Entropic repulsion for a class of Gaussian interface models in high dimensions

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Abstract

Consider the centred Gaussian field on the lattice $\mathbb{Z}^d$, d large enough, with covariances given by the inverse of $\sum_{j=k}^{K} q_j (-\Delta)^j$, where Δ is the discrete Laplacian and $q_j \in \mathbb{R}$, $k \le j \le K$, the q_j satisfying certain additional conditions. We extend a previously known result to show that the probability that all spins are nonnegative on a box of side-length N has an exponential decay at a rate of order $N^{d-2k} \log N$. The constant is given in terms of a higher-order capacity of the unit cube, analogously to the known case of the lattice free field. This result then allows us to show that, if we condition the field to stay positive in the N-box, the local sample mean of the field is pushed to a height of order $\sqrt{\log N}$.
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1. Introduction and results

We study the entropic repulsion of a class of real valued Gaussian random fields $\varphi = \{\varphi_x\}_{x \in \mathbb{Z}^d}$, which can be interpreted as a d-dimensional (discrete) interface in a $(d + 1)$-dimensional space. Entropic repulsion refers to the fact that the presence of a wall forces the random surface to move away from the wall, in order to gain space for local fluctuations (cf. [7]). In our case, the wall is simply the d-dimensional coordinate hyperplane, and the effect of the wall is described by requiring the field $\{\varphi_x\}$ to be positive in a certain region.

A basic object to study is the asymptotics of the probability $P(\varphi_x \ge 0, \ x \in V)$, $V \subset \mathbb{Z}^d$ finite, when $V \uparrow \mathbb{Z}^d$. Its behaviour is well understood in the case where $\{\varphi_x\}$ is the (lattice) Gaussian

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free field in dimension $d \geq 3$ (see [3]), which is the Gibbs measure with formal Hamiltonian $H(\varphi) = \sum_{|x-y|=1}(\varphi_x - \varphi_y)^2$. The free field has a simple random walk representation of the covariances, which enables one to calculate various conditional distributions in an easy way. The main aim of the present paper is to extend the analysis to a class of fields with a more general Hamiltonian which includes the so-called "membrane models". The crucial difference is that these models do not possess a random walk representation.

The only mathematically rigorous result on these models we are aware of is the paper by Sakagawa [10], who derived lower and upper bounds for $P(\varphi_x \geq 0, \; x \in V)$. His bounds however don't match. We derive here an upper bound which asymptotically matches Sakagawa's lower bound, and therefore we prove that his lower bound gives the correct leading order of the asymptotics. This first result then enables us to compute the exact height to which the average of the field is pushed by the wall.

Let $\Omega = \mathbb{R}^{\mathbb{Z}^d}$. We consider the (formal) Hamiltonian $H : \Omega \to \mathbb{R}$ given by

$$H(\varphi) = \sum_{j=1}^{K} q_j \sum_{x \in \mathbb{Z}^d} ((-\Delta)^{\frac{j}{2}} \varphi_x)^2, \tag{1}$$

with $q_j \in \mathbb{R}$, for $1 \leq j \leq K$. The discrete Laplacian Δ is the operator on $L^2(\mathbb{Z}^d)$ defined by $\Delta f(x) = \left(\frac{1}{2d} \sum_{y \in \mathbb{Z}^d, |x-y|=1} f(y) \right) - f(x)$. If j is odd, we set $\sum_{x \in \mathbb{Z}^d}((-\Delta)^{\frac{j}{2}}\varphi_x)^2 = \sum_{i=1}^{d} \sum_{x \in \mathbb{Z}^d}((-\Delta)^{\frac{j-1}{2}}\nabla_i \varphi_x)^2$, where ∇_i denotes the discrete gradient in the i-th direction. The free field is thus the special case $K = 1$, $q_1 = 1$. We make the following assumptions:

(a) $d \geq 2k + 1$, where $k = \min\{j : q_j \neq 0\}$,
(b) $q = \{q_j\}_{1 \leq j \leq K} \in \mathbb{R}^K$ satisfies $q(r) := \sum_{j=k}^{K} q_j r^j > 0$ for $0 < r \leq 2$.

Under assumptions (a) and (b), the infinite-volume Gibbs measure corresponding to H exists (see [10], Section 2). It can be described as follows: For $\varepsilon \geq 0$ and $x, y \in \mathbb{Z}^d$ set $J_\varepsilon(x, y) = q(\varepsilon I - \Delta)(x, y)$, where I is the identity matrix on $\mathbb{Z}^d$, and Δ the matrix Laplacian defined by

$$\Delta(x, y) = \begin{cases} -1 & \text{if } x = y, \\ \dfrac{1}{2d} & \text{if } |x - y| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The above assumptions ensure that, for ε small enough, the matrix J_ε is positive definite with positive definite inverse J_ε^{-1}. We set $J(x, y) = J_0(x, y)$ and $G(x, y) = J^{-1}(x, y)$. From [4], Chapter 13, we know that the centred Gaussian field with covariance matrix G exists. We denote its law by P. It is characterized by the following DLR-equation as an infinite-volume Gibbs measure, and corresponds to the Hamiltonian (1):

