stochastic processes and their applications

Stochastic Processes and their Applications 78 (1998) 47–68

# Hamiltonians on random walk trajectories

Pablo A. Ferrari [a,*], Servet Martínez [b]

[a] *Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, 05315-970 São Paulo, Brazil*

[b] *Departamento de Ingeniería Matemática, Universidad de Chile, Facultad de Ciencias Físicas y Matemáticas, Casilla 170-3, Correo 3, Santiago, Chile*

Received 9 September 1997; received in revised form 12 June 1998; accepted 16 June 1998

**Abstract**

We consider Gibbs measures on the set of paths of nearest-neighbors random walks on $\mathbb{Z}_+$. The basic measure is the uniform measure on the set of paths of the simple random walk on $\mathbb{Z}_+$ and the Hamiltonian awards each visit to site $x \in \mathbb{Z}_+$ by an amount $\alpha_x \in \mathbb{R}$, $x \in \mathbb{Z}_+$. We give conditions on $(\alpha_x)$ that guarantee the existence of the (infinite volume) Gibbs measure. When comparing the measures in $\mathbb{Z}_+$ with the corresponding measures in $\mathbb{Z}$, the so-called entropic repulsion appears as a counting effect. 

*AMS classification*: 82B; 82C; 60K35

*Keywords:* Solid on solid models; Entropic repulsion; Pinning surfaces; Interface; Random walks

## 0. Introduction

We consider trajectories $\sigma = (\sigma_i)_{i\in\mathbb{Z}}$ of nearest-neighbors random walks on $\mathbb{Z}_+^{\mathbb{Z}}$, so that $\sigma_i \in \mathbb{Z}_+$ and $|\sigma_i - \sigma_{i-1}| = 1$. Let $\mathcal{N}_x(\sigma) = \sum_i \delta(\sigma_i, x)$, the number of times that the path $\sigma$ visits $x$, where we put $\delta(y,z) = 1$ if $y = z$ and zero otherwise. Let $\alpha_x$ be given real numbers, and $H$ be the Hamiltonian

$$H(\sigma) = \sum_x \alpha_x \mathcal{N}_x(\sigma) = \sum_i \alpha(\sigma_i).$$

(Of course the sum is well defined only if $i$ runs on a finite set.) We also study random walks on $\mathbb{Z}$ with symmetric interactions under sign permutation.

The Hamiltonian corresponding to the simple random walk on $\mathbb{Z}_+$ with probabilities $p$ and $1-p$ for jumping one unit to the right and left, respectively, and with reflection at the origin is the one that awards with $\alpha_0 = -\log p$ each visit to the origin and does not award the other sites: $\alpha_x = 0$ for $x > 0$. In this case, we have that for $\alpha_0 > \log 2$,

* Corresponding author. Fax: +55 11 814 4135; e-mail: pablo@ime.usp.br.

0304-4149/98/$ – see front matter 
PII: S0304-4149(98)00052-0

there is a Gibbs measure associated to this Hamiltonian – the one that corresponds to the stationary Markov chain with $p<\frac{1}{2}$ – and that for $\alpha_0 \leqslant \log 2$ there are no Gibbs measures associated to this Hamiltonian – in this case the Markov chain is null recurrent or transient. We show this in Section 1.

Consider a Hamiltonian that awards $\alpha_x$ each visit to state $x$. In Theorems 1.2 and 1.4 below we give sufficient conditions on $(\alpha_x)$ that guarantee the existence of a Gibbs measure when $\alpha_x$ is constant for $x$ sufficiently large. When $\alpha_{x+2} \leqslant \alpha_x$ for $x$ sufficiently large we give only sufficient conditions. These conditions are given in function of continued fractions.

There is a natural relation between Hamiltonians awarding visits to sites with Hamiltonians awarding visits to edges: each time that the path jumps from $x$ to $x+1$ the award is $b_x$ and each time that the path jumps from $x+1$ to $x$ the award is $c_x$. The relation between $(\alpha_x)$ and $(b_x, c_x)$ is given in Lemma 1.1 below. The Hamiltonian awarding jumps gives rise naturally to the so-called transfer matrix $Q=(q_{x,y})$, where $q_{x,x+1}=e^{b_x}$ and $q_{x+1,x}=e^{c_x}$, $x \geqslant 0$; $q_{x,y}=0$ otherwise (see Georgii, 1988, Ch. 3 for a discussion on transfer matrices for one-dimensional lattices with a finite state set, in particular, the one-dimensional Ising model). If the transfer matrix turns out to be stochastic (i.e. the sum of each row is one), then the problem reduces to the study of recurrence properties of birth and death chains. In the general case, the transfer matrix is still bi-diagonal but not stochastic.

Hence, we consider a positive bi-diagonal matrix $Q$ with the only positive entrances $q_{x,x+1}=e^{b(x)}$ and $q_{x+1,x}=e^{c(x)}$, $x \geqslant 0$. Let $R$ be the common convergence radius of $Q$ (see definition in next section) and $f$ be the right eigenvector with eigenvalue $R$. The matrix $Q$ is said to be $R$-positive if the Markov chain with transition probabilities

$$p(x,y)=R^{-1}q_{x,y}\frac{f_y}{f_x}$$

is positive recurrent. Kesten (1976) proved that there exists a unique Gibbs state corresponding to a given Hamiltonian if and only if the corresponding matrix $Q$ is $R$-positive. The problem left is then to find conditions for a positive bidiagonal matrix to be $R$-positive.

When $q_{x,x\pm 1}$ is constant for $x$ sufficiently large, we give necessary and sufficient conditions on the matrix $Q$ to guarantee $R$-positivity. These conditions involve careful analysis of continued fractions.

We then prove that if for $\alpha_x$ constant when $x>M$ there is a Gibbs state, then for any sequence identical to $\alpha_x$ for $x \leqslant M$ and non-increasing for $x>M$ there is also a Gibbs state. To show this we jointly construct particle systems having Gibbs states as invariant (and reversible) measures in such a way that one of the systems dominates the other, coordinate by coordinate. This is called coupling. As a marginal result we obtain a stationary infinite particle system on the set of nearest-neighbors random walks (contained in $\mathbb{Z}^{\mathbb{Z}}$) for which a drift in direction to the origin is present only in a finite number of $x$. For periodic boundary conditions the corresponding finite systems are null recurrent. See details about this in Section 3.

A model that shares properties with our model is the so-called solid-on-solid (SOS) model. The state space is the same as ours but the Hamiltonian is given by

$$H(\sigma) = -\sum_x |\sigma_x - \sigma_{x+1}| + h|\sigma_x|.$$

The first term awards those configurations that have nearest neighbors as close as possible and the second term awards those configurations that stay close to the axis 0. In our process we force the distance between neighbors to be one and only award visits to some points: the field does not increase with the distance. The Gibbs measure on trajectories of random walk we study share a property with SOS models: For SOS models on $\mathbb{Z}^{\mathbb{Z}}$ Leeuwen and Hilhorst (1981) showed that any positive award to the origin guarantees the existence of a Gibbs state, while when the model lives in $\mathbb{Z}_+^{\mathbb{Z}}$, the award must superate a critical value. This fact is known as entropic repulsion of the wall $x=0$. In our case the entropic repulsion appears as a counting effect. If there is a Gibbs state in $\mathbb{Z}_+^{\mathbb{Z}}$ for some $(\alpha_x,\ x\geqslant 0)$, then there is a Gibbs state in $\mathbb{Z}^{\mathbb{Z}}$ for $(\alpha'_x,\ x\in\mathbb{Z})$, where $\alpha'_0=\alpha_0-\log 2$ and $\alpha'_x=\alpha_{|x|}$ for $x\neq 0$. In particular, since we saw that $\log 2$ is the critical value above which there is a Gibbs state in $\mathbb{Z}_+^{\mathbb{Z}}$ in the random walk case, any positive award to the origin and no award to the other points guarantees the existence of a Gibbs state in $\mathbb{Z}^{\mathbb{Z}}$.

Cesi and Martinelli (1996a, b) and Lebowitz and Mazel (1996) study SOS models interacting with a wall to which the surface is attracted. The main result is that at low enough temperature the process shows “layering transition”. In words, this means that depending on the strength of the field, the surface chooses one level $k$ and most of the heights localize at $k$ with small fluctuations. Probably this transition is not present in the measure we study, but this has not been proven.

## 1. Results

We consider measures in the space of trajectories $\Omega$ of a nearest-neighbors non-negative random walk,

$$\Omega = \{w\in\mathbb{Z}_+^{\mathbb{Z}}:\ |w_i - w_{i+1}| = 1\ \ \forall i\in\mathbb{Z}\}.$$

Let $\boldsymbol{\alpha}=(\alpha_x:\ x\in\mathbb{Z}_+)$, $\boldsymbol{b}=(b_x:\ x\in\mathbb{Z}_+)$, $\boldsymbol{c}=(c_x:\ x\in\mathbb{Z}_+)$ be fixed sequences. We associated to them pair interaction potentials $\phi^{\boldsymbol{\alpha}}$ and $\phi^{\boldsymbol{b},\boldsymbol{c}}$ on configurations $w\in\Omega$, then $(\phi^{\boldsymbol{\alpha}})_A(w)=(\phi^{\boldsymbol{b},\boldsymbol{c}})_A(w)=0$ for $A\neq\{i,i+1\}$. $\phi^{\boldsymbol{\alpha}}$ is the interaction with weight $\alpha_x$ on site $x$, and $\phi^{\boldsymbol{b},\boldsymbol{c}}$ is the interaction with weights $b_x$ on the edge $(x,x+1)$ and $c_x$ on the edge $(x+1,x)$. Then

$$(\phi^{\boldsymbol{\alpha}})_{\{i,i+1\}}(w) = \sum_{x\in\mathbb{Z}_+} \alpha_x\delta(w_{i+1},x),$$

$$(\phi^{\boldsymbol{b},\boldsymbol{c}})_{\{i,i+1\}}(w) = \sum_{x\in\mathbb{Z}_+} (b_x\delta(w_iw_{i+1},x(x+1)) + c_x\delta(w_iw_{i+1},(x+1)x)).$$

For $i\leqslant j$ we put $w[i,j]=(w_i,\dots,w_j)$ and by $\Omega[i,j]=\{\sigma=w[i,j]:\ w\in\mathbb{Z}_+^{\mathbb{Z}}\}$ we mean the set of blocks $[i,j]$ in $\Omega$. For $x,y\in\mathbb{Z}_+$ and a block $\sigma\in\Omega[i,j]$ we denote by

$$\mathcal{N}_x(\sigma)=\sum_{k=i}^{j}\delta(\sigma_k,x)\quad\text{and}\quad \mathcal{N}_{x,y}(\sigma)=\sum_{k=i}^{j-1}\delta(\sigma_k\sigma_{k+1},xy),$$

the number of times $\sigma$ visits $x$ and the number of times $\sigma$ passes through the edge $xy$, respectively, this last quantity vanishing if $|x-y|\neq 1$.

