martingale M with $M_0 = 0$ and a predictable càdlàg increasing process K with $K_0 = 0$ such that for every $a \geq 0$,*

$$Y_t = Y_{T\wedge a} + \int_t^{T\wedge a} dV_r + \int_t^{T\wedge a} dK_r - \int_t^{T\wedge a} dM_r, \quad t \in [0, T\wedge a],$$

where $Y_t = \sup_{n\geq 1} Y^n_t$, $t \in [0, T\wedge a]$, $a \geq 0$. Moreover, if $T < \infty$ then the pair (Y, M) is a solution of $\mathrm{BSDE}^T(\xi, dV + dK)$.

Proof. It is enough to repeat the arguments between (2.22)–(2.28) in the proof of [17, Theorem 2.13] with $\overline{X} = \overline{Y}$ and with $(\int_0^{\cdot} f(r, Y^n_r)\,dr, V^n)$ replaced by $(V^n, 0)$. □

3.2. Systems with oblique reflection

Theorem 3.11. *Let the assumptions (A1)–(A5), (A8) hold, and let (A7) be satisfied with some processes $\underline{Y}, \overline{Y}$. Then there exists a minimal solution (Y, M, K) of (1.5) such that $\underline{Y} \leq Y \leq \overline{Y}$. Moreover,*

$$E\int_0^T |f(r, Y_r)|\,dr + E|K|_T < \infty \tag{3.8}$$

and M is a uniformly integrable martingale.

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Proof. Let $(Y^0, M^0, K^0) := (\underline{Y}, \underline{M}, 0)$. We define $(Y^{j,n}, M^{j,n}, K^{j,n})$ to be a solution of $\mathrm{RBSDE}^T(\xi^j, f^j(\cdot, Y^{n-1}; \cdot) + dV, H^j(\cdot, Y^{n-1}))$ (see Definition 3.4). It exists by Theorem 3.9. For every $a \geq 0$ we have

$$\begin{cases} Y^{n,j}_t = Y^{n,j}_{T\wedge a} + \int_t^{T\wedge a} f^j(r, Y^{n-1}_r; Y^{n,j}_r)\,dr + \int_t^{T\wedge a} dV^j_r \\ \qquad\qquad + \int_t^{T\wedge a} dK^{n,j}_r - \int_t^{T\wedge a} dM^{n,j}_r, \\ Y^{n,j}_t \geq H^j(t, Y^{n-1}_t), \quad t \in [0, T\wedge a], \\ \int_0^{T\wedge a} (Y^{n,j}_{t-} - H^j_{t-}(\cdot, Y^{n-1}))\,dK^{n,j}_t = 0. \end{cases} \tag{3.9}$$

Moreover by (A2), (A3), (A8) and Proposition 3.7,

$$Y^n \leq Y^{n+1} \leq \overline{Y}, \quad n \geq 0. \tag{3.10}$$

By Proposition 3.10 there exists an increasing predictable càdlàg process K with $K_0 = 0$ and a local martingale M with $M_0 = 0$ such that for every $a \geq 0$,

$$Y^j_t = Y^j_{T\wedge a} + \int_t^{T\wedge a} f^j(r, Y_r)\,dr + \int_t^{T\wedge a} dV^j_r + \int_t^{T\wedge a} dK^j_r - \int_t^{T\wedge a} dM^j_r, \quad t \in [0, T\wedge a], \tag{3.11}$$

where $Y_t = \sup_{n\geq 0} Y^n_t$. By (3.9) and (A8) we also have $Y^j \geq H^j(\cdot, Y)$. Let $(\overline{X}^j, \overline{N}^j, \overline{K}^j)$ denote a solution of $\mathrm{RBSDE}^T(\xi^j, f^j(\overline{Y}; \cdot) + dV^j, H^j(\cdot, \overline{Y}))$ and $(\underline{X}^j, \underline{N}^j, \underline{K}^j)$ denote a solution of $\mathrm{RBSDE}^T(\xi^j, f^j(\underline{Y}; \cdot) + dV^j, H^j(\cdot, \underline{Y}))$. By (3.10) and Proposition 3.7, $\underline{X}_t \leq Y^n_t \leq \overline{X}_t$, $t \in [0, T\wedge a]$, $a \geq 0$. This implies that $Y_{T\wedge a} \to \xi$ as $a \to \infty$. What is left is to show that K satisfies the minimality condition. Set

$$\tau_k = \inf\Big\{t \geq 0 : \sum_{j=1}^N \int_0^t |f^j(r, \overline{Y}_r; \underline{Y}^j_r)| + |f^j(r, \underline{Y}_r; \overline{Y}^j_r)|\,dr \geq k\Big\} \wedge T.$$

Then on the interval $[0, \tau_k]$ we have

$$Y^j_t = \operatorname*{ess\,sup}_{t\leq\tau} E\Big(\int_t^{\tau_k\wedge\tau} f^j(r, Y_r)\,dr + \int_t^{\tau_k\wedge\tau} dV_r + H^j(\tau, Y_\tau)\mathbf{1}_{\{\tau<\tau_k\}} + Y^j_{\tau_k}\mathbf{1}_{\{\tau_k\wedge\tau=\tau_k\}}|\mathcal{F}_t\Big). \tag{3.12}$$

Indeed, by Remark 3.6,

$$Y^{n,j}_t = \operatorname*{ess\,sup}_{t\leq\tau} E\Big(\int_t^{\tau_k\wedge\tau} f^j(r, Y^{n-1}_r; Y^{n,j}_r)\,dr + \int_t^{\tau_k\wedge\tau} dV_r + H^j(\tau, Y^{n-1}_\tau)\mathbf{1}_{\{\tau<\tau_k\}} + Y^{n,j}_{\tau_k}\mathbf{1}_{\{\tau_k\wedge\tau=\tau_k\}}|\mathcal{F}_t\Big),$$

so by (A3) and (A8),

$$Y^{n,j}_t \leq \operatorname*{ess\,sup}_{t\leq\tau} E\Big(\int_t^{\tau_k\wedge\tau} f^j(r, Y_r; Y^{n,j}_r)\,dr + \int_t^{\tau_k\wedge\tau} dV_r + H^j(\tau, Y_\tau)\mathbf{1}_{\{\tau<\tau_k\}} + Y^j_{\tau_k}\mathbf{1}_{\{\tau_k\wedge\tau=\tau_k\}}|\mathcal{F}_t\Big).$$

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Letting $n \to \infty$ and using (A4) we see that Y^j is less than or equal to the right-hand side of (3.12). The opposite inequality follows from the fact that the process $Y^j + \int_0^{\cdot} f(r, Y_r)\,dr + \int_0^{\cdot} dV_r$ is a supermartingale which dominates the process $L = \int_0^{\cdot} f(r, Y_r)\,dr + \int_0^{\cdot} dV_r + H^j(\cdot, Y)\mathbf{1}_{\{\cdot<\tau_k\}} + Y^j_{\tau_k}\mathbf{1}_{\{\cdot=\tau_k\}}$. Thus (3.12) is proved. By (3.12) and Remark 3.6,

$$\int_0^{\tau_k} (Y^j_{t-} - H^j_{t-}(\cdot, Y))\,dK^j_t = 0.$$

Letting $k \to \infty$ gives the above inequality on each interval $[0, T \wedge a]$, $a \geq 0$. Let (Y^*, M^*, K^*) be a solution of (1.5) such that $\underline{Y} \leq Y^* \leq \underline{Y}$. By Proposition 3.7, $Y^n \leq Y^*$, $n \geq 0$. Hence $Y \leq Y^*$. To get (3.8) it is enough to apply Theorem 3.9 with $X = \overline{Y}^j$ because by (A3), and (A7), $E\int_0^T |f^j(Y_r; \overline{Y}^j_r)|\,dr < \infty$, and by (A7) and (A8), $\overline{Y}^j \geq H^j(\cdot, Y)$. □

