



Q -processes and asymptotic properties of Markov processes conditioned not to hit moving boundaries

William Oçafrain

Institut de Mathématiques de Toulouse, UMR 5219; Université de Toulouse, CNRS, UPS IMT, F-31062, Toulouse Cedex 9, France

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Abstract

We investigate some asymptotic properties of general Markov processes conditioned not to be absorbed by the moving boundaries. We first give general criteria involving an exponential convergence towards the Q -process, that is the law of the considered Markov process conditioned never to reach the moving boundaries. This exponential convergence allows us to state the existence and uniqueness of the quasi-ergodic distribution considering either boundaries moving periodically or stabilizing boundaries. We also state the existence and uniqueness of a quasi-limiting distribution when absorbing boundaries stabilize. We finally deal with some examples such as diffusions which are coming down from infinity. © 2019 Published by Elsevier B.V.

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1. Introduction

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $(X_t)_{t \in I}$ be a time-homogeneous Markov process (where $I = \mathbb{Z}_+$ or \mathbb{R}_+) defined on a metric state space (E, d) . We associate with E a σ -algebra \mathcal{E} . For any $t \in I$, denote by $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$ the σ -field generated by $(X_s)_{0 \leq s \leq t \in I}$. For any subset $F \subset E$, denote by $\mathcal{M}_1(F)$ the set of probability measures defined on F and $\mathcal{B}(F)$ the set of the bounded measurable function $f: F \rightarrow \mathbb{R}$.

We define, for each time $t \in I$, a subset $A_t \in \mathcal{E}$ called *absorbing subset at time t* and we denote by E_t the complement set of A_t called *survival subset at time t* . We will call $t \mapsto A_t$

E-mail address: william.oacafrain@math.univ-toulouse.fr.

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the moving absorbing subset or the moving absorbing boundary. We denote by

$$\tau_A := \inf\{t \in I : X_t \in A_t\}$$

the reaching time of $(A_t)_{t \in I}$ by the process $(X_t)_{t \in I}$. In all what follows, we will assume that τ_A is a stopping time for the filtration $(\mathcal{F}_t)_{t \in I}$. This assumption holds when, for example, the Markov process $(X_t)_{t \in I}$ is continuous and all the sets $(A_t)_{t \in I}$ are closed.

Even though the process $(X_t)_{t \in I}$ is time-homogeneous, we will associate to this process a family of probability measures $(\mathbb{P}_{s,x})_{s \in I, x \in E}$ such that, for any $s \in I$ and for any $x \in E$, $\mathbb{P}_{s,x}(X_s = x) = 1$ and, for any measure μ on E , define $\mathbb{P}_{s,\mu} = \int \mathbb{P}_{s,x} d\mu(x)$. We denote by $\mathbb{E}_{s,x}$ and $\mathbb{E}_{s,\mu}$ the corresponding expectations. When the starting time is not needed, we will prefer the notation $\mathbb{P}_\mu := \mathbb{P}_{0,\mu}$ and $\mathbb{E}_\mu := \mathbb{E}_{0,\mu}$.

In this paper, we will deal with the so-called *Q-process*, *quasi-limiting distribution* and *quasi-ergodic distribution*, defined as below:

Definition 1.

- (i) We say that there is a *Q-process* if there exists a family of probability measures $(\mathbb{Q}_{s,x})_{s \in I, x \in E_s}$ such that for any $s \leq t$, $x \in E_s$,

$$\mathbb{P}_{s,x}(X_{[s,t]} \in \cdot | \tau_A > T) \xrightarrow[T \in I, T \rightarrow \infty]{(d)} \mathbb{Q}_{s,x}(X_{[s,t]} \in \cdot),$$

where, for any $u, v \in I$, $X_{[u,v]}$ is the trajectory of $(X_t)_{t \in I}$ between times u and v and where (d) refers to the weak convergence of probability measures.

- (ii) We say that $\alpha \in \mathcal{M}_1(E)$ is a *quasi-limiting distribution* if, for some $\mu \in \mathcal{M}_1(E_0)$,

$$\mathbb{P}_\mu(X_t \in \cdot | \tau_A > t) \xrightarrow[t \in I, t \rightarrow \infty]{(d)} \alpha. \quad (1)$$

- (iii) We say that $\beta \in \mathcal{M}_1(E)$ is a *quasi-ergodic distribution* if there exists $\mu \in \mathcal{M}_1(E_0)$ such that,

•

$$\frac{1}{n} \sum_{k=0}^n \mathbb{P}_\mu(X_k \in \cdot | \tau_A > n) \xrightarrow[n \rightarrow \infty]{(d)} \beta$$

if $I = \mathbb{Z}_+$,

•

$$\frac{1}{t} \int_0^t \mathbb{P}_\mu(X_s \in \cdot | \tau_A > t) ds \xrightarrow[t \rightarrow \infty]{(d)} \beta$$

if $I = \mathbb{R}_+$.

For Markov processes absorbed by non-moving boundaries (i.e. $A_t = A_0$ for any $t \in I$), the notions of *Q-process*, *quasi-limiting distribution* and *quasi-ergodic distribution* are dealt with by the theory of *quasi-stationarity*, which studies the asymptotic behavior of such processes conditioned not to be absorbed. In particular, the main object of this theory is the *quasi-stationary distribution*, which is defined as a probability measure α such that, for all $t \in I$,

$$\mathbb{P}_\alpha(X_t \in \cdot | \tau_A > t) = \alpha. \quad (2)$$

In the time-homogeneous setting, it is well known that the notions of quasi-stationary distributions and quasi-limiting distributions are equivalent. The interested reader can see [16]

and [10] for an overview of the theory. In particular, these monographs give some results about the existence of quasi-limiting distributions and Q -processes for several processes: Markov chains on finite state space and countable space, birth and death processes, diffusion processes and others. In a same way, existence of quasi-ergodic distributions has been also shown for such processes. The reader can see [4,12,18] for the study on quasi-ergodic distributions in a very general framework.

In this article, we will be interested in the existence of a Q -process, a quasi-limiting distribution and a quasi-ergodic distribution when $(A_t)_{t \in I}$ depends on the time. More precisely, we want to generalize the results presented in [17], which were only obtained for discrete-time Markov chains defined on finite state space. In particular, this paper showed, in a first time, that the notion of quasi-stationary distribution as defined by the relation (2), considering that the boundary $(A_n)_{n \in \mathbb{Z}_+}$ is moving, is not well-defined. If moreover the boundary moves periodically, then the notion of quasi-limiting distribution is not well-defined either. Finally, it is shown in [17] that, still considering periodic moving boundaries, the probability measure

$$\frac{1}{n} \sum_{i=1}^n \mathbb{P}_\mu(X_k \in \cdot | \tau_A > n)$$

converges weakly towards a quasi-ergodic distribution β if the initial measure μ satisfies some assumptions (see [17, Theorem 3]). Moreover, the Q -process is well-defined.

Hence, the main goal of this paper is to recover these results for a more wide class of Markov processes, such as diffusion processes. In particular, we want to know if the quasi-ergodic distribution is still well-defined for such processes when the moving boundary $(A_t)_{t \in I}$ is periodic.

The main assumption that $(X_t)_{t \in I}$ will satisfy in this paper will be based on a Champagnat–Villemonais type condition. When A does not depend on t , Champagnat and Villemonais introduce in [6] the following assumption: there exists $\nu \in \mathcal{M}_1(E)$ such that

(A1) there exist $t_0 \geq 0$ and $c_1 > 0$ such that

$$\forall x \in E_0, \quad \mathbb{P}_x(X_{t_0} \in \cdot | \tau_A > t_0) \geq c_1 \nu;$$

(A2) there exists $c_2 > 0$ such that: $\forall x \in E_0, \forall t \geq 0$,

$$\mathbb{P}_\nu(\tau_A > t) \geq c_2 \mathbb{P}_x(\tau_A > t).$$

In particular, (A1) can be seen as a conditional version of Doeblin's condition. Then the authors show that (A1)–(A2) are equivalent to an exponential uniform convergence of the total variation distance between the conditional probability $\mathbb{P}_\mu(X_t \in \cdot | \tau_A > t)$ and the unique quasi-stationary distribution. Moreover, one has, under these assumptions, the existence of a Q -process, as well as the existence and the uniqueness of the quasi-ergodic distribution (see [8] for this last result).

Champagnat and Villemonais also adapt the assumptions (A1)–(A2) to the time-inhomogeneous setting in the paper [9]. This time-inhomogeneous version will be used to our purpose; we refer the reader to Section 3 for more details about it. In particular, the Assumption (A'), which is introduced in Section 2, is a particular case of their time-inhomogeneous conditions. In this paper, the existence of a Q -process will be proved, as well as the exponential convergence in total variation of the probability measure $\mathbb{P}_{s,x}(X_{[s,t]} \in \cdot | \tau_A > T)$ towards the Q -process, when T goes to infinity. In the same way as in the paper [8], this exponential convergence implies that the existence and the uniqueness of the quasi-ergodic distribution is equivalent to an ergodic theorem for the Q -process. In particular, this corollary

will be applied for periodic moving boundaries to show the existence and the uniqueness of a quasi-ergodic distribution.

Moreover, the case of a non-increasing converging moving boundary (the notion of convergence will be defined further) will be dealt with. In this case, one can expect an asymptotic homogeneity of the conditional probability $\mathbb{P}_{s,x}(X_{s+t} \in \cdot | \tau_A > s+t)$ when s goes to infinity (in the meaning of Proposition 3 in Section 4.2), and use this property to show the existence of a quasi-limiting distribution. It will be therefore shown in this paper that, under the Champagnat–Villemonais condition and some extra assumptions, there exists a unique quasi-limiting distribution for which the weak convergence (1) holds for any initial law μ .

This paper ends with an application of these results to a one-dimensional diffusion process *coming down from infinity*, that is to say, for some $t \geq 0$ and $y \in \mathbb{R}_+$,

$$\lim_{x \rightarrow +\infty} \mathbb{P}_x(\tau_y < t) > 0,$$

where τ_y is the hitting time of y by $(X_t)_{t \in I}$. It will be shown that, under additional assumptions, the diffusion process $(X_t)_{t \geq 0}$ satisfies the time-inhomogeneous Champagnat–Villemonais conditions.

2. Assumptions and general results

From now on, assume that $(A_t)_{t \in I}$ could depend on time and for any $s \in I$ and $x \in E_s$,

$$\mathbb{P}_{s,x}(\tau_A < \infty) = 1,$$

and, in order to make sense of the conditioning, we will assume that for any $s \leq t$ and any $x \in E_s$,

$$\mathbb{P}_{s,x}(\tau_A > t) > 0.$$

We introduce now the main assumption adapted from the Champagnat–Villemonais conditions introduced in [6]:

Assumption (A'). There exist $(\nu_s)_{s \in I}$ a sequence of probability measures ($\nu_s \in \mathcal{M}_1(E_s)$ for each $s \in I$), and $t_0, c_1, c_2 > 0$ such that

$$(A'1) \quad \forall s \in I, \forall x \in E_s,$$

$$\mathbb{P}_{s,x}(X_{s+t_0} \in \cdot | \tau_A > s+t_0) \geq c_1 \nu_{s+t_0};$$

$$(A'2) \quad \forall s \leq t, \forall x \in E_s,$$

$$\mathbb{P}_{s,\nu_s}(\tau_A > t) \geq c_2 \mathbb{P}_{s,x}(\tau_A > t).$$

In this section, the main results and contributions in this paper are presented. Let us recall that the *total variation distance* between two probability measures μ and ν on E is defined by

$$\|\mu - \nu\|_{TV} := \sup_{f \in \mathcal{B}_1(E)} |\mu(f) - \nu(f)|,$$

where $\mathcal{B}_1(E) := \{f \in \mathcal{B}(E) : \|f\|_\infty \leq 1\}$ and where the notation

$$\mu(f) := \int_E f(x) \mu(dx)$$

is used. Then let us state our main result:

Theorem 1. Under Assumption (A'), there exists a Q -process (Definition 1(i)). Furthermore, there exists $C, \lambda > 0$ such that, for any $s \leq t \leq T$ and $x \in E_s$,

$$\|\mathbb{P}_{s,x}(X_{[s,t]} \in \cdot | \tau_A > T) - \mathbb{Q}_{s,x}(X_{[s,t]} \in \cdot)\|_{TV} \leq Ce^{-\lambda(T-t)}.$$

Explicit formulae will be provided later in Theorem 5, whose statement is more precise than the one of the previous theorem.

As written in the introduction, two specific behaviors will be studied in this paper:

- Periodic moving boundaries,
- Non-increasing converging moving boundaries, i.e. $A_t \subset A_s$ for all $s \leq t$ and

$$A_\infty := \bigcap_{t \in I} A_t \neq \emptyset. \quad (3)$$

In the periodic case, the following theorem is shown in Section 4.1.

Theorem 2. If $(X_t)_{t \geq 0}$ satisfies Assumption (A'), then there exists a unique probability measure β such that, for any $\mu \in \mathcal{M}_1(E_0)$,

$$\frac{1}{t} \int_0^t \mathbb{P}_\mu(X_s \in \cdot | \tau_A > t) ds \xrightarrow[t \rightarrow \infty]{(d)} \beta.$$

The expression of the quasi-ergodic distribution β is spelled out later in Theorem 6.

For converging non-increasing moving boundaries, some extra assumptions are needed to state the theorems. The following assumptions will be useful to show the asymptotic homogeneity of the conditional probability $\mathbb{P}_{s,x}(X_{s+t} \in \cdot | \tau_A > s+t)$:

Assumption (H_{hom}).

- (a) **Strong Markov property:** For any τ stopping time of $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$ and for any $x \in E$,

$$\mathbb{P}_x((X_{\tau+t})_{t \in I} \in \cdot, \tau < \infty | \mathcal{F}_\tau) = \mathbb{1}_{\tau < \infty} \mathbb{P}_{X_\tau}((X_t)_{t \in I} \in \cdot);$$

- (b) **Convergence in law for the hitting times:** For any $x \in E_0$ and for any $t \in I$,

$$\mathbb{P}_{s,x}(\tau_A > s+t) \xrightarrow{s \rightarrow +\infty} \mathbb{P}_x(\tau_{A_\infty} > t),$$

where $\tau_{A_\infty} := \inf\{t \geq 0 : X_t \in A_\infty\}$;

- (c) **Time-continuity:** For any $x \in E_0$ and $s \geq 0$, the functions $t \rightarrow \mathbb{P}_{s,x}(\tau_A > t)$ and $t \rightarrow \mathbb{P}_x(\tau_{A_\infty} > t)$ are continuous;
- (d) **State-continuity:** For any $t \in I$, the function $x \rightarrow \mathbb{P}_x(\tau_{A_\infty} > t)$ is continuous.

Moreover, defining E_∞ as the complement of A_∞ , let us set the additional following assumption:

Assumption (H_∞). There exists a unique probability measure $\alpha_\infty \in \mathcal{M}_1(E_\infty)$ such that, for any $\mu \in \mathcal{M}_1(E_\infty)$ and $t \geq 0$,

$$\|\mathbb{P}_\mu(X_t \in \cdot | \tau_{A_\infty} > t) - \alpha_\infty\|_{TV} \leq C_\infty e^{-\gamma_\infty t},$$

where $C_\infty, \gamma_\infty > 0$.

