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A critical branching process with immigration in random environment☆

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Abstract

A Galton–Watson branching process with immigration evolving in a random environment is considered. Its associated random walk is assumed to be oscillating. We prove a functional limit theorem in which the process under consideration is normalized by a random coefficient depending on the random environment only. The distribution of the limiting process is described in terms of a strictly stable Levy process and a sequence of independent and identically distributed random variables which is independent of this process.

Keywords: Branching process in random environment; Branching process with immigration; Functional limit theorem

1. Introduction and statement of main result

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and Δ be the space of probability measures on $\mathbf{N}_0 := \{0, 1, \ldots\}$ equipped with the metric of total variation. We consider random elements $Q_1, Q_2, \ldots$, mapping the space $(\Omega, \mathcal{F}, \mathbf{P})$ into Δ^2. It means that Q_n, for each $n \in \mathbf{N}$, has the form (F_n, G_n), where F_n, G_n are probability measures on $\mathbf{N}_0$. The sequence $\Pi = \{Q_n,\ n \in \mathbf{N}\}$ is called *a random environment.*

A sequence of non-negative integer random variables $\{Z_n,\ n \in \mathbf{N}_0\}$ is called *a branching process with immigration in random environment* (BPIRE) if $Z_0 = 0$ and

$$Z_{n+1} = \sum_{i=1}^{Z_n+\eta_n} \xi_i^{(n)}, \qquad n \in \mathbf{N}_0,$$

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where it is assumed that, conditioned on the random environment Π, the random variables $\left\{\xi_i^{(n)}, \eta_n : n \in \mathbf{N}_0,\ i \in \mathbf{N}\right\}$ are independent. Moreover, for any fixed $n \in \mathbf{N}$, the variables $\xi_1^{(n)}, \xi_2^{(n)}, \ldots$ are identically distributed with the distribution F_{n+1} and the variable η_n has the distribution G_{n+1}. In the language of branching processes Z_n is the size of the nth generation without the immigrants which joined this generation; η_n is the number of immigrants which joined the nth generation; $\xi_i^{(n)}$ is the offspring number of the ith particle from the set consisting of nth generation particles and immigrants who joined them.

In other words, the random process $\{Z_n,\ n \in \mathbf{N}_0\}$, conditioned on the random environment Π, is an inhomogeneous branching Galton–Watson process with immigration (see [9], Chapter 6, § 7) and, for each $n \in \mathbf{N}_0$, the number of immigrants joining the nth generation has the distribution G_{n+1} and the offspring reproduction law of particles of the nth generation (including joined immigrants) is F_{n+1}. Let $f_n(\cdot)$ and $g_n(\cdot)$ be the generating functions of the (random) distributions F_n and G_n respectively.

We consider this model under the assumption that the random elements $Q_1, Q_2, \ldots$ are independent and identically distributed.

Set for $i \in \mathbf{N}$

$$X_i = \ln f_i'(1), \quad \mu_i = g_i'(1)$$

(we assume that $0 < f_1'(1) < +\infty$, $0 < g_i'(1) < +\infty$ a.s.). Introduce the so-called *associated random walk*:

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, \quad n \in \mathbf{N}.$$

It is clear that the random vectors $(X_1, \mu_1), (X_2, \mu_2), \ldots$ are independent and identically distributed under our assumptions.

We impose the following restriction on the distribution of X_1.

Assumption A. The distribution of X_1 belongs without centering to the domain of attraction of a stable law with index $\alpha \in (0, 2]$ and the limit law is not a one-sided stable law.

Under Assumption A the Skorokhod functional limit theorem is valid (see, for instance, [16], Chapter 16): there are such positive normalizing constants C_n that, as $n \to \infty$,

$$W_n \xrightarrow{D} W, \tag{1}$$

where $W_n = \left\{C_n^{-1} S_{\lfloor nt \rfloor},\ t \geq 0\right\}$, the process $W = \{W(t),\ t \geq 0\}$ is a strictly stable Lévy process with index $\alpha \in (0, 2]$ and the symbol $\xrightarrow{D}$ means convergence in distribution (in this case, in the space $D[0, +\infty)$ with Skorokhod topology). Moreover,

$$C_n = n^{1/\alpha} l(n),$$

where $\{l(n),\ n \in \mathbf{N}\}$ is a slowly varying sequence. It is known that the finite-dimensional distributions of the process W are absolutely continuous. Note that $\rho := \mathbf{P}(W(1) > 0) \in (0, 1)$ under Assumption A. Thus, the Spitzer–Doney condition is satisfied:

$$\lim_{n\to\infty} \mathbf{P}(S_n > 0) = \rho \in (0, 1). \tag{2}$$

The Spitzer–Doney condition means that the random walk $\{S_n\}$ is oscillating.

The aim of this paper is to prove a functional limit theorem for the process $\left\{Z_{\lfloor nt \rfloor},\ t \geq 0\right\}$, as $n \to \infty$ (see Theorem 1).

We need some notation and definitions to formulate the theorem. Let for $n \in \mathbf{N}$

$$M_n = \max_{1 \le i \le n} S_i, \quad L_n = \min_{0 \le i \le n} S_i.$$

If the Spitzer–Doney condition (2) is satisfied, then, as $n \to \infty$,

$$\{(Q_i, S_i, \mu_i),\ i \in \mathbf{N} \mid L_n \ge 0\} \xrightarrow{D} \left\{\left(Q_i^+, S_i^+, \mu_i^+\right),\ i \in \mathbf{N}\right\}, \tag{3}$$

$$\{(Q_i, S_i, \mu_i),\ i \in \mathbf{N} \mid M_n < 0\} \xrightarrow{D} \left\{\left(Q_i^-, S_i^-, \mu_i^-\right),\ i \in \mathbf{N}\right\}, \tag{4}$$

where $\left\{\left(Q_i^+, S_i^+, \mu_i^+\right)\right\}$, $\left\{\left(Q_i^-, S_i^-, \mu_i^-\right)\right\}$ are some random sequences. Moreover: a) the sequences $\left\{Q_i^+,\ i \in \mathbf{N}\right\}$, $\left\{Q_i^-,\ i \in \mathbf{N}\right\}$ can be viewed as random environments; b) the sequences $\left\{S_i^+,\ i \in \mathbf{N}\right\}$, $\left\{S_i^-,\ i \in \mathbf{N}\right\}$ are the corresponding associated random walks ($S_0^+ = S_0^- = 0$); c) the sequences $\left\{\mu_i^+,\ i \in \mathbf{N}\right\}$ and $\left\{\mu_i^-,\ i \in \mathbf{N}\right\}$ are positive and constructed from $\left\{Q_i^+,\ i \in \mathbf{N}\right\}$ and $\left\{Q_i^-,\ i \in \mathbf{N}\right\}$, respectively, the same as the sequence $\{\mu_i,\ i \in \mathbf{N}\}$ is constructed from $\{Q_i,\ i \in \mathbf{N}\}$. Suppose that the sequences $\left\{Q_i^+,\ i \in \mathbf{N}\right\}$ and $\left\{Q_i^-,\ i \in \mathbf{N}\right\}$ are defined on the same probability space $(\Omega^*, \mathcal{F}^*, \mathbf{P}^*)$ and are independent (below we denote by $\mathbf{E}^*$ the expectation on this probability space).

Relations (3) and (4) follow from Lemma 2.5 of [8] (it is necessary to fix $i \in \mathbf{N}$ and to apply an arbitrary bounded and continuous function $\varphi : \mathbf{R}^{3i} \to \mathbf{R}$ to the random element $((Q_1, S_1, \mu_1), \ldots, (Q_i, S_i, \mu_i))$, and then to apply the mentioned lemma to the obtained random variable). We note also that the sequences $\left\{S_i^+,\ i \in \mathbf{N}\right\}$ and $\left\{S_i^-,\ i \in \mathbf{N}\right\}$ are Markov chains (see Section 2 of [8] and Section 3 of [14] where these facts and the distribution of $\left\{Q_i^+,\ i \in \mathbf{N}\right\}$, conditioned on the sequence $\left\{S_i^+,\ i \in \mathbf{N}\right\}$, are discussed).

We now come back to our initial BPIRE. Set $\mathbf{N}_i = \{i, i+1, \ldots\}$ for $i \in \mathbf{Z}$. Fix $i \in \mathbf{N}_0$ and, for $n \in \mathbf{N}_i$, denote by $Z_{i,n}$ the total number of particles in the nth generation which are the descendants of the immigrants joined the ith generation (we assume that $Z_{i,i} = \eta_i$ and $Z_{i,n} = 0$ for $i > n$ and $i < 0$). Note that the random sequence $\left\{Z_{i,n},\ n \in \mathbf{N}_i\right\}$ is a standard (without immigration) branching process in the random environment $\{G_{i+1};\ F_n,\ n \in \mathbf{N}_{i+1}\}$ and, if this random environment is fixed, G_{i+1} is the distribution of the number of particles in the initial generation. Set for $n \in \mathbf{N}_i$

$$a_{i,n} = e^{-(S_n - S_i)}.$$

If the random environment $\{G_{i+1};\ F_n,\ n \in \mathbf{N}_{i+1}\}$ is fixed, the random sequence $\left\{a_{i,n} Z_{i,n},\ n \in \mathbf{N}_i\right\}$ is a non-negative martingale with respect to the natural filtration of the sequence $\left\{Z_{i,n},\ n \in \mathbf{N}_i\right\}$. Hence (without assuming that the random environment is fixed), there is a finite limit $\lim_{n\to\infty} a_{i,n} Z_{i,n}$ $\mathbf{P}$-a.s.

Set

$$Q_i^* = \begin{cases} Q_i^+, & i \in \mathbf{N}, \\ Q_{-i+1}^-, & i \in \mathbf{Z} \setminus \mathbf{N}, \end{cases} \tag{5}$$

$$S_i^* = \begin{cases} S_i^+, & i \in \mathbf{N}, \\ -S_{-i}^-, & i \in \mathbf{Z} \setminus \mathbf{N}, \end{cases} \tag{6}$$

$$\mu_i^* = \begin{cases} \mu_i^+, & i \in \mathbf{N}, \\ \mu_{-i+1}^-, & i \in \mathbf{Z} \setminus \mathbf{N}. \end{cases} \tag{7}$$

The sequence $\Pi^* := \left\{Q_k^*,\ k \in \mathbf{Z}\right\}$ can be considered as a random environment (we denote the components of Q_k^* by G_k^* and F_k^*). We assume that the probability space $(\Omega^*, \mathcal{F}^*, \mathbf{P}^*)$ is rich enough to accommodate a branching process with immigration in the random environment Π^*.

Fix $i \in \mathbf{Z}$ and, for $j \in \mathbf{N}_i$, denote by $Z^*_{i,j}$ the total number of particles in the jth generation being descendants of immigrants which joined the ith generation (we denote the number of such immigrants by η^*_i and assume that $Z^*_{i,i} = \eta^*_i$). Note that the random sequence $\left\{Z^*_{i,j},\ j \in \mathbf{N}_i\right\}$ is a branching process in the random environment $\left\{G^*_{i+1};\ F^*_j,\ j \in \mathbf{N}_{i+1}\right\}$. The sequence $\left\{S^*_j - S^*_i,\ j \in \mathbf{N}_i\right\}$ is the associated random walk and, given the environment, the random variable μ^*_i is the mean of the random variable η^*_i. Set

$$a^*_{i,j} = e^{-\left(S^*_j - S^*_i\right)}.$$

In accordance with the above the limit

$$\lim_{j\to\infty} a^*_{i,j} Z^*_{i,j} =: \zeta^*_i \tag{8}$$

exists $\mathbf{P}^*$-a.s. Furthermore, $\mathbf{P}^*\left(\zeta^*_i > 0\right) > 0$ for $i \in \mathbf{N}_0$ (see [8], Proposition 3.1).

Introduce the following random series:

$$\Sigma_1 := \sum_{i\in\mathbf{Z}} \mu^*_{i+1} e^{-S^*_i}, \quad \Sigma_2 := \sum_{i\in\mathbf{Z}} \zeta^*_i e^{-S^*_i} \tag{9}$$

It is clear that $\Sigma_1 > 0$ $\mathbf{P}^*$-a.s. and $\mathbf{P}^*(\Sigma_2 > 0) > 0$. Both series converge $\mathbf{P}^*$-a.s. under certain restrictions (see Lemma 4).

Let W be a strictly stable Lévy process with index α (in the sequel we call W simply *the Lévy process*). We define *the (lower) level* $L = \{L(t),\ t \geq 0\}$ of the Lévy process as

$$L(t) = \inf_{s\in[0,t]} W(s).$$

Let, further, $\gamma_1, \gamma_2, \ldots$ be an independent of W sequence of independent random variables distributed as the random variable Σ_2/Σ_1.

With the help of these ingredients we define finite-dimensional distributions of a random process $Y = \{Y(t),\ t \geq 0\}$ which plays an important role in the sequel. First we set $Y(0) = 0$. Consider an arbitrary $m \in \mathbf{N}$ and arbitrary times $t_1, t_2, \ldots, t_m$: $0 = t_0 < t_1 < t_2 < \cdots < t_m$. The random vector $(Y(t_1), \ldots, Y(t_m))$ coincides in distribution with the following vector $\widehat{Y} := \left(\widehat{Y}_1, \ldots, \widehat{Y}_m\right)$. We first describe the possible values of the vector $\widehat{Y}$. Its first several coordinates coincide with γ_1, the next several coordinates coincide with γ_2 and so on up to the mth coordinate. The coordinates of the vector $\widehat{Y}$ are specified according to the level L of the Lévy process W. The first coordinate $\widehat{Y}_1$ is equal to γ_1. Let the coordinate $\widehat{Y}_k$ for some $k < m$ be known. For instance, $\widehat{Y}_k = \gamma_l$ for some $l \in \mathbf{N}$. If the level of the Lévy process at the time t_{k+1} remains the same as at the time t_k, i.e. $L(t_{k+1}) = L(t_k)$, then $\widehat{Y}_{k+1} = \gamma_l$. If the level of the Lévy process at the time t_{k+1} is changed, i.e. $L(t_{k+1}) < L(t_k)$, then $\widehat{Y}_{k+1} = \gamma_{l+1}$.

Set for $n \in \mathbf{N}_0$

$$a_n = e^{-S_n}, \quad b_n = \sum_{i=0}^{n-1} \mu_{i+1} e^{-S_i}\ (b_0 = 0).$$

Introduce for each $n \in \mathbf{N}$ the random process $Y_n = \{Y_n(t),\ t \geq 0\}$, where

$$Y_n(0) = 0, \quad Y_n(t) = \frac{a_{\lfloor nt\rfloor}}{b_{\lfloor nt\rfloor}} Z_{\lfloor nt\rfloor}, \quad t > 0.$$

Note that for $k \in \mathbf{N}$ the ratio b_k/a_k is equal to the mean of Z_k, conditioned on the random environment Π.

Let the symbol $\Rightarrow$ mean convergence of random processes in the sense of finite-dimensional distributions and $\ln^+ x = \max(0, \ln x)$ for $x > 0$.

Theorem 1. *If Assumption A is valid and* $\mathbf{E}\left(\ln^+ \mu_1\right)^{\alpha+\varepsilon} < +\infty$ *for some* $\varepsilon > 0$, *then, as* $n \to \infty$,

$$Y_n \Rightarrow Y.$$

A detailed description of the theory of critical branching processes in random environment is available in [8,17].