Remark 3.12. If K^n, K, V from the proof of Theorem 3.11 are continuous, then $Y^n \nearrow Y$, $K^n \to K$ in ucp. Indeed, in this case

$$^pY^n_t = Y^n_{t-}, \qquad ^pY_t = Y_{t-}, \tag{3.13}$$

where $^pY^n$, ^pY denote predictable projections of Y^n and Y, respectively. It is known that $Y^n \nearrow Y$ implies that $^pY^n \nearrow {}^pY$. By this and (3.13), $Y^n_{t-} \nearrow Y_{t-}$, $t \in [0, T \wedge a]$, $a \geq 0$. Therefore by the generalized Dini theorem (see [4, p. 185]), $Y^n \nearrow Y$ in ucp. The convergence of $\{K^n\}$ now follows from [13, Theorem 1.8] (for details see the reasoning at the beginning of page 4220 in [17]).

Remark 3.13. In Theorem 3.11 assume additionally that h is strictly increasing with respect to y ((A8) implies only that it is nondecreasing), and the following condition considered in [10] is satisfied:

(A9) there are no $(y_1, \ldots, y_k) \in \mathbb{R}^k$ and $j_2 \in A_{j_1}, \ldots, j_k \in A_{j_{k-1}}$, $j_1 \in A_{j_k}$ such that for some $t \in [0, T]$ we have

$$y_1 = h_{j_1,j_2}(t, y_2),\ \ y_2 = h_{j_2,j_3}(t, y_3), \ldots, y_{k-1} = h_{j_{k-1},j_k}(t, y_k),\ \ y_k = h_{j_k,j_1}(t, y_1).$$

Moreover, assume that the underlying filtration is quasi-left continuous and V is continuous. Then K is continuous. Indeed, since the filtration is quasi-left continuous and V is continuous, $\Delta K_\tau = -\Delta Y_\tau$ for every predictable stopping time τ. Therefore in the same way as in Step 4 of the proof of [10, Theorem 3.2] one can show that $\Delta K_\tau = 0$. Since K is predictable and τ is an arbitrary predictable stopping time, applying the predictable cross-section theorem (see [3, Theorem 86, p. 138]) shows that K is continuous.

3.3. Approximation via penalization

Let us consider the following system of BSDEs

$$\begin{aligned} Y^{j,n}_t &= \xi^j + \int_t^T f^j(r, Y^n_r)\,dr + \int_t^T dV^j_r \\ &\quad + \int_t^T n(Y^{j,n}_r - H^j(r, Y^n_r))^-\,dr - \int_t^T dM^{j,n}_r. \end{aligned} \tag{3.14}$$

Let us put $f^j_n(t, y) := f^j(t, y) + n(y^j - H^j(t, y))^-$. We see that (Y^n, M^n) is a solution to $\mathrm{BSDE}^T(\xi, f_n + dV)$.

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Theorem 3.14. *Let (A1)–(A5), (A8) hold, and let (A7) be satisfied with some processes* $\underline{Y}, \overline{Y}$. *Then there exists a minimal solution* (Y^n, M^n) *of* (3.14) *such that* $\underline{Y} \le Y^n \le \overline{Y}$. *Moreover,* $Y^n_t \nearrow Y_t$, $t \in [0, T \wedge a]$, $a \ge 0$, *where* (Y, M, K) *is the minimal solution of* (1.5) *such that* $\underline{Y} \le Y \le \overline{Y}$.

Proof. Observe that by (A7), $n(\overline{Y}^j_t - H^j(t, \overline{Y}_t))^- = 0$, so $(\overline{Y}, \overline{M})$ is a supersolution of (3.14). It is clear that $(\underline{Y}, \underline{M})$ is a subsolution of (3.14). Now we will show that (A6) is satisfied for Eq. (3.14) with $\underline{Y}, \overline{Y}$ and with $S = \overline{Y}$. By (A8) and the fact that $\underline{Y} \le \overline{Y}$ we have

$$f^j_n(t, \underline{Y}_t; S^j_t) = f^j(t, \underline{Y}_t; \overline{Y}^j_t), \qquad f^j_n(t, \overline{Y}_t; S^j_t) = f^j(t, \overline{Y}_t).$$

By this and (A7),

$$\sum_{i=1}^N E \int_0^T |f^j_n(r, \underline{Y}_r; S^j_r)|\, dr + \sum_{i=1}^N E \int_0^T |f^j_n(r, \overline{Y}_r; S^j_r)|\, dr < \infty.$$

Since the other assumptions of Theorem 2.13 are also satisfied for equation (3.14), there exists a minimal solution (Y^n, M^n) of (3.14) such that $\underline{Y} \le Y^n \le \overline{Y}$. By Corollary 2.14, $Y^n \le Y^{n+1}$. Therefore repeating step by step the arguments from the proof of Theorem 3.11 (see also the end of the proof of Theorem 3.9) we show that there exists a local martingale $\tilde{M}$ and an increasing càdlàg process $\tilde{K}$ such that the triple $(\tilde{Y}, \tilde{M}, \tilde{K})$, where $\tilde{Y}_t = \lim_{n\to\infty} Y^n_t$, $t \in [0, T \wedge a]$, $a \ge 0$, is a solution of (1.5). What is now left is to show that $\tilde{Y} = Y$, where (Y, M, K) is the minimal solution of (1.5) such that $\underline{Y} \le Y \le \overline{Y}$. Of course, $Y \le \tilde{Y}$. Moreover, since $Y^j \ge H^j(\cdot, Y)$, we have

$$Y^j_t = \xi^j + \int_t^T f^j(r, Y_r)\, dr + \int_t^T dV^j_r + \int_t^T dK^j_r$$
$$+ \int_t^T n(Y^j_r - H^j(r, Y_r))^-\, dr - \int_t^T dM^j_r.$$

By this and Corollary 2.14, $Y^n \le Y$. Hence $\tilde{Y} \le Y$, which completes the proof. $\square$

Remark 3.15. Set

$$K^{n,j}_t = \int_0^t n(Y^{n,j}_r - H^j(r, Y^n_r))^-\, dr.$$

If the processes K, V from Theorem 3.11 are continuous, then $Y^n \nearrow Y$ and $K^n \to K$ in ucp. This follows by the same method as in Remark 3.12.

4. Switching problem

In what follows by a strategy we mean a pair $\mathcal{S} = (\{\xi_n, n \ge 1\}, \{\tau_n, n \ge 1\})$, where $\{\tau_n, n \ge 1\}$ is an increasing sequence of stopping times such that

$$P(\tau_n < T, \forall\, n \ge 1) = 0,$$

and $\{\xi_n, n \ge 1\}$ is a sequence of random variables taking values in $\{1, \dots, N\}$ such that ξ_n is $\mathcal{F}_{\tau_n}$-measurable for each $n \in \mathbb{N}$.

