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Stochastic Processes and their Applications 117 (2007) 1928–1959

stochastic processes and their applications

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Cumulants of the maximum of the Gaussian random walk

A.J.E.M. Janssen[a], J.S.H. van Leeuwaarden[b,*]

[a] *Digital Signal Processing Group, Philips Research, WO-02, 5656 AA Eindhoven, The Netherlands*
[b] *Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands*

Received 8 November 2006; received in revised form 16 March 2007; accepted 21 March 2007
Available online 30 March 2007

Abstract

Let $X_1, X_2, \ldots$ be independent variables, each having a normal distribution with negative mean $-\beta < 0$ and variance 1. We consider the partial sums $S_n = X_1 + \cdots + X_n$, with $S_0 = 0$, and refer to the process $\{S_n : n \geq 0\}$ as the Gaussian random walk. This paper is concerned with the cumulants of the maximum $M_\beta = \max\{S_n : n \geq 0\}$.

We express all cumulants of M_β in terms of Taylor series about β at 0 with coefficients that involve the Riemann zeta function. Building upon the work of Chang and Peres [J.T. Chang, Y. Peres, Ladder heights, Gaussian random walks and the Riemann zeta function, Ann. Probab. 25 (1997) 787–802] on $\mathbb{P}(M_\beta = 0)$ and Bateman's formulas on Lerch's transcendent, expressions of this type for the first and second cumulants of M_β have been previously obtained by the authors [A.J.E.M. Janssen, J.S.H. van Leeuwaarden, On Lerch's transcendent and the Gaussian random walk, Ann. Appl. Probab. 17 (2007) 421–439]. The method is systemized in this paper to yield similar Taylor series expressions for all cumulants.

The key idea in obtaining the Taylor series for the kth cumulant is to differentiate its Spitzer-type expression (involving the normal distribution) $k + 1$ times, rewrite the resulting expression in terms of Lerch's transcendent, and integrate $k + 1$ times. The major issue then is to determine the $k + 1$ integration constants, for which we invoke Euler–Maclaurin summation, among other things.

Since the Taylor series are only valid for $\beta < 2\sqrt{\pi}$, we obtain alternative series expansions that can be evaluated for all $\beta > 0$. We further present sharp bounds on $\mathbb{P}(M_\beta = 0)$ and the first two moments of M_β. We show how the results in this paper might find important applications, particularly for queues in heavy traffic, the limiting overshoot in boundary crossing problems and the equidistant sampling of Brownian motion.

* Corresponding author. Tel.: +31 40 2472813.
E-mail addresses: a.j.e.m.janssen@philips.com (A.J.E.M. Janssen), j.s.h.v.leeuwaarden@tue.nl (J.S.H. van Leeuwaarden).

0304-4149/$ - see front matter
doi:10.1016/j.spa.2007.03.006

MSC: 11M06; 30B40; 60G50; 60G51; 65B15

Keywords: Gaussian random walk; All-time maximum; Cumulants; Riemann zeta function; Lerch's transcendent; Spitzer's identity; Euler–Maclaurin summation; Queues in heavy traffic; Equidistant sampling of Brownian motion

1. Introduction

Let $X_1, X_2, \ldots$ be independent variables, each having a normal distribution with mean $-\beta < 0$ and variance 1. We consider the partial sums $S_n = X_1 + \cdots + X_n$, with $S_0 = 0$, and refer to the process $\{S_n : n \geq 0\}$ as the Gaussian random walk. In this paper we present explicit expressions for all moments (in terms of the cumulants) of the maximum

$$M_\beta = \max\{S_n : n \geq 0\}. \tag{1.1}$$

These explicit expressions hold for $0 < \beta < 2\sqrt{\pi}$ and are in terms of Taylor series about β at 0 with coefficients that involve the Riemann zeta function. The explicit expressions facilitate the derivation of accurate asymptotic approximations for small values of β. We also present sharp bounds on $\mathbb{P}(M_\beta = 0)$, $\mathbb{E}M_\beta$ and $\operatorname{Var} M_\beta$.

The distribution of M_β plays an important role in several areas of applied probability, like *queueing theory*, *risk theory* and *sequential analysis*. In queueing theory, the most famous model is the $GI/G/1$ queue. Determining the stationary waiting time distribution requires the solution of a linear integral equation of Wiener–Hopf type known as Lindley's equation (see [23]). In general, solving Lindley's equation is challenging, both analytically and numerically. However, sharp approximations can be obtained in a regime called *heavy traffic* (see [3,20,22,28]), in which the load is just below its critical level, and so the queue is only just stable with relatively large waiting times. Kingman showed [22] that in heavy traffic, the waiting time of a scaled version of the $GI/G/1$ queue can be very well approximated by the maximum of the Gaussian random walk with $\beta \downarrow 0$. In risk theory, the counterpart of heavy traffic is *small-safety loading*, a regime in which the premium charged is close to the typical pay-out for claims. In these classical heavy-traffic or small-safety loading settings, one is thus typically interested in the distribution of M_β for values of β close to zero.

In queueing theory, M_β for β away from zero plays a key role in a heavy-traffic scaling regime known as the Halfin–Whitt regime, see [17,20]. Under this regime, the stationary waiting time is in fact identical in distribution to M_β. Contrary to the classical heavy-traffic regime, though, β need not be small, but instead is an important decision variable (see Borst et al. [5] and the references therein).

Other important applications involving M_β stem from the fact that the Gaussian random walk can be obtained from a Brownian motion by equidistant sampling. For this reason, M_β shows up in a range of applications, such as sequentially testing for the drift of a Brownian motion [12], corrected diffusion approximations [24], simulation of Brownian motion [4,8], option pricing [6], and the thermodynamics of a polymer chain [14].

In all the above-mentioned applications, the moments of the maximum of the Gaussian random walk are often the principal targets of investigation. In [18] we obtained explicit expressions for the first two moments of M_β. We built upon the work of Chang and Peres [10] on the first descending ladder height S_τ with $\tau = \inf\{n \geq 1 : S_n < 0\}$. They derived an exact expression for $\mathbb{E}S_\tau$, therewith complementing first order approximations of Siegmund [24] and Chang [9]. By the relation $\mathbb{E}S_\tau = -\beta/\mathbb{P}(M_\beta = 0)$ (see Asmussen [3], p. 225), the result leads to an exact expression for $\mathbb{P}(M_\beta = 0)$. Chang and Peres start from a Spitzer-type expression for

$\mathbb{P}(M_\beta = 0)$. The key idea of Chang and Peres is then to differentiate this expression with respect to β, and rewrite the resulting expression in terms of the *polylogarithm* $\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} n^{-s} z^n$, with $z = \exp(-\beta^{2/2})$, case $s = 1/2$. Subsequently, Chang and Peres present an analytic continuation of $\mathrm{Li}_s(z)$, which results in Taylor series about $\beta = 0$ with coefficients that involve the Riemann zeta function. The final result for $\mathbb{E}S_\tau$ is then obtained by integration. As pointed out in [18], $\mathrm{Li}_s(z)$ is a special case of *Lerch's transcendent*, see (4.9), for which the matter of analytic continuation has been established in full generality by Bateman (and/or the staff of the Bateman Manuscript Project), see [15], Section 1.11(8).

While the first two moments were studied in [18] on their own merits, we now generalize the approach in [18] to obtain a systematic method to compute all cumulants (i.e. moments). The key idea in obtaining the Taylor series for the kth cumulant is to differentiate its Spitzer-type expression $k + 1$ times, rewrite the resulting expression in terms of Lerch's transcendent, and integrate $k + 1$ times. The major issue then is to determine the $k + 1$ integration constants, which will be done using Euler–Maclaurin summation, among other things.

The paper is structured as follows. We present our main results in Section 2. These concern expressions for all cumulants of M_β in terms of Taylor series about $\beta = 0$ with coefficients that involve the Riemann zeta function, analytic continuation of these series, and sharp bounds on $\mathbb{P}(M_\beta = 0)$, $\mathbb{E}M_\beta$ and $\mathrm{Var}\, M_\beta$ for small values of β. In Section 3 we discuss three applications: equidistant sampling of Brownian motion, the limiting overshoot in boundary crossing problems and a discrete queue under Halfin–Whitt scaling. An outline of the proof of the Taylor series result is provided in Section 4, while the details are presented in Section 5. The analytic continuation of these series is outlined in Section 6. The bounds are proved in Section 7.

2. Main results

Spitzer's identity leads to (see Thm. 3.1 in [26], and e.g. [1,21])

$$\mathbb{E}(\mathrm{e}^{sM_\beta}) = \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n}\mathbb{E}(\mathrm{e}^{sS_n^+} - 1)\right\}, \quad \mathrm{Re}\, s \le 0, \tag{2.1}$$

with $x^+ = \max\{0, x\}$. The kth cumulant of a random variable A is defined as the kth derivative of $\log \mathbb{E}\mathrm{e}^{sA}$ evaluated at $s = 0$. We then see that

$$\log \mathbb{E}(\mathrm{e}^{sM_\beta}) = \sum_{n=1}^{\infty} \frac{1}{n}\mathbb{E}(\mathrm{e}^{sS_n^+} - 1) = \sum_{n=1}^{\infty} \frac{1}{n}\int_0^{\infty} \left(sx + \frac{1}{2}s^2x^2 + \cdots\right) f_{S_n^+}(x)\mathrm{d}x, \tag{2.2}$$

with $f_{S_n^+}$ the density function of S_n^+, and so the kth cumulant of M_β equals

$$\frac{\mathrm{d}^k}{(\mathrm{d}s)^k} \log \mathbb{E}(\mathrm{e}^{sM_\beta})|_{s=0} = \sum_{n=1}^{\infty} \frac{1}{n}\mathbb{E}((S_n^+)^k) =: J_k(\beta), \quad k = 1, 2, \ldots. \tag{2.3}$$

Recall that the first cumulant is the mean, the second cumulant is the variance, the third cumulant is the central moment $\mathbb{E}(M_\beta - \mathbb{E}M_\beta)^3$, and the fourth cumulant is $\mathbb{E}(M_\beta - \mathbb{E}M_\beta)^4 - 3\mathbb{E}(M_\beta - \mathbb{E}M_\beta)^2$. Using the normality of S_n, it follows immediately from (2.3) that the quantities $J_k(\beta)$ can be expressed as

$$J_k(\beta) = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{2\pi}} \int_{\beta\sqrt{n}}^{\infty} (\sqrt{n}x - \beta n)^k \mathrm{e}^{-x^2/2}\mathrm{d}x. \tag{2.4}$$

In the above form, the definition of $J_k(\beta)$ extends to the case $k = 0$, for which it obviously holds that $J_0(\beta) = \sum_{n=1}^{\infty} \frac{1}{n}\mathbb{P}(S_n > 0)$. From Spitzer's identity we then know that $J_0(\beta) = -\ln \mathbb{P}(M_\beta = 0)$ (see Sec. 8.5 in Chung [13]).

The main contribution in this paper is then the following result for $J_k(\beta)$ (with $\zeta(z)$ the Riemann zeta function):

Theorem 1. *There holds*

$$J_0(\beta) = -\ln\beta - \frac{1}{2}\ln 2 - \frac{\zeta(1/2)}{\sqrt{2\pi}}\beta - \frac{1}{\sqrt{2\pi}}\sum_{r=1}^{\infty}\frac{\zeta(-r+1/2)(-1/2)^r\beta^{2r+1}}{r!(2r+1)}, \tag{2.5}$$

and for $k = 1, 2, \ldots$

$$J_k(\beta) = \frac{(k-1)!}{2^k}\beta^{-k} + \sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j\Gamma(\frac{k-j+1}{2})}{\sqrt{2\pi}}\zeta\left(-\frac{1}{2}k - \frac{1}{2}j + 1\right)2^{\frac{k-j-1}{2}}\beta^j$$
$$+\frac{(-1)^{k+1}k!}{\sqrt{2\pi}}\sum_{r=0}^{\infty}\frac{\zeta(-k-r+1/2)(-1/2)^r\beta^{2r+k+1}}{r!(2r+1)\cdots(2r+k+1)}, \tag{2.6}$$

when $0 < \beta < 2\sqrt{\pi}$.

Theorem 1 generalizes some previously obtained results. For $\mathbb{P}(M_\beta = 0)$ we get

$$\mathbb{P}(M_\beta = 0) = \sqrt{2}\beta\exp\left\{\frac{\beta}{\sqrt{2\pi}}\sum_{r=0}^{\infty}\frac{\zeta(1/2-r)}{r!(2r+1)}\left(\frac{-\beta^2}{2}\right)^r\right\}, \tag{2.7}$$

a result that was already obtained by Chang and Peres [10], Thm. 1.1 on p. 788. An alternative proof of (2.7) was presented in [18], along with the derivations of (2.6) for $k = 1, 2$, yielding

$$\mathbb{E}M_\beta = \frac{1}{2\beta} + \frac{\zeta(1/2)}{\sqrt{2\pi}} + \frac{1}{4}\beta + \frac{\beta^2}{\sqrt{2\pi}}\sum_{r=0}^{\infty}\frac{\zeta(-1/2-r)}{r!(2r+1)(2r+2)}\left(\frac{-\beta^2}{2}\right)^r, \tag{2.8}$$

and

$$\mathrm{Var}\,M_\beta = \frac{1}{4\beta^2} - \frac{1}{4} - \frac{2\zeta(-1/2)}{\sqrt{2\pi}}\beta - \frac{\beta^2}{24} - \frac{2\beta^3}{\sqrt{2\pi}}\sum_{r=0}^{\infty}\frac{\zeta(-3/2-r)(-\beta^2/2)^r}{r!(2r+1)(2r+2)(2r+3)}. \tag{2.9}$$

We need not necessarily rely on Theorem 1 to obtain exact results on the moments of M_β, since the the normality of S_n leads to (2.4). In comparing (2.6) and (2.4) it is evident that (2.6) provides more qualitative insight into the role of β. For $\beta \downarrow 0$, (2.6) is a powerful result that clearly shows the difference between $J_k(\beta)$ and its limiting value $(k-1)!(2\beta)^{-k}$. For moderate values of β, (2.6) provides valuable information on the influence of β.