A particular case of a subcritical BPIRE (when the offspring generating function $f_n(\cdot)$ is fractional-linear and $g_n(s) \equiv s$ for each $n \in \mathbf{N}$) was considered in [18]. The main attention there was paid to obtaining an exponential estimate for the tail distribution of the so-called life period of this process (i.e., the time until the first extinction). A more general class of subcritical BPIRE was analyzed in [19] where a limit theorem describing the population size at a distant moment was proved and an exponential estimate for the tail distribution of the life period was established. A strong law of large numbers and a central limit theorem for a wide class of subcritical BPIRE were proved in [21].

A critical BPIRE was considered in [10] where sufficient conditions of transience and recurrence were obtained. The author of [3], studying a random walk in random environment, proved a particular case of Theorem 1 (when the offspring generating function $f_n(\cdot)$ is fractional-linear and $g_n(s) \equiv s$ for each $n \in \mathbf{N}$). We would like to stress that the proof used in the present paper differs significantly from the one given in [3]. An interesting observation is made in [3] (for the mentioned special case). It is shown there that the random variable γ_1 has the exponential distribution with parameter 1 and, therefore, in view of Theorem 1, the random variable Σ_2/Σ_1 has the same distribution. Along with [3] we also mention papers [4–6] in which critical and supercritical processes (with stopped immigration) are considered under some restrictions on their lifetimes.

Recent papers [13,20] contain exact asymptotic formulae for the tail distribution of the life period for critical and subcritical BPIRE.

It should be noted that limit processes similar in the spirit to our process Y arise already in the study of a standard (without immigration) intermediate subcritical branching process in a random environment (see [2] and [7]), as well as in the study (in the special case described above) of a supercritical BPIRE with stopped immigration (see [5]). The construction of such limit processes is based on the so-called *conditional Lévy processes* rather than on standard Lévy processes.

2. Auxiliary statements

Let τ_n be the first moment when the minimum of the random walk $S_0, \ldots, S_n$ is attained:

$$\tau_n = \min\{i : S_i = L_n,\ 0 \le i \le n\}.$$

Set for $n \in \mathbf{N}$

$$S'_{i,n} = \begin{cases} S_{\tau_n+i} - S_{\tau_n}, & i \in \mathbf{N}_{(-\tau_n)}, \\ 0, & i \in \mathbf{Z} \setminus \mathbf{N}_{(-\tau_n)}. \end{cases} \tag{10}$$

For positive integer numbers $n_1 < n_2$ set

$$L_{n_1,n_2} = \min_{n_1 \le i \le n_2} S_i.$$

Lemma 1. *If the Spitzer–Doney condition (2) is satisfied, then, as $n \to \infty$,*

$$\left\{S'_{i,n}, \quad i \in \mathbf{Z}\right\} \xrightarrow{D} \left\{S^*_i, \quad i \in \mathbf{Z}\right\}.$$

Proof. First, we show that for any fixed $i \in \mathbf{N}$, as $n \to \infty$,

$$\left(S'_{1,n}, \ldots, S'_{i,n}\right) \xrightarrow{D} \left(S^*_1, \ldots, S^*_i\right). \tag{11}$$

Let A be a $\left(S^*_1, \ldots, S^*_i\right)$-continuous (relative to the measure $\mathbf{P}^*$) Borel set from $\mathbf{R}^i$. Then for $n \in \mathbf{N}_i$

$$\begin{aligned}
&\mathbf{P}\left(\left(S'_{1,n}, \ldots, S'_{i,n}\right) \in A, \quad \tau_n + i \le n\right) \\
&= \sum_{k=0}^{n-i} \mathbf{P}\left(\left(S'_{1,n}, \ldots, S'_{i,n}\right) \in A, \quad \tau_n = k\right) \\
&= \sum_{k=0}^{n-i} \mathbf{P}\left((S_{k+1} - S_k, \ldots, S_{k+i} - S_k) \in A, \quad S_k < L_{k-1}, \quad S_k \le L_{k+1,n}\right)
\end{aligned}$$

and by the Markov property of random walks we have that

$$\begin{aligned}
&\mathbf{P}\left((S_{k+1} - S_k, \ldots, S_{k+i} - S_k) \in A, \quad S_k < L_{k-1}, \quad S_k \le L_{k+1,n}\right) \\
&= \mathbf{P}(S_k < L_{k-1})\,\mathbf{P}\left((S_{k+1} - S_k, \ldots, S_{k+i} - S_k) \in A, \quad S_k \le L_{k+1,n}\right) \\
&= \mathbf{P}(S_k < L_{k-1})\,\mathbf{P}\left((S_1, \ldots, S_i) \in A, \quad L_{n-k} \quad \ge 0\right) \\
&= \mathbf{P}\left((S_1, \ldots, S_i) \in A \mid L_{n-k} \quad \ge 0\right)\mathbf{P}(S_k < L_{k-1})\,\mathbf{P}(L_{n-k} \quad \ge 0) \\
&= \mathbf{P}\left((S_1, \ldots, S_i) \in A \mid L_{n-k} \quad \ge 0\right)\mathbf{P}(\tau_n = k).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbf{P}\left(\left(S'_{1,n}, \ldots, S'_{i,n}\right) \in A, \quad \tau_n + i \le n\right) \\
&= \sum_{k=0}^{n-i} \mathbf{P}\left((S_1, \ldots, S_i) \in A \mid L_{n-k} \quad \ge 0\right)\mathbf{P}(\tau_n = k).
\end{aligned} \tag{12}$$

If the Spitzer–Doney condition is satisfied, then the following generalized arcsine law is valid (see, for instance, [12], Chapter 8, Theorem 8.9.9): for $x \in [0, 1]$

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\tau_n}{n} \le x\right) = \frac{\sin(\pi\rho)}{\pi} \int_0^x u^{\rho-1} (1-u)^{-\rho}\, du. \tag{13}$$

We pass to the limit in formula (12), as $n \to \infty$. Due to (13)

$$\lim_{n\to\infty} \mathbf{P}(\tau_n + i \le n) = \lim_{n\to\infty} \mathbf{P}(\tau_n/n \le 1 - i/n) = 1.$$

Therefore the limit of the left-hand side of (12) coincides with the limit of probability $\mathbf{P}\left(\left(S'_{1,n}, \ldots, S'_{i,n}\right) \in A\right)$, as $n \to \infty$, if at least one of these limits exists.

If $\varepsilon \in (0, 1)$ and n is large enough, then by (12)

$$\mathbf{P}\left(\left(S'_{1,n}, \ldots, S'_{i,n}\right) \in A, \quad \tau_n + i \le n\right) = P_1(n, \varepsilon) + P_2(n, \varepsilon), \tag{14}$$

where

$$P_1(n, \varepsilon) = \sum_{k=0}^{\lfloor (1-\varepsilon)n \rfloor} \mathbf{P}\left((S_1, \ldots, S_i) \in A \mid L_{n-k} \quad \ge 0\right)\mathbf{P}(\tau_n = k),$$

$$P_2(n,\varepsilon)=\sum_{k=\lfloor(1-\varepsilon)n\rfloor+1}^{n-i}\mathbf{P}\left((S_1,\ldots,S_i)\in A\mid L_{n-k}\geq 0\right)\mathbf{P}(\tau_n=k).$$

Clearly,

$$\begin{aligned}P_2(n,\varepsilon)&\leq\sum_{k=\lfloor(1-\varepsilon)n\rfloor+1}^{n}\mathbf{P}(\tau_n=k)=\mathbf{P}(\tau_n>\lfloor(1-\varepsilon)n\rfloor)\\&\underset{n\to\infty}{\to}1-\frac{\sin(\pi\rho)}{\pi}\int_0^{1-\varepsilon}u^{\rho-1}(1-u)^{-\rho}\,du\underset{\varepsilon\to 0}{\to}0.\end{aligned}$$

Therefore

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}P_2(n,\varepsilon)=0. \tag{15}$$

In view of (3) the probability $\mathbf{P}\left((S_1,\ldots,S_i)\in A\mid L_{n-k}\geq 0\right)$ tends, as $n\to\infty$, to

$$\mathbf{P}^*\left(\left(S_1^+,\ldots,S_i^+\right)\in A\right)=\mathbf{P}^*\left(\left(S_1^*,\ldots,S_i^*\right)\in A\right)$$

uniformly over $0\leq k\leq\lfloor(1-\varepsilon)n\rfloor$. Consequently,

$$\begin{aligned}\lim_{n\to\infty}P_1(n,\varepsilon)&=\mathbf{P}^*\left(\left(S_1^*,\ldots,S_i^*\right)\in A\right)\lim_{n\to\infty}\sum_{k=0}^{\lfloor(1-\varepsilon)n\rfloor}\mathbf{P}(\tau_n=k)\\&=\mathbf{P}^*\left(\left(S_1^*,\ldots,S_i^*\right)\in A\right)\lim_{n\to\infty}\mathbf{P}(\tau_n\leq\lfloor(1-\varepsilon)n\rfloor)\\&=\mathbf{P}^*\left(\left(S_1^*,\ldots,S_i^*\right)\in A\right)\frac{\sin(\pi\rho)}{\pi}\int_0^{1-\varepsilon}u^{\rho-1}(1-u)^{-\rho}\,du\\&\underset{\varepsilon\to 0}{\to}\mathbf{P}^*\left(\left(S_1^*,\ldots,S_i^*\right)\in A\right)\end{aligned}$$

implying

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}P_1(n,\varepsilon)=\mathbf{P}^*\left(\left(S_1^*,\ldots,S_i^*\right)\in A\right). \tag{16}$$

It follows from relations (14)–(16) that for $i\in\mathbf{N}_0$

$$\lim_{n\to\infty}\mathbf{P}\left(\left(S'_{1,n},\ldots,S'_{i,n}\right)\in A,\quad\tau_n+i\leq n\right)=\mathbf{P}^*\left(\left(S_1^*,\ldots,S_i^*\right)\in A\right).$$

Thus,

$$\lim_{n\to\infty}\mathbf{P}\left(\left(S'_{1,n},\ldots,S'_{i,n}\right)\in A\right)=\mathbf{P}^*\left(\left(S_1^*,\ldots,S_i^*\right)\in A\right).$$

This proves the desired relation (11).

We now show that for any fixed $i\in\mathbf{N}$, as $n\to\infty$,

$$\left(S'_{-1,n},\ldots,S'_{-i,n}\right)\xrightarrow{D}\left(S^*_{-1},\ldots,S^*_{-i}\right). \tag{17}$$

Let B be a $\left(S^*_{-1},\ldots,S^*_{-i}\right)$-continuous (relative to the measure $\mathbf{P}^*$) Borel set from $\mathbf{R}^i$. Then for $n\in\mathbf{N}_i$

$$\begin{aligned}&\mathbf{P}\left(\left(S'_{-1,n},\ldots,S'_{-i,n}\right)\in B,\quad\tau_n-i\geq 0\right)\\&=\sum_{k=i}^{n}\mathbf{P}\left(\left(S'_{-1,n},\ldots,S'_{-i,n}\right)\in B,\quad\tau_n=k\right)\\&=\sum_{k=i}^{n}\mathbf{P}\left((S_{k-1}-S_k,\ldots,S_{k-i}-S_k)\in B,\quad S_k<L_{k-1},\quad S_k\leq L_{k+1,n}\right),\end{aligned}$$

and by the Markov property and the duality property of random walks we have that

$$\begin{aligned}
&\mathbf{P}\left((S_{k-1}-S_k,\ldots,S_{k-i}-S_k)\in B,\quad S_k<L_{k-1},\quad S_k\le L_{k+1,n}\right)\\
&=\mathbf{P}\left((S_{k-1}-S_k,\ldots,S_{k-i}-S_k)\in B,\quad S_k<L_{k-1}\right)\mathbf{P}\left(S_k\le L_{k+1,n}\right)\\
&=\mathbf{P}\left((-S_1,\ldots,-S_i)\in B,\quad M_k\quad <0\right)\mathbf{P}\left(S_k\le L_{k+1,n}\right)\\
&=\mathbf{P}\left((-S_1,\ldots,-S_i)\in B\mid M_k\quad <0\right)\mathbf{P}\left(M_k\quad <0\right)\mathbf{P}\left(S_k\le L_{k+1,n}\right)\\
&=\mathbf{P}\left((-S_1,\ldots,-S_i)\in B\mid M_k\quad <0\right)\mathbf{P}\left(\tau_n=k\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbf{P}\left(\left(S'_{-1,n},\ldots,S'_{-i,n}\right)\in B,\quad \tau_n-i\ge 0\right)\\
&=\sum_{k=i}^{n}\mathbf{P}\left((-S_1,\ldots,-S_i)\in B\mid M_k\quad <0\right)\mathbf{P}\left(\tau_n=k\right),
\end{aligned}\tag{18}$$

therefore, if $\varepsilon\in(0,1)$ and n is large enough, then

$$\mathbf{P}\left(\left(S'_{-1,n},\ldots,S'_{-i,n}\right)\in B,\quad \tau_n-i\ge 0\right)=P_3(n,\varepsilon)+P_4(n,\varepsilon),$$

where

$$P_3(n,\varepsilon)=\sum_{k=i}^{\lfloor\varepsilon n\rfloor}\mathbf{P}\left((-S_1,\ldots,-S_i)\in B\mid M_k\quad <0\right)\mathbf{P}\left(\tau_n=k\right),$$

$$P_4(n,\varepsilon)=\sum_{k=\lfloor\varepsilon n\rfloor+1}^{n}\mathbf{P}\left((-S_1,\ldots,-S_i)\in B\mid M_k\quad <0\right)\mathbf{P}\left(\tau_n=k\right).$$

It is not difficult to show (see our proof of relation (15)) that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}P_3(n,\varepsilon)=0.$$

Due to (4) the probability $\mathbf{P}\left((-S_1,\ldots,-S_i)\in B\mid M_k\quad <0\right)$ tends, as $n\to\infty$, to

$$\mathbf{P}^*\left(\left(-S^-_{-1},\ldots,-S^-_{-i}\right)\in B\quad\right)=\mathbf{P}^*\left(\left(S^*_{-1},\ldots,S^*_{-i}\right)\in B\quad\right)$$

uniformly over $\lfloor\varepsilon n\rfloor<k\le n$. Therefore,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}P_4(n,\varepsilon)=\mathbf{P}^*\left(\left(S^*_{-1},\ldots,S^*_{-i}\right)\in B\quad\right).$$

As a result, we obtain that

$$\lim_{n\to\infty}\mathbf{P}\left(\left(S'_{-1,n},\ldots,S'_{-i,n}\right)\in B,\quad \tau_n-i\ge 0\right)=\mathbf{P}^*\left(\left(S^*_{-1},\ldots,S^*_{-i}\right)\in B\quad\right).$$

Hence,

$$\lim_{n\to\infty}\mathbf{P}\left(\left(S'_{-1,n},\ldots,S'_{-i,n}\right)\in B\right)=\mathbf{P}^*\left(\left(S^*_{-1},\ldots,S^*_{-i}\right)\in B\quad\right).$$

Thus, the desired relation (17) is proved.

It can be shown, by the arguments similar to those used in proving of (11) and (17), that these two relations can be combined into one. To justify this statement one should take into account that for $k\in\{0,1,\ldots,n\}$

$$\begin{aligned}
&\mathbf{P}\left(\left(S'_{1,n},\ldots,S'_{i,n}\right)\in A,\quad \left(S'_{-1,n},\ldots,S'_{-i,n}\right)\in B,\quad \tau_n=k\right)\\
&=\mathbf{P}\left((S_1,\ldots,S_i)\in A\mid L_{n-k}\quad\ge 0\right)\\
&\quad\times\mathbf{P}\left((-S_1,\ldots,-S_i)\in B\mid M_k\quad <0\right)\mathbf{P}\left(\tau_n=k\right).
\end{aligned}$$

The lemma is proved.