The set of all strategies we denote by $\mathbf{A}$. For every $\alpha \in \mathcal{T}$ by $\mathbf{A}_\alpha$ we denote the set of all strategies $\mathcal{S} \in \mathbf{A}$ such that $\tau_1 \ge \alpha$. For $\mathcal{S} \in \mathbf{A}$ we set

$$w^j_t = j\mathbf{1}_{[0,\tau_1)}(t) + \sum_{n\ge1} \xi_n \mathbf{1}_{[\tau_n, \tau_{n+1})}(t).$$

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Remark 4.1. Let L be an adapted càdlàg process of class (D), and let ξ be an integrable random variable such that $\limsup_{a\to\infty} L_{T\wedge a} \le \xi$. Set

$$Y_t = \operatorname*{ess\,sup}_{\tau\ge t} E(L_\tau \mathbf{1}_{\{\tau<T\}} + \xi\mathbf{1}_{\{\tau=T\}}|\mathcal{F}_t).$$

By Remark 3.6 and Theorem 3.9, Y is the first component of a solution (Y, M, K) of RBSDE$^T(\xi, 0, L)$. Observe that if K is continuous then

$$Y_t = E(L_{\tau_t^*}\mathbf{1}_{\{\tau_t^*<T\}} + \xi\mathbf{1}_{\{\tau_t^*=T\}}|\mathcal{F}_t), \tag{4.1}$$

where

$$\tau_t^* = \inf\{s \ge t : Y_s = L_s\} \wedge T.$$

Indeed, by the definition of τ_t^* and the definition of a solution of RBSDE$^T(\xi, 0, L)$,

$$Y_t = E\Big(\int_t^{\tau_t^*} dK_r + L_{\tau_t^*}\mathbf{1}_{\{\tau_t^*<T\}} + \xi\mathbf{1}_{\{\tau_t^*=T\}}|\mathcal{F}_t\Big) \tag{4.2}$$

and, since K is continuous,

$$\int_0^{T\wedge a} (Y_r - L_r)\, dK_r = 0, \quad a \ge 0.$$

This implies that $\int_t^{\tau_t^*} dK_r = 0$, which when combined with (4.2) yields (4.1).

In the rest of this section we assume that

$$H^j(t, y) = \max_{i\in A_j}(-c_{j,i}(t) + y^i), \tag{4.3}$$

where $c_{j,i}$ are continuous adapted process such that for some constant $c > 0$,

$$c_{j,i}(t) \ge c, \quad i \in A_j, \quad t \in [0, T \wedge a], \quad a \ge 0, \quad j = 1, \dots, N.$$

Remark 4.2. Assume that the underlying filtration $\mathbb{F}$ is quasi-left continuous, H is of the form (4.3) and V is continuous. Then (A9) is satisfied. Indeed, in this case $-\Delta K_\tau^j = \Delta Y_\tau^j$ for every predictable stopping time τ. Therefore repeating step by step the proof of [5, Proposition 2] we get the desired result.

Theorem 4.3. *Assume that f does not depend on y and H^j are of the form (4.3). If $E(\int_0^T d|V|_r + \int_0^T |f(r)|\, dr) < \infty$ then there exists a solution (Y, M, K) of (1.5). Moreover, if K is continuous, then for every $\alpha \in \mathcal{T}$*

$$\begin{aligned} Y_\alpha^j = \operatorname*{ess\,sup}_{\mathcal{S}\in\mathbf{A}_\alpha} E\Big(&\int_\alpha^T f^{w_r^j}(r)\, dr + \int_\alpha^T dV_r^{w_r^j} \\ &- \sum_{n\ge1} c_{w^j_{\tau_{n-1}}, w^j_{\tau_n}}(\tau_n)\mathbf{1}_{\{\tau_n<T\}} + \xi^{w_T^j}|\mathcal{F}_\alpha\Big), \end{aligned} \tag{4.4}$$

and the optimal strategy $\mathcal{S}^*$ for Y^j is given by

$$\tau_0^{j,*} = \alpha, \quad \xi_0^{j,*} = j, \quad \tau_k^{j,*} = \inf\{t \ge \tau_{k-1}^{j,*} : Y_t^{\xi_{k-1}^{j,*}} = H_t^{\xi_{k-1}^{j,*}}\} \wedge T, \quad k \ge 1,$$

$$\xi_k^{j,*} = \max\{i \in A_{\xi_{k-1}^{j,*}};\ H_{\tau_k}^{\xi_{k-1}^{j,*}} = -c_{\xi_{k-1}^{j,*}, i}(\tau_k^{j,*}) + Y^i_{\tau_k^{j,*}}\}, \quad k \ge 1,$$

where $H_t^j := H^j(t, Y_t)$.

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Proof. The existence part follows from Theorem 3.11, because assumptions (A1)–(A5), (A8) are clearly satisfied, and (A7) is satisfied with $(\overline{Y}, \overline{M})$ and $(\underline{Y}, \underline{M})$ defined as follows:

$$\overline{Y}^1 = \cdots = \overline{Y}^N, \quad \overline{M}^1 = \cdots = \overline{M}^N, \qquad \underline{Y}^1 = \cdots = \underline{Y}^N, \quad \underline{M}^1 = \cdots = \underline{M}^N,$$

and $(\overline{Y}^1, \overline{M}^1)$ (resp. $(\underline{Y}^1, \underline{M}^1)$) is a solution of BSDE$^T(\xi^1 \vee \cdots \vee \xi^N, f^1 \vee \cdots \vee f^N + dV^1 \vee \cdots \vee dV^N)$ (resp. BSDE$^T(\xi^1 \wedge \cdots \wedge \xi^N, f^1 \wedge \cdots \wedge f^N + dV^1 \wedge \cdots \wedge dV^N)$). Thanks to Remark 4.1, to get (4.4) it suffices now to repeat step by step the proof of [5, Theorem 1]. □

Corollary 4.4. *Let (A1), (A2) be satisfied. Assume that f^j depends only on y^j, H^j are of the form (4.3), $\mathbb{F}$ is quasi-left continuous and V is continuous. Then there exists at most one solution (Y, M, K) of (1.5) such that $E\int_0^T |f(r, Y_r)|\,dr < \infty$.*