For a block $\sigma\in\Omega[i,j]$ we put

$$H^{\boldsymbol{\alpha}}(\sigma)=\sum_{x\in\mathbb{Z}_+}\alpha_x\mathcal{N}_x(\sigma)\quad\text{and}\quad H^{\boldsymbol{b},\boldsymbol{c}}(\sigma)=\sum_{x\in\mathbb{Z}_+}(b_x\mathcal{N}_{x,x+1}(\sigma)+c_x\mathcal{N}_{x+1,x}(\sigma)). \tag{1.1}$$

With this notation the Hamiltonians associated to the interactions $\phi^{\boldsymbol{\alpha}}$, $\phi^{\boldsymbol{b},\boldsymbol{c}}$ on the intervals $[i,j]$, for configurations $w\in\Omega$, are

$$H^{\boldsymbol{\alpha}}_{[i,j]}(w)=H^{\boldsymbol{\alpha}}(w[i,j+1])\quad\text{and}\quad H^{\boldsymbol{b},\boldsymbol{c}}_{[i,j]}(w)=H^{\boldsymbol{b},\boldsymbol{c}}(w[i-1,j+1]). \tag{1.2}$$

Now, for the Hamiltonians $H=H^{\boldsymbol{\alpha}}$ and $H=H^{\boldsymbol{b},\boldsymbol{c}}$, the probability measures associated to them are (see Georgii, 1988, Definition 2.9)

$$\mu^{H}_{[i,j]}(w)(\sigma)=(Z_{[i,j]}(w))^{-1}\sum_{\substack{w'[i-1,j+1]\,:\,w'[k,l]=\sigma\\ w'_{i-1}=w_{i-1},\,w'_{j+1}=w_{j+1}}}\mathrm{e}^{H_{[i,j]}(w')}\quad\text{for }\sigma\in\Omega[k,l], \tag{1.3}$$

where

$$i<k,\ \ j>l,\ \text{being }Z_{[i,j]}(w)=\sum_{\substack{w'[i-1,j+1]\,:\\ w'_{i-1}=w_{i-1},\,w'_{j+1}=w_{j+1}}}\mathrm{e}^{H_{[i,j]}(w')}$$

the partition function of $w\in\Omega$ in $[i,j]$. We have the following relation between both Hamiltonians and measures.

**Lemma 1.1.** *Assume that the sequences* $\boldsymbol{\alpha}$, $\boldsymbol{b}$, $\boldsymbol{c}$ *verify the equalities*

$$\alpha_x+\alpha_{x+1}=b_x+c_x\quad\textit{for } x\in\mathbb{Z}_+. \tag{1.4}$$

*Then there exists a real function* $\gamma(n,m,p)$ *defined in* $\mathbb{Z}_+^3$ *such that*

$$H^{\boldsymbol{b},\boldsymbol{c}}_{[i,j]}(w)=H^{\boldsymbol{\alpha}}_{[i,j]}(w)+\gamma(w_{i-1},w_{j+1},j-i). \tag{1.5}$$

*In particular, this relation implies*

$$\mu^{H^{\boldsymbol{b},\boldsymbol{c}}}_{[i,j]}(w)(\sigma)=\mu^{H^{\boldsymbol{\alpha}}}_{[i,j]}(w)(\sigma)\quad\textit{for any }\sigma\in\Omega[k,l],\ \textit{with }[k,l]\subseteq[i+1,j-1]. \tag{1.6}$$

Its proof will be given in the appendix.

Our results concern particular classes of sequences. We say that the sequence $\boldsymbol{a}$ is *ultimately constant* (respectively, *ultimately non-increasing*) if there exists $M = M(\boldsymbol{a})$ such that the sequence $(a_x:\ x > M)$ is constant (respectively, non-increasing). For $\boldsymbol{a}$ ultimately constant we denote $a = a_{M(\boldsymbol{a})+1}$ so $a_x = a\ \forall x > M(\boldsymbol{a})$. Below we will also use the following notation: $\boldsymbol{a} > 0$ if $a_x > 0\ \forall x \in \mathbb{Z}_+$.

One of our main results is:

**Theorem 1.2.** *Let $\boldsymbol{\alpha}$ and $\boldsymbol{b}+\boldsymbol{c}$ be ultimately constant sequences. Then there exists a unique translational invariant Gibbs state*

(i) $\mu^{\boldsymbol{\alpha}}$ *for* $H^{\boldsymbol{\alpha}}$ *if and only if for some* $x = 0,\ldots,M = M(\boldsymbol{\alpha})$ *it is verified*:

$$1 - \cfrac{e^{\alpha_x+\alpha_{x+1}-2\alpha}/4}{1 - \cfrac{e^{\alpha_{x+1}+\alpha_{x+2}-2\alpha}/4}{\cfrac{\cdots}{1 - \cfrac{e^{\alpha_{M-1}+\alpha_M-2\alpha}/4}{1 - e^{\alpha_M-\alpha}/2}}}} \in [-\infty, 0), \tag{1.7}$$

*where for $x = M$ it means* $1 - e^{\alpha_M-\alpha}/2 < 0$.

(ii) $\mu^{\boldsymbol{b},\boldsymbol{c}}$ *for* $H^{\boldsymbol{b},\boldsymbol{c}}$ *if and only if for some* $x = 0,\ldots,M = M(\boldsymbol{b}+\boldsymbol{c})$ *it is verified*:

$$1 - \cfrac{e^{b_x+c_x-(b+c)}/4}{1 - \cfrac{e^{b_{x+1}+c_{x+1}-(b+c)}/4}{\cfrac{\cdots}{1 - \cfrac{e^{b_{M-1}+c_{M-1}-(b+c)}/4}{1 - e^{b_M+c_M-(b+c)}/2}}}} \in [-\infty, 0), \tag{1.8}$$

*for $x = M$ it means* $1 - e^{b_M+c_M-(b+c)}/2 < 0$.

*The measures $\mu^{\boldsymbol{\alpha}}$, $\mu^{\boldsymbol{b},\boldsymbol{c}}$ define translational invariant positive recurrent birth and death chain on $\mathbb{Z}_+$. Moreover, the transition probabilities $p(x, x+1)$, $p(x+1, x)$ of these chains are constants for $x > M$.*

Let us show that part (i) can be deduced from (ii). Let us pick $\boldsymbol{b}, \boldsymbol{c}$ such that

$$\alpha_x + \alpha_{x+1} = b_x + c_x.$$

Since $\alpha_x = \alpha$ for $x > M(\boldsymbol{\alpha})$, $\boldsymbol{b}+\boldsymbol{c}$ is ultimately constant with $M(\boldsymbol{b}+\boldsymbol{c}) = M(\boldsymbol{\alpha})$. Then the conditions of Lemma 1.1 are verified for $\boldsymbol{\alpha}, \boldsymbol{b}, \boldsymbol{c}$. Now we apply (ii) to the sequences $\boldsymbol{b}$, $\boldsymbol{c}$. Since $\alpha_x + \alpha_{x+1} - 2\alpha = b_x + c_x - (b+c)$ for $x = 0,\ldots,M$ we find that (i) follows from (ii).

In the proof of (ii) we will characterize $R$-positivity of a class of 2-periodic irreducible matrices (see Vere-Jones, 1962, Vere-Jones, 1967). Below we precise this notion. We will deal with matrices $Q = (q_{x,y}:\ x, y \in \mathbb{Z}_+)$ verifying

$$q_{x,y} = 0 \quad \text{if } |x-y| \neq 1 \quad \text{and} \quad q_{x,y} > 0 \quad \text{if } |x-y| = 1. \tag{1.9}$$

Observe that irreducibility implies that $R = \limsup_{N\to\infty} (Q^{2N}(u,u))^{1/2N}$ is a common convergence radius, i.e. it is independent of $u \in \mathbb{Z}_+$. The matrix $Q$ is $R$-positive if there exists a solution to the eigenvector problem:

$$Qf = Rf \quad \text{where } f = (f_x\colon x \in \mathbb{Z}_+) > 0$$

and such that the following stochastic matrix $P^{(R)} = (p(x,y)\colon x, y \in \mathbb{Z}_+)$, with

$$p(x,y) = R^{-1} q_{x,y} \frac{f_y}{f_x}, \quad x, y \in \mathbb{Z}_+,$$

defines a positive recurrent Markov chain. We notice that the particular shape of $Q$ implies that the matrix $P^{(R)}$ defines a birth and death chain because $p(x,y) = 0$ if $|x-y| \neq 1$. We will consider the sequences $\boldsymbol{p}$, $\boldsymbol{q}$ defined by

$$p_x \doteq q_{x,x+1} \quad \text{and} \quad q_x \doteq q_{x+1,x} \quad \text{for } x \in \mathbb{Z}_+.$$

A key result is the following characterization of $R$-positivity when $\boldsymbol{p}\cdot\boldsymbol{q} = (p_x q_x\colon x \in \mathbb{N})$ is ultimately constant.

**Theorem 1.3.** *Let $Q = (q_{x,y}\colon x, y \in \mathbb{Z}_+)$ be a positive matrix such that $q_{x,y} = 0$ if $|x-y| \neq 1$ and $q_{x,y} > 0$ if $|x-y| = 1$. Assume $\boldsymbol{p}\cdot\boldsymbol{q}$ is ultimately constant. Then $Q$ is $R$-positive if and only if for some $x = 0, \dots, M = M(\boldsymbol{p}\cdot\boldsymbol{q})$, we have*

$$1 - \cfrac{p_x q_x / 4pq}{1 - \cfrac{p_{x+1} q_{x+1}/4pq}{\cdots \atop 1 - \cfrac{p_{M-1} q_{M-1}/4pq}{1 - p_M q_M / 2pq}}} \in [-\infty, 0). \tag{1.10}$$

*For $x = M$ it means $1 - p_M q_M / 2pq < 0$. Moreover, the transition probabilities $p(x, x+1)$, $p(x+1, x)$ of the matrix $P^{(R)}$ associated to $Q$ are constants for $x > M$.*

Let us introduce the main steps to deduce Theorem 1.2 from Theorem 1.3. Let $Q = (q_{x,y}\colon x, y \in \mathbb{Z}_+)$ be the following irreducible matrix:

$$q_{x,y} = 0 \quad \text{if } |x-y| \neq 1 \quad \text{and} \quad q_{x,x+1} = \mathrm{e}^{b_x}, \quad q_{x+1,x} = \mathrm{e}^{c_x} \quad \text{for } x \in \mathbb{Z}_+. \tag{1.11}$$

From definition we get

$$\mu_N^{H^{\boldsymbol{b},\boldsymbol{c}}}(w)(\sigma) = \frac{Q^{N-k+1}(w_{-(N+1)}, \sigma_{-k}) \prod_{i=-k}^{k-1} Q(\sigma_i, \sigma_{i+1}) \cdot Q^{N-k+1}(\sigma_k, w_{N+1})}{Q^{2N+2}(w_{-(N+1)}, w_{N+1})}.$$

From Theorem 1 in Kesten (1976) for strictly positive matrices and extended in Theorem C in Gurevich (1984) for irreducible matrices, we have that there exists a unique translational invariant Gibbs state for the Hamiltonian $H^{\boldsymbol{b},\boldsymbol{c}}$ if and only if $Q$ is an $R$-positive matrix. By using notation (1.11), we have $p_x q_x / pq = \mathrm{e}^{b_x + c_x - (b+c)}$. Therefore, Kesten's theorem allows to reduce Theorem 1.2 from Theorem 1.3 and Lemma 1.1.

When the sequences verify some ultimately decreasing properties we get sufficient conditions for the existence of translational invariant Gibbs states.