$$P(\cdot \mid \mathcal{F}_{\{x\}^c})(\varphi) = \mathcal{N}\left(-J(0,0) \cdot \sum_{y \neq x} J(x, y)\varphi_y, \; J(0, 0)^{-1}\right) \quad P\text{-a.s.,} \tag{2}$$

where we use the notation $\mathcal{F}_A = \sigma(\varphi_y : y \in A)$ for the σ-field generated by $\{\varphi_y : y \in A\}$, $A \subset \mathbb{Z}^d$, and $\mathcal{N}(\mu, \sigma^2)$ is the Gaussian distribution with mean μ and variance σ^2. If $G(x, y) \geq 0$ for all x, y, then P satisfies the FKG-inequalities (see [9]). We make the additional assumption

(c) There exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $J_{\varepsilon_n}^{-1}(x, y) \geq 0$ for all $n \in \mathbb{N}$ and all $x, y \in \mathbb{Z}^d$.

Throughout the paper we will always assume that (a), (b) and (c) hold.

Now set $V = [-1, 1]^d$ and $V_N = NV \cap \mathbb{Z}^d$. We consider the entropic repulsion event

$$\Omega_N^+ = \{\varphi \in \Omega : \varphi_x \geq 0 \ \forall x \in V_N\}.$$

Theorem 2.1 of [10] states that there exist constants $C_1, C_2 > 0$ such that

$$-C_1 \leq \liminf_{N\to\infty} \frac{1}{N^{d-2k}\log N} \log P(\Omega_N^+) \leq \limsup_{N\to\infty} \frac{1}{N^{d-2k}\log N} \log P(\Omega_N^+) \leq -C_2 \quad (3)$$

holds. Moreover, the constant C_1 has been identified to be $C_1 = 2k \cdot q_k \cdot G(0, 0) \cdot C_k(V)$, where

$$C_k(V) = \inf \left\{ \frac{1}{(2d)^k} \int_{\mathbb{R}^d} |(-\nabla)^k h|^2 \, dx; \, h \in H^k(V), h \geq 1 \text{ on } V \right\} \quad (4)$$

is the k-th order capacity of the unit cube V, k being the minimal degree of the polynomial q. This lower bound was proved using a relative entropy argument and the FKG-property of P. Assumption (c) above is necessary for this proof. For the case of the free field, the statement (3) was proved before in [3]. There it was shown in addition that in this case the constants C_1 and C_2 of the upper and the lower bound coincide. Our first result shows that this is still true for our model:

Theorem 1.1. *For $d \geq 2k + 1$,*

$$\lim_{N\to\infty} \frac{1}{N^{d-2k}\log N} \log P\left(\Omega_N^+\right) = -2k \cdot q_k \cdot G \cdot C_k(V), \quad (5)$$

where $C_k(V)$ is given by (4), *and $G = G(0, 0)$.*

In the next section, we will prove the upper bound of (5). Together with (3) and (4) this proves Theorem 1.1. Thus the decay of $P(\Omega_N^+)$ for $k \geq 2$ is completely analogous to the case for $k = 1$. Using Theorem 1.1, we can then prove the height estimate for the averaged field:

Theorem 1.2. *Let $\varepsilon > 0$ and $\eta > 0$. Then*

$$\lim_{N\to\infty} \sup_{\substack{z\in V_N, \\ V_{N,\varepsilon}(z)\subset V_N}} P\left(\left| \frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} - \sqrt{4kG} \right| \geq \eta \, \middle| \, \Omega_N^+ \right) = 0, \quad (6)$$

where $\overline{\varphi}_{N,\varepsilon}(z) = \frac{1}{|V_{N,\varepsilon}(z)|} \sum_{x\in V_{N,\varepsilon}(z)} \varphi_x$ and $V_{N,\varepsilon}(z) = \{x \in V_N : \max_{1\leq i\leq d} |x_i - z_i| \leq \varepsilon N\}$.

The lower bound for this height estimate was obtained in [10]. Our exact result in Theorem 1.1 allows us now to give the correct upper bound. This means that, as expected, the local sample mean of the field is pushed to $\sqrt{4kG \cdot \log N}$ by the hard wall.

2. Proof of the upper bound in Theorem 1.1

We follow a strategy introduced in [1], which was used in [5] for the case $k = K = 1$, $q_k = 1$. The idea is to use a conditioning argument on larger boxes than those of the proof of [10]. The main difficulty – when trying to follow the proof for the free field – arises when considering the expectations of φ_x conditioned on the boundary of a box of side-length L. While in the

case of the harmonic crystal, we know by the random walk representation, that on Ω_N^+ the conditional expectations are nonnegative, in our more general case they can be strictly negative. We overcome this difficulty by estimating the proportion of conditional expectations that are of order $-N^\lambda$, $\lambda \in \mathbb{N}$. Then we prove that this proportion is negligible if we let N tend to infinity.