From a numerical viewpoint, (2.6) has advantages over (2.4) as well. It is clear that the infinite series in (2.6) converge more rapidly for smaller values of β, while the contrary holds for their Gaussian-type counterparts (2.4) (for a comparison of speed of convergence, see Sec. 6 of [18]). An advantage of (2.4) is that it holds for all $\beta > 0$, and that it can be used to derive the bounds presented in Theorem 3 below.

The series over r in (2.5) and (2.6) converge for $|\beta| < 2\sqrt{\pi}$ while it is clear from (2.4) that $J_k(\beta)$ makes sense for all $\beta > 0$. In Section 6 we present alternative series expansions for $J_k(\beta)$ that can be evaluated for all $\beta > 0$. The alternative expansions for $J_k(\beta)$, $k = 0, 1, 2$, lead to the result below. The result is in the same spirit but more explicit than what was developed by Chang and Peres in Section 2 of [10].

Theorem 2. *We have for* $\beta > 0$

$$\mathbb{P}(M_\beta = 0) = \sqrt{2}\beta \exp\left\{\frac{\zeta(1/2)}{\sqrt{2\pi}}\beta + \frac{\beta}{\pi}\mathrm{Re}\left[\mathrm{e}^{\frac{\pi \mathrm{i}}{4}} S_0\left(\frac{-\mathrm{i}\beta^2}{4\pi}\right)\right]\right\}, \tag{2.10}$$

$$\mathbb{E}M_\beta = \frac{1}{2\beta} + \frac{\zeta(1/2)}{\sqrt{2\pi}} + \frac{1}{4}\beta + \frac{\beta^2}{2\pi^2}\mathrm{Re}\left[-\mathrm{e}^{\frac{\pi \mathrm{i}}{4}} S_1\left(\frac{-\mathrm{i}\beta^2}{4\pi}\right)\right], \tag{2.11}$$

$$\mathrm{Var}\, M_\beta = \frac{1}{4\beta^2} - \frac{1}{4} - \frac{2\zeta(-1/2)}{\sqrt{2\pi}}\beta - \frac{\beta^2}{24} - \frac{\beta^3}{4\pi^3}\mathrm{Re}\left[\mathrm{e}^{\frac{\pi \mathrm{i}}{4}} S_2\left(\frac{-\mathrm{i}\beta^2}{4\pi}\right)\right], \tag{2.12}$$

in which

$$S_0(b) = \frac{\sqrt{\pi}}{\sqrt{b}}\sum_{n=1}^{\infty}\left(\arcsin(b/n)^{1/2} - (b/n)^{1/2}\right), \tag{2.13}$$

$$S_1(b) = \frac{\sqrt{\pi}}{2b}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sqrt{n} - \sqrt{n-b}\right), \tag{2.14}$$

$$S_2(b) = \frac{\sqrt{\pi}}{4b}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sqrt{n} - \sqrt{n-b}\right). \tag{2.15}$$

Remark. For larger values of β, the terms in (2.4) for $n = 2, 3, \ldots$ are dominated by the term for $n = 1$. Upon some rewriting we get from (2.4) that

$$\frac{1}{n\sqrt{2\pi}}\int_{\beta\sqrt{n}}^{\infty}(\sqrt{n}x - \beta n)^k \mathrm{e}^{-x^2/2}\mathrm{d}x = (2n)^{\frac{1}{2}k-1} k!\, \mathrm{i}^k \mathrm{erfc}\, z, \tag{2.16}$$

with $\mathrm{i}^n\mathrm{erfc}$ the n times repeated integral of the complementary error function, see Abramowitz and Stegun [2], 7.2 on pp. 299–300. From [2], 7.2.14 on p. 300, we then get the asymptotic series

$$\begin{aligned}\frac{1}{n\sqrt{2\pi}}\int_{\beta\sqrt{n}}^{\infty}(\sqrt{n}x - \beta n)^k \mathrm{e}^{-x^2/2}\mathrm{d}x &\sim \frac{2}{\sqrt{\pi}}(2n)^{-3/2}\beta^{-k-1}\mathrm{e}^{-\frac{1}{2}n\beta^2}\sum_{m=0}^{\infty}\frac{(-1)^m(2m+k)!}{m!(2n\beta^2)^m} \\ &= \frac{\beta^{-k-1}}{n\sqrt{2\pi n}}\mathrm{e}^{-\frac{1}{2}n\beta^2}\left[k! - \frac{(k+2)!}{2n\beta^2} + \frac{(k+4)!}{8n^2\beta^4} - \cdots\right].\end{aligned} \tag{2.17}$$

When we apply this, for instance, with $\beta = 10$, we see that the term in the series in (2.4) for $J_k(\beta)$ with $n = 2$ is about $2^{-3/2}\mathrm{e}^{-50}$ times the term with $n = 1$: this second term and all subsequent terms are totally negligible. For this value of β, the accuracy of

$$\left.\frac{\beta^{-k-1}\mathrm{e}^{-\frac{1}{2}n\beta^2}}{n\sqrt{2\pi n}}\right|_{n=1} \tag{2.18}$$

as an approximation of the first term is of order $(k+2)(k+1)/200$ (relative error).

We shall now present some bounds and approximations. The expressions (2.5)–(2.9) all involve infinite series, comprising the Riemann zeta function, that converge absolutely for

$0 < \beta < 2\sqrt{\pi}$. For small values of β the terms involving higher powers of β are quite small. It follows, for instance, from (2.7) that

$$\mathbb{P}(M_\beta = 0) = \sqrt{2}\beta \exp\left\{\frac{\zeta(1/2)}{\sqrt{2\pi}}\beta + \mathcal{O}(\beta^3)\right\}, \tag{2.19}$$

where $\zeta(1/2)/\sqrt{2\pi} \approx -0.5826$ and the constant implied by the $\mathcal{O}$-symbol is of the order $|\zeta(-1/2)/6\sqrt{2\pi}| \approx 0.0138$ when $\beta > 0$ is away from $2\sqrt{\pi}$ (see [18], (6.3) where $\zeta(1/2-r)$ is estimated). This is in fact Chang's result on the expected first descending ladder height $\mathbb{E}S_\tau$ for the Gaussian family (we recall the relation $\mathbb{E}S_\tau = -\beta/\mathbb{P}(M_\beta = 0)$), see Chang [9], Thm. 4.2 on p. 732 (see also Siegmund [24], Lemma 2 on p. 705). Likewise, we get from (2.8)

$$\mathbb{E}M_\beta = \frac{1}{2\beta} + \frac{\zeta(1/2)}{\sqrt{2\pi}} + \frac{1}{4}\beta + \mathcal{O}(\beta^2), \tag{2.20}$$

which is a refinement of Kingman's result, [22], (51) on p. 156, and Siegmund's result, [24], Thm. 1 on p. 704, in the sense that it is more specific about the terms after the constant $\zeta(1/2)/\sqrt{2\pi}$. Similarly, we have

$$\operatorname{Var} M_\beta = \frac{1}{4\beta^2} - \frac{1}{4} - \frac{2\zeta(-1/2)}{\sqrt{2\pi}}\beta - \frac{1}{24}\beta^2 + \mathcal{O}(\beta^3). \tag{2.21}$$

More generally, the expression on the first line on the right-hand side of (2.6) provides an approximation of $J_k(\beta)$ whose absolute and relative error decays quickly with k when $\beta > 0$ is well below $2\sqrt{\pi}$. This is so since for these values of β the term on the second line of the right-hand side of (2.6) is of order $(k-1)!(\beta/2\pi)^{k+1}$, compare [18], Sec. 6.

We now present some sharp bounds on the expressions in (2.7)–(2.9) that rely solely on β and do not contain the Riemann zeta function.

Theorem 3. (i) *There holds for* $0 < \beta \le \sqrt{2/\pi}$

$$\mathbb{P}(M_\beta = 0) \le 2\left(1 - \mathrm{e}^{-\beta^2/2}\right)^{1/2} \exp\left\{-\frac{\beta}{\sqrt{\pi}} + \frac{1}{8}\beta^2\right\}, \tag{2.22}$$

$$\mathbb{P}(M_\beta = 0) \ge 2\left(1 - \mathrm{e}^{-\beta^2/2}\right)^{1/2} \exp\left\{-\frac{3\beta}{2\sqrt{2\pi}} + \frac{1}{8}\beta^2 - \frac{\beta^3}{9\sqrt{2\pi}}\right\}. \tag{2.23}$$

(ii) *There holds for* $\beta > 0$

$$\mathbb{E}M_\beta \le \frac{1}{2\beta} - \frac{1}{\sqrt{\pi}} + \frac{1}{4}\beta - \frac{\beta^2}{12\sqrt{\pi}} + \frac{\beta^4}{240\sqrt{\pi}}, \tag{2.24}$$

$$\mathbb{E}M_\beta \ge \frac{1}{2\beta} - \frac{3}{2\sqrt{2\pi}} + \frac{1}{4}\beta - \frac{\beta^2}{12\sqrt{2\pi}} - \frac{\beta^4}{24\sqrt{2\pi}}. \tag{2.25}$$

(iii) *There holds for* $\beta > 0$

$$\operatorname{Var} M_\beta \le \frac{1}{4\beta^2} - \frac{1}{4} + \frac{\beta}{3\sqrt{\pi}} - \frac{1}{16}\beta^2 + \frac{\beta^3}{60\sqrt{\pi}}, \tag{2.26}$$

$$\operatorname{Var} M_\beta \ge \frac{1}{4\beta^2} - \frac{1}{4} + \frac{\beta}{3\sqrt{2\pi}} - \frac{\beta^3}{30\sqrt{2\pi}} - \frac{\beta^5}{63\sqrt{2\pi}}. \tag{2.27}$$

Table 1
$\mathbb{P}(M_\beta = 0)$ for various values of β

β	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	0.7000	0.7900
$\mathbb{P}(M_\beta = 0)$	0.1334	0.2518	0.3564	0.4485	0.5293	0.6000	0.6615	0.7099
(2.23)	0.1332	0.2509	0.3541	0.4441	0.5215	0.5873	0.6421	0.6827
(2.22)	0.1337	0.2527	0.3582	0.4515	0.5335	0.6053	0.6678	0.7169
(2.19)	0.1334	0.2517	0.3562	0.4480	0.5283	0.5981	0.6582	0.7049

The exact values of $\mathbb{P}(M_\beta = 0)$ are approximated by truncating the infinite series in (2.7) at $r = 50$. In (2.19) the term $\mathcal{O}(\beta^3)$ has been omitted.

Table 2
$\mathbb{E}M_\beta$ for various values of β

β	0.1000	0.2500	0.5000	0.7500	1.0000	1.5000	2.0000
$\mathbb{E}M_\beta$	4.4420	1.4773	0.5321	0.2484	0.1264	0.0347	0.0090
(2.25)	4.4263	1.4619	0.5172	0.2318	0.1017	−0.0490	−0.2474
(2.24)	4.4603	1.4954	0.5492	0.2643	0.1411	0.0503	0.0354
(2.20)	4.4424	1.4799	0.5424	0.2716	0.1674	0.1257	0.1674

The exact values of $\mathbb{E}M_\beta$ are approximated by truncating the infinite series in (2.8) at $r = 50$. In (2.20) the term $\mathcal{O}(\beta^2)$ has been omitted.

For Theorem 3(i) we note that $2(1 - \mathrm{e}^{-\beta^2/2})^{1/2} = \sqrt{2}\beta(1 + \mathcal{O}(\beta^2))$, see (2.7). Furthermore, comparing Theorem 3(i) with (2.7) and Theorem 3(ii) with (2.8), we note that

$$0.5642 \approx \frac{1}{\sqrt{\pi}} < -\frac{\zeta(1/2)}{\sqrt{2\pi}} < \frac{3}{2\sqrt{2\pi}} \approx 0.5984 \tag{2.28}$$

with $-\zeta(1/2)/\sqrt{2\pi} \approx 0.5826$. Finally, comparing Theorem 3(iii) with (2.9), we note that

$$0.1330 \approx \frac{1}{3\sqrt{2\pi}} < -\frac{2\zeta(-1/2)}{\sqrt{2\pi}} < \frac{1}{3\sqrt{\pi}} \approx 0.1881 \tag{2.29}$$

with $-2\zeta(-1/2)/\sqrt{2\pi} \approx 0.1659$.

Tables 1–3 display the bounds and/or approximations for $\mathbb{P}(M = 0)$, $\mathbb{E}M$ and $\mathrm{Var}\, M$, respectively, for various values of β. In Figs. 1–3 we have plotted $\mathbb{P}(M = 0)$, $\beta\mathbb{E}M$ and $\beta^2\mathrm{Var}\, M$, respectively, as a function of β.

3. Applications

3.1. Equidistant sampling of Brownian motion

Let the process $\{B_t : t \geq 0\}$ be a Brownian motion with negative drift coefficient $-\beta$ and variance parameter σ^2, so that

$$B_t = B_0 - \beta t + \sigma W_t, \tag{3.1}$$

where $\{W_t : t \geq 0\}$ is a Wiener process (standard Brownian motion). Let

$$\tilde{M} = \max\{B_t : t \geq 0\}. \tag{3.2}$$

Table 3
Var M_β for various values of β

β	0.1000	0.2500	0.5000	0.7500	1.0000	1.5000	2.0000
Var M_β	24.7662	3.7889	0.8229	0.2969	0.1276	0.0280	0.0062
(2.27)	24.7633	3.7830	0.8146	0.2871	0.1134	−0.0324	−0.2306
(2.26)	24.7682	3.7933	0.8296	0.3043	0.1350	0.0343	0.0139
(2.21)	24.7662	3.7889	0.8225	0.2954	0.1242	0.0162	−0.0224

The exact values of Var M_β are approximated by truncating the infinite series in (2.9) at $r = 50$. In (2.21) the term $\mathcal{O}(\beta^3)$ has been omitted.