Remark 1. It is not difficult to verify (see Lemma 1 of [1] or the proof of Theorem 1.5 of [8]) that, under the condition $\{L_n \geq 0\}$ (or under the condition $\{M_n < 0\}$), the random sequence $\{S_i, \quad i \in \mathbf{N}_0\}$ and the random process W_n are asymptotically independent, as $n \to \infty$. Similarly to relations (12) and (18), one can show that for any $a \leq 0$ and $b > 0$

$$\begin{aligned}
&\mathbf{P}\left(\left(S'_{1,n}, \ldots, S'_{i,n}\right) \in A, \quad \tau_n + i \leq n, \quad \frac{L_n}{C_n} \leq a, \quad \frac{S_n - L_n}{C_n} \leq b\right) \\
&= \sum_{k=0}^{n-i} \mathbf{P}\left(\left(S_1, \ldots, S_i\right) \in A, \quad \frac{S_{n-k}}{C_n} \leq b \;\middle|\; L_{n-k} \quad \geq 0\right) \\
&\quad \times \mathbf{P}\left(\frac{S_k}{C_n} \leq a \;\middle|\; M_k < 0\right) \mathbf{P}\left(\tau_n = k\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{P}\left(\left(S'_{-1,n}, \ldots, S'_{-i,n}\right) \in B, \quad \tau_n - i \geq 0, \quad \frac{L_n}{C_n} \leq a, \quad \frac{S_n - L_n}{C_n} \leq b\right) \\
&= \sum_{k=i}^{n} \mathbf{P}\left(\left(-S_1, \ldots, -S_i\right) \in B, \quad \frac{S_k}{C_n} \leq a \;\middle|\; M_k \quad < 0\right) \\
&\quad \times \mathbf{P}\left(\frac{S_{n-k}}{C_n} \leq b \;\middle|\; L_{n-k} \quad \geq 0\right) \mathbf{P}\left(\tau_n = k\right).
\end{aligned}$$

Using these facts and repeating the reasonings given in the proof of Lemma 1, one can show that for any $a \leq 0$ and $b > 0$, as $n \to \infty$,

$$\left\{S'_{i,n}, \quad i \in \mathbf{Z} \;\middle|\; \frac{L_n}{C_n} \leq a, \quad \frac{S_n - L_n}{C_n} \leq b\right\} \xrightarrow{D} \left\{S_i^*, \quad i \in \mathbf{Z}\right\}.$$

Recall that $(\Omega, \mathcal{F}, \mathbf{P})$ is the underlying probability space. We introduce new probability measures $\mathbf{P}^+$ and $\mathbf{P}^-$. With this aim we define the following functions:

$$u\left(x\right) = 1 + \sum_{n=1}^{\infty} \mathbf{P}\left(-S_n \leq x, \quad M_n < 0\right), \quad x \geq 0,$$

$$v\left(x\right) = 1 + \sum_{n=1}^{\infty} \mathbf{P}\left(-S_n > x, \quad L_n \geq 0\right), \quad x \leq 0.$$

Set

$$I_n^{(2)} := \{(i, j) : i, j \in \{0, \ldots, n\} \text{ and } i \leq j\}.$$

Let $\mathcal{F}_n$, $n \in \mathbf{N}_0$, denote the σ-algebra generated by the segment of the random environment $Q_1, \ldots, Q_n$ and the random variables $Z_{i,j}$ for $(i, j) \in I_n^{(2)}$. We now introduce a probability measure $\mathbf{P}^+$ on the σ-algebra $\mathcal{F}_\infty := \sigma\left(\cup_{n=1}^{\infty} \mathcal{F}_n\right)$, defined for each $n \in \mathbf{N}_0$ and each $\mathcal{F}_n$-measurable non-negative random variable β by the formula

$$\mathbf{E}^+ \beta = \mathbf{E}\left(\beta u\left(S_n\right); \quad L_n \geq 0\right). \tag{19}$$

Similarly, we also introduce a probability measure $\mathbf{P}^-$ on the σ-algebra $\mathcal{F}_\infty$, defined for each $n \in \mathbf{N}_0$ and each $\mathcal{F}_n$-measurable non-negative random variable β by the formula

$$\mathbf{E}^- \beta = \mathbf{E}\left(\beta v\left(S_n\right); \quad M_n < 0\right). \tag{20}$$

The existence of measures $\mathbf{P}^{+}$ and $\mathbf{P}^{-}$ follows from Lemma 2.4 of [8] (note that it may be necessary to change the underlying probability space). Thus, three measures $\mathbf{P}, \mathbf{P}^{+}, \mathbf{P}^{-}$ are defined on one and the same measurable space $(\Omega, \mathcal{F}_{\infty})$. To explicitly indicate the measure on $(\Omega, \mathcal{F}_{\infty})$ according to which we consider this or those random elements we use the measure symbol as a lower index.

For instance, it is shown in Lemma 2.5 of [8] that

$$\left\{\left(Q_i^{+}, S_i^{+}, \mu_i^{+}\right), \quad i \in \mathbf{N}\right\} \stackrel{D}{=} \{(Q_i, S_i, \mu_i), \quad i \in \mathbf{N}\}_{\mathbf{P}^{+}} . \tag{21}$$

Similarly (see Lemma 5.2 of [17]),

$$\left\{\left(Q_i^{-}, S_i^{-}, \mu_i^{-}\right), \quad i \in \mathbf{N}\right\} \stackrel{D}{=} \{(Q_i, S_i, \mu_i), \quad i \in \mathbf{N}\}_{\mathbf{P}^{-}} . \tag{22}$$

Due to (21), (22) and our assumption about the independence of the left-hand sides of these relations, the product of probability spaces $\left(\Omega, \mathcal{F}_{\infty}, \mathbf{P}^{+}\right)$ and $\left(\Omega, \mathcal{F}_{\infty}, \mathbf{P}^{-}\right)$ may be considered as a probability space $(\Omega^{*}, \mathcal{F}^{*}, \mathbf{P}^{*})$ and, consequently, the direct product of the measures $\mathbf{P}^{+}$ and $\mathbf{P}^{-}$ may be treated as the measure $\mathbf{P}^{*}$.

Remark 2. If a random element ξ is given on the space $\left(\Omega, \mathcal{F}_{\infty}, \mathbf{P}^{+}\right)$ we can define the random element ξ^{+}, specified on the product of the spaces $\left(\Omega, \mathcal{F}_{\infty}, \mathbf{P}^{+}\right)$ and $\left(\Omega, \mathcal{F}_{\infty}, \mathbf{P}^{-}\right)$ by means of the formula $\xi^{+}(\omega_1, \omega_2) = \xi(\omega_1)$ for $(\omega_1, \omega_2) \in \Omega \times \Omega$. It is clear that the element ξ^{+} coincides in distribution with the element ξ. Similarly, if a random element ξ is given on the space $\left(\Omega, \mathcal{F}_{\infty}, \mathbf{P}^{-}\right)$ we can define the random element ξ^{-}, specified on the product of the spaces $\left(\Omega, \mathcal{F}_{\infty}, \mathbf{P}^{+}\right)$ and $\left(\Omega, \mathcal{F}_{\infty}, \mathbf{P}^{-}\right)$ by means of the formula $\xi^{-}(\omega_1, \omega_2) = \xi(\omega_2)$ for $(\omega_1, \omega_2) \in \Omega \times \Omega$, and the element ξ^{-} coincides in distribution with the element ξ. In accordance with the agreement we can consider the random elements standing in the left-hand sides of formulae (21) and (22) as generated by the random elements $\{(Q_i, S_i, \mu_i), \quad i \in \mathbf{N}\}_{\mathbf{P}^{+}}$ and $\{(Q_i, S_i, \mu_i), \quad i \in \mathbf{N}\}_{\mathbf{P}^{-}}$ respectively.

Lemma 2. *If the Spitzer–Doney condition (2) is satisfied, then, as $n \to \infty$,*

$$\left\{a_{i,n} Z_{i,n}, \quad i \in \mathbf{N}_0 \quad \middle| \quad L_n \quad \geq 0\right\} \xrightarrow{D} \left\{\zeta_i^{*}, \quad i \in \mathbf{N}_0\right\},$$

where $\left\{\zeta_i^{*}, \quad i \in \mathbf{N}_0\right\}$ is the random sequence defined by relation (8) .

Proof. By virtue of the first part of Lemma 2.5 from [8] for any fixed $i \in \mathbf{N}_0$ and $k \in \mathbf{N}_i$, as $n \to \infty$,

$$\left\{a_{0,k} Z_{0,k}, \ldots, a_{i,k} Z_{i,k} \quad \middle| \quad L_n \quad \geq 0\right\} \xrightarrow{D} \left\{\left(a_{0,k} Z_{0,k}, \ldots, a_{i,k} Z_{i,k}\right)\right\}_{\mathbf{P}^{+}} .$$

Note (see [8], Section 3) that in view of (19) for any fixed $j \in \mathbf{N}_0$ the random sequence $\left\{Z_{j,k}, \ k \in \mathbf{N}_j\right\}_{\mathbf{P}^{+}}$ is a branching process in the random environment $\left\{G_{j+1}; F_k, \ k \in \mathbf{N}_{j+1}\right\}_{\mathbf{P}^{+}}$. Hence, the sequence $\left\{Z_{j,k}, \ k \in \mathbf{N}_j\right\}_{\mathbf{P}^{+}}$ coincides in distribution with the sequence $\left\{Z_{j,k}^{*}, \ k \in \mathbf{N}_j\right\}$. Moreover,

$$\left(a_{0,k} Z_{0,k}, \ldots, a_{i,k} Z_{i,k}\right)_{\mathbf{P}^{+}} \stackrel{D}{=} \left(a_{0,k}^{*} Z_{0,k}^{*}, \ldots, a_{i,k}^{*} Z_{i,k}^{*}\right) .$$

Thus, as $n \to \infty$,

$$\left\{a_{0,k} Z_{0,k}, \ldots, a_{i,k} Z_{i,k} \quad \middle| \quad L_n \quad \geq 0\right\} \xrightarrow{D} \left(a_{0,k}^{*} Z_{0,k}^{*}, \ldots, a_{i,k}^{*} Z_{i,k}^{*}\right) .$$

Due to (8) $\mathbf{P}^*$-a.s.

$$\lim_{k\to\infty}\left(a_{0,k}^* Z_{0,k}^*,\ldots,a_{i,k}^* Z_{i,k}^*\right)=\left(\zeta_0^*,\ldots,\zeta_i^*\right).$$

It means, in view of the second part of Lemma 2.5 from [8], that, as $n\to\infty$,

$$\left\{a_{0,n}Z_{0,n},\ldots,a_{i,n}Z_{i,n}\quad\middle|\quad L_n\quad\geq 0\right\}\xrightarrow{D}\left(\zeta_0^*,\ldots,\zeta_i^*\right).$$

The lemma is proved.

Set for $n\in\mathbf{N}$

$$Z'_{i,n}=\begin{cases} Z_{\tau_n+i,n}, & i\in\mathbf{N}_{(-\tau_n)},\\ 0, & i\in\mathbf{Z}\setminus\mathbf{N}_{(-\tau_n)},\end{cases}\tag{23}$$

$$a'_{i,n}=\begin{cases} a_n/a_{\tau_n+i}, & i\in\mathbf{N}_{(-\tau_n)},\\ 1, & i\in\mathbf{Z}\setminus\mathbf{N}_{(-\tau_n)}.\end{cases}\tag{24}$$

Lemma 3. *If the Spitzer–Doney condition (2) is satisfied, then, as $n\to\infty$,*

$$\left\{a'_{i,n}Z'_{i,n},\quad i\in\mathbf{Z}\right\}\xrightarrow{D}\left\{\zeta_i^*,\quad i\in\mathbf{Z}\right\},$$

where $\left\{\zeta_i^*,\ i\in\mathbf{N}_0\right\}$ is the random sequence defined by relation (8).

Proof. First, we show that for any fixed $i\in\mathbf{N}_0$, as $n\to\infty$,

$$\left(a'_{0,n}Z'_{0,n},\ldots,a'_{i,n}Z'_{i,n}\right)\xrightarrow{D}\left(\zeta_0^*,\ldots,\zeta_i^*\right).\tag{25}$$

Let A be an arbitrary one-dimensional $\left(\zeta_0^*,\ldots,\zeta_i^*\right)$-continuous (relative to the measure $\mathbf{P}^*$) Borel set from $\mathbf{R}^{i+1}$. Set for $k\in\mathbf{N}_0$, $n\in\mathbf{N}_k$

$$\widehat{Z}_{k,n}=a_{k,n}Z_{k,n}$$

and for $k\in\mathbf{N}_0$, $n\in\mathbf{N}_{k+i}$

$$\widehat{\mathbf{Z}}_{k,n}=\left(\widehat{Z}_{k,n},\ldots,\widehat{Z}_{k+i,n}\right).$$

Then for $n\in\mathbf{N}_i$

$$\begin{aligned}
&\mathbf{P}\left(\left(a'_{0,n}Z'_{0,n},\ldots,a'_{i,n}Z'_{i,n}\right)\in A,\quad \tau_n+i\leq n\right)\\
&=\sum_{k=0}^{n-i}\mathbf{P}\left(\left(a'_{0,n}Z'_{0,n},\ldots,a'_{i,n}Z'_{i,n}\right)\in A,\quad \tau_n=k\right)\\
&=\sum_{k=0}^{n-i}\mathbf{P}\left(\widehat{\mathbf{Z}}_{k,n}\in A,\quad S_k<L_{k-1},\quad S_k\leq L_{k+1,n}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{P}\left(\widehat{\mathbf{Z}}_{k,n}\in A,\quad S_k<L_{k-1},\quad S_k\leq L_{k+1,n}\right)\\
&=\mathbf{P}\left(S_k<L_{k-1}\right)\mathbf{P}\left(\widehat{\mathbf{Z}}_{k,n}\in A,\quad S_k\leq L_{k+1,n}\right)\\
&=\mathbf{P}\left(S_k<L_{k-1}\right)\mathbf{P}\left(\widehat{\mathbf{Z}}_{0,n-k}\in A,\quad L_{n-k}\quad\geq 0\right)\\
&=\mathbf{P}\left(\widehat{\mathbf{Z}}_{0,n-k}\in A\;\middle|\; L_{n-k}\quad\geq 0\right)\mathbf{P}\left(S_k<L_{k-1}\right)\mathbf{P}\left(L_{n-k}\quad\geq 0\right)\\
&=\mathbf{P}\left(\widehat{\mathbf{Z}}_{0,n-k}\in A\;\middle|\; L_{n-k}\quad\geq 0\right)\mathbf{P}\left(\tau_n=k\right).
\end{aligned}$$

Thus,

$$\mathbf{P}\left(\left(a'_{0,n}Z'_{0,n},\ldots,a'_{i,n}Z'_{i,n}\right)\in A,\quad \tau_n+i\le n\right)$$
$$=\sum_{k=0}^{n-i}\mathbf{P}\left(\widehat{\mathbf{Z}}_{0,n-k}\in A\;\middle|\; L_{n-k}\ \ge 0\right)\mathbf{P}\left(\tau_n=k\right),$$

Therefore, if $\varepsilon\in(0,1)$ and n is large enough, then

$$\mathbf{P}\left(\left(a'_{0,n}Z'_{0,n},\ldots,a'_{i,n}Z'_{i,n}\right)\in A,\quad \tau_n+i\le n\right)=P_1\left(n,\varepsilon\right)+P_2\left(n,\varepsilon\right),$$

where

$$P_1\left(n,\varepsilon\right)=\sum_{k=0}^{\lfloor(1-\varepsilon)n\rfloor}\mathbf{P}\left(\widehat{\mathbf{Z}}_{0,n-k}\in A\;\middle|\; L_{n-k}\ \ge 0\right)\mathbf{P}\left(\tau_n=k\right),$$
$$P_2\left(n,\varepsilon\right)=\sum_{k=\lfloor(1-\varepsilon)n\rfloor+1}^{n-i}\mathbf{P}\left(\widehat{\mathbf{Z}}_{0,n-k}\in A\;\middle|\; L_{n-k}\ \ge 0\right)\mathbf{P}\left(\tau_n=k\right).$$

It is easy to show (see the proof of relation (15)) that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}P_2\left(n,\varepsilon\right)=0.$$

By Lemma 2

$$\lim_{n\to\infty}\mathbf{P}\left(\widehat{\mathbf{Z}}_{0,n-k}\in A\;\middle|\; L_{n-k}\quad \ge 0\right)=\mathbf{P}^*\left(\left(\zeta_0^*,\ldots,\zeta_i^*\right)\in A\quad\right)$$

uniformly over $0\le k\le\lfloor(1-\varepsilon)n\rfloor$. Therefore

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}P_1\left(n,\varepsilon\right)=\mathbf{P}^*\left(\left(\zeta_0^*,\ldots,\zeta_i^*\right)\in A\quad\right).$$

As a result, we obtain that

$$\lim_{n\to\infty}\mathbf{P}\left(\left(a'_{0,n}Z'_{0,n},\ldots,a'_{i,n}Z'_{i,n}\right)\in A,\quad \tau_n+i\le n\right)=\mathbf{P}^*\left(\left(\zeta_0^*,\ldots,\zeta_i^*\right)\in A\quad\right).$$

Hence,

$$\lim_{n\to\infty}\mathbf{P}\left(\left(a'_{0,n}Z'_{0,n},\ldots,a'_{i,n}Z'_{i,n}\right)\in A\right)=\mathbf{P}^*\left(\left(\zeta_0^*,\ldots,\zeta_i^*\right)\in A\quad\right).$$

Thus, the desired relation (25) is proved.