Proof of the upper bound of Theorem 1.1. Fix a natural number $L > K + 1$ such that $L - K$ is even, and let $\overline{\Lambda} = (L, L, \ldots, L) + L\mathbb{Z}^d$. For $x \in \overline{\Lambda}$ denote by $\partial B(x) = \{y \in \mathbb{Z}^d : \max_{i=1,\ldots,d} |x_i - y_i| \in [\frac{L-K}{2}, \frac{L+K}{2}]\}$ the boundary of the box $B(x) := \{y \in \mathbb{Z}^d : \max_{i=1,\ldots,d} |x_i - y_i| < \frac{L-K}{2}\}$. Let $\tilde{\Lambda} = \{x \in \overline{\Lambda} : \partial B(x) \subset V_N\}$ and $\Lambda = \cup_{x \in \tilde{\Lambda}} \partial B(x)$.

Since $J(x, y) = 0$ for $|x - y| > K$, the field $\{\varphi_x\}_{x \in \tilde{\Lambda}}$ is Markovian, in the sense that $P(\cdot \mid \mathcal{F}_{B(x)^c}) = P(\cdot \mid \mathcal{F}_{\partial B(x)})$ for all $x \in \tilde{\Lambda}$, and thus (see [4], Proposition 13.13), under $P(\cdot \mid \mathcal{F}_{B(x)^c})$, the φ_x, $x \in \tilde{\Lambda}$, are independent normally distributed random variables. For the mean and the variance we write

$$m_x = E\left(\varphi_x \mid \mathcal{F}_{B(x)^c}\right) \quad \text{and} \quad G_L = \text{var}\left(\varphi_x \mid \mathcal{F}_{B(x)^c}\right)$$

respectively. Note that $\lim_{L \to \infty} G_L = G$ (see [4], Section 13.1). For any subset A of $\mathbb{Z}^d$ let Ω_A^+ denote the event $\{\varphi_x \geq 0 \; \forall x \in A\}$. Because of the independence we have

$$P\left(\Omega_N^+\right) \leq P\left(\Omega_\Lambda^+ \cap \Omega_{\tilde{\Lambda}}^+\right) \leq E\left[\prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 \mid \mathcal{F}_{\partial B(x)}) \cdot 1_{\Omega_\Lambda^+}\right]. \tag{7}$$

As in [5], we use a decomposition of V on a larger scale: Let $\theta > 0$, $r \in \mathbb{R}^d$ and set $A_r = r + [0, \theta)^d$, and $I = \{r \in \theta\mathbb{Z}^d : A_r \subset V, \partial A_r \cap \partial V = \emptyset\}$. Set $\tilde{B}_r = NA_r \cap \tilde{\Lambda}$, the box containing the centres of the smaller boxes $B(x)$, with $x \in NA_r$. Note that $B := |\tilde{B}_r| = O(N^d)$.

Let $0 < \delta < 1$ and $0 < \gamma < 1$. For $\kappa > 0$, define $a_N = \sqrt{4k(G - \kappa)\log N}$ and consider the following events:

$$E_{\delta,\kappa} = \left\{\varphi : \text{there is } r \in I \text{ such that } |\{x \in \tilde{B}_r : m_x \leq a_N\}| \geq \delta B\right\},$$

$$E_\delta^{-\lambda} = \left\{\varphi : \text{there is } r \in I \text{ such that } |\{x \in \tilde{B}_r : m_x \leq -N^\lambda\}| \geq \frac{\delta}{N^{2k+(2-\gamma)\lambda-\gamma}} B\right\},$$

where λ is a nonnegative integer. Note that for $\lambda \geq \lambda_{\max} = \lfloor (d - 2k + \gamma)/(2 - \gamma) \rfloor + 1$ (where $\lfloor \cdot \rfloor$ denotes the integer part), we have $N^{-2k-(2-\gamma)\lambda+\gamma} \delta B < 1$. For these λs, $E_\delta^{-\lambda}$ equals $\{\varphi : \text{there is } r \in I \text{ such that } \{x \in \tilde{B}_r : m_x \leq -N^\lambda\} \neq \emptyset\}$, and these $E_\delta^{-\lambda}$s are all contained in $E_\delta^{-\lambda_{\max}}$. Set

$$F_\delta = \bigcup_{\lambda=0}^{\lambda_{\max}} E_\delta^{-\lambda}.$$

The estimate (7) now gives

$$P(\Omega_N^+) \leq E\left[\prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 \mid \mathcal{F}_{\partial B(x)}) \cdot 1_{\Omega_\Lambda^+ \cap F}\right]$$

$$+ E\left[\prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 \mid \mathcal{F}_{\partial B(x)}) \cdot 1_{\Omega_\Lambda^+ \cap F^c}\right],$$

where $F = E_{\delta,\kappa} \cup F_\delta$. The following lemma shows that we can estimate

$$\prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 \mid \mathcal{F}_{\partial B(x)})$$

uniformly on F:

Lemma 2.1. *Let* $0 < \gamma < 1$. *The following hold:*

(a) *For L large enough, there exists a constant* $c_1 > 0$ *such that*

$$E\left[\prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 \mid \mathcal{F}_{\partial B(x)}) \cdot 1_{\Omega_\Lambda^+ \cap E_{\delta,\kappa}}\right] \leq \exp\left(-c_1 N^{d-2k+\gamma}\right). \tag{8}$$

(b) *For N large enough, there exists a constant* c_2 *such that*

$$E\left[\prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 \mid \mathcal{F}_{\partial B(x)}) \cdot 1_{\Omega_\Lambda^+ \cap F_\delta}\right] \leq \exp\left(-c_2 N^{d-2k+\gamma}\right). \tag{9}$$

Both constants depend on L, θ *and* δ *but not on* N.