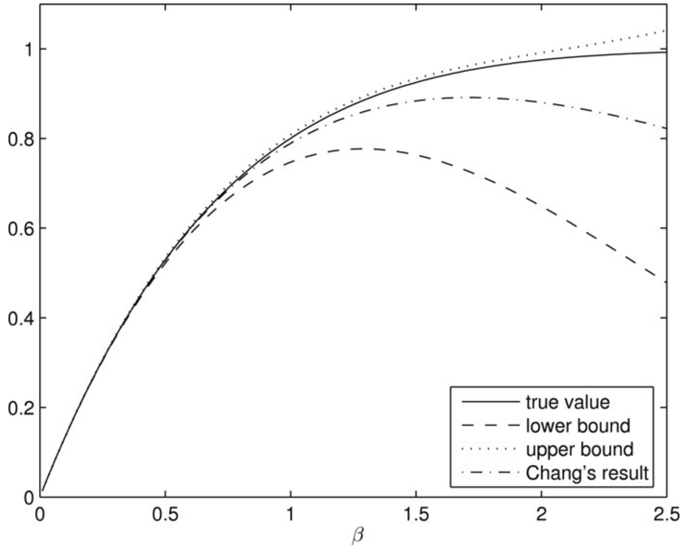

Fig. 1. Plot of $\mathbb{P}(M_\beta = 0)$, along with the bounds (2.24) and (2.25), and Chang's result (2.19) with $\mathcal{O}(\beta^3)$ deleted. The exact values of $\mathbb{P}(M_\beta = 0)$ are approximated by truncating the infinite series in (2.7) at $r = 50$.

We take $B_0 = 0$, and then it is well known that $\mathbb{P}(\tilde{M} \geq x) = \mathrm{e}^{-(2\beta/\sigma^2)x}$ (exponential distribution with rate $2\beta/\sigma^2$, see e.g. Chen and Yao [11], Lemma 5.5 on p. 102). It thus follows that

$$\mathbb{E}(\mathrm{e}^{s\tilde{M}}) = \frac{2\beta/\sigma^2}{2\beta/\sigma^2 - s}, \tag{3.3}$$

and so the kth cumulant of $\tilde{M}$ equals

$$\frac{(k-1)!}{2^k \beta^k \sigma^{-2k}}. \tag{3.4}$$

We set σ to 1 (without loss of generality). One way then to see the Gaussian random walk is the (equidistantly) sampled version of this Brownian motion, and by increasing the sampling frequency the Gaussian random walk will converge to the Brownian motion. How fast is this convergence? To address this question we first extend our definition of the Gaussian random walk. Let the Gaussian random walk be defined by

$$S(\beta, \nu) := \{S_n(\beta, \nu) : n = 0, 1, \ldots\}, \tag{3.5}$$

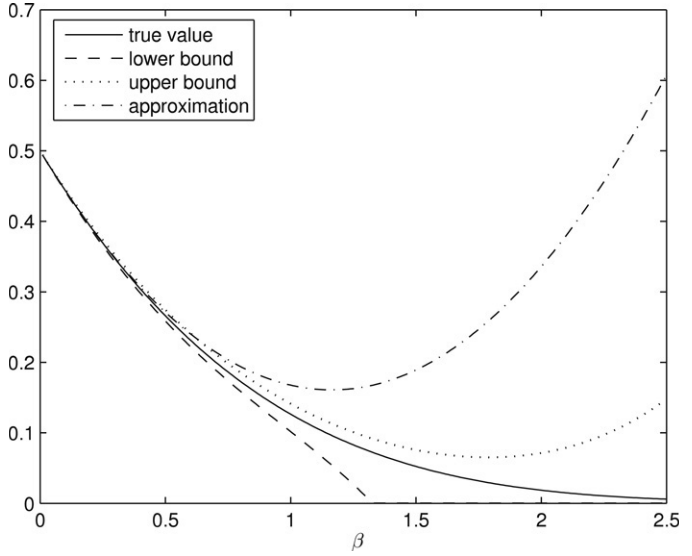

Fig. 2. Plot of $\beta \mathbb{E} M_\beta$, along with β times the bounds (2.24) and (2.25), and the approximation (2.20) with $\mathcal{O}(\beta^2)$ deleted. The exact values of $\mathbb{E} M_\beta$ are approximated by truncating the infinite series in (2.8) at $r = 50$.

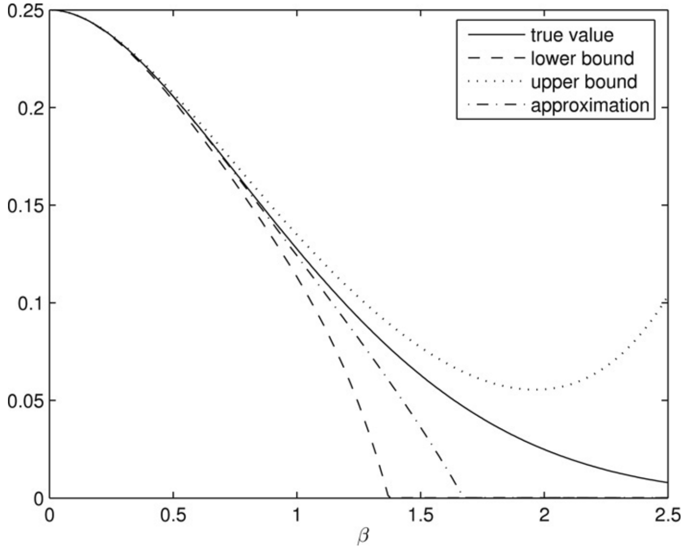

Fig. 3. Plot of $\beta^2 \operatorname{Var} M_\beta$, along with β^2 times the bounds (2.26) and (2.27), and the approximation (2.21) with $\mathcal{O}(\beta^3)$ deleted. The exact values of $\operatorname{Var} M_\beta$ are approximated by truncating the infinite series in (2.9) at $r = 50$.

where $S_n(\beta, \nu) = 0$ and $S_n(\beta, \nu) = X_{\nu,1} + \cdots + X_{\nu,n}$, with $X_{\nu,1}, X_{\nu,2}, \ldots$ independent variables, each having a normal distribution with mean $-\beta/\nu < 0$ and variance $1/\nu$. Let

$$M_{\nu,\beta} = \max\{S_n(\beta, \nu) : n = 0, 1, \ldots\}. \tag{3.6}$$

Our earlier definition of the Gaussian random walk corresponds to $S(\beta, 1)$ with its associated maximum $M_{1,\beta} =: M_\beta$. Since

$$M_{\nu,\beta} \overset{d}{=} \nu^{-1/2} M_{\nu^{-1/2}\beta}, \tag{3.7}$$

where $\overset{d}{=}$ denotes equality in distribution, all characteristics of $M_{\nu,\beta}$ can be expressed in those of M_β.

We now sample the Brownian motion at points $0, 1/\nu, 2/\nu, \ldots$, with ν some positive integer, and use as a measure of convergence the difference in all-time maximum between the Brownian motion and its sampled version (where we know that $\mathbb{E}\tilde{M} = 1/(2\beta)$). From our results on $\mathbb{E}M_\beta$ for the Gaussian random walk (2.8), together with (3.4) and (3.7), we find that

$$\mathbb{E}\tilde{M} - \mathbb{E}M_{\nu,\beta} = -\frac{\zeta(1/2)}{\sqrt{2\pi\nu}} + \mathcal{O}(1/\nu). \tag{3.8}$$

Results similar to (3.8), in slightly different settings, have been presented in Asmussen et al. [4], Thm. 2 on p. 884, and Calvin [8], Thm. 1 on p. 611. A crucial difference is that our result (3.8) is obtained from the exact expression for $\mathbb{E}M_{\nu,\beta}$, while the results in [4,8] are derived from considering the Brownian motion in a finite time interval, and estimating its maximum by Euler–Maclaurin summation.

The leading part of the right-hand side of (3.8) does not depend on the drift β. However, from Theorem 1 we can easily obtain higher-order asymptotics that do involve the drift, like

$$\mathbb{E}\tilde{M} - \mathbb{E}M_{\nu,\beta} = -\frac{\zeta(1/2)}{\sqrt{2\pi\nu}} + \frac{\beta}{4\nu} + \mathcal{O}(\nu^{-3/2}). \tag{3.9}$$

Moreover, our exact analysis of M_β leads to asymptotic expressions up to any order, for all cumulants of the maximum. For example, it readily follows from (2.9) that (with $\operatorname{Var}\tilde{M} = 1/(4\beta^2)$)

$$\operatorname{Var}\tilde{M} - \operatorname{Var}M_{\nu,\beta} = -\frac{1}{4\nu} - \frac{2\zeta(-1/2)}{\sqrt{2\pi}}\frac{\beta}{\nu^{3/2}} + \mathcal{O}(\nu^{-2}), \tag{3.10}$$

where $-2\zeta(-1/2)/\sqrt{2\pi} \approx 0.1659$.

3.2. *Limiting overshoot in boundary crossing problems*

The first (descending) ladder height $\tau = \inf\{n \geq 1 : S_n < 0\}$ fulfils a crucial role in applications of random walk theory (see e.g. Asmussen [3], Feller [16] and Siegmund [24]). One important application is the asymptotic analysis of boundary crossing problems [25]. In the latter case, a quantity of interest is the *expected limiting overshoot* which arises in e.g. sequential analysis [12,31], corrected diffusion approximations [24] and option pricing [6].

Define the overshoot R_a by

$$R_a = -S_{\tau_a} - a, \tag{3.11}$$

where $\tau(a)$ is the first passage time $\inf\{n \geq 1 : S_n < -a\}$. Hence, R_a is the excess of the random walk over the boundary $-a$ at the time it first downcrosses $-a$. Standard results from renewal theory say that R_a converges in distribution to a random variable R_∞ we refer to as the limiting overshoot (see [3], Thm. 2.1 on p. 224). For the expected limiting overshoot

$\mathbb{E}R_\infty = \lim_{a\to\infty} \mathbb{E}R_a =: \rho(\beta)$ it is known that $\rho(\beta) = -\mathbb{E}S_\tau^2/2\mathbb{E}S_\tau$ (see e.g. Woodroofe [31], corollary 2.2 on p. 20).

A relation between the moments of τ and the moments of M_β can be found in Asmussen [3], Thm. 2.2 on p. 270,

$$\sum_{k=0}^{n} \binom{n+1}{k} \mathbb{E}M_\beta^k \mathbb{E}X^{n+1-k} = \frac{\mathbb{E}S_\tau^{n+1}}{\mathbb{E}\tau}, \quad n = 1, 2, \ldots. \tag{3.12}$$

From (3.12) for $n = 1$ we get $\mathbb{E}S_\tau^2 = \mathbb{E}\tau(1 + \beta^2 - 2\beta\mathbb{E}M_\beta)$ which together with $\mathbb{E}S_\tau = -\beta\mathbb{E}\tau$ yields

$$\rho(\beta) = \frac{1+\beta^2}{2\beta} - \mathbb{E}M_\beta. \tag{3.13}$$

Combining (3.13) and (2.8) immediately leads to the result below.

Corollary 1. *There holds*

$$\rho(\beta) = -\frac{\zeta(1/2)}{\sqrt{2\pi}} + \frac{1}{4}\beta - \frac{\beta^2}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-1/2-r)}{r!(2r+1)(2r+2)} \left(\frac{-\beta^2}{2}\right)^r, \tag{3.14}$$

when $0 < \beta < 2\sqrt{\pi}$.

Corollary 1 complements results obtained earlier by several authors. Chernoff [12] showed that $\rho(0) = -\zeta(1/2)/\sqrt{2\pi}$, Siegmund [24], Problem 10.2 on p. 227, showed that $\rho'(0) = 1/4$ (see also Wijsman [29]), and Chang and Peres [10], p. 801, showed that $\rho''(0) = \zeta(3/2)/2(2\pi)^{3/2}$ (which equals $-\zeta(-1/2)/\sqrt{2\pi}$ by Riemann's relation, cf. (6.3)).

3.3. A discrete queue under Halfin–Whitt scaling

The proof of Theorem 1 consists of finding an analytic expression in terms of Lerch's transcendent, see (4.9) below, of the quantity

$$T_{k,i}(\beta) = \sum_{n=1}^{\infty} n^{k+1/2} \int_\beta^\infty e^{-\frac{1}{2}nx^2} x^i \,dx \tag{3.15}$$

with $i = 0, 1, \ldots$ and $k \in \mathbb{Z}$. This has an application in the analysis of a specific queue in heavy traffic. Consider the process

$$W_0 = 0; \qquad W_{m+1} = (W_m + A_m - s)^+, \quad m = 0, 1, \ldots, \tag{3.16}$$

in which $x^+ = \max\{0, x\}$ and the A_m are i.i.d. according to a random variable A having a Poisson distribution with mean λ (the arrival rate) and s (service capacity) is a positive integer larger than λ. Denote by W the random variable following the stationary distribution of the process defined in (3.16).