**Theorem 1.4.** (i) *If $\boldsymbol{\alpha}$ verifies that the sequence $\boldsymbol{A}=(\alpha_x+\alpha_{x+1}\colon x\in\mathbb{Z}_+)$ is ultimately decreasing then there exists a translational invariant Gibbs state $\mu^{\boldsymbol{\alpha}}$ for $H^{\boldsymbol{\alpha}}$ when for some $x=0,\dots,M=M(\boldsymbol{A})$ it is verified*:

$$1-\cfrac{\mathrm{e}^{\alpha_x+\alpha_{x+1}-\alpha_{M+1}-\alpha_{M+2}}/4}{1-\cfrac{\mathrm{e}^{\alpha_{x+1}+\alpha_{x+2}-\alpha_{M+1}-\alpha_{M+2}}/4}{\cdots\atop 1-\cfrac{\mathrm{e}^{\alpha_{M-1}+\alpha_M-\alpha_{M+1}-\alpha_{M+2}}/4}{1-\mathrm{e}^{\alpha_M-\alpha_{M+2}}/2}}}\in[-\infty,0),$$

*where for $x=M$ it means $1-\mathrm{e}^{\alpha_M-\alpha_{M+2}}/2<0$.*

(ii) *If $\boldsymbol{b},\boldsymbol{c}$ verify that the sequence $\boldsymbol{B}=(b_x+c_x\colon x\in\mathbb{Z}_+)$ is ultimately decreasing then there exists a translational invariant Gibbs state $\mu^{\boldsymbol{b},\boldsymbol{c}}$ for $H^{\boldsymbol{b},\boldsymbol{c}}$ when for some $x=0,\dots,M=M(\boldsymbol{B})$ it is verified*:

$$1-\cfrac{\mathrm{e}^{b_x+c_x-b_{M+1}-c_{M+1}}/4}{1-\cfrac{\mathrm{e}^{b_{x+1}+c_{x+1}-b_{M+1}-c_{M+1}}/4}{\cdots\atop 1-\cfrac{\mathrm{e}^{b_{M-1}+c_{M-1}-b_{M+1}-c_{M+1}}/4}{1-\mathrm{e}^{b_M+c_M-b_{M+1}-c_{M+1}}/2}}}\in[-\infty,0),$$

*for $x=M$ it means $1-\mathrm{e}^{b_M+c_M-b_{M+1}-c_{M+1}}/2<0$.*

As before part (i) follows from (ii) by taking $\boldsymbol{b},\boldsymbol{c}$ such that $b_x+c_x=\alpha_x+\alpha_{x+1}$. The hypothesis $\alpha_{x+2}+\alpha_{x+1}\leqslant\alpha_{x+1}+\alpha_x$ for $x>M(\boldsymbol{\alpha})$ is equivalent to $b_{x+1}+c_{x+1}\leqslant b_x+c_x$ for $x>M(\boldsymbol{\alpha})$.

Now, we discuss the role of the entropic repulsion of the reflecting wall at $x=0$. Consider the space of trajectories of a simple random walk with nearest-neighbor jumps without the constraint that it must be positive. The space of configurations is $\boldsymbol{\Omega}=\{\boldsymbol{w}\in\mathbb{Z}^{\mathbb{Z}}\colon |\boldsymbol{w}_i-\boldsymbol{w}_{i+1}|=1\ \forall i\in\mathbb{Z}\}$ and the set of blocks $[i,j]$ is $\boldsymbol{\Omega}[i,j]=\{\boldsymbol{\sigma}=\boldsymbol{w}[i,j]\colon \boldsymbol{w}\in\boldsymbol{\Omega}\}$. The sequence $\boldsymbol{\alpha}=(\boldsymbol{\alpha}_x\colon x\in\mathbb{Z})$ is restricted to be symmetric with respect to sign permutation, i.e. it verifies $\boldsymbol{\alpha}_x=\boldsymbol{\alpha}_{-x}$, for $x\in\mathbb{Z}$. The Hamiltonian $\boldsymbol{H}^{\boldsymbol{\alpha}}$ is defined analogously as in Eq. (1.1) but the sum is over $\mathbb{Z}$ instead of over $\mathbb{Z}_+$. Also $\boldsymbol{\mu}^{\boldsymbol{H}^{\boldsymbol{\alpha}}}$ is defined analogously as in Eq. (1.3). For a block $\boldsymbol{w}[i,j]\in\boldsymbol{\Omega}[i,j]$ we set $|\boldsymbol{w}[i,j]|=\{|\boldsymbol{w}_k|\colon i\leqslant k\leqslant j\}$ which is a block in $\Omega[i,j]$. For $\boldsymbol{\sigma}\in\boldsymbol{\Omega}[i-1,j+1]$ we have

$$\sum_{\substack{\boldsymbol{\sigma}'\in\boldsymbol{\Omega}[i-1,j+1]:\\ |\boldsymbol{\sigma}'_{i-1}|=|\boldsymbol{\sigma}_{i-1}|,\,|\boldsymbol{\sigma}'_{j+1}|=|\boldsymbol{\sigma}_{j+1}|}}\delta(|\boldsymbol{\sigma}'|,|\boldsymbol{\sigma}|)=2^{\mathcal{N}_0(|\boldsymbol{\sigma}[i,j]|)}\sum_{\substack{\sigma'\in\Omega[i,j]:\\ \sigma'_{i-1}=|\boldsymbol{\sigma}_{i-1}|,\,\sigma'_{j+1}=|\boldsymbol{\sigma}_{j+1}|}}\delta(\sigma',|\boldsymbol{\sigma}|).$$

Hence, for $i<j$, $k>l$, $\boldsymbol{\sigma}\in\boldsymbol{\Omega}[k,l]$, $\boldsymbol{w}\in\boldsymbol{\Omega}$ we have

$$\boldsymbol{\mu}^{\boldsymbol{H}^{\boldsymbol{\alpha}}}_{[i,j]}(\boldsymbol{w})(\boldsymbol{\sigma})=\left(\sum_{\tilde{\boldsymbol{\sigma}}\in\boldsymbol{\Omega}[k,l]}\delta(|\tilde{\boldsymbol{\sigma}}|,|\boldsymbol{\sigma}|)\right)^{-1}\mu^{H^{\boldsymbol{\alpha}}}_{[i,j]}(|\boldsymbol{w}|)(|\boldsymbol{\sigma}|),$$

where $\boldsymbol{\alpha}=(\alpha_x\colon x\in\mathbb{Z}_+)$ verifies $\alpha_x=\boldsymbol{\alpha}_x$ for $x\geqslant 1$ and $\alpha_0=\boldsymbol{\alpha}_0+\log 2$. Hence, we have proved the following result.

**Theorem 1.5.** *If $\boldsymbol{\alpha}$ is invariant under sign permutation and ultimately constant then there exists a translational invariant Gibbs measure defined on the product $\sigma$-field of $\mathbb{Z}^{\mathbb{Z}}$ if and only if condition* (1.7) *holds for $\boldsymbol{\alpha}=(\boldsymbol{\alpha}_0+\log 2,\boldsymbol{\alpha}_x$ for $x\geqslant 1)$.*

This shows, in the simplest way, the role of the entropic repulsion of the wall at $x=0$. The number 2 to the power (number of visits to the wall) is the extra probability that each path in the free system obtains with respect to the corresponding path in the system with the wall.

Observe that Theorem 1.2 can also be written in terms of thermodynamic limits (see Georgii, 1988, Theorem 7.12). In this purpose define for fixed $u,v\in\mathbb{Z}_+$ the following functional for $H=H^{\boldsymbol{\alpha}}$ or $H=H^{\boldsymbol{b},\boldsymbol{c}}$,

$$\mu^H_{N,u,v}(\sigma)=\frac{\sum_{\sigma'\in\Omega[-N,N]\colon\,\sigma'_{-N}=u,\,\sigma'_N=v,\,\sigma'[-k,k]=\sigma}\mathrm{e}^{H(\sigma')}}{\sum_{\sigma'\in\Omega[-N,N]\colon\,\sigma'_{-N}=u,\,\sigma'_N=v}\mathrm{e}^{H(\sigma')}}\quad\text{for }\sigma\in\Omega[-k,k].\tag{1.12}$$

The theorem asserts that $\lim_{N\to\infty}\mu^H_{N,u,v}(\sigma)$ exists and defines a translational invariant probability measure on $\Omega$ if and only if it holds condition (1.7) for $H=H^{\boldsymbol{\alpha}}$, or condition (1.8) for $H=H^{\boldsymbol{b},\boldsymbol{c}}$. Analogously for Theorem 1.5.

We can give probabilistic interpretation of Theorem 1.2 in some special cases. In this purpose consider the birth and death chain $(X_n)$ with transition probabilities $Q=(q_{x,y})$, where $q_{x,y}$ is the probability of jumping from $x$ to $y$, where only the transitions $x$ to $x+1$ and $x$ to $x-1$ are allowed. The state 0 is a reflecting barrier i.e. $q_{0,1}=1$.

If the chain is positive recurrent, $\pi$ is the stationary probability vector, $\pi=\pi Q$, and $\mathbb{P}_\pi$ is the translational invariant probability measure on $\mathbb{Z}_+^{\mathbb{Z}}$, for $\sigma\in\Omega[i,j]$ we have

$$\mathbb{P}_\pi\{(X_i,\dots,X_j)=\sigma\}=\pi_{\sigma_i}\prod_{k=i}^{j-1}q_{\sigma_k,\sigma_{k+1}},$$

then for $u,v\in\mathbb{Z}_+$ fixed it is verified

$$\lim_{N\to\infty}\mathbb{P}_\pi\{(X_i,\dots,X_j)=\sigma|X_{-N}=u,\ X_N=v\}=\mathbb{P}_\pi\{(X_i,\dots,X_j)=\sigma\}.$$

Now, assume $((q_{x,x+1})\colon\ x\in\mathbb{Z}_+)$ is constant for $x\geqslant 1$, so $q_{x,x+1}=p\in[0,1]$ for $x>0$ and $q_{0,1}=1$. This chain is positive recurrent if and only if $p<\frac{1}{2}$. We set $q=1-p$. We have

$$\mathbb{P}((X_i,\dots,X_j)=\sigma|X_i=\sigma_i)=(pq)^{j-i/2}\left(\frac{p}{q}\right)^{(\sigma_j-\sigma_i)/2}p^{-\mathcal{N}_{0,1}(\sigma)}.$$

where $\mathcal{N}_{0,1}(\sigma)$ is the number of times $\sigma_k=0$, as $k$ varies in $\{i,\dots,j-1\}$. Putting $b_0=-\log p$ and using the above expression, for a fixed couple $u,v\in\mathbb{Z}_+$ and $\sigma\in$

$\Omega[-k,k]$, we have

$$\mathbb{P}\{(X_{-k},\ldots,X_k)=\sigma | X_{-N}=u,\ X_N=v\}$$

$$=\frac{\sum_{\sigma'\in\Omega[-N,N]:\ \sigma'_{-N}=u,\ \sigma'_N=v,\ \sigma'[-k,k]=\sigma} \mathrm{e}^{b_0 \mathcal{N}_{0,1}(\sigma')}}{\sum_{\sigma'\in\Omega[-N,N]:\ \sigma'_{-N}=u,\ \sigma'_N=v} \mathrm{e}^{b_0 \mathcal{N}_{0,1}(\sigma')}}=\mu_{N,u,v}^{H^{\boldsymbol{b},\boldsymbol{c}}}(\sigma),$$

with $\boldsymbol{b}=(b_0,0,\ldots,0,\ldots)$, $\boldsymbol{c}=(0,\ldots,0,\ldots)$. By Lemma 1.1 last quantity is equal to $\mu_{N,u,v}^{H^{\boldsymbol{\alpha}}}(\sigma)$ with $\boldsymbol{\alpha}=(b_0,0,\ldots,0,\ldots)$. Hence $\lim_{N\to\infty}\mu_{N,u,v}^{H^{\boldsymbol{b},\boldsymbol{c}}}(\sigma)$ is strictly positive and equal to $\mathbb{P}_\pi\{(X_{-k},\ldots,X_k)=\sigma\}$ if $p\in(0,\frac{1}{2})$ and equal to 0 if $p\geqslant\frac{1}{2}$. This is exactly condition (1.7), respectively, Eq. (1.8), for the sequences $\boldsymbol{\alpha}$, respectively, $\boldsymbol{b},\boldsymbol{c}$, for $M=0$. In fact they correspond to $\alpha_0>\log 2$, respectively, $b_0>\log 2$. In this random walk case $b_0=-\log p$ is restricted to be positive. Observe that if $q_{x,x+1}=p$ for $x>M$, then

$$b_0=-\log p,\quad b_x=\log\frac{q_{x,x+1}}{p},\quad c_x=\log\frac{1-q_{x,x+1}}{1-p}=\log\frac{1-\mathrm{e}^{b_x-b_0}}{1-\mathrm{e}^{-b_0}}\quad \text{for } x\leqslant M$$

and

$$b_x=c_x=0\quad \text{for } x>M.$$

In this case also $\lim_{N\to\infty}\mu_{N,u,v}^{H^{\boldsymbol{\alpha}}}(\sigma)$ defined a probability measure if and only if $b_0>\log 2$.