Proof. In both cases, we use standard estimates on the centred Gaussian variables $m_x - \varphi_x$ under $P(\cdot \mid \mathcal{F}_{\partial B(x)})$.

(a) Since $G_L \longrightarrow G$, we have that, for L large enough, $4k(G - \kappa)/2G_L \leq 2k - \gamma$. We therefore get on $E_{\delta,\kappa}$

$$\prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 \mid \mathcal{F}_{\partial B(x)}) \leq P\left(\varphi_0 - m_0 \leq a_N \mid \mathcal{F}_{\partial B(x)}\right)^{\delta B}$$

$$\leq \left(1 - \frac{\sqrt{G_L}}{a_N} \exp\left(-\frac{a_N^2}{2G_L}\right)\right)^{\delta B}$$

$$\leq \exp\left(-c_1 N^{d-2k+\gamma}\right).$$

(b) On F_δ we have for some constants $C > 0, c_2 > 0$, and for N large enough

$$\prod_{x \in \tilde{\Lambda}} P(\varphi_x \geq 0 \mid \mathcal{F}_{\partial B(x)}) \leq \sum_{\lambda=0}^{\infty} P(\varphi_0 - m_0 \geq N^\lambda \mid \mathcal{F}_{\partial B(x)})^{\delta N^{-2k-(2-\gamma)\lambda+\gamma} B}$$

$$\leq \sum_{\lambda=0}^{\infty} \left(\exp\left(-\frac{N^{2\lambda}}{2G_L}\right)\right)^{\delta N^{-2k-(2-\gamma)\lambda+\gamma} B}$$

$$\leq \sum_{\lambda=0}^{\infty} \exp\left(-C N^{d-2k+\gamma}\right)^{N^{\gamma\lambda}}$$

$$\leq \exp\left(-c_2 N^{d-2k+\gamma}\right). \quad \square$$

Thus we only need to consider F^c, where we can easily bound $\sum_{x \in \tilde{B}_r} m_x$. Write

$$\sum_{x \in \tilde{B}_r} m_x = \sum_{x: m_x > a_N} m_x + \sum_{x: -1 < m_x \leq a_N} m_x + \sum_{\lambda=0}^{\lambda_{\max}} \sum_{x: -N^{\lambda+1} < m_x \leq N^\lambda} m_x$$

and bound the three parts separately: On $E_{\delta,\kappa}^c$, at least $(1 - \delta)$ of the m_x are at height at least a_N, so for the first part we get

$$\sum_{m_x > a_N} m_x \geq (1 - \delta) B a_N. \tag{10}$$

The second term can be estimated easily by writing

$$\sum_{-1 < m_x \leq a_N} m_x \geq -B. \tag{11}$$

Finally, since on F_δ^c there is

$$|\{x \in \tilde{B}_r : -N^{\lambda+1} < m_x \leq -N^\lambda\}| \leq |\{x \in \tilde{B}_r : m_x \leq -N^\lambda\}| \leq \frac{\delta}{N^{2k+(2-\gamma)\lambda-\gamma}} B,$$

we get

$$\sum_{\lambda=0}^{\lambda_{\max}} \left[\sum_{-N^{\lambda+1} < m_x \leq N^\lambda} m_x \right] \geq -\sum_{\lambda=0}^{\lambda_{\max}} B \cdot \delta \cdot N^{-2k-(2-\gamma)\lambda+\gamma} \cdot N^{\lambda+1}$$

$$= -B \cdot \delta \cdot N^{-2k+\gamma+1} \sum_{\lambda=0}^{\lambda_{\max}} N^{-(1-\gamma)\lambda}$$

$$\geq -c \cdot B \cdot N^{-2k+\gamma+1}. \tag{12}$$

The three estimates (10), (11), (12) together give

$$\frac{1}{B} \sum_{x \in \tilde{B}_r} m_x \geq (1 - \delta) a_N + O(1) \tag{13}$$

on F^c. Let $f_r \geq 0$ $(r \in I)$. Then (13) implies

$$P(\Omega_N^+ \cap F^c) \leq P \left(\sum_{r \in I} f_r \frac{1}{B} \sum_{x \in \tilde{B}_r} m_x > (1 - \delta) a_N \sum_{r \in I} f_r + O(1) \right)$$

$$\leq \exp \left(\frac{-(1-\delta)^2 a_N^2 \left(\sum_{r \in I} f_r \right)^2 + O(\sqrt{\log N})}{2 \mathrm{var} \left(\sum_{r \in I} f_r \frac{1}{B} \sum_{x \in \tilde{B}_r} m_x \right)} \right). \tag{14}$$