We now show that for any fixed $i\in\mathbf{N}$, as $n\to\infty$,

$$\left(a'_{-1,n}Z'_{-1,n},\ldots,a'_{-i,n}Z'_{-i,n}\right)\xrightarrow{D}\left(\zeta_{-1}^*,\ldots,\zeta_{-i}^*\right). \tag{26}$$

With this aim we use the continuity theorem for Laplace transform. Let $\lambda_1,\ldots,\lambda_i\in\mathbf{R}_+:=\{x\in\mathbf{R}:\quad x\ge 0\}$. Note that for $n\in\mathbf{N}_i$

$$\mathbf{E}\left(\exp\left(-\sum_{j=1}^{i}\lambda_j a'_{-j,n}Z'_{-j,n}\right);\quad \tau_n-i\ge 0\right)$$
$$=\sum_{k=i}^{n}\mathbf{E}\left(\exp\left(-\sum_{j=1}^{i}\lambda_j a'_{-j,n}Z'_{-j,n}\right);\quad \tau_n=k\right)$$
$$=\sum_{k=i}^{n}\mathbf{E}\left(\exp\left(-\Sigma_{k,n}\right);\quad S_k<L_{k-1},\quad S_k\le L_{k+1,n}\right), \tag{27}$$

where

$$\Sigma_{k,n}=\sum_{j=1}^{i}\lambda_j\widehat{Z}_{k-j,n}.$$

Let the generation numbering of our initial BPIRE determine the time scale. Consider (without reference to the initial BPIRE) a branching process in the random environment $\{F_{k+1},F_{k+2},\ldots\}$ generated at time $k\in\mathbf{N}_0$ by a fixed set of particles which is divided into i pairwise non-intersecting subsets whose cardinalities are $l_1,\ldots,l_i\in\mathbf{N}_0$, respectively. For $n\in\mathbf{N}_k$, we denote $Z_{k,n}\left(l_j\right)$ the total number of particles at time n being descendants of the particles from the jth subset (we assume that $Z_{k,k}\left(l_j\right)=l_j$).

Set for $k\in\mathbf{N}_0$, $n\in\mathbf{N}_k$, $\mathbf{l}=(l_1,\ldots,l_i)\in\mathbf{N}_0^i$

$$\mathbf{Z}_{k,n}(\mathbf{l})=\left(Z_{k,n}(l_1),\ldots,Z_{k,n}(l_i)\right),\quad \mathbf{S}_{k,n}=(S_{k+1}-S_k,\ldots,S_n-S_k)$$

and for $k\in\mathbf{N}_0$, $n\in\mathbf{N}_k$

$$\mathbf{Z}_{k,-i,n}=\left(Z_{k-1,n},\ldots,Z_{k-i,n}\right),\quad \mathbf{a}_{k,-i,n}=\left(a_{k-1,n},\ldots,a_{k-i,n}\right).$$

Let $\mathbf{P}_\Pi(A):=\mathbf{P}(A\mid\Pi)$, $\mathbf{E}_\Pi\xi:=\mathbf{E}(\xi\mid\Pi)$, where $A\in\mathcal{F}$ and ξ is a random variable defined on $(\Omega,\mathcal{F},\mathbf{P})$. Denote by I_A the indicator of $A\in\mathcal{F}$.

Let $k\in\mathbf{N}_i$ and $n\in\mathbf{N}_k$. Given the random environment Π the distribution of the random vector $\mathbf{Z}_{k,-i,n}$ conditioned on the event $\left\{\mathbf{Z}_{k,-i,k}=\mathbf{l}\right\}$, where $\mathbf{l}\in\mathbf{N}_0^i$, coincides with the distribution of the random vector $\mathbf{Z}_{k,n}(\mathbf{l})$. Hence, taking into account the equality $\mathbf{a}_{k,-i,n}=a_{k,n}\mathbf{a}_{k,-i,k}$, we obtain (in the same way as in [8], p. 659) that for each bounded and measurable numerical function $f(\mathbf{x},\mathbf{y},\mathbf{z})$, $\mathbf{x}\in\mathbf{N}_0^i$, $\mathbf{y}\in\mathbf{R}_+^i$, $\mathbf{z}\in\mathbf{R}^{n-k}$,

$$\begin{aligned}
&\mathbf{E}\left(f\left(\mathbf{Z}_{k,-i,n},\mathbf{a}_{k,-i,n},\mathbf{S}_{k,n}\right);\quad \mathbf{Z}_{k,-i,k}=\mathbf{l},\quad \mathbf{a}_{k,-i,k}\in d\mathbf{y},\quad \mathbf{S}_{0,k}\in d\mathbf{z}\right)\\
&=\mathbf{E}\left(\mathbf{E}_\Pi\left(f\left(\mathbf{Z}_{k,-i,n},a_{k,n}\mathbf{y},\mathbf{S}_{k,n}\right);\quad \mathbf{Z}_{k,-i,k}=\mathbf{l}\right);\quad \mathbf{a}_{k,-i,k}\in d\mathbf{y},\quad \mathbf{S}_{0,k}\in d\mathbf{z}\right)\\
&=\mathbf{E}\left(\mathbf{E}_\Pi f\left(\mathbf{Z}_{k,n}(\mathbf{l}),a_{k,n}\mathbf{y},\mathbf{S}_{k,n}\right)\mathbf{P}_\Pi\left(\mathbf{Z}_{k,-i,k}=\mathbf{l}\right)I_{\{\mathbf{a}_{k,-i,k}\in d\mathbf{y},\quad \mathbf{S}_{0,k}\in d\mathbf{z}\}}\right)\\
&=\mathbf{E}f\left(\mathbf{Z}_{k,n}(\mathbf{l}),a_{k,n}\mathbf{y},\mathbf{S}_{k,n}\right)\mathbf{P}\left(\mathbf{Z}_{k,-i,k}=\mathbf{l},\quad \mathbf{a}_{k,-i,k}\in d\mathbf{y},\quad \mathbf{S}_{0,k}\in d\mathbf{z}\right)
\end{aligned}$$

(the latter equality is justified by the fact that the (random) expectation

$$\mathbf{E}_\Pi f\left(\mathbf{Z}_{k,n}(\mathbf{l}),a_{k,n}\mathbf{y},\mathbf{S}_{k,n}\right)$$

is expressed through the environment $Q_{k+1},\ldots,Q_n$ and the random variable

$$\mathbf{P}_\Pi\left(\mathbf{Z}_{k,-i,k}=\mathbf{l}\right)I_{\{\mathbf{a}_{k,-i,k}\in d\mathbf{y},\quad \mathbf{S}_{0,k}\in d\mathbf{z}\}}$$

is expressed through the environment $Q_{k-i+1},\ldots,Q_k$). Thus,

$$\begin{aligned}
&\mathbf{E}\left(f\left(\mathbf{Z}_{k,-i,n},\mathbf{a}_{k,-i,n},\mathbf{S}_{k,n}\right)\;\middle|\;\mathbf{Z}_{k,-i,k},\quad \mathbf{a}_{k,-i,k},\quad \mathbf{S}_{0,k}\right)\\
&=\left(\mathbf{E}f\left(\mathbf{Z}_{k,n}(\mathbf{l}),a_{k,n}\mathbf{y},\mathbf{S}_{k,n}\right)\right)_{\mathbf{l}=\mathbf{Z}_{k,-i,k},\quad \mathbf{y}=\mathbf{a}_{k,-i,k}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbf{E}\left(\exp\left(-\Sigma_{k,n}\right);\quad S_k<L_{k-1},\quad S_k\le L_{k+1,n}\right)\\
&=\mathbf{E}\left(\mathbf{E}\left(\exp\left(-\Sigma_{k,n}\right)I_{\{S_k<L_{k-1},\quad S_k\le L_{k+1,n}\}}\;\middle|\;\mathbf{Z}_{k,-i,k},\quad \mathbf{a}_{k,-i,k},\quad \mathbf{S}_{0,k}\right)\right)\\
&=\mathbf{E}\left(\mathbf{E}\left(\exp\left(-\Sigma_{k,n}\right)I_{\{S_k\le L_{k+1,n}\}}\;\middle|\;\mathbf{Z}_{k,-i,k},\quad \mathbf{a}_{k,-i,k},\quad \mathbf{S}_{0,k}\right);\quad S_k<L_{k-1}\right)\\
&=\mathbf{E}\left(U\left(\mathbf{Z}_{k,-i,k},\quad \mathbf{a}_{k,-i,k}\right);\quad S_k<L_{k-1}\right),
\end{aligned}$$

where

$$U(\mathbf{l},\mathbf{y})=\mathbf{E}\left(\exp\left(-\sum_{j=1}^{i}\lambda_j y_j\widehat{Z}_{k,n}(l_j)\right);\quad S_k\le L_{k+1,n}\right)$$

for $\mathbf{l}=(l_1,\dots,l_i)\in\mathbf{N}_0^i$ and $\mathbf{y}=(y_1,\dots,y_i)\in\mathbf{R}_+^i$. Note that

$$U(\mathbf{l},\mathbf{y})=\mathbf{E}\left(\exp\left(-\sum_{j=1}^{i}\lambda_j y_j\widehat{Z}_{0,n-k}(l_j)\right);\quad L_{n-k}\quad\ge 0\right).$$

For each $n\in\mathbf{N},$ we introduce the function

$$H_n(\mathbf{l},\mathbf{y})=\mathbf{E}\left(\exp\left(-\sum_{j=1}^{i}\lambda_j y_j\widehat{Z}_{0,n}(l_j)\right)\,\middle|\; L_n\quad\ge 0\right)$$

for $\mathbf{l}=(l_1,\dots,l_i)\in\mathbf{N}_0^i$ and $\mathbf{y}=(y_1,\dots,y_i)\in\mathbf{R}_+^i$. As a result, we obtain that

$$\begin{aligned}&\mathbf{E}\left(\exp\left(-\Sigma_{k,n}\right);\quad S_k<L_{k-1},\quad S_k\le L_{k+1,n}\right)\\&=\mathbf{E}\left(H_{n-k}\left(\mathbf{Z}_{k,-i,k},\quad\mathbf{a}_{k,-i,k}\right)\,\middle|\; S_k<L_{k-1}\right)\mathbf{P}\left(S_k<L_{k-1}\right)\mathbf{P}\left(L_{n-k}\quad\ge 0\right)\\&=\mathbf{E}\left(H_{n-k}\left(\mathbf{Z}_{k,-i,k},\quad\mathbf{a}_{k,-i,k}\right)\,\middle|\; S_k<L_{k-1}\right)\mathbf{P}\left(\tau_n=k\right).\end{aligned}\tag{28}$$

Set $\mathbf{Q}_{l,m}=(Q_l,\dots,Q_m)$ for $l,m\in\mathbf{N}$. In the sequel, we explicitly include a random environment in the notation. For example, we write $\mathbf{Z}_{k,-i,k}\left\langle\mathbf{Q}_{k-i+1,k}\right\rangle$ instead of $\mathbf{Z}_{k,-i,k}$. Set

$$Q_k=\widetilde{Q}_1,\dots,Q_{k-i+1}=\widetilde{Q}_i,\dots,Q_1=\widetilde{Q}_k$$

and consider the random environment $\widetilde{Q}_1,\dots,\widetilde{Q}_k$. Then

$$\begin{aligned}&\mathbf{E}\left(H_{n-k}\left(\mathbf{Z}_{k,-i,k}\left\langle\mathbf{Q}_{k-i+1,k}\right\rangle,\mathbf{a}_{k,-i,k}\right)\,\middle|\; S_k<L_{k-1}\right)\\&=\mathbf{E}\left(H_{n-k}\left(\mathbf{Z}_{0,-i,0}\left\langle\widetilde{\mathbf{Q}}_{i,1}\right\rangle,(\widetilde{a}_1,\dots,\widetilde{a}_i)\right)\,\middle|\;\widetilde{M}_k\quad<0\right),\end{aligned}$$

where the symbols $\widetilde{a}_1,\dots,\widetilde{a}_i,\widetilde{M}_k,\widetilde{Q}_{i,1}$ have the same meaning for the random environment $\widetilde{Q}_1,\dots,\widetilde{Q}_k$ as the symbols $a_1,\dots,a_i,M_k,Q_{i,1}$ mean for the random environment $Q_1,\dots,Q_k$. Further, the random environments $\widetilde{Q}_1,\dots,\widetilde{Q}_k$ and $Q_1,\dots,Q_k$ are identically distributed. Therefore

$$\begin{aligned}&\mathbf{E}\left(H_{n-k}\left(\mathbf{Z}_{0,-i,0}\left\langle\widetilde{\mathbf{Q}}_{i,1}\right\rangle,(\widetilde{a}_1,\dots,\widetilde{a}_i)\right)\,\middle|\;\widetilde{M}_k\quad<0\right)\\&=\mathbf{E}\left(H_{n-k}\left(\mathbf{Z}_{0,-i,0}\left\langle\mathbf{Q}_{i,1}\right\rangle,(a_1,\dots,a_i)\right)\,\middle|\; M_k\quad<0\right).\end{aligned}$$

As a result, setting $\boldsymbol{\psi}_i=\mathbf{Z}_{0,-i,0}\left\langle\mathbf{Q}_{i,1}\right\rangle$ and $\mathbf{a}_i=(a_1,\dots,a_i)$, we obtain that

$$\begin{aligned}&\mathbf{E}\left(H_{n-k}\left(\mathbf{Z}_{k,-i,k}\left\langle\mathbf{Q}_{k-i+1,k}\right\rangle,\mathbf{a}_{k,-i,k}\right)\,\middle|\; S_k<L_{k-1}\right)\\&=\mathbf{E}\left(H_{n-k}\left(\boldsymbol{\psi}_i,\mathbf{a}_i\right)\,\middle|\; M_k\quad<0\right).\end{aligned}\tag{29}$$