Finally, in the context of Theorem 1.3 we point out that for general $Q$ verifying Eq. (1.9), and in the absence of any other condition, it was shown in Ferrari, Martínez (1994), by using Theorem 11.2 of Wall (1948) that the chain $P^{(R)}$ is recurrent (but not necessarily positive recurrent) if and only if the following condition on continued fractions holds

$$1-\cfrac{p_0q_0/R^2}{1-\cfrac{p_1q_1/R^2}{\cdots\,1-\cfrac{p_xq_x/R^2}{\cdots}}}=0.$$

## 2. Proof of Theorem 1.3

Let $Q=(q_{x,y})$ be a non-negative matrix on $\mathbb{Z}_+$. Let us consider the general eigenvalue problem:

$$Q\boldsymbol{f}=r\boldsymbol{f}\quad \text{for } r>0,\ \boldsymbol{f}>0 \tag{2.1}$$

where $\boldsymbol{f}=(f_x:\ x\in\mathbb{Z}_+)$. Observe that for $r>0$ there is at most a unique, up to a homothetic transformation, $\boldsymbol{f}>0$ verifying Eq. (2.1). In this case the matrix $P^{(r)}=(p(x,y):\ x,y\in\mathbb{Z}_+)$ defined by

$$p(x,y)=r^{-1}\frac{f_y}{f_x}q_{x,y}\quad \text{for } x,y\in\mathbb{Z}_+^* \tag{2.2}$$

is a stochastic matrix. Moreover, $P^{(r)}$ defines a birth–death chain with $\{0\}$ being a reflecting state: $p(x,y)=0$ if $|x-y|\neq 1$, $p(0,1)=1$. We put

$$w_x \doteq p(x,x+1) = r^{-1}\frac{f_{x+1}}{f_x}p_x. \tag{2.3}$$

From Eq. (2.2) we get that $(w_x\colon x\in\mathbb{Z}_+)$ verifies the equation:

$$w_0 = 1 \quad \text{and} \quad w_{x+1} = 1 - \frac{r^{-2}p_x q_x}{w_x} \quad \text{for } x\in\mathbb{Z}_+. \tag{2.4}$$

Reciprocally, it is direct to prove that if the sequence $\boldsymbol{w}=(w_x\colon x\in\mathbb{Z}_+)$ given by Eq. (2.3) verifies $\boldsymbol{w}>0$ then $\boldsymbol{f}$ defined by

$$f_0>0, \quad f_{x+1} = f_0 r^{-(x+1)}\prod_{y=0}^{s}\frac{w_y}{p_y} \quad \text{for } x\in\mathbb{Z}_+,$$

verifies Eq. (2.1). Observe that if Eq. (2.1) is verified then necessarily $r\geqslant R$ and if $r>R$ then $P^{(r)}$ is transient. Hence, if we are able to show that $P^{(r_0)}$ is positive recurrent for some $r_0>0$ we get $R=r_0$. Then $Q$ is $R$-positive if and only if $P^{(r_0)}$ is positive recurrent for some $r_0>0$.

At this point it is convenient to introduce some new notation and a definition. First, for $a>0$ we consider the following continuous and on to strictly increasing function $\varphi_a:(0,\infty]\to(-\infty,1]$:

$$\varphi_a(w) = 1-\frac{a}{w}.$$

**Definition 2.1.** Let $\boldsymbol{a}=(a_x>0\colon x\in\mathbb{Z}_+)>0$ be a strictly positive fixed sequence. It is said to be allowed if it verifies

$$\varphi_{a_x}\circ\cdots\circ\varphi_{a_0}(1)>0 \quad \forall x\in\mathbb{Z}_+. \tag{2.5}$$

Then if we consider the sequence $\boldsymbol{a}$ with $a_x=p_xq_x$ for $x\in\mathbb{Z}_+$, from Eq. (2.4) we find that $Q$ is $R$-positive if and only if for some $r_0$, the sequence $r_0^2\boldsymbol{a}$ is allowed and $P^{(r_0)}$ is positive recurrent. Hence the proof of the theorem is reduced to show that condition (1.10) is equivalent to this last property. This follows from the study of allowed sequences, which we will now develop.

Observe that the inverse of $\varphi_a$, $\varphi_a^{-1}(w)=a/(1-w)$, satisfies analogous properties as $\varphi_a$. Also from the definition we get

$$\text{if } w>0 \text{ and } \varphi_a(w)>0 \text{ then } \varphi_a(w)\in(0,1). \tag{2.6}$$

The first part of the next result was already proved in Ferrari et al., 1992, and we supply it for completeness of this work.

**Lemma 2.2.** *The strictly positive sequence* $\boldsymbol{a}$ *is allowed if and only if it verifies*

$$\forall x\in\mathbb{Z}_+,\quad \forall y\geqslant x\colon\ \varphi_{a_x}^{-1}\circ\cdots\circ\varphi_{a_y}^{-1}(0)<1. \tag{2.7}$$

*Moreover, if* $\boldsymbol{a}>0$ *and* $\boldsymbol{d}>0$ *then*

$$\boldsymbol{d}\leqslant\boldsymbol{a} \text{ and } \boldsymbol{a} \text{ is allowed implies } \boldsymbol{d} \text{ is allowed.} \tag{2.8}$$

**Proof.** Assume $\boldsymbol{a}$ is allowed. Define

$$w_0=1 \quad \text{and} \quad w_{x+1}=\varphi_{a_x}\circ\cdots\circ\varphi_{a_0}(1) \quad \text{for } x\in\mathbb{Z}_+. \tag{2.9}$$

From Eq. (2.6) we deduce that $w_x\in(0,1)$, $\forall x\in\mathbb{Z}_+^*$. Now, since $\varphi_{a_y}^{-1}$ is increasing in $(-\infty,1)$ we find $\varphi_{a_y}^{-1}(0)<\varphi_{a_y}^{-1}(w_{y+1})=w_y$, and $w_y<1$ if $y\geqslant 1$, $w_0=1$. Therefore by induction

$$\varphi_{a_x}^{-1}\circ\cdots\circ\varphi_{a_y}^{-1}(0)<w_x\leqslant 1 \quad \text{for } x\in\mathbb{Z}_+,\ y\geqslant x.$$

Then condition (2.5) is verified. Let us show the reciprocal, i.e. Eq. (2.7) implies $\boldsymbol{a}$ is allowed. We denote

$$h_{\boldsymbol{a}}(x,y)\doteq\varphi_{a_x}^{-1}\circ\cdots\circ\varphi_{a_y}^{-1}(0) \quad \text{for } x\in\mathbb{Z}_+,\ y\geqslant x. \tag{2.10}$$

With this notation our hypothesis is $h_{\boldsymbol{a}}(x,y)<1$. Let us show that

$$h_{\boldsymbol{a}}(x,y)\in(0,1) \quad \text{for } x\in\mathbb{Z}_+,\ y\geqslant x. \tag{2.11}$$

First, observe that $h_{\boldsymbol{a}}(y,y)=\varphi_{a_y}^{-1}(0)>0$. Now, assume that for some couple $y>x$, $x\in\mathbb{Z}_+$ we have $h_{\boldsymbol{a}}(x,y)<0$. Since $h_{\boldsymbol{a}}(x,y)=\varphi_{a_x}^{-1}(h_{\boldsymbol{a}}(x+1,y))$, we get from the shape of $\varphi_{a_x}^{-1}$ that $h_{\boldsymbol{a}}(x+1,y)>1$ contradicting the hypotheses. Then Eq. (2.11) is verified. Now, since $\varphi_a$ is increasing in $[0,\infty)$ we get

$$0=\varphi_{a_0}(\varphi_{a_0}^{-1}(0))=\varphi_{a_0}(h_{\boldsymbol{a}}(1,1))<\varphi_{a_0}(1)=w_1.$$

Again from Eq. (2.6), since $w_1>0$ we deduce $w_1\in(0,1)$. Analogously, $0<\varphi_{a_0}^{-1}\circ\cdots\circ\varphi_{a_y}^{-1}(0)<1$ implies that

$$0<\varphi_{a_1}^{-1}\circ\cdots\circ\varphi_{a_y}^{-1}(0)<\varphi_{a_0}(1) \text{ and then } \varphi_{a_0}(1)<1.$$

By recurrence $0<\varphi_{a_y}^{-1}(0)<\varphi_{a_{y-1}}\circ\cdots\circ\varphi_{a_0}(1)$, then $w_y<1$ and we get $0<\varphi_{a_y}\circ\cdots\circ\varphi_{a_0}(1)$. Then Eq. (2.5) is verified.

Let us show Eq. (2.8). Observe that for $x\in\mathbb{Z}_+^*$ and $w<1$ fixed, $\varphi_{a_x}^{-1}(w)$ is increasing in $a_x$. Then $h_{\boldsymbol{d}}(y,y)=\varphi_{d_y}^{-1}(0)\leqslant\varphi_{a_y}^{-1}(0)\leqslant h_{\boldsymbol{a}}(y,y)<1$. Hence,

$$h_{\boldsymbol{d}}(y-1,y)\leqslant\varphi_{d_{y-1}}^{-1}(\varphi_{d_y}^{-1}(0))\leqslant\varphi_{d_{y-1}}^{-1}(\varphi_{a_y}^{-1}(0))\leqslant\varphi_{a_{y-1}}^{-1}(\varphi_{ay}^{-1}(0))=h_{\boldsymbol{a}}(y-1,y)<1.$$

By recurrence $h_{\boldsymbol{d}}(x,y)<1$ $\forall x\in\mathbb{Z}_+,\ y\geqslant x$, then the result follows from Eq. (2.7). □

Assume $\boldsymbol{a}$ is allowed then $\varphi_{a_{y+1}}^{-1}(0)\in(0,1)$ and by the increasing property we get

$$h_{\boldsymbol{a}}(x,y)=\varphi_{a_x}^{-1}\circ\cdots\circ\varphi_{a_y}^{-1}(0)<\varphi_{a_x}^{-1}\circ\cdots\circ\varphi_{a_{y+1}}^{-1}(0)=h_{\boldsymbol{a}}(x,y+1),$$

i.e. the sequence $h_{\boldsymbol{a}}(x,y)$ is strictly increasing in $y\in\mathbb{Z}_+^*$. Then the following limit exists and verifies:

$$h_{\boldsymbol{a}}(x,\infty)=\lim_{y\nearrow\infty}h_{\boldsymbol{a}}(x,y)\leqslant 1.$$

Observe that

$$h_{\mathbf{a}}(x,y)=\cfrac{a_x}{1-\cfrac{a_{x+1}}{1-\cfrac{a_{x+2}}{\cfrac{\cdots}{1-a_y}}}},$$

then $h_{\mathbf{a}}(x,\infty)$ is a continued fraction.