Then, considering non-increasing converging moving boundaries, one has the following statement:

Theorem 3. Under the Assumptions (A') , (H_{hom}) and (H_∞) , for any $\mu \in \mathcal{M}_1(E_0)$,

$$\mathbb{P}_\mu(X_t \in \cdot | \tau_A > t) \xrightarrow[t \rightarrow \infty]{(d)} \alpha_\infty,$$

where α_∞ is the quasi-stationary distribution defined in the Assumption (H_∞) .

The existence and the uniqueness of the quasi-ergodic distribution is also shown in Section 4.2.

3. Exponential convergence towards Q -process and quasi-ergodic distribution

First, we recall Proposition 3.1. and Theorem 3.3. of [9]. In their paper, N. Champagnat and D. Villemonais took a time-inhomogeneous Markov process and $(Z_{s,t})_{s \leq t}$ a collection of multiplicative nonnegative random variables (i.e. satisfying $Z_{s,r} Z_{r,t} = Z_{s,t}$, $\forall s \leq r \leq t$) such that, for any $s \leq t \in I$ and $x \in E_s$, $\mathbb{E}_{s,x}(Z_{s,t}) > 0$ and $\sup_{y \in E_s} \mathbb{E}_{s,y}(Z_{s,t}) < \infty$. In our case, $(X_t)_{t \in I}$ is time-homogeneous, however the penalization $(Z_{s,t})_{s \leq t}$ we shall use is given by

$$Z_{s,t} = \mathbb{1}_{\tau_A > t}, \quad \forall s \leq t.$$

and is time-inhomogeneous because $(A_t)_{t \in I}$ depends on t . For any $s \leq t$, define

$$\phi_{t,s}: \mu \mapsto \mathbb{P}_{s,\mu}(X_t \in \cdot | \tau_A > t).$$

Then, by Markov property, the family $(\phi_{t,s})_{s \leq t}$ is a semi-flow, that is: for any $r \leq s \leq t$,

$$\phi_{t,r} = \phi_{t,s} \circ \phi_{s,r}. \quad (4)$$

Let $t_0 \in I$. For any $s \geq t_0$ and $x_1, x_2 \in E_{s-t_0}$, define v_{s,x_1,x_2} and v_s as follows:

$$v_{s,x_1,x_2} = \min_{j=1,2} \phi_{s,s-t_0}(\delta_{x_j}); \quad (5)$$

$$v_s = \min_{x \in E_{s-t_0}} \phi_{s,s-t_0}(\delta_x), \quad (6)$$

where the minimum of several measures is understood as the largest measure smaller than all the considered measures. Finally, for any $s \geq t_0$, define

$$d_s = \inf_{t \geq 0, x_1, x_2 \in E_{s-t_0}} \frac{\mathbb{P}_{s,v_{s,x_1,x_2}}(\tau_A > t+s)}{\sup_{x \in E_s} \mathbb{P}_{s,x}(\tau_A > t+s)}; \quad (7)$$

$$d'_s = \inf_{t \geq 0} \frac{\mathbb{P}_{s,v_s}(\tau_A > s+t)}{\sup_{x \in E_s} \mathbb{P}_{s,x}(\tau_A > s+t)}. \quad (8)$$

In particular, $v_s \leq v_{s,x_1,x_2}$ and $d'_s \leq d_s$. We can now state Proposition 3.1. and Theorem 3.3. of [9] in our situation (see [9] for a more general framework):

Proposition 1 (Proposition 3.1. [9]). For any $s \in I$ such that $d'_s > 0$ and $y \in E_s$, there exists a finite constant $C_{s,y}$ only depending on s and y such that, for all $x \in E_s$ and $t, u \geq s + t_0$ with $t \leq u$,

$$\left| \frac{\mathbb{P}_{s,x}(\tau_A > t)}{\mathbb{P}_{s,y}(\tau_A > t)} - \frac{\mathbb{P}_{s,x}(\tau_A > u)}{\mathbb{P}_{s,y}(\tau_A > u)} \right| \leq C_{s,y} \inf_{v \in [s+t_0, t]} \frac{1}{d'_v} \prod_{k=0}^{\lfloor \frac{u-s}{t_0} \rfloor - 1} (1 - d_{v-k}). \quad (9)$$

In particular, if

$$\liminf_{t \in I, t \rightarrow \infty} \frac{1}{d'_t} \prod_{k=0}^{\lfloor \frac{t-s}{t_0} \rfloor - 1} (1 - d_{t-k}) = 0, \quad (10)$$

for all $s \geq 0$, there exists a positive bounded function $\eta_s : E_s \rightarrow (0, \infty)$ such that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}_{s,x}(\tau_A > t)}{\mathbb{P}_{s,y}(\tau_A > t)} = \frac{\eta_s(x)}{\eta_s(y)}, \quad \forall x, y \in E_s,$$

where, for any fixed y , the convergence holds uniformly in x . η_s satisfies for all $x \in E_s$ and $s \leq t \in I$,

$$\mathbb{E}_{s,x}(\mathbb{1}_{\tau_A > t} \eta_t(X_t)) = \eta_s(x).$$

In addition, the function $s \rightarrow \|\eta_s\|_\infty$ is locally bounded on $[0, \infty)$.

Theorem 4 (Theorem 3.3 [9]). Assume that

$$\liminf_{t \in I, t \rightarrow \infty} \frac{1}{d'_t} \prod_{k=0}^{\lfloor \frac{t-s}{t_0} \rfloor - 1} (1 - d_{t-k}).$$

Then there exists $(\mathbb{Q}_{s,x})_{s \in I, x \in E_s}$ such that

$$\mathbb{P}_{s,x}(X_{[s,s+t]} \in \cdot | \tau_A > T) \xrightarrow[T \in I, T \rightarrow \infty]{(d)} \mathbb{Q}_{s,x}(X_{[s,s+t]} \in \cdot), \quad \forall s, t \in I, x \in E_s,$$

and $\mathbb{Q}_{s,x}$ is given by, for all $s \leq t$ and $x \in E_s$,

$$\mathbb{Q}_{s,x}(X_{[s,t]} \in \cdot) = \mathbb{E}_{s,x} \left(\mathbb{1}_{X_{[s,t]} \in \cdot} \frac{\mathbb{1}_{\tau_A > t} \eta_t(X_t)}{\mathbb{E}_{s,x}(\mathbb{1}_{\tau_A > t} \eta_t(X_t))} \right) = \mathbb{E}_{s,x} \left(\mathbb{1}_{X_{[s,t]} \in \cdot, \tau_A > t} \frac{\eta_t(X_t)}{\eta_s(x)} \right). \quad (11)$$

Furthermore, under $(\mathbb{Q}_{s,x})_{s \in I, x \in E_s}$, $(X_t)_{t \in I}$ is a time-inhomogeneous Markov process. Finally, this process is asymptotically mixing in the sense that, for any $s \leq t$ and for any $\mu, \pi \in \mathcal{M}_1(E_s)$,

$$\|\mathbb{Q}_{s,\mu}(X_t \in \cdot) - \mathbb{Q}_{s,\pi}(X_t \in \cdot)\|_{TV} \leq 2 \prod_{k=0}^{\lfloor \frac{t-s}{t_0} \rfloor - 1} (1 - d_{t-k}),$$

where

$$\mathbb{Q}_{s,\mu}(\cdot) := \int_{E_s} \mathbb{Q}_{s,x}(\cdot) \mu(dx). \quad (12)$$

Remark 1. Note that, by the definition (12), when μ is not a Dirac mass,

$$\mathbb{Q}_{s,\mu} \neq \lim_{T \rightarrow +\infty} \mathbb{P}_{s,\mu}(\cdot | \tau_A > T).$$

However, using the notation

$$f * \mu(dx) := \frac{f(x) \mu(dx)}{\mu(f)}, \quad \forall \mu \in \mathcal{M}_1(E_s), \quad \forall f \in \mathcal{B}(E_s), \quad (13)$$

one has

$$\lim_{T \rightarrow +\infty} \mathbb{P}_{s,\mu}(\cdot | \tau_A > T) = \mathbb{Q}_{s,\eta_s * \mu}.$$

Remark 2. We emphasize that, in [9], Proposition 3.1. and Theorem 3.3 are stated for any penalizations $(Z_{s,t})_{s \leq t}$. In particular, instead of considering absorbed Markov process, it is possible to work on renormalized Feynman–Kac semi-group taking

$$Z_{s,t} = e^{\int_s^t g(X_u) du},$$

for some measurable functions g . Indeed, the specific choice of $Z_{s,t}$ we did in [Proposition 1](#) and [Theorem 4](#) does not play a role in the proofs.

Under Assumption (A'), and considering $t_0 \in I$ as defined in Assumption (A'), one has, for any $s \in I$,

$$d_s \geq d'_s \geq c_1 c_2 > 0. \quad (14)$$

Hence, by [Proposition 1](#), (10) is satisfied and, for any $s < s + t_0 \leq t \leq u$ and $x, y \in E_s$,

$$\left| \frac{\mathbb{P}_{s,x}(\tau_A > t)}{\mathbb{P}_{s,y}(\tau_A > t)} - \frac{\mathbb{P}_{s,x}(\tau_A > u)}{\mathbb{P}_{s,y}(\tau_A > u)} \right| \leq C_{s,y} \times \frac{1}{c_1 c_2} (1 - c_1 c_2)^{\lfloor \frac{t-s}{t_0} \rfloor}. \quad (15)$$

From this last equation, we can expect an exponential convergence of the family of probability measures $(\mathbb{P}_{s,x}(X_{[s,t]} \in \cdot | \tau_A > T))_{T \geq t}$ towards the Q -process. Let us now reformulate [Theorem 1](#), in a more precise manner:

Theorem 5. *Let $(X_t)_{t \geq 0}$ be a Markov process satisfying Assumption (A').*

1. *Then, for any $s \leq t \leq T$ and $x \in E_s$,*

$$\|\mathbb{P}_{s,x}(X_{[s,t]} \in \cdot | \tau_A > T) - \mathbb{Q}_{s,x}(X_{[s,t]} \in \cdot)\|_{TV} \leq \frac{1}{(c_1 c_2)^3} (1 - c_1 c_2)^{\lfloor \frac{T-t}{t_0} \rfloor},$$

where $\mathbb{Q}_{s,x}$ is defined by [\(11\)](#) in [Theorem 4](#).

2. *If the Q -process satisfies an ergodic theorem, i.e. there exists a probability measure β such that for any $x \in E_0$,*

$$\frac{1}{t} \int_0^t \mathbb{Q}_{0,x}(X_s \in \cdot) ds \xrightarrow[t \rightarrow \infty]{(d)} \beta, \quad (16)$$

then for any $\mu \in \mathcal{M}_1(E_0)$,

$$\frac{1}{t} \int_0^t \mathbb{P}_\mu(X_s \in \cdot | \tau_A > t) ds \xrightarrow[t \rightarrow \infty]{(d)} \beta.$$

The statement of this theorem is implicitly written for $I = \mathbb{R}_+$. Obviously, the statement holds when $I = \mathbb{Z}_+$ and, from now, we will confuse integral and sum to deal with quasi-ergodic distributions when the time space I will not be specified.

Proof of Theorem 5. First we will show the exponential convergence towards the Q -process essentially thanks to [\(15\)](#). In the second step, we will show the existence and uniqueness of the quasi-ergodic distribution using a method similar to that used in [\[8\]](#).

Step 1 : Exponential convergence towards the Q -process

We may extend [\(15\)](#) to general initial law μ and π : putting moreover $1/c_1 c_2$ inside the constant, there exists $C_{s,\pi} > 0$ only depending on s and π such that, for any $s \leq t \leq u$,

$$\left| \frac{\mathbb{P}_{s,\mu}(\tau_A > u)}{\mathbb{P}_{s,\pi}(\tau_A > u)} - \frac{\mathbb{P}_{s,\mu}(\tau_A > t)}{\mathbb{P}_{s,\pi}(\tau_A > t)} \right| \leq C_{s,\pi} (1 - c_1 c_2)^{\lfloor \frac{t-s}{t_0} \rfloor}.$$

Thus, by [Theorem 4](#) and letting $u \rightarrow \infty$,

$$\left| \frac{\mu(\eta_s)}{\pi(\eta_s)} - \frac{\mathbb{P}_{s,\mu}(\tau_A > t)}{\mathbb{P}_{s,\pi}(\tau_A > t)} \right| \leq C_{s,\pi} (1 - c_1 c_2)^{\lfloor \frac{t-s}{t_0} \rfloor}. \quad (17)$$

Using Markov property, for any $s \leq t \leq T$ and for any $x \in E_s$,

$$\mathbb{P}_{s,x}(X_{[s,t]} \in \cdot | \tau_A > T)$$

$$\begin{aligned}
 &= \mathbb{E}_{s,x} \left(\mathbb{1}_{X_{[s,t]} \in \cdot} \frac{\mathbb{1}_{\tau_A > t} \mathbb{P}_{t,X_t}(\tau_A > T)}{\mathbb{P}_{s,x}(\tau_A > T)} \right) \\
 &= \mathbb{E}_{s,x} \left(\mathbb{1}_{X_{[s,t]} \in \cdot} \frac{\mathbb{1}_{\tau_A > t} \mathbb{P}_{t,X_t}(\tau_A > T)}{\mathbb{E}_{s,x}(\mathbb{1}_{\tau_A > t} \mathbb{P}_{t,X_t}(\tau_A > T))} \right) \\
 &= \mathbb{E}_{s,x} \left(\mathbb{1}_{X_{[s,t]} \in \cdot} \frac{\mathbb{1}_{\tau_A > t} \mathbb{P}_{t,X_t}(\tau_A > T)}{\mathbb{P}_{s,x}(\tau_A > t) \mathbb{E}_{s,x}(\mathbb{P}_{t,X_t}(\tau_A > T) | \tau_A > t)} \right) \\
 &= \mathbb{E}_{s,x} \left(\mathbb{1}_{X_{[s,t]} \in \cdot} \frac{\mathbb{1}_{\tau_A > t} \mathbb{P}_{t,X_t}(\tau_A > T)}{\mathbb{P}_{s,x}(\tau_A > t) \mathbb{P}_{t,\phi_{t,s}(\delta_x)}(\tau_A > T)} \right).
 \end{aligned}$$

Using this last equality and (11), for any $s \leq t \leq T$, for any $x \in E_s$ and any $B \in \mathcal{E}$,

$$\begin{aligned}
 &|\mathbb{P}_{s,x}(X_{[s,t]} \in B | \tau_A > T) - \mathbb{Q}_{s,x}(X_{[s,t]} \in B)| \\
 &= \left| \mathbb{E}_{s,x} \left(\frac{\mathbb{1}_{X_{[s,t]} \in B} \mathbb{1}_{\tau_A > t}}{\mathbb{P}_{s,x}(\tau_A > t)} \left(\frac{\mathbb{P}_{t,X_t}(\tau_A > T)}{\mathbb{P}_{t,\phi_{t,s}(\delta_x)}(\tau_A > T)} - \frac{\eta_t(X_t)}{\phi_{t,s}(\delta_x)(\eta_t)} \right) \right) \right| \\
 &\leq C_{t,\phi_{t,s}(\delta_x)}(1 - c_1 c_2) \left\lfloor \frac{T-t}{t_0} \right\rfloor \mathbb{E}_{s,x} \left(\frac{\mathbb{1}_{X_{[s,t]} \in B} \mathbb{1}_{\tau_A > t}}{\mathbb{P}_{s,x}(\tau_A > t)} \right),
 \end{aligned}$$

where the last inequality follows from (17). Moreover, for any $s \leq t$,

$$\mathbb{E}_{s,x} \left(\frac{\mathbb{1}_{X_{[s,t]} \in B} \mathbb{1}_{\tau_A > t}}{\mathbb{P}_x(\tau_A > t)} \right) = \mathbb{P}_{s,x}(X_{[s,t]} \in B | \tau_A > t) \leq 1, \quad \forall B \in \mathcal{E}.$$

Hence, for any $s \leq t$, $x \in E_s$ and $B \in \mathcal{E}$,

$$|\mathbb{P}_{s,x}(X_{[s,t]} \in B | \tau_A > T) - \mathbb{Q}_{s,x}(X_{[s,t]} \in B)| \leq C_{t,\phi_{t,s}(\delta_x)}(1 - c_1 c_2) \left\lfloor \frac{T-t}{t_0} \right\rfloor.$$

Without loss of generality, one can assume $t - s \geq t_0$, since for any $t \leq s + t_0$,

$$\{X_{[s,t]} \in B\} = \{X_{[s,s+t_0]} \in \tilde{B}\},$$

where $\tilde{B} := \{\omega : [s, s + t_0] \rightarrow E : \omega_{[s,t]} \in B\}$ is a measurable set.