We then consider a heavy-traffic regime in which the arrival rate is just below the service capacity according to $s = \lambda + \beta\sqrt{\lambda}$, with $\beta > 0$ fixed and $\lambda \to \infty$. This regime has a long history in queueing theory (see Borst et al. [5] for an overview), and is referred to as the Halfin–Whitt regime or square-root safety staffing, see [5,17,20]. It is readily seen (see [19]) that the distribution of $W/\sqrt{\lambda}$ converges to that of M_β as $\lambda \to \infty$. The analysis of this equilibrium distribution for finite λ is, however, far more complicated than in the case of the Gaussian random walk. We show in [19] that

$$-\ln \mathbb{P}(W=0) = \sum_{n=1}^{\infty} \frac{p(ns)}{n^{1/2}} \frac{1}{\sqrt{2\pi}} \int_{\hat{\beta}}^{\infty} \mathrm{e}^{-\frac{1}{2}nx^2} \varphi(x/\sqrt{s}) \mathrm{d}x \tag{3.17}$$

in which

$$\hat{\beta} = \left(-2s\left(1 - \frac{\lambda}{s} + \ln \frac{\lambda}{s}\right)\right)^{1/2} \approx \frac{s-\lambda}{\sqrt{\lambda}} = \beta, \tag{3.18}$$

where $\approx$ is sharp for large values of λ. Furthermore, $p(n) = n^n \mathrm{e}^{-n}\sqrt{2\pi n}/n!$ and φ is a function analytic in $|x| < 2\sqrt{\pi}$ with $\varphi(0) = 1$. For p there is Stirling's formula, see Abramowitz and Stegun [2], 6.1.37 on p. 257,

$$p(n) \sim 1 - \frac{1}{12n} + \frac{1}{288n^2} + \cdots = \sum_{l=0}^{\infty} \frac{p_l}{n^l}, \quad n \to \infty, \tag{3.19}$$

and for φ there is the power series representation

$$\varphi(x) = 1 - \frac{2}{3}x + \frac{1}{12}x^2 + \cdots = \sum_{i=0}^{\infty} b_i x^i, \quad |x| < 2\sqrt{\pi}. \tag{3.20}$$

Thus for $-\ln \mathbb{P}(W=0)$ there is the asymptotics

$$-\ln \mathbb{P}(W=0) \sim \frac{1}{\sqrt{2\pi}} \sum_{l,i=0}^{\infty} p_l b_i s^{-l-i/2} T_{-l-1,i}(\hat{\beta}), \quad s \to \infty, \tag{3.21}$$

with T defined in (3.15).

Similar expressions, though somewhat more complicated than the one in (3.17), exist for $\mathbb{E}W$ and Var W and these give rise to asymptotic expansions as in (3.21) involving $T_{k,i}$ with $i = 0, 1, \ldots$; $k = 0, -1, -2 \ldots$ and $k = 1, 0, -1 \ldots$, respectively. Hence, for $0 < \beta < 2\sqrt{\pi}$, we can use the analytic expression for $T_{k,i}$ as found in Section 5 in the asymptotic formula (3.21) and its counterparts for $\mathbb{E}W$ and Var W.

4. Proof of Theorem 1

Starting from (2.4), we can express $J_k(\beta)$ as

$$\begin{aligned} J_k(\beta) &= \sum_{n=1}^{\infty} \frac{n^{k-1/2}}{\sqrt{2\pi}} \int_{\beta}^{\infty} (y-\beta)^k \mathrm{e}^{-\frac{1}{2}ny^2} \mathrm{d}y \\ &= \frac{1}{\sqrt{2\pi}} \sum_{i=0}^{k} \binom{k}{i} (-\beta)^{k-i} T_{k-1,i}(\beta), \end{aligned} \tag{4.1}$$

where

$$T_{k-1,i}(\beta) = \sum_{n=1}^{\infty} n^{k-1/2} \int_{\beta}^{\infty} \mathrm{e}^{-\frac{1}{2}nx^2} x^i \mathrm{d}x. \tag{4.2}$$

In Section 5 we shall conduct a detailed study of the quantities $T_{k-1,i}(\beta)$, leading to

$$T_{k-1,i}(\beta) = \frac{\Gamma(k+1/2)}{2k-i} 2^{k+1/2} \beta^{i-2k} + L_{k-1,i} - \sum_{r=0}^{\infty} \frac{\zeta(-k-r+1/2)(-1/2)^r \beta^{2r+i+1}}{r!(2r+i+1)} \tag{4.3}$$

for $i = 0, 1, \ldots, i \neq 2k$, and

$$T_{k-1,2k}(\beta) = -\Gamma\left(k+\frac{1}{2}\right) 2^{k+1/2} \ln\beta + L_{k-1,2k} - \sum_{r=0}^{\infty} \frac{\zeta(-k-r+1/2)(-1/2)^r \beta^{2r+2k+1}}{r!(2r+2k+1)}, \tag{4.4}$$

where

$$L_{k-1,i} = \frac{1}{2}\Gamma\left(\frac{i+1}{2}\right) 2^{\frac{i+1}{2}} \zeta\left(-k+\frac{1}{2}i+1\right), \quad i = 0, 1, \ldots, \ i \neq 2k, \tag{4.5}$$

$$L_{k-1,2k} = 2^{k+1/2}\Gamma(k+1/2)\left(\sum_{j=0}^{k-1} \frac{1}{2j+1} - \frac{1}{2}\ln 2\right). \tag{4.6}$$

For $k = 0$ we have directly from (4.4) and (4.6) that $J_0(\beta) = -\ln \mathbb{P}(M_\beta = 0) = \frac{1}{\sqrt{2\pi}} T_{-1,0}$. For $k = 1, 2, \ldots$ we observe that

$$J_k^{(k+1)}(\beta) = (-1)^{k+1} k! \sum_{n=1}^{\infty} \frac{n^{k-1/2}}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}n\beta^2}, \tag{4.7}$$

where we have differentiated the expression on the second line of (4.1) $k+1$ times with respect to β, using

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\left[\int_\beta^\infty f(y,\beta)\mathrm{d}y\right] = -f(\beta,\beta) + \int_\beta^\infty \frac{\partial f}{\partial \beta}(y,\beta)\mathrm{d}y. \tag{4.8}$$

The right-hand side of (4.7) can be expressed in terms of Lerch's transcendent Φ, defined as the analytic continuation of the series

$$\Phi(z,s,v) = \sum_{n=0}^{\infty} (v+n)^{-s} z^n, \tag{4.9}$$

which converges for any real number $v \neq 0, -1, -2, \ldots$ if z and s are any complex numbers with either $|z| < 1$, or $|z| = 1$ and $\mathrm{Re}(s) > 1$. Note that $\zeta(s) = \Phi(1, s, 1)$. Indeed,

$$J_k^{(k+1)}(\beta) = \frac{(-1)^{k+1} k!}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\beta^2} \Phi\left(z = \mathrm{e}^{-\frac{1}{2}\beta^2}, s = \frac{1}{2} - k, v = 1\right). \tag{4.10}$$

We then use the important result derived by Bateman [15], Section 1.11(8) (with $\zeta(s, v) := \Phi(1, s, v)$ the Hurwitz zeta function)

$$\Phi(z,s,v)=\frac{\Gamma(1-s)}{z^v}(\ln 1/z)^{s-1}+z^{-v}\sum_{r=0}^{\infty}\zeta(s-r,v)\frac{(\ln z)^r}{r!}, \tag{4.11}$$

which holds for $|\ln z|<2\pi$, $s\neq 1,2,3,\dots$, and $v\neq 0,-1,-2,\dots$, to obtain

$$J_k^{(k+1)}(\beta)=\frac{(-1)^{k+1}k!}{\sqrt{2\pi}}\left(\Gamma(k+1/2)\left(\frac{2}{\beta^2}\right)^{k+1/2}+\sum_{r=0}^{\infty}\zeta\left(-k-r+\frac{1}{2}\right)\frac{(-\frac{1}{2}\beta^2)^r}{r!}\right) \tag{4.12}$$

for $0<\beta<2\sqrt{\pi}$. Hence

$$\begin{aligned}&J_k^{(k+1)}(\beta)-\frac{(-1)^{k+1}k!}{\sqrt{2\pi}}\Gamma(k+1/2)\frac{2^{k+1/2}}{\beta^{2k+1}}\\&\quad=\frac{(-1)^{k+1}k!}{\sqrt{2\pi}}\sum_{r=0}^{\infty}\zeta\left(-k-r+\frac{1}{2}\right)\frac{(-\frac{1}{2}\beta^2)^r}{r!}.\end{aligned} \tag{4.13}$$

The right-hand side of (4.13) is a well-behaved function of β. We integrate identity (4.13) from 0 to β and use dominated convergence of the series on the right-hand side of (4.13) to interchange sum and integral, see (4.11). This results in

$$\begin{aligned}&J_k(\beta)-\frac{(-1)^{k+1}k!}{\sqrt{2\pi}}\Gamma(k+1/2)2^{k+1/2}\frac{\beta^{-k}}{-2k(-2k+1)\cdot\dots\cdot -k}\\&\quad=L_0+L_1\beta+\dots+L_k\beta^k+\frac{(-1)^{k+1}k!}{\sqrt{2\pi}}\\&\qquad\times\sum_{r=0}^{\infty}\frac{\zeta(-k-r+1/2)(-1/2)^r\beta^{2r+k+1}}{r!(2r+1)(2r+2)\cdot\dots\cdot(2r+k+1)},\end{aligned} \tag{4.14}$$

where $L_k, L_{k-1},\dots,L_0$ are integration constants that appear subsequently when integrating (4.13) $k+1$ times. We observe that

$$\frac{(-1)^{k+1}k!}{\sqrt{2\pi}}\frac{\Gamma(k+1/2)2^{k+1/2}}{-2k(-2k+1)\cdot\dots\cdot -k}=\frac{(k-1)!}{2^k}, \tag{4.15}$$

and we are left with determining the $L_0,\dots,L_k$. Thus from (4.1) and (4.2) (with $C_{z^j}[f(z)]$ denoting the coefficient of z^j in $f(z)$)

$$\begin{aligned}L_j&=\frac{1}{\sqrt{2\pi}}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}C_{\beta^j}\left[\beta^{k-i}T_{k-1,i}\right]=\frac{1}{\sqrt{2\pi}}\binom{k}{k-j}(-1)^jL_{k-1,k-j}\\&=\frac{1}{\sqrt{2\pi}}\binom{k}{j}(-1)^j\frac{1}{2}\Gamma\left(\frac{k-j+1}{2}\right)2^{\frac{k-j+1}{2}}\zeta\left(-\frac{1}{2}k-\frac{1}{2}j+1\right),\end{aligned} \tag{4.16}$$

for $j=0,1,\dots,k$, where it has been used that $\beta^{k-i}T_{k-1,i}$ has non-zero coefficients for the terms β^{-k}, β^{k-i} and β^{k+2r+1}, $r=0,1,\dots$, only. From (4.14)–(4.16) we then get Theorem 1.

An alternative proof of Theorem 1 starts from the last line expression in (4.1) for $J_k(\beta)$ and uses the full result (4.2) for $i=0,1,\dots,k$. Thus

$$J_k(\beta) = \frac{1}{\sqrt{2\pi}}\Gamma(k+1/2)2^{k+1/2}\beta^{-k}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{k-i}}{2k-i}$$
$$+\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\frac{\Gamma(\frac{i+1}{2})}{\sqrt{2\pi}}\zeta\left(-k+\frac{1}{2}i+1\right)2^{\frac{i-1}{2}}\beta^{k-i}$$
$$-\frac{1}{\sqrt{2\pi}}\sum_{r=0}^{\infty}\frac{\zeta(-k-r+1/2)(-1/2)^r}{r!}\beta^{2r+k+1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{k-i}}{2r+i+1}. \tag{4.17}$$

To complete this proof of Theorem 1 we only need to establish that

$$\frac{1}{\sqrt{2\pi}}\Gamma(k+1/2)2^{k+1/2}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{k-i}}{2k-i} = \frac{(k-1)!}{2^k}, \tag{4.18}$$

i.e., that

$$\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{k-i}}{2k-i} = \frac{(k-1)!k!}{(2k)!}, \tag{4.19}$$

and that

$$\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{k-i}}{2r+i+1} = \frac{-(-1)^{k+1}k!}{(2r+1)\cdot\cdots\cdot(2r+k+1)}, \qquad r = 0, 1, \ldots. \tag{4.20}$$

The identities (4.19) and (4.20) follow from the relation

$$\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{k-i}}{x-i} = \frac{k!}{x(x-1)\cdot\cdots\cdot(x-k)}, \tag{4.21}$$

by plugging in $x = 2k > k$ and $x = -2r - 1 < 0$, respectively. The identity in (4.21) is readily obtained by partial fraction expansion of the right-hand side.

5. Proof of the result for $T_{k,i}$

We shall conduct a study of the quantities

$$T_{k,i}(a) = \sum_{n=1}^{\infty} n^{k+1/2}\int_a^{\infty} \mathrm{e}^{-\frac{1}{2}nx^2}x^i\,\mathrm{d}x, \tag{5.1}$$

for integer k and $i = 0, 1, \ldots$ which is required (with $a = \beta$ and $k - 1$ instead of k) in (4.2). The main result is (5.4) and (5.5) with $L_{k,i}$ and $L_{k,2k+2}$ given in (5.53) and (5.54), respectively. We intend to use this result in a different setting (Halfin–Whitt scaling, see Section 3.3) and this is why we passed to a neutral variable a (instead of β) and shifted the integer k by one unit.