For $a>0$ denote by $a_\pm=\frac{1}{2}(1\pm\sqrt{1-4a})$ the fixed points of $\varphi_a$. Observe that $a_\pm(1-a_\pm)=a$. These points are reals (and in this case lie in $[0,1]$) if and only if $a\leqslant\frac{1}{4}$. Notice that since $\varphi_a$ is increasing in $(0,\infty)$ we get for $d>0$

$$\text{if } d<a_- \text{ then } \varphi_a(d)<\varphi_a(a_-)=a_-$$

and

$$\text{if } d>a_+ \text{ then } \varphi_a(d)>\varphi_a(a_+)=a_+. \tag{2.12}$$

**Lemma 2.3.** *Let $a>0$ and $d\in(0,1]$. Assume $\varphi_a^n(d)>0$ for all $n\in\mathbb{Z}_+$. Then $a\leqslant\frac{1}{4}$ and $d\in[a_-,1]$. Moreover, $\varphi_a^n(a_-)=a_-$ $\forall n$ and*

$$\lim_{n\to\infty}\varphi_a^n(d)=a_+ \quad \textit{if } d\in(a_-,1].$$

*In particular, a constant sequence $\mathbf{a}>0$ is allowed if and only if $a\leqslant\frac{1}{4}$ and in this case $h_{\mathbf{a}}(1,\infty)=a_-$.*

**Proof.** We shall firstly prove:

$$\text{if } d(1-d)\leqslant a \text{ then } \varphi_a^{(n)}(d) \text{ decreases with } n\in\mathbb{Z}_+ \tag{2.13}$$

and

$$\text{if } d(1-d)\geqslant a \text{ then } \varphi_a^{(n)}(d) \text{ increases with } n\in\mathbb{Z}_+. \tag{2.14}$$

Let us prove Eq. (2.13). We have

$$\varphi_a^{(n+1)}(d)=1-\frac{a}{\varphi_a^n(d)}.$$

Now, $\varphi_a(d)=1-a/d\leqslant d$ if and only if $d(1-d)\leqslant a$. For $n>1$ we have $\varphi_a^{(n+1)}(d)\leqslant\varphi_a^{(n)}(d)$ if and only if

$$1-\frac{a}{\varphi_a^n(d)}\leqslant 1-\frac{a}{\varphi_a^{(n-1)}(d)},$$

i.e. if and only if $\varphi_a^n(d)\leqslant\varphi_a^{(n-1)}(d)$. Then by recurrence (2.13) is verified. This argument also proves Eq. (2.14). Hence if $d(1-d)=a$ we have $\varphi_a^{(n)}(d)=d$ for $n\in\mathbb{Z}_+^*$.

Since $\varphi_a^{(n)}(d)$ is monotone, denote its limit by $\eta=\lim_{n\to\infty}\varphi_a^{(n)}(d)$. From the hypothesis and Eq. (2.6) we get $\eta\in[0,1]$. If $\eta=0$ we have $\varphi_a^{(n+1)}(d)=\varphi_a(\varphi_a^{(n)}(d))$ which is a contradiction because $\varphi_a^{(n)}(d)\to_{n\to\infty}0$ and $\varphi_a(0)=-\infty$. Then $\eta\in(0,1]$. Hence, from continuity of $\varphi_a$ on $\eta>0$ we get $\varphi_a(\eta)=\lim_{n\to\infty}\varphi_a^{(n+1)}(d)$. Then, $\eta=\varphi_a(\eta)$, i.e. $\eta(1-\eta)=a$. Therefore $\eta\in\{a_-,a_+\}$. Since $\eta(1-\eta)\leqslant\frac{1}{4}$ we find $a\leqslant\frac{1}{4}$.

Assume $d(1-d)<a$. Then $d<a_-$ or $d>a_+$. In this case the sequence $\varphi_a^n(d)$ is strictly decreasing. Since $x(1-x)$ is increasing for $x<\frac{1}{2}$ we necessarily have $\varphi_a^{(n)}(d)(1-\varphi_a^{(n)}(d))\leqslant d(1-d)<a$. Hence there is a contradiction with $\lim_{n\to\infty}\varphi_a^{(n)}(d)=\eta\in\{a_+,a_-\}$, i.e. the hypothesis $\varphi_a^{(n)}(d)>0\ \forall n\in\mathbb{Z}_+$ is not verified. Then $d\geqslant a_+$. From Eq. (2.12) we get that $\varphi_a(d)\geqslant a_+$ and by recurrence we also get that $\varphi_a^{(n)}(d)\geqslant a_+$ for $n\in\mathbb{Z}_+^*$. On the other hand, since $\lim_{n\to\infty}\varphi_a^{(n)}(d)=\eta\in\{a_-,a_+\}$ we conclude $\lim_{n\to\infty}\varphi_a^{(n)}(d)=a_+$.

Now assume $d(1-d)>a$. Then $d\in(a_-,a_+)$. In this case the sequence $\varphi_a^{(n)}(d)$ is increasing. We conclude $\lim_{n\to\infty}\varphi_a^{(n)}(d)=a_+$.

Finally, if the constant sequence $\boldsymbol{a}>0$ is allowed we necessarily have $a\leqslant\frac{1}{4}$. Reciprocally if $a\leqslant\frac{1}{4}$ and $d=1$, Eqs. (2.13) and (2.12) imply $\varphi^{(n)}(1)$ is decreasing and $\varphi^{(n)}(1)\geqslant a_+$, so $\boldsymbol{a}$ is allowed. Also by previous analysis $\varphi^{(n)}(1)\to_{n\to\infty}a_+$, then $h_{\boldsymbol{a}}(1,\infty)=1-\lim_{n\to\infty}\varphi^{(n)}(1),=a_-$. □

We remark that the last part of previous lemma can be deduced, by using monotonic properties, from Wall (Theorem 8.2, p. 39).

Let $\boldsymbol{a}$ be a fixed strictly positive sequence. We denote by

$$r\boldsymbol{a}=(ra_x\colon x\in\mathbb{Z}_+)\quad\text{for } r>0\quad\text{and}\quad\mathscr{I}(\boldsymbol{a})=\{r>0\colon r\boldsymbol{a}\text{ is allowed}\}.$$

**Lemma 2.4.** *If* $\boldsymbol{a}>0$ *then* $\mathscr{I}(\boldsymbol{a})=\phi$ *or* $\mathscr{I}(\boldsymbol{a})=(0,r^*]$ *for some* $0<r^*\leqslant(4a)^{-1}$. *If* $\boldsymbol{a}$ *is ultimately constant then* $\mathscr{I}(\boldsymbol{a})\neq\phi$ i.e. $\mathscr{I}(\boldsymbol{a})=(0,r^*]$ *for some* $0<r^*\leqslant(4a)^{-1}$.

**Proof.** Assume $\mathscr{I}(\boldsymbol{a})=\phi$. From the second part of Lemma 2.2 we have: $0<r'<r$, $r\in\mathscr{I}(\boldsymbol{a})$ implies $r'\in\mathscr{I}(\boldsymbol{a})$. Then $\mathscr{I}(\boldsymbol{a})=(0,r^*]$ for $0<r^*\leqslant\infty$. Since $r\in\mathscr{I}(\boldsymbol{a})$ implies $\varphi_{ra_0}^{-1}(0)=ra_0<1$ we find $r^*$ is finite. Let us show $r^*=\sup\mathscr{I}(\boldsymbol{a})$ belongs to $\mathscr{I}(\boldsymbol{a})$. Let $r_0\in\mathscr{I}(\boldsymbol{a})$ and denote $\mathscr{I}_0=\mathscr{I}(\boldsymbol{a})\cap[r_0,\infty)$. We also put $h_r(x,y)\doteq h_{r\boldsymbol{a}}(x,y)$. Let us show that

$$\forall y\geqslant x,\quad x\geqslant 1,\quad H(x,y)=\sup_{r\in\mathscr{I}_0}h_r(x,y)<1.$$

In fact, if for some couple $H(x,y)=1$ we can take $(r_n)\in\mathscr{I}_0$ with $\lim_{n\to\infty}h_{r_n}(x,y)=1$ then $\lim_{n\to\infty}\varphi_{r_na_{x-1}}^{-1}(h_{r_n}(x,y))\geqslant\lim_{n\to\infty}\varphi_{r_0a_{x-1}}^{-1}(h_{r_n}(x,y)=\infty$, which contradicts $\sup_{r\in\mathscr{I}_0}h_r(x-1,y)\leqslant 1$.

We have $r^*=\sup\mathscr{I}_0$. By continuity of $\varphi_{ra_x}^{-1}(w)$ as a function of $r$, for $x\in\mathbb{Z}_+$ and $w<1$ fixed, and since $h_r(x,y)\leqslant H(x,y)<1$ for $r\in\mathscr{I}_0$, we deduce by recurrence $h_{r^*}(x,y)\leqslant H(x,y)<1$ for all $y\geqslant x$, $x\geqslant 1$.

Now, by continuity we have $h_{r^*}(0,y)\leqslant 1$ for all $y\in\mathbb{Z}_+$. If $h_{r^*}(0,y)<1$ for some $y\in\mathbb{Z}_+$ we should have $h_{r^*}(0,y)=1$. Since $\varphi_{r_0a_y}^{-1}(0)\in(0,1)$ for $y\in\mathbb{Z}_+$ we get: $\varphi_{r_0a_y}^{-1}(0)<\varphi_{r_0a_y}^{-1}(\varphi_{r_0a_{y+1}}^{-1}(0))$. Hence by induction we obtain $h_{r^*}(0,y)<h_{r^*}(0,y+1)\leqslant 1$, then $h_{r^*}(0,y)=1<h_{r^*}(0,y+1)\leqslant 1$, which is a contradiction. Then $h_{r^*}(x,y)<1\ \forall x\in\mathbb{Z}_+$, $y\geqslant x$. We conclude $r^*\in\mathscr{I}(\boldsymbol{a})$.

Now, let us show the second part of the lemma. Then assume that $\boldsymbol{a}$ is ultimately constant and $M=M(\boldsymbol{a})$. First, observe that if $M=0$, i.e. $\boldsymbol{a}$ is constant then Lemma 2.3 implies $\mathscr{I}(\boldsymbol{a})=(0,(4a)^{-1}]$. Now let us consider the general case. Take $r\in(0,(4a)^{-1}]$

small, we have

$$h_r(M+1,\infty)=(ar)_+ \quad \text{with } (ar)_+=\tfrac{1}{2}(1-\sqrt{1-4ar})$$

and $h_r(x,y)=(ar)_+$ for $y\geqslant x\geqslant M+1$. We have $h_r(M+1,\infty)=(1+\varepsilon')ar<1$ for $r$ small enough and $\varepsilon'>0$ constant (not depending on $r$). Then

$$h_r(M,\infty)=\varphi^{-1}_{a_M r}((ar)_+)\leqslant(1+\varepsilon'')ar<1$$

for $r$ small enough and $\varepsilon''$ constant. Also $h_r(x,y)\leqslant(1+\varepsilon'')ar$ for $y\geqslant x\geqslant M$. By recurrence we show $h_r(0,\infty)=(1+\varepsilon)ar$ for $r$ small enough and $\varepsilon$ constant. Also $h_r(x,y)\leqslant(1+\varepsilon)ar$ for $y\geqslant x\geqslant 0$. Hence $r\in\mathscr{I}(\boldsymbol{a})$ for $r$ small enough. From the first part of this lemma we deduce $\mathscr{I}(\boldsymbol{a})=(0,r^*]$ for some $r^*\in(0,(4a)^{-1}]$. $\square$

Now, remind notation (2.9), $w_0=1$, $w_{x+1}=\varphi_{a_x}\circ\cdots\circ\varphi_{a_0}(1)$ for $x\in\mathbb{Z}_+$.