Proof. Suppose that (Y, M, K), $(\tilde{Y}, \tilde{M}, \tilde{K})$ are two solutions of (1.5). Let $\tau_k^j = \inf\{t \ge 0 : Y_t^j - \tilde{Y}_t^j < -\frac{1}{k}\} \wedge T$ and $\sigma_k^j = \inf\{t \ge \tau_k^j : Y_t^j = \tilde{Y}_t^j\} \wedge T$. Of course, (Y, M, K) (resp. $(\tilde{Y}, \tilde{M}, \tilde{K})$) is also a solution of (1.5) on the interval $[0, \sigma_k^j]$ with terminal condition $\xi^j = Y^j_{\sigma_k^j}$ (resp. $\xi^j = \tilde{Y}^j_{\sigma_k^j}$). Hence, by Theorem 4.3 (see also Remarks 3.13 and 4.2), for every $\alpha \le \sigma_k^j$ we have

$$Y_\alpha^j = \operatorname*{ess\,sup}_{\mathcal{S}\in\mathbf{A}_\alpha} E\Big(\int_\alpha^{\sigma_k^j} f^{w_r^j}(r, Y_r^j)\,dr + \int_\alpha^{\sigma_k^j} dV_r^{w_r^j} - \sum_{n\ge1} c_{w^j_{\tau_{n-1}}, w^j_{\tau_n}}(\tau_n)\mathbf{1}_{\{\tau_n<\sigma_k^j\}} + Y^{w^j_{\sigma_k^j}}_{\sigma_k^j}|\mathcal{F}_\alpha\Big), \tag{4.5}$$

and

$$\tilde{Y}_\alpha^j = \operatorname*{ess\,sup}_{\mathcal{S}\in\mathbf{A}_\alpha} E\Big(\int_\alpha^{\sigma_k^j} f^{w_r^j}(r, \tilde{Y}_r^j)\,dr + \int_\alpha^{\sigma_k^j} dV_r^{w_r^j} - \sum_{n\ge1} c_{w^j_{\tau_{n-1}}, w^j_{\tau_n}}(\tau_n)\mathbf{1}_{\{\tau_n<\sigma_k^j\}} + \tilde{Y}^{w^j_{\sigma_k^j}}_{\sigma_k^j}|\mathcal{F}_\alpha\Big). \tag{4.6}$$

Let $\mathcal{S}, \tilde{\mathcal{S}} \in \mathbf{A}_\alpha$ be an optimal strategy for (Y, M, K), $(\tilde{Y}, \tilde{M}, \tilde{K})$, respectively, with $\alpha = \tau_k^j$. Then by (4.5), (4.6) and (A2),

$$Y^j_{\tau_k^j} - \tilde{Y}^j_{\tau_k^j} \ge E\Big(\int_{\tau_k^j}^{\sigma_k^j} f^{\tilde{w}_r^j}(r, Y_r^j)\,dr - \int_{\tau_k^j}^{\sigma_k^j} f^{\tilde{w}_r^j}(r, \tilde{Y}_r^j)\,dr|\mathcal{F}_{\tau_k^j}\Big) \ge 0.$$

From this we conclude that $P(\tau_k^j < T) = 0$. Since $k \in \mathbb{N}$ was arbitrary, we see that $Y^j \ge \tilde{Y}^j$. A similar argument applied to $\tilde{Y}^j - Y^j$ shows that $Y^j \le \tilde{Y}^j$. □

Corollary 4.5. *Let (A1) be satisfied. Assume that f^j satisfies (2.13), T is bounded, H^j are of the form (4.3), $\mathbb{F}$ is quasi-left continuous and V is continuous. Then there exists at most one solution (Y, M, K) of (1.5) such that $E\int_0^T |f(r, Y_r)|\,dr < \infty$.*

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Proof. Let (Y, M, K), $(\tilde{Y}, \tilde{M}, \tilde{K})$ be solutions to (1.5). By the representation (4.4) and (2.13) we easily get

$$E|Y_t - \tilde{Y}_t| \le NLE \int_t^T |Y_r - \tilde{Y}_r|\, dr.$$

Since T is bounded, applying Gronwall's lemma shows that $Y = \tilde{Y}$. □

5. Systems of elliptic quasi-variational inequalities

In this section E is a locally compact separable metric space, m is a Radon measure on E such that $\mathrm{supp}[m] = E$, and $(\mathcal{E}, D[\mathcal{E}])$ is a regular transient semi-Dirichlet form on $L^2(E; m)$. By $(L, D(L))$ we denote the generator associated with $(\mathcal{E}, D[\mathcal{E}])$ (see [25, Chapter 1]).

Let us recall that $(\mathcal{E}, D[\mathcal{E}])$ is called semi-Dirichlet if $D[\mathcal{E}]$ is dense in $L^2(E; m)$ and $\mathcal{E}$ is a bilinear form on $D[\mathcal{E}] \times D[\mathcal{E}]$ satisfying the conditions $(\mathcal{E}1)$–$(\mathcal{E}4)$ below:

$(\mathcal{E}1)$ $\mathcal{E}$ is lower bounded, i.e. there exists $\alpha_0 \ge 0$ such that

$$\mathcal{E}_{\alpha_0}(u, u) \ge 0, \quad u \in D[\mathcal{E}],$$

where $\mathcal{E}_{\alpha_0}(u, v) = \mathcal{E}(u, v) + \alpha_0(u, v)$,

$(\mathcal{E}2)$ $\mathcal{E}$ satisfies the sector condition, i.e. there exists $K > 0$ such that

$$|\mathcal{E}(u, v)| \le K\mathcal{E}_{\alpha_0}(u, u)^{1/2}\mathcal{E}_{\alpha_0}(v, v)^{1/2}, \quad u, v \in D[\mathcal{E}],$$

$(\mathcal{E}3)$ $\mathcal{E}$ is closed, i.e. for every $\alpha > \alpha_0$ the space $D[\mathcal{E}]$ equipped with the inner product $\mathcal{E}^{(s)}_\alpha(u, v) := \frac{1}{2}(\mathcal{E}_\alpha(u, v) + \mathcal{E}_\alpha(v, u))$ is a Hilbert space,

$(\mathcal{E}4)$ $\mathcal{E}$ has the Markov property, i.e. for every $a \ge 0$,

$$\mathcal{E}(u \wedge a, u \wedge a) \le \mathcal{E}(u \wedge a, u), \quad u \in D[\mathcal{E}].$$

Note that $(\mathcal{E}4)$ is equivalent to the fact that the semigroup $\{T_t,\ t \ge 0\}$ associated with $(\mathcal{E}, D[\mathcal{E}])$ is sub-Markov (see [25, Theorem 1.1.5]). Recall also that $\mathcal{E}$ is said to have the dual Markov property if

$(\mathcal{E}5)$ for every $a \ge 0$,

$$\mathcal{E}(u \wedge a, u \wedge a) \le \mathcal{E}(u, u \wedge a), \quad u \in D[\mathcal{E}].$$

Condition $(\mathcal{E}5)$ is equivalent to the fact that associated dual semigroup $\{\hat{T}_t, t \ge 0\}$ associated with $(\mathcal{E}, D[\mathcal{E}])$ is sub-Markov (see [25, Theorem 1.1.5]). For the notions of transiency and regularity see [25, Section 1.2, Section 1.3].

Let Cap denote the capacity associated with $(\mathcal{E}, D[\mathcal{E}])$ (see [25, Chapter 2]), and let $\mathbb{X} = (\{X_t, t \ge 0\}, \{P_x, x \in E\})$ be a Hunt process with life time ζ associated with $(\mathcal{E}, D[\mathcal{E}])$ (see [25, Chapter 3]). We say that some property holds quasi-everywhere (q.e. for short) if there is a set $B \subset E$ such that $\mathrm{Cap}(B) = 0$ and it holds on the set $E \setminus B$. A set $B \subset E$ such that $\mathrm{Cap}(B) = 0$ is called exceptional.