We have when $\frac{1}{2}a^2 < 2\pi$ by (4.11)

$$T'_{k,i}(a) = -\sum_{n=1}^{\infty} n^{k+1/2}\mathrm{e}^{-\frac{1}{2}na^2}a^i = -a^i\mathrm{e}^{-\frac{1}{2}a^2}\,\Phi\left(z = \mathrm{e}^{-\frac{1}{2}a^2}, s = -k - \frac{1}{2}, v = 1\right)$$
$$= -\Gamma(k+3/2)2^{k+3/2}a^{i-2k-3} - \sum_{r=0}^{\infty}\frac{\zeta(-k-r-\frac{1}{2})(-1/2)^r a^{2r+i}}{r!}. \tag{5.2}$$

Hence

$$T'_{k,i}(a) + \Gamma(k+3/2)2^{k+3/2}a^{i-2k-3} = -\sum_{r=0}^{\infty} \frac{\zeta(-k-r-\frac{1}{2})(-1/2)^r a^{2r+i}}{r!}, \tag{5.3}$$

where the right-hand side of (5.3) is well-behaved as $a \downarrow 0$. Therefore, upon integration from 0 to a,

$$T_{k,i}(a) + \frac{\Gamma(k+3/2)2^{k+3/2}}{i-2k-2}a^{i-2k-2} = L_{k,i} - \sum_{r=0}^{\infty} \frac{\zeta(-k-r-\frac{1}{2})(-1/2)^r a^{2r+i+1}}{r!(2r+i+1)} \tag{5.4}$$

when $i \neq 2k+2$, and

$$\begin{aligned} &T_{k,2k+2}(a) + \Gamma(k+3/2)2^{k+3/2}\ln a \\ &\quad = L_{k,2k+2} - \sum_{r=0}^{\infty} \frac{\zeta(-k-r-\frac{1}{2})(-1/2)^r a^{2r+2k+3}}{r!(2r+2k+3)}, \end{aligned} \tag{5.5}$$

where

$$L_{k,i} = \lim_{a\downarrow 0}\left[T_{k,i}(a) + \frac{\Gamma(k+3/2)2^{k+3/2}}{i-2k-2}a^{i-2k-2}\right], \quad i = 0, 1, \ldots, \; i \neq 2k+2, \tag{5.6}$$

and

$$L_{k,2k+2} = \lim_{a\downarrow 0}\left[T_{k,2k+2}(a) + \Gamma(k+3/2)2^{k+3/2}\ln a\right]. \tag{5.7}$$

Below we shall determine the $L_{k,i}$, and to that end we distinguish between the cases

I. $i > 2k+2$,
II. $i = 2k+2$,
III.a $i = 1, 3, \ldots, 2k+1$,
III.b $i = 0, 2, \ldots, 2k$.

Note that when $k \leq 0$, some or all the cases in II and III are degenerate since we restrict to $i = 0, 1, \ldots$. Furthermore, for Theorem 1 it is only necessary to consider $T_{k-1,i}$ with $k = 0, 1, \ldots$ and $i = 0, 1, \ldots, k$. However, this does not significantly ease the problem at hand, and for future work on the queueing model sketched in Section 3.3, it is necessary to solve the full problem.

Case I. We have for $i > 2k+2$ that

$$\begin{aligned} L_{k,i} &= \lim_{a\downarrow 0} T_{k,i}(a) = \sum_{n=1}^{\infty} n^{k+1/2}\int_0^{\infty} \mathrm{e}^{-\frac{1}{2}nx^2}x^i\,\mathrm{d}x \\ &= \frac{1}{2}\Gamma\left(\frac{i+1}{2}\right)2^{\frac{i+1}{2}}\sum_{n=1}^{\infty}\frac{1}{n^{i/2-k}} = \frac{1}{2}\Gamma\left(\frac{i+1}{2}\right)2^{\frac{i+1}{2}}\zeta\left(\frac{i}{2}-k\right), \end{aligned} \tag{5.8}$$

and this is a finite, positive number since $i > 2k+2$.

Case II. We assume that $2k+2 \geq 0$ and we determine $L_{k,2k+2}$. To that end we write $T_{k,2k+2}(a)$ as

$$\begin{aligned} T_{k,2k+2}(a) &= \sum_{n=1}^{\infty} n^{k+1/2}\int_a^{\infty} \mathrm{e}^{-\frac{1}{2}nx^2}x^{2k+2}\mathrm{d}x \\ &= 2^{k+3/2}\sum_{n=1}^{\infty}\frac{1}{n}\int_{\sqrt{n\delta}}^{\infty}\mathrm{e}^{-u^2}u^{2k+2}\mathrm{d}u, \end{aligned} \tag{5.9}$$

where $\delta = \frac{1}{2}a^2$. Now $\int_0^\infty e^{-u^2} u^{2k+2} du = \frac{1}{2}\Gamma(k+3/2)$, and this leads to writing the series on the second line of (5.9) as

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{\sqrt{n\delta}}^{\infty} e^{-u^2} u^{2k+2} du = \delta \sum_{n=1}^{\infty} \frac{1}{n\delta} \left(\int_{\sqrt{n\delta}}^{\infty} e^{-u^2} u^{2k+2} du - \frac{1}{2}\Gamma(k+3/2) e^{-n\delta} \right)$$
$$- \frac{1}{2}\Gamma(k+3/2) \ln(1 - e^{-\delta}). \tag{5.10}$$

The function

$$x > 0 \longmapsto \frac{1}{x} \left(\int_{\sqrt{x}}^{\infty} e^{-u^2} u^{2k+2} du - \frac{1}{2}\Gamma(k+3/2) e^{-x} \right) \tag{5.11}$$

decays exponentially as $x \to \infty$, is $\mathcal{O}(x^{k+1/2})$ as $x \downarrow 0$, and is smooth everywhere else on $(0, \infty)$. It is elementary to show that the first expression on the second line of (5.10) tends to

$$\int_0^{\infty} \frac{1}{x} \left(\int_{\sqrt{x}}^{\infty} e^{-u^2} u^{2k+2} du - \frac{1}{2}\Gamma(k+3/2) e^{-x} \right) dx \;=:\; I_k \tag{5.12}$$

as $\delta \downarrow 0$. Since $\ln(1 - e^{-\delta}) = \ln \delta + \mathcal{O}(\delta)$ as $\delta = \frac{1}{2}a^2 \downarrow 0$, we thus see that

$$T_{k,2k+2}(a) + 2^{k+1/2}\Gamma(k+3/2) \ln\left(\frac{1}{2}a^2\right) \longrightarrow 2^{k+3/2} I_k \tag{5.13}$$

as $a \downarrow 0$. We finally compute I_k by partial integration as

$$I_k = -\int_0^{\infty} \ln x \left(-\frac{1}{2} x^{-1/2} e^{-x} x^{k+1} + \frac{1}{2}\Gamma(k+3/2) e^{-x} \right) dx$$
$$= \frac{1}{2}\Gamma'(k+3/2) - \frac{1}{2}\Gamma(k+3/2)\Gamma'(1) = \Gamma(k+3/2) \left(-\ln 2 + \sum_{j=0}^{k} \frac{1}{2j+1} \right), \tag{5.14}$$

where Abramowitz and Stegun [2], 6.3.2 and 6.3.4 on p. 258, has been used. Therefore,

$$\begin{aligned} L_{k,2k+2} &= \lim_{a \downarrow 0} \left[T_{k,2k+2}(a) + 2^{k+3/2}\Gamma(k+3/2) \ln a \right] \\ &= \lim_{a \downarrow 0} \left[2^{k+3/2} I_k + 2^{k+1/2}\Gamma(k+3/2) \ln 2 \right] \\ &= 2^{k+3/2}\Gamma(k+3/2) \left(\sum_{j=0}^{k} \frac{1}{2j+1} - \frac{1}{2}\ln 2 \right). \end{aligned} \tag{5.15}$$

Case III.a. We assume $k \geq 0$, and we let $i = 2m+1$ with $m = 0, 1, \ldots, k$. We rewrite

$$\int_a^{\infty} e^{-\frac{1}{2}nx^2} x^{2m+1} dx = \frac{1}{2} \left(\frac{2}{n} \right)^{m+1} \int_{\frac{1}{2}na^2}^{\infty} e^{-v} v^m dv. \tag{5.16}$$

From Szegö [27], Section 1, we have

$$\int_y^{\infty} e^{-v} v^m dv = m! e^{-y} \sum_{r=0}^{m} \frac{y^r}{r!}. \tag{5.17}$$

Consequently,

$$T_{k,2m+1}(a) = \sum_{n=1}^{\infty} n^{k+1/2} \frac{1}{2} \left(\frac{2}{n}\right)^{m+1} m! \mathrm{e}^{-\frac{1}{2}na^2} \sum_{r=0}^{m} \frac{(\frac{1}{2}na^2)^r}{r!}$$
$$= m! 2^m \sum_{r=0}^{m} \frac{(\frac{1}{2}a^2)^r}{r!} \sum_{n=1}^{\infty} n^{k-m+r-1/2} \mathrm{e}^{-\frac{1}{2}na^2}. \tag{5.18}$$

Now, by Bateman's result (4.11), we have for $\frac{1}{2}a^2 < 2\pi$ that

$$\sum_{n=1}^{\infty} n^{k-m+r-1/2} \mathrm{e}^{-\frac{1}{2}na^2} = \mathrm{e}^{-\frac{1}{2}a^2} \Phi\left(z = \mathrm{e}^{-\frac{1}{2}a^2}, s = -k+m-r+\frac{1}{2}, v = 1\right)$$
$$= \Gamma(k-m+r+1/2) \left(\frac{1}{2}a^2\right)^{-k+m-r-1/2}$$
$$+ \sum_{r'=0}^{\infty} \frac{\zeta(-k+m-r+\frac{1}{2}-r')(-1/2)^{r'} a^{2r'}}{(r')!}$$
$$= \Gamma(k-m+r+1/2) 2^{k-m+r+1/2} a^{-2k+2m-2r-1}$$
$$+ \zeta\left(-k+m+\frac{1}{2}\right) + \mathcal{O}(a^2). \tag{5.19}$$

Hence,

$$T_{k,2m+1}(a) = m! 2^m \sum_{r=0}^{m} \frac{2^{-r} a^{2r}}{r!} \Gamma(k-m+r+1/2) 2^{k-m+r+1/2} a^{-2k+2m-2r-1}$$
$$+ m! 2^m \zeta\left(-k+m+\frac{1}{2}\right) + \mathcal{O}(a^2)$$
$$= 2^{k+1/2} a^{-2k+2m-1} \sum_{r=0}^{m} \frac{m!}{r!} \Gamma(k-m+r+1/2) + m! 2^m \zeta\left(-k+m+\frac{1}{2}\right)$$
$$+ \mathcal{O}(a^2). \tag{5.20}$$

Lemma 1.

$$\sum_{r=0}^{m} \frac{m!}{r!} \Gamma(k-m+r+1/2) = \frac{\Gamma(k+3/2)}{k-m+1/2}. \tag{5.21}$$

Proof. From (5.6) we have that

$$T_{k,2m+1}(a) + \frac{\Gamma(k+3/2)}{2m-2k+1} 2^{k+3/2} a^{2m-2k-1} \tag{5.22}$$

has a finite limit as $a \downarrow 0$. Then (5.20) immediately gives the result. Of course, a direct proof is also possible. Using $\Gamma(x+1) = x\Gamma(x)$ repeatedly, one rewrites the identity to be proved as

$$\sum_{r=0}^{m} \frac{m!}{r!}(x+r-1)(x+r-2)\cdot\cdots\cdot x = (x+m)(x+m-1)\cdot\cdots\cdot(x+1), \tag{5.23}$$

and this is readily proved by induction. □

From the lemma we have that

$$L_{k,2m+1} = m!2^m \zeta\left(-k+m+\frac{1}{2}\right) \tag{5.24}$$

as $L_{k,2m+1}$ is the limit of (5.22) as $a \downarrow 0$ (see the last line of (5.20)).

Case III.b. We assume $k \geq 0$ and we let $i = 2m$, $m = 0, 1, \ldots, k$. We have now with $\delta = \frac{1}{2}a^2$ and the substitution $u = x\sqrt{n/2}$ that

$$\begin{aligned} T_{k,2m}(a) &= \sum_{n=1}^{\infty} n^{k+1/2} \int_a^{\infty} \mathrm{e}^{-\frac{1}{2}nx^2} x^{2m} \mathrm{d}x \\ &= 2^{m+1/2} \sum_{n=1}^{\infty} n^{k-m} \int_{\sqrt{n\delta}}^{\infty} \mathrm{e}^{-u^2} u^{2m} \mathrm{d}u \\ &= 2^{m+1/2} \delta^{m-k} \sum_{n=1}^{\infty} (n\delta)^{k-m} \int_{\sqrt{n\delta}}^{\infty} \mathrm{e}^{-u^2} u^{2m} \mathrm{d}u. \end{aligned} \tag{5.25}$$

Set

$$f(x) = g(\delta x); \qquad g(y) = y^{k-m} \int_{\sqrt{y}}^{\infty} \mathrm{e}^{-u^2} u^{2m} \mathrm{d}u. \tag{5.26}$$

We apply the Euler–Maclaurin summation formula (see De Bruijn [7], Sec. 3.6, pp. 40–42)

$$\begin{aligned} \sum_{n=1}^{N} f(n) = \int_1^N f(x)\mathrm{d}x + \frac{1}{2}f(1) + \frac{1}{2}f(N) \\ + \sum_{j=1}^{p} \frac{B_{2j}}{(2j)!}(f^{(2j-1)}(N) - f^{(2j-1)}(1)) + R_{p,N}, \end{aligned} \tag{5.27}$$

in which

$$R_{p,N} = -\int_1^N f^{(2p)}(x) \frac{B_{2p}(x - \lfloor x \rfloor)}{(2p)!} \mathrm{d}x, \tag{5.28}$$

where the $B_n(t)$ denote the Bernoulli polynomials, defined by

$$\frac{z\mathrm{e}^{zt}}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(t)z^n}{n!}, \tag{5.29}$$

and the $B_n = B_n(0)$ denote the Bernoulli numbers. Since f and all its derivatives decay exponentially fast as $x \to \infty$, we can let $N \to \infty$ in (5.27), and we obtain

$$\sum_{n=1}^{\infty} f(n) = \int_1^{\infty} f(x)\mathrm{d}x + \frac{1}{2}f(1) - \sum_{j=1}^{p} \frac{B_{2j}}{(2j!)} f^{(2j-1)}(1) + R_p, \tag{5.30}$$

where $R_p = \lim_{N\to\infty} R_{p,N}$. In terms of g, see (5.25) and (5.26), we have

$$T_{k,2m}(a) = 2^{m+1/2}\delta^{m-k}\left\{\int_1^\infty g(\delta x)\mathrm{d}x + \frac{1}{2}g(\delta) - \sum_{j=1}^{p}\frac{B_{2j}}{(2j!)}\delta^{2j-1}g^{(2j-1)}(\delta) + R_p\right\}, \tag{5.31}$$

where $\delta = \frac{1}{2}a^2$ and

$$R_p = -\delta^{2p}\int_1^\infty g^{(2p)}(x)\frac{B_{2p}\,(x-\lfloor x\rfloor)}{(2p)!}\mathrm{d}x. \tag{5.32}$$