Now we can conclude the proof of the upper bound as in [3]. Since m_x is the conditional expectation $E(\varphi_x \mid \mathcal{F}_{\partial B(x)}) = E(\varphi_x \mid \mathcal{F}_\Lambda)$, we have by Jensen's inequality

$$\mathrm{var} \left(\sum_{r \in I} f_r \frac{1}{B} \sum_{x \in \tilde{B}_r} m_x \right) \leq \mathrm{var} \left(\sum_{r \in I} f_r \frac{1}{B} \sum_{x \in \tilde{B}_r} \varphi_x \right).$$

Define $f_\theta : \mathbb{R}^d \to \mathbb{R}$ by $f_\theta(t) = \sum_{r \in I} f_r 1_{A_r}(t)$. One easily sees that

$$\sum_{r \in I} f_r \frac{1}{B} \sum_{x \in \tilde{B}_r} \varphi_x = \frac{1}{B} \sum_{x \in \tilde{\Lambda}} f_\theta \left(\frac{x}{N} \right) \varphi_x \quad \text{and} \quad \sum_{r \in I} f_r = \frac{1}{B} \sum_{x \in \tilde{\Lambda}} f_\theta \left(\frac{x}{N} \right),$$

and consequently

$$\text{var} \left(\frac{1}{B} \sum_{r \in I} f_r \sum_{x \in \tilde{B}_r} \varphi_x \right) = \frac{1}{B^2} \sum_{x,y \in \tilde{\Lambda}} f_\theta \left(\frac{x}{N} \right) f_\theta \left(\frac{y}{N} \right) G(x, y).$$

Thus we obtain, using Lemma 2.1 and (14),

$$\limsup_{N \to \infty} \frac{1}{N^{d-2k} \log N} \log P(\Omega_N^+)$$

$$\leq \limsup_{N \to \infty} \frac{1}{N^{d-2k} \log N} \log P(\Omega_\Lambda^+ \cap F^c)$$

$$\leq \limsup_{N \to \infty} \frac{1}{N^{d-2k} \log N} \frac{-(1-\delta)^2 4k(G-\kappa) \log N \left(\sum_{r \in I} f_r \right)^2}{2\text{var} \left(\sum_{r \in I} f_r \frac{1}{B} \sum_{x \in \tilde{B}_r} m_x \right)}$$

$$= -(1-\delta)^2 2k(G-\kappa) \limsup_{N \to \infty} \frac{1}{N^{d-2k}} \cdot \frac{\left(\sum_{x \in \tilde{\Lambda}} f_\theta(\frac{x}{N}) \right)^2}{\sum_{x,y \in \tilde{\Lambda}} f_\theta(\frac{x}{N}) f_\theta(\frac{y}{N}) G(x, y)}.$$

As in [3], the proof is now concluded by applying Proposition 4.3, taking the supremum over all possible f_θ and letting $\kappa \to 0$ and $\delta \to 0$.

3. Proof of the height estimate

To prove Theorem 1.2, there are two directions to show. The first was proved in Theorem 2.2 of [10]: For any $\varepsilon > 0$, $\eta > 0$, and $z \in V_N$, such that $V_{N,\varepsilon}(z) \subset V_N$,

$$\lim_{N \to \infty} P \left(\frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \leq \sqrt{4kG} - \eta \,\middle|\, \Omega_N^+ \right) = 0. \tag{15}$$

We will now use Theorem 1.1 to show the other bound:

Proposition 3.1. *For any $\varepsilon > 0$, $\eta > 0$ and $z \in V_N$, with $V_{N,\varepsilon}(z) \subset V_N$*

$$\lim_{N \to \infty} P \left(\frac{\overline{\varphi}_{N,\varepsilon}(z)}{\sqrt{\log N}} \geq \sqrt{4kG} + \eta \,\middle|\, \Omega_N^+ \right) = 0. \tag{16}$$

The proof for the lattice free field in [3] uses the FKG-inequality for the conditional measure, which does not hold in our case. Similarly to in Section 2, we can handle this problem by carefully estimating the probability that, on Ω_N^+, the local sample mean of the field is higher

than $\sqrt{4kG \cdot \log N}$. This is done by comparing $\overline{\varphi}_{N,\varepsilon}(z)$ with the average of the conditional expectations m_x.

Proof. First, let $z = 0$, set $\overline{\varphi}_{N,\varepsilon} := \overline{\varphi}_{N,\varepsilon}(0)$, and $V_{N,\varepsilon} := V_{N,\varepsilon}(0)$. Fix L as in Section 2 and recall the definition of the subgrid $\overline{\Lambda}$, the boxes $B(x)$ and their K-boundary $\partial B(x)$. In this section, $\tilde{\Lambda}$ denotes the set $\{x \in \overline{\Lambda} : \partial B(x) \subset V_{N,\varepsilon}\}$, and $\Lambda = \cup_{x \in \tilde{\Lambda}} \partial B(x)$. For $r \in \mathbb{R}^d$ and $0 < \theta < 1$ let A_r be defined as in Section 2, and set $I = \{r \in \theta \mathbb{Z}^d : A_r \subset V_\varepsilon, \partial A_r \cap \partial V_\varepsilon = \emptyset\}$, where $V_\varepsilon = [-\varepsilon, \varepsilon]^d$. Set $B_r = N A_r$, and $\tilde{B}_r = B_r \cap \tilde{\Lambda}$. As before, set $m_x := E(\varphi_x \mid \mathcal{F}_{\partial B(x)})$ for $x \in \tilde{B}_r$.