Let μ be a signed measure E. By μ^+ (resp. μ^-) we denote its positive (resp. negative) part, and we set $|\mu| = \mu^+ + \mu^-$. A Borel signed measure μ on E is called smooth if μ charges no exceptional sets and there exists an increasing sequence $\{F_n\}$ of closed subsets of E such that $|\mu|(F_n) < \infty$ for $n \ge 1$, and for every compact $K \subset E$.

$$\mathrm{Cap}(K \setminus F_n) \to 0.$$

In the sequel the set of all signed smooth measures on E such that $\|\mu\|_{TV} := |\mu|(E) < \infty$ will be denoted by $\mathcal{M}_{0,b}$.

It is known (see [25, Section 4.1]) that there is one-to-one correspondence (the Revuz duality) between positive continuous additive functionals (PCAFs for short) of $\mathbb{X}$ and positive smooth measures. By A^μ we denote the unique PCAF of $\mathbb{X}$ associated with positive smooth measure μ. For a signed smooth measure μ we set $A^\mu = A^{\mu^+} - A^{\mu^-}$. By $\mathbb{M}$ we denote the set of all smooth measures μ on E such that

$$E_x \int_0^\zeta dA_r^{|\mu|} < \infty$$

for q.e. $x \in E$, where E_x denotes the expectation with respect to P_x. For a fixed positive measurable function f and a positive Borel measure μ we denote by $f \cdot \mu$ the measure defined as

$$(f \cdot \mu)(\eta) = \int_E \eta f \, d\mu, \quad \eta \in \mathcal{B}^+(E).$$

We write $f \in \mathbb{M}$ if $f \cdot m \in \mathbb{M}$. By [25, Corollary 1.3.6], if $(\mathcal{E}, D[\mathcal{E}])$ has the dual Markov property then

$$\mathcal{M}_{0,b} \subset \mathbb{M}. \tag{5.1}$$

By $qL^1(E;m)$ we denote the set of all measurable real functions f on E such that $A_\zeta^{f\cdot m} < \infty$ for every $t \geq 0$. By (5.1),

$$L^1(E;m) \subset qL^1(E;m).$$

Note that in general the form associated with the operator defined by (1.2) does not have the dual Markov property. Nevertheless, for this form (5.1) holds true.

Recall that a set $U \subset E$ is called quasi-open if for every $\varepsilon > 0$ there exists an open set $U \subset U_\varepsilon \subset E$ such that $\mathrm{Cap}(U_\varepsilon \setminus U) < \varepsilon$. The family of quasi-open sets induces the quasi-topology on E. We say that a function u on E is quasi-continuous if it is continuous with respect to the quasi-topology.

5.1. Existence and approximation of solutions

For $i, j = 1, \dots, N$ let $h_{j,i}, f^j : E \times \mathbb{R}^N \to \mathbb{R}$ be measurable functions, μ^j be smooth measures on E, and let $A_j \subset \{1, \dots, j-1, j+1, \dots, N\}$. We maintain the notation $f^j(x, y; a)$ introduced at the beginning of Section 2, and we set

$$H^j(x, y) = \max_{i \in A_j} h_{j,i}(x, y^i), \qquad H = (H^1, \dots, H^N),$$

$$f = (f^1, \dots, f^N), \qquad \mu = (\mu^1, \dots, \mu^N).$$

We will make the following hypotheses:

(H1) $\mu^j \in \mathbb{M}$, $j = 1, \dots, N$,
(H2) for $j = 1, \dots, N$ the function $a \mapsto f^j(x, y; a)$ is nonincreasing for all $x \in E$, $y \in \mathbb{R}^N$,
(H3) f is off-diagonal nondecreasing, i.e. for $j = 1, \dots, N$ we have $f^j(x, y; a) \leq f^j(x, \bar{y}; a)$ for all $y, \bar{y} \in \mathbb{R}^N$ such that $y \leq \bar{y}$ and $a \in \mathbb{R}$,

(H4) $y \mapsto f(x, y)$ is continuous for every $x \in E$,
(H5) $f^j(\cdot, y) \in qL^1(E; m)$ for all $y \in \mathbb{R}^N$, $j = 1, \dots, N$.

Consider the following system of equations

$$-Lu = f(x, u) + \mu. \tag{5.2}$$

Following [19,22] we adopt the following definition of a solution of (5.2).

Definition 5.1. We say that a measurable function $u = (u^1, \dots, u^N) : E \to \mathbb{R}^N$ is a solution of (5.2) (PDE($f + d\mu$) for short) if $f^j(\cdot, u) \in \mathbb{M}$, $j = 1, \dots, N$, and for q.e. $x \in E$,

$$u^j(x) = E_x\Big(\int_0^\zeta f^j(X_r, u(X_r))\, dr + \int_0^\zeta dA_r^{\mu^j}\Big), \quad j = 1, \dots, N. \tag{5.3}$$

Remark 5.2. A measurable function $u : E \to \mathbb{R}^N$ satisfying (5.3) may be called a probabilistic solution of (5.2). Note that if $f^j(\cdot, u) \in L^1(E; m)$ and $\mu^j \in \mathcal{M}_b$ then u^j is a renormalized solution of (5.2) (see [21]).

Remark 5.3. (i) If u is a solution of (5.2) in the sense of Definition 5.1 then by [19, Theorem 4.7] the pair $(u(X), M)$, where

$$M_t^j = E_x\Big(\int_0^\zeta f^j(X_r, u(X_r))\, dr + \int_0^\zeta dA_r^{\mu^j} |\mathcal{F}_t\Big), \quad t \ge 0, \tag{5.4}$$

is a solution of BSDE$^\zeta(0, f(X, \cdot) + dA^\mu)$ under the measure P_x for q.e. $x \in E$ (in fact, M in (5.4) is an independent of x version of the right-hand side of Eq. (5.4); such a version always exists, see [7, Section A.3]).
(ii) If (Y, M) is a solution of BSDE$^\zeta(0, f(X, \cdot) + dA^\mu)$ under the measure P_x for q.e. $x \in E$, and there exists a function u such that $u(X) = Y$ under the measure P_x for q.e. $x \in E$ and $f^j(\cdot, u) \in \mathbb{M}$, $j = 1, \dots, N$, then u is a solution of (5.2). This follows directly from Remark 2.2.

Definition 5.4. We say that a measurable function $u : E \to \mathbb{R}^N$ is a subsolution (resp. supersolution) of (5.2) if there exist $\underline{\mu} \in \mathbb{M}$ (resp. $\overline{\mu} \in \mathbb{M}$) and $\underline{f}$ (resp. $\overline{f}$) satisfying (H2) such that $\underline{\mu} \le \mu$ (resp. $\mu \le \overline{\mu}$), $\underline{f}(x, y) \le f(x, y)$, $x \in E$, $y \in \mathbb{R}$ (resp. $f(x, y) \le \overline{f}(x, y)$, $x \in E$, $y \in \mathbb{R}$) and u is a solution of PDE($\underline{f} + \underline{\mu}$) (resp. PDE($\overline{f} + \overline{\mu}$)).

Remark 5.5. By Remark 5.3, if u is a subsolution (resp. supersolution) of (5.2) then $u(X)$ is the first component of a subsolution (resp. supersolution) of the equation BSDE$^\zeta(0, f(X, \cdot) + dA^\mu)$ under the measure P_x for q.e. $x \in E$.

Definition 5.6. We say that a quasi-continuous function u on E is a solution of (1.3) if there exist positive measures $\nu^1, \dots, \nu^N \in \mathbb{M}$ such that u is a solution of PDE($f + d\mu + d\nu$) with $\nu = (\nu^1, \dots, \nu^N)$, and the second and the third condition in (1.3) are satisfied.

