Stochastic Processes and their Applications 84 (2000) 333–354

stochastic processes and their applications

www.elsevier.com/locate/spa

Entropic repulsion for massless fields

Jean-Dominique Deuschel[a,b,*], Giambattista Giacomin[a,b]

[a] TU-Berlin, Fachbereich Mathematik, Strasse des 17 juni 136, D-10623 Berlin, Germany

[b] Dipartimento di Matematica, Università di Milano, via C. Saldini 50, 20133 Milano, Italy

Received 13 August 1999; received in revised form 7 March 2000; accepted 7 March 2000

Abstract

We consider the *anharmonic crystal*, or *lattice massless field*, with 0-boundary conditions outside $D_N = ND \cap \mathbb{Z}^d$, $D \subseteq \mathbb{R}^d$ and N a large natural number, that is the finite volume Gibbs measure $\mathbb{P}_N$ on $\{\varphi \in \mathbb{R}^{\mathbb{Z}^d}: \varphi_x = 0$ for every $x \notin D_N\}$ with Hamiltonian $\sum_{x\sim y} V(\varphi_x - \varphi_y)$, V a strictly convex even function. We establish various bounds on $\mathbb{P}_N(\Omega^+(D_N))$, where $\Omega^+(D_N) = \{\varphi: \varphi_x \geqslant 0$ for all $x \in D_N\}$. Then we extract from these bounds the asymptotics ($N \to \infty$) of $\mathbb{P}_N(\cdot|\Omega^+(D_N))$: roughly speaking we show that the field is repelled by a hard-wall to a height of $O(\sqrt{\log N})$ in $d \geqslant 3$ and of $O(\log N)$ in $d = 2$. If we interpret φ_x as the height at x of an interface in a $(d+1)$-dimensional space, our results on the conditioned measure $\mathbb{P}_N(\cdot|\Omega^+(D_N))$ clarify some aspects of the effect of a hard-wall on an interface. Besides classical techniques, like the FKG inequalities and the Brascamp–Lieb inequalities for log-concave measures, we exploit a representation of the random field in term of a random walk in dynamical random environment (Helffer–Sjöstrand representation).

MSC: 60K35; 82C24

Keywords: Gibbs measures; Massless fields; Entropic repulsion; Random surfaces; Random walk in random environment

1. Introduction

Let $D \subset \mathbb{R}^d$ be a connected bounded domain with non-void interior and piecewise smooth boundary (this class of sets will be denoted by $\mathscr{D}$). For $N \in \mathbb{Z}^+$ let $D_N = ND \cap \mathbb{Z}^d$ be the basis of the *random interface* $\varphi = \{\varphi_x, x \in D_N\} \in \Omega_N = \mathbb{R}^{D_N}$. Consider the formal Hamiltonian $\mathscr{H}_N$

$$\mathscr{H}_N(\varphi) = \sum_{x\sim y} V(\varphi_x - \varphi_y), \tag{1.1}$$

where the summation is over the nearest neighbors of $\mathbb{Z}^d$, each couple of nearest-neighbor points is counted only once, and we set 0 boundary condition on $D_N^{\complement}$: $\varphi_x \equiv 0$,

* Corresponding author. Fax: 49-30-31-421-695.

E-mail addresses: deuschel@math.tu-berlin.de (J.-D. Deuschel), giacomin@mat.unimi.de (G. Giacomin).

0304-4149/00/$ - see front matter
PII: S0304-4149(00)00030-2

$x \notin D_N$. The corresponding finite volume Gibbs state with 0 boundary condition is given by

$$\mathbb{P}_N(\mathrm{d}\varphi) \equiv \frac{1}{Z_N}\exp(-\mathscr{H}_N(\varphi)) \prod_{x\in D_N} \mathrm{d}\varphi_x. \tag{1.2}$$

We make the following assumptions on the potential V:

A1. *Smoothness.* $V \in C^{2,\delta}(\mathbb{R})$, the set of C^2 functions with δ-Hölder continuous second derivatives, $\delta > 0$.

A2. *Symmetry.* $V(r) = V(-r), \quad r \in \mathbb{R}$.

A3. *Strict convexity.* There exists $\underline{c} > 0$ such that

$$V'' \geqslant \underline{c}. \tag{1.3}$$

A4. *Boundedness.* There exists $\bar{c} \in \mathbb{R}$ such that

$$V'' \leqslant \bar{c}. \tag{1.4}$$

Sometimes this last condition can be replaced by the weaker constraint on the growth of V'' of at infinity:

A5. *α-growth at infinity*. For some $\alpha \geqslant 1$,

$$\limsup_{r\to\infty} \frac{V''(r)}{r^{2\alpha-2}} \leqslant c_\alpha < \infty. \tag{1.5}$$

The prototype of interactions satisfying the above conditions is the quadratic or *harmonic* potential [1]

$$V^*(\varphi_x - \varphi_y) = \frac{1}{2d}(\varphi_x - \varphi_y)^2.$$

In this case the measure $\mathbb{P}^*_N$ is *Gaussian* and it is called *the* (*finite volume*) *massless free field*: it is fully characterized by the mean

$$\mathbb{E}^*_N[\varphi_x] = 0, \quad x \in D_N \tag{1.6}$$

and the covariances

$$\mathrm{cov}^*_N(\varphi_x, \varphi_y) = G^*_{D_N}(x, y), \quad x, y \in D_N, \tag{1.7}$$

where $G^*_{D_N}$ is the Green function of the simple random walk, killed as it exits D_N (cf. Bolthausen and Deuschel, 1993). The behavior as $N \nearrow \infty$ of $G^*_{D_N}$ is known in detail (see e.g. Lawler, 1991). In particular in the *recurrent dimension* $d = 2$ we have a logarithmic divergence, $\mathrm{var}^*_N(\varphi_x) = \mathrm{O}(\log N)$, whereas in the *transient dimensions* $d \geqslant 3$, the variance remains bounded: $\mathrm{var}^*_N(\varphi_x) = \mathrm{O}(1)$. In $d \geqslant 3$, $G^*_{D_N}$ converges to the infinite volume Green function G^*, independently of D, and one has therefore existence of the weak limit $\mathbb{P}^*$ of $\mathbb{P}^*_N$. In the Gaussian case *all* the infinite volume massless Gibbs states are known (see e.g. Georgii, 1998).

In the general *non-harmonic* case, by using the Brascamp–Lieb (B–L) inequality (cf. Brascamp and Lieb, 1976; Brascamp et al. 1976), one sees that the variances of the harmonic and non-harmonic case are comparable (see the appendix for the precise statements). This has immediate consequences: in $d = 2$ we have the same logarithmic

[1] We use the superscript $*$ to denote this special case.

divergence we encountered in the Gaussian case and $\mathrm{var}_N(\varphi_x) = \mathrm{O}(1)$ for N large in $d \geqslant 3$. In the latter case, as a consequence, the family of measures $\{\mathbb{P}_N, N \in \mathbb{Z}^+\}$ is tight and we denote by $\mathbb{P}$ any limit point of this sequence: the results that we will present hold for any such a $\mathbb{P}$ with the same constant. The structure of the set of massless Gibbs measures is not yet fully understood: the most advanced results have been obtained, under A4, by Funaki and Spohn (1997).

The substantial limitation of the B–L inequalities is that they yield only diagonal estimates. Recently, however Helffer and Sjöstrand have introduced a representation similar to (1.7) for the covariances, but this time in terms of the Green function of a symmetric random walk in a dynamical random environment (cf. Helffer and Sjöstrand, 1994; Deuschel et al., 2000). This representation, which gives a direct derivation of B–L inequalities, allows for example off-diagonal estimates, see Section 3.

We remark here that we will repeatedly exploit another well known tool for continuous spin systems with convex interactions: the FKG inequality. As well as the B–L inequalities, the FKG inequality can be extracted from the H–S representation (Helffer and Sjöstrand, 1994; Naddaf and Spencer, 1997).

The fundamental event (entropic repulsion event) that we will analyze is

$$\Omega^+(A_N) = \{\varphi \in \Omega_N : \varphi_x \geqslant 0 \text{ for all } x \in A_N\},$$

where $A \in \mathscr{D}$ with $A \subset D$. In particular, we want to describe the asymptotic behavior of $\mathbb{P}_N(\Omega^+(A_N))$ and of the conditional measure $\mathbb{P}_N(\cdot\,|\Omega^+(A_N))$ as $N \to \infty$. Our main objective is to generalize the result recently obtained for the Gaussian case to the non-harmonic situation (Bolthausen et al., 1995; Deuschel, 1996). Examples of situations in which these type of results are of interest are

E1. The construction of droplets on a hard wall with fixed volume (cf. Bolthausen and Ioffe, 1997; Deuschel et al., 2000);
E2. The investigation of the wetting transition (cf. Bricmont et al., 1986; Bolthausen et al., 2000a);
E3. Questions related to quasi-locality of random fields (cf. van Enter et al., 1993; Lebowitz and Maes, 1987).

We start our analysis by showing that, for sites x in the *interior of* D_N, that is at distance $\mathrm{dist}(x, D_N^{\complement}) > \delta N$, $\delta > 0$, under the hard wall condition $\Omega^+(D_N)$, the random interface φ_x is repelled at height $\mathrm{O}(\log N)$ in the recurrent dimension $d = 2$, respectively $\mathrm{O}(\sqrt{\log N})$ in the transient dimensions $d \geqslant 3$. This phenomenon, called entropic repulsion, is due to the relative *stiffness* of the interface and to the local fluctuations which push the random interface to infinity in presence of the hard wall condition (see Corollaries 2.6 and 2.8 for the precise statements). This result follows from a careful investigation of the probability of $\Omega^+(A_N)$ for $A \subseteq D$ with $\mathrm{dist}(A, D^{\complement}) > \delta$: we prove the existence of $0 < C_1 \leqslant C_2 < \infty$ such that

$$\begin{aligned} -C_2 &\leqslant \liminf_{N\to\infty} \frac{1}{N^{d-2}\log_d(N)} \mathbb{P}_N(\Omega^+(A_N)), \\ &\leqslant \limsup_{N\to\infty} \frac{1}{N^{d-2}\log_d(N)} \mathbb{P}_N(\Omega^+(A_N)) \leqslant -C_1, \end{aligned} \tag{1.8}$$

where we write $\log_d(N) = \log N$ for $d \geqslant 3$ and $\log_2(N) = (\log N)^2$. This result can be extended to infinite Gibbs state(s) $\mathbb{P}$ in transient dimensions. Probability estimates for the events considered in (1.8) will be referred to as estimates in the *interior* (or in the *bulk*).