We have from (27)–(29) that

$$\begin{aligned}&\mathbf{E}\left(\exp\left(-\sum_{j=1}^{i}\lambda_1 a'_{-j,n}Z'_{-j,n}\right);\quad\tau_n-i\ge 0\right)\\&=\sum_{k=i}^{n}\mathbf{E}\left(H_{n-k}\left(\boldsymbol{\psi}_i,\mathbf{a}_i\right)\,\middle|\; M_k\quad<0\right)\mathbf{P}\left(\tau_n=k\right).\end{aligned}$$

Therefore, if $\varepsilon \in (0,1)$ and n is large enough, then

$$\mathbf{E}\left(\exp\left(-\sum_{j=1}^{i}\lambda_1 a'_{-j,n}Z'_{-j,n}\right); \quad \tau_n - i \ge 0\right) = E_1(n,\varepsilon) + E_2(n,\varepsilon) + E_3(n,\varepsilon), \tag{30}$$

where

$$E_1(n,\varepsilon) = \sum_{k=i}^{\lfloor \varepsilon n\rfloor}\mathbf{E}\left(H_{n-k}\left(\boldsymbol{\psi}_i,\mathbf{a}_i\right) \mid M_k \quad < 0\right)\mathbf{P}(\tau_n = k),$$

$$E_2(n,\varepsilon) = \sum_{k=\lfloor(1-\varepsilon)n\rfloor+1}^{n}\mathbf{E}\left(H_{n-k}\left(\boldsymbol{\psi}_i,\mathbf{a}_i\right) \mid M_k \quad < 0\right)\mathbf{P}(\tau_n = k),$$

$$E_3(n,\varepsilon) = \sum_{k=\lfloor \varepsilon n\rfloor+1}^{\lfloor(1-\varepsilon)n\rfloor}\mathbf{E}\left(H_{n-k}\left(\boldsymbol{\psi}_i,\mathbf{a}_i\right) \mid M_k \quad < 0\right)\mathbf{P}(\tau_n = k).$$

Similar to relation (15) we conclude that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} E_1(n,\varepsilon) = 0, \tag{31}$$

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} E_2(n,\varepsilon) = 0. \tag{32}$$

Let $\mathbf{l} = (l_1,\ldots,l_i) \in \mathbf{N}_0^i$ be fixed. It is not difficult to demonstrate that for each $j \in \{1,\ldots,i\}$

$$\lim_{n\to\infty}\widehat{Z}_{0,n}\left(l_j\right) =: \zeta_0\left(l_j\right) \tag{33}$$

exists a.s. on the probability space $\left(\Omega,\mathcal{F}_\infty,\mathbf{P}^+\right)$. By the arguments analogous to those used in Lemma 2 one can show, as $n \to \infty$,

$$\left\{\widehat{Z}_{0,n}(l_1),\ldots,\quad \widehat{Z}_{0,n}(l_i) \mid L_n \quad \ge 0\right\} \xrightarrow{D} (\zeta_0(l_1),\ldots,\quad \zeta_0(l_i))_{\mathbf{P}^+}. \tag{34}$$

For $\mathbf{y} = (y_1,\ldots,y_i) \in \mathbf{R}_+^i$ set

$$H(\mathbf{l},\mathbf{y}) = \mathbf{E}^+\left(\exp\left(-\sum_{j=1}^{i}\lambda_j y_j \zeta_0\left(l_j\right)\right)\right).$$

It follows from (34) by the dominated convergence theorem that for each $\mathbf{y} = (y_1,\ldots,y_i) \in \mathbf{R}_+^i$

$$\lim_{n\to\infty} H_n(\mathbf{l},\mathbf{y}) = H(\mathbf{l},\mathbf{y}). \tag{35}$$

Note that $H(\mathbf{l},\mathbf{y})$ is continuous in $y_1,\ldots,y_i$ and, for each $n \in \mathbf{N}$, the function $H_n(\mathbf{l},\mathbf{y})$ is not increasing with respect to these variables. This means that the convergence in (35) is uniform on every compact subset of $\mathbf{R}_+^i$.

By Lemma 2.5 of [8], as $n \to \infty$,

$$\left\{\mathbf{Z}_{0,-i,0}\left\langle\mathbf{Q}_{i,1}\right\rangle,\mathbf{a}_i \mid M_n \quad < 0\right\} \xrightarrow{D} \left(\mathbf{Z}_{0,-i,0}\left\langle\mathbf{Q}_{i,1}\right\rangle,\mathbf{a}_i\right)_{\mathbf{P}^-}.$$

In view of (20), when the passage from the measure $\mathbf{P}$ to the measure $\mathbf{P}^-$, a branching process in the random environment $\mathbf{Q}_{i,1}$ transforms into a branching process in the same environment.

By formula (22) (see also (5) and (6))

$$\left(\mathbf{Q}_{i,1}\right)_{\mathbf{P}^-} \stackrel{D}{=} \left(Q_i^-, \dots, Q_1^-\right) = \left(Q_{-i+1}^*, \dots, Q_0^*\right),$$

$$\left(\mathbf{a}_i\right)_{\mathbf{P}^-} \stackrel{D}{=} \left(e^{-S_1^-}, \dots, e^{-S_i^-}\right) = \left(e^{S_{-1}^*}, \dots, e^{S_{-i}^*}\right) = \left(a_{-1,0}^*, \dots, a_{-i,0}^*\right),$$

therefore

$$\left(\mathbf{Z}_{0,-i,0}\left\langle \mathbf{Q}_{i,1}\right\rangle, \mathbf{a}_i\right)_{\mathbf{P}^-} \stackrel{D}{=} \left(\mathbf{Z}_{0,-i,0}^*, \mathbf{a}_{0,-i,0}^*\right),$$

where for $k \in \mathbf{Z}$ and $l \in \mathbf{N}_k$

$$\mathbf{Z}_{k,-i,l}^* = \left(Z_{k-1,l}^*, \dots, Z_{k-i,l}^*\right), \quad \mathbf{a}_{k,-i,l}^* = \left(a_{k-1,l}^*, \dots, a_{k-i,l}^*, a_{k,l}^*\right).$$

Thus, as $n \to \infty$,

$$\left\{ \boldsymbol{\psi}_i, \mathbf{a}_i \mid M_n < 0\right\} \xrightarrow{D} \left(\mathbf{Z}_{0,-i,0}^*, \mathbf{a}_{0,-i,0}^*\right). \tag{36}$$

By (35), (36) and Theorem 5.5 of [11] we see that for each fixed $\mathbf{l} \in \mathbf{N}_0^i$, as $n \to \infty$,

$$\left\{ H_{m(n)}\left(\mathbf{l}, \mathbf{a}_i\right) \mid M_n < 0\right\} \xrightarrow{D} H\left(\mathbf{l}, \mathbf{a}_{0,-i,0}^*\right) \tag{37}$$

if $m(n) \to +\infty$. Since $0 \le H_n(\mathbf{l}, \mathbf{y}) \le 1$ for $n \in \mathbf{N}$, $\mathbf{l} \in \mathbf{N}_0^i$ and $\mathbf{y} \in \mathbf{R}_+^i$, it follows from (37) by the dominated convergence theorem that

$$\lim_{n\to\infty} \mathbf{E}\left(H_{n-k}\left(\mathbf{l}, \mathbf{a}_i\right) \mid M_k < 0\right) = \mathbf{E}^* H\left(\mathbf{l}, \mathbf{a}_{0,-i,0}^*\right) \tag{38}$$

uniformly in $\lfloor \varepsilon n\rfloor < k \le \lfloor (1-\varepsilon) n\rfloor$.

Now we show that

$$\lim_{n\to\infty} \mathbf{E}\left(H_{n-k}\left(\boldsymbol{\psi}_i, \mathbf{a}_i\right) \mid M_k < 0\right) = \mathbf{E}^* H\left(\mathbf{Z}_{0,-i,0}^*, \mathbf{a}_{0,-i,0}^*\right) \tag{39}$$

uniformly in $\lfloor \varepsilon n\rfloor < k \le \lfloor (1-\varepsilon) n\rfloor$. For $N > 0$ we write

$$\mathbf{E}\left(H_{n-k}\left(\boldsymbol{\psi}_i, \mathbf{a}_i\right) \mid M_k < 0\right) = E_4(k,n,N) + E_5(k,n,N), \tag{40}$$

where

$$E_4(k,n,N) = \mathbf{E}\left(H_{n-k}\left(\boldsymbol{\psi}_i, \mathbf{a}_i\right) I_{\{|\boldsymbol{\psi}_i| \le N\}} \,\Big|\, M_k < 0\right),$$

$$E_5(k,n,N) = \mathbf{E}\left(H_{n-k}\left(\boldsymbol{\psi}_i, \mathbf{a}_i\right) I_{\{|\boldsymbol{\psi}_i| > N\}} \,\Big|\, M_k < 0\right)$$

(here $|\mathbf{a}|$ is the norm of $\mathbf{a} \in \mathbf{R}^i$). Since

$$E_5(k,n,N) \le \mathbf{P}\left(\left|\boldsymbol{\psi}_i\right| > N \mid M_k < 0\right),$$

it follows from (36) that

$$\lim_{N\to\infty} \limsup_{n\to\infty} E_5(k,n,N) = 0 \tag{41}$$

uniformly in $\lfloor \varepsilon n\rfloor < k \le \lfloor (1-\varepsilon) n\rfloor$. Clearly,

$$E_4(k,n,N) = \sum_{\mathbf{l}\in\mathbf{N}_0^i:\ |\mathbf{l}|\le N} \mathbf{E}\left(H_{n-k}\left(\mathbf{l}, \mathbf{a}_i\right) I_{\{\boldsymbol{\psi}_i = \mathbf{l}\}} \,\Big|\, M_k < 0\right). \tag{42}$$

We can deduce from (36) a stronger relation than (37): as $n \to \infty$,

$$\left\{ \boldsymbol{\psi}_i, \quad H_{m(n)}\left(\mathbf{l}, \mathbf{a}_i\right) \mid M_n < 0\right\} \xrightarrow{D} \left(\mathbf{Z}_{0,-i,0}^*, \quad H\left(\mathbf{l}, \mathbf{a}_{0,-i,0}^*\right)\right)$$

if $m(n) \to +\infty$. Therefore, by the dominated convergence theorem

$$\lim_{n\to\infty} \mathbf{E}\left(H_{n-k}(\mathbf{l}, \mathbf{a}_i) I_{\{\psi_i = \mathbf{l}\}} \;\middle|\; M_k \quad < 0 \right) = \mathbf{E}^* \left(H\left(\mathbf{l}, \mathbf{a}^*_{0,-i,0}\right); \quad \mathbf{Z}^*_{0,-i,0} = \mathbf{l} \right) \tag{43}$$

uniformly in $\lfloor \varepsilon n \rfloor < k \le \lfloor (1-\varepsilon) n \rfloor$. From (42) and (43) we find that

$$\lim_{N\to\infty} \lim_{n\to\infty} E_4(k, n, N) = \mathbf{E}^* H\left(\mathbf{Z}^*_{0,-i,0}, \mathbf{a}^*_{0,-i,0}\right) \tag{44}$$

uniformly in $\lfloor \varepsilon n \rfloor < k \le \lfloor (1-\varepsilon) n \rfloor$. From (40), (41) and (44) we obtain the desired relation (39).

It follows from (39) that

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} E_3(n, \varepsilon) = \mathbf{E}^* H\left(\mathbf{Z}^*_{0,-i,0}, \mathbf{a}^*_{0,-i,0}\right). \tag{45}$$

Now (30)–(32) and (45) imply

$$\lim_{n\to\infty} \mathbf{E}\left(\exp\left(-\sum_{j=1}^{i} \lambda_j a'_{-j,n} Z'_{-j,n} \right); \quad \tau_n - i \ge 0 \right) = \mathbf{E}^* H\left(\mathbf{Z}^*_{0,-i,0}, \mathbf{a}^*_{0,-i,0}\right).$$

Hence,

$$\lim_{n\to\infty} \mathbf{E} \exp\left(-\sum_{j=1}^{i} \lambda_j a'_{-j,n} Z'_{-j,n} \right) = \mathbf{E}^* H\left(\mathbf{Z}^*_{0,-i,0}, \mathbf{a}^*_{0,-i,0}\right). \tag{46}$$

We now analyze a branching process in the random environment $\{F_1^*, F_2^*, \ldots\}$ initiated at time 0 by a fixed set of particles which is divided into i pairwise non-intersecting subsets whose cardinalities are $l_1, \ldots, l_i \in \mathbf{N}_0$, respectively. For $n \in \mathbf{N}_0$, we denote $Z^*_{0,n}(l_j)$ the total number of particles at time n being descendants of the particles from the jth subset (we assume that $Z^*_{0,0}(l_j) = l_j$). Suppose that the random environment Π^* is fixed, then the distribution of the random vector $\mathbf{Z}^*_{0,-i,n}$ conditioned on the event $\{\mathbf{Z}^*_{0,-i,0} = \mathbf{l}\}$, where $\mathbf{l} \in \mathbf{N}_0^i$, coincides with the distribution of the random vector $\left(Z^*_{0,n}(l_1), \ldots, Z^*_{0,n}(l_i)\right)$. For each $n \in \mathbf{N}$, we introduce the function

$$V_n(\mathbf{l}, \mathbf{y}) = \mathbf{E}^* \exp\left(-\sum_{j=1}^{i} \lambda_j y_j a^*_{0,n} Z^*_{0,n}(l_j) \right)$$

for $\mathbf{l} = (l_1, \ldots, l_i) \in \mathbf{N}_0^i$ and $\mathbf{y} = (y_1, \ldots, y_i) \in \mathbf{R}_+^i$. Taking now into account properties of conditional expectation we find that

$$\mathbf{E}^* \exp\left(-\sum_{j=1}^{i} \lambda_j a^*_{-j,n} Z^*_{-j,n} \right) = \mathbf{E}^* V_n\left(\mathbf{Z}^*_{0,-i,0}, \mathbf{a}^*_{0,-i,0}\right). \tag{47}$$

Note that

$$\left(a^*_{0,n} Z^*_{0,n}(l_1), \ldots, a^*_{0,n} Z^*_{0,n}(l_i)\right) \stackrel{D}{=} \left(\widehat{Z}_{0,n}(l_1), \ldots, \widehat{Z}_{0,n}(l_i)\right)_{\mathbf{P}+}. \tag{48}$$

In view of (48)

$$V_n(\mathbf{l}, \mathbf{y}) = \mathbf{E}^+ \exp\left(-\sum_{j=1}^{i} \lambda_j y_j \widehat{Z}_{0,n}(l_j) \right). \tag{49}$$

Recall (see (8) and (33)) that $\mathbf{P}^*$-a.s.

$$\lim_{n\to\infty}\left(a^*_{-1,n}Z^*_{-1,n},\dots,a^*_{-i,n}Z^*_{-i,n}\right)=\left(\zeta^*_{-1},\dots,\zeta^*_{-i}\right) \tag{50}$$

and $\mathbf{P}^+$-a.s.

$$\lim_{n\to\infty}\left(\widehat{Z}_{0,n}(l_1),\dots,\widehat{Z}_{0,n}(l_i)\right)=\left(\zeta_0(l_1),\dots,\zeta_0(l_i)\right). \tag{51}$$

Using (49) and (51) and applying the dominated convergence theorem we see that for $\mathbf{l}\in\mathbf{N}_0^i$ and $\mathbf{y}\in\mathbf{R}_+^i$

$$\lim_{n\to\infty}V_n(\mathbf{l},\mathbf{y})=H(\mathbf{l},\mathbf{y}), \tag{52}$$

where, recall, $H(\mathbf{l},\mathbf{y})=\mathbf{E}^+\exp\left(-\sum_{j=1}^i\lambda_jy_j\zeta_0\left(l_j\right)\right)$. Applying the dominated convergence theorem again we obtain from (47), (50) and (52) that

$$\mathbf{E}^*\exp\left(-\sum_{j=1}^i\lambda_j\zeta^*_{-j}\right)=\mathbf{E}^*H\left(\mathbf{Z}^*_{0,-i,0},\mathbf{a}^*_{0,-i,0}\right). \tag{53}$$

It follows from (46) and (53) that for $\lambda_1,\dots,\lambda_i\in\mathbf{R}_+$

$$\lim_{n\to\infty}\mathbf{E}\exp\left(-\sum_{j=1}^i\lambda_ja'_{-j,n}Z'_{-j,n}\right)=\mathbf{E}^*\exp\left(-\sum_{j=1}^i\lambda_j\zeta^*_{-j}\right). \tag{54}$$

Clearly (54) implies (26).