We use (5.31) with a p such that $\delta^{m-k}R_p \to 0$ as $\delta = \frac{1}{2}a^2 \downarrow 0$. It turns out that any $p \geq 1$ with $2p-1 > k-m$ achieves this goal. To see this we note that g decays exponentially as $y \to \infty$ and

$$\begin{aligned} g(y) &= y^{k-m}\left(\int_0^\infty \mathrm{e}^{-u^2}u^{2m}\mathrm{d}u - \int_0^{\sqrt{y}} \mathrm{e}^{-u^2}u^{2m}\mathrm{d}u\right) \\ &= \frac{1}{2}\Gamma(m+1/2)y^{k-m} - \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{y^{k+j+1/2}}{2m+2j+1}, \quad y > 0. \end{aligned} \tag{5.33}$$

As a consequence of (5.33) we have that

$$g^{(2p)}(y) = \mathcal{O}(y^{k+1/2-2p}), \quad y \downarrow 0, \tag{5.34}$$

since $k-m < 2p-1 < 2p$. Therefore

$$\begin{aligned} R_p &= -\delta^{2p-1}\left(\int_\delta^1 + \int_1^\infty\right) g^{(2p)}(y)\frac{B_{2p}\,(y/\delta - \lfloor y/\delta\rfloor)}{(2p)!}\mathrm{d}y \\ &= \mathcal{O}\left(\delta^{2p-1}\int_\delta^1 y^{k+1/2-2p}\mathrm{d}y\right) + \mathcal{O}(\delta^{2p-1}) = \mathcal{O}(\delta^q) + \mathcal{O}(\delta^{2p-1}), \end{aligned} \tag{5.35}$$

where $q = 2p-1$ when $k+\frac{1}{2}-2p > -1$ and $q = k+\frac{1}{2}$ when $k+\frac{1}{2}-2p < -1$. In either case we have $q \geq k-m+\frac{1}{2}$, whence

$$\delta^{m-k}R_p = \mathcal{O}(\delta^{1/2}), \quad \delta \downarrow 0, \tag{5.36}$$

and our goal, to show that $\delta^{m-k}R_p \to 0$ as $\delta \downarrow 0$, has been achieved.

We next consider the terms

$$\int_1^\infty g(\delta x)\mathrm{d}x, \qquad \frac{1}{2}g(\delta), \quad \text{and} \quad \delta^{2j-1}g^{(2j-1)}(\delta), \quad j = 1,\ldots,p, \tag{5.37}$$

that occur on the right-hand side of (5.31) with $p \geq 1$ such that $2p-1 > k-m$. We have

$$\int_1^\infty g(\delta x)\mathrm{d}x = \frac{1}{\delta}\int_0^\infty g(y)\mathrm{d}y - \frac{1}{\delta}\int_0^\delta g(y)\mathrm{d}y. \tag{5.38}$$

Now from (5.33)

$$\frac{1}{\delta}\int_0^{\delta} g(y)\mathrm{d}y = \frac{1}{2}\Gamma(m+1/2)\frac{\delta^{k-m}}{k-m+1} + \mathcal{O}(\delta^{k+1/2}). \tag{5.39}$$

Also, by partial integration,

$$\begin{aligned}\int_0^{\infty} g(y)\mathrm{d}y &= \int_0^{\infty} y^{k-m}\left(\int_{\sqrt{y}}^{\infty} \mathrm{e}^{-u^2}u^{2m}\mathrm{d}u\right)\mathrm{d}y \\ &= \frac{1}{2(k-m+1)}\int_0^{\infty} y^{k+1/2}\mathrm{e}^{-y}\mathrm{d}y = \frac{\Gamma(k+3/2)}{2(k-m+1)}.\end{aligned} \tag{5.40}$$

Thus

$$\int_1^{\infty} g(\delta x)\mathrm{d}x = \frac{\Gamma(k+3/2)\delta^{-1}}{2(k-m+1)} - \frac{\Gamma(m+1/2)}{2(k-m+1)}\delta^{k-m} + \mathcal{O}(\delta^{k+1/2}). \tag{5.41}$$

Next we have from (5.33)

$$\frac{1}{2}g(\delta) = \frac{1}{4}\Gamma(m+1/2)\delta^{k-m} + \mathcal{O}(\delta^{k+1/2}). \tag{5.42}$$

Finally, from (5.33) for $j = 1, \ldots, p$

$$\delta^{2j-1}g^{(2j-1)}(\delta) = \frac{1}{2}\Gamma(m+1/2)\frac{(k-m)!\delta^{k-m}}{(k-m-2j+1)!} + \mathcal{O}(\delta^{k+1/2}). \tag{5.43}$$

Note that the first quantity on the right-hand side of (5.43) should be read as 0 when $2j-2 \geq k-m$.

Combining (5.36) and (5.41)–(5.43) we see from (5.31) that

$$\begin{aligned}T_{k,2m}(a) = 2^{m+1/2}\delta^{m-k}\Bigg\{&\frac{\Gamma(k+3/2)\delta^{-1}}{2(k-m+1)} - \frac{\Gamma(m+1/2)}{2(k-m+1)}\delta^{k-m} + \mathcal{O}(\delta^{k+1/2}) \\ &+ \frac{1}{4}\Gamma(m+1/2)\delta^{k-m} + \mathcal{O}(\delta^{k+1/2}) \\ &- \frac{1}{2}\Gamma(m+1/2)\sum_{j=1}^{p}\frac{B_{2j}}{(2j)!}\frac{(k-m)!\delta^{k-m}}{(k-m-2j+1)!}\Bigg\} + \mathcal{O}(\delta^{1/2}).\end{aligned} \tag{5.44}$$

That is,

$$T_{k,2m}(a) = 2^{m+1/2}\frac{\Gamma(k+3/2)\delta^{m-k-1}}{2(k-m+1)} - \frac{1}{2}\Gamma(m+1/2)2^{m+1/2}C_{k-m,p} + \mathcal{O}(\delta^{1/2}), \tag{5.45}$$

where for $n = 0, 1, \ldots,$

$$C_{n,p} = \frac{-1}{n+1} + \frac{1}{2} - \sum_{j=1}^{p}\frac{B_{2j}}{(2j)!}\frac{n!}{(n-2j+1)!}. \tag{5.46}$$

Lemma 2. *There holds*

$$C_{n,p} = \zeta(-n), \quad \textit{for } n = 0, 1, \ldots, \; n \leq 2p-2. \tag{5.47}$$

Proof. First consider the case that $n = 0$. Then $n!/(n-2j+1)! = 0$ by convention, and the $\sum_{j=1}^{p}$ on the right-hand side of (5.46) vanishes altogether. From $\zeta(0) = -1/2$, see Abramowitz and Stegun [2], 23.2.11 on p. 807, we conclude that (5.47) holds for $n = 0$.

When $n = 1, 2, \ldots$ we have that all terms j with $2j - 1 > n$ in the series on the right-hand side of (5.46) vanish. Also, $B_{2j+1} = 0$ for $j = 1, 2, \ldots$. Therefore, since $n \le 2p - 2$,

$$\sum_{j=1}^{p} \frac{B_{2j}}{(2j)!} \frac{n!}{(n-2j+1)!} = \sum_{i=2}^{n+1} \frac{B_i}{i!} \frac{n!}{(n+1-i)!}$$
$$= \frac{-B_0}{n+1} - B_1 + \frac{1}{n+1} \sum_{i=0}^{n+1} \binom{n+1}{i} B_i. \tag{5.48}$$

By Abramowitz and Stegun [2], 23.1.3 and 23.1.7 on p. 804 ($x = h = 0$), we have

$$B_0 = 1, \qquad B_1 = -\frac{1}{2}; \qquad \sum_{i=0}^{n+1} \binom{n+1}{i} B_i = B_{n+1}, \quad n = 0, 1, \ldots. \tag{5.49}$$

Furthermore, by Abramowitz and Stegun [2], 23.2.14-15 on p. 807, we have

$$\frac{B_{n+1}}{n+1} = -\zeta(-n), \quad n = 1, 2, \ldots. \tag{5.50}$$

The result then follows for $n = 1, 2, \ldots$ by inserting (5.49) and (5.50) into (5.48). $\square$

Restoring the variable a via $\delta = \frac{1}{2}a^2$, we see from (5.45) and the lemma that for $m = 0, 1, \ldots, k$

$$T_{k,2m}(a) = \frac{\Gamma(k+3/2)}{2k-2m+2} 2^{k+3/2} a^{2m-2k-2} + \frac{1}{2}\Gamma(m+1/2) 2^{m+1/2} \zeta(-k+m) + \mathcal{O}(a). \tag{5.51}$$

Hence, we have for $m = 0, 1, \ldots, k$

$$L_{k,2m} = \lim_{a \downarrow 0} \left[T_{k,2m}(a) + \frac{\Gamma(k+3/2)}{2m-2k-2} 2^{k+3/2} a^{2m-2k-2} \right]$$
$$= \frac{1}{2}\Gamma(m+1/2) 2^{m+1/2} \zeta(-k+m). \tag{5.52}$$

Combining (5.8), (5.24) and (5.52) we have for integer $i \ge 0$

$$L_{k,i} = \frac{1}{2}\Gamma\left(\frac{i+1}{2}\right) 2^{\frac{i+1}{2}} \zeta\left(-k + \frac{1}{2}i\right), \quad i \ne 2k+2, \tag{5.53}$$

while from (5.15) we have

$$L_{k,2k+2} = 2^{k+3/2}\Gamma(k+3/2) \left(\sum_{j=0}^{k} \frac{1}{2j+1} - \frac{1}{2}\ln 2 \right). \tag{5.54}$$

For $k < 0$, the right-hand side of (5.54) equals $-2^{k+1/2}\Gamma(k+3/2)\ln 2$, and the case that $k = -1$ yields $-\frac{1}{2}\sqrt{2\pi}\ln 2$, as should.

Comment on the proof. Despite the fact that the validity range of (5.53) contains $i = 0, 1, \ldots, 2k+1$, we have not been able to find an argument that works both for odd and for even such i. Clearly, one cannot use the argument of III.a for even i, the formula (5.17) being crucial. The argument of III.b cannot be used for odd i since in that case the g that would appear in (5.26) has leading order behaviour $\frac{1}{2}m!y^{k-m-1/2}$, and no high-order $2p$ derivative of this latter function vanishes (as was the case in (5.33) for $i = 2m$, even).

6. Alternative series expressions for the Bateman series

In this section we shall prove Theorem 2. Consider for $k = 0$ the series

$$Q_0(\beta) = \frac{-1}{\sqrt{2\pi}} \sum_{r=1}^{\infty} \frac{\zeta(-r+1/2)(-1/2)^r \beta^{2r+1}}{r!(2r+1)} \tag{6.1}$$

and for $k = 1, 2, \ldots$ the series

$$Q_k(\beta) = \frac{(-1)^{k+1}k!}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \frac{\zeta(-k-r+1/2)(-1/2)^r \beta^{2r+k+1}}{r!(2r+1)\cdots(2r+k+1)} \tag{6.2}$$

that occur in the expression (2.5) and (2.6) for $J_0(\beta)$ and $J_k(\beta)$, $k = 1, 2, \ldots$, respectively. Note that $Q_k(\beta)$ in (6.2) with $k = 0$ differs from $Q_0(\beta)$ in (6.1) only in that in the latter the term with $r = 0$ is excluded. The series in (6.1) and (6.2) converge for $|\beta| < 2\sqrt{\pi}$ while it is clear from (2.4) that $J_k(\beta)$ makes sense for all $\beta > 0$. In this section we present alternative series expansions for $J_k(\beta)$ that can be evaluated for all $\beta > 0$; these alternative expressions are intimately related to Lerch's transformation formula, Bateman [15], 1.11(7) on p. 29.

Using Riemann's relation, see Whittaker and Watson [30], Section 13.151 on p. 269,

$$\zeta(1-s) = \frac{2}{(2\pi)^s} \Gamma(s)\zeta(s)\cos\pi s \tag{6.3}$$

with $s = k + r + 1/2$, for $k, r = 0, 1, \ldots$, we have

$$\zeta(-k-r+1/2) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\Gamma(k+r+1/2)\zeta(k+r+1/2)}{(2\pi)^{k+r}} \cos\frac{1}{2}\pi(k+r+1/2). \tag{6.4}$$

Therefore, for $k = 0, 1, \ldots$ from (6.4) and using that

$$\cos\frac{1}{2}\pi(k+r+1/2) = \mathrm{Re}[\mathrm{e}^{\frac{\pi\mathrm{i}}{4}} i^k i^r], \tag{6.5}$$

we get

$$Q_k(\beta) = 2\left(\frac{-\beta}{2\pi}\right)^{k+1} \mathrm{Re}\left[\mathrm{e}^{\frac{\pi\mathrm{i}}{4}} i^k S_k\left(\frac{\beta^2}{4\pi\mathrm{i}}\right)\right], \tag{6.6}$$

in which

$$S_k(b) = \sum_r \frac{\Gamma(k+r+1/2)\zeta(k+r+1/2)}{r!(2r+1)\cdots(2r+k+1)} b^r, \quad |b| < 1. \tag{6.7}$$

The summations over r in (6.7) are from 1 to ∞ and from 0 to ∞ for the cases $k = 0$ and $k = 1, 2, \ldots$, respectively.