Note that [9] provides an explicit formula of $C_{s,y}$ in the proof of Proposition 3.1. for s and y fixed. Adapting this formula for a general probability measure π and recalling that we put the term $1/c_1 c_2$ inside $C_{s,\pi}$, one explicit formula of $C_{s,\pi}$ for $s \in I$ can be

$$C_{s,\pi} = \frac{1}{c_1 c_2} \frac{\sup_{z \in E_s} \mathbb{P}_{s,z}(\tau_A > v_s)}{d'_{v_s} \mathbb{P}_{s,\pi}(\tau_A > v_s)}, \quad (18)$$

where $v_s \in I$ is the smaller time $v \geq s + t_0$ such that $d'_v > 0$ (with d'_v as defined in (8)). Then, by (14), $v_s = s + t_0$ and $d'_{s+t_0} \geq c_1 c_2$, so

$$C_{s,\pi} \leq C \frac{\sup_{z \in E_s} \mathbb{P}_{s,z}(\tau_A > s + t_0)}{\mathbb{P}_{s,\pi}(\tau_A > s + t_0)}, \quad \forall s \geq 0, \forall \pi \in \mathcal{M}_1(E_s),$$

where we set $C := \frac{1}{(c_1 c_2)^2}$. Thus, for any $x \in E_s$,

$$C_{t,\phi_{t,s}(\delta_x)} \leq C \frac{\sup_{z \in E_t} \mathbb{P}_{t,z}(\tau_A > t + t_0)}{\mathbb{P}_{t,\phi_{t,s}(\delta_x)}(\tau_A > t + t_0)}.$$

Now, the following lemma is needed:

Lemma 1. For any $s \leq t$ such that $t - s \geq t_0$, for any $x \in E_s$,

$$\phi_{t,s}(\delta_x) \geq c_1 v_t. \quad (19)$$

In particular, the condition (A'1) holds replacing t_0 by any time t_1 greater than t_0 . The proof of this lemma is postponed at the end of this proof. Then, by Lemma 1 and using (A'2),

$$\begin{aligned}\mathbb{P}_{t,\phi_{t,s}(\delta_x)}(\tau_A > t + t_0) &\geq c_1 \mathbb{P}_{t,v_t}(\tau_A > t + t_0) \\ &\geq c_1 c_2 \sup_{z \in E_t} \mathbb{P}_{t,z}(\tau_A > t + t_0).\end{aligned}$$

As a result $C_{t,\phi_{t,s}(\delta_x)} \leq 1/(c_1 c_2)^3$, and

$$|\mathbb{P}_{s,x}(X_{[s,t]} \in B | \tau_A > T) - \mathbb{Q}_{s,x}(X_{[s,t]} \in B)| \leq \frac{1}{(c_1 c_2)^3} (1 - c_1 c_2) \lfloor \frac{T-t}{t_0} \rfloor.$$

This concludes the first step.

Step 2 : Convergence towards the quasi-ergodic distribution

We just proved that for any $0 \leq s \leq t$ and $x \in E_0$,

$$\|\mathbb{P}_x(X_s \in \cdot | \tau_A > t) - \mathbb{Q}_{0,x}(X_s \in \cdot)\|_{TV} \leq \frac{1}{(c_1 c_2)^3} (1 - c_1 c_2) \lfloor \frac{t-s}{t_0} \rfloor.$$

Note that, in the same way, it was possible to consider a general initial law μ instead of a Dirac measure δ_x , so that the inequality

$$\|\mathbb{P}_\mu(X_s \in \cdot | \tau_A > t) - \mathbb{Q}_{0,\eta_0 * \mu}(X_s \in \cdot)\|_{TV} \leq \frac{1}{(c_1 c_2)^3} (1 - c_1 c_2) \lfloor \frac{t-s}{t_0} \rfloor \quad (20)$$

holds for any probability measure μ on E_0 (the notation $\eta_0 * \mu$ is defined in (13)).

As a result for any $0 \leq s \leq t$, for any $\mu \in \mathcal{M}_1(E_0)$,

$$\begin{aligned}&\left\| \frac{1}{t} \int_0^t \mathbb{P}_\mu(X_s \in \cdot | \tau_A > t) ds - \frac{1}{t} \int_0^t \mathbb{Q}_{0,\eta_0 * \mu}(X_s \in \cdot) ds \right\|_{TV} \\ &\leq \frac{1}{(c_1 c_2)^3 t} \int_0^t (1 - c_1 c_2) \lfloor \frac{t-s}{t_0} \rfloor ds \\ &\leq \frac{1}{(c_1 c_2)^3 t} \int_0^t (1 - c_1 c_2)^{\frac{t-s}{t_0} - 1} ds \\ &= \left(-\frac{t_0}{(c_1 c_2)^3 (1 - c_1 c_2) \log(1 - c_1 c_2)} \right) \times \frac{1 - (1 - c_1 c_2)^{\frac{t}{t_0}}}{t}.\end{aligned}$$

Let β be as defined in (16). Then for any $\mu \in \mathcal{M}_1(E_0)$ and $f \in \mathcal{B}(E)$,

$$\begin{aligned}&\left| \frac{1}{t} \int_0^t \mathbb{E}_\mu(f(X_s) | \tau_A > t) ds - \beta(f) \right| \\ &\leq \left\| \frac{1}{t} \int_0^t \mathbb{P}_\mu(X_s \in \cdot | \tau_A > t) ds - \frac{1}{t} \int_0^t \mathbb{Q}_{0,\eta_0 * \mu}(X_s \in \cdot) ds \right\|_{TV} \\ &\quad + \left| \frac{1}{t} \int_0^t \mathbb{E}_{0,\eta_0 * \mu}^\mathbb{Q}(f(X_s)) ds - \beta(f) \right| \\ &\leq \left(-\frac{t_0}{(c_1 c_2)^3 (1 - c_1 c_2) \log(1 - c_1 c_2)} \right) \times \frac{1 - (1 - c_1 c_2)^{\frac{t}{t_0}}}{t} \\ &\quad + \left| \frac{1}{t} \int_0^t \mathbb{E}_{0,\eta_0 * \mu}^\mathbb{Q}(f(X_s)) ds - \beta(f) \right|,\end{aligned}$$

where $\mathbb{E}_{0,\eta_0*\mu}^{\mathbb{Q}}$ is the expectation with respect to $\mathbb{Q}_{0,\eta_0*\mu}$. Then, using the ergodic theorem for the Q -process,

$$\left| \frac{1}{t} \int_0^t \mathbb{E}_{\mu}(f(X_s) | \tau_A > t) ds - \beta(f) \right| \xrightarrow[t \rightarrow \infty]{} 0. \quad \square$$

Proof of Lemma 1. Applying the condition (A'1) to the starting time $t - t_0$,

$$\mathbb{P}_{t-t_0,y}(X_t \in \cdot, \tau_A > t) \geq c_1 v_t(\cdot) \mathbb{P}_{t-t_0,y}(\tau_A > t), \quad \forall y \in E_{t-t_0}.$$

Then, for any probability measure μ , integrating the last inequality over $\mu(dx)$ and dividing by $\mathbb{P}_{t-t_0,\mu}(\tau_A > t)$, one obtains

$$\mathbb{P}_{t-t_0,\mu}(X_t \in \cdot | \tau_A > t) \geq c_1 v_t, \quad \forall \mu \in \mathcal{M}_1(E_{t-t_0}).$$

Hence, using the semi-flow property (4) of $(\phi_{t,s})_{s \leq t}$, for any $s \geq 0$ and $t \geq s + t_0$,

$$\phi_{t,s}(\delta_x) = \phi_{t,t-t_0} \circ \phi_{t-t_0,s}(\delta_x) = \mathbb{P}_{t-t_0,\phi_{t-t_0,s}(\delta_x)}(X_t \in \cdot | \tau_A > t) \geq c_1 v_t,$$

which is (19). \square

Remark 3. The time-homogeneity of the Markov process $(X_t)_{t \in I}$ does not play a particular role in the previous proof. In particular, Theorem 5 can be applied to time-inhomogeneous Markov process. However, in the next section, the time-homogeneity of $(X_t)_{t \in I}$ will be needed.

4. Some behaviors of moving boundaries and quasi-ergodicity

In this section, we will focus on two types of behavior for the moving boundaries

1. when A is γ -periodic with $\gamma > 0$;
2. when A is non-increasing and converges at infinity towards $A_{\infty} \neq \emptyset$.

Under Assumption (A'), the existence of the Q -process is provided by Theorem 4 (Theorem 3.3, [9]) and we get moreover an exponential convergence towards the Q -process provided by Theorem 5. Now we want to investigate on the existence of a quasi-ergodic distribution in the two cases described above.

4.1. Quasi-ergodic distribution when A is γ -periodic

In this subsection, we will work on periodic moving boundaries and we will assume that the Markov process $(X_t)_{t \geq 0}$ satisfies the Assumption (A'). In particular, considering Assumption (A') for $s = 0$, for any $x \in E_0$ and $t \in I$,

1. $\mathbb{P}_x(X_{t_0} \in \cdot | \tau_A > t_0) \geq c_1 v_{t_0}$;
2. $\mathbb{P}_{v_0}(\tau_A > t) \geq c_2 \mathbb{P}_x(\tau_A > t)$.

As Lemma 1 claims, any time t_1 greater than t_0 is suitable for the condition (A'1). Hence, without loss of generality, t_0 will be taken such that $t_0 = n_0 \gamma$ with $n_0 \in \mathbb{N}$. Moreover, by periodicity of A , it is easy to see that $(v_s)_{s \geq 0}$ can be chosen as a γ -periodic sequence. As a result, one has

$$v_{t_0} = v_{n_0 \gamma} = v_0.$$

In all what follows, we will consider such a choice of $(v_s)_{s \geq 0}$. The aim is to obtain the convergence of $\frac{1}{t} \int_0^t \mathbb{P}_{\mu}(X_s \in \cdot | \tau_A > t) ds$ towards a quasi-ergodic distribution which will

be unique. Let us state the following result, which is the more precise version of [Theorem 2](#) introduced in [Section 2](#):

Theorem 6. Assume A is γ -periodic with $\gamma > 0$, and assume that Assumption (A') is satisfied. Then for any $\mu \in \mathcal{M}_1(E_0)$,

$$\frac{1}{t} \int_0^t \mathbb{P}_\mu(X_s \in \cdot | \tau_A > t) ds \xrightarrow[t \rightarrow \infty]{(d)} \frac{1}{\gamma} \int_0^\gamma \mathbb{Q}_{0, \beta_\gamma}(X_s \in \cdot) ds,$$

where β_γ is the invariant measure of $(X_{n\gamma})_{n \in \mathbb{N}}$ under $\mathbb{Q}_{0, \cdot}$, i.e.

$$\forall n \in \mathbb{N}, \quad \beta_\gamma = \mathbb{Q}_{0, \beta_\gamma}(X_{n\gamma} \in \cdot) = \int_{E_0} \beta_\gamma(dx) \mathbb{Q}_{0, x}(X_{n\gamma} \in \cdot).$$

Proof of Theorem 6. We want to show an ergodic theorem for the time-inhomogeneous Markov process $(X_t)_{t \geq 0}$ under $(\mathbb{Q}_{s, x})_{s \geq 0, x \in E_s}$. Since $(A_t)_{t \geq 0}$ is γ -periodic, for any $0 \leq s \leq t$, for any $x \in E_s$,

$$\mathbb{Q}_{s+k\gamma, x}(X_{t+k\gamma} \in \cdot) = \mathbb{Q}_{s, x}(X_t \in \cdot), \quad \forall k \in \mathbb{Z}_+. \quad (21)$$

Moreover, for any $n \in \mathbb{Z}_+$,

$$\begin{aligned} \mathbb{Q}_{0, x}(X_{n\gamma} \in \cdot) &= \lim_{t \rightarrow \infty} \mathbb{P}_x(X_{n\gamma} \in \cdot | \tau_A > t) \\ &= \lim_{m \in \mathbb{Z}_+, m \rightarrow \infty} \mathbb{P}_x(X_{n\gamma} \in \cdot | \tau_A > m\gamma) \\ &= \lim_{m \in \mathbb{Z}_+, m \rightarrow \infty} \mathbb{P}_x(Y_n \in \cdot | \tau_\partial > m), \end{aligned}$$

where τ_∂ is defined by

$$\tau_\partial = \begin{cases} \inf\{n \geq 1 : \exists t \in ((n-1)\gamma, n\gamma], X_t \in A_t\} & \text{if } Y_0 \in E_0 \\ 0 & \text{if } Y_0 \in A_0 \end{cases}$$

and $(Y_n)_{n \in \mathbb{Z}_+}$ is the time-homogeneous Markov chain defined by

$$Y_n = \begin{cases} X_{n\gamma} & \text{for } n < \tau_\partial \\ \partial & \text{otherwise} \end{cases}$$

where ∂ plays the role of an absorbing state for $(Y_n)_{n \in \mathbb{Z}_+}$. In other words, τ_∂ is an absorbing time for $(Y_n)_{n \in \mathbb{Z}_+}$ and, under $(\mathbb{Q}_{0, x})_{x \in E_0}$, the chain $(X_{n\gamma})_{n \in \mathbb{Z}_+}$ is the Q -process of $(Y_n)_{n \in \mathbb{Z}_+}$.