We want to estimate

$$P\left(\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N} \ \Big| \ \Omega_N^+\right)$$

$$= \frac{1}{P(\Omega_N^+)} P\left(\{\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N}\} \cap \Omega_N^+\right).$$

Recall F from Section 3, fix $t > 0$ and set $D_t := \{\varphi : \text{there is } r \in I \text{ such that } \frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} (\varphi_x - m_x) < -t\}$. Then we can write

$$P\left(\{\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N}\} \cap \Omega_N^+\right)$$

$$= P\left(\{\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N}\} \cap \Omega_N^+ \cap F\right)$$

$$+ P\left(\{\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N}\} \cap \Omega_N^+ \cap F^c \cap D_t\right)$$

$$+ P\left(\{\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N}\} \cap \Omega_N^+ \cap F^c \cap D_t^c\right).$$

We have seen in the last section that the first term is negligible compared to $P(\Omega_N^+)$. For the second part, recall that conditioned on $\mathcal{F}_{\partial B(x)}$, the $\varphi_x - m_x, x \in \tilde{B}_r$, are independent centred Gaussian variables with variance G_L. Thus for the variance of the average we get

$$\text{var}\left(\frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} (\varphi_x - m_x) \Bigg| \mathcal{F}_\Lambda\right) = \frac{1}{|\tilde{B}_r|^2} \sum_{x \in \tilde{B}_r} \text{var}\left(\varphi_x - m_x \mid \mathcal{F}_{\partial B(x)}\right) = \frac{1}{|\tilde{B}_r|} \cdot G_L.$$

We can therefore find constants $c_1 > 0$, and $c_2 = c_2(\theta) > 0$ such that

$$P\left(D_t \cap F^c \cap \Omega_N^+\right) \leq c_2 E\left(P\left(\frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} (\varphi_x - m_x) < -t \Bigg| \mathcal{F}_\Lambda\right) \cdot 1_{F^c \cap \Omega_N^+}\right)$$

$$\leq c_2 \exp\left(\frac{-t^2 \cdot c_1 N^d}{2 G_L}\right), \tag{17}$$

which is also negligible compared with $P(\Omega_N^+)$. Therefore we only need to estimate

$$\limsup_{N \to \infty} \frac{1}{P(\Omega_N^+)} P\left(\{\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N}\} \cap \Omega_N^+ \cap F^c \cap D_t^c\right).$$

For this purpose we bound $\frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} \varphi_x$ from below on $\Omega_N^+ \cap F^c \cap D_t^c$. Write

$$\frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} \varphi_x = \frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} (\varphi_x - m_x) + \frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} m_x \tag{18}$$

and recall from the last section that on $\Omega_N^+ \cap F^c$

$$\frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} m_x \geq (1 - \delta)a_N + O(1) \tag{19}$$

for any $\delta > 0$ and $\kappa > 0$. This implies that on $\Omega_N^+ \cap F^c \cap D_t^c$, we have

$$\frac{1}{|\tilde{B}_r|} \sum_{x \in \tilde{B}_r} \varphi_x \geq (1 - \delta)a_N + O(1).$$

Since we can repeat this argument with any shift of the subgrid Λ, and average over all shifts, we conclude that on $\Omega_N^+ \cap F^c \cap D_t^c$

$$\frac{1}{|B_r|} \sum_{x \in B_r} \varphi_x \geq (1 - \delta)a_N + O(1). \tag{20}$$

From now on we will abbreviate $\frac{1}{|B_r|} \sum_{x \in B_r} \varphi_x$ as $\overline{\varphi}_r$. For $\kappa' > 0$, set $C_{\kappa'} := \{\varphi : $ there exists r_0 such that $\overline{\varphi}_{r_0} \geq (\sqrt{4k(G - \kappa)} + \kappa')\sqrt{\log N}\}$. It follows from (20) that, on $\Omega_N^+ \cap F^c \cap D_t^c$, for every $\eta > 0$ we can find $\kappa' > 0$ and $r_0 \in I$ such that, for $N \to \infty$,

$$P\left(\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG} + \eta)\sqrt{\log N}\right) \leq P\left(\{\overline{\varphi}_r \geq (1 - \delta)a_N \ \forall r \in I\} \cap C_{\kappa'}\right).$$

Let $f_r > 0, r \in I$.