We speak of *boundary estimates* when dealing with probability estimates for $\Omega^+(D_N)$, i.e. the repulsion goes all the way to the boundary. The behavior is then characterized by a pure surface order: for $L \in \mathbb{N}$ let $\partial_L D_N = \{x \in D_N \colon \operatorname{dist}(x, D_N^{\complement}) \leqslant L\}$ then

$$\lim_{L\to\infty} \limsup_{N\to\infty} \frac{1}{N^{d-1}} |\log \mathbb{P}_N(\Omega^+(D_N)) - \log \mathbb{P}_N(\Omega^+(\partial_L D_N))| = 0. \tag{1.9}$$

That is, only the spins at finite distance to the boundary, are responsible for the leading surface order of the decay of $\mathbb{P}_N(\Omega^+(D_N)) = \exp(-\mathrm{O}(N^{d-1}))$.

The paper is organized as follows. In Section 2 we present the results in the interior of D_N. Section 3 is devoted to the behavior at the boundary. In the appendix we recall the B–L inequalities in our framework.

2. Results in the interior of D_N

2.1. Harmonic results

Let us briefly recall the results for the Gaussian case where one has a very precise picture of the entropic repulsion.

Theorem 2.1. *Let $A, D \in \mathscr{D}$ with $A \subset D$ and $\operatorname{dist}(A, D^{\complement}) > 0$, then*

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log_d(N)} \log \mathbb{P}_N^*(\Omega^+(A_N)) = -2\boldsymbol{G}_d^* \operatorname{cap}_D(A), \tag{2.1}$$

where

$$\boldsymbol{G}_d^* = \begin{cases} \lim_{N\to\infty} G_{D_N}^*(0,0), & d \geqslant 3, \\ \lim_{N\to\infty} G_{D_N}^*(0,0)/\log N, & d = 2 \end{cases} \tag{2.2}$$

and

$$\operatorname{cap}_D(A) = \inf\left\{\tfrac{1}{2}\|\nabla h\|^2_{L^2(D)} \colon h \in H_0^1(D), h \geqslant \mathbf{1}_A\right\} \tag{2.3}$$

is the Newtonian capacity of A in D. Next, for each $\varepsilon > 0$ and $\delta \in (0,1)$,

$$\lim_{N\to\infty} \inf_{x \in D_{\delta N}} \mathbb{P}_N^*\left(\left|\frac{\varphi_x}{\sqrt{\log_d(N)}} - \sqrt{4\boldsymbol{G}_d^*}\right| \leqslant \varepsilon \,\middle|\, \Omega^+(D_N)\right) = 1. \tag{2.4}$$

Finally, for $d \geqslant 3$, (2.1) and (2.4) also hold with $\mathbb{P}_N^*$ replaced by the infinite volume free field $\mathbb{P}^*$ and $\operatorname{cap}_D(A)$ replaced by $\operatorname{cap}_{\mathbb{R}^d}(A)$.

Proof. For transient dimensions $d \geqslant 3$, (2.1) and (2.4) are proved in Deuschel (1996) and, respectively, Bolthausen et al. (1995) for the infinite volume measure. Actually in this case one can even take $\delta = 1$ for domain D satisfying an exterior cone property (cf. Deuschel and Giacomin, 1999). The quoted result for $d = 2$ is part of

Bolthausen et al. (2000b). We remark that the existence of the limit that defines G_2^* in (2.2) is proven in Lawler (1991). □

2.2. Lower bounds

We prove here the lower bound for the probability of the entropic repulsion and give an upper bound for the height of the repulsion. We recall that the notation $x \sim y$ means that $x, y \in \mathbb{Z}^d$ are nearest neighbors, i.e. $|x-y|=1$.

Theorem 2.2. *Assume* A1–A3 *and* A5, *take* $A, D \in \mathscr{D}$ *with* $A \subset D$ *and* $\text{dist}(A, D^{\complement}) > 0$, *then*

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}\log_d(N)} \log \mathbb{P}_N(\Omega^+(A_N)) \geqslant -\frac{dc_\alpha C_V}{\underline{c}} \boldsymbol{G}_d^* \operatorname{cap}_D(A),$$

where

$$C_V = \limsup_{N\to\infty} \sup_{x\sim y} \mathbb{E}_N[(\varphi_x - \varphi_y)^{2\alpha-2}] < \infty.$$

Proof. In the proofs we will use explicitly the *continuum symmetry* of the model: given $\tilde{\varphi} \in \Omega_N$, let us denote by $T_{\tilde{\varphi}}$ the map from Ω_N to Ω_N defined by $(T_{\tilde{\varphi}}\varphi)_x = \varphi_x + \tilde{\varphi}_x$. Let us also set $\mathbb{P}_N^{\tilde{\varphi}} = \mathbb{P}_N \circ T_{\tilde{\varphi}}^{-1}$. We use a change of measure argument, as in Lemma 2.3 of Bolthausen et al. (1995). The changed measure will be simply obtained by translating the original field, in a way close in spirit to Fröhlich and Pfister (1981), in which one can find the following lemma.

Lemma 2.3. *Given* $\tilde{\varphi} \in \Omega_N$, *we have that* $\mathbb{P}_N$-*a.s.*

$$\log \frac{\mathrm{d}\mathbb{P}_N^{\tilde{\varphi}}}{\mathrm{d}\mathbb{P}_N}(\varphi) = \sum_{x\sim y} [V(\varphi_x - \varphi_y) - V(\varphi_x - \varphi_y - (\tilde{\varphi}_x - \tilde{\varphi}_y))].$$

Proof. The result is a direct consequence of the definition of the measure. □

We will use the following inequality, which is a consequence of Jensen's inequality (see e.g. Lemma 5.4.21 in Deuschel and Stroock (1989)):

$$\log\left(\frac{\mathbb{P}_N(\Omega^+(A_N))}{\mathbb{P}_N^{\tilde{\varphi}}(\Omega^+(A_N))}\right) \geqslant -\frac{1}{\mathbb{P}_N^{\tilde{\varphi}}(\Omega^+(A_N))}[\boldsymbol{H}(\mathbb{P}_N^{\tilde{\varphi}}|\mathbb{P}_N) + (1/e)] \tag{2.5}$$

for any $\tilde{\varphi}$. Here $\boldsymbol{H}$ is the relative entropy, i.e. $\boldsymbol{H}(\mathbb{P}_N^{\tilde{\varphi}}|\mathbb{P}_N) = \mathbb{E}_N^{\tilde{\varphi}}[\log(\mathrm{d}\mathbb{P}_N^{\tilde{\varphi}}/\mathrm{d}\mathbb{P}_N)]$. An upper bound on the relative entropy is provided by the following lemma.

Lemma 2.4. *Let* $\tilde{\varphi}_x = a(N)\psi(x/N)$, *where* $a(N) = \sqrt{a\log_d(N)}, a > 0$, *and* $\psi \in C_0^2(D)$, *then*

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}\log_d(N)} \boldsymbol{H}(\mathbb{P}_N^{\tilde{\varphi}}|\mathbb{P}_N) \leqslant \frac{ac_\alpha C_V}{2} \|\nabla\psi\|_{L^2(D)}^2. \tag{2.6}$$

Proof. By means of Lemma 2.3, expanding the sum in Taylor series, we have

$$\begin{aligned}\boldsymbol{H}(\mathbb{P}_N^{\tilde\varphi}|\mathbb{P}_N) &= \mathbb{E}_N^{\tilde\varphi}\left[\sum_{x\sim y}(V(\varphi_x-\varphi_y)-V(\varphi_x-\varphi_y-(\tilde\varphi_x-\tilde\varphi_y)))\right]\\ &= \mathbb{E}_N\left[\sum_{x\sim y}(V(\varphi_x-\varphi_y+(\tilde\varphi_x-\tilde\varphi_y))-V(\varphi_x-\varphi_y))\right]\\ &= \sum_{x\sim y}\left[(\tilde\varphi_x-\tilde\varphi_y)\mathbb{E}_N[V'(\varphi_x-\varphi_y)]\right.\\ &\quad\left.+(\tilde\varphi_x-\tilde\varphi_y)^2\int_0^1\int_0^t\mathbb{E}_N[V''(\varphi_x-\varphi_y+s(\tilde\varphi_x-\tilde\varphi_y))]\,\mathrm{d}s\,\mathrm{d}t\right].\end{aligned}$$

We now use the fact that V' is odd, so that $\mathbb{E}_N[V'(\varphi_x-\varphi_y)]=0$. Moreover, since ψ has bounded derivatives we have that there exists a constant c such that

$$\sup_{x\sim y}\frac{|\tilde\varphi_x-\tilde\varphi_y|}{a(N)/N}\leqslant c \tag{2.7}$$

and therefore for every $s\in[-1,1]$

$$\begin{aligned}\limsup_{N\to\infty}\sup_{x\sim y}\mathbb{E}_N[V''(\varphi_x-\varphi_y+s(\tilde\varphi_x-\tilde\varphi_y))] &= \limsup_{N\to\infty}\sup_{x\sim y}\mathbb{E}_N[V''(\varphi_x-\varphi_y)]\\ &= c_\alpha C_V<\infty.\end{aligned} \tag{2.8}$$

We can then pass to the limit to obtain

$$\begin{aligned}\limsup_{N\to\infty}\frac{1}{N^{d-2}\log_d(N)}\boldsymbol{H}(\mathbb{P}_N^{\tilde\varphi}|\mathbb{P}_N) &\leqslant \frac{ac_\alpha C_V}{2}\lim_{N\to\infty}\frac{1}{N^d}\sum_{x\sim y}N^2[\psi(x/N)-\psi(y/N)]^2\\ &\leqslant \frac{ac_\alpha C_V}{2}\|\nabla\psi\|^2_{L^2(D)}. \qquad\square\end{aligned}$$

We go back to the proof of the lower bound in Theorem 2.2. Observe that there exists $C=C(A)$ such that

$$\mathbb{P}_N^{\tilde\varphi}((\Omega^+(A_N))^{\complement})\leqslant\sum_{x\in A_N}\mathbb{P}_N^{\tilde\varphi}(\varphi_x<0)\leqslant CN^d\sup_{x\in A_N}\mathbb{P}_N(\varphi_x>a(N)). \tag{2.9}$$

Next, by the B–L inequality (4.4)

$$\mathbb{P}_N(\varphi_x>a(N))\leqslant\exp\left(-\frac{\underline{c}a\log_d(N)}{2G^*_{D_N}(x,x)}\right). \tag{2.10}$$

Using $G^*_{D_N}(x,x)\leqslant\boldsymbol{G}^*_d$ for $d\geqslant3$, immediate consequence of the simple random walk representation (1.7), and $\limsup_{N\to\infty}\sup_{x\in A_N}G^*_{D_N}(x,x)/\log N\leqslant\boldsymbol{G}^*_2$ for $d=2$, we conclude that for each $a>2d\boldsymbol{G}^*_d/\underline{c}$

$$\lim_{N\to\infty}\mathbb{P}_N^{\tilde\varphi}(\Omega^+(A_N))\geqslant1-\lim_{N\to\infty}CN^{d-a\underline{c}/2\boldsymbol{G}^*_d}=1. \tag{2.11}$$

The lower bound then follows from the inequality (2.5), Lemma 2.4, (2.11) and the fact that

$$\mathrm{cap}_D(A)=\inf\{\|\nabla\psi\|^2_{L^2(D)}:\psi\in C_0^2(D),\psi\geqslant\mathbf{1}_A\}. \qquad\square$$

In transient dimensions $d\geqslant 3$, we can formulate the corresponding result for the infinite measure $\mathbb{P}$: the proof is just a slight modification of the proof of Theorem 2.2 and we leave the details to the interested reader.