It can be shown, by repeating the reasonings used to prove (25) and (26), that these two relations can be combined into one.

The lemma is proved.

Remark 3. It is not difficult to verify, using Remark 1, that Lemma 3 admits the following generalization: for any $a\le 0$ and $b>0$, as $n\to\infty$,

$$\left\{a'_{i,n}Z'_{i,n},\quad i\in\mathbf{Z}\;\middle|\;\frac{L_n}{C_n}\le a,\quad\frac{S_n-L_n}{C_n}\le b\right\}\overset{D}{\to}\left\{\zeta^*_i,\quad i\in\mathbf{Z}\right\}.$$

Lemma 4. *If the conditions of Theorem 1 are satisfied, then* $\mathbf{P}^*$*-a.s.*

$$\Sigma_1<+\infty,\quad \Sigma_2<+\infty.$$

Proof. It follows from the proof of Lemma 2.7 of [8] that, if the conditions of Theorem 1 are satisfied, then the series $\sum_{i=0}^\infty\mu_{i+1}e^{-S_i}$ converges $\mathbf{P}^+$-a.s. Hence, the series $\sum_{i=0}^\infty\mu^+_{i+1}e^{-S_i^+}$ converges $\mathbf{P}^*$-a.s. Similarly we can prove that the series $\sum_{i=1}^\infty\mu^-_ie^{S_i^-}$ converges $\mathbf{P}^*$-a.s. As a result, we obtain (see (6) and (7)) that the series $\sum_{i\in\mathbf{Z}}\mu^*_{i+1}e^{-S_i^*}$ converges $\mathbf{P}^*$-a.s. Thus, $\Sigma_1<+\infty$ $\mathbf{P}^*$-a.s. (see (9)).

Fix $i\in\mathbf{Z}$. If the random environment Π^* is fixed, the random sequence $\left\{a^*_{i,j}Z^*_{i,j},\ j\in\mathbf{N}_i\right\}$ is a martingale. Therefore

$$\mathbf{E}^*\left(a^*_{i,j}Z^*_{i,j}\;\middle|\;\Pi^*\right)=\mu^*_{i+1} \tag{55}$$

for $j\in\mathbf{N}_i$. By (8), (55) using Fatou's lemma we obtain that

$$\mathbf{E}^*\left(\zeta^*_i\;\middle|\;\Pi^*\right)\le\liminf_{j\to\infty}\mathbf{E}^*\left(a^*_{i,j}Z^*_{i,j}\;\middle|\;\Pi^*\right)=\mu^*_{i+1}$$

and, consequently,

$$\mathbf{E}^{*}\left(\zeta_{i}^{*}e^{-S_{i}^{*}}\;\middle|\;\Pi^{*}\right)=e^{-S_{i}^{*}}\mathbf{E}^{*}\left(\zeta_{i}^{*}\;\middle|\;\Pi^{*}\right)\leq\mu_{i+1}^{*}e^{-S_{i}^{*}}. \tag{56}$$

We have proved that the series $\sum_{i\in\mathbf{Z}}\mu_{i+1}^{*}e^{-S_{i}^{*}}$ converges $\mathbf{P}^{*}$-a.s. This fact combined with (56) implies convergence of the series $\sum_{i\in\mathbf{Z}}\mathbf{E}^{*}\left(\zeta_{i+1}^{*}e^{-S_{i}^{*}}\;\middle|\;\Pi^{*}\right)$ $\mathbf{P}^{*}$-a.s. Since the random variables $\zeta_{i}^{*}e^{-S_{i}^{*}}$ are non-negative, it follows that the series $\sum_{i\in\mathbf{Z}}\zeta_{i}^{*}e^{-S_{i}^{*}}$ converges a.s. if the random environment Π^{*} is fixed. Hence, $\Sigma_{2}<+\infty$ $\mathbf{P}^{*}$-a.s. (see (9)).

The lemma is proved.

Set

$$\Sigma_{1}^{(1)}=\sum_{i=0}^{\infty}\mu_{i+1}^{+}e^{-S_{i}^{+}}=\sum_{i\in\mathbf{N}_{0}}\mu_{i+1}^{*}e^{-S_{i}^{*}}, \tag{57}$$

$$\Sigma_{1}^{(2)}=\sum_{i=1}^{\infty}\mu_{i}^{-}e^{S_{i}^{-}}=\sum_{i\in\mathbf{Z}\setminus\mathbf{N}_{0}}\mu_{i+1}^{*}e^{-S_{i}^{*}}. \tag{58}$$

Clearly,

$$\Sigma_{1}=\Sigma_{1}^{(1)}+\Sigma_{1}^{(2)} \tag{59}$$

and by virtue of Lemma 4 $\mathbf{P}^{*}$-a.s.

$$\Sigma_{1}^{(1)}<+\infty,\quad \Sigma_{1}^{(2)}<+\infty. \tag{60}$$

Lemma 5. *If the conditions of Theorem 1 are satisfied, then $\mathbf{P}^{*}$-a.s., as $n\to\infty$,*

$$\left\{\sum_{i=0}^{n-1}\mu_{i+1}e^{-S_{i}}\;\middle|\;L_{n}\geq 0\right\}\xrightarrow{D}\Sigma_{1}^{(1)}, \tag{61}$$

$$\left\{\sum_{i=1}^{n-1}\mu_{i}e^{S_{i}}\;\middle|\;M_{n}<0\right\}\xrightarrow{D}\Sigma_{1}^{(2)}. \tag{62}$$

Proof. Let $f:\mathbf{R}\to\mathbf{R}$ be a bounded and continuous function. By virtue of (3) for fixed $k\in\mathbf{N}$

$$\left\{f\left(\sum_{i=0}^{k}\mu_{i+1}e^{-S_{i}}\right)\;\middle|\;L_{n}\geq 0\right\}\xrightarrow{D}f\left(\sum_{i=0}^{k}\mu_{i+1}^{+}e^{-S_{i}^{+}}\right)$$

as $n\to\infty$. Recalling (60) we conclude that

$$\lim_{k\to\infty}f\left(\sum_{i=0}^{k}\mu_{i+1}^{+}e^{-S_{i}^{+}}\right)=f\left(\Sigma_{1}^{(1)}\right)$$

$\mathbf{P}^{*}$-a.s. From these two facts, in view of Lemma 2.5 of [8], it follows that

$$\left\{f\left(\sum_{i=0}^{n-1}\mu_{i+1}e^{-S_{i}}\right)\;\middle|\;L_{n}\geq 0\right\}\xrightarrow{D}f\left(\Sigma_{1}^{(1)}\right).$$

Thus, relation (61) is true. Relation (62) can be proved by similar arguments.

The lemma is proved.

Remark 4. It is not difficult to verify that if we combine the left-hand sides of relations (3) and (61) (or (4) and (62)), then the respective statements concerning convergence in distribution of the four dimensional tuples of the random elements given $L_n \geq 0$ (or $M_n < 0$) are still in force.

Set for $n \in \mathbf{N}$

$$\mu'_{i,n} = \begin{cases} \mu_{\tau_n+i}, & i \in \mathbf{N}_{(-\tau_n)}, \\ 0, & i \in \mathbf{Z} \setminus \mathbf{N}_{(-\tau_n)}. \end{cases} \tag{63}$$

Let

$$\Sigma_1^{(1)}(n) = \sum_{j=0}^{n-1-\tau_n} \mu'_{j+1,n} e^{-S'_{j,n}}, \quad \Sigma_1^{(2)}(n) = \sum_{j=1}^{\tau_n} \mu'_{-j+1,n} e^{-S'_{-j,n}}. \tag{64}$$

Lemma 6. *If the conditions of Theorem 1 are satisfied, then* $\mathbf{P}^*$*-a.s., as* $n \to \infty$,

$$\left(\left\{\left(\mu'_{i,n}, S'_{i,n}\right), \ i \in \mathbf{Z}\right\}, \Sigma_1^{(1)}(n), \Sigma_1^{(2)}(n)\right) \xrightarrow{D} \left(\left\{\left(\mu_i^*, S_i^*\right), \ i \in \mathbf{Z}\right\}, \Sigma_1^{(1)}, \Sigma_1^{(2)}\right).$$

Proof. Fix $i \in \mathbf{N}$. Let $f : \mathbf{R}^{2i+1} \to \mathbf{R}$ be a bounded and continuous function. Similarly to (12), one can show that for $n \in \mathbf{N}_i$

$$\begin{aligned} &\mathbf{E}\left(f\left(\mu'_{1,n}, S'_{1,n}, \ldots, \mu'_{i,n}, S'_{i,n}, \Sigma_1^{(1)}(n)\right); \quad \tau_n + i \leq n\right) \\ &= \sum_{k=0}^{n-i} \mathbf{E}\left(f\left(\mu_1, S_1, \ldots, \mu_i, S_i, b_{n-k}\right) \mid L_{n-k} \geq 0\right) \mathbf{P}\left(\tau_n = k\right), \end{aligned} \tag{65}$$

where, recall, $b_n = \sum_{i=0}^{n-1} \mu_{i+1} e^{-S_i}$ for $n \in \mathbf{N}$. Repeating the arguments of Lemma 1 and using Lemma 5 and Remark 4, we can deduce from (65) (see also (6), (7), (57)) that

$$\begin{aligned} &\lim_{n\to\infty} \mathbf{E} f\left(\mu'_{1,n}, S'_{1,n}, \ldots, \mu'_{i,n}, S'_{i,n}, \Sigma_1^{(1)}(n)\right) \\ &= \mathbf{E}^* f\left(\mu_1^+, S_1^+, \ldots, \mu_i^+, S_i^+, \sum_{j\in\mathbf{N}_0} \mu_{j+1}^+ e^{-S_j^+}\right) \\ &= \mathbf{E}^* f\left(\mu_1^*, S_1^*, \ldots, \mu_i^*, S_i^*, \Sigma_1^{(1)}\right). \end{aligned}$$

Thus, as $n \to \infty$,

$$\left(\mu'_{1,n}, S'_{1,n}, \ldots, \mu'_{i,n}, S'_{i,n}, \Sigma_1^{(1)}(n)\right) \xrightarrow{D} \left(\mu_1^*, S_1^*, \ldots, \mu_i^*, S_i^*, \Sigma_1^{(1)}\right). \tag{66}$$

Fix $i \in \mathbf{N}_0$. Let $f : \mathbf{R}^{2i+3} \to \mathbf{R}$ be a bounded and continuous function. It is easy to show (see the proof of relation (18)) that for any $n \in \mathbf{N}_i$

$$\begin{aligned} &\mathbf{E}\left(f\left(\mu'_{0,n}, S'_{0,n}, \ldots, \mu'_{-i,n}, S'_{-i,n}, \Sigma_1^{(2)}(n)\right); \quad \tau_n - i \geq 0\right) \\ &= \sum_{k=i}^{n} \mathbf{E}\left(f\left(\mu_1, -S_0, \ldots, \mu_{i+1}, -S_i, \sum_{j=1}^{k} \mu_j e^{S_j}\right) \,\middle|\, M_k < 0\right) \mathbf{P}\left(\tau_n = k\right) \end{aligned}$$

and therefore (see Lemma 5, Remark 4 and (6), (7), (58)).

$$\begin{aligned}&\lim_{n\to\infty}\mathbf{E}f\left(\mu'_{0,n}, S'_{0,n}, \dots, \mu'_{-i,n}, S'_{-i,n}, \Sigma_1^{(2)}(n)\right)\\&=\mathbf{E}^*f\left(\mu_1^-, -S_0^-, \dots, \mu_{i+1}^-, -S_i^-, \sum_{j=1}^{\infty}\mu_j^- e^{S_j^-}\right)\\&=\mathbf{E}^*f\left(\mu_0^*, S_0^*, \dots, \mu_{-i}^*, S_{-i}^*, \Sigma_1^{(2)}\right).\end{aligned}$$

Thus, as $n\to\infty$,

$$\left(\mu'_{0,n}, S'_{0,n}, \dots, \mu'_{-i,n}, S'_{-i,n}, \Sigma_1^{(2)}(n)\right) \stackrel{D}{\to} \left(\mu_0^*, S_0^*, \dots, \mu_{-i}^*, S_{-i}^*, \Sigma_1^{(2)}\right). \tag{67}$$

Since the left-hand sides of (66) and (67) are asymptotically independent and their right-hand sides are independent, these two relations can be combined into one.

The lemma is proved.

Remark 5. It is not difficult to verify, using Remark 1, that statement of Lemma 6 admits the following generalization: for any $a\le 0$ and $b>0$, as $n\to\infty$,

$$\left(\left\{\left(\mu'_{i,n}, S'_{i,n}\right),\quad i\in\mathbf{Z}\right\}, \Sigma_1^{(1)}(n), \Sigma_1^{(2)}(n)\quad\middle|\quad \frac{L_n}{C_n}\le a,\quad \frac{S_n-L_n}{C_n}\le b\right)\stackrel{D}{\to}$$
$$\left(\left\{\left(\mu_i^*, S_i^*\right),\quad i\in\mathbf{Z}\right\}, \Sigma_1^{(1)}, \Sigma_1^{(2)}\right).$$

Lemma 7. *If the conditions of Theorem 1 are satisfied, then, as $n\to\infty$,*

$$\left\{\frac{b_n-b_{\tau_n+i}}{b_n},\quad i\in\mathbf{N}_0\right\}\stackrel{D}{\to}\left\{\frac{\sum_{j=i}^{\infty}\mu_{j+1}^+\exp\left(-S_j^+\right)}{\Sigma_1},\quad i\in\mathbf{N}_0\right\},$$

$$\left\{\frac{b_{\tau_n-i}}{b_n},\quad i\in\mathbf{N}\right\}\stackrel{D}{\to}\left\{\frac{\sum_{j=i+1}^{\infty}\mu_j^-\exp\left(S_j^-\right)}{\Sigma_1},\quad i\in\mathbf{N}\right\},$$

$$\left\{\frac{a_{\tau_n+i}}{b_n},\quad i\in\mathbf{N}_0\right\}\stackrel{D}{\to}\left\{\frac{\exp\left(-S_i^+\right)}{\Sigma_1},\quad i\in\mathbf{N}_0\right\},$$

$$\left\{\frac{a_{\tau_n-i}}{b_n},\quad i\in\mathbf{N}\right\}\stackrel{D}{\to}\left\{\frac{\exp\left(S_i^-\right)}{\Sigma_1},\quad i\in\mathbf{N}\right\}.$$

Proof. To simplify the presentation we check the first statement only. If $j\in\mathbf{N}_0$ and $\tau_n+j\le n$ then (see (10), (63) and (64))

$$\begin{aligned}\frac{b_{\tau_n+j}}{b_n}&=\frac{\sum_{k=0}^{\tau_n+j-1}\mu_{k+1}\exp\left(-S_k\right)}{\sum_{k=0}^{n-1}\mu_{k+1}\exp\left(-S_k\right)}=\frac{\sum_{k=0}^{\tau_n+j-1}\mu_{k+1}\exp\left(-\left(S_k-S_{\tau_n}\right)\right)}{\sum_{k=0}^{n-1}\mu_{k+1}\exp\left(-\left(S_k-S_{\tau_n}\right)\right)}\\&=\frac{\sum_{k=0}^{j-1}\mu'_{k+1,n}\exp\left(-S'_{k,n}\right)+\Sigma_1^{(2)}(n)}{\Sigma_1^{(1)}(n)+\Sigma_1^{(2)}(n)}.\end{aligned}$$

Since the last expression is a bounded continuous function of the random element mentioned in Lemma 6, it follows that

$$\left(\frac{b_{\tau_n}}{b_n}, \ldots, \frac{b_{\tau_n+i}}{b_n}\right) \xrightarrow{D} \frac{\left(\Sigma_1^{(2)}, \ldots, \Sigma_1^{(2)} + \sum_{j=0}^{i-1} \mu_{j+1}^{+} \exp\left(-S_j^{+}\right)\right)}{\Sigma_1^{(1)} + \Sigma_1^{(2)}}$$

as $n \to \infty$. Whence, taking into account (59) we obtain the required relation. The remaining three statements may be proved by similar arguments.