We will also need the following hypotheses:

(H6) There exist a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of (5.2) such that

$$\underline{u} \le \overline{u}, \quad H(\cdot, \overline{u}) \le \overline{u}, \quad \sum_{j=1}^N (|f^j(\cdot, \underline{u}; \overline{u}^j)| + |f^j(\cdot, \overline{u})|) \in \mathbb{M},$$

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(H7) H^j is continuous on $E \times \mathbb{R}^N$ equipped with the product topology consisting of quasi-topology on E and the Euclidean topology on $\mathbb{R}^N$ and nondecreasing with respect to y.

In the proof of the next theorem we will use some result from [18] on solutions of the usual obstacle problem for single equation and one quasi-continuous barrier $h : E \to \mathbb{R}$. For the convenience of the reader we recall below the definition of a solution.

Definition 5.7. Let $N = 1$. We say that a pair (u, ν) is a solution of the obstacle problem for L with lower barrier h and the right-hand side $f + d\mu$ (OP$(f + d\mu, h)$ for short) if u is quasi-continuous, ν is a positive measure such that $\nu \in \mathbb{M}$, $u \geq h$ q.e., and

$$-Lu = f(x, u) + \mu + \nu, \qquad \int_E (u - h)\, d\nu = 0.$$

In the sequel, for $\mu = (\mu^1, \dots, \mu^N)$ we write $|\mu| = \sum_{j=1}^N |\mu^j|$, $\|\mu\|_{TV} = \sum_{j=1}^N \|\mu^j\|_{TV}$.

Theorem 5.8. *Assume (H1)–(H7). Then there exists a minimal solution of (1.3) such that $\underline{u} \leq u \leq \overline{u}$.*

Proof. We first observe that the data $f(X, \cdot)$, $H^j(X, \cdot)$, $\xi := 0$, $T := \zeta$, $\underline{Y} := \underline{u}(X)$, $\overline{Y} := \overline{u}(X)$ satisfy assumptions (A1)–(A5), (A7), (A8) under the measure P_x for q.e. $x \in E$ (see Remark 5.3). Set $Y^0 := \underline{Y}$. By Theorem 3.11, for q.e. $x \in E$ there exist a solution $(Y^{n,j}, M^{n,j}, K^{n,j})$ of RBSDE$^\zeta(0, f^j(X, Y^{n-1}; \cdot) + dA^{\mu^j}, H^j(X, Y^{n-1}))$, $n \geq 1$, $j = 1, \dots, N$, and a solution (Y, M, K) of the following RBSDE with oblique reflection

$$\begin{cases} Y_t^j = \int_t^\zeta f^j(X_r, Y_r)\, dr + \int_t^\zeta dA_r^{\mu^j} + \int_t^\zeta dK_r^j - \int_t^\zeta dM_r^j, \quad t \in [0, \zeta], \\ Y_t^j \geq H^j(X_t, Y_t), \quad t \in [0, \zeta], \\ \int_0^\zeta (Y_t^j - H^j(X_t, Y_t))\, dK_t^j = 0, \; j = 1, \dots, N \end{cases} \tag{5.5}$$

under the measure P_x, and moreover, $Y_t^n \nearrow Y_t$, $t \in [0, \zeta]$, P_x-a.s. Set $u_0 = \underline{u}$. By [18, Theorem 3.2], for every $n \geq 1$,

$$u_n^j(X_t) = Y_t^{n,j}, \qquad A_t^{\nu_n^j} = K_t^{n,j},$$

where (u_n^j, ν_n^j) is a solution of OP$(f^j(\cdot, u_{n-1}; \cdot) + d\mu^j, H^j(\cdot, u_{n-1}))$. Since $Y^n \leq Y^{n+1}$ P_x-a.s. for q.e. $x \in E$, we have $u_n \leq u_{n+1}$ q.e. Therefore putting $u^j = \sup_{n \geq 0} u_n^j$ and $u = (u^1, \dots, u^N)$ we obtain

$$u(X_t) = Y_t, \quad t \in [0, \zeta], \quad P_x\text{-a.s.}$$

for q.e. $x \in E$. By Theorem 3.11, $f^j(\cdot, u) \in \mathbb{M}$. Observe that the triple (Y^j, M^j, K^j) is a solution to RBSDE$^\zeta(0, f^j(X, u(X); \cdot) + dA^{\mu^j}, H^j(X, u(X)))$, $j = 1, \dots, N$. By [18, Theorem 3.2] there exists $\nu^j \in \mathbb{M}$ such that $K^j = A^{\nu^j}$ and (u^j, ν^j) is a solution to OP$(f^j(\cdot, u; \cdot) + d\mu^j, H^j(\cdot, u))$. This implies that the pair (u, ν), where $\nu = (\nu^1, \dots, \nu^N)$, is a solution of (1.3). □

Remark 5.9. Assume that $(\mathcal{E}, D[\mathcal{E}])$ has the dual Markov property, (H2) and (H3) are satisfied, and hypotheses (H1), (H6) hold true with $\mathbb{M}$ replaced by $\mathcal{M}_{0,b}$. Assume also that $\overline{\mu} \in \mathcal{M}_{0,b}$ $\overline{f}(\cdot, \overline{u}) \in L^1(E; m)$. Let (u, ν) be a solution of (1.3). Then $\nu^j \in \mathcal{M}_{0,b}$ and $f^j(\cdot, u) \in L^1(E; m)$,

$j=1,\dots,N$. Indeed, since $u\le\bar u$, we have

$$E_x\int_0^\zeta dA^{\nu^j}_r\le E_x\int_0^\zeta f^{j,-}(X_r,u(X_r))\,dr+E_x\int_0^\zeta \overline{f}^{j,+}(X_r,\bar u(X_r))\,dr + E_x\int_0^\zeta dA^{\mu^{j,-}}+E_x\int_0^\zeta dA^{\bar\mu^{j,+}}_r.$$

By [22, Lemma 2.6], the above inequality implies that

$$\|\nu^j\|_{TV}\le\|f^{j,-}(\cdot,u)\|_{L^1}+\|\overline{f}^{j,+}(\cdot,\bar u)\|_{L^1}+\|\mu^{j,-}\|_{TV}+\|\bar\mu^{j,+}\|_{TV}.$$

By our assumptions, $\|\overline{f}^{j,+}(\cdot,\bar u)\|_{L^1}+\|\mu^{j,-}\|_{TV}+\|\bar\mu^{j,+}\|_{TV}<\infty$. Observe that by (H2) and (H3), $f^j(\cdot,u)\ge f^j(\cdot,\underline{u};\bar u^j)$ q.e. Hence, by (H6), $\|f^{j,-}(\cdot,u)\|_{L^1}<\infty$. Therefore $\nu^j\in\mathcal{M}_{0,b}$, $j=1,\dots,N$. Observe that $f^j(\cdot,u;\bar u^j)\in L^1(E;m)$ by (H3) and (H6). Now the integrability of $f(\cdot,u)$ follows from [22, Proposition 3.10].