By Assumption (A') and recalling that we chose $(\nu_s)_{s \geq 0}$ as γ -periodic, $(Y_n)_{n \in \mathbb{Z}_+}$ satisfies the following Champagnat–Villemonais type condition:

1.

$$\forall x \in E_0, \quad \mathbb{P}_x(Y_{n_0} \in \cdot | \tau_\partial > n_0) \geq c_1 \nu_0;$$

2.

$$\forall x \in E_0, \forall n \in \mathbb{Z}_+, \quad \mathbb{P}_{\nu_0}(\tau_\partial > n) \geq c_2 \mathbb{P}_x(\tau_\partial > n).$$

where we recall that $n_0 = \frac{t_0}{\gamma}$. Hence, by Theorem 3.1 in [\[6\]](#), there exists $\beta_\gamma \in \mathcal{M}_1(E_0)$, $C > 0$ and $\rho \in (0, 1)$ such that for any $n \in \mathbb{Z}_+$,

$$\|\mathbb{Q}_{0, x}(X_{n\gamma} \in \cdot) - \beta_\gamma\|_{TV} \leq C\rho^n, \quad \forall x \in E_0.$$

This implies that, under $\mathbb{Q}_{0,\cdot}$, $(X_{n\gamma})_{n \in \mathbb{N}}$ is Harris recurrent. We can therefore apply Theorem 2.1 in [13] and deduce that, for any nonnegative function f ,

$$\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow[t \rightarrow \infty]{} \mathbb{E}_{0,\beta_\gamma}^{\mathbb{Q}} \left(\frac{1}{\gamma} \int_0^\gamma f(X_s) ds \right), \quad \mathbb{Q}_{0,x}\text{-almost surely, } \forall x \in E_0,$$

where $\mathbb{E}_{0,\mu}^{\mathbb{Q}}(G) = \int G d\mathbb{Q}_{0,\mu}$ for any measurable nonnegative function G and $\mu \in \mathcal{M}_1(E_0)$. It extends to $f \in \mathcal{B}(E)$ using $f = f_+ - f_-$ with f_+, f_- non negative functions. Thus, by bounded Lebesgue's convergence theorem, for any $x \in E_0$ and for any $f \in \mathcal{B}(E)$,

$$\frac{1}{t} \int_0^t \mathbb{E}_{0,x}^{\mathbb{Q}}(f(X_s)) ds \xrightarrow[t \rightarrow \infty]{} \mathbb{E}_{0,\beta_\gamma}^{\mathbb{Q}} \left(\frac{1}{\gamma} \int_0^\gamma f(X_s) ds \right).$$

Hence the condition (16) is satisfied. We conclude the proof using the second part of Theorem 5. \square

Remark 4. In [13], Höpfner and Kutoyants claimed their results for Markov processes with continuous paths. It is easy to see using their arguments that the statement in Theorem 2.1. can be generalized to any time-inhomogeneous Markov processes $(X_t)_{t \in I}$ such that the condition of periodicity (21) is satisfied and the chain $(X_{n\gamma})_{n \in \mathbb{Z}_+}$ is Harris recurrent. See also Proposition 5 of [14].

4.2. Quasi-ergodic distribution when A converges at infinity

In this subsection, we assume that A is non-increasing and let A_∞ be as defined in (3). In what follows, we will first state the existence and uniqueness of a quasi-limiting distribution under these assumptions. Then we will deal with quasi-ergodic distribution.

4.2.1. Quasi-limiting distribution

First we state the following proposition which will be useful to prove the theorem on the existence and the uniqueness of the quasi-limiting distribution.

Proposition 2. Under Assumptions (A'), for any $B \in \mathcal{E}$, the quantities

$$\limsup_{t \rightarrow \infty} \mathbb{P}_{s,\mu}(X_t \in B | \tau_A > t) \text{ and } \liminf_{t \rightarrow \infty} \mathbb{P}_{s,\mu}(X_t \in B | \tau_A > t)$$

do not depend on any couple (s, μ) such that $\mu \in \mathcal{M}_1(E_s)$.

Proof of Proposition 2. We recall the statement of [9, Theorem 2.1], which is adapted to our case:

Theorem 7 (Theorem 2.1., [9]). For any $s \in I$, for any $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$, for any $t \geq s + t_0$,

$$\|\mathbb{P}_{s,\mu_1}(X_t \in \cdot | \tau_A > t) - \mathbb{P}_{s,\mu_2}(X_t \in \cdot | \tau_A > t)\|_{TV} \leq 2(1 - c_1 c_2) \left\lfloor \frac{t-s}{t_0} \right\rfloor. \quad (22)$$

Let $B \in \mathcal{E}$. First we remark that, for s fixed, $\limsup_{t \rightarrow \infty} \mathbb{P}_\mu(X_t \in B | \tau_{A \circ \theta_s} > t)$ does not depend on $\mu \in \mathcal{M}_1(E_s)$. This is straightforward since, thanks to (22), for any $s \geq 0$ and any $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$,

$$\|\mathbb{P}_{s,\mu_1}(X_t \in \cdot | \tau_A > t) - \mathbb{P}_{s,\mu_2}(X_t \in \cdot | \tau_A > t)\|_{TV} \xrightarrow[t \rightarrow 0]{} 0,$$

which implies that, for any $s \geq 0$ and $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$,

$$\limsup_{t \rightarrow \infty} \mathbb{P}_{s, \mu_1}(X_t \in B | \tau_A > t) = \limsup_{t \rightarrow \infty} \mathbb{P}_{s, \mu_2}(X_t \in B | \tau_A > t). \quad (23)$$

Now for any $u \geq 0$, recalling the notation $\phi_{t,s}(\mu) = \mathbb{P}_{s,\mu}(X_{s+u} \in \cdot | \tau_A > s + u)$ for any $s \leq t$ and $\mu \in \mathcal{M}_1(E_s)$,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \mathbb{P}_{s, \mu}(X_t \in B | \tau_A > t) &= \limsup_{t \rightarrow \infty} \mathbb{P}_{s, \mu}(X_{t+u} \in B | \tau_A > t + u) \\ &= \limsup_{t \rightarrow \infty} \mathbb{P}_{s+u, \phi_{t,s}(\mu)}(X_t \in B | \tau_A > t) \\ &= \limsup_{t \rightarrow \infty} \mathbb{P}_{s+u, \nu}(X_t \in B | \tau_A > t), \end{aligned} \quad (24)$$

where we used first the semi-flow property of $(\phi_{t,s})_{s \leq t}$, and then (23) with a given probability measure $\nu \in \mathcal{M}_1(E_{s+u})$.

Hence (24) show that $\limsup_{t \rightarrow \infty} \mathbb{P}_{s, \mu}(X_t \in B | \tau_A > t)$ does not depend on any couple (s, μ) satisfying $s \in I$ and $\mu \in \mathcal{M}_1(E_s)$. A similar reasoning shows that $\liminf_{t \rightarrow \infty} \mathbb{P}_{s, \mu}(X_t \in B | \tau_A > t)$ does not depend on s and μ either. \square

Before showing the existence of a quasi-limiting and a quasi-ergodic distribution, let us state the following proposition providing a uniform-in-time convergence of the time-inhomogeneous conditioned semi-group towards the time-homogeneous limit semi-group.

Proposition 3. *Under Assumptions (H_{hom}) and (H_∞) , for any $x \in E_0$,*

$$\lim_{s \rightarrow \infty} \sup_{0 \leq t \leq T} \|\mathbb{P}_{s,x}(X_{t+s} \in \cdot | \tau_A > s + T) - \mathbb{P}_x(X_t \in \cdot | \tau_{A_\infty} > T)\|_{TV} = 0. \quad (25)$$

Remark 5. Taking $T = t$, (25) implies that, for any $x \in E_0$,

$$\lim_{s \rightarrow \infty} \sup_{t \geq 0} \|\mathbb{P}_{s,x}(X_{t+s} \in \cdot | \tau_A > s + t) - \mathbb{P}_x(X_t \in \cdot | \tau_{A_\infty} > t)\|_{TV} = 0.$$

This is actually a stronger version than the definition of asymptotic pseudotrajectories as introduced by Benaïm and Hirsch in [3], for which the supremum is usually only taken on a compact set of time. In a practical way, it is difficult to use the weak version to show the convergence of the time-inhomogeneous semi-flow; considering instead a uniform convergence on \mathbb{R}_+ will be useful for our purpose. The interested reader can see [2] for more details about asymptotic pseudotrajectories.

Proof of Proposition 3. Let $x \in E_0$. Then for any $0 \leq t \leq T$, $s \geq 0$ and $B \in \mathcal{E}$,

$$\begin{aligned} &|\mathbb{P}_{s,x}(X_{t+s} \in B | \tau_A > s + T) - \mathbb{P}_x(X_t \in B | \tau_{A_\infty} > T)| \\ &= |\mathbb{P}_{s,x}(X_{t+s} \in B | \tau_A > s + T) - \mathbb{P}_{s,x}(X_{s+t} \in B | \tau_{A_\infty} > s + T)| \\ &= \left| \frac{\mathbb{P}_{s,x}(\tau_{A_\infty} > s + T)}{\mathbb{P}_{s,x}(\tau_A > s + T)} \frac{\mathbb{P}_{s,x}(X_{t+s} \in B, \tau_A > s + T)}{\mathbb{P}_{s,x}(\tau_{A_\infty} > s + T)} - \frac{\mathbb{P}_{s,x}(X_{s+t} \in B, \tau_{A_\infty} > s + T)}{\mathbb{P}_{s,x}(\tau_{A_\infty} > s + T)} \right| \\ &\leq \left| \frac{\mathbb{P}_{s,x}(\tau_{A_\infty} > s + T)}{\mathbb{P}_{s,x}(\tau_A > s + T)} \frac{\mathbb{P}_{s,x}(X_{s+t} \in B, \tau_A > s + T)}{\mathbb{P}_{s,x}(\tau_{A_\infty} > s + T)} - \frac{\mathbb{P}_{s,x}(X_{t+s} \in B, \tau_A > s + T)}{\mathbb{P}_{s,x}(\tau_{A_\infty} > s + T)} \right| \\ &\quad + \left| \frac{\mathbb{P}_{s,x}(X_{s+t} \in B, \tau_A > s + T)}{\mathbb{P}_{s,x}(\tau_{A_\infty} > s + T)} - \frac{\mathbb{P}_{s,x}(X_{s+t} \in B, \tau_{A_\infty} > s + T)}{\mathbb{P}_{s,x}(\tau_{A_\infty} > s + T)} \right| \\ &\leq \left| \frac{\mathbb{P}_{s,x}(\tau_{A_\infty} > s + T)}{\mathbb{P}_{s,x}(\tau_A > s + T)} - 1 \right| \times \frac{\mathbb{P}_{s,x}(X_{s+t} \in B, \tau_A > s + T)}{\mathbb{P}_{s,x}(\tau_{A_\infty} > s + T)} \end{aligned}$$

$$+ \left| \frac{\mathbb{P}_{s,x}(X_{s+t} \in B, \tau_A > s+T) - \mathbb{P}_{s,x}(X_{s+t} \in B, \tau_{A_\infty} > s+T)}{\mathbb{P}_{s,x}(\tau_{A_\infty} > s+T)} \right| \\ \leq \left| \frac{\mathbb{P}_{s,x}(\tau_{A_\infty} > s+T)}{\mathbb{P}_{s,x}(\tau_A > s+T)} - 1 \right| + \frac{\mathbb{P}_{s,x}(\tau_A \leq s+T < \tau_{A_\infty})}{\mathbb{P}_{s,x}(\tau_{A_\infty} > s+T)},$$

where we used several times the fact that $A_\infty \subset A_t$ for any t (in particular to say that $\mathbb{P}_{s,x}(\tau_A > s+u) \leq \mathbb{P}_{s,x}(\tau_{A_\infty} > s+u)$ for any $u \geq 0$). Hence it is enough to prove that

$$\sup_{t \geq 0} \frac{\mathbb{P}_{s,x}(\tau_A \leq s+t < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > t)} \xrightarrow{s \rightarrow \infty} 0. \quad (26)$$

As a matter of fact, (26) is equivalent to

$$\sup_{t \geq 0} \left| \frac{\mathbb{P}_{s,x}(\tau_A > s+t)}{\mathbb{P}_x(\tau_{A_\infty} > t)} - 1 \right| \xrightarrow{s \rightarrow \infty} 0.$$

and it is easy to check that, for general functions $(s, t) \rightarrow f(s, t)$, $(f(s, \cdot))_{s \geq 0}$ converges uniformly towards the constant function equal to 1 if and only if $\left(\frac{1}{f(s, \cdot)}\right)_{s \geq 0}$ also converges uniformly towards 1.

Fix $t \geq 0$. Since A is non-increasing, for any $s < s'$,

$$\frac{\mathbb{P}_{s,x}(\tau_A \leq s+t < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > t)} \geq \frac{\mathbb{P}_{s',x}(\tau_A \leq s'+t < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > t)}.$$

Moreover, using the convergence in law for the hitting times of Assumption (H_{hom}) , one has, for any $t \geq 0$,

$$\frac{\mathbb{P}_{s,x}(\tau_A \leq s+t < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > t)} \xrightarrow{s \rightarrow \infty} 0.$$

Finally, by the strong Markov property of Assumption (H_{hom}) , for any $t \geq 0$,

$$\mathbb{P}_{s,x}(\tau_A \leq s+t < \tau_{A_\infty}) = \mathbb{E}_{s,x}(\mathbb{1}_{\tau_A \leq s+t} \phi(X_{\tau_A}, \tau_A, t+s)),$$

where $\phi(\cdot, \cdot, \cdot)$ is defined as follows

$$\forall z \in E_\infty, \forall 0 \leq u \leq t, \quad \phi(z, u, t) = \mathbb{P}_z(\tau_{A_\infty} > t-u).$$

By Assumption (H_∞) , there exists a unique quasi-stationary distribution α_∞ for the process $(X_t)_{t \in I}$ absorbed at A_∞ . Then we recall (see [10,16]) that there exists $\lambda_\infty > 0$ such that

$$\mathbb{P}_{\alpha_\infty}(\tau_{A_\infty} > t) = e^{-\lambda_\infty t}, \quad \forall t \geq 0.$$

In [6] it is shown (Proposition 2.3.) that, under (H_∞) , there exists a positive bounded function η_∞ defined on $(0, \infty)$ such that

$$\eta_\infty(x) = \lim_{t \rightarrow \infty} e^{\lambda_\infty t} \mathbb{P}_x(\tau_{A_\infty} > t), \quad \forall x \in E_\infty. \quad (27)$$

where the convergence holds for the uniform norm on E_∞ . Thus, by [Proposition 2.3, [6]], $\frac{\mathbb{1}_{\tau_A \leq t+s} \phi(X_{\tau_A}, \tau_A, t+s)}{\mathbb{P}_x(\tau_{A_\infty} > t)}$ is uniformly bounded and converges almost surely towards $\mathbb{1}_{\tau_A < \infty} e^{\lambda_\infty \tau_A} \frac{\eta_\infty(X_{\tau_A})}{\eta_\infty(x)}$. Then, by the bounded Lebesgue's convergence theorem, for any $s \geq 0$,