The lemma is proved.

Remark 6. The random element constructed from the left-hand sides of all the relations included in Lemmas 3 and 7 converges in distribution to the random element constructed from the respective right-hand sides of the relations included in these lemmas. Moreover (see Remarks 3 and 5), the random element constructed from the left-hand sides is asymptotically independent, as $n \to \infty$, of the random event

$$\left\{C_n^{-1} L_n \leq a, \quad C_n^{-1}\left(S_n - L_n\right) \leq b\right\}$$

for any $a \leq 0$ and $b > 0$.

Remark 7. Lemma 7 (for $i = 0$) implies the following statement: if the conditions of Theorem 1 are satisfied, then, as $n \to \infty$,

$$\frac{a_{\tau_n}}{b_n} = \frac{\exp\left(-L_n\right)}{b_n} \xrightarrow{D} \frac{1}{\Sigma_1}.$$

Note that this statement, first, substantially generalizes the main result of [15] and, secondly, shows that one may choose for scaling of $Z_{\lfloor nt \rfloor}$ in Theorem 1 a more simple coefficient $\exp\left(-\left(S_{\lfloor nt \rfloor} - L_{\lfloor nt \rfloor}\right)\right)$ instead of $a_{\lfloor nt \rfloor}/b_{\lfloor nt \rfloor}$. In view of (1) this leads to the hypothesis that, as $n \to \infty$,

$$\left\{C_n^{-1} \ln\left(Z_{\lfloor nt \rfloor} + 1\right), \quad t \geq 0\right\} \xrightarrow{D} \left\{W(t) - L(t), \quad t \geq 0\right\}.$$

A particular case of this result is established by Theorem 2 of [3].

3. Proof of the main result

First part. We establish convergence of one-dimensional distributions: if $t > 0$, then, as $n \to \infty$,

$$\frac{a_{\lfloor nt \rfloor}}{b_{\lfloor nt \rfloor}} Z_{\lfloor nt \rfloor} \xrightarrow{D} \frac{\Sigma_2}{\Sigma_1}. \tag{68}$$

Set for $r \in \mathbf{N}$

$$U_r^{(i)} = \sum_{j=\tau_r - i}^{\tau_r + i - 1} Z_{j,r},$$

$$V_r^{(i)} = \sum_{j=0}^{\tau_r - i - 1} Z_{j,r} + \sum_{j=\tau_r + i}^{r-1} Z_{j,r}.$$

It is clear that for $i \in \mathbf{N}$

$$Z_{\lfloor nt \rfloor} = \sum_{j=0}^{\lfloor nt \rfloor - 1} Z_{j,\lfloor nt \rfloor} = U_{\lfloor nt \rfloor}^{(i)} + V_{\lfloor nt \rfloor}^{(i)}. \tag{69}$$

Note that

$$\mathbf{E}\left(a_{j,\lfloor nt\rfloor} Z_{j,\lfloor nt\rfloor} \;\middle|\; Q_{1,\lfloor nt\rfloor}\right) = \mu_{j+1}, \tag{70}$$

if $0 \le j < \lfloor nt\rfloor$. Observing that $a_{\lfloor nt\rfloor} = a_j a_{j,\lfloor nt\rfloor}$ for $0 \le j < \lfloor nt\rfloor$ we obtain by (70) that

$$\begin{aligned}
&\mathbf{E}\left(\frac{a_{\lfloor nt\rfloor}}{b_{\lfloor nt\rfloor}} V^{(i)}_{\lfloor nt\rfloor}\right)\\
&= \mathbf{E} b^{-1}_{\lfloor nt\rfloor}\left(\sum_{j=0}^{\tau_{\lfloor nt\rfloor}-i-1} a_j a_{j,\lfloor nt\rfloor} Z_{j,\lfloor nt\rfloor} + \sum_{j=\tau_{\lfloor nt\rfloor}+i}^{\lfloor nt\rfloor-1} a_j a_{j,\lfloor nt\rfloor} Z_{j,\lfloor nt\rfloor}\right)\\
&= \mathbf{E} b^{-1}_{\lfloor nt\rfloor}\left(\sum_{j=0}^{\tau_{\lfloor nt\rfloor}-i-1} \mu_{j+1} a_j + \sum_{j=\tau_{\lfloor nt\rfloor}+i}^{\lfloor nt\rfloor-1} \mu_{j+1} a_j\right)\\
&= \mathbf{E}\frac{b_{\tau_{\lfloor nt\rfloor}-i} + \left(b_{\lfloor nt\rfloor} - b_{\tau_{\lfloor nt\rfloor}+i}\right)}{b_{\lfloor nt\rfloor}}.
\end{aligned} \tag{71}$$

Applying Lemma 7 to the right-hand side of (71), we conclude that

$$\lim_{n\to\infty} \mathbf{E}\left(\frac{a_{\lfloor nt\rfloor}}{b_{\lfloor nt\rfloor}} V^{(i)}_{\lfloor nt\rfloor}\right) = \mathbf{E}\frac{\sum_{j=i}^{\infty} \mu^{+}_{j+1} \exp\left(-S^{+}_{j}\right) + \sum_{j=i+1}^{\infty} \mu^{-}_{j} \exp\left(S^{-}_{j}\right)}{\Sigma_1}$$

and, therefore (see (60)),

$$\lim_{i\to\infty}\lim_{n\to\infty} \mathbf{E}\left(\frac{a_{\lfloor nt\rfloor}}{b_{\lfloor nt\rfloor}} V^{(i)}_{\lfloor nt\rfloor}\right) = 0. \tag{72}$$

By Markov inequality for any $\varepsilon > 0$

$$\mathbf{P}\left(\frac{a_{\lfloor nt\rfloor}}{b_{\lfloor nt\rfloor}} V^{(i)}_{\lfloor nt\rfloor} \ge \varepsilon\right) \le \varepsilon^{-1}\mathbf{E}\left(\frac{a_{\lfloor nt\rfloor}}{b_{\lfloor nt\rfloor}} V^{(i)}_{\lfloor nt\rfloor}\right).$$

Hence, taking into account (72) we obtain that

$$\lim_{i\to\infty}\lim_{n\to\infty} \mathbf{P}\left(\frac{a_{\lfloor nt\rfloor}}{b_{\lfloor nt\rfloor}} V^{(i)}_{\lfloor nt\rfloor} \ge \varepsilon\right) = 0. \tag{73}$$

Observe that we may assume in the sequel that $i \le \tau_{\lfloor nt\rfloor} < \lfloor nt\rfloor - i$ (see the proof of Lemma 1). Note that (see (23), (24))

$$U^{(i)}_{\lfloor nt\rfloor} = \sum_{j=-i}^{i-1} Z_{\tau_{\lfloor nt\rfloor}+j,\lfloor nt\rfloor} = \sum_{j=-i}^{i-1} Z'_{j,\lfloor nt\rfloor}$$

and, therefore,

$$\frac{a_{\lfloor nt\rfloor}}{b_{\lfloor nt\rfloor}} U^{(i)}_{\lfloor nt\rfloor} = \sum_{j=-i}^{i-1} \frac{a_{\tau_{\lfloor nt\rfloor}+j}}{b_{\lfloor nt\rfloor}} a'_{j,\lfloor nt\rfloor} Z'_{j,\lfloor nt\rfloor}. \tag{74}$$

Applying Lemmas 3, 7 and the first part of Remark 6 to (74), we obtain that, as $n \to \infty$,

$$\frac{a_{\lfloor nt\rfloor}}{b_{\lfloor nt\rfloor}} U^{(i)}_{\lfloor nt\rfloor} \xrightarrow{D} \frac{1}{\Sigma_1} \sum_{j=-i}^{i-1} \zeta^{*}_{j} e^{-S^{*}_{j}}. \tag{75}$$

Hence, for any fixed $i \in \mathbf{N}$ and for all but a countable set of $x \geq 0$

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{a_{\lfloor nt\rfloor}}{b_{\lfloor nt\rfloor}} U^{(i)}_{\lfloor nt\rfloor} \leq x\right) = \mathbf{P}\left(\frac{1}{\Sigma_1}\sum_{j=-i}^{i-1} \zeta_j^* e^{-S_j^*} \leq x\right). \tag{76}$$

In view of Lemma 4 (see (9))

$$\lim_{i\to\infty} \mathbf{P}\left(\frac{1}{\Sigma_1}\sum_{j=-i}^{i-1} \zeta_i^* e^{-S_i^*} \leq x\right) = \mathbf{P}\left(\frac{\Sigma_2}{\Sigma_1} \leq x\right). \tag{77}$$

We obtain by (76) and (77) that

$$\lim_{i\to\infty}\lim_{n\to\infty} \mathbf{P}\left(\frac{a_{\lfloor nt\rfloor}}{b_{\lfloor nt\rfloor}} U^{(i)}_{\lfloor nt\rfloor} \leq x\right) = \mathbf{P}\left(\frac{\Sigma_2}{\Sigma_1} \leq x\right). \tag{78}$$

It follows from (69), (73) and (78) that for all but a countable set of $x \geq 0$

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{a_{\lfloor nt\rfloor}}{b_{\lfloor nt\rfloor}} Z_{\lfloor nt\rfloor} \leq x\right) = \mathbf{P}\left(\frac{\Sigma_2}{\Sigma_1} \leq x\right).$$

This proves (68).

Remark 8. It is not difficult to verify (see Remark 6) that relation (75) admits the following generalization: for any $a \leq 0$ and $b > 0$, as $n \to \infty$,

$$\left\{\frac{a_{\lfloor nt\rfloor}}{b_{\lfloor nt\rfloor}} U^{(i)}_{\lfloor nt\rfloor} \;\middle|\; \frac{L_{\lfloor nt\rfloor}}{C_n} \leq a, \quad \frac{S_{\lfloor nt\rfloor} - L_{\lfloor nt\rfloor}}{C_n} \leq b\right\} \xrightarrow{D} \frac{1}{\Sigma_1}\sum_{j=-i}^{i+1} \zeta_j^* e^{-S_j^*}.$$

Second part. Now we establish convergence of two-dimensional distributions. Select $0 < t_1 < t_2$, fix an $\varepsilon > 0$ and introduce the following random events:

$$A_{n,\varepsilon} = \left\{L_{\lfloor nt_1\rfloor} > L_{\lfloor nt_1\rfloor,\lfloor nt_2\rfloor} + \varepsilon C_n\right\},$$
$$B_{n,\varepsilon} = \left\{L_{\lfloor nt_1\rfloor} < L_{\lfloor nt_1\rfloor,\lfloor nt_2\rfloor} - \varepsilon C_n\right\},$$
$$D_{n,\varepsilon} = \left\{\left|L_{\lfloor nt_1\rfloor} - L_{\lfloor nt_1\rfloor,\lfloor nt_2\rfloor}\right| \leq \varepsilon C_n\right\}.$$

We show that, as $n \to \infty$,

$$\left\{\frac{a_{\lfloor nt_1\rfloor}}{b_{\lfloor nt_1\rfloor}} Z_{\lfloor nt_1\rfloor}, \frac{a_{\lfloor nt_2\rfloor}}{b_{\lfloor nt_2\rfloor}} Z_{\lfloor nt_2\rfloor} \;\middle|\; A_{n,\varepsilon}\right\} \xrightarrow{D} (\gamma_1, \gamma_2), \tag{79}$$

$$\left\{\frac{a_{\lfloor nt_1\rfloor}}{b_{\lfloor nt_1\rfloor}} Z_{\lfloor nt_1\rfloor}, \frac{a_{\lfloor nt_2\rfloor}}{b_{\lfloor nt_2\rfloor}} Z_{\lfloor nt_2\rfloor} \;\middle|\; B_{n,\varepsilon}\right\} \xrightarrow{D} (\gamma_1, \gamma_1), \tag{80}$$

where γ_1, γ_2 are independent random variables and $\gamma_1 \stackrel{d}{=} \gamma_2 \stackrel{d}{=} \Sigma_2/\Sigma_1$.

First we establish (79). With this aim we prove that, for any fixed $i \in \mathbf{N}$ and for all but a countable set of $(x_1, x_2) \in \mathbf{R}^2_+$,

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{a_{\lfloor nt_1\rfloor}}{b_{\lfloor nt_1\rfloor}} U^{(i)}_{\lfloor nt_1\rfloor} \leq x_1, \quad \frac{a_{\lfloor nt_2\rfloor}}{b_{\lfloor nt_2\rfloor}} U^{(i)}_{\lfloor nt_2\rfloor} \leq x_2 \;\middle|\; A_{n,\varepsilon}\right)$$
$$= \mathbf{P}\left(\frac{1}{\Sigma_1}\sum_{j=-i}^{i+1} \zeta_j^* e^{-S_j^*} \leq x_1\right)\mathbf{P}\left(\frac{1}{\Sigma_1}\sum_{j=-i}^{i+1} \zeta_j^* e^{-S_j^*} \leq x_2\right). \tag{81}$$

By virtue of Remark 7, as $n \to \infty$,

$$\frac{b_{\lfloor nt_1 \rfloor}}{\exp\left(-L_{\lfloor nt_1 \rfloor}\right)} \xrightarrow{D} \chi_1, \tag{82}$$

$$\frac{b_{\lfloor nt_2 \rfloor} - b_{\lfloor nt_1 \rfloor}}{\exp\left(-\left(L_{\lfloor nt_1 \rfloor, \lfloor nt_2 \rfloor}\right)\right)} \stackrel{D}{=} \frac{b_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}}{\exp\left(-\left(L_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}\right)\right)} \xrightarrow{D} \chi_2, \tag{83}$$

where χ_1 and χ_2 are positive identically distributed random variables. The left-hand sides of relations (82) and (83) are independent. Set $L(t_1, t_2) = \inf_{t \in [t_1, t_2]} W(t)$. Taking into account the estimate (see (1))

$$\lim_{n \to \infty} \mathbf{P}\left(A_{n,\varepsilon}\right) = \mathbf{P}\left(L(t_1) > L(t_1, t_2) + \varepsilon\right) > 0,$$

we obtain by (82) and (83) that, as $n \to \infty$,

$$\left\{ \frac{b_{\lfloor nt_1 \rfloor}}{b_{\lfloor nt_2 \rfloor} - b_{\lfloor nt_1 \rfloor}} \;\middle|\; A_{n,\varepsilon} \right\} \xrightarrow{D} 0. \tag{84}$$

Therefore, if the event $A_{n,\varepsilon}$ occurred, we can replace the coefficient $a_{\lfloor nt_2 \rfloor}/b_{\lfloor nt_2 \rfloor}$ for $U^{(i)}_{\lfloor nt_2 \rfloor}$ at the left-hand part of (81) by the coefficient

$$\frac{a_{\lfloor nt_2 \rfloor}}{b_{\lfloor nt_2 \rfloor} - b_{\lfloor nt_1 \rfloor}} = \frac{\exp\left(-\left(S_{\lfloor nt_2 \rfloor} - S_{\lfloor nt_1 \rfloor}\right)\right)}{\sum_{i=\lfloor nt_1 \rfloor}^{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor} \mu_{i+1} \exp\left(-\left(S_i - S_{\lfloor nt_1 \rfloor}\right)\right)} = \frac{\widetilde{a}_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}}{\widetilde{b}_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}},$$

where the values $\widetilde{a}_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}$ and $\widetilde{b}_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}$ are constructed from the random environment $\widetilde{Q}_i := Q_{\lfloor nt_1 \rfloor + i}$, $i = 1, \ldots, \lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor$, just as the values $a_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}$ and $b_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}$ are constructed from the random environment $Q_{1, \lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}$.