Theorem 2.5. *Let* $d\geqslant 3$ *and assume* A1–A3 *and* A5, *take* $A\in\mathscr{D}$, *then*

$$\liminf_{N\to\infty}\frac{1}{N^{d-2}\log N}\log\mathbb{P}(\Omega^+(A_N))\geqslant -\frac{dc_\alpha C_V'}{\underline{c}}\boldsymbol{G}_d^*\,\mathrm{cap}_{\mathbb{R}^d}(A),$$

where, for any $x\sim y$

$$C_V'=\sup_{x\sim y}\mathbb{E}[(\varphi_x-\varphi_y)^{2\alpha-2}]<\infty. \tag{2.12}$$

As an immediate consequence of FKG and B–L, we have the following *upper* bound for the repulsion:

Corollary 2.6. *For any* $C>2d\boldsymbol{G}_d^*/\underline{c}$ *we have*

$$\limsup_{N\to\infty}\sup_{x\in D_N}\mathbb{P}_N(\varphi_x>\sqrt{C\log_d(N)}|\Omega^+(D_N))=0.$$

Proof. By the B–L inequality, which can also be applied to the conditioned measure $\mathbb{P}_N^+(\cdot)=\mathbb{P}_N(\cdot|\Omega^+(D_N))$ (see the appendix) we have

$$\mathbb{P}_N^+(\varphi_x>\sqrt{C\log_d(N)})\leqslant\exp\left(-\frac{\underline{c}((\sqrt{C\log_d(N)}-\mathbb{E}_N^+[\varphi_x])\vee 0)^2}{2G_{D_N}^*(x,x)}\right).$$

Thus it suffices to prove that

$$\limsup_{N\to\infty}\sup_{x\in D_N}\frac{\mathbb{E}_N^+[\varphi_x]}{\sqrt{C\log_d(N)}}<1.$$

Take $\tilde\varphi_x\equiv a(N), x\in D_N$, where $a(N)=\sqrt{a\log_d(N)}, a>0$, and introduce the conditioned probability $\mathbb{P}_N^{\tilde\varphi,+}=\mathbb{P}_N^{\tilde\varphi}(\cdot|\Omega^+(D_N))$. Then by FKG and B–L (4.2), we have

$$\mathbb{E}_N^+[\varphi_x]\leqslant\mathbb{E}_N^{\tilde\varphi,+}[\varphi_x]\leqslant\frac{\mathbb{E}_N^{\tilde\varphi}[|\varphi_x|]}{\mathbb{P}_N^{\tilde\varphi}(\Omega^+(D_N))}\leqslant\frac{a(N)+\sqrt{(\underline{c}d)^{-1}G_{D_N}^*(x,x)}}{\mathbb{P}_N^{\tilde\varphi}(\Omega^+(D_N))}.$$

Choosing $a>2d\boldsymbol{G}_d^*/\underline{c}$ as above implies

$$\mathbb{P}_N^{\tilde\varphi}(\Omega^+(D_N))^{\complement}\leqslant\sum_{x\in D_N}\mathbb{P}_N^{\tilde\varphi}(\varphi_x<0)\leqslant CN^d\sup_{x\in D_N}\mathbb{P}_N(\varphi_x>a(N))\to 0,$$

as $N\to\infty$, thus $\lim_{N\to\infty}\mathbb{P}_N^{\tilde\varphi}(\Omega^+(D_N))=1$ and shows the result. □

2.3. Upper bounds

Theorem 2.7. *Assume* A1–A4, *take* $A,D\in\mathscr{D}$ *with* $A\subset D$, *then there exists* $C_1=C_1(D,A)>0$ *such that*

$$\limsup_{N\to\infty}\frac{1}{N^{d-2}\log_d(N)}\log\mathbb{P}_N(\Omega^+(A_N))\leqslant -C_1. \tag{2.13}$$

Replacing the condition A4 *with* A5, *we have*, *in transient dimensions* $d \geqslant 3$

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}(\log N)^{1/\alpha}} \log \mathbb{P}_N(\Omega^+(A_N)) \leqslant -C_1. \tag{2.14}$$

Proof. We give a different argument for the transient and recurrent dimensions. Let us start with the transient dimensions assuming A1–A3 and A5: let $A_N^{\mathrm{e}}, A_N^{\mathrm{o}}$ denote the even and odd sites of A_N and write $\mathscr{F}^{\mathrm{e}} = \sigma\{\varphi_x : x \in A_N^{\mathrm{e}}\}$. Then, conditioning on even sites

$$\begin{aligned}\mathbb{P}_N(\Omega^+(A_N)) &= \mathbb{E}_N[\mathbb{P}_N(\Omega^+(A_N^{\mathrm{o}})|\mathscr{F}^{\mathrm{e}}); \Omega^+(A_N^{\mathrm{e}})] \\ &= \mathbb{E}_N\left[\prod_{x\in A_N^{\mathrm{o}}} \mathbb{P}_N(\varphi_x \geqslant 0|\mathscr{F}^{\mathrm{e}}); \Omega^+(A_N^{\mathrm{e}})\right].\end{aligned}$$

Next, let $m_x = (1/2d)\sum_{y:\,|y-x|=1}\varphi_y$ be the arithmetical mean of the neighbors of φ_x. Note that $m_x \geqslant 0$ on $\Omega^+(A_N^{\mathrm{e}})$. For given $a>0$ define

$$B_N \equiv \{x \in A_N^{\mathrm{o}} : m_x \leqslant a(N)\} \quad \text{where } a(N) = (a\log N)^{1/2\alpha},$$

then, for each $\varepsilon \in (0,1)$, in view of the above

$$\begin{aligned}\mathbb{P}_N(\Omega^+(A_N)) \leqslant\ & \mathbb{E}_N\left[\prod_{x\in A_N^{\mathrm{o}}} \mathbb{P}_N(\varphi_x \geqslant 0|\mathscr{F}^{\mathrm{e}}); \{|B_N| \geqslant \varepsilon |A_N^{\mathrm{o}}|\} \cap \Omega^+(A_N^{\mathrm{e}})\right] \\ &+ \mathbb{P}_N(\{|B_N| < \varepsilon|A_N^{\mathrm{o}}|\} \cap \Omega^+(A_N^{\mathrm{e}})).\end{aligned}$$

Take a site $x \in B_N$, then on $\Omega^+(A_N^{\mathrm{e}})$, we have $\varphi_y \leqslant 2da(N)$, $\forall y$ with $|y-x|=1$. Note that A5 implies

$$\limsup_{r\to\infty} \frac{V(r)}{r^{2\alpha}} \leqslant \frac{c_\alpha}{2}. \tag{2.15}$$

Now by FKG and A5, using the explicit expression for $\mathbb{P}_N(\cdot|\mathscr{F}^{\mathrm{e}})$ we get, with $\varrho = c_\alpha(2d)^{2\alpha}$

$$\begin{aligned}\mathbb{P}_N(\varphi_x \geqslant 0|\mathscr{F}^{\mathrm{e}}) &\leqslant \mathbb{P}_N(\varphi_x \geqslant 0|\varphi_y = 2da(N), |y-x|=1) \\ &\leqslant 1 - C\exp(-\varrho a(N)^{2\alpha})\end{aligned}$$

for some $C>0$. Thus on $\{|B_N| \geqslant \varepsilon|A_N^{\mathrm{o}}|\} \cap \Omega^+(A_N^{\mathrm{e}})$ we have the a priori estimate

$$\prod_{x\in A_N^{\mathrm{o}}} \mathbb{P}_N(\varphi_x \geqslant 0\,|\,\mathscr{F}^{\mathrm{e}}) \leqslant (1 - C\exp(-\varrho a\log N))^{\varepsilon|A_N^{\mathrm{o}}|} \leqslant \exp(-C'\varepsilon N^{d-a\varrho}). \tag{2.16}$$

This term can be neglected, as soon as $a < 2/\varrho$. On the other hand, we have

$$\{|B_N| < \varepsilon|A_N^{\mathrm{o}}|\} \cap \Omega^+(A_N^{\mathrm{e}}) \subset \left\{\frac{1}{|A_N^{\mathrm{o}}|}\sum_{x\in A_N^{\mathrm{o}}} m_x \geqslant (1-\varepsilon)a(N)\right\}. \tag{2.17}$$

But using the B–L inequality (4.4), we have

$$\log \mathbb{P}_N\left(\frac{1}{|A_N^{\mathrm{o}}|}\sum_{x\in A_N^{\mathrm{o}}} m_x \geqslant (1-\varepsilon)a(N)\right) \leqslant -\frac{(1-\varepsilon)^2 \underline{c} a(N)^2}{2\,\mathrm{var}_N^*(1/|A_N^{\mathrm{o}}|\sum_{x\in A_N^{\mathrm{o}}} m_x)}. \tag{2.18}$$

Using the convergence

$$\lim_{N\to\infty} N^{d-2}\mathrm{var}_N^*\left(\frac{1}{|A_N^\circ|}\sum_{x\in A_N^\circ} m_x\right) = (\mathbf{1}_A, \mathscr{G}_D\mathbf{1}_A)_{\mathbb{R}^d}, \tag{2.19}$$

where $\mathscr{G}_D$ is the Green function of the Laplacian in $\mathbb{R}^d$ with Dirichlet boundary conditions, we complete the proof of the upper bound for transient dimensions.