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}_{s,x}(\tau_A \leq s+t < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > t)} = \lim_{t \rightarrow \infty} \mathbb{E}_{s,x} \left(\frac{\mathbb{1}_{\tau_A \leq s+t} \phi(X_{\tau_A}, \tau_A, t+s)}{\mathbb{P}_x(\tau_{A_\infty} > t)} \right)$$

$$= \mathbb{E}_{s,x} \left(\mathbb{1}_{\tau_A < \infty} e^{\lambda_\infty \tau_A} \frac{\eta_\infty(X_{\tau_A})}{\eta_\infty(x)} \right).$$

For any $s \geq 0$, we can therefore define $f_s : t \rightarrow \frac{\mathbb{P}_{s,x}(\tau_A \leq s+t < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > t)}$ on the Alexandroff extension $\mathbb{R}_+ \cup \{\infty\}$ setting

$$f_s(\infty) := \lim_{t \rightarrow \infty} \frac{\mathbb{P}_{s,x}(\tau_A \leq s+t < \tau_{A_\infty})}{\mathbb{P}_x(\tau_{A_\infty} > t)} = \mathbb{E}_{s,x} \left(\mathbb{1}_{\tau_A < \infty} e^{\lambda_\infty \tau_A} \frac{\eta_\infty(X_{\tau_A})}{\eta_\infty(x)} \right).$$

Then, like any $t \in \mathbb{R}$, $(f_s(\infty))_{s \geq 0}$ is non-increasing and since η_∞ vanishes on A_∞ and η_∞ is continuous (this is due to the *state-continuity* of Assumption (H_{hom}) and the uniform convergence (27)),

$$\lim_{s \rightarrow \infty} f_s(\infty) = \lim_{s \rightarrow \infty} \mathbb{E}_{s,x} \left(\mathbb{1}_{\tau_A < \infty} e^{\lambda_\infty \tau_A} \frac{\eta_\infty(X_{\tau_A})}{\eta_\infty(x)} \right) = 0.$$

We conclude to the uniform convergence (26) using Dini's theorem for a non-increasing sequence of functions. \square

Now one will prove Theorem 3 stated in Section 2, which is recalled below:

Theorem 8. Under Assumptions (A') , (H_{hom}) and (H_∞) , for any $\mu \in \mathcal{M}_1(E_0)$,

$$\mathbb{P}_\mu(X_t \in \cdot | \tau_A > t) \xrightarrow[t \rightarrow \infty]{(d)} \alpha_\infty,$$

where α_∞ is the quasi-stationary distribution defined in Assumption (H_∞) .

Proof of Theorem 8 (Theorem 3). Fix $B \in \mathcal{E}$ and note that, by Assumption (H_∞) , for any $\mu \in \mathcal{M}_1(E_\infty)$,

$$\limsup_{t \rightarrow \infty} \mathbb{P}_\mu(X_t \in B | \tau_{A_\infty} > t) = \liminf_{t \rightarrow \infty} \mathbb{P}_\mu(X_t \in B | \tau_{A_\infty} > t) = \alpha_\infty(B),$$

where we recall that α_∞ is the quasi-stationary distribution of $(X_t)_{t \in I}$ absorbed at A_∞ . By Proposition 2, for a given $s \in I$, $\limsup_{t \rightarrow \infty} \mathbb{P}_{s,\mu}(X_t \in B | \tau_A > t)$ and $\liminf_{t \rightarrow \infty} \mathbb{P}_{s,\mu}(X_t \in B | \tau_A > t)$ do not depend on $\mu \in \mathcal{M}_1(E_s)$. Denote therefore by F_{sup} and F_{inf} the functions defined by, for any $s \geq 0$ and any $\mu \in \mathcal{M}_1(E_s)$,

$$F_{sup}(s) := \limsup_{t \rightarrow \infty} \mathbb{P}_{s,\mu}(X_{s+t} \in B | \tau_A > s+t) = \limsup_{t \rightarrow \infty} \mathbb{P}_{s,x}(X_{s+t} \in B | \tau_A > s+t)$$

and

$$F_{inf}(s) := \liminf_{t \rightarrow \infty} \mathbb{P}_{s,\mu}(X_{s+t} \in B | \tau_A > s+t) = \liminf_{t \rightarrow \infty} \mathbb{P}_{s,x}(X_{s+t} \in B | \tau_A > s+t)$$

for a given $x \in E_0$. Then F_{sup} and F_{inf} do not depend on s either (by Proposition 2), hence for any $s \geq 0$,

$$F_{sup}(s) = \lim_{u \rightarrow \infty} F_{sup}(u),$$

and

$$F_{inf}(s) = \lim_{u \rightarrow \infty} F_{inf}(u).$$

Moreover, by the uniform convergence (25) of Proposition 3,

$$\lim_{u \rightarrow \infty} F_{sup}(u) = \lim_{u \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P}_{u,x}(X_{u+t} \in B | \tau_A > u+t)$$

$$\begin{aligned} &= \limsup_{t \rightarrow \infty} \mathbb{P}_x(X_t \in B | \tau_{A_\infty} > t) \\ &= \alpha_\infty(B). \end{aligned}$$

Similarly,

$$\lim_{u \rightarrow \infty} F_{inf}(u) = \alpha_\infty(B).$$

Hence, for any $s \geq 0$ and $\mu \in \mathcal{M}_1(E_s)$,

$$\limsup_{t \rightarrow \infty} \mathbb{P}_{s,\mu}(X_t \in B | \tau_A > t) = \liminf_{t \rightarrow \infty} \mathbb{P}_{s,\mu}(X_t \in B | \tau_A > t) = \alpha_\infty(B). \quad \square$$

Remark 6. It can be interesting to compare this result and this proof with the one of [1, Theorem 3.11] obtained by Bansaye et al. In particular, they used a different property of asymptotic homogeneity, which is uniform-in-state in their case.

4.2.2. Quasi-ergodic distribution

Now we can state the existence and uniqueness of the quasi-ergodic distribution:

Theorem 9. Under the assumptions of Theorem 8, for any $\mu \in \mathcal{M}_1(E_0)$,

$$\frac{1}{t} \int_0^t \mathbb{P}_x(X_s \in \cdot | \tau_A > t) ds \xrightarrow[t \rightarrow \infty]{(d)} \beta_\infty,$$

where β_∞ is the unique invariant measure of the Q -process of $(X_t)_{t \geq 0}$ absorbed by A_∞ .

Proof of Theorem 9. We will show that the Q -process converges weakly towards a probability measure. Fix $B \in \mathcal{E}$. Since we have the following inequality shown in Theorem 3.3 of [9]

$$\|\mathbb{Q}_{s,\mu_1}(X_t \in \cdot) - \mathbb{Q}_{s,\mu_2}(X_t \in \cdot)\|_{TV} \leq 2(1 - c_1 c_2)^{\lfloor \frac{t-s}{T_0} \rfloor},$$

for any $\mu_1, \mu_2 \in \mathcal{M}_1(E_s)$. We get therefore that

$$\limsup_{t \rightarrow \infty} \mathbb{Q}_{s,\mu_1}(X_t \in B) = \limsup_{t \rightarrow \infty} \mathbb{Q}_{s,\mu_2}(X_t \in B),$$

and we can therefore use the reasoning of the proof of Proposition 2 to show that, for any $s, u \in I$, for any $\mu, \nu \in \mathcal{M}_1(E_s) \times \mathcal{M}_1(E_{s+u})$,

$$\limsup_{t \rightarrow \infty} \mathbb{Q}_{s,\mu}(X_t \in B) = \limsup_{t \rightarrow \infty} \mathbb{Q}_{s+u,\nu}(X_t \in B).$$

In particular, for any $s \geq 0$, $\mu \in \mathcal{M}_1(E_s)$ and $x \in E_0$,

$$\limsup_{t \rightarrow \infty} \mathbb{Q}_{s,\mu}(X_t \in B) = \lim_{u \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{Q}_{u,x}(X_t \in B).$$

By the uniform convergence (25) of Proposition 3, for any $s \geq 0$, $\mu \in \mathcal{M}_1(E_s)$ and $x \in E_0$,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \mathbb{Q}_{s,\mu}(X_t \in B) &= \lim_{u \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{Q}_{u,x}(X_{u+t} \in B) \\ &= \lim_{u \rightarrow \infty} \limsup_{t \rightarrow \infty} \lim_{T \rightarrow \infty} \mathbb{P}_{u,x}(X_{u+t} \in B | \tau_A > u + T) \\ &= \limsup_{t \rightarrow \infty} \lim_{T \rightarrow \infty} \mathbb{P}_x(X_t \in B | \tau_{A_\infty} > T) \\ &= \limsup_{t \rightarrow \infty} \mathbb{Q}_x^\infty(X_t \in B), \end{aligned}$$

where, for any $x \in E_\infty$,

$$\mathbb{Q}_x^\infty(X_t \in B) = \lim_{T \rightarrow \infty} \mathbb{P}_x(X_t \in B | \tau_{A_\infty} > T)$$

is well-defined by [6, Theorem 3.1] under Assumption (H_∞) . This theorem states moreover that $(X_t)_{t \in I}$ admits under $(\mathbb{Q}_x^\infty)_{x \in E_\infty}$ a unique invariant measure β_∞ and for any $x \in E_\infty$,

$$\lim_{t \rightarrow \infty} \mathbb{Q}_x^\infty(X_t \in \cdot) = \beta_\infty.$$

Thus, for any $B \in \mathcal{E}$, $s \geq 0$ and $x \in E_s$,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \mathbb{Q}_{s,x}(X_t \in B) &= \beta_\infty(B) \\ &= \liminf_{t \rightarrow \infty} \mathbb{Q}_{s,x}(X_t \in B). \end{aligned}$$

Finally, thanks to the convergence in law of the Q -process we just prove, we can deduce the weak ergodic theorem using Cesaro's rule

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{Q}_{0,x}(X_s \in \cdot) ds = \beta_\infty.$$

Hence the condition (16) holds. As a result we can apply the second part of Theorem 5 and conclude the proof. \square

5. Example: Diffusion on \mathbb{R}

Let $(X_t)_{t \geq 0}$ be a diffusion on \mathbb{R} satisfying the following stochastic differential equation

$$dX_t = dW_t - V(X_t)dt, \quad (28)$$

where $(W_t)_{t \in \mathbb{R}_+}$ is Brownian motion on \mathbb{R} and $V \in \mathcal{C}^1(\mathbb{R})$. We assume that, under \mathbb{P}_x , there exists a strongly unique non explosive solution of (28) such that $X_0 = x$ almost surely.

Let h be a positive bounded \mathcal{C}^1 -function. We define τ_h the random time defined by

$$\tau_h = \inf\{t \geq 0 : X_t \leq h(t)\}.$$

Let us also recall the definition of the semi-flow $(\phi_{t,s})_{s \leq t}$ when the absorbing boundary is h :

$$\phi_{t,s} : \mu \mapsto \mathbb{P}_{s,\mu}(X_t \in \cdot | \tau_h > t), \quad \forall s \leq t.$$

5.1. Preliminaries on one-dimensional diffusion processes coming down from infinity

We assume that $(X_t)_{t \in \mathbb{R}_+}$ comes down from infinity (in the sense given in [5]), that is, there exists $y > h_{\max} := \sup_{s \geq 0} h(s)$ and $t > 0$ such that

$$\lim_{x \rightarrow \infty} \mathbb{P}_x(\tau_y < t) > 0, \quad (29)$$

where, for any $z \in \mathbb{R}$,

$$\tau_z := \inf\{t \geq 0 : X_t = z\}.$$

In this case, as remarked in the subsection 4.5.2. of [7], $(X_t)_{t \geq 0}$ satisfies then

$$\int_0^1 \frac{1}{x} \left(\sup_{y \in (z, \Lambda_z^{-1}(x)]} \frac{1}{\Lambda_z(y)} \int_z^y \Lambda_z(\xi)^2 m(d\xi) \right) dx < \infty,$$

where, for any $z \geq 0$, Λ_z is the scale function of X satisfying $\Lambda_z(z) = 0$ and defined by

$$\Lambda_z(x) = \int_z^x e^{2 \int_0^y V(\xi) d\xi} dy, \quad \forall x \geq z \quad (30)$$

and m is the speed measure of $(X_t)_{t \geq 0}$ defined by

$$m(d\xi) = 2e^{-2 \int_0^\xi V(\xi') d\xi'} d\xi.$$

In particular, for any $z \geq 0$, the process $Y^z := (\Lambda_z(X_t))_{t \geq 0}$ is a local martingale and, since X is solution of (28), by Itô's formula, for any $t \geq 0$,

$$Y_t^z = Y_0^z + \int_0^t \Lambda'_z(\Lambda_z^{-1}(Y_s^z)) dW_s.$$

Note that $\Lambda'_z = \Lambda'_0 = e^{2 \int_0^\cdot V(\xi) d\xi}$ for any z . So denoting for any $x, z \geq 0$ $\sigma_z(x) := \Lambda'_z(\Lambda_z^{-1}(x)) = \Lambda'_0(\Lambda_z^{-1}(x))$, one has

$$dY_t^z = \sigma_z(Y_t^z) dW_t.$$

Adapting [7, Theorem 4.6] for general diffusion processes, we deduce that for any $t > 0$, there exists $A_t^z < \infty$ such that

$$\mathbb{P}_x(t < \tau_z) \leq A_t^z \Lambda_z(x), \quad \forall x \geq z.$$

So let $u_1 \geq 0$ be arbitrarily chosen. One has for any $z \geq 0$,

$$\mathbb{P}_x(u_1 < \tau_z) \leq A_{u_1}^z \Lambda_z(x), \quad \forall x \geq z \quad (31)$$

or, equivalently,

$$\mathbb{P}_{\Lambda_z^{-1}(x)}(u_1 < \tau_z) \leq A_{u_1}^z x, \quad \forall x \geq 0.$$

Denoting for any $r \geq 0$ and for any process $(R_t)_{t \geq 0}$ $\tau_r(R) := \inf\{t \geq 0 : R_t = r\}$, one has for any $z \geq 0$ and $x \geq r$,

$$\mathbb{P}_{\Lambda_z^{-1}(x)}(u_1 < \tau_{\Lambda_z^{-1}(r)}) = \mathbb{P}(\tau_r(Y^z) > u_1 | Y_0^z = x).$$

Since $z \rightarrow \Lambda_z^{-1}(x)$ is increasing for any $x > 0$, then, for any $x > 0$ and for any $z \geq z'$,

$$\sigma_z(x) \geq \sigma_{z'}(x). \quad (32)$$

Thus, using the same reasoning as in the proof of Lemma 4.2. in the paper [9], it is possible to show that (32) implies that, for any $z \geq z'$ and $x \geq r$

$$\mathbb{P}(\tau_r(Y^{z'}) > u_1 | Y_0^{z'} = x) \geq \mathbb{P}(\tau_r(Y^z) > u_1 | \tilde{Y}_0^z = x)$$

or, equivalently,

$$\mathbb{P}_{\Lambda_z^{-1}(x)}(u_1 < \tau_{\Lambda_z^{-1}(r)}) \leq \mathbb{P}_{\Lambda_{z'}^{-1}(x)}(u_1 < \tau_{\Lambda_{z'}^{-1}(r)}). \quad (33)$$

Taking $z' = r = 0$, for any $x \geq 0$,

$$\mathbb{P}_{\Lambda_z^{-1}(x)}(u_1 < \tau_z) \leq \mathbb{P}_{\Lambda_0^{-1}(x)}(u_1 < \tau_0) \leq A_{u_1}^0 x, \quad \forall x \geq 0.$$

In conclusion, one has, for any $z \geq 0$,

$$\mathbb{P}_x(u_1 < \tau_z) \leq A_{u_1}^0 \Lambda_z(x), \quad \forall x \geq z. \quad (34)$$

One sets $A := A_{u_1}^0$. Let us now state and prove the following lemma.