Further, given $A_{n,\varepsilon}$, the inequality $\tau_{\lfloor nt_2 \rfloor} > \tau_{\lfloor nt_1 \rfloor}$ is true. We may assume that $\lfloor nt_2 \rfloor - i > \tau_{\lfloor nt_2 \rfloor} > \tau_{\lfloor nt_1 \rfloor} + i$. Thus, if the random environment $\{Q_n, \quad n \in \mathbf{N}\}$ is fixed, the distribution of the random variable $U^{(i)}_{\lfloor nt_1 \rfloor}$ is completely determined by the random environment $Q_{1, \lfloor nt_1 \rfloor}$ and the distribution of the random variable $U^{(i)}_{\lfloor nt_2 \rfloor}$ is completely determined by the random environment $Q_{\lfloor nt_1 \rfloor + 1, \lfloor nt_2 \rfloor}$. Moreover, $U^{(i)}_{\lfloor nt_2 \rfloor} = \widetilde{U}^{(i)}_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}$, where $\widetilde{U}^{(i)}_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}$ has the same meaning for the environment $\widetilde{Q}_i$, $i = 1, \ldots, \lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor$, as $U^{(i)}_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}$ has for the environment $Q_{1, \lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}$.

Summarizing the arguments above, we see that to prove (81) it is sufficient to show that

$$\begin{aligned} &\lim_{n \to \infty} \mathbf{P}\left(\frac{a_{\lfloor nt_1 \rfloor}}{b_{\lfloor nt_1 \rfloor}} U^{(i)}_{\lfloor nt_1 \rfloor} \le x_1, \quad \frac{\widetilde{a}_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}}{\widetilde{b}_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}} \widetilde{U}^{(i)}_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor} \le x_2 \;\middle|\; A_{n,\varepsilon} \right) \\ &= \mathbf{P}\left(\frac{1}{\Sigma_1} \sum_{j=-i}^{i+1} \zeta_j^* e^{-S_j^*} \le x_1 \right) \mathbf{P}\left(\frac{1}{\Sigma_1} \sum_{j=-i}^{i+1} \zeta_j^* e^{-S_j^*} \le x_2 \right). \end{aligned} \tag{85}$$

Note that

$$\begin{aligned} &\mathbf{P}\left(\frac{a_{\lfloor nt_1 \rfloor}}{b_{\lfloor nt_1 \rfloor}} U^{(i)}_{\lfloor nt_1 \rfloor} \le x_1, \quad \frac{\widetilde{a}_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}}{\widetilde{b}_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}} \widetilde{U}^{(i)}_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor} \le x_2, \quad A_{n,\varepsilon} \right) \\ &= \int_{-\infty}^{0} \int_{0}^{+\infty} \mathbf{P}\left(\frac{a_{\lfloor nt_1 \rfloor}}{b_{\lfloor nt_1 \rfloor}} U^{(i)}_{\lfloor nt_1 \rfloor} \le x_1, \quad \frac{L_{\lfloor nt_1 \rfloor}}{C_n} \in da, \quad \frac{S_{\lfloor nt_1 \rfloor} - L_{\lfloor nt_1 \rfloor}}{C_n} \in db \right) \\ &\times \mathbf{P}\left(\frac{a_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}}{b_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}} U^{(i)}_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor} \le x_2, \quad \frac{L_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}}{C_n} < b - a - \varepsilon \right). \end{aligned}$$

Hence, taking into account Remark 8 we deduce that, as $n \to \infty$,

$$\begin{aligned}
&\mathbf{P}\left(\frac{a_{\lfloor nt_1\rfloor}}{b_{\lfloor nt_1\rfloor}}U^{(i)}_{\lfloor nt_1\rfloor} \le x_1, \quad \frac{\widetilde{a}_{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}}{\widetilde{b}_{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}}U^{(i)}_{\lfloor nt_2\rfloor} \le x_2, \quad A_{n,\varepsilon}\right)\\
&\sim \mathbf{P}\left(\frac{1}{\Sigma_1}\sum_{j=-i}^{i+1}\zeta_j^* e^{-S_j^*} \le x_1\right)\mathbf{P}\left(\frac{1}{\Sigma_1}\sum_{j=-i}^{i+1}\zeta_j^* e^{-S_j^*} \le x_2\right)\\
&\times \int_{-\infty}^{0}\int_{0}^{+\infty}\mathbf{P}\left(\frac{L_{\lfloor nt_1\rfloor}}{C_n}\in da, \quad \frac{S_{\lfloor nt_1\rfloor}-L_{\lfloor nt_1\rfloor}}{C_n}\in db\right)\\
&\times \mathbf{P}\left(\frac{L_{\lfloor nt_2\rfloor-\lfloor nt_1\rfloor}}{C_n} < b-a-\varepsilon\right).
\end{aligned}$$

Since the last integral is equal to $\mathbf{P}\left(A_{n,\varepsilon}\right)$, we obtain (85) and, as a result, the required relation (81).

It follows from (81) that (see (78))

$$\begin{aligned}
&\lim_{i\to\infty}\lim_{n\to\infty}\mathbf{P}\left(\frac{a_{\lfloor nt_1\rfloor}}{b_{\lfloor nt_1\rfloor}}U^{(i)}_{\lfloor nt_1\rfloor} \le x_1, \quad \frac{b_{\lfloor nt_2\rfloor}}{a_{\lfloor nt_2\rfloor}}U^{(i)}_{\lfloor nt_2\rfloor} \le x_2 \;\middle|\; A_{n,\varepsilon}\right)\\
&= \mathbf{P}\left(\frac{\Sigma_2}{\Sigma_1}\le x_1\right)\mathbf{P}\left(\frac{\Sigma_2}{\Sigma_1}\le x_2\right).
\end{aligned} \tag{86}$$

Applying now the same arguments which we have used in First part of the proof to establish (68) from (78), we obtain (79) from (86).

We now prove (80). To this aim we check that, for any fixed $i \in \mathbf{N}$ and for all but a countable set of $(x_1, x_2) \in \mathbf{R}_+^2$,

$$\begin{aligned}
&\lim_{n\to\infty}\mathbf{P}\left(\frac{a_{\lfloor nt_1\rfloor}}{b_{\lfloor nt_1\rfloor}}U^{(i)}_{\lfloor nt_1\rfloor} \le x_1, \quad \frac{a_{\lfloor nt_2\rfloor}}{b_{\lfloor nt_k\rfloor}}U^{(i)}_{\lfloor nt_2\rfloor} \le x_2 \;\middle|\; B_{n,\varepsilon}\right)\\
&= \mathbf{P}\left(\frac{1}{\Sigma_1}\sum_{j=-i}^{i+1}\zeta_j^* e^{-S_j^*} \le \min\left(x_1, x_2\right)\right).
\end{aligned} \tag{87}$$

Set

$$\begin{aligned}
Z'_{i,n}(m) &= Z_{\tau_n+i,m},\\
U^{(i)}_n(m) &= \sum_{j=-i}^{i+1} Z'_{j,n}(m).
\end{aligned}$$

Similarly to relation (84), it can be shown that, as $n \to \infty$,

$$\left\{\frac{b_{\lfloor nt_2\rfloor}-b_{\lfloor nt_1\rfloor}}{b_{\lfloor nt_1\rfloor}} \;\middle|\; B_{n,\varepsilon}\right\} \xrightarrow{D} 0.$$

Therefore, if the event $B_{n,\varepsilon}$ occurred, we can replace the coefficient $a_{\lfloor nt_2\rfloor}/b_{\lfloor nt_2\rfloor}$ for $U^{(i)}_{\lfloor nt_2\rfloor}$ at the left-hand part of (87) by the coefficient $a_{\lfloor nt_2\rfloor}/b_{\lfloor nt_1\rfloor}$. Given that the random event $B_{n,\varepsilon}$ occurred, $\tau_{\lfloor nt_2\rfloor} = \tau_{\lfloor nt_1\rfloor}$, and hence,

$$U^{(i)}_{\lfloor nt_1\rfloor} = \sum_{j=-i}^{i+1} Z'_{i,\lfloor nt_1\rfloor}\left(\lfloor nt_1\rfloor\right) = U^{(i)}_{\lfloor nt_1\rfloor}\left(\lfloor nt_1\rfloor\right),$$

$$U_{\lfloor nt_2\rfloor}^{(i)}=\sum_{j=-i}^{i+1}Z'_{i,\lfloor nt_1\rfloor}\left(\lfloor nt_2\rfloor\right)=U_{\lfloor nt_1\rfloor}^{(i)}\left(\lfloor nt_2\rfloor\right).$$

Thus, to prove (87) it is sufficient to show that

$$\lim_{n\to\infty}\mathbf{P}\left(\frac{a_{\lfloor nt_1\rfloor}}{b_{\lfloor nt_1\rfloor}}U_{\lfloor nt_1\rfloor}^{(i)}\left(\lfloor nt_1\rfloor\right)\le x_1,\quad \frac{a_{\lfloor nt_2\rfloor}}{b_{\lfloor nt_1\rfloor}}U_{\lfloor nt_1\rfloor}^{(i)}\left(\lfloor nt_2\rfloor\right)\le x_2\;\middle|\; B_{n,\varepsilon}\right)$$
$$=\mathbf{P}\left(\frac{1}{\Sigma_1}\sum_{j=-i}^{i+1}\zeta_j^{*}e^{-S_j^{*}}\le\min\left(x_1,x_2\right)\right).\tag{88}$$

Applying the arguments similar to those used to establish relation (26), we can show that

$$\left\{a'_{i,m}Z'_{i,n}\left(m\right),\quad i\in\mathbf{Z}\right\}\xrightarrow{D}\left\{\zeta_i^{*},\quad i\in\mathbf{Z}\right\},$$

as $m\ge n\to\infty$. Moreover,

$$\left\{\left(a'_{i,n}Z'_{i,n}\left(n\right),a'_{i,m}Z'_{i,n}\left(m\right)\right),\quad i\in\mathbf{Z}\right\}\xrightarrow{D}\left\{\left(\zeta_i^{*},\zeta_i^{*}\right),\quad i\in\mathbf{Z}\right\}\tag{89}$$

and (see Remark 3) the left-hand side of this relation is asymptotically independent from the random event $\left\{C_n^{-1}L_n\le a,\quad C_n^{-1}\left(S_n-L_n\right)\le b\right\}$ for any $a\le 0$ and $b>0$. It follows from (89) that (see the proof of (75))

$$\left(\frac{a_n}{b_n}U_n^{(i)}\left(n\right),\frac{a_m}{b_n}U_n^{(i)}\left(m\right)\right)\xrightarrow{D}\frac{1}{\Sigma_1}\left(\sum_{j=-i}^{i+1}\zeta_j^{*}e^{-S_j^{*}},\sum_{j=-i}^{i+1}\zeta_j^{*}e^{-S_j^{*}}\right),\tag{90}$$

as $m\ge n\to\infty$. From (90) we obtain the desired relation (88) and, as a result, (87). Now statement (80) follows from (87) in a standard way.

Finally, according to (1)

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\mathbf{P}\left(A_{n,\varepsilon}\right)=\mathbf{P}\left(L\left(t_1\right)>L\left(t_1,t_2\right)\right)=\mathbf{P}\left(L\left(t_1\right)>L\left(t_2\right)\right),\tag{91}$$

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\mathbf{P}\left(B_{n,\varepsilon}\right)=\mathbf{P}\left(L\left(t_1\right)<L\left(t_1,t_2\right)\right)=\mathbf{P}\left(L\left(t_1\right)=L\left(t_2\right)\right)\tag{92}$$

and

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\mathbf{P}\left(D_{n,\varepsilon}\right)=0.\tag{93}$$

By the total probability formula

$$\mathbf{P}\left(\frac{a_{\lfloor nt_1\rfloor}}{b_{\lfloor nt_1\rfloor}}Z_{\lfloor nt_1\rfloor}\le x_1,\quad \frac{a_{\lfloor nt_2\rfloor}}{b_{\lfloor nt_2\rfloor}}Z_{\lfloor nt_2\rfloor}\le x_2\right)$$
$$=\mathbf{P}\left(\frac{a_{\lfloor nt_1\rfloor}}{b_{\lfloor nt_1\rfloor}}Z_{\lfloor nt_1\rfloor}\le x_1,\quad \frac{a_{\lfloor nt_2\rfloor}}{b_{\lfloor nt_2\rfloor}}Z_{\lfloor nt_2\rfloor}\le x_2\;\middle|\; A_{n,\varepsilon}\right)\mathbf{P}\left(A_{n,\varepsilon}\right)$$
$$+\mathbf{P}\left(\frac{a_{\lfloor nt_1\rfloor}}{b_{\lfloor nt_1\rfloor}}Z_{\lfloor nt_1\rfloor}\le x_1,\quad \frac{a_{\lfloor nt_2\rfloor}}{b_{\lfloor nt_2\rfloor}}Z_{\lfloor nt_2\rfloor}\le x_2\;\middle|\; B_{n,\varepsilon}\right)\mathbf{P}\left(B_{n,\varepsilon}\right)$$
$$+\mathbf{P}\left(\frac{a_{\lfloor nt_1\rfloor}}{b_{\lfloor nt_1\rfloor}}Z_{\lfloor nt_1\rfloor}\le x_1,\quad \frac{a_{\lfloor nt_2\rfloor}}{b_{\lfloor nt_2\rfloor}}Z_{\lfloor nt_2\rfloor}\le x_2\;\middle|\; D_{n,\varepsilon}\right)\mathbf{P}\left(D_{n,\varepsilon}\right),\tag{94}$$

where $(x_1, x_2) \in \mathbf{R}_+^2$. Combining (79), (80) and (91)–(94) we deduce that

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{a_{\lfloor nt_1 \rfloor}}{b_{\lfloor nt_1 \rfloor}} Z_{\lfloor nt_1 \rfloor} \le x_1, \quad \frac{a_{\lfloor nt_2 \rfloor}}{b_{\lfloor nt_2 \rfloor}} Z_{\lfloor nt_2 \rfloor} \le x_2\right)$$
$$= \mathbf{P}(\gamma_1 \le x_1, \quad \gamma_2 \le x_2)\, \mathbf{P}(L(t_1) > L(t_2))$$
$$+ \mathbf{P}(\gamma_1 \le x_1, \quad \gamma_1 \le x_2)\, \mathbf{P}(L(t_1) = L(t_2))$$

for all but a countable set of $(x_1, x_2) \in \mathbf{R}_+^2$. This gives the desired convergence of two-dimensional distributions.

Third part. The proof of convergence of multidimensional distributions (for dimensions exceeding two) is carried out by induction using the reasonings of Second part of the proof.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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