**Lemma 2.5.** *Let $\boldsymbol{a}>0$ be allowed and ultimately constant with $a_x=a$ for $x>M=M(\boldsymbol{a})$. Then $w_M\geqslant a_-$ and*

(i) *if $r^*=(4a)^{-1}$ then $w_{n+M}\geqslant\frac{1}{2}$ $\forall n\geqslant 0$;*

(ii) *if $r^*<(4a)^{-1}$ then* $\begin{cases} \lim_{n\to\infty} w_n=a_+>\frac{1}{2} & \text{if } w_M>a_-;\\ w_{n+M}=a_-<\frac{1}{2} & \forall n\geqslant 0 \text{ if } w_M=a_- \\ & \quad \textit{or equivalently} \\ & \quad \textit{if } \varphi^{-1}_{a_0}\circ\cdots\circ\varphi^{-1}_{a_M}(a_-)=1.\end{cases}$

**Proof.** If $w_M<a_-$, Lemma 2.3 implies that $w_{M+n}=\varphi_a^{(n)}(w_M)$ is not strictly positive. Let $a=\frac{1}{4}$, then $a_+=a_-=\frac{1}{2}$. If $w_M>\frac{1}{2}$, we get from Eq. (2.12) that $w_{M+n}=\varphi_a^{(n)}(w_M)>\frac{1}{2}$ for every $n\in\mathbb{Z}_+$. If $w_M=\frac{1}{2}$, then $w_{M+n}=\varphi_a^{(n)}(w_M)=\frac{1}{2}$.

Let $a<\frac{1}{4}$. If $w_M>a_-$ then, from Lemma 2.3 we find $\lim_{n\to\infty}w_{M+n}=\lim_{n\to\infty}\varphi_a^n(w_M)=a_+>\frac{1}{2}$. If $w_M=a_-$, $w_{M+n}=a_-<\frac{1}{2}$ for $n\in\mathbb{Z}_+$. $\square$

**Lemma 2.6.** *Let $\boldsymbol{a}>0$ be ultimately constant with $a_x=a$ for $x>M=M(\boldsymbol{a})$ and such that $\boldsymbol{a}$ is allowed. Let $r^*=\sup\mathscr{I}(\boldsymbol{a})$. Then*

(i) *if $r^*=(4a)^{-1}$ then $w_{n+M}\geqslant\frac{1}{2}$ $\forall n\geqslant 0$;*

(ii) *if $r^*<(4a)^{-1}$ then $w_{n+M}=a_-<\frac{1}{2}$ $\forall n\geqslant 0$.*

*Moreover, we have the equivalence*

$$r^*<(4a)^{-1} \textit{ if and only if } \varphi^{-1}_{a_x/4a}\circ\cdots\circ\varphi^{-1}_{a_M/4a}(\tfrac{1}{2})\in(1,+\infty] \textit{ for some } x=0,\dots,M. \tag{2.15}$$

**Proof.** Part (i) follows from Lemma 2.5(i). Let us show that $r^*<(4a)^{-1}$ is equivalent to conditions (2.15). In this purpose we shall evaluate $r^*$. Let us denote $\tilde r=(4a)^{-1}$ and $\tilde a_x=a_x\tilde r^2=a_x/4a$ for $0\leqslant x\leqslant M$. From Lemmas 2.2, 2.4 and 2.5 we get

$$h_r(x,y)<1 \quad \text{for } y\geqslant x\geqslant M+1 \quad \text{if and only if } r\leqslant\tilde r.$$

In this case $h_r(x,y)<h_r(M+1,\infty)=\frac{1}{2}$ for $y\geqslant x\geqslant M+1$.

Since $\varphi^{-1}_{\tilde a_M}$ is increasing in $(-\infty,1)$ we get that $h_{\tilde r}(x,y)<1$ for all $y\geqslant x\geqslant M$ if and only if $\varphi^{-1}_{\tilde a_M}(\frac{1}{2})\leqslant 1$. Assume $\varphi^{-1}_{\tilde a_M}(\frac{1}{2})\leqslant 1$. If $M>1$ use again the increasing property of

$\varphi^{-1}_{\tilde a_{M-1}}$ to find that $h_{\tilde r}(M,\infty)=\varphi^{-1}_{\tilde a_M}(\frac12)<1$ and

$$h_{\tilde r}(x,y)<1\ \forall y\geqslant x\geqslant M-0 \text{ if and only if } \varphi^{-1}_{\tilde a_M}(\tfrac12)<1$$
$$\text{and } h_{\tilde r}(M-1,\infty)=\varphi^{-1}_{\tilde a_{M-1}}\circ\varphi^{-1}_{\tilde a_M}(\tfrac12)\leqslant 1.$$

By a recurrence argument we find $h_{\tilde r}(x,y)<1\ \forall y\geqslant x\geqslant 1$ if and only if

$$\varphi^{-1}_{\tilde a_x}\circ\cdots\circ\varphi^{-1}_{\tilde a_M}(\tfrac12)<1 \quad \text{for } x=1,\dots,M \quad \text{and} \quad \varphi^{-1}_{\tilde a_0}\circ\cdots\circ\varphi^{-1}_{\tilde a_M}(\tfrac12)\leqslant 1.$$

If this condition holds we get $r^*=\tilde r=(4a)^{-1}$. Hence if $r^*<\tilde r$ condition (2.15) is verified.

Now let us show part (b). From Lemma 2.5(ii) we must show that conditions (2.15) implies $\varphi^{-1}_{a_0r^*}\circ\cdots\circ\varphi^{-1}_{a_Mr^*}((r^*a)_+)=1$. Let us first assume that $\varphi^{-1}_{\tilde a_x}\circ\cdots\circ\varphi^{-1}_{\tilde a_M}(\frac12)\leqslant 1$ for $x=1,\dots,M$ and $\varphi^{-1}_{\tilde a_0}\circ\cdots\circ\varphi^{-1}_{\tilde a_M}(\frac12)\in(1,+\infty]$. Observe that necessarily $\varphi^{-1}_{\tilde a_0}\circ\cdots\circ^{-1}_{\tilde a_M}(\frac12)<1$ for $x>2$ because if it is $=1$ then $\varphi^{-1}_{\tilde a_{x-1}}\circ\cdots\circ\varphi^{-1}_{\tilde a_M}(\frac12)=\infty$. Now, since $\boldsymbol a$ is ultimately constant we have $\mathscr I(\boldsymbol a)=(0,r^*]$ with $r^*<\tilde r$. From Lemmas 2.2 and 2.4 and since $\varphi_b^{-1}(w)$ is increasing in $b>0$ when $w<1$ is fixed we get $h_{r^*}(x,y)<h_{\tilde r}(x,y)<1\ \forall y\geqslant x\geqslant 0$. Hence $\varphi^{-1}_{a_0r^*}\circ\cdots\circ\varphi^{-1}_{a_Mr^*}((r^*a)_+)\leqslant 1$.

Assume $\varphi^{-1}_{a_0r^*}\circ\cdots\circ\varphi^{-1}_{a_Mr^*}((r^*a)_+)<1$. Remind that for $r^*<r<\tilde r$ we have $h_r(x,y)<h_{\tilde r}(x,y)<1\ \forall y\geqslant x\geqslant 0$. Hence $h_r(1,y)>0$ and since $(ra)_+$ is continuous in $r$ we also get that $h_r(0,y)$ in continuous on $r$. Also it is increasing in $r$. Since

$$h_{\tilde r}(1,\infty)=\varphi^{-1}_{\tilde r a_0}\circ\cdots\circ\varphi^{-1}_{\tilde r a_M}(1)>1 \quad\text{and}\quad h_{r^*}(1,\infty)$$
$$=\varphi^{-1}_{a_0r^*}\circ\cdots\circ\varphi^{-1}_{a_Mr^*}((r^*a)_+)<1$$

we find that there exists $r\in(r^*,\tilde r)$ such that $h_r(0,\infty)=\varphi^{-1}_{a_0r}\circ\cdots\circ\varphi^{-1}_{a_Mr}((ra)_+)=1$. We get $h_r(0,y)<h_r(0,\infty)=1\ \forall y\in\mathbb Z_+$, then $r\in\mathscr I(\boldsymbol a)$ which contradicts $r^*=\sup\mathscr I(\boldsymbol a)$. Hence $\varphi^{-1}_{r^*a_0}\circ\cdots\circ\varphi^{-1}_{r^*a_M}(a_+)=1$.

Now assume $h_{\tilde r}(x,\infty)=\varphi^{-1}_{\tilde a_x}\circ\cdots\circ\varphi^{-1}_{\tilde a_M}(\frac12)\in(1,\infty]$ for some $1\leqslant x\leqslant M$. Let $x$ the biggest one for which it occurs. Now, by applying Lemma 2.4 to the sequence $\boldsymbol a^{(x-1)}=(a_y\colon y\geqslant x-1)$ we have $\mathscr I(\boldsymbol a^{(x-1)})=(0,r^*_{x-1}]$ with $r^*_{x-1}<\tilde r$, this last relation because

$$h_{\tilde r}(x-1,\infty)=\lim_{y\to\infty}\frac{a_{x-1}}{1-h_{\tilde r}(1,y)}<0$$

and condition (2.11). Since $h_r(x,y)>0$ for all $y\geqslant x$, we use continuity of $h_r(x,\infty)$ in $r\in(0,\tilde r)$ to get that $h_{r^*_{x-1}}(x,\infty)=1$. If $x>1$ we find $h_{r^*_{x-1}}(x-1,\infty)=+\infty$, so we can apply the same argument to $x-1$ to arrive finally to $h_{r^*}(1,\infty)=\varphi^{-1}_{r^*a_0}\circ\cdots\circ\varphi^{-1}_{r^*a_M}((r^*a)_+)=1$. As we pointed out before, Lemma 2.4(ii) implies the result. $\square$

Let us develop conditions (2.15). It is

$$\cfrac{a_x/4a}{1-\cfrac{a_{x+1}/4a}{\cdots\atop 1-\cfrac{a_{M-1}/4a}{1-a_M/2a}}}\in(1,+\infty] \quad \text{for some } x=0,\dots,M, \tag{2.16}$$

where, for $x=M$ it means $a_M/2a>1$.

From Lemma 2.6 we deduce the theorem. In fact from Lemma 2.6 we get that a necessary and sufficient condition in order that $P^{(r_0)}$ is positive recurrent for some $r_0>0$ is that condition (2.16) is verified. But this condition is exactly the same as the condition of the theorem because of the choice $a_x=p_xq_x$ and the fact that $a_x/a=p_xq_x/pq$.

The last assertion of Theorem 1.3, i.e. that the sequences $(p(x,x+1)),(p(x+1,x))$ of the matrix $P^{(r_0)}$ are constants for $x>M$, follows from Lemma 2.6(ii). □

## 3. Monotonicity and Proof of Theorem 1.4

In this section we show Theorem 1.4. The key of the proof is the following property of monotonicity.