Lemma 2. *There exists $u_0 \geq 0$, $\kappa > 0$ a family of probability measures $(\psi_z)_{z \in [0, h_{\max}]}$ such that, for any $z \in [0, h_{\max}]$,*

$$\mathbb{P}_x(X_u \in \cdot | \tau_z > u) \geq \kappa \psi_z, \quad \forall x > z, \forall u > u_0. \quad (35)$$

The difference between this lemma and [7, Theorem 4.1] is that the time u_0 and the constant κ do not depend on z . The sketch of the proof is inspired from the proof of the Theorem 4.1 presented in [7, Subsection 5.1].

Proof of Lemma 2. The following proof is divided into two steps.

Step 1.: Mimicking the Step 1 in the proof of [7, Theorem 4.1]

The aim of this first step is to prove that there exist $\epsilon, c > 0$ not depending on z such that

$$\mathbb{P}_x(\Lambda_z(X_{u_1}) \geq \epsilon | \tau_z > u_1) \geq c, \quad \forall x > z. \quad (36)$$

Since, for any $z \in [0, h_{\max}]$, $\Lambda_z(X)$ is a local martingale, one has for any $x \in (z, \Lambda_z^{-1}(1))$,

$$\begin{aligned} \Lambda_z(x) &= \mathbb{E}_x(\Lambda_z(X_{u_1 \wedge \tau_{\Lambda_z^{-1}(1)}})) \\ &= \mathbb{P}_x(\tau_z > u_1) \mathbb{E}_x(\Lambda_z(X_{u_1 \wedge \tau_{\Lambda_z^{-1}(1)}}) | \tau_z > u_1) + \mathbb{P}_x(\tau_{\Lambda_z^{-1}(1)} < \tau_z \leq u_1). \end{aligned}$$

By Markov property,

$$\begin{aligned} \mathbb{P}_x(\tau_{\Lambda_z^{-1}(1)} < \tau_z \leq u_1) &\leq \mathbb{E}_x(\mathbb{1}_{\tau_{\Lambda_z^{-1}(1)} < \tau_z \wedge u_1} \mathbb{P}_{\Lambda_z^{-1}(1)}(\tau_z \leq u_1)) \\ &\leq \mathbb{P}_x(\tau_{\Lambda_z^{-1}(1)} < \tau_z) \mathbb{P}_{\Lambda_z^{-1}(1)}(\tau_z \leq u_1) \\ &= \Lambda_z(x) \mathbb{P}_{\Lambda_z^{-1}(1)}(\tau_z \leq u_1), \end{aligned}$$

where the following identity is used

$$\mathbb{P}_x(\tau_a < \tau_b) = \frac{\Lambda_z(x) - \Lambda_z(b)}{\Lambda_z(a) - \Lambda_z(b)}, \quad \forall x \in [a, b].$$

As a result, using (34), one has, for any $x \in (z, \Lambda_z^{-1}(1))$,

$$\mathbb{E}_x(1 - \Lambda_z(X_{1 \wedge \tau_{\Lambda_z^{-1}(1)}}) | 1 < \tau_z) \leq 1 - \frac{1}{A'_z},$$

where $A'_z := A / \mathbb{P}_{\Lambda_z^{-1}(1)}(u_1 < \tau_z)$. But, since $z \in [0, h_{\max}]$, the inequality (33) applied to $r = 0$ implies that

$$\mathbb{P}_{\Lambda_z^{-1}(1)}(u_1 < \tau_z) \geq \mathbb{P}_{\Lambda_{h_{\max}}^{-1}(1)}(u_1 < \tau_{h_{\max}}).$$

So, defining $A' := A / \mathbb{P}_{\Lambda_{h_{\max}}^{-1}(1)}(u_1 < \tau_{h_{\max}})$, one has

$$\mathbb{E}_x(1 - \Lambda_z(X_{u_1 \wedge \tau_{\Lambda_z^{-1}(1)}}) | u_1 < \tau_z) \leq 1 - \frac{1}{A'}, \quad \forall x \in (z, \Lambda_1^{-1}(1)).$$

Thus, using Markov's inequality,

$$\mathbb{P}_x \left(\Lambda_z(X_{u_1 \wedge \tau_{\Lambda_z^{-1}(1)}}) \leq \frac{1}{2A' - 1} \middle| \tau_z > u_1 \right) \leq 1 - \frac{1}{2A'}.$$

Then, since $A' > 1$ by (34), $1/(2A' - 1) < 1$. Thus, for any $\epsilon \in (0, 1/(2A' - 1))$ and $x \in (z, \Lambda_z^{-1}(1/(2A' - 1)))$,

$$\begin{aligned} & \mathbb{P}_x(\Lambda_z(X_{u_1}) \geq \epsilon, \tau_z > u_1) \\ & \geq \mathbb{P}_x(\tau_{\Lambda_z^{-1}(1/(2A' - 1))} < u_1 \wedge \tau_z, \tau_{\Lambda_z^{-1}(\epsilon)} \circ \theta_{\tau_{\Lambda_z^{-1}(1/(2A' - 1))}} > u_1 + \tau_{\Lambda_z^{-1}(1/(2A' - 1))}) \\ & = \mathbb{P}_x(\tau_{\Lambda_z^{-1}(1/(2A' - 1))} < u_1 \wedge \tau_z) \mathbb{P}_{\Lambda_z^{-1}(1/(2A' - 1))}(\tau_{\Lambda_z^{-1}(\epsilon)} > u_1) \\ & \geq \mathbb{P}_x \left(\Lambda_z(X_{u_1 \wedge \tau_{\Lambda_z^{-1}(1)}} \wedge \tau_z) \geq 1/(2A' - 1) \right) \mathbb{P}_{\Lambda_z^{-1}(1/(2A' - 1))}(\tau_{\Lambda_z^{-1}(\epsilon)} > u_1) \\ & \geq \frac{\mathbb{P}_x(\tau_z > u_1)}{2A'} \mathbb{P}_{\Lambda_z^{-1}(1/(2A' - 1))}(\tau_{\Lambda_z^{-1}(\epsilon)} > u_1) \\ & \geq \frac{\mathbb{P}_x(\tau_z > u_1)}{2A'} \mathbb{P}_{\Lambda_{hmax}^{-1}(1/(2A' - 1))}(\tau_{\Lambda_{hmax}^{-1}(\epsilon)} > u_1), \end{aligned}$$

where (33) is used again. So, if ϵ is chosen such that $\mathbb{P}_{\Lambda_{hmax}^{-1}(1/(2A' - 1))}(\tau_{\Lambda_{hmax}^{-1}(\epsilon)} > u_1) > 0$ (it is possible since $\mathbb{P}_{\Lambda_{hmax}^{-1}(1/(2A' - 1))}(\tau_z > u_1) > 0$), then there exist $\epsilon \in (0, 1/(2A' - 1))$ and $c > 0$ (not depending on z) such that, for any $x \in (z, \Lambda_z^{-1}(1/(2A' - 1)))$,

$$\mathbb{P}_x(\Lambda_z(X_{u_1}) \geq \epsilon | \tau_z > u_1) \geq c.$$

For $x \geq \Lambda_z^{-1}(1/(2A' - 1))$,

$$\begin{aligned} \mathbb{P}_x(\Lambda_z(X_{u_1}) > \epsilon | \tau_z > u_1) & \geq \mathbb{P}_x(\Lambda_z(X_{u_1}) > \epsilon, \tau_z > u_1) \\ & \geq \mathbb{P}_x(\tau_{\Lambda_z^{-1}(\epsilon)} > u_1) \\ & \geq \mathbb{P}_{\Lambda_z^{-1}(1/(2A' - 1))}(\tau_{\Lambda_z^{-1}(\epsilon)} > u_1) \\ & \geq \mathbb{P}_{\Lambda_{hmax}^{-1}(1/(2A' - 1))}(\tau_{\Lambda_{hmax}^{-1}(\epsilon)} > u_1) > 0. \end{aligned}$$

Finally, there exist $\epsilon \in (0, 1/(2A' - 1))$ and $c > 0$ (not depending on z) such that, for any $x \geq z$,

$$\mathbb{P}_x(\Lambda_z(X_{u_1}) \geq \epsilon | \tau_z > u_1) \geq c.$$

Step 2. Mimicking the steps 2 and 3 in the proof of [7, Theorem 4.1].

Now, taking the exact same reasoning as the one presented in the second step of the proof of Theorem 4.1 [7, Subsection 5.1], one can prove that, for any $z \in [0, h_{max}]$, for all $x \geq \epsilon$,

$$\mathbb{P}_{\Lambda_z^{-1}(\epsilon)}(\Lambda_z(X_{u_{2,z}}) \in \cdot, \tau_z > u_{2,z}) \geq c_{1,z} \psi_z,$$

where

- $u_{2,z}$ can be any time satisfying $c'_{1,z} := \inf_{y > z} \mathbb{P}_y(\tau_z < u_{2,z}) > 0$,
- $c_{1,z} := c'_{1,z} \mathbb{P}_{\Lambda_z^{-1}(\epsilon)}(\tau_z > u_{2,z})$,
- and $\tilde{v}_z := \mathbb{P}_{\Lambda_z^{-1}(\epsilon)}(\Lambda_z(X_{u_{2,z}}) \in \cdot | \tau_z > u_{2,z})$.

In particular, for $z = 0$, one chooses $u_{2,0}$ such that

$$\inf_{y > 0} \mathbb{P}_y(\tau_0 < u_{2,0}) > 0. \quad (37)$$

Hence, for any $z \in [0, h_{\max}]$ and $x > z$,

$$\mathbb{P}_x(\tau_z < u_{2,0}) \geq \mathbb{P}_x(\tau_0 < u_{2,0}) \geq \inf_{y>0} \mathbb{P}_y(\tau_0 < u_{2,0}) = c'_{1,0}.$$

Hence, for any $z \in [0, h_{\max}]$,

$$c'_{1,z} = \inf_{x>z} \mathbb{P}_x(\tau_z < u_{2,0}) > c'_{1,0}.$$

In other words, we can set for any $z \in [0, h_{\max}]$

$$u_{2,z} = u_{2,0}.$$

Hence, one can define for any $z \in [0, h_{\max}]$,

$$c_{1,z} := c'_{1,z} \mathbb{P}_{\Lambda_z^{-1}(\epsilon)}(\tau_z > u_{2,0}), \quad \tilde{v}_z := \mathbb{P}_{\Lambda_z^{-1}(\epsilon)}(\Lambda_z(X_{u_{2,0}}) \in \cdot | \tau_z > u_{2,0}).$$

As a result, doing the same computation as those presented in Step 3 of the proof of Theorem 4.1 in [7], and defining $u_0 := u_1 + u_{2,0}$, for any $x > z$,

$$\mathbb{P}_x(\Lambda_z(X_{u_0}) \in \cdot | \tau_z > u_0) \geq c_{1,z} c \tilde{v}_z \geq c c'_{1,0} \mathbb{P}_{\Lambda_{h_{\max}}^{-1}(\epsilon)}(\tau_{h_{\max}} > u_{2,0}) \tilde{v}_z.$$

In conclusion, to get (almost) (35), one has to set $\kappa := c c'_{1,0} \mathbb{P}_{\Lambda_{h_{\max}}^{-1}(\epsilon)}(\tau_{h_{\max}} > u_{2,0})$ and $\psi_z := \mathbb{P}_{\Lambda_z^{-1}(\epsilon)}(X_{u_{2,0}} \in \cdot | \tau_z > u_{2,0})$ and one has

$$\mathbb{P}_x(X_{u_0} \in \cdot | \tau_z > u_0) \geq \kappa \psi_z, \quad \forall x > z. \quad (38)$$

To get (35) exactly, just note that Lemma 1 (seen in the proof of Theorem 5) can be applied to the conditional probability $\mathbb{P}_x(X_u \in \cdot | \tau_z > u)$, in such a way that the inequality (38) holds for any u greater than u_0 . \square

5.2. Periodic absorbing function

Before showing that the Assumption (A') is satisfied when h is periodic or converging, we will need to give some hypothesis on the function V as defined in (28). In both cases we will deal with, the absorbing function h will be Lipschitz, i.e.

$$L := \sup_{s \leq t} \frac{|h(t) - h(s)|}{|t - s|} < \infty.$$

Now we state the assumption we need on the function V

Assumption 1 (Hypothesis on V).

- V is such that the process X satisfying (28) comes down from infinity.
- V is positive and increasing on $[-Lu_0, \infty)$ (where u_0 is mentioned in Lemma 2).
- $\sup_{x \in \mathbb{R}} V'(x) - V^2(x) < \infty$.

Note that the functions $V : x \rightarrow (x - c)^\alpha$ with $\alpha > 1$ and $c > 0$ are suitable functions.

Now the following proposition is stated and proved:

Proposition 4. *Let $(X_t)_{t \geq 0}$ be a diffusion process following (28), such that Assumption 1 is satisfied. Assume moreover that h is a periodic function, with period $\gamma > 0$.*

Then Assumption (A') holds. In particular, there exists a probability measure β_γ such that, for any $x > h(0)$,

$$\frac{1}{t} \int_0^t \mathbb{P}_x(X_s \in \cdot | \tau_A > t) ds \xrightarrow[t \rightarrow \infty]{} \beta_\gamma.$$

Proof of Proposition 4. We will show that the two points in Assumption (A') are satisfied.

1. Denote by \mathcal{T}_{max} the set defined by

$$\mathcal{T}_{max} = \{t \geq 0 : h(t) = h_{max}\}.$$

where we recall that $h_{max} = \sup_{s \geq 0} h(s)$. The main part of this proof is to show that there exists $C_{max} > 0$ such that, for any $s \in \mathcal{T}_{max}$ and any $u \in [u_0, u_0 + \gamma]$

$$\mathbb{P}_{s,x}(X_{s+u} \in \cdot | \tau_h > s + u) \geq C_{max} \psi_{h_{max}}, \quad \forall x > h_{max}. \quad (39)$$

where u_0 and $\psi_{h_{max}}$ are defined in Lemma 2. Then we will generalize (39) to any $s \geq 0$ using Markov property.