In order to prove the upper bound in the recurrent dimension $d=2$ under A4, let us modify the conditioning argument as follows: for $\Delta = 2[N^\gamma]$ with $\gamma \in (0,1)$ and let us cover A_N with a grid of mesh[2] Δ and pick one point in the middle of each square:

$$\Lambda = ((\Delta\mathbb{Z}\times\mathbb{Z})\cup(\mathbb{Z}\times\Delta\mathbb{Z}))\cap A_N, \qquad I = (\Delta\mathbb{Z}^2 + (\Delta/2,\Delta/2))\cap A_N.$$

Next, let $\mathscr{F}_\Lambda = \sigma(\varphi_x : x\in\Lambda)$ and set

$$m_x = \mathbb{E}_N[\varphi_x \mid \mathscr{F}_\Lambda], \quad \sigma_x^{2,*} = \mathrm{var}_N^*(\varphi_x \mid \mathscr{F}_\Lambda), \qquad x\in I.$$

Then, since the $\varphi_x, x\in I$, are independent under $\mathscr{F}_\Lambda$, we have

$$\mathbb{P}_N(\Omega^+(A_N)) \leqslant \mathbb{E}_N[\mathbb{P}_N(\Omega^+(I)|\mathscr{F}_\Lambda);\Omega^+(\Lambda))] = \mathbb{E}_N\left[\prod_{x\in I}\mathbb{P}_N(\varphi_x\geqslant 0|\mathscr{F}_\Lambda);\Omega^+(\Lambda)\right].$$

For fixed $a>0$, let $B_N = \{x\in I : m_x(\varphi) < a\log N\}$, then for each $\varepsilon>0$,

$$\begin{aligned}
&\mathbb{E}_N\left[\prod_{x\in I}\mathbb{P}_N(\varphi_x\geqslant 0 \mid \mathscr{F}_\Lambda);\Omega^+(\Lambda)\right]\\
&\quad\leqslant \mathbb{E}_N\left[\prod_{x\in I}\mathbb{P}_N(\varphi_x\geqslant 0 \mid \mathscr{F}_\Lambda);\Omega^+(\Lambda)\cap\{|B_N|\geqslant\varepsilon|I|\}\right]\\
&\qquad + \mathbb{P}_N(\Omega^+(\Lambda)\cap\{|B_N|<\varepsilon|I|\}).
\end{aligned}$$

Using the reversed B–L inequality (4.5) and the fact that $\sigma_x^{2,*}\geqslant\rho'\log\Delta$, we can find $C<1/2$ and $\rho>0$ such that

$$\mathbb{P}_N(\varphi_x\geqslant 0\mid\mathscr{F}_\Lambda) = 1-\mathbb{P}_N(\varphi_x - m_x < -m_x \mid \mathscr{F}_\Lambda) \leqslant 1 - C\exp\left(-\frac{\rho m_x^2}{\log\Delta}\right).$$

Thus on $\Omega^+(\Lambda)\cap\{|B_N|\geqslant\varepsilon|I|\}$,

$$\begin{aligned}
\prod_{x\in I}\mathbb{P}_N(\varphi_x\geqslant 0\mid\mathscr{F}_\Lambda) &\leqslant \left(1 - C\exp\left(-\frac{\rho a^2(\log N)^2}{\log\Delta}\right)\right)^{\varepsilon|I|}\\
&\leqslant \exp(-\varepsilon C' N^{2-2\gamma}N^{-\rho a^2/\gamma})
\end{aligned} \tag{2.20}$$

which can be neglected if $a^2<\gamma(2-2\gamma)/\rho$. On the other hand, on $\Omega^+(\Lambda)\cap\{|B_N|<\varepsilon|I|\}$, we have

$$\frac{1}{|I|}\sum_{x\in I} m_x \geqslant (1-\varepsilon)a\log N.$$

[2] The reason for choosing Δ growing with N is to increase the variance $\sigma_x^{2,*}$ in the reversed B–L inequality (4.5).

Next, since m_x is a conditional expectation, we have, for each $\gamma \in \mathbb{R}^+$, by Jensen's inequality and again by the B–L inequality (4.3)

$$\mathbb{E}_N\left[\exp\left(\frac{\gamma}{|I|}\sum_{x\in I} m_x\right)\right] \leqslant \mathbb{E}_N\left[\exp\left(\frac{\gamma}{|I|}\sum_{x\in I}\varphi_x\right)\right] \leqslant \exp\left(\frac{\gamma^2}{2\underline{c}}\mathrm{var}_N^*\left(\frac{1}{|I|}\sum_{x\in I}\varphi_x\right)\right)$$

with

$$\limsup_{N\to\infty} \mathrm{var}_N^*\left(\frac{1}{|I|}\sum_{x\in I}\varphi_x\right) = C < \infty$$

which follows from the estimate

$$\mathrm{cov}_N^*(\varphi_x, \varphi_y) \leqslant \mathrm{O}\left(\log\left(\frac{N}{1+|x-y|}\right)\right)$$

(cf. Lawler, 1991). This shows, via Chebychev's inequality,

$$\limsup_{N\to\infty} \frac{1}{(\log N)^2}\log \mathbb{P}_N(\Omega^+(\Lambda)\cap\{|B_N| < \varepsilon|I|\}) \leqslant -\frac{(1-\varepsilon)^2 a^2 \underline{c}}{2C}$$

and concludes the proof. □

Corollary 2.8. *Assume* A1–A4, *then there exists* $K > 0$, *such that for each* $\delta \in (0,1)$,

$$\lim_{N\to\infty} \sup_{x\in D_{\delta N}} \mathbb{P}_N(\varphi_x < \sqrt{K\log_d(N)}\,|\Omega^+(D_N)) = 0.$$

Moreover, assuming A5 *instead of* A4, *we have in transient dimensions* $d \geqslant 3$

$$\lim_{N\to\infty} \sup_{x\in D_{\delta N}} \mathbb{P}_N(\varphi_x < K(\log N)^{1/2\alpha}|\Omega^+(D_N)) = 0.$$

This corollary follows from FKG, once we prove the following Lemma (cf. Deuschel, 1996):

Lemma 2.9. *Assume* A1–A4, *there exists* $K > 0$, *such that for all* $\varepsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}_N(|\{x\in D_{\delta N}: \varphi_x < \sqrt{K\log_d(N)}\}| \geqslant \varepsilon|D_{\delta N}|\,|\,\Omega^+(D_{\delta N})) = 0.$$

Moreover, assuming A5 *instead of* A4, *we have in transient dimensions* $d \geqslant 3$

$$\lim_{N\to\infty} \mathbb{P}_N(|\{x\in D_{\delta N}: \varphi_x < K(\log N)^{1/2\alpha}\}| \geqslant \varepsilon|D_{\delta N}|\,|\,\Omega^+(D_{\delta N})) = 0.$$

Proof. Set $A = \delta D$. Again the proof is slightly different for the transient and recurrent dimensions. Let us start with the transient case under A5. In view of the proof of the previous theorem, cf. (2.16), and the lower bound, we can find $a > 0$ such that for each $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}_N(|\{x\in A_N^{\circ}: m_x(\varphi) < (a\log N)^{1/2\alpha}\}| \geqslant \varepsilon|A_N^{\circ}|\,|\,\Omega^+(A_N)) = 0, \tag{2.21}$$

where $m_x(\varphi) = (1/2d)\sum_{y:|x-y|=1}\varphi_y$. Next, let us show that

$$\limsup_{N\to\infty} N^d \log \mathbb{P}_N\left(\left|\left\{x\in A_N^{\circ}: |\varphi_x - m_x(\varphi)| > \frac{(a\log N)^{1/2\alpha}}{2}\right\}\right| \geqslant \varepsilon|A_N^{\circ}|\right) = -\infty. \tag{2.22}$$

Note that (2.21) and (2.22), together with Theorem 2.2, imply

$$\lim_{N\to\infty} \mathbb{P}_N\left(\left|\left\{x\in A_N^{\mathrm{o}}\colon \varphi_x<\frac{(a\log N)^{1/2\alpha}}{2}\right\}\right|\geqslant\varepsilon|A_N^{\mathrm{o}}|\,|\Omega^+(A_N)\right)=0.$$

Of course, we can repeat this argument for the even sites. This shows the result since

$$\begin{aligned}&\left\{\left|\left\{x\in A_N\colon \varphi_x<\frac{(a\log N)^{1/2\alpha}}{2}\right\}\right|\geqslant\varepsilon|A_N|\right\}\\&\quad\subseteq\left\{\left|\left\{x\in A_N^{\mathrm{o}}\colon \varphi_x<\frac{(a\log N)^{1/2\alpha}}{2}\right\}\right|\geqslant\frac{\varepsilon}{2}|A_N^{\mathrm{o}}|\right\}\\&\quad\cup\left\{\left|\left\{x\in A_N^{\mathrm{e}}\colon \varphi_x<\frac{(a\log N)^{1/2\alpha}}{2}\right\}\right|\geqslant\frac{\varepsilon}{2}|A_N^{\mathrm{e}}|\right\}.\end{aligned}$$

In order to show (2.22), simply note that

$$(\varphi_x-m_x(\varphi))^2=\left(\frac{1}{2d}\sum_{y:|y-x|=1}(\varphi_x-\varphi_y)\right)^2\leqslant\frac{1}{2d}\sum_{y:|y-x|=1}(\varphi_x-\varphi_y)^2.$$

Moreover, under the assumption A3, we can find $\kappa>0$ such that

$$\begin{aligned}&\limsup_{N\to\infty}\frac{1}{N^d}\log\mathbb{E}_N\left[\exp\left(\kappa\sum_{x\in A_N^{\mathrm{o}}}(\varphi_x-m_x(\varphi))^2\right)\right]\\&\quad\leqslant\limsup_{N\to\infty}\frac{1}{N^d}\log\mathbb{E}_N\left[\exp\left(\frac{\kappa}{2d}\sum_{x\in A_N^{\mathrm{o}},y:|y-x|=1}(\varphi_x-\varphi_y)^2\right)\right]<\infty\end{aligned}\tag{2.23}$$

which implies (2.22) by the Chebychev inequality.