Remark 5.10. Under the assumptions of Remark 5.9 the functions u^j, $j=1,\dots,N$, have the property that $T_k(u^j)\in D_e[\mathcal{E}]$ for $k\ge0$, where $T_k(y)=\max(\min(y,k),-k)$. This follows from Remark 5.9 and [19, Proposition 5.9]. Therefore under the assumptions of Remark 5.9 the function u^j is a solution of the first equation in (1.3) in the sense of Stampacchia, or, in different terminology, are solution in the sense of duality (see [19, Proposition 5.3]). Equivalently, it is a renormalized solution of this equation (see [21]).

Proposition 5.11. *Let $N=1$. Assume (H1), (H3), (H4), (H5). Moreover, assume that there exists a real valued measurable function v on E such that $Lv\in\mathbb{M}$ and $f(\cdot,v)\in\mathbb{M}$. Then there exists a solution u of PDE$(f+d\mu)$.*

Proof. Set $\beta=-Lv$. Observe that the data $f(X,\cdot)$, $V:=A^\mu$, $\xi:=0$, $S:=v(X)$, $T:=\zeta$ satisfy the assumptions of Theorem 2.9 under the measure P_x for q.e. $x\in E$. From the proof of Theorem 2.9 it follows that there exists a solution (Y,M) of BSDE$^\zeta(0,f(X,\cdot)+dA^\mu)$, and that $Y=\tilde Y+S$, where $(\tilde Y,\tilde M)$ is a solution of BSDE$^T(0,f_S+dA^\mu-dA^\beta)$ under measure P_x for q.e. $x\in E$. By [19, Theorem 4.7] there exists a solution $\tilde u$ of PDE$(f_v+d\mu-d\beta)$ with $f_v(x,y)=f(x,v(x)+y)$, and $\tilde u(X)=\tilde Y$. Hence $Y=\tilde u(X)+v(X)$. It is clear (see Remark 2.2) that $u:=\tilde u+v$ is a solution of PDE$(f+d\mu)$. □

In the next proposition we will need the following hypothesis.

(H8) There exist a subsolution $\underline{u}$ and a supersolution $\bar u$ of (5.2), and a measurable function $v:E\to\mathbb{R}^N$ such that $Lv^j\in\mathbb{M}$, $j=1,\dots,N$, and

$$\underline{u}\le\bar u,\qquad\sum_{j=1}^N|f^j(\cdot,\bar u;v^j)|+|f^j(\cdot,\underline{u};v^j)|\in\mathbb{M}.$$

Proposition 5.12. *Assume (H1)–(H5), (H8). Then there exists a minimal solution u of (5.2) such that $\underline{u}\le u\le\bar u$.*

Proof. Observe that the data $f(X,\cdot)$, $\underline{Y}:=\underline{u}(X)$, $\overline{Y}:=\bar u(X)$, $S:=v(X)$, $V:=A^\mu$, $\xi:=0$, $T:=\zeta$ satisfy the assumptions of Theorem 3.11 under the measure P_x for q.e. $x\in E$. Set $u_0=\underline{u}$. By Theorem 3.11, for q.e. $x\in E$ there exist a unique solution (Y^n,M^n) of BSDE$^\zeta(0,f(X,Y^{n-1})+dA^\mu)$ and a minimal solution (Y,M) of BSDE$^\zeta(0,f(X,\cdot)+dA^\mu)$ such that $\underline{Y} \le Y \le \overline{Y}$ under the measure P_x. By Proposition 5.11 (see also Remark 5.3), $Y^{n,j} = u_n^j(X)$ P_x-a.s. for q.e. $x \in E$, where u_n^j is the solution of PDE$(f^j(\cdot, u_{n-1}; \cdot) + d\mu^j)$. From the proof of Theorem 3.11 it follows that $Y^n \le Y^{n+1}$. Hence $u_n \le u_{n+1}$ q.e. Set $u = \sup_{n\ge1} u_n$. It is clear that $Y = u(X)$ P_x-a.s. for q.e. $x \in E$. Hence, by Remark 5.3, u is a minimal solution of PDE$(f + d\mu)$ such that $\underline{u} \le u \le \overline{u}$. □

Theorem 5.13. *Assume (H1)–(H7). Then there exists a minimal solution u_n of the system*

$$-Lu_n^j = f^j(\cdot, u_n) + n(u_n^j - H^j(\cdot, u_n))^- + \mu^j \tag{5.6}$$

such that $\underline{u} \le u_n \le \overline{u}$. Moreover, $u_n \nearrow u$ q.e., where u is the minimal solution of (1.3) such that $\underline{u} \le u \le \overline{u}$.

Proof. Observe that $\overline{u}$ is a supersolution of (5.6), whereas $\underline{u}$ is a subsolution of (5.6). Moreover, $f^j(\cdot, \overline{u}) + n(\overline{u}^j - H^j(\cdot, \overline{u}))^- = f^j(\cdot, \overline{u}) \in \mathbb{M}$ and $f^j(\cdot, \underline{u}; \overline{u}^j) + n(\overline{u}^j - H^j(\cdot, \underline{u}))^- = f^j(\cdot, \underline{u}; \overline{u}^j) \in \mathbb{M}$ by (H6). Therefore (H8) is satisfied for (5.6) with $v := \overline{u}$. Hence, by Proposition 5.12, there exists a minimal solution u_n to (5.6) such that $\underline{u} \le u \le \overline{u}$ q.e. By Remark 5.3, $u_n(X)$ is the first component of the solution of BSDE$^\zeta(0, f_n(X, \cdot) + dA^\mu)$ with $f_n^j(t, y) = f^j(X_t, y) + n(y^j - H^j(X_t, y))^-$. By the construction (see Proposition 5.12), it is the minimal solution of BSDE$^\zeta(0, f_n(X, \cdot) + dA^\mu)$. By Theorem 3.14, the sequence $\{u_n(X)\}$ is nondecreasing and $u_n(X) \nearrow Y$, where Y is the first component of the minimal solution of (5.5) such that $\underline{u}(X) \le Y \le \overline{u}(X)$ under the measure P_x for q.e. $x \in E$. In particular $u_n \le u_{n+1}$ q.e., $n \ge 1$. Set $\tilde{u} := \sup_{n\ge1} u_n$. It is clear that $Y = \tilde{u}(X)$ P_x-a.s. for q.e. $x \in E$. On the other hand, by Theorem 5.8, there exists the minimal solution u to (1.3) such that $\underline{u} \le u \le \overline{u}$, and by the proof of Theorem 5.8, for q.e. $x \in E$ the process $u(X)$ is the first component of the minimal solution to (5.5) under the measure P_x. Hence $\tilde{u}(X) = u(X)$ P_x-a.s. for q.e. $x \in E$, which implies that $\tilde{u} = u$ q.e. Of course, this implies that $u_n \nearrow u$ q.e. □

5.2. *Application to the switching problem*

In the theorem below we keep the notation introduced in Section 4, and we assume that

$$H^j(x, y) = \max_{i\in A_j}(-c_{j,i}(x) + y^i), \tag{5.7}$$

where $c_{j,i}$ are quasi-continuous functions on E such that for some constant $c > 0$,

$$c_{j,i}(x) \ge c, \quad x \in E, \quad i \in A_j, \quad j = 1, \dots, N.$$

Proposition 5.14. *Let (H1), (H2) be satisfied. Assume that f^j does not depend on y^i, $i \ne j$, and H^j is of the form (5.7). Then there exists at most one solution to (1.3).*