**Proposition 3.1.** *Let $\boldsymbol{b},\boldsymbol{c},\boldsymbol{b}',\boldsymbol{c}'$ be sequences such that*

$$b_{x+1}+c_{x+1}-(b_x+c_x)\leqslant b'_{x+1}+c'_{x+1}-(b'_x+c'_x)$$

*for all $x$. For each $N,u,v$, call $\mu_N=\mu^{H^{\boldsymbol{b},\boldsymbol{c}}}_{N,u,v}$ and $\mu'_N=\mu^{H^{\boldsymbol{b}',\boldsymbol{c}'}}_{N,u,v}$. Then it follows:*

$$\mu_N\leqslant\mu'_N$$

*stochastically. Furthermore, if there exists a unique measure $\mu'=\lim_{N\to\infty}\mu'_N$, then there exists a unique measure $\mu=\lim_{N\to\infty}\mu_N$* and *$\mu\leqslant\mu'$ stochastically.*

We prove the proposition using coupling. First fix $u$ and $v$ in $\mathbb{Z}^+$. Then fix $N$ such that $2N+1$ and $u+v$ have the same parity and $|u-v|\leqslant 2N+1$. We construct a continuous time Markov process $\sigma(t)(=\sigma^N(t))$ on the set of paths

$$\mathscr{C}_N=\mathscr{C}_{2N+1}(u,v)=\{\sigma\in\mathbb{Z}_+^{\{-N,\dots,N\}}:\ \sigma_{-N}=u,\sigma_N=v\}$$

for which $\mu_N$ is reversible. The notation $\sigma_i(t)$ indicates the value at coordinate $i$ of the process at time $t$. We put

$$\beta(x)=\frac{e^{b_x+c_x}}{e^{b_x+c_x}+e^{b_{x-1}+c_{x-1}}}.$$

Let

$$\sigma_y^{i\pm}=\begin{cases}\sigma_i\pm 2 & \text{if } y=i \text{ and } \sigma_{i-1}=\sigma_{i+1}=\sigma_i\pm 1,\\ \sigma_y & \text{otherwise.}\end{cases}$$

Let the process $\sigma(t)$ be the Markov process in the finite state space $\mathscr{C}_N$ with generator

$$L_Nf(\sigma)=\sum_i[\beta(\sigma_{i-1})[f(\sigma^{i+})-f(\sigma)]+(1-\beta(\sigma_{i-1}))[f(\sigma^{i-})-f(\sigma)]].$$

In words, at site $i$ the value of $\sigma$ can change only if both neighbors of $i$ have the same value (say) $x$. In this case there are only two possible moves at each site $i$:

$$x\ x-1\ x\to x\ x+1\ x\quad\text{at rate }\beta(x),$$
$$x\ x+1\ x\to x\ x-1\ x\quad\text{at rate }1-\beta(x),$$

where the value of the site $i$ is the one in the middle of each group of 3. It is a simple matter to show that $\mu_N$ is reversible for the process with generator $L_N$. On the other hand, since we are dealing with a finite state ergodic Markov process, we have that starting with any configuration $\eta \in \mathscr{C}_N$, the process converges in distribution to $\mu_N$. We can say the same of the process induced by $\beta'(x)$ instead of $\beta(x)$, where

$$\beta'(x) = \frac{e^{b'_x + c'_x}}{e^{b'_x + c'_x} + e^{b'_{x-1} + c'_{x-1}}}.$$

Call $L'_N$, $S'_N(t)$ and $\sigma'(t)(= \sigma'^N(t))$ the corresponding generator, semigroup and process.

We perform now the Harris graphical construction of the process. With this aim we attach to each site $i$ in $\{-N+1, \ldots, N-1\}$ a Poisson process of rate 1. To each event of each Poisson process attach a uniform random variable in $[0,1]$. Call $U_{i,k}$ the uniform random variable in $[0,1]$ attached to the $k$th event of the Poisson process of $i$. Call $\tau(i,k)$ the time of occurrence of the $k$th event of the Poisson process of $i$. All these processes and variables are mutually independent.

Define $T_0 = 0$ and for $n \geqslant 1$,

$$\begin{aligned}
&T_n = \inf\{\tau(i,k) > T_{n-1}\colon\ i \in \{-N, \ldots, N\},\ k \geqslant 1\},\\
&K_n = k \ \text{ if } \ T_n = \tau(i,k),\\
&I_n = i \ \text{ if } \ T_n = \tau(i,k)
\end{aligned}$$

(hence $T_n = \tau(I_n, K_n)$) and

$$\sigma(T_n) = \begin{cases} (\sigma(T_{n-1}))^{I_n+} & \text{if } \sigma_{I_n-1}(T_{n-1}-1) = \sigma_{I_n+1}(T_{n-1}) = \sigma_{I_n}(T_{n-1}) \\ & \quad \text{and } U(I_n, K_n) < \beta(\sigma_{I_n-1}(T_{n-1})); \\ (\sigma(T_{n-1}))^{I_n-} & \text{if } \sigma_{I_n-1}(T_{n-1}+1) = \sigma_{I_n+1}(T_{n-1}) = \sigma_{I_n}(T_{n-1}) \\ & \quad \text{and } U(I_n, K_n) \geqslant \beta(\sigma_{I_n-1}(T_{n-1})); \\ \sigma(T_{n-1}) & \text{otherwise.} \end{cases}$$

Define $\sigma(t) = \sigma(T_n)$ for $t \in [T_n, T_{n+1})$. It is immediate to check that the above construction gives the process with generator $L_N$.

The process $\sigma'(t)$ is defined using the same Poisson processes and uniform random variables but substituting $\beta$ with $\beta'$ where it corresponds. In this way, we have constructed a coupling between the processes $\sigma(t)$ and $\sigma'(t)$.

**Lemma 3.2.** *With the above coupling, if*

$$b_{x+1} + c_{x+1} - b_x - c_x \leqslant b'_{x+1} + c'_{x+1} - b'_x - c'_x \tag{3.1}$$

*for all $x$ and*

$$\sigma_i(0) \leqslant \sigma'_i(0)$$

*for all $i$, then*

$$P(\sigma_i(t) \leqslant \sigma'_i(t) \text{ for all } i) = 1.$$

*In other words, if the increments of* $b_x+c_x$ *are dominated by the increments of* $b'_x+c'_x$*, then the coupling conserves order for initial configurations in the same sublattice.*

**Proof.** Notice first that under Eq. (3.1), for all $x$ it holds

$$\beta(x)\leqslant\beta'(x).$$

We prove for all $n$ that, if $\sigma(T_{n-1})\leqslant\sigma'(T_{n-1})$, then $\sigma(T_n)\leqslant\sigma'(T_n)$. Assume that at the Poisson time $T_n$, $I_n=i$, $U(I_n,K_n)=u$ and call $x(i,n)=\sigma_i(T_n)$, $x'(i,n)=\sigma'_i(T_n)$. Since by hypothesis $x(n-1,i)$ and $x'(n-1,i)$ have the same parity and the length of the jump cannot exceed 2, we have to consider only two cases: (1) $x(n-1,i)=x'(n-1,i)$ and (2) $x(n-1,i)+2=x'(n-1,i)$.

In case (1) we have

$$\begin{aligned}\sigma_i(T_n) &= x(n-1,i)+2\mathbf{1}\{u<\beta(x(n-1,i-1))\}\\ &\qquad\times\mathbf{1}\{x(n-1,i-1)=x(n-1,i+1)=x(n-1,i)+1\}\\ &\qquad\quad-2\mathbf{1}\{u>\beta(x(n-1,i-1))\}\\ &\qquad\times\mathbf{1}\{x(n-1,i-1)=x(n-1,i+1)=x(n-1,i)-1\}\\ &\leqslant x(n-1,i)+2\mathbf{1}\{u<\beta'(x'(n-1,i-1))\}\\ &\qquad\times\mathbf{1}\{x'(n-1,i-1)=x'(n-1,i+1)=x(n-1,i)+1\}\\ &\qquad\quad-2\mathbf{1}\{u>\beta'(x'(n-1,i-1))\}\\ &\qquad\times\mathbf{1}\{x'(n-1,i-1)=x'(n-1,i+1)=x(n-1,i)-1\}\\ &=\sigma'_i(T_n).\end{aligned}$$

The inequality is obtained for the positive terms because
(a) since for all $x$, $\beta(x)\leqslant\beta'(x)$, in the set $\{x(n-1,i-1)=x(n-1,i+1)=x(n-1,i)-1=x'(n-1,i-1)=x'(n-1,i+1)=x'(n-1,i)-1\}$,

$$\mathbf{1}\{u<\beta(x(n-1,i-1))\}\leqslant\mathbf{1}\{u<\beta'(x'(n-1,i-1))\};$$

(b) in the set $\{x(n-1,i)=x'(n-1,i)\}$, if $x(n-1,i-1)<x'(x-1,i-1)$, then $\mathbf{1}\{x(n-1,i-1)=x(n-1,i+1)=x(n-1,i)+1\}=0$ while $\mathbf{1}\{x'(n-1,i-1)=x'(n-1,i+1)=x(n-1,i)+1\}\geqslant 0$; if $x(n-1,i+1)<x'(x-1,i+1)$, then $\mathbf{1}\{x(n-1,i-1)=x(n-1,i+1)=x(n-1,i)+1\}=0$ while $\mathbf{1}\{x'(n-1,i-1)=x'(n-1,i+1)=x(n-1,i)+1\}\geqslant 0$.

Analogous reasons show that the negative terms are non-increasing.

To show case (2), we need only to show that a jump up for $x(n-1,i)$ and a jump down for $x'(n-1,i)=x(n-1,i)+2$ cannot occur simultaneously. This could happen in the set $\{x(n-1,i-1)=x(n-1,i+1)=x(n-1,i)+1=x'(n-1,i-1)=x'(n-1,i+1)=x'(n-1,i)-1\}$. But the jump up occurs if $u<\beta(x(n-1,i-1))$, while the jump down occurs if $u \geqslant \beta'(x(n-1,i-1))$. Since for all $x$, $\beta(x) \leqslant \beta'(x)$, the two jumps cannot occur simultaneously. □

**Proof of Proposition 3.1.** Call $S_N(t)$ and $S'_N(t)$ the semigroups corresponding to the processes $\sigma(t)$ and $\sigma'(t)$. If $\nu_N$ and $\nu'_N$ are measures on $\mathscr{C}_N$ such that $\nu_N \leqslant \nu'_N$ stochastically, then there exists a measure on $(\mathscr{C}_N)^2$ with marginals $\nu_N$ and $\nu'_N$ concentrating mass in the set $\{(\sigma,\sigma') \in (\mathscr{C}_N)^2\colon \sigma \leqslant \sigma'$ and $\sigma_i, \sigma'_i$ have the same parity$\}$. Applying Lemma 3.2 to the initial configurations $\sigma, \sigma'$ with distributions $\nu_N$ and $\nu'_N$, respectively, and such that $\sigma \leqslant \sigma'$, we obtain that $\sigma(t) \leqslant \sigma'(t)$. This implies that

$$\nu_N S_N(t) \leqslant \nu'_N S'_N(t)$$

for all $t$ and, since $\mu_N = \lim_{t\to\infty} \nu_N S_N(t)$ and $\mu'_N = \lim_{t\to\infty} \nu'_N S'_N(t)$, we have $\mu_N \leqslant \mu'_N$.