Let us now give the proof for the recurrent dimension $d=2$. Choose $\Delta=2[N^\gamma]$ as in the proof of Theorem 2.7 and set $Q_\Delta=[-\Delta/2,\Delta/2]^2\cap\mathbb{Z}^2$. Let

$$I(k)=I+k,\quad \Lambda(k)=\Lambda+k,\quad m_x(k)=\mathbb{E}[\varphi_x\,|\,\mathscr{F}_{\Lambda(k)}],\qquad k\in Q_\Delta.$$

Next set $B_N(k)=\{x\in I(k)\colon m_x(k)<a\log N\}$ and $\bar B_N(k)=\{x\in I(k)\colon \varphi_x<(a/2)\log N\}$ Now in view of (2.20) and the lower bound, replacing ε by $\Delta^{-2}\varepsilon>0$ and choosing $a^2<\gamma(1-2\gamma)/\Delta$, implies

$$\lim_{N\to\infty}\mathbb{P}_N\left(\bigcup_{k\in Q_\Delta}\{|B_N(k)|\geqslant\varepsilon\Delta^{-2}|I(k)|\}|\Omega^+(A_N)\right)=0.\tag{2.24}$$

Next, let us prove that we can find $\gamma\in(0,1/4)$ and $\beta>0$, such that

$$\limsup_{N\to\infty}N^{-\beta}\log\mathbb{P}_N\left(\left|\left\{x\in I(k)\colon\varphi_x-m_x(k)<-\frac{a}{2}\log N\right\}\right|\geqslant\varepsilon|I(k)|\right)=-\infty.\tag{2.25}$$

Indeed, (2.25) implies that we can replace $B_N(k)$ by $\bar B_N(k)$ in (2.24) and shows the result since

$$\{|\{x\in A_N\colon\varphi_x<(a/2)\log N\}|\geqslant\varepsilon|A_N|\}\subseteq\bigcup_{k\in Q_\Delta}\{|\bar B_N(k)|\geqslant\varepsilon\Delta^{-2}|I(k)|\}.$$

In order to prove (2.25), it suffices to show that for all $\alpha > 0$

$$\limsup_{N\to\infty} N^{-\beta}\log \mathbb{E}_N\left[\exp\left(\alpha N^\beta \Delta^2 |I(k)|^{-1}\sum_{x\in I(k)} \mathbf{1}_{\{\varphi_x - m_x(k) < -(a/2)\log N\}}\right)\right] = 0. \tag{2.26}$$

Using the B–L inequality (4.4) there exists $\rho' > 0$, independent of γ such that

$$\begin{aligned}\mathbb{P}_N(\varphi_x - m_x(k) < (a/2)\log N \mid \mathscr{F}_{\Lambda(k)}) &\leqslant \exp\left(-\frac{a^2(\log N)^2}{2\underline{c}\sigma_x^{2,*}}\right)\\ &\leqslant \exp(-\rho' \log N) = N^{-\rho'}.\end{aligned}$$

Thus we get

$$\begin{aligned}&N^{-\beta}\log \mathbb{E}_N\left[\exp\left(\alpha N^\beta \Delta^2 |I(k)|^{-1}\sum_{x\in I(k)} \mathbf{1}_{\{\varphi_x - m_x(k) < -(a/2)\log N\}}\right)\right]\\
&= N^{-\beta}\log \mathbb{E}_N \prod_{x\in I(k)} \mathbb{E}_N[\exp\left(\alpha N^\beta \Delta^2 |I(k)|^{-1}\mathbf{1}_{\{\varphi_x - m_x(k) < -(a/2)\log N\}}\right) \mid \mathscr{F}_{\Lambda(k)}]\\
&= N^{-\beta}\log \mathbb{E}_N \prod_{x\in I(k)} ((\mathrm{e}^{\alpha N^\beta \Delta^2 |I(k)|^{-1}} - 1)\mathbb{P}_N(\varphi_x - m_x(k) < (a/2)\log N \mid \mathscr{F}_{\Lambda(k)}) + 1)\\
&\leqslant N^{-\beta}|I(k)|((\exp(\alpha N^\beta \Delta^2 |I(k)|^{-1}) - 1)\exp(-\rho' \log N) + 1)\\
&\leqslant CN^{2-2\gamma-\beta}\log((\exp(\alpha C N^{\beta+4\gamma-2}) - 1)N^{-\rho'} + 1).\end{aligned}$$

Choosing now $\gamma < \rho'/2$ and $\beta < 2 - 4\gamma$ shows (2.26). □

Remark 2.10. The results of this section, Theorem 2.7 and Corollary 2.8, can be reformulated in transient dimensions $d \geqslant 3$ for the infinite Gibbs state $\mathbb{P}$.

Remark 2.11. Note that in transient dimensions $d \geqslant 3$ under the assumption A5 there is a gap between the lower bound at height $O((\log N)^{1/2\alpha})$, cf. Corollary 2.8, and the upper bound at height $O(\sqrt{\log N})$, (cf. Corollary 2.6). The problem is in this case we do not know the correct exponential tail behavior of the massless field: namely the B–L inequality predicts an upper bound with quadratic tail, $\mathbb{P}_N(\varphi_x \geqslant L) \leqslant c_1 \exp(-c_2 L^2)$, on the other hand the corresponding reversed B–L inequality valid under A4, (4.5), is missing under A5. The best lower bound we can get, is the following: for all $L \geqslant 1$

$$\mathbb{P}_N(\varphi_x \geqslant L) \geqslant \begin{cases} c_3 \exp(-c_4 L^{2\alpha}), & \alpha \leqslant \frac{d}{2},\ x \in D_N,\\ c_3 \exp(-c_4 L^d), & \alpha > \frac{d}{2},\ \mathrm{dist}(x, D_N) \geqslant L.\end{cases} \tag{2.27}$$

In order to prove (2.27) in the case $\alpha > d/2$, we assume that $x = 0$ and proceed as in the proof of Lemma 2.4, choosing this time $\tilde{\varphi}_x = Lf(x/L)$, $f \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^+)$, compactly supported in the unit ball with $f(0) = 1$. Choose N sufficiently large with respect to L:

by Lemma 2.3 and using the fact that V' is odd we arrive at

$$\begin{aligned} \boldsymbol{H}(\mathbb{P}_N^{\tilde\varphi}|\mathbb{P}_N) &= \sum_{x\sim y}(\tilde\varphi_x-\tilde\varphi_y)^2\int_0^1\int_0^t \mathbb{E}_N[V''(\varphi_x-\varphi_y+s(\tilde\varphi_x-\tilde\varphi_y))]\,\mathrm{d}s\,\mathrm{d}t \\ &\leqslant \frac{1}{2}\sum_{x\sim y}L^2[f(x/L)-f(y/L)]^2\sup_{|t|\leqslant 1}\mathbb{E}_N[V''(\varphi_x-\varphi_y+t)] \\ &\leqslant c_\alpha 2^{2\alpha-2}\sum_{x\sim y}L^2[f(x/L)-f(y/L)]^2(\mathbb{E}_N[(\varphi_x-\varphi_y)^{2\alpha-2}]+1) \end{aligned}$$

which holds for L sufficiently large. Therefore, since by the B–L inequality (4.4),

$$\sup_N\sup_{x\sim y}\mathbb{E}_N[(\varphi_x-\varphi_y)^{2\alpha-2}]=C_V<\infty$$

we can conclude that there exists c such that

$$\sup_N \boldsymbol{H}(\mathbb{P}_N^{\tilde\varphi}|\mathbb{P}_N)\leqslant cL^d$$

for every L. Since $\mathbb{P}_N^{\tilde\varphi}(\varphi_0\geqslant L)=\frac{1}{2}$, using the entropy inequality (2.5) we conclude.

In case $\alpha\leqslant d/2$ simply take $\tilde\varphi_x=L\delta_0(x)$, then $\mathbb{P}_N^{\tilde\varphi}(\varphi_x\geqslant L)=\frac{1}{2}$ and in view of (2.15) and the above, we see that $\sup_N \boldsymbol{H}(\mathbb{P}_N^{\tilde\varphi}|\mathbb{P}_N)\leqslant cL^{2\alpha}$, which implies the result by (2.5).

Besides being relevant for our purposes, the question about the tail behavior of the massless field (say in infinite volume for $d\geqslant 3$), in spite of looking rather basic, is, to our knowledge, open.

3. Behavior at the boundary

In this section we investigate the behavior of the repulsion all the way to the boundary. In contrast to the previous section, based on B–L inequalities and the continuum symmetry of the model, we use here explicitly the more refined H–S random walk representation (cf. Lemma 3.3) and work under the assumption A4.

Theorem 3.1. *Assume* A1–A4, *then for each* $D\in\mathscr{D}$, *there exists* $0<C_1\leqslant C_2<\infty$ *such that*

$$\begin{aligned} -C_2 &\leqslant \liminf_{N\to\infty}\frac{1}{N^{d-1}}\log\mathbb{P}_N(\Omega^+(D_N)) \\ &\leqslant \limsup_{N\to\infty}\frac{1}{N^{d-1}}\log\mathbb{P}_N(\Omega^+(D_N))\leqslant -C_1. \end{aligned}\tag{3.1}$$

Moreover

$$\lim_{L\to\infty}\limsup_{N\to\infty}\frac{1}{N^{d-1}}|\log\mathbb{P}_N(\Omega^+(D_N))-\log\mathbb{P}_N(\Omega^+(\partial_L D_N))|=0. \tag{3.2}$$

The basic step in the proof of the above is the following refinement of the lower bound

Lemma 3.2. *Let $D_N(L) \equiv D_N \setminus \partial_L D_N$. There exists $L_0 > 0$ and $c_1 < \infty$ such that*

$$\mathbb{P}_N(\Omega^+(D_N(L))) \geqslant \exp\left(-c_1 N^{d-1} \frac{\log_d(L)}{L}\right), \qquad \forall L \geqslant L_0,\ \forall N \geqslant L. \tag{3.3}$$

Proof. *Notation*. In what follows $c_1, c_2, \dots$ are positive constants, not depending on N or L, which values may differ from line to line. If the constants depend on L, we write $c_1(L), c_2(L), \dots$.