First step: Proof of (39)

Let $s \in \mathcal{T}_{max}$. For any $x > h_{max}$, for any $t \geq 0$,

$$\mathbb{P}_{s,x}(X_{t+s} \in \cdot | \tau_h > s + t) \geq \frac{\mathbb{P}_x(\tau_{h_{max}} > t)}{\mathbb{P}_{s,x}(\tau_h > s + t)} \mathbb{P}_x(X_t \in \cdot | \tau_{h_{max}} > t).$$

Using the Champagnat–Villemonais type condition (35) for $z = h_{max}$, for any $u \geq u_0$,

$$\mathbb{P}_x(X_u \in \cdot | \tau_{h_{max}} > u) \geq \kappa \psi_{h_{max}}, \quad \forall x \in (h_{max}, \infty)$$

Then we obtain for any $u_0 \leq u \leq u_0 + \gamma$,

$$\begin{aligned} \mathbb{P}_{s,x}(X_{s+u} \in \cdot | \tau_h > s + u) &\geq \frac{\mathbb{P}_x(\tau_{h_{max}} > u)}{\mathbb{P}_{s,x}(\tau_h > s + u)} \kappa \psi_{h_{max}} \\ &\geq \frac{\mathbb{P}_x(\tau_{h_{max}} > u_0 + \gamma)}{\mathbb{P}_{s,x}(\tau_h > s + u_0)} \kappa \psi_{h_{max}}. \end{aligned}$$

Recalling that h is Lipschitz and that we defined $L = \sup_{s \leq t} \frac{|h(t) - h(s)|}{|t - s|}$, for any $x \in (h_{max}, \infty)$,

$$\frac{\mathbb{P}_x(\tau_{h_{max}} > u_0 + \gamma)}{\mathbb{P}_{s,x}(\tau_h > s + u_0)} \geq \frac{\mathbb{P}_x(\tau_{h_{max}} > u_0 + \gamma)}{\mathbb{P}_x(\tau_{u \rightarrow h_{max} - Lu} > u_0)},$$

where

$$\tau_{u \rightarrow h_{max} - Lu} := \inf\{t \geq 0 : X_t = h_{max} - Lt\}.$$

To show that

$$\inf_{x \in (h_{max}, \infty)} \frac{\mathbb{P}_x(\tau_{h_{max}} > u_0 + \gamma)}{\mathbb{P}_x(\tau_{u \rightarrow h_{max} - Lu} > u_0)} > 0$$

using a continuity argument, it is enough to show that

$$\liminf_{x \rightarrow h_{max}} \frac{\mathbb{P}_x(\tau_{h_{max}} > u_0 + \gamma)}{\mathbb{P}_x(\tau_{u \rightarrow h_{max} - Lu} > u_0)} > 0 \quad (40)$$

and

$$\liminf_{x \rightarrow \infty} \frac{\mathbb{P}_x(\tau_{h_{max}} > u_0 + \gamma)}{\mathbb{P}_x(\tau_{u \rightarrow h_{max} - Lu} > u_0)} > 0. \quad (41)$$

(41) is obvious since

$$\liminf_{x \rightarrow \infty} \frac{\mathbb{P}_x(\tau_{h_{\max}} > u_0 + \gamma)}{\mathbb{P}_x(\tau_{u \rightarrow h_{\max} - Lu} > u_0)} \geq \lim_{x \rightarrow \infty} \mathbb{P}_x(\tau_{h_{\max}} > u_0 + \gamma) > 0.$$

Thus let us focus on (40). Our strategy will be to reduce the study to the case of a Brownian motion. Denote by $(M_t)_{t \geq 0}$ the exponential local martingale defined by, for any t ,

$$\begin{aligned} M_t &:= \exp \left(- \int_0^t V(W_s) dW_s - \frac{1}{2} \int_0^t V^2(W_s) ds \right) \\ &= \exp \left(F(W_0) - F(W_t) + \frac{1}{2} \int_0^t (V'(W_s) - V^2(W_s)) ds \right), \end{aligned}$$

where F is a primitive of V that we choose as a positive function on $[-Lu_0, \infty)$ (it is possible since F is necessarily non-decreasing by the assumptions on V). Under \mathbb{P}_x for $x \in (h_{\max}, h_{\max} + 1]$, $W_0 = x$ almost surely. Moreover denote by $\tau_{h_{\max}}^W$ and $\tau_{u \rightarrow h_{\max} - Lu}^W$ the following random times:

$$\tau_{h_{\max}}^W := \inf\{t \geq 0 : W_t = h_{\max}\},$$

$$\tau_{u \rightarrow h_{\max} - Lu}^W := \inf\{t \geq 0 : W_t = h_{\max} - Lt\}.$$

Thus, since F is non-decreasing, the stopped local martingale $(M_{t \wedge u_0 \wedge \tau_{u \rightarrow h_{\max} - Lu}^W})_{t \geq 0}$ is almost surely bounded by $\exp(F(h_{\max} + 1) + \frac{u_0}{2} \sup_{y \in \mathbb{R}} V'(y) - V^2(y))$ and is therefore a martingale. Likewise, the stopped local martingale $(M_{t \wedge u_0 + \gamma \wedge \tau_{h_{\max}}^W})_{t \geq 0}$ is also a martingale. By Girsanov's theorem,

$$\begin{aligned} \frac{\mathbb{P}_x(\tau_{h_{\max}} > u_0 + \gamma)}{\mathbb{P}_x(\tau_{u \rightarrow h_{\max} - Lu} > u_0)} &= \frac{\mathbb{E}_x \left(\mathbb{1}_{\tau_{h_{\max}}^W > u_0 + \gamma} M_{u_0 + \gamma \wedge \tau_{h_{\max}}^W} \right)}{\mathbb{E}_x \left(\mathbb{1}_{\tau_{u \rightarrow h_{\max} - Lu}^W > u_0} M_{u_0 \wedge \tau_{u \rightarrow h_{\max} - Lu}^W} \right)} \\ &= \frac{\mathbb{E}_x \left(\mathbb{1}_{\tau_{h_{\max}}^W > u_0 + \gamma} M_{u_0 + \gamma} \right)}{\mathbb{E}_x \left(\mathbb{1}_{\tau_{u \rightarrow h_{\max} - Lu}^W > u_0} M_{u_0} \right)}. \end{aligned}$$

For any $x \in (h_{\max}, h_{\max} + 1]$,

$$\mathbb{E}_x(\mathbb{1}_{\tau_{h_{\max}}^W > u_0 + \gamma} M_{u_0 + \gamma}) \geq \mathbb{E}_x \left(\mathbb{1}_{\tau_{h_{\max}}^W > u_0 + \gamma} M_{u_0 + \gamma} \mathbb{1}_{\sup_{s \in [0, u_0 + \gamma]} W_s \leq h_{\max} + 2} \right).$$

On the event $\{\sup_{s \in [0, u_0 + \gamma]} W_s \leq h_{\max} + 2\}$,

$$M_{u_0 + \gamma} \geq \exp \left(-F(h_{\max} + 2) + \frac{u_0 + \gamma}{2} \inf_{s \in [h_{\max}, h_{\max} + 2]} (V'(s) - V^2(s)) \right) =: \tilde{M}_{u_0 + \gamma}.$$

As a result,

$$\begin{aligned} \mathbb{E}_x(\mathbb{1}_{\tau_{h_{\max}}^W > u_0 + \gamma} M_{u_0 + \gamma}) &\geq \tilde{M}_{u_0 + \gamma} \mathbb{E}_x \left(\mathbb{1}_{\tau_{h_{\max}}^W > u_0 + \gamma} \mathbb{1}_{\sup_{s \in [0, u_0 + \gamma]} W_s \leq h_{\max} + 2} \right) \\ &\geq \tilde{M}_{u_0 + \gamma} \mathbb{P}_x(\tau_{h_{\max}}^W > u_0 + \gamma) \inf_{y \in (h_{\max}, h_{\max} + 1]} \mathbb{P}_y \\ &\quad \times \left(\sup_{s \in [0, u_0 + \gamma]} W_s \leq h_{\max} + 2 \mid \tau_{h_{\max}}^W > u_0 + \gamma \right). \end{aligned}$$

Noting that

$$\lim_{y \rightarrow h_{\max}} \mathbb{P}_y \left(\sup_{s \in [0, u_0 + \gamma]} W_s \leq h_{\max} + 2 \mid \tau_{h_{\max}}^W > u_0 + \gamma \right) = \mathbb{P} \left(\sup_{s \in [0, u_0 + \gamma]} W_s^+ \leq 2 \right) > 0,$$

where $(W_t^+)_{t \geq 0}$ is a Brownian meander (see [11], Theorem 2.1.), we deduce finally that there exists $c > 0$ such that for any $x \in (h_{\max}, h_{\max} + 1]$

$$\mathbb{E}_x(\mathbb{1}_{\tau_{h_{\max}}^W > u_0 + \gamma} M_{u_0 + \gamma}) \geq c \mathbb{P}_x(\tau_{h_{\max}}^W > u_0 + \gamma).$$

On the other side, as we said before, $(M_{t \wedge u_0 \wedge \tau_{u \rightarrow h_{\max} - Lu}^W})_{t \geq 0}$ is almost surely bounded by $\exp(F(h_{\max} + 1) + \frac{u_0}{2} \sup_{y \in \mathbb{R}} V'(y) - V^2(y))$. Hence there exists $d > 0$ such that, for any $x \in (h_{\max}, h_{\max} + 1]$,

$$\mathbb{E}_x(\mathbb{1}_{\tau_{u \rightarrow h_{\max} - Lu}^W > u_0} M_{u_0}) \leq d \mathbb{P}_x(\tau_{u \rightarrow h_{\max} - Lu}^W > u_0).$$

As a result, for any $(h_{\max}, h_{\max} + 1]$,

$$\frac{\mathbb{P}_x(\tau_{h_{\max}}^W > u_0 + \gamma)}{\mathbb{P}_x(\tau_{u \rightarrow h_{\max} - Lu}^W > u_0)} \geq \frac{c}{d} \frac{\mathbb{P}_x(\tau_{h_{\max}}^W > u_0 + \gamma)}{\mathbb{P}_x(\tau_{u \rightarrow h_{\max} - Lu}^W > u_0)}.$$

For any $x > h_{\max}$, denote by $p_{h_{\max}}^W(x, \cdot)$ and $p_{u \rightarrow h_{\max} - Lu}^W(x, \cdot)$ the density functions of $\tau_{h_{\max}}^W$ and $\tau_{u \rightarrow h_{\max} - Lu}^W$ which are known to be equal to

$$p_{h_{\max}}^W(x, t) = \frac{x - h_{\max}}{\sqrt{2\pi t^3}} \exp\left(-\frac{(x - h_{\max})^2}{2t}\right)$$

and

$$p_{u \rightarrow h_{\max} - Lu}^W(x, t) = \frac{x - h_{\max}}{\sqrt{2\pi t^3}} \exp\left(-\frac{1}{2t}(x - h_{\max} + Lt)^2\right).$$

Then, for any $x \in (h_{\max}, h_{\max} + 1]$,

$$\frac{\mathbb{P}_x(\tau_{h_{\max}}^W > u_0 + \gamma)}{\mathbb{P}_x(\tau_{u \rightarrow h_{\max} - Lu}^W > u_0)} = \frac{\int_{u_0 + \gamma}^{\infty} p_{h_{\max}}^W(x, t) dt}{\int_{u_0}^{\infty} p_{u \rightarrow h_{\max} - Lu}^W(x, t) dt}.$$

By l'Hôpital's rule,

$$\begin{aligned} \lim_{x \rightarrow h_{\max}} \frac{\mathbb{P}_x(\tau_{h_{\max}}^W > u_0 + \gamma)}{\mathbb{P}_x(\tau_{u \rightarrow h_{\max} - Lu}^W > u_0)} &= \lim_{x \rightarrow h_{\max}} \frac{\int_{u_0 + \gamma}^{\infty} \partial_x p_{h_{\max}}^W(x, t) dt}{\int_{u_0}^{\infty} \partial_x p_{u \rightarrow h_{\max} - Lu}^W(x, t) dt} \\ &= \frac{\int_{u_0 + \gamma}^{\infty} \partial_x p_{h_{\max}}^W(h_{\max}, t) dt}{\int_{u_0}^{\infty} \partial_x p_{u \rightarrow h_{\max} - Lu}^W(h_{\max}, t) dt} > 0. \end{aligned}$$

As a result,

$$\liminf_{x \rightarrow h_{\max}} \frac{\mathbb{P}_x(\tau_{h_{\max}}^W > u_0 + \gamma)}{\mathbb{P}_x(\tau_{u \rightarrow h_{\max} - Lu}^W > u_0)} \geq \frac{c}{d} \lim_{x \rightarrow h_{\max}} \frac{\mathbb{P}_x(\tau_{h_{\max}}^W > u_0 + \gamma)}{\mathbb{P}_x(\tau_{u \rightarrow h_{\max} - Lu}^W > u_0)} > 0.$$

In conclusion, $\inf_{x \in (h_{\max}, \infty)} \frac{\mathbb{P}_x(\tau_{h_{\max}}^W > u_0 + \gamma)}{\mathbb{P}_x(\tau_{u \rightarrow h_{\max} - Lu}^W > u_0)} > 0$ and (39) holds with $C_{\max} = \kappa \times \inf_{x \in (h_{\max}, \infty)} \frac{\mathbb{P}_x(\tau_{h_{\max}}^W > u_0 + \gamma)}{\mathbb{P}_x(\tau_{u \rightarrow h_{\max} - Lu}^W > u_0)}$.

Second step: Generalization and conclusion

Now let $s \geq 0$. Then there exists $s' \geq 0$ such that $s + s' \in \mathcal{T}_{max}$. As a result we can construct a function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ as follows

$$g(s) = \inf\{s' \geq 0 : s + s' \in \mathcal{T}_{max}\}. \quad (42)$$

In particular, $g(s) = 0$ if $s \in \mathcal{T}_{max}$. Since h is a continuous function, $s + g(s) \in \mathcal{T}_{max}$ for any $s \geq 0$. Moreover, since h is γ -periodic, then for any $s \geq 0$, $g(s) \leq \gamma$.

