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Gaussian fields and Gaussian sheets with generalized Cauchy covariance structure

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Abstract

Two types of Gaussian processes, namely the Gaussian field with generalized Cauchy covariance (GFGCC) and the Gaussian sheet with generalized Cauchy covariance (GSGCC) are considered. Some of the basic properties and the asymptotic properties of the spectral densities of these random fields are studied. The associated self-similar random fields obtained by applying the Lamperti transformation to GFGCC and GSGCC are studied.

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1. Introduction

Generalization of some well-known stochastic processes indexed by a single parameter to processes indexed by two parameters has attracted considerable interest recently. For example, Houdre and Villa [24] generalized fractional Brownian motion parametrized by a single Hurst index to the bifractional Brownian motion characterized by two indices. Another example is provided by the multidimensional stationary Gaussian fields with generalized Cauchy covariance indexed by two parameters introduced by Gneiting and Schlather [21]. These processes can be regarded as extension of the Gaussian processes with Cauchy covariance used in geostatistics.

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For simplicity, we call such processes the Gaussian field with generalized Cauchy covariance (GFGCC). Here we would like to point out that one should not confuse such a process with a stable process with Cauchy marginals.

In general, processes parametrized by two indices can provide more flexibility in their applications in modeling physical phenomena. In particular, the GFGCC model has an additional nice and useful property as it allows separate characterization of fractal dimension and long range dependence (LRD) by two different indices. This is in contrast to models based on fractional Brownian motion (or fractional Brownian noise) which use a single index to characterize these two properties. Models based on a stochastic process or field parametrized by a single index seem inadequate. A detailed analysis on network traffic carried by Park et al. [43] shows that fractional Brownian motion/fractional Gaussian noise model is inadequate for description of network traffic for all scales since at very small time scales the traffic fluctuations are no longer statistically self-similar. The need to replace global scaling by local scaling is essential for processes such as multifractional Brownian motion introduced independently by Peltier and Vehel [44] and Benassi, Jaffard and Roux [7].

The main aim of this paper is to study GFGCC and its anisotropic counterpart, which we call Gaussian sheet with generalized Cauchy covariance (GSGCC). In view of the fact that GFGCC is widely used in geostatistics and other applications [10,8,47,20,49,39,40,48], it will be useful to consider its properties in more detail. In these existing applications, usually only the covariance structure of GFGCC are used, and the sample path properties of GFGCC are rarely mentioned. However, a better understanding of the sample properties of GFGCC and GSGCC will render more versatility and flexibility to their applications. Our approach to this subject is mainly from a physical viewpoint. Basic sample properties such as the long range dependence and the local self-similarity properties of GFGCC and GSGCC are investigated. A simpler method is used to derive the asymptotic properties of the spectral densities of GFGCC and GSGCC. By generalizing the Lamperti transformation to n-dimensional processes, new types of random field and random sheet with global self-similar property associated with GFGCC and GSGCC are obtained. Properties of these random field and random sheet are also studied.

2. Isotropic Gaussian field with generalized cauchy covariance

In this section we consider GFGCC, which is a multidimensional isotropic Gaussian random field in n-dimensional Euclidean space. We first introduce some notations and state some basic definitions and properties of GFGCC. Denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{R}_+$ the sets of positive integers, integers, real numbers and positive real numbers respectively. Let $t = (t_1, \ldots, t_n)$ and $s = (s_1, \ldots, s_n)$ be two vectors in $\mathbb{R}^n$, and $\|t\| = \sqrt{\sum_{i=1}^n t_i^2}$, $n, j \in \mathbb{N}$ be its Euclidean norm. By $t \to 0^+$ and $t \to \infty$, we mean $t_i \to 0^+$ and $t_i \to \infty$ respectively for all $i = 1, \ldots, n$.

Definition 2.1. A random field $X_{\alpha,\beta}(t)$ on $\mathbb{R}^n$ is called a Gaussian field with generalized Cauchy covariance (or GFGCC) if it is a stationary Gaussian field with mean zero and covariance given by

$$C_{\alpha,\beta}(\tau) = \langle X_{\alpha,\beta}(t+\tau) X_{\alpha,\beta}(t)\rangle = \left(1 + \|\tau\|^{\alpha}\right)^{-\beta}, \tag{2.1}$$

where $\alpha \in (0, 2]$ and $\beta > 0$.

Note that (2.1) has the same functional form as the characteristic function of the generalized multivariate Linnik distribution first studied by Anderson [2]. $C_{\alpha,\beta}(\tau)$ is positive-definite for the above ranges of α and β, and it is completely monotone for $0 < \alpha \leq 1$, $\beta > 0$. $X_{\alpha,\beta}(t)$ becomes the Gaussian field with usual Cauchy covariance when $\alpha = 2$, $\beta = 1$.