Let

$$W_N(L) = \{x \in D_N : L \leqslant \operatorname{dist}(x, D_N^{\complement}) \leqslant 2L\}.$$

Then it suffices to prove that

$$\mathbb{P}_N(\Omega^+(W_N(L))) \geqslant \exp\left(-c_2 N^{d-1} \frac{\log_d(L)}{L}\right), \qquad \forall L \geqslant L_0,\ \forall N \geqslant L. \tag{3.4}$$

for some $c_2 < \infty$. In fact, since $D_N(L) \subseteq \bigcup_{\ell=0}^{m} W_N(2^\ell L)$, for some $m \leqslant \log_2(N/L) - 1$, we have by FKG in view of (3.4),

$$\begin{aligned}\log \mathbb{P}_N(\Omega^+(D_N(L))) &\geqslant \sum_{\ell=0}^{m} \log \mathbb{P}_N(\Omega^+(W_N(2^\ell L)))\\ &\geqslant -c_2 N^{d-1} L^{-1} \sum_{\ell=0}^{m} 2^{-\ell} \log_d(2^\ell L) \geqslant -c_1 N^{d-1} L^{-1} \log_d(L).\end{aligned}$$

In order to prove (3.4), we use the same conditioning argument as in the proof of the corresponding result in the Gaussian case (cf. Deuschel, 1996). Let $V_N(L) = \{x \in D_N : L/3 \leqslant \operatorname{dist}(x, D_N^{\complement}) \leqslant 3L\}$ and, for $K \in (0,1)$ to be chosen later, set $\Lambda = V_N(L) \cap \Delta_d \mathbb{Z}^d$, where

$$\Delta_d = \begin{cases} [KL^{2/d}], & d \geqslant 3,\\ [K\frac{L}{(\log L)^{1/2}}], & d = 2. \end{cases} \tag{3.5}$$

Next, write $\mathscr{F}_\Lambda = \sigma(\varphi_x : x \in \Lambda)$ and $W_N'(L) = W_N(L) \setminus \Lambda$. Let $a_d(L) = \sqrt{a \log_d(L)}, a > 0$, then by FKG

$$\begin{aligned}\mathbb{P}_N(\Omega^+(W_N(L))) &\geqslant \mathbb{E}_N[\mathbb{P}_N(\Omega^+(W_N'(L)) | \mathscr{F}_\Lambda); \Omega^+(\Lambda)]\\ &\geqslant \mathbb{E}_N\left[\prod_{x \in W_N'(L)} \mathbb{P}_N(\varphi_x \geqslant 0 | \mathscr{F}_\Lambda); \bigcap_{x \in \Lambda} \{\varphi_x \geqslant a_d(L)\}\right].\end{aligned} \tag{3.6}$$

Let $m_x = \mathbb{E}_N[\varphi_x | \mathscr{F}_\Lambda]$. We will see below, Lemma 3.3, that we can choose $K > 0$, and $L_0 > 1$, such that for all $N \geqslant L \geqslant L_0$, on the set $\bigcap_{x \in \Lambda} \{\varphi_x \geqslant a_d(L)\}$ we have

$$m_x \geqslant \tfrac{1}{2} a_d(L) \quad \text{for all } x \in W_N'(L). \tag{3.7}$$

Let $\sigma_x^{2,*} = \operatorname{var}_N^*(\varphi_x | \mathscr{F}_\Lambda)$. Note that by the B–L inequality (4.2) and (4.6)

$$\sup_{x \in W_N'(L)} \operatorname{var}_N(\varphi_x | \mathscr{F}_\Lambda) \leqslant \sup_{x \in W_N'(L)} \sigma_x^{2,*} \leqslant \begin{cases} c_3, & d \geqslant 3,\\ c_3 \log L & d = 2. \end{cases}$$

Thus, by the B–L inequality, (4.4) and (4.6), we have, on $\bigcap_{x\in\Lambda}\{\varphi_x \geqslant a_d(L)\}$

$$\begin{aligned}\mathbb{P}_N(\varphi_x \geqslant 0|\mathscr{F}_\Lambda) &= 1 - \mathbb{P}_N(\varphi_x - m_x < -m_x|\mathscr{F}_\Lambda)\\ &\geqslant 1 - \exp\left(-\frac{m_x^2}{2\underline{c}\sigma_x^{2,*}}\right) \geqslant 1 - \exp(-c_4 a\log L).\end{aligned}$$

Putting this in the above estimate and using FKG, we get

$$\begin{aligned}\mathbb{P}_N(\Omega^+(W_N(L))) &\geqslant (1-\exp(-c_4 a\log L))^{|W_N'(L)|}\,\mathbb{P}_N\left(\bigcap_{x\in\Lambda}\{\varphi_x \geqslant a_d(L)\}\right)\\ &\geqslant \exp(-c_5 N^{d-1}L^{1-c_4 a})\prod_{x\in\Lambda}\mathbb{P}_N(\varphi_x \geqslant \sqrt{a\log_d(L)}).\end{aligned}$$

Choose $a > 2/c_4$, so that, in view of the result we want to prove, we can get rid of the first factor. Next using the reversed B–L inequality (4.5) and the fact that

$$\inf_{x\in\Lambda}\sigma_x^{2,*} \geqslant \begin{cases} c_6, & d\geqslant 3,\\ c_6\log L & d=2\end{cases}$$

we have

$$\mathbb{P}_N(\varphi_x \geqslant \sqrt{a\log_d(L)}) \geqslant C\exp\left(-D\,\frac{a\log_d(L)}{\sigma_x^{2,*}}\right) \geqslant \exp(-c_7 a\log L).$$

Thus

$$\prod_{x\in\Lambda}\mathbb{P}_N(\varphi_x \geqslant \sqrt{a\log_d(L)}) \geqslant \exp(-|\Lambda|c_7 a\log L) \geqslant \exp(-c_8 AK^{-2}N^{d-1}L^{-1}\log_d(L))$$

which concludes the proof, once (3.7) is proved. □

We postpone the proof of (3.7) at the end of this subsection. A direct consequence of the above lemma is

Proof of Theorem 3.1. By FKG, we have, for all $L \geqslant 1$

$$\mathbb{P}_N(\Omega^+(\partial_L D_N))\mathbb{P}_N(\Omega^+(D_N(L))) \leqslant \mathbb{P}_N(\Omega^+(D_N)) \leqslant \mathbb{P}_N(\Omega^+(\partial_L D_N)), \tag{3.8}$$

where $\mathbb{P}_N(\Omega^+(\partial_L D_N)) \geqslant \exp(-(\log 2)|\partial_L D_N|)$. In view of the above Lemma this implies (3.2) and the lower bound in (3.1). In order to prove the upper bound in (3.1), it suffices to show for fixed $L \geqslant 1$ that

$$\limsup_{N\to\infty}\frac{1}{N^{d-1}}\log\mathbb{P}_N(\Omega^+(\partial_L D_N)) \leqslant -c_1(L).$$

The argument, which works in both transient and recurrent dimension, is just a rerun of the first part of the proof of Theorem 2.7: let $W_N^{\mathrm{e}}, W_N^{\mathrm{o}}$ denote the even and odd sites of $W_N(L)$ and write $\mathscr{F}^{\mathrm{e}} = \sigma\{\varphi_x : x \in W_N^{\mathrm{e}}\}$. For given $a > 0$, let $B_N \equiv \{x \in W_N^{\mathrm{o}}: m_x = \mathbb{E}_N[\varphi_x\,|\mathscr{F}^{\mathrm{e}}] \leqslant a\}$, then, conditioning on even sites

$$\begin{aligned}\mathbb{P}_N(\Omega^+(W_N)) &\leqslant \mathbb{E}_N\left[\prod_{x\in W_N^{\mathrm{o}}}\mathbb{P}_N(\varphi_x \geqslant 0\,|\mathscr{F}^{\mathrm{e}});\{|B_N| \geqslant (1/2)|W_N^{\mathrm{o}}|\}\cap\Omega^+(W_N^{\mathrm{e}})\right]\\ &\quad+\mathbb{P}_N(\{|B_N| < (1/2)|W_N^{\mathrm{o}}|\}\cap\Omega^+(W_N^{\mathrm{e}})).\end{aligned}$$

The reversed B–L inequality, (4.5) and (4.6), implies on $\{|B_N|\geqslant\frac{1}{2}|W_N^{\mathrm{o}}|\}\cap\Omega^+(W_N^{\mathrm{e}})$ the a priori estimate

$$\prod_{x\in W_N^{\mathrm{o}}}\mathbb{P}_N(\varphi_x\geqslant 0\,|\,\mathscr{F}^{\mathrm{e}})\leqslant(1-C\exp(-4dDa))^{(1/2)|W_N^{\mathrm{o}}|}\leqslant\exp\left(-\frac{\log 2}{2}|W_N^{\mathrm{o}}|\right)$$

as soon as $a<\log(2C)/(4dD)$. On the other hand, proceeding as in the proof of Theorem 2.7 and using the estimate

$$\lim_{N\to\infty}|W_N^{\mathrm{o}}|\mathrm{var}_N^*\left(\frac{1}{|W_N^{\mathrm{o}}|}\sum_{x\in W_N^{\mathrm{o}}}\varphi_x\right)\leqslant c_2(L),$$

cf. [10], one shows

$$\begin{aligned}&\limsup_{N\to\infty}\frac{1}{|W_N^{\mathrm{o}}|}\log\mathbb{P}_N(\{|B_N|<(1/2)|W_N^{\mathrm{o}}|\}\cap\Omega^+(W_N^{\mathrm{e}}))\\&\quad\leqslant\limsup_{N\to\infty}\frac{1}{|W_N^{\mathrm{o}}|}\log\mathbb{P}_N\left(\frac{1}{|W_N^{\mathrm{o}}|}\sum_{x\in W_N^{\mathrm{o}}}m_x\geqslant\frac{a}{2}\right)\leqslant -c_3(L)\end{aligned}$$

for some $c_3(L)>0$, which completes the proof of the upper bound. □

We conclude this section with the proof of the crucial inequality (3.7): recall the definitions $\Delta_d=[KL^{2/d}]$, for $d\geqslant 3$, and $\Delta_2=[KL(\log L)^{-1/2}]$, $\Lambda=V_N(L)\cap\Delta_d\mathbb{Z}^d$, where $V_N(L)=\{x\in D_N: L/3\leqslant\mathrm{dist}(x,D_N^{\complement})\leqslant 3L\}$ and $W_N'(L)=W_N(L)\setminus\Lambda$ with $W_N(L)=\{x\in D_N: L\leqslant\mathrm{dist}(x,D_N^{\complement})\leqslant 2L\}$.