Thus, by the semi-flow property of $(\phi_{t,s})_{s \leq t}$, one has for any $x \in E_s$,

$$\begin{aligned} \mathbb{P}_{s,x}(X_{s+u_0+\gamma} \in \cdot | \tau_h > s + u_0 + \gamma) &= \phi_{s+u_0+\gamma,s}(\delta_x) \\ &= \phi_{s+u_0+\gamma,s+g(s)} \circ \phi_{s+g(s),s}(\delta_x) \\ &= \mathbb{P}_{s+g(s),\phi_{s+g(s),s}(\delta_x)}(X_{s+u_0+\gamma} \in \cdot | \tau_h > s + u_0 + \gamma). \end{aligned}$$

Now by (39), for any $x > h(s)$,

$$\mathbb{P}_{s+g(s),\phi_{s+g(s),s}(\delta_x)}(X_{s+u_0+\gamma} \in \cdot | \tau_h > s + u_0 + \gamma) \geq C_{max} \psi_{h_{max}}.$$

since $u_0 + \gamma - g(s) \in [u_0, u_0 + \gamma]$. Hence, for any $s \geq 0$ and $x > h(s)$,

$$\mathbb{P}_{s,x}(X_{s+u_0+\gamma} \in \cdot | \tau_h > s + u_0 + \gamma) \geq C_{max} \psi_{h_{max}}.$$

As a result the first condition in Assumption (A') holds denoting for any $s \geq 0$,

$$\begin{aligned} \nu_s &= \psi_{h_{max}}, \\ t_0 &= \gamma + u_0, \\ c_1 &= C_{max}. \end{aligned} \quad (43)$$

- For the second condition of Assumption (A'), we will use some part of the proof of [7, Theorem 4.1]. First we recall [7, Lemma 5.1] :

Lemma 3 (Lemma 5.1., [7]). *There exists $a > h_{max}$ such that $\psi_{h_{max}}([a, \infty)) > 0$ and, for any $k \in \mathbb{N}$,*

$$\mathbb{P}_a(X_{ku_0 \wedge \tau_{h_{max}}} \geq a) \geq e^{-\rho k u_0},$$

with $\rho > 0$.

So let a be as in the previous lemma. It is shown in [7] that we can choose $b > a$ large enough such that

$$\sup_{x \geq b} \mathbb{E}_x(e^{\rho \tau_b}) < \infty. \quad (44)$$

Using Markov property, for any $s \geq 0$, $t \geq 0$, and for any $s_0 = k_0 \gamma$ with $k_0 \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}_{s,a}(\tau_h > s + t) &\geq \mathbb{P}_{s,a}(\tau_h > s + s_0 \wedge \tau_{h_{max}} + t) \\ &\geq \mathbb{P}_{s,a}(X_{s+s_0 \wedge \tau_{h_{max}}} \geq b, \tau_h > s + s_0 \wedge \tau_{h_{max}} + t) \\ &\geq \mathbb{P}_a(X_{s_0 \wedge \tau_{h_{max}}} \geq b) \mathbb{P}_{s+s_0,b}(s + s_0 + t < \tau_h) \\ &\geq \mathbb{P}_a(X_{s_0 \wedge \tau_{h_{max}}} \geq b) \mathbb{P}_{s,b}(s + t < \tau_h). \end{aligned}$$

Then, for $s_0 > 0$ fixed, $C := 1/\mathbb{P}_a(X_{s_0 \wedge \tau_{h_{max}}} \geq b) < \infty$, and for any $s \leq t$,

$$\mathbb{P}_{s,b}(t < \tau_h) \leq C \mathbb{P}_{s,a}(t < \tau_h).$$

Thanks to Markov property again, for any $u \leq t \in \mathbb{R}_+$

$$\mathbb{P}_a(X_{u \wedge \tau_{h_{\max}}} \geq a) \mathbb{P}_{s+u,a}(s+t < \tau_h) \leq \mathbb{P}_{s,a}(s+t < \tau_h). \quad (45)$$

According to Markov property, for any $u \in \mathbb{R}_+$,

$$\begin{aligned} \mathbb{P}_a(X_{u \wedge \tau_{h_{\max}}} \geq a) &\geq \mathbb{P}_a(X_{\lfloor \frac{u}{u_0} \rfloor u_0 \wedge \tau_{h_{\max}}} \geq a) \mathbb{P}_a(X_{(u - \lfloor \frac{u}{u_0} \rfloor u_0) \wedge \tau_{h_{\max}}} \geq a) \\ &\geq C' \mathbb{P}_a(X_{\lfloor \frac{u}{u_0} \rfloor u_0 \wedge \tau_{h_{\max}}} \geq a), \end{aligned} \quad (46)$$

where

$$C' := \inf_{v \in [0, u_0]} \mathbb{P}_a(X_{v \wedge \tau_{h_{\max}}} \geq a) > 0$$

since $v \rightarrow \mathbb{P}_a(X_{v \wedge \tau_{h_{\max}}} \geq a)$ is continuous and $\mathbb{P}_a(X_{v \wedge \tau_{h_{\max}}} \geq a) > 0$ for any $v \in [0, u_0]$. Gathering all these inequalities and using also Lemma 3, for any $x \geq b$,

$$\begin{aligned} \mathbb{P}_{s,x}(t+s < \tau_h) &\leq \mathbb{P}_x(\tau_b > t) + \int_0^t \mathbb{P}_{s+u,b}(t+s < \tau_h) \mathbb{P}_x(\tau_b \in du) \\ &\leq \sup_{x \geq b} \mathbb{E}_x(e^{\rho \tau_b}) e^{-\rho t} + C \int_0^t \mathbb{P}_{s+u,a}(t+s < \tau_h) \mathbb{P}_x(\tau_b \in du) \\ &\leq \sup_{x \geq b} \mathbb{E}_x(e^{\rho \tau_b}) e^{-\rho \lfloor t/u_0 \rfloor u_0} \\ &\quad + \frac{C}{C'} \mathbb{P}_{s,a}(s+t < \tau_h) \int_0^t \frac{1}{\mathbb{P}_a(X_{\lfloor u/u_0 \rfloor u_0 \wedge \tau_{h_{\max}}} \geq a)} \\ &\quad \times \mathbb{P}_x(\tau_b \in du) \\ &\leq \sup_{x \geq b} \mathbb{E}_x(e^{\rho \tau_b}) e^{\rho u_0} e^{-\rho \lfloor \frac{t}{u_0} \rfloor + 1} u_0 \\ &\quad + \frac{C}{C'} \mathbb{P}_{s,a}(t+s < \tau_h) \int_0^t e^{\rho u} \mathbb{P}_x(\tau_b \in du) \\ &\leq \sup_{x \geq b} \mathbb{E}_x(e^{\rho \tau_b}) e^{\rho u_0} \mathbb{P}_a(X_{(\lfloor t/u_0 \rfloor + 1)u_0 \wedge \tau_{h_{\max}}} \geq a) \\ &\quad + \frac{C}{C'} \mathbb{P}_{s,a}(t+s < \tau_h) \int_0^t e^{\rho u} \mathbb{P}_x(\tau_b \in du) \\ &\leq \sup_{x \geq b} \mathbb{E}_x(e^{\rho \tau_b}) e^{\rho u_0} \mathbb{P}_{s,a}(\tau_h > s+t) \\ &\quad + \frac{C}{C'} \mathbb{P}_{s,a}(t+s < \tau_h) \int_0^t e^{\rho u} \mathbb{P}_x(\tau_b \in du). \end{aligned} \quad (48)$$

We deduce from (44) that, for any $t \geq 0$,

$$\sup_{x \geq b} \mathbb{P}_{s,x}(t+s < \tau_h) \leq C'' \mathbb{P}_{s,a}(t+s < \tau_h),$$

where

$$C'' = \left(e^{\rho u_0} + \frac{C}{C'} \right) \sup_{x \geq b} \mathbb{E}_x(e^{\rho \tau_b}) < \infty.$$

Since $\psi_{h_{\max}}([a, \infty)) > 0$, we conclude the point 2. of Assumption (A') setting

$$c_2 = \frac{1}{C''}. \quad \square$$

5.3. When h is decreasing and converges at infinity

Let us now state the main proposition of this subsection:

Proposition 5. *Let $(X_t)_{t \geq 0}$ be a diffusion process following (28), such that Assumption 1 is satisfied. Assume moreover that h is a decreasing C^1 -function going to 0 as t goes to infinity. Then Assumption (A') holds.*

Since this is a diffusion process on \mathbb{R}_+ , $(X_t)_{t \geq 0}$ satisfies the strong Markov property and the two assumptions of continuity presented in Assumption (H_{hom}) . Moreover, since $t \rightarrow X_t$ is continuous almost surely and, for any $s \geq 0$, $\tau_{h(s)}$ is the hitting time of the closed set $[-1, h(s)]$, then $\tau_{h(s)} \xrightarrow{s \rightarrow \infty} \tau_0$ almost surely, which entails the convergence in law of the hitting times of Assumption (H_{hom}) . In other words, Assumption (H_{hom}) is satisfied for such a process and Proposition 5 entails therefore the existence of a probability measure α on $(0, +\infty)$ such that, for any $x > h(0)$,

$$\mathbb{P}_\mu(X_t \in \cdot | \tau_h > t) \xrightarrow[t \rightarrow \infty]{(d)} \alpha.$$

Proof of Proposition 5.

1. Adapting exactly the same reasoning as Proposition 4, we can show that for any $s \geq 0$ and any $x > h(s)$,

$$\mathbb{P}_{s,x}(X_{s+u_0} \in \cdot | \tau_h > s + u_0) \geq \tilde{d}_s \kappa_0 \psi_{h(s)},$$

where we recall that u_0, κ_0 and ψ_z are such that (35) holds, and where \tilde{d}_s is defined by

$$\tilde{d}_s = \frac{\mathbb{P}_x(\tau_{h(s)} > u_0)}{\mathbb{P}_x(\tau_{u \rightarrow h(s)-Lu} > u_0)}.$$

We have therefore to show that

$$\inf_{s \geq 0} \tilde{d}_s > 0.$$

For any $z \in [0, h(0)]$ define $(X_t^{(z)})_{t \geq 0}$ by the solution of

$$dX_t^{(z)} = dW_t - V(X_t^{(z)} + z)dt.$$

In particular, $X^{(0)} \stackrel{(d)}{=} X$. Likewise, for any $y \in \mathbb{R}$ and $z \in [0, h(0)]$, we denote $\tau_y^{(z)} := \inf\{t \geq 0 : X_t^{(z)} = y\}$ and $\tau_{u \rightarrow y-Lu}^{(z)} := \inf\{t \geq 0 : X_t^{(z)} = y - Lt\}$. Since V is positive and increasing on $[-Lu_0, \infty)$, then, using Theorem 1.1 in [[15], Chapter VI, p.437], we can show that for any $x > 0$ and $z \in [0, h(0)]$,

$$\mathbb{P}_x(\tau_0^{(z)} > u_0) \geq \mathbb{P}_x(\tau_0^{(h(0))} > u_0)$$

and that

$$\mathbb{P}_x(\tau_{u \rightarrow -Lu}^{(z)} > u_0) \leq \mathbb{P}_x(\tau_{u \rightarrow -Lu}^{(0)} > u_0).$$

Then, for any $x > 0$ and $s \geq 0$,

$$\begin{aligned} \mathbb{P}_{x+h(s)}(\tau_{h(s)} > u_0) &= \mathbb{P}_x(\tau_0^{(h(s))} > u_0) \\ &\geq \mathbb{P}_x(\tau_0^{(h(0))} > u_0) \\ &\geq \frac{\mathbb{P}_x(\tau_0^{(h(0))} > u_0)}{\mathbb{P}_x(\tau_{u \rightarrow -Lu}^{(0)} > u_0)} \mathbb{P}_{x+h(s)}(\tau_{u \rightarrow h(s)-Lu} > u_0). \end{aligned}$$

To conclude, it is enough to see that $\inf_{x>0} \frac{\mathbb{P}_x(\tau_0^{(h(0))} > u_0)}{\mathbb{P}_x(\tau_{u \rightarrow -Lu}^{(0)} > u_0)} > 0$ using the same techniques as the point 1 of [Proposition 4](#).

As a result the first hypothesis of Assumption (A') holds setting for any $s \geq 0$,

$$v_s = \begin{cases} \psi_{h(0)} & \text{if } s \leq u_0 \\ \psi_{h(s-u_0)} & \text{if } s > u_0 \end{cases}$$

$$t_0 = u_0,$$

$$c_1 = \kappa_0 \times \inf_{s \geq 0} \tilde{d}_s.$$

2. The reasoning is the same as the point 2. in the proof of [Proposition 4](#) and the technical computations could be hidden if they are already explicitly written for the periodic case. Noting that, for any $z \in [0, h(0)]$ and any $y \geq h(0)$, $\psi_z([y, \infty)) > 0$, then, by Lemma 3, there exists $a > h(0)$ such that, for any $z \in [0, h(0)]$, $\psi_z([a, \infty)) > 0$ and for any $k \in \mathbb{N}$

$$\mathbb{P}_a(X_{ku_0 \wedge \tau_{h(0)}} \geq a) \geq e^{-\rho k u_0},$$

where $\rho > 0$. We deduce that for any $s \geq 0$

$$\mathbb{P}_a(X_{ku_0 \wedge \tau_{h(s)}} \geq a) \geq e^{-\rho k u_0}.$$

As in the proof of [Proposition 4](#), we can choose $b > a$ large enough such that

$$\sup_{x \geq b} \mathbb{E}_x(e^{\rho \tau_b}) < \infty.$$

Since h is non-increasing, for any $s, t \geq 0$ and $s_0 \geq 0$,

$$\mathbb{P}_{s+s_0, b}(\tau_h > s + s_0 + t) \geq \mathbb{P}_{s, b}(\tau_h > s + t).$$

Hence, according to Markov property,

$$\begin{aligned} \mathbb{P}_{s, a}(s + t < \tau_h) &\geq \mathbb{P}_a(X_{s_0 \wedge \tau_{h(s)}} \geq b) \mathbb{P}_{s, b}(s + t < \tau_h) \\ &\geq \mathbb{P}_a(X_{s_0 \wedge \tau_{h(0)}} \geq b) \mathbb{P}_{s, b}(s + t < \tau_h). \end{aligned}$$

for any $t \geq 0$ and any $s_0 \geq 0$. Hence, for s_0 fixed, $C := \frac{1}{\mathbb{P}_a(X_{s_0 \wedge \tau_{h(0)}} \geq b)} < \infty$, and for any $s \leq t$,

$$\mathbb{P}_{s, b}(t < \tau_h) \leq C \mathbb{P}_{s, a}(t < \tau_h).$$

Likewise, one finds an analog of the inequality [\(45\)](#)

$$\mathbb{P}_a(X_{u \wedge \tau_{h(s)}} \geq a) \mathbb{P}_{s+u, a}(t + s < \tau_h) \leq \mathbb{P}_{s, a}(t < \tau_h),$$

and using the same reasoning as for the inequality [\(46\)](#),

$$\mathbb{P}_a(X_{u \wedge \tau_{h(s)}} \geq a) \geq C' \mathbb{P}_a(X_{\lfloor \frac{u}{u_0} \rfloor u_0 \wedge \tau_{h(s)}} \geq a),$$

with

$$C' := \inf_{v \in [0, u_0]} \mathbb{P}_a(X_{v \wedge \tau_{h(0)}} \geq a) > 0.$$

Hence, using these previous inequalities and doing again the array of computation [\(47\)–\(48\)](#), we deduce that, for any $s \leq t$,

$$\sup_{x \geq b} \mathbb{P}_{s, x}(t < \tau_h) \leq C'' \mathbb{P}_{s, a}(t < \tau_h),$$

where

$$C'' := \left(e^{\rho u_0} + \frac{C}{C'} \right) \sup_{x \geq b} \mathbb{E}_x(e^{\rho \tau_b}) < \infty.$$

Since $\psi_{h(s)}([a, \infty)) > 0$ for any $s \geq 0$, we conclude the proof of the point 2 setting

$$c_2 = \frac{1}{C''}. \quad \square$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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