We shall express $S_k(b)$ in a form that can be used to evaluate $Q_k(\beta)$ for all $\beta > 0$. We start by using

$$\zeta(k+r+1/2) = \sum_{n=1}^{\infty} n^{-k-r-1/2}, \tag{6.8}$$

where we need that $k + r \geq 1$ (explaining why for $Q_0(\beta)$ the term with $r = 0$ was deleted). This yields

$$S_k(b) = \sum_{n=1}^{\infty} \frac{1}{n^{k+1/2}} T_k(b/n), \tag{6.9}$$

in which

$$T_k(t) = \sum_{r} \frac{\Gamma(k+r+1/2)t^r}{r!(2r+1)\cdots(2r+k+1)}, \quad |t| < 1, \tag{6.10}$$

with the same convention for the summation over r as before. Let U_n denote the Chebyshev polynomial of the second kind and degree $n = 0, 1, \ldots,$ see Abramowitz and Stegun [2], item 22.3.7 in Table 22.3 on p. 775. We define an R-operation for a function $f(t)$ having a Laurent series $\sum_{j=-\infty}^{\infty} c_j t^j$ in $0 < |t| < 1$ by

$$\mathrm{R}[f(t)] = \sum_{j=0}^{\infty} c_j t^j = f(t) - \sum_{j=-\infty}^{-1} c_j t^j. \tag{6.11}$$

Proposition 1. (i) *We have for* $|t| < 1$

$$T_0(t) = \sqrt{\pi}\left(\frac{\arcsin\sqrt{t}}{\sqrt{t}} - 1\right). \tag{6.12}$$

(ii) *We have for* $k = 1, 2, \ldots, |t| < 1$

$$T_k(t) = \frac{-\sqrt{\pi}}{k2^k}\mathrm{R}\left[\frac{(1-t)^{1/2}U_{k-1}(\sqrt{t})}{(t^{1/2})^{k+1}}\right]. \tag{6.13}$$

Proof. (i) By analyticity it is sufficient to consider $t = x^2$ with $0 \leq x < 1$. There holds

$$\begin{aligned} T_0(x^2) &= \frac{1}{x}\sum_{r=1}^{\infty} \frac{\Gamma(r+1/2)x^{2r+1}}{r!(2r+1)} = \frac{\sqrt{\pi}}{x}\int_0^x \sum_{r=1}^{\infty} \binom{-1/2}{r}(-y^2)^r \mathrm{d}y \\ &= \frac{\sqrt{\pi}}{x}\int_0^x \left(\frac{1}{\sqrt{1-y^2}} - 1\right)\mathrm{d}y = \sqrt{\pi}\left(\frac{\arcsin x}{x} - 1\right). \end{aligned} \tag{6.14}$$

(ii) We first write

$$\psi_k(t) := \frac{-\sqrt{\pi}}{k2^k}\frac{(1-t)^{1/2}U_{k-1}(\sqrt{t})}{(t^{1/2})^{k+1}} = \sum_{j=-\infty}^{\infty} c_j t^j, \tag{6.15}$$

where we note that U_{k-1} is odd when $k-1$ is odd and even when $k-1$ is even, so that $\psi_k(t)$ in (6.15) has indeed a Laurent expansion in powers of t. Writing again $t = x^2$ and denoting with

$C_{x^n}[f(x)]$ the coefficient of x^n in $f(x)$, we have that (6.13) is equivalent with

$$\frac{-\sqrt{\pi}}{k2^k}C_{x^{2j+k+1}}[(1-x^2)^{1/2}U_{k-1}(x)] = \frac{\Gamma(k+j+1/2)}{j!(2j+1)\cdots(2j+k+1)}, \quad j=0,1,\ldots. \tag{6.16}$$

We first verify (6.16) for $k=1,2$. Since $U_0(x)=1$, we must show that

$$-\frac{1}{2}\sqrt{\pi}C_{x^{2j+1}}[(1-x^2)^{1/2}] = \frac{\Gamma(j+3/2)}{j!(2j+1)(2j+2)}, \quad j=0,1,\ldots. \tag{6.17}$$

The left-hand side of (6.17) equals $-\frac{1}{2}\sqrt{\pi}(-1)^{j+1}\binom{1/2}{j+1}$ while the right-hand side equals

$$\begin{aligned}\frac{1}{4}\frac{\Gamma(j+1/2)}{(j+1)!} &= \frac{1}{4}\frac{(j-1/2)(j-3/2)\cdot\cdots\cdot -1/2\cdot\Gamma(-1/2)}{(j+1)!}\\ &= \frac{1}{4}(-1)^{j+1}\binom{1/2}{j+1}\cdot -2\sqrt{\pi}.\end{aligned} \tag{6.18}$$

Next $U_1(x)=2x$, whence we should verify whether

$$-\frac{1}{4}\sqrt{\pi}C_{x^{2j+3}}[(1-x^2)^{1/2}x] = \frac{\Gamma(j+5/2)}{j!(2j+1)(2j+2)(2j+3)}, \quad j=0,1,\ldots. \tag{6.19}$$

The left-hand side of (6.19) equals $-\frac{1}{4}\sqrt{\pi}(-1)^{j+1}\binom{1/2}{j+1}$, while the right-hand side equals $\frac{1}{8}\Gamma(j+1/2)/(j+1)!$, and this gives (6.19) from (6.18).

We now assume that we have established (6.16) for $k=1,2,\ldots,n+1$ (for $n=1,2,\ldots$). Using that $U_{n+1}(x)=2xU_n(x)-U_{n-1}(x)$, see Abramowitz and Stegun [2], item 22.7.5 in Table 22.7 on p. 782, we write the left-hand side of (6.16) for $k=n+2$ as

$$\begin{aligned}&\frac{-\sqrt{\pi}}{(n+2)2^{n+2}}C_{x^{2j+n+3}}[(1-x^2)^{1/2}U_{n+1}(x)]\\ &\quad= \frac{n+1}{n+2}\frac{\Gamma(n+j+3/2)}{j!(2j+1)\cdots(2j+n+2)} - \frac{n}{4(n+2)}\frac{\Gamma(n+j+3/2)}{(j+1)!(2j+3)\cdots(2j+n+3)}.\end{aligned} \tag{6.20}$$

Here validity of (6.16) for $k=n+1$ and for $k=n$ (with $j+1$ instead of j) has been used. Some standard manipulations show that the right-hand side of (6.20) equals

$$\frac{\Gamma(n+j+5/2)}{j!(2j+1)(2j+2)\cdots(2j+n+3)}. \tag{6.21}$$

This establishes (6.16) for $k=n+2$, and the proof is complete. □

Proposition 2. *For the function ψ_k defined in (6.15) we find that*

$$\mathrm{R}[\psi_k(t)] = \psi_k(t) - \frac{1}{2^{k+1}}\sum_{r=0}^{\lfloor\frac{k-1}{2}\rfloor}\frac{\Gamma(k-\frac{1}{2}-r)\Gamma(-\frac{1}{2}-r)}{\Gamma(\frac{1}{2}k-r)\Gamma(\frac{1}{2}k+\frac{1}{2}-r)}t^{-1-r}. \tag{6.22}$$

Proof. Following the proof of Proposition 1(ii) we can show that (6.16) also holds for $j = -1, -2, \ldots, -\lfloor \frac{k+1}{2} \rfloor$. For this the denominator $j!(2j+1)\cdots(2j+k+1)$ of the right-hand side of (6.16) is interpreted as $2^{k+1}(j + \lfloor \frac{k+1}{2} \rfloor)!(j+\frac{1}{2})(j+\frac{3}{2})\cdots(j+\lfloor \frac{k}{2} \rfloor + \frac{1}{2})$, and then the result follows upon some administration with Γ-functions. An alternative proof for both Propositions 1 and 2 follows from expressing $T_k(t)$ in terms of hypergeometric functions, and using [2], items 15.1.15 and 15.1.17 on p. 556, for the cases of even and odd k, respectively. □

Combining (2.7), (6.1), (6.6) and (6.9) and Proposition 1(i) yields the result for $\mathbb{P}(M_\beta = 0)$ in Theorem 2. By the explicit regularization in Proposition 2, similar results can be obtained for $S_k(b)$, $k = 1, 2, \ldots$, leading to expressions for the cumulants $J_k(\beta)$ through (6.6), (6.2) and (2.6) that are valid for all values of $\beta > 0$. We get, for instance,

$$S_1(b) = \frac{\sqrt{\pi}}{2b} \sum_{n=1}^{\infty} \frac{1}{n} \left(\sqrt{n} - \sqrt{n-b} \right), \tag{6.23}$$

$$S_2(b) = \frac{\sqrt{\pi}}{4b} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sqrt{n} - \sqrt{n-b} \right), \tag{6.24}$$

$$S_3(b) = \frac{-\sqrt{\pi}}{24b^2} \sum_{n=1}^{\infty} \frac{1}{n^3} \left(\sqrt{n} \left(n - \frac{9}{2} b \right) + \sqrt{n-b}(4b - n) \right). \tag{6.25}$$

We note that the series in (2.13) and (6.23)–(6.25) have terms that are analytic in a set that allows evaluation of $S_k(b)$, $k = 0, 1, 2, 3$, for all points $b = \beta^{2/4}\pi\mathrm{i}$ with $\beta > 0$. The series converge for these values of b, although convergence is slow, especially for the series (2.13) and (6.23). Nevertheless, the series can be evaluated conveniently by using a dedicated form of Euler–Maclaurin summation.

An alternative to using Euler–Maclaurin summation techniques is as follows. We can do the developments of this section equally well with series $Q_{kR}(\beta)$ where the subscript R refers to the fact that the terms with index $r \le R$ have been omitted in the series (6.1) and (6.2). This gives rise to functions $T_{kR}(t)$, by correspondingly deleting terms in the series (6.10), and functions $S_{kR}(b)$ associated with T_{kR} as in (6.9). Now these $T_{kR}(t)$ are $\mathcal{O}(t^{R+1})$ as $t \to 0$, whence the terms in the series (6.9) for S_{kR} are $\mathcal{O}(n^{-k-R-3/2})$, $n \to \infty$. Thus, by moving an appropriate number of terms from the Bateman series to the polynomial part of the representation (2.6) for $J_k(\beta)$ we achieve that the remaining infinite series can still be evaluated for all $\beta > 0$ in the form of an infinite series with explicitly given terms having any desired decay rate.

7. Proofs of the bounds on $\mathbb{P}(M_\beta = 0)$, $\mathbb{E}M_\beta$ and Var M_β

In this section we present the proof of Theorem 3 on bounds for $\mathbb{P}(M_\beta = 0)$, $\mathbb{E}M_\beta$ and $\mathrm{Var}\, M_\beta$. It turns out that these three quantities can be expressed in terms of a simple analytic expression together with a series of the form

$$\delta \sum_{n=1}^{\infty} f(\delta n) =: I_f(\delta), \qquad \delta > 0, \tag{7.1}$$

with $\delta = \frac{1}{2}\beta^2$ typically small and f a rapidly decaying, positive, decreasing, convex function on $(0, \infty)$ for which $\int_0^\infty f(x)\mathrm{d}x =: I_f$ is finite (f does not need to be bounded at $x = 0$). For such functions f there is the following result.

Proposition 3. *There holds, for f as above,*

$$I_f + \frac{1}{2}\delta f(\delta) - \int_0^{\delta} f(x)\mathrm{d}x \le I_f(\delta) \le I_f - \int_0^{\delta/2} f(x)\mathrm{d}x, \quad \delta > 0. \tag{7.2}$$

Furthermore, $I_f(\delta)$ decreases from $I_f = \int_0^\infty f(x)\mathrm{d}x$ to 0 as δ increases from 0 to ∞.

The proof of the inequality (7.2) uses basic facts from advanced calculus. The monotonicity is established by basic facts as well, but is not entirely trivial. Monotonicity can be checked easily for the special case where f is of the form

$$f_t(x) = (1 - x/t)^+, \quad x \ge 0, \tag{7.3}$$

for some $t > 0$. Then writing a general f as

$$f(x) = \int_0^\infty t f''(t) f_t(x)\mathrm{d}t, \quad x > 0, \tag{7.4}$$

we get monotonicity of $I_f(\delta)$ for general f as above. In (7.4) we assume that f is twice differentiable on $[x, \infty)$ with $tf''(t)$ and $f'(t)$ absolutely integrable on $[x, \infty)$. We may note here that the monotonicity result fails to hold when the definition of $I_f(\delta)$ is changed into $\delta \sum_{n=1}^{\infty} f(\delta(n - \alpha))$ with α positive but arbitrarily small.

Note that the difference between the far right-hand side and the far left-hand side of (7.2) equals $\int_{\delta/2}^{\delta}(f(x) - f(\delta))\mathrm{d}x$ and that this number can be bounded by $\frac{1}{4}\delta(f(\delta/2) - f(\delta))$.

The bounds in (7.2) on $I_f(\delta)$ are in terms of the "global" quantity $I_f = \int_0^\infty f(x)\mathrm{d}x$ and the "local" quantities δ, $f(\delta)$, $\int_0^a f(x)\mathrm{d}x$ with $a = \delta/2$. In the cases at hand we are able to evaluate the global quantity, and to estimate and bound the local quantities. We shall now present the details for the three cases.