Lemma 3.3. *There exists $K\in(0,1)$ and $L_0>1$, such that, for all $M>0$, $N\geqslant L\geqslant L_0$, on $\{\varphi_x\geqslant M: x\in\Lambda\}$*

$$\inf_{x\in W_N'(L)}m_x=\inf_{x\in W_N'(L)}\mathbb{E}_N[\varphi_x\,|\,\mathscr{F}_\Lambda]\geqslant M/2. \tag{3.9}$$

Proof. Using the H–S random walk representation (Helffer and Sjöstrand, 1994; Deuschel et al., 2000) the point is to show that a non-degenerate symmetric random walk is more likely to get trapped at Λ before exiting the domain D_N: for fixed $\psi\in\mathbb{R}^\Lambda$ let $\mathscr{H}_N^\psi(\varphi)$ be the Hamiltonian on $D_N'=D_N\setminus\Lambda$ with boundary conditions $\varphi_x=\psi_x, x\in\Lambda$, and, as usual, $\varphi_x=0, x\notin D_N$. Next consider the diffusion generator on $\Omega_N'=\mathbb{R}^{D_N'}$

$$L^\psi=\sum_{x\in D_N'}\left(\frac{\partial^2}{\partial\varphi_x^2}-\frac{\partial}{\partial\varphi_x}\mathscr{H}_N^\psi(\varphi)\frac{\partial}{\partial\varphi_x}\right)$$

and consider the Markov process $(\Phi(t),X(t))_{t\in[0,\infty)}$ on $\mathbb{R}^{D_N'}\times\mathbb{Z}^d$ generated by

$$\mathscr{L}^\psi=L^\psi+\sum_{i=1}^d\nabla_i^*[a_i^\psi(\cdot,\varphi)\nabla_i], \tag{3.10}$$

where ∇_i is the discrete gradient, ∇_i^* its adjoint on $\ell^2(\mathbb{Z}^d)$ and

$$a_i^\psi(x,\varphi)=\begin{cases}V''(\varphi_{x+e_i}-\varphi_x), & x,x+e_i\in D_N',\\ V''(\varphi_x-\psi_{x+e_i}), & x\in D_N', x+e_i\in\Lambda,\\ V''(\varphi_x), & x\in D_N', x+e_i\notin D_N,\\ \underline{c}, & x,x+e_i\notin D_N.\end{cases}$$

We refer to Deuschel et al. (2000), Section 2 for a detailed construction of this process: we just remark that, by the structure of the generator (3.10) one can first construct the $\mathbb{R}^{D'_N}$-valued process $\{\Phi(t)\}_{t\geqslant 0}$ generated by L^ψ and then the associated jump process $\{X(t)\}_{t\geqslant 0}$, which has time dependent, in fact Φ-dependent, inhomogeneous rates. Let $\boldsymbol{P}^{\psi}_{(\varphi,x)}$ denote the law of the Markov process $\{(\Phi(t),X(t))\}_{t\geqslant 0}$ with initial condition $(\Phi(0),X(0))$ in (φ,x) and write $\boldsymbol{P}^{\psi}_{x}=\int_{\Omega'_N}\boldsymbol{P}^{\psi}_{(\varphi,x)}\mathbb{P}^{\psi}_{N}(\mathrm{d}\varphi)$, where $\mathbb{P}^{\psi}_{N}=\mathbb{P}_N(\cdot\mid\mathscr{F}_\Lambda)$. Finally, $\boldsymbol{P}^{\psi}_{(\Phi,x)}$ will denote the law of the $\{X(t),t\geqslant 0\}$ with frozen diffusion path $\{\Phi(t),t\geqslant 0\}$. By *frozen* we mean that we fix the realization of the process $\{\Phi(t)\}_{t\geqslant 0}$ and look at the evolution of the random process $\{X(t)\}_{t\geqslant 0}$: all the estimate with *frozen diffusion* Φ are uniform in $\{\Phi(t)\}_{t\geqslant 0}$.

Introduce the stopping times $\tau_N=\inf\{t\geqslant 0: X(t)\notin D_N\}$, $\tau_\Lambda=\inf\{t\geqslant 0: X(t)\in\Lambda\}$ and $\tau=\tau_N\wedge\tau_\Lambda$. Then, using the result of Deuschel et al. (2000) Section 2, we have the following representation:

$$m_x=\mathbb{E}_N[\varphi_x|\mathscr{F}_\Lambda]=\sum_{y\in\Lambda}\left(\int_0^1\boldsymbol{P}^{\varepsilon\psi}_x(X(\tau)=y)\,\mathrm{d}\varepsilon\right)\psi_y.$$

Thus if $\psi_x\geqslant M$ for every $x\in\Lambda$ we have that

$$\begin{aligned}m_x &\geqslant M\left(\int_0^1\sum_{y\in\Lambda}\boldsymbol{P}^{\varepsilon\psi}_x(X(\tau)=y)\,\mathrm{d}\varepsilon\right)=M\left(\int_0^1\boldsymbol{P}^{\varepsilon\psi}_x(X(\tau)\in\Lambda)\,\mathrm{d}\varepsilon\right)\\ &=M\left(\int_0^1\boldsymbol{P}^{\varepsilon\psi}_x(\tau_\Lambda<\tau_N)\,\mathrm{d}\varepsilon\right).\end{aligned}\tag{3.11}$$

Now the result will follow once we prove that we can choose $K>0$, $L_0>1$ such that

$$\boldsymbol{P}^{\varepsilon\psi}_x(\tau_\Lambda<\tau_N)\geqslant\tfrac{1}{2},\qquad\forall\varepsilon\in[0,1],x\in W'_N(L),\qquad\forall N\geqslant L\geqslant L_0.\tag{3.12}$$

The proof of (3.12) goes as follows. For any $T>0$, we have

$$\boldsymbol{P}^{\varepsilon\psi}_x(\tau_\Lambda\geqslant\tau_N)\leqslant\boldsymbol{P}^{\varepsilon\psi}_x(\tau_N<T)+\boldsymbol{P}^{\varepsilon\psi}_x(\tau_\Lambda>T).\tag{3.13}$$

We quote the following Aronson estimates, which are proven in Appendix B of Giacomin et al. (1999): consider the transition kernel of the random walk with frozen diffusion path, then there exist $C,\delta\in(0,\infty)$ depending only on $d,\underline{c},\bar{c}$ such that

$$\boldsymbol{P}^{\varepsilon\psi}_{(\Phi,x)}(X(s)=y)\leqslant\frac{C}{1\vee s^{d/2}}\exp\left(-\frac{|x-y|}{C(1\vee s^{1/2})}\right)\tag{3.14}$$

for every $x,y\in\mathbb{Z}^d$ and $s\geqslant 0$, and

$$\boldsymbol{P}^{\varepsilon\psi}_{(\Phi,x)}(X(s)=y)\geqslant\frac{\delta}{1\vee s^{d/2}}\tag{3.15}$$

for every $s>0$ and $x,y\in\mathbb{Z}^d$ such that $|x-y|^2\leqslant s$. From the upper bound (3.14), one first shows that

$$\boldsymbol{P}^{\varepsilon\psi}_x(\tau_N<T)\leqslant\boldsymbol{P}^{\varepsilon\psi}_x\left(\sup_{t\in[0,T]}|X(t)|>L\right)\leqslant c_1\exp\left(-\frac{L}{c_1T^{1/2}}\right)\tag{3.16}$$

for some $1 < c_1 < \infty$, (see Proposition 6.5 in Bass (1998, p.179)). Next, following the argument of the proof of Lemma A.7 of Bolthausen and Deuschel (1993), let us show that

$$\boldsymbol{P}_x^{\varepsilon\psi}(\tau_\Lambda > T) \leqslant \exp(-c_2 T/K^2L^2), \quad x \in W_N'(L) \tag{3.17}$$

for any $T > 2\Delta^2$. These two inequalities imply the result: setting $T = L^2/(c_1 \log(4c_1))^2$, for $L \geqslant L_0$ such that $T > 2\Delta^2$, and choosing $K > 0$ sufficiently small implies $\boldsymbol{P}_x^{\varepsilon\psi}(\tau_\Lambda \geqslant \tau_N) \leqslant 1/2$.

In view of the lower bound (3.15), we have, for each $x \in \tilde{W}_N(L) \equiv \{x \in \mathbb{Z}^d : L/2 \leqslant \mathrm{dist}(x, D_N^\complement) \leqslant 5L/2\} \setminus \Lambda$

$$\boldsymbol{P}_{(\Phi,x)}^{\varepsilon\psi}(X(s) \in \Lambda) = \sum_{y \in \Lambda} \boldsymbol{P}_{(\Phi,x)}^{\varepsilon\psi}(X(s) = y) \geqslant \frac{\delta}{2^{d/2}} \Delta^{-d}, \quad \Delta^2 \leqslant s \leqslant 2\Delta^2. \tag{3.18}$$

On the other hand, using again the Aronson estimate (3.14) and evaluating the expression, we get

$$\sup_{y \in \Lambda, \Phi} \boldsymbol{E}_{(\Phi,y)}^{\varepsilon\psi} \left[\int_0^{2\Delta^2} \mathbf{1}_\Lambda(X(s))\, \mathrm{d}s \right] \leqslant c_d(\Delta), \tag{3.19}$$

where $c_d(\Delta) \leqslant c_3$ for $d \geqslant 3$ and $c_d(\Delta) \leqslant c_3 \log \Delta$, for $d = 2$. Thus, using the strong Markov property we have, for each $x \in \tilde{W}_N(L)$

$$\begin{aligned}
\frac{\delta}{2^{d/2}} \Delta^{-d+2} &\leqslant \int_{\Delta^2}^{2\Delta^2} \boldsymbol{P}_{(\Phi,x)}^{\varepsilon\psi}(X(s) \in \Lambda)\, \mathrm{d}s \leqslant \boldsymbol{E}_{(\Phi,x)}^{\varepsilon\psi} \left[\int_0^{2\Delta^2} \mathbf{1}_\Lambda(X(s))\, \mathrm{d}s \right] \\
&\leqslant \boldsymbol{P}_{(\Phi,x)}^{\varepsilon\psi}(\tau_\Lambda \leqslant 2\Delta^2) \sup_{y \in \Lambda, \Phi} \boldsymbol{E}_{(\Phi,y)}^{\varepsilon\psi} \left[\int_0^{2\Delta^2} \mathbf{1}_\Lambda(X(s))\, \mathrm{d}s \right] \\
&\leqslant c_d(\Delta) \boldsymbol{P}_{(\Phi,x)}^{\varepsilon\psi}(\tau_\Lambda \leqslant 2\Delta^2).
\end{aligned}$$