Recall that a random field $X(t)$ is H-self-similar (Hss) if $X(ct) =_d c^H X(t)$, where $=_d$ denotes equality in the sense of finite-dimensional distributions of X. Self-similar property requires scale invariance to hold for all scales. This is rather too restrictive for many applications. We also know from Samorodnitsky and Taqqu [46] that a stationary Gaussian random field such as $X_{\alpha,\beta}(t)$ can not be a self-similar field. However, $X_{\alpha,\beta}(t)$ satisfies a weaker self-similar property known as local self-similarity considered by Kent and Wood [27].

Definition 2.2. Let $\alpha \in (0, 2]$. A centered stationary Gaussian field is locally self-similar (lss) of order $\alpha/2$ if for $\|\tau\| \to 0^+$, its covariance $C(\tau)$ satisfies

$$C(\tau) = A - B\|\tau\|^\alpha \left[1 + \mathrm{O}\left(\|\tau\|^\delta\right)\right] \tag{2.2}$$

for some positive constants A, B and δ.

Since as $\|\tau\| \to 0^+$,

$$C_{\alpha,\beta}(\tau) = 1 - \beta\|\tau\|^\alpha \left[1 + \|\tau\|^\alpha\right], \tag{2.3}$$

the GFGCC $X_{\alpha,\beta}(t)$ is $\alpha/2$–lss, with $A = 1$, $B = \beta$ and $\delta = \alpha$. Adler [1] called the class of Gaussian fields which satisfy (2.2) the indexed-α fields. These processes are also known as the Adler processes according to some authors, for example Lang and Roueff [33]. They form a very rich class of Gaussian random fields, which include the centered Gaussian field $\Xi_{\alpha,\beta}(t)$ with powered exponential covariance

$$\left\langle \Xi_{\alpha,\beta}(t+\tau)\Xi_{\alpha,\beta}(t)\right\rangle = \mathrm{e}^{-\beta\|\tau\|^\alpha}, \tag{2.4}$$

which have the same functional form as the characteristic function of the multivariate symmetric stable distribution as given in Kotz, Kozubowski and Podgorski [30], and Garoni and Frankel [16]. Instead of using (2.2) to characterize local self-similarity, one can also use the definition of locally asymptotically self-similar (lass) property first introduced by Benassi, Jaffard and Roux [7] for multifractional Brownian motion $B_{H(t)}(t)$, which is a generalization of fractional Brownian motion with the Hurst index replaced by the Hurst function $H(t)$, $0 < H(t) < 1$. It can be shown that under some regularity conditions on $H(t)$, the multifractional Brownian motion $B_{H(t)}(t)$ is lass. This property can be adapted to GFGCC if we take $H(t)$ as constant with its value in (0, 1).

Definition 2.3. A stochastic process $X(t)$ is lass at a point t_0 with order κ if

$$\lim_{\varepsilon\to 0^+} \left\{\frac{X(t_0+\varepsilon u) - X(t_0)}{\varepsilon^\kappa}\right\}_{u\in\mathbb{R}^n} =_d T_{t_0}(u), \tag{2.5}$$

and $T_{t_0}(u)$ is nontrivial. Here the convergence and equality are in the sense of finite dimensional distributions, and $T_{t_0}(u)$ is called the tangent field of $X(t)$ at the point t_0.

Proposition 2.4. *GFGCC is a lass random field of order $\alpha/2$; and its tangent field is Lévy fractional Brownian field of index $\alpha/2$.*

Proof. By using (2.3), the covariance of the increment field $\Delta_\tau X_{\alpha,\beta}(t) := X_{\alpha,\beta}(t+\tau) - X_{\alpha,\beta}(t)$, for $\rho, \sigma \to 0^+$ is given by

$$\left\langle \Delta_\rho X_{\alpha,\beta}(t)\Delta_\sigma X_{\alpha,\beta}(t)\right\rangle = \beta\left(\|\rho\|^\alpha + \|\sigma\|^\alpha - \|\rho-\sigma\|^\alpha\right) + \mathrm{O}\left(\|\rho\|^{2\alpha}, \|\sigma\|^{2\alpha}\right).$$