To show the second part of the proposition, observe that both $\mu_N$ and $\mu'_N$ define birth and death processes on $\mathbb{Z}_+$ conditioned to fixed starting and ending points. But since the thermodynamic limit $\mu'$ defines a stationary positive recurrent birth and death process and any weak limit $\mu$ of $\mu_N$ satisfies $\mu \leqslant \mu'$, then $\mu$ defines a stationary positive recurrent birth and death process. Hence $\mu$ must be unique. □

**Proof of Theorem 1.4.** We only need to show part (ii). Assume that $\boldsymbol{b}, \boldsymbol{c}$ are sequences such that there exists an $M > 0$ such that $b_{x+1} + c_{x+1} - b_x - c_x \leqslant 0$ for all $x \geqslant M$. Let sequences $\boldsymbol{b}', \boldsymbol{c}'$ be defined by

$$b'_x = \begin{cases} b_x & \text{if } x \leqslant M, \\ b_M & \text{if } x > M, \end{cases} \qquad c'_x = \begin{cases} c_x & \text{if } x \leqslant M, \\ c_M & \text{if } x > M. \end{cases}$$

Then $\boldsymbol{b}, \boldsymbol{c}$ satisfy conditions (1.8) of Theorem 1.2. Now apply Proposition 3.1 to the sequences $\boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{b}', \boldsymbol{c}'$. This proves (ii). □

**Comment about the invariant measure for the infinite particle system.** The process $\sigma^N(t)$ with generator $L_N$ defined on $\mathscr{C}_N$ converges, as $N \to \infty$, to a process $\sigma^\infty(t)$ on $\mathbb{Z}^{\mathbb{Z}}$. In fact, since the rates of flipping are bounded by 1 and the process is one dimensional, it is a simple matter to show that

$$\lim_{N\to\infty} S_N(t) = S(t),$$

a semigroup corresponding to the generator $L = \lim_N L_N$. See Liggett (1985) for a discussion about generators and semigroups. On the other hand, since $\mu_N$ is invariant for $S_N(t)$,

$$\mu_N S_N(t) = \mu_N.$$

This implies that if the weak limit

$$\lim_{N\to\infty} \mu_N = \mu$$

exists, then $\mu$ is invariant for $S(t)$. Notice that this process corresponds to interacting random walks or birth and death chains, one for each $i$. Each one of these walks jumps two units at rate at most 1.

Now, assume that $(\alpha_x)$ is ultimately constant and satisfies the conditions of Theorem 1.2. Let

$$\bar{\mathscr{C}}_N = \mathscr{C}_{2N+1}(u,v) = \{\sigma \in \mathbb{Z}_+^{\{-N,\dots,N\}}: \sigma_{-N} = \sigma_N\}$$

The space $\bar{\mathscr{C}}_N$ corresponds to periodic boundary conditions. For $\sigma \in \bar{\mathscr{C}}_N$ consider the generator

$$\bar{L}_N f(\sigma) = \sum_i [\bar{\beta}(\sigma_{i-1})[f(\sigma^{i+}) - f(\sigma)] + (1 - \bar{\beta}(\sigma_{i-1}))[f(\sigma^{i-}) - f(\sigma)]].$$

defining $\bar{\beta}$ as $\beta$ but assuming $\bar{\sigma}(N+1) = \bar{\sigma}(-N+1)$. Let $\bar{S}_N(t)$ the corresponding semigroup. For the process with generator $\bar{L}_N$, there is a drift in the direction of the origin at most for a finite number of configurations. It is not hard to show that under periodic boundary conditions the countable state process with generator $\bar{L}_N$ is null recurrent.

On the other hand, the infinite volume process with generator $S(t)$ can be obtained as the limit of any one of the semigroups $S_N(t)$ or $\bar{S}_N(t)$.

Under the conditions of Theorem 1.2, the process with generator $L_N$ accepts an invariant measure $\mu_N$ and this measure converges to $\mu$, as $N \to \infty$. Theorem 1.2 guarantees that this measure is a Gibbs state. Hence the infinite volume process has $\mu$ as invariant measure. This invariant measure is obtained when pinning the extremes at the finite values $u$ and $v$.

The curious fact is that the pinning disappears in the infinite volume limit but the invariant measure persists. The invariant measure cannot be obtained using the limit of the periodic system.

## Acknowledgements

We thank Roberto Fernandez for very enlightening discussions about motivations and references concerning this work and to Joachim Krug for clarifying us about the role of the entropic repulsion. This work is partially supported by FONDECYT 1970506 and FONDAP-CONICYT, Chile, FAPESP, CNPq and PRONEX, Brazil.

## Appendix. Proof of Lemma 1.1

Obviously Eq. (1.6) follows from relation (1.5) because the $\gamma$ term simplifies in the measures $\mu^{\mathrm{H}}$. Let us show this last equality. First let us introduce some notation. For $i \leqslant j$ in $\mathbb{Z}$ put $\sigma = w[i-1, j+1]$. Also it is convenient to set $\sigma(-1) = w_{i-1}$, $\sigma(+1) = w_{j+1}$, in fact, the boundary terms will appear in the sums with a periodicity, with this notation we have $\sigma((-1)^z) = w_{i-1}$ if $z$ is odd, and $\sigma((-1)^z) = w_{j+1}$ if $z$ is even. We shall use recurrence to get relations between $\mathscr{N}_x, \mathscr{N}_{x,y}$, then we put the boundary condition

$$\mathscr{N}_{-1}(\sigma) = \mathscr{N}_{-1,0}(\sigma) = \mathscr{N}_{0,-1}(\sigma) = 0. \tag{A.1}$$

Observe that the following relations are verified:

$$\mathcal{N}_x(\sigma) = \mathcal{N}_{x,x+1}(\sigma) + \mathcal{N}_{x,x-1}(\sigma) + \delta(x, \sigma(+1))$$

and

$$\mathcal{N}_x(\sigma) = \mathcal{N}_{x+1,x}(\sigma) + \mathcal{N}_{x-1,x}(\sigma) + \delta(x, \sigma(-1)) \quad \text{for any } x \in \mathbb{Z}_+. \tag{A.2}$$

By recurrence, it is direct to check that Eq. (A.2) with boundary relations (A.1) have the unique solution:

$$\begin{aligned} \mathcal{N}_{x+1,x}(\sigma) &= \sum_{y=0}^{x} (-1)^{x+y} \mathcal{N}_y(\sigma) + \sum_{y=0}^{x} (-1)^{x+y+1} \delta(y, \sigma((-1)^{x+y+1})), \\ \mathcal{N}_{x,x+1}(\sigma) &= \sum_{y=0}^{x} (-1)^{x+y} \mathcal{N}_y(\sigma) + \sum_{y=0}^{x} (-1)^{x+y+1} \delta(y, \sigma((-1)^{x+y})). \end{aligned} \tag{A.3}$$

We set $\overline{H} = H^{\boldsymbol{b},\boldsymbol{c}}_{[i,j]}(w)$ and $H = H^{\alpha}_{[i,j]}(w)$. Let $K = (w_{i-1} \vee w_{j+1}) + (j-i+4)$, then $w_x < K$ for $x \in [i-1, j+1]$. Hence $\mathcal{N}_x(\sigma) = 0$, $\mathcal{N}_{x,x+1}(\sigma) = 0$, $\mathcal{N}_{x+1,x}(\sigma) = 0$ for $x > K$. From Eqs. (1.1), (1.2) and (A.3) we can decompose $\overline{H} = \overline{H}^1 + \overline{H}^2$, with

$$\overline{H}^1 = \sum_{x \in \mathbb{Z}_+} (b_x + c_x) \left( \sum_{y=0}^{x} (-1)^{x+y} \mathcal{N}_y(\sigma) \right) = \sum_{y=0}^{K} \left( \sum_{x=y}^{K} (-1)^{x+y} (b_x + c_x) \right) \mathcal{N}_y(\sigma)$$

and

$$\overline{H}^2 = \sum_{x \in \mathbb{Z}_+} \left( b_x \sum_{y=0}^{x} (-1)^{x+y+1} \delta(y, \sigma((-1)^{x+y})) + c_x \sum_{y=0}^{x} (-1)^{x+y+1} \delta(y, \sigma((-1)^{x+y+1})) \right).$$

Then

$$\begin{aligned} \overline{H}^2 &= \sum_{y=0}^{K} \left( \sum_{x=y}^{K} (-1)^{x+y+1} (b_x \delta(y, \sigma((-1)^{x+y})) + c_x \delta(y, \sigma((-1)^{x+y+1}))) \right) \\ &= \gamma_1(w_{i-1}, w_{j+1}, K). \end{aligned}$$

On the other hand, by using relations (1.4) we get

$$\begin{aligned} \overline{H}^1 &= \sum_{y=0}^{K} \left( \sum_{x=y}^{K} (-1)^{x+y} (\alpha_x + \alpha_{x+1}) \right) \mathcal{N}_y(\sigma) = \sum_{y=0}^{K} (\alpha_y + (-1)^{y+K} \alpha_{K+1}) \mathcal{N}_y(\sigma) \\ &= H + \left( \sum_{y=0}^{K} (-1)^{y+K} \mathcal{N}_y(\sigma) \right) \alpha_{K+1} = H + \mathcal{N}_{K,K+1}(\sigma) \alpha_{K+1} + \gamma_2(w_{i-1}, w_{j+1}, K), \end{aligned}$$

with $\gamma_2(w_{i-1}, w_{j+1}, K) = -\sum_{y=0}^{K} (-1)^{x_y+1} \delta(y, \sigma((-1)^{x+y})) \alpha_{K+1}$. This last equality following from Eq. (A.3). Since $\mathcal{N}_{K,K+1}(\sigma) = 0$ we find $\overline{H} = H + \gamma(w_{i-1}, w_{j+1}, j-i)$ with $\gamma(w_{i-1}, w_{j+1}, j-i) = \gamma_1(w_{i-1}, w_{j+1}, K) + \gamma_2(w_{i-1}, w_{j+1}, K)$. Then Eq. (1.5) is shown.

## References


Cesi, F., Martinelli, F., 1996a. On the layering transition of an SOS surface interacting with a wall. II. The Glauber dynamics. Comm. Math. Phys. 177 (1) 173–201.

Cesi, F., Martinelli, F., 1996b. On the layering transition of an SOS surface interacting with a wall. I. Equilibrium results. J. Statist. Phys. 82 (3-4) 823–913.

Ferrari, P., Martínez, S., Picco P., 1992. Existence of non trivial quasi-stationary distributions in the birth and death chains. Adv. Appl. Probab. 24, 795–813.

Ferrari, P., Martínez, S., 1994. Quasi-stationary distributions: a continued fraction criterion for recurrence. Resenhas 1, 321–333.

Georgii, H.-O., 1988. Gibbs measures and phase transitions. de Gruyter Studies in Mathematics, 9. Walter de Gruyter & Co., Berlin-New York, 1988.

Gurevich. B.M., 1984. A variational characterization of one-dimensional countable Gibbs random fields. Z. Wahrscheinlichkeitstheorie verw. Gabiete 68, 205–242.

Kesten, H., 1976. Existence and uniqueness of countable one-dimensional random fields. Ann. Probab. 4, 557–569.

Lebowitz, J.L., Mazel, A.E., 1996. A remark on the low-temperature behavior of the SOS interface in half-space. J. Statist. Phys. 84 (3–4) 379–397.

van Leeuwen, J.M.J., Hilhorst, H.J. 1981. Pinning of a rough interface by an external potential. Phys. A 107 (2) 319–329.

Vere-Jones, D., 1962. Geometric ergodicity in denumerable Markov chains. Quart J. Math. Oxford Ser. 2, 7–28.

Vere-Jones, D., 1967. Ergodic properties of non-negative matrices I. Pacific J. Math. 22, 361–386.

Wall, H.C., 1948. Analytic Theory of Continued Fractions. D. Van Nostrand Co. Inc.

Liggett, T.L., 1985. Interacting Particle Systems. Springer Verlag.