That is

$$\inf_{x \in \tilde{W}_N(L), \Phi} \boldsymbol{P}_{(\Phi,x)}^{\varepsilon\psi}(\tau_\Lambda \leqslant 2\Delta^2) \geqslant \begin{cases} c_4 \Delta^{-d+2}, & d \geqslant 3, \\ c_4 (\log \Delta)^{-1}, & d = 2. \end{cases}$$

Let $\tilde{\tau}_N = \inf\{s \geqslant 0 : X(s) \notin \tilde{W}_N(L)\}$, then by the Markov property and the usual renewal argument, for any $T > 2\Delta^2$ and $x \in W_N'(L)$

$$\begin{aligned}
&\boldsymbol{P}_{(\Phi,x)}^{\varepsilon\psi}(\tau_\Lambda \geqslant T) \leqslant \boldsymbol{P}_{(\Phi,x)}^{\varepsilon\psi}(\tau_\Lambda \geqslant 2[T/2\Delta^2]\Delta^2; \tilde{\tau}_N \geqslant T) + \boldsymbol{P}_{(\Phi,x)}^{\varepsilon\psi}(\tilde{\tau}_N < T) \leqslant \boldsymbol{P}_{(\Phi,x)}^{\varepsilon\psi}(\tilde{\tau}_N < T) \\
&+ \begin{cases} (1 - c_4 \Delta^{-d+2})^{[T/2\Delta^2]} \leqslant \exp\left(-c_4 \left[\dfrac{T}{2\Delta^2}\right] \Delta^{-d+2}\right) \leqslant \exp(-c_5 T/\Delta^{-d}), & d \geqslant 3, \\[2ex] (1 - c_4 (\log \Delta)^{-1})^{[T/2\Delta^2]} \leqslant \exp\left(-c_4 \left[\dfrac{T}{2\Delta^2}\right] (\log \Delta)^{-1}\right) \exp(-c_5 T/(\Delta^2 \log \Delta)), & d = 2, \end{cases}
\end{aligned}$$

where as above, possibly changing c_1,

$$\boldsymbol{P}_{(\Phi,x)}^{\varepsilon\psi}(\tilde{\tau}_N < T) \leqslant c_1 \exp\left(-\frac{T}{c_1 L^2}\right).$$

This proves (3.17) by the definition of Δ. $\square$

Remark 3.4. In view of Lemma 3.1, using precisely the same argument as in the proof of Proposition 3.7 in Deuschel (1996), with the corresponding adaptations for the recurrent dimensions, cf. proof of Corollary 2.8, one shows that at distance L from the boundary of D_N, the spins are repelled at height $\mathrm{O}(\log_d(L))$.

Remark 3.5. In view of (3.2), it is natural to ask whether, for each fixed $L>0$, the limit

$$\lim_{N\to\infty}\frac{1}{N^{d-1}}\log\mathbb{P}_N(\Omega^+(\partial_L D_N))=-\kappa_D(L) \tag{3.20}$$

exists. This is a delicate question involving the influence of the boundary conditions. For simplicity, let us consider a cube $D_N=[-N,N]^d\cap\mathbb{Z}^d$, and denote by $\partial_L^i D_N$, $i=-d,\ldots,d$ the different pieces of the L boundary:

$$\partial_L^i D_N=\{x\in D_N\colon N-L\leqslant x_i\leqslant N\}\quad\text{and}$$
$$\partial_L^{-i} D_N=\{x\in D_N\colon -N\leqslant x_i\leqslant -N+L\}.$$

Then, a subadditive argument, based on the estimate (3.3) shows the convergence

$$\lim_{N\to\infty}\frac{1}{N^{d-1}}\log\mathbb{P}_N(\Omega^+(\partial_L^i D_N))=-\kappa^i(L). \tag{3.21}$$

The main difficulty is to prove that the different pieces of the boundary are asymptotically independent:

$$\lim_{N\to\infty}\frac{1}{N^{d-1}}\log\mathbb{P}_N\left(\bigcap_i\Omega^+(\partial_L^i D_N)\right)=\sum_i\lim_{N\to\infty}\frac{1}{N^{d-1}}\log\mathbb{P}_N(\Omega^+(\partial_L^i D_N)).$$

The lower bound follows from FKG

$$\mathbb{P}_N(\Omega^+(\partial_L D_N))=\mathbb{P}_N(\cap_i\Omega^+(\partial_L^i D_N))\geqslant\prod_i\mathbb{P}_N(\Omega^+(\partial_L^i D_N))$$

and therefore,

$$\liminf_{N\to\infty}\frac{1}{N^{d-1}}\log\mathbb{P}_N(\Omega^+(\partial_L D_N))\geqslant-\sum_i\kappa^i(L). \tag{3.22}$$

We expect that, as in the Gaussian case (cf. Deuschel, 1996) the corresponding upper bound holds

$$\limsup_{N\to\infty}\frac{1}{N^{d-1}}\log\mathbb{P}_N(\Omega^+(\partial_L D_N))\leqslant-\sum_i\kappa^i(L).$$

This with (3.22) would imply (3.21), and, using (3.2)

$$\lim_{N\to\infty}\frac{1}{N^{d-1}}\log\mathbb{P}_N(\Omega^+(D_N))=-\sum_{i=-d}^{d}\kappa^i,\quad\text{where }\kappa^i=\lim_{L\to\infty}\kappa^i(L). \tag{3.23}$$

4. Appendix A. The Brascamp–Lieb inequalities

We briefly review the B–L inequalities for log-concave measures in the context of massless fields. These inequalities were originally discovered in Brascamp and Lieb (1976) and applied for the first time to massless fields in Brascamp et al. (1976).

In the same set up of the introduction, let us consider the probability measure on $\mathbb{R}^{D_N}$

$$\mathbb{P}_N^v(\mathrm{d}\varphi) \equiv \frac{1}{Z_N^v} \exp(-\mathscr{H}_N(\varphi) - v(\varphi)) \prod_{x \in D_N} \mathrm{d}\varphi_x, \tag{4.1}$$

where $v: \mathbb{R}^{D_N} \to \mathbb{R}$ is a C^2 convex function which is either non-negative or $v(\varphi) \geqslant -c\sum_x |\varphi_x|$ for some $c \geqslant 0$. Then the following two inequalities hold for every choice of $f: \mathbb{Z}^d \to \mathbb{R}$ (Brascamp and Lieb, 1976; Deuschel et al. 2000; Helffer and Sjöstrand, 1994):

$$\mathrm{var}_N^v\left(\sum_x f(x)\varphi(x)\right) \leqslant \frac{1}{\underline{c}d} \mathbb{E}_N^*\left[\left(\sum_x f(x)\varphi(x)\right)^2\right] = \frac{1}{\underline{c}d} \langle f, G_{D_N}^* f\rangle, \tag{4.2}$$

$$\begin{aligned}
&\mathbb{E}_N^v\left[\exp\left(\sum_x f(x)\varphi(x) - \mathbb{E}_N^v\left(\sum_x f(x)\varphi(x)\right)\right)\right] \\
&\quad \leqslant \mathbb{E}_N^*\left[\exp\left((\underline{c}d)^{-1/2}\sum_x f(x)\varphi(x)\right)\right] \\
&\quad = \exp\left(\frac{1}{2\underline{c}d}\langle f, G_{D_N}^* f\rangle\right),
\end{aligned} \tag{4.3}$$

where $\langle f, G_{D_N}^* f\rangle = \sum_{x,y} f(x) G_{D_N}^*(x,y) f(y)$. Using Chebychev's inequality, this implies the following concentration:

$$\mathbb{P}_N^v\left(\left|\sum_x f(x)\varphi(x) - \mathbb{E}_N^v\left[\sum_x f(x)\varphi(x)\right]\right| \geqslant L\right) \leqslant 2\exp\left(-\frac{d\underline{c}L^2}{2\langle f, G_{D_N}^* f\rangle}\right). \tag{4.4}$$

In the case $v(\varphi) = \sum_x g(x)\varphi(x)$ we have a corresponding reversed inequality (Deuschel et al. 2000) Lemma 2.9:

$$\begin{aligned}
&\mathbb{P}_N^v\left(\left|\sum_x f(x)\varphi(x) - \mathbb{E}_N^v\left[\sum_x f(x)\varphi(x))\right]\right| \geqslant L\right) \\
&\quad \geqslant c(\bar{c})\exp\left(-\frac{2d\bar{c}^2L^2}{\underline{c}\langle f, G_{D_N}^* f\rangle}\right)
\end{aligned} \tag{4.5}$$

and $c(\bar{c})$ can be chosen equal to $\exp(-2\log 2\bar{c}d - 2)$. The first inequality in the formulas (4.2) and (4.3) as well as inequalities (4.4) and (4.5) hold also if we are conditioning to the event $E_\psi = \{\varphi: \varphi_x = \psi_x \text{ for every } x \in A\}$, $A \subset D_N$ and $\psi \in \mathbb{R}^{D_N}$, that is if we make in the first line of (4.2)–(4.5) the replacements

$$\mathbb{E}_N^v(\cdot) \to \mathbb{E}_N^v(\cdot|\mathscr{F}_A)(\psi) \quad \text{and} \quad \mathbb{E}_N^*(\cdot) \to \mathbb{E}_N^*(\cdot|\mathscr{F}_A)(\psi_0), \tag{4.6}$$

where $\psi_0 \equiv 0$.

As already observed in Deuschel and Giacomin (1999), (4.2)–(4.4) hold also if $\mathbb{P}_N^v(\mathrm{d}\varphi)$ is replaced by the conditioned measure $\mathbb{P}_N(\mathrm{d}\varphi|\Omega^+(A))$, $A \subset \mathbb{Z}^d$. This is an immediate consequence of (4.2), and (4.4), because we can approximate $\mathbb{P}_N(\mathrm{d}\varphi|\Omega^+(A))$ by choosing $v(\varphi)=k\sum_{x\in A}\varphi_x^4\mathbf{1}_{(-\infty,0)}(\varphi_x)$, $k\geqslant 0$, and letting $k\nearrow\infty$.

Finally we remark that the extension of the B–L inequalities to the infinite volume cases (quickly) considered in this paper is straightforward.

Acknowledgements

G.G. has been partly supported by the Swiss National Fund, Project 20–41′925.94 and by COFIN99 (MURST, Italy). Part of the work has been done while one of the authors (G.G.) was at the department of mathematics of the EPFL (CH), *Chaire de Probabilités Appliquées*.

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