7.1. *Details for $\mathbb{P}(M_\beta = 0)$*

We have by Spitzer's identity (Thm. 3.1 in [26]) that

$$-\ln \mathbb{P}(M_\beta = 0) = \delta \sum_{n=1}^{\infty} \frac{1}{n\delta} \frac{1}{\sqrt{\pi}} \int_{\sqrt{\delta n}}^{\infty} \mathrm{e}^{-u^2}\mathrm{d}u, \quad \delta = \frac{1}{2}\beta^2. \tag{7.5}$$

The right-hand side of (7.5) is of the form $I_f(\delta)$, but it tends to ∞ when $\delta \downarrow 0$. In order to apply the above approach, we write (7.5) as

$$-\ln \mathbb{P}(M_\beta = 0) = -\delta \sum_{n=1}^{\infty} f(\delta n) - \frac{1}{2} \ln\left(1 - \mathrm{e}^{-\delta}\right), \tag{7.6}$$

where we have set

$$f(x) = \frac{\mathrm{e}^{-x}}{2x} - \frac{1}{x\sqrt{\pi}} \int_{\sqrt{x}}^{\infty} \mathrm{e}^{-u^2}\mathrm{d}u, \quad x > 0, \tag{7.7}$$

and where we have used that $-\ln(1 - \mathrm{e}^{-\delta}) = \sum_{n=1}^{\infty} n^{-1}\mathrm{e}^{-n\delta}$. Note that f is rapidly decaying and that $f(x) = \mathcal{O}(1/\sqrt{x})$, $x \downarrow 0$, whence $I_f = \int_0^\infty f(x)\mathrm{d}x$ is finite.

Proposition 4. *The f defined in (7.7) is positive, decreasing and convex on $(0, \infty)$.*

Proof. We use the inequality, see Abramowitz and Stegun [2], 7.1.13 on p. 298,

$$e^{y^2}\int_y^\infty e^{-u^2}du < \frac{1}{y+\sqrt{y^2+4/\pi}}, \quad y>0. \tag{7.8}$$

From (7.8), with $y=\sqrt{x}$, the positivity of the f easily follows. Next, we compute for $x>0$

$$f'(x) = \frac{-e^{-x}}{2x^2}\left(1+x-\frac{1}{\sqrt{\pi}}x^{1/2}\right)+\frac{1}{x^2\sqrt{\pi}}\int_{\sqrt{x}}^\infty e^{-u^2}du, \tag{7.9}$$

$$f''(x) = \frac{e^{-x}}{2x^3}\left(2+2x+x^2-\frac{3}{2\sqrt{\pi}}x^{1/2}-\frac{3}{2\sqrt{\pi}}x^{3/2}\right)-\frac{2}{x^3\sqrt{\pi}}\int_{\sqrt{x}}^\infty e^{-u^2}du. \tag{7.10}$$

Thus for $y>0$

$$f'(y^2)<0 \Leftrightarrow e^{y^2}\int_y^\infty e^{-u^2}du < \frac{1}{2}\sqrt{\pi}\left(1+y^2-\frac{1}{\sqrt{\pi}}y\right), \tag{7.11}$$

$$f''(y^2)>0 \Leftrightarrow e^{y^2}\int_y^\infty e^{-u^2}du < \frac{1}{2}\sqrt{\pi}\left(1+y^2+\frac{1}{2}y^4-\frac{3}{4\sqrt{\pi}}y-\frac{3}{4\sqrt{\pi}}y^3\right). \tag{7.12}$$

Both inequalities in the right-hand side statements in (7.11) and (7.12) follow easily from (7.8), and the proof is complete. □

Proposition 5. *We have that* $I_f=\int_0^\infty f(x)dx=\ln 2$.

Proof. See [18], (3.11). □

To bound $I_f(\delta)$ according to (7.2) we need to approximate $f(x)$ and $\int_0^a f(x)dx$. To that end we note that for $x>0$

$$f(x)=\frac{1}{\sqrt{\pi x}}-\frac{1}{2}+E(x); \quad E(x)=\frac{1}{x\sqrt{\pi}}\int_0^{\sqrt{x}}\left(1-e^{-u^2}\right)du-\frac{e^{-x}-(1-x)}{2x}, \tag{7.13}$$

and that by Taylor expansion

$$E(x)=\sum_{l=2}^\infty \frac{(-1)^l x^{l-3/2}}{(l-1)!}\left[\frac{1}{(2l-1)\sqrt{\pi}}-\frac{x^{1/2}}{2l}\right], \quad x>0. \tag{7.14}$$

It is easily seen that the terms in the latter series have alternating signs and decrease in modulus to 0 when $0\le x\le 1/\pi$. Hence, retaining only 0 or 1 term in this series, we get for $x,a\in[0,1/\pi]$

$$0\le E(x)\le \frac{x^{1/2}}{3\sqrt{\pi}}-\frac{1}{4}x, \tag{7.15}$$

and

$$0\le \int_0^a E(x)dx \le \frac{2a^{3/2}}{9\sqrt{\pi}}-\frac{1}{8}a^2. \tag{7.16}$$

Using (7.15) and (7.16) in (7.13), we get from (7.6) with Propositions 3–5 the inequalities

$$\ln \mathbb{P}(M_\beta = 0) \le \frac{1}{2}\ln(1 - \mathrm{e}^{-\delta}) + \ln 2 - \left(\frac{2\delta}{\pi}\right)^{1/2} + \frac{1}{4}\delta, \tag{7.17}$$

$$\ln \mathbb{P}(M_\beta = 0) \ge \frac{1}{2}\ln(1 - \mathrm{e}^{-\delta}) + \ln 2 - \left(\frac{9\delta}{4\pi}\right)^{1/2} + \frac{1}{4}\delta - \frac{2\delta^{3/2}}{9\sqrt{\pi}} + \frac{1}{8}\delta^2, \tag{7.18}$$

for $0 < \delta \le 1/\pi$. From (7.17) and (7.18) we get Theorem 3(i) on restoring $\beta \le \sqrt{2/\pi}$ according to $\delta = \frac{1}{2}\beta^2$ (in the resulting inequality (2.23) the $\frac{1}{8}\delta^2$ on the right-hand side of (7.18) has been omitted).

7.2. *Details for $\mathbb{E}M_\beta$ and Var M_β*

We have from (2.4) by the substitutions $u = \frac{1}{2}ny^2$, $\delta = \frac{1}{2}\beta^2$

$$\begin{aligned}\beta^k J_k(\beta) &= \sum_{n=1}^{\infty} \frac{\beta^k n^{k-1/2}}{\sqrt{2\pi}} \int_\beta^\infty (y-\beta)^k \mathrm{e}^{-\frac{1}{2}ny^2} \mathrm{d}y \\ &= \frac{2^k}{\sqrt{\pi}}\delta \sum_{n=1}^{\infty} (\delta n)^{\frac{k}{2}-1} \int_{\sqrt{\delta n}}^\infty (u - \sqrt{\delta n})^k \mathrm{e}^{-u^2} \mathrm{d}u. \end{aligned} \tag{7.19}$$

Thus

$$\beta^k J_k(\beta) = \frac{2^k}{\sqrt{\pi}}\delta \sum_{n=1}^{\infty} f_k(\delta n) \tag{7.20}$$

with

$$f_k(x) = x^{\frac{k}{2}-1} \int_{\sqrt{x}}^\infty (u - \sqrt{x})^k \mathrm{e}^{-u^2} \mathrm{d}u. \tag{7.21}$$

Proposition 6. *f_1 and f_2 are positive, decreasing and convex on* $(0, \infty)$.

Proof. Clearly f_k is positive for $k = 1, 2, \ldots$. From

$$f_k(x) = x^{\frac{k}{2}-1} \int_0^\infty u^k \mathrm{e}^{-(u+\sqrt{x})^2} \mathrm{d}u, \quad x > 0, \tag{7.22}$$

we compute

$$f_k'(x) = x^{\frac{k}{2}-2} \int_0^\infty u^k \left[\frac{k}{2} - 1 - x - u\sqrt{x}\right] \mathrm{e}^{-(u+\sqrt{x})^2} \mathrm{d}u, \quad x > 0. \tag{7.23}$$

Clearly, $f_k'(x) < 0$ when $k = 1, 2$ and $x > 0$. More particularly,

$$f_1'(x) = -\int_0^\infty u\left(\frac{1}{2x^{3/2}} + \frac{1}{x^{1/2}} + \frac{u}{x}\right) \mathrm{e}^{-(u+\sqrt{x})^2} \mathrm{d}u, \tag{7.24}$$

and this increases in $x > 0$ since both

$$\frac{1}{2x^{3/2}} + \frac{1}{x^{1/2}} + \frac{u}{x} \quad \text{and} \quad \mathrm{e}^{-(u+\sqrt{x})^2} \tag{7.25}$$

are positive and decreasing in $x > 0$ when $u \geq 0$. Similarly,

$$f_2'(x) = -\int_0^\infty u^2\left(1 + \frac{u}{x^{1/2}}\right)\mathrm{e}^{-(u+\sqrt{x})^2}\mathrm{d}u \tag{7.26}$$

increases in $x > 0$. This completes the proof. $\square$

Proposition 7. *We have* $\int_0^\infty f_k(x)\mathrm{d}x = (k-1)!4^{-k}\sqrt{\pi}$.

Proof. This follows from Theorem 1 and the fact that $\delta\sum_{n=1}^\infty f_k(\delta n)$ tends to $\int_0^\infty f_k(x)\mathrm{d}x$ when $\delta = \frac{1}{2}\beta^2 \downarrow 0$. $\square$

We finally need to approximate $f_k(x)$ and $\int_0^a f_k(x)\mathrm{d}x$. To that end we observe that for $k = 1, 2, \ldots$

$$f_k(x) = x^{\frac{k}{2}-1}\int_0^\infty (u-\sqrt{x})^k \mathrm{e}^{-u^2}\mathrm{d}u - (-1)^k E_k(x), \quad x > 0, \tag{7.27}$$

where

$$E_k(x) = x^{\frac{k}{2}-1}\int_0^{\sqrt{x}} (\sqrt{x}-u)^k \mathrm{e}^{-u^2}\mathrm{d}u, \quad x > 0. \tag{7.28}$$

The term comprising the integral on the right-hand side of (7.27) can be evaluated,

$$x^{\frac{k}{2}-1}\int_0^\infty (u-\sqrt{x})^k\mathrm{e}^{-u^2}\mathrm{d}u = \frac{1}{2}\sum_{j=0}^k \binom{k}{j}(-1)^j x^{\frac{1}{2}k+\frac{1}{2}j-1}\Gamma\left(\frac{k-j+1}{2}\right), \tag{7.29}$$

simply by expanding $(u-\sqrt{x})^k$.

Proposition 8. *We have for* $x > 0$

$$\frac{x^{k-1/2}}{k+1}\left(1 - \frac{1}{3}x\right) \leq E_k(x) \leq \frac{x^{k-1/2}}{k+1}. \tag{7.30}$$

Proof. The second inequality in (7.30) follows from $\mathrm{e}^{-u^2} \leq 1$ and computing the resulting integral on the right-hand side of (7.28). As to the first inequality we note that both $(\sqrt{x}-u)^k$ and e^{-u^2} are non-negative and decreasing for $u \in [0, \sqrt{x}]$, whence the average of the product $(\sqrt{x}-u)^k\mathrm{e}^{-u^2}$ over $[0, \sqrt{x}]$ is at least equal to the product of the averages of $(\sqrt{x}-u)^k$ and e^{-u^2} over $[0, \sqrt{x}]$. Then the first inequality in (7.30) follows upon using $\mathrm{e}^{-u^2} \geq 1 - u^2$ in the latter average and computing the resulting integrals. $\square$

As a consequence of Proposition 8 we have for $a > 0$

$$\frac{a^{k+1/2}}{(k+1)(k+1/2)}\left(1 - \frac{1}{3}\frac{k+1/2}{k+3/2}a\right) \leq \int_0^a E_k(x)\mathrm{d}x \leq \frac{a^{k+1/2}}{(k+1)(k+1/2)}. \tag{7.31}$$

Using (7.30) and (7.31) in (7.27), and combining that with (7.28) and (7.29), we get from (7.20) with Propositions 3, 6 and 7, for $k = 1, 2$ the inequalities

$$\beta J_1(\beta) \leq \frac{1}{2} - \left(\frac{2\delta}{\pi}\right)^{1/2} + \frac{1}{2}\delta - \frac{\delta^{3/2}}{3\sqrt{2\pi}} + \frac{\delta^{5/2}}{30\sqrt{2\pi}}, \tag{7.32}$$

$$\beta J_1(\beta) \geq \frac{1}{2} - \left(\frac{9\delta}{4\pi}\right)^{1/2} + \frac{1}{2}\delta - \frac{\delta^{3/2}}{6\sqrt{\pi}} - \frac{\delta^{5/2}}{6\sqrt{\pi}}, \tag{7.33}$$

and

$$\beta^2 J_2(\beta) \leq \frac{1}{4} - \frac{1}{2}\delta + \frac{4\delta^{3/2}}{3\sqrt{2\pi}} - \frac{1}{4}\delta^2 + \frac{2\delta^{5/2}}{15\sqrt{2\pi}}, \tag{7.34}$$

$$\beta^2 J_2(\beta) \geq \frac{1}{4} - \frac{1}{2}\delta + \frac{2\delta^{3/2}}{3\sqrt{\pi}} - \frac{2\delta^{5/2}}{15\sqrt{2\pi}} - \frac{8\delta^{7/2}}{63\sqrt{\pi}}, \tag{7.35}$$

and this holds for all $\delta > 0$. From (7.32)–(7.35) we get Theorem 3(ii) and (iii) on restoring $\beta > 0$ according to $\delta = \frac{1}{2}\beta^2$ and remembering that $J_1(\beta) = \mathbb{E}M_\beta$ and $J_2(\beta) = \operatorname{Var} M_\beta$.

Acknowledgement

The second author was supported by the Netherlands Organisation for Scientific Research (NWO).

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