$$P\left(\overline{\varphi}_r \geq (1 - \delta)a_N \quad \forall r \in I, \overline{\varphi}_{r_0} \geq (\sqrt{4k(G - \kappa)} + \kappa')\sqrt{\log N}\right)$$

$$\leq P\left(\sum_{r \in I} f_r \overline{\varphi}_r \geq (1 - \delta)a_N \cdot \sum_{r \in I} f_r + \kappa' f_{r_0}\sqrt{\log N}\right)$$

$$\leq \exp\left(\frac{-\left((1 - \delta)a_N \sum_{r \in I} f_r + \kappa' f_{r_0}\sqrt{\log N}\right)^2}{2\mathrm{var}\left(\sum_{r \in I} f_r \overline{\varphi}_r\right)}\right).$$

Defining f_θ as in the last section, we have

$$\sum_{r \in I} f_r = \frac{1}{|B_r|} \sum_{x \in V_{N,\varepsilon}} f_\theta\left(\frac{x}{N}\right)$$

and

$$\mathrm{var}\left(\sum_{r \in I} f_r \overline{\varphi}_r\right) = \frac{1}{|B_r|^2} \sum_{r \in I} \sum_{s \in I} f_r f_s \sum_{x \in B_r} \sum_{y \in B_s} G(x, y)$$

$$= \frac{1}{|B_r|^2} \sum_{x, y \in V_{N,\varepsilon}} f_\theta\left(\frac{x}{N}\right) f_\theta\left(\frac{x}{N}\right) G(x, y)$$

$$= O(N^{-d+2k}).$$

Similarly to the end of Section 2, we can then optimize over f_θ, use Proposition 4.3, and let κ and δ tend to 0. Then we see that there is a constant $c > 0$, such that

$$\limsup_{N\to\infty} \frac{1}{N^{d-2k}\log N} \log P\left(\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG}+\eta)\sqrt{\log N}\right) \leq -4k\cdot G\cdot C_k(V) - c.$$

Now we apply Theorem 1.1, and obtain

$$\limsup_{N\to\infty} \frac{1}{N^{d-2k}\log N} \log P\left(\overline{\varphi}_{N,\varepsilon} \geq (\sqrt{4kG}+\eta)\sqrt{\log N} \ \Big|\ \Omega_N^+\right) \leq -c,$$

which proves the claim in the case $z = 0$. The case of an arbitrary z is obtained by repeating the same arguments on a shifted grid. $\square$

Theorem 1.2 now follows immediately from (15) and Proposition 3.1. This proves the height estimate.

4. Green's function and k-capacity

In this section, we prove that there are several equivalent expressions for the capacity $C_k(V)$, as in the case $k = 1$. A crucial step is the decay of the Green's function $G(x, y)$ as $|x - y|$ tends to infinity. In the case of the free field, the local central limit theorem for the simple random walk yields a decay of order $|x|^{-d+2}$ (see [2,6]). In our model, we do not have a random walk representation, since the entries of the "transition matrix" $I - J$ can be negative. Nevertheless, using the methods of Section 3 of [8], one can, without the use of a local central limit theorem, obtain a decay of G of order $|x|^{2k-d}$. This was done by Sakagawa:

Lemma 4.1 ([10], Lemma 5.1). *Let $d \geq 2k + 1$. Then there is a constant η_k such that*

$$\lim_{|x|\to\infty} \frac{G(0, x)}{|x|^{2k-d}} = \frac{1}{q_k}\eta_k.$$

Define now a function $g_k : \mathbb{R}^d \to \mathbb{R}$ by $g_k(x) = \frac{\eta_k}{q_k}|x|^{2k-d}$ and a positive compact operator K_k on $L^2(V)$ by

$$K_k f(x) = \int_V g_k(x - y) f(y)\,\mathrm{d}y \quad (x \in V).$$

From the above lemma we get, for $|x - y| \to \infty$,

$$|g_k(x - y) - G(x, y)| = o(|x - y|^{-d+2k}). \tag{21}$$

In this section we use the short notation $\langle f, g\rangle_V := \int_V f(x)g(x)\,\mathrm{d}x$, for suitable functions f, g and $V \subset \mathbb{R}^d$. Note first the following (see also [10], Lemma 5.2):

Lemma 4.2. *Let $d \geq 2k + 1$. For all $h, f \in H^k(V)$,*

$$\frac{q_k}{(2d)^k}\left\langle h, K_k \frac{1}{(2d)^k}(-\Delta)^k f\right\rangle_V = \langle h, f\rangle_V.$$

Proof. In order to distinguish between the discrete and the continuous Laplacian, we denote them by Δ_d and Δ_c respectively. Using (21) we obtain

$$\frac{q_k}{(2d)^k} \langle h, K_k(-\Delta_c)^k f \rangle_V$$

$$= \lim_{N \to \infty} \frac{1}{N^{2d}} \sum_{x \in V_N} \sum_{y \in V_N} h\left(\frac{x}{N}\right) \cdot g_k\left(\frac{x}{N} - \frac{y}{N}\right) \cdot \frac{q_k}{(2d)^k} \cdot \left((-\Delta_c)^k f\right)\left(\frac{y}{N}\right)$$

$$= \lim_{N \to \infty} \frac{1}{N^{2d}} \sum_{x \in V_N} h\left(\frac{x}{N}\right) \sum_{y \in V_N} N^{d-2k} \cdot G\left(\frac{x}{N}, \frac{y}{N}\right) \cdot \frac{1}{N^{-2k}} q_k \left((-\Delta_d)^k f\right)\left(\frac{y}{N}\right)$$

$$= \lim_{N \to \infty} \frac{1}{N^d} \sum_{x \in V_N} h\left(\frac{x}{N}\right) \cdot f\left(\frac{x}{N}\right) = \frac{1}{q_k} \langle h, f \rangle_V. \quad \square$$