Proof. By the definition of a solution u to (1.3), the process $u(X)$ is the first component of the solution to (5.5) under the measure P_x for q.e. $x \in E$ (see also Remark 5.3) and $E_x \int_0^\zeta |f(X_r, u(X_r))|\,dr < \infty$ for q.e. $x \in E$. Since $\mathbb{F}$ is quasi-left continuous and A^μ is continuous, the desired result follows from Corollary 4.4. □

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Theorem 5.15. *Assume that f does not depend on y, H^j are of the form (5.7), and $f^j, \mu^j \in \mathbb{M}$, $j = 1, \dots, N$. Then there exists a unique solution u of (1.3). Moreover,*

$$u^j(X_t) = \sup_{\mathcal{S}\in\mathbf{A}} E_x\Big(\int_0^\zeta f^{w_r^j}(X_r)\,dr + \int_0^\zeta dA_r^{\mu^{w_r^j}} - \sum_{n\ge1} c_{w^j_{\tau_{n-1}},w^j_{\tau_n}}(X_{\tau_n})\mathbf{1}_{\{\tau_n<\zeta\}}\Big)$$

and

$$u^j(x) = E_x\Big(\int_0^\zeta f^{w_r^{j,*}}(X_r)\,dr + \int_0^\zeta dA_r^{\mu^{w_r^{j,*}}} - \sum_{n\ge1} c_{w^{j,*}_{\tau_{n-1}},w^{j,*}_{\tau_n}}(X_{\tau_n})\mathbf{1}_{\{\tau_n<\zeta\}}\Big),$$

where

$$w_t^{j,*} = j\mathbf{1}_{[0,\tau_1^*)}(t) + \sum_{n\ge1}\xi_n^{j,*}\mathbf{1}_{[\tau_n^*,\tau_{n+1}^*)}(t)$$

and

$$\tau_0^{j,*} = 0, \quad \tau_k^{j,*} = \inf\{t \ge \tau_{k-1}^{j,*} : u^{\xi_{k-1}^{j,*}}(X_t) = H^{\xi_{k-1}^{j,*}}(X_t, u(X_t))\} \wedge \zeta, \quad k \ge 1,$$

$$\xi_0^{j,*} = j, \quad \xi_k^{j,*} = \max\{i \in A_{\xi_{k-1}^{j,*}};\ u^{\xi_{k-1}^{j,*}}(X_{\tau_k}) = -c_{\xi_{k-1}^{j,*},i}(X_{\tau_k}) + u^i(X_{\tau_k})\}, \quad k \ge 1.$$

Proof. We know that $\mathbb{F}$ is quasi-left continuous and A^μ is continuous. Therefore the theorem follows from Theorem 4.3, Remarks 3.13, 4.2 and Proposition 5.12. □

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References

[1] Ph. Briand, B. Delyon, Y. Hu, E. Pardoux, L. Stoica, L^p solutions of backward stochastic differential equations, Stochastic Process. Appl. 108 (2003) 109–129.

[2] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, Renormalized solutions of elliptic equations with general measure data, Ann. Sc. Norm. Super. Pisa Cl. Sci. 28 (4) (1999) 741–808.

[3] C. Dellacherie, P.-A. Meyer, Probabilities and Potential, North-Holland Publishing Co., Amsterdam, 1978.

[4] C. Dellacherie, P.-A. Meyer, Probabilities and Potential B. Theory of Martingales, North-Holland Publishing Co., Amsterdam, 1982.

[5] B. Djehiche, S. Hamadène, A. Popier, The finite horizon optimal multiple switching problem, SIAM J. Control Optim. 48 (2009) 2751–2770.

[6] B. El Asri, S. Hamadène, The finite horizon optimal multi-modes switching problem: the viscosity solution approach, Appl. Math. Optim. 60 (2009) 213–235.

[7] M. Fukushima, Y. Oshima, M. Takeda, Dirichlet Forms and Symmetric Markov Processes, Walter de Gruyter, Berlin, 1994.

[8] S. Hamadène, M. Jeanblanc, On the starting and stopping problem: application in reversible investments, Math. Oper. Res. 32 (2007) 182–192.

[9] S. Hamadène, M.A. Morlais, Viscosity solutions of systems of PDEs with interconnected obstacles and multi-modes switching problem, Stochastics 88 (2016) 632–649.

[10] S. Hamadène, J. Zhang, Switching problem and related system of reflected backward SDEs, Stochastic Process. Appl. 120 (2010) 403–426.

[11] S. Hamadène, X. Zhao, Systems of integro-PDEs with interconnected obstacles and multi-modes switching problem driven by Lévy process, NoDEA Nonlinear Differential Equations Appl. 22 (2015) 1607–1660.

[12] Y. Hu, S. Tang, Multi-dimensional BSDE with oblique reflection and optimal switching, Probab. Theory Related Fields 147 (2010) 89–121.

[13] J. Jacod, Convergence en loi de semimartingales et variation quadratique, Lecture Notes in Math. 850 (1981) 547–560.
[14] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and their Applications, Academic Press, 1980.
[15] T. Klimsiak, BSDEs with monotone generator and two irregular reflecting barriers, Bull. Sci. Math. 137 (2013) 268–321.
[16] T. Klimsiak, Cauchy problem for semilinear parabolic equation with time-dependent obstacles: a BSDEs approach, Potential Anal. 39 (2013) 99–140.
[17] T. Klimsiak, Reflected BSDEs on filtered probability spaces, Stochastic Process. Appl. 125 (2015) 4204–4241.
[18] T. Klimsiak, Quasi-regular Dirichlet forms and the obstacle problem for elliptic equations with measure data. Available at arXiv:1503.06402.
[19] T. Klimsiak, A. Rozkosz, Dirichlet forms and semilinear elliptic equations with measure data, J. Funct. Anal. 265 (2013) 890–925.
[20] T. Klimsiak, A. Rozkosz, Obstacle problem for semilinear parabolic equations with measure data, J. Evol. Equ. 15 (2015) 457–491.
[21] T. Klimsiak, A. Rozkosz, Renormalized solutions of semilinear equations involving measure data and operator corresponding to Dirichlet form, NoDEA Nonlinear Differential Equations Appl. 22 (2015) 1911–1934.
[22] T. Klimsiak, A. Rozkosz, Semilinear elliptic equations with measure data and quasi-regular Dirichlet forms, Colloq. Math. 145 (2016) 35–67.
[23] N. Lundström, K. Nyström, M. Olofsson, Systems of variational inequalities for non-local operators related to optimal switching problems: existence and uniqueness, Manuscripta Math. 145 (2014) 407–432.
[24] N. Lundström, K. Nyström, M. Olofsson, Systems of variational inequalities in the context of optimal switching problems and operators of Kolmogorov type, Ann. Mat. Pura Appl. (4) 193 (2014) 1213–1247.
[25] Y. Oshima, Semi-Dirichlet Forms and Markov Processes, Walter de Gruyter, Berlin, 2013.
[26] S. Peng, M. Xu, The smallest g-supermartingale and reflected BSDE with single and double L2 obstacles, Ann. Inst. H. Poincaré Probab. Statist. 41 (2005) 605–630.
[27] A. Rozkosz, L. Słomiński, Stochastic representation of entropy solutions of semilinear elliptic obstacle problems with measure data, Electron. J. Probab. 17 (40) (2012) 1–27.