We can now prove the equivalence of several expressions for the k-th order capacity $C_k(V)$. Proposition 4.3 below was used implicitly in Section 5 of [10] (Lemma 5.2). As we are not aware of a reference, we include the proof here.

Proposition 4.3. *Let* $V = [-1, 1]^d$, $d \geq 2k + 1$. *Then*

$$\inf\left\{\frac{q_k}{(2d)^k} \int_{\mathbb{R}^d} |(-\nabla)^k h|^2 \, dx : H \in H_0^k(\mathbb{R}^d), h \geq 1_V\right\}$$

$$= \sup\left\{2\langle f, 1_V \rangle_V - \langle f, K_k f \rangle_V : f \in L_2(V)\right\}$$

$$= \sup\left\{\frac{\langle f, 1_V \rangle_V^2}{\langle f, K_k f \rangle_V} : f \in L_2(V)\right\}.$$

Proof. Let us prove the first equality. Notice that $M := \{h \in H_0^k(\mathbb{R}^d) : h \geq 1_V\}$ is a closed convex subset of the Hilbert space $H_0^k(\mathbb{R}^d)$, and thus has a minimizer h_0 for the Sobolev-norm on $H_0^k(\mathbb{R}^d)$. But this means exactly that h_0 minimizes $\int_{\mathbb{R}^d} |(-\nabla)^k h|^2 \, dx$ for $h \in M$. It is immediate that $h_0 = 1$ on V. Furthermore, $(-\Delta)^k h_0 = 0$ outside V. To see this, set $g(\varepsilon) = \int_{\mathbb{R}^d} |(-\nabla)^k h + \varepsilon \varphi|^2 \, dx$ for some $\varphi \in C_c^\infty(\mathbb{R}^d \setminus V)$. Then $\frac{dg}{d\varepsilon}\big|_{\varepsilon=0} = 0$, because h_0 is a minimizer of the integral. But this implies $\langle(-\Delta)^k h_0, \varphi\rangle_{\mathbb{R}^d \setminus V} = \langle(-\nabla)^k h_0, (-\nabla)^k \varphi\rangle_{\mathbb{R}^d \setminus V} = 0$ for all $\varphi \in C_c^\infty(\mathbb{R}^d \setminus V)$ and thus $(-\Delta)^k h_0 = 0$ on $\mathbb{R}^d \setminus V$.

There exist $\tau_n \in C_0^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$ such that $\lim_{n \to \infty}\langle h_0 - \tau_n, (-\Delta)^k(h_0 - \tau_n)\rangle_{\overset{\circ}{V}} = 0$ and $\tau_n = h_0$ on $\mathbb{R}^d \setminus V$, where $\overset{\circ}{V}$ denotes the interior of V. Set

$$f_n = \frac{q_k}{(2d)^k}(-\Delta)^k \tau_n.$$

For every n, f_n belongs to $L_2(\mathbb{R}^d)$, and, by the fact that $f_n = 0$ outside V, Lemma 4.2 and partial integration yield

$$2\langle f_n, \tau_n \rangle_V - \langle f_n, K_k f_n \rangle_V = \frac{q_k}{(2d)^k}\langle(-\Delta)^k \tau_n, \tau_n\rangle_{\mathbb{R}^d} = \frac{q_k}{(2d)^k}\int_{\mathbb{R}^d}|(-\nabla)^k \tau_n|^2 \, dx.$$

Moreover, as in [2], $\lim_{n \to \infty}|\langle f_n, 1_V - \tau_n\rangle_{L^2(\mathbb{R}^d)}| = 0$. Together with the above this yields

$$\sup\{2\langle f, 1_V \rangle_V - \langle f, K_k f \rangle_V\} \geq \limsup_{n \to \infty}\{2\langle f_n, 1_V \rangle_V - \langle f_n, K_k f_n \rangle_V\}$$

$$= \frac{q_k}{(2d)^k}\int_{\mathbb{R}^d}|(-\nabla)^k h_0| \, dx,$$

which gives one direction in the first equation. The other direction is an elementary calculation based on Lemma 4.2.

The second equation follows by expanding f in a basis of eigenvectors of the compact positive operator K_k. Maximizing shows that both sides are equal to $\sum_{i \in \mathbb{N}} \frac{\langle e_i, 1_V \rangle^2}{\lambda_i}$, where the e_i are the eigenvectors of K_k and λ_i the corresponding eigenvalues.

Acknowledgements

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