Let $\rho = \varepsilon u$ and $\sigma = \varepsilon v$, then

$$\lim_{\varepsilon\to 0^+}\left\langle \frac{\Delta_{\varepsilon u}X_{\alpha,\beta}(t)}{\varepsilon^{\alpha/2}}\frac{\Delta_{\varepsilon v}X_{\alpha,\beta}(t)}{\varepsilon^{\alpha/2}}\right\rangle = \lim_{\varepsilon\to 0^+}\left\{\beta\left(\|u\|^\alpha + \|v\|^\alpha - \|u-v\|^\alpha\right) + \mathrm{O}(\varepsilon^\alpha)\right\}$$
$$= 2\beta\left\langle B_{\alpha/2}(u)B_{\alpha/2}(v)\right\rangle,$$

where $B_{\alpha/2}(u)$ is the index-$\alpha/2$ Lévy fractional Brownian field with zero mean and covariance given by

$$\left\langle B_{\alpha/2}(u)B_{\alpha/2}(v)\right\rangle = \frac{1}{2}\left(\|u\|^\alpha + \|v\|^\alpha - \|u-v\|^\alpha\right), \quad u, v \in \mathbb{R}^n. \tag{2.6}$$

Thus up to a multiplicative constant $\sqrt{2\beta}$, the tangent field of GFGCC at any point $t_0 \in \mathbb{R}^n$ is the Lévy fractional Brownian field indexed by $\alpha/2$. □

The tangent field of $X_{\alpha,\beta}(t)$ at the point t_0 reflects the local structure of the random field at t_0. In other words, GFGCC behaves locally like a Lévy fractional Brownian field. This provides an example to the general results on tangent fields considered by Falconer [14].

The fractal dimension of the graph of a random field $X(t)$ depends on the local property of the random field. The local irregularities of the graph are measured by the parameter α, which can be regarded as the fractal index of the random field. Thus the behavior of the covariance function at the origin to a great extent determines the roughness of the random field. The results on the fractal dimension of an lss field are treated by Adler [1], Kent and Wood [27], Davies and Hall [11]. A nice property of $\alpha/2$–lss fields is that their fractal dimension is determined by α.

Definition 2.5. Let $X(t)$ be a stationary Gaussian field and let $\sigma^2(\tau) = \left\langle \Delta_\tau X(t)^2\right\rangle$ be the variance of the increment process $\Delta_\tau X(t) := X(t+\tau) - X(t)$. If there exists $\alpha \in (0, 2]$ satisfying

$$\alpha = \sup\left\{\varsigma : \sigma^2(\tau) = \mathrm{o}\left(\|\tau\|^\varsigma\right) \quad \text{as } \|\tau\| \to 0\right\}$$
$$= \inf\left\{\varsigma : \|\tau\|^\varsigma = \mathrm{o}\left(\sigma^2(\tau)\right) \quad \text{as } \|\tau\| \to 0\right\}, \tag{2.7}$$

then $\alpha/2$ is called the fractal index of the random field $X(t)$. Equivalently, $\alpha/2$ is the local Hölder index of the random field.

Clearly, condition (2.7) is fulfilled by random field which satisfies (2.2) such as the GFGCC $X_{\alpha,\beta}(t)$. Adler had shown that $\alpha/2$ is the upper bound of the indices for which, with probability one, the graph of $X(t)$ satisfies a global regularity of the same order. Thus, α characterizes the roughness of the sample path. The fractal dimension of GFGCC can be obtained by using the following result for the fractal (Hausdorff) dimension of an lss field as given in Adler [1], chapter 8.

Proposition 2.6. *The fractal dimension D of the graph of a locally self-similar field $X(t)$, $t \in \mathbb{R}^n$, of fractal index $\alpha/2$, over a hyperrectangle $\mathcal{C} = \prod_{i=1}^n [a_i, b_i]$, is given by*

$$D = n + 1 - \frac{\alpha}{2}. \tag{2.8}$$

The estimation of α for lss field has been studied extensively, see for example Wood and Chan [50], Istas and Lang [25], Kent and Wood [27], Lang and Roueff [33]. The parameter β is also known as topothesy in the studies of roughness of surfaces by Wood and Chan [50], and Davies and Hall [11]. The topothesy of a cross section provides a measure of the roughness which is scale-dependent in contrast to fractal dimension which is scale-invariant.

The Gaussian random field $X_{\alpha,\beta}(t)$ can have SRD (short range dependence) or LRD (long range dependence), depending on the values of the parameters α and β. For this purpose we make use of the following definition which is a generalization of the one-dimensional case considered by Flandrin et al. [15], Lim and Muniandy [36]:

Definition 2.7. A stationary centered Gaussian field with covariance $C(\tau)$ is said to be a long range dependent process if

$$\int_{\mathbb{R}^n_+} |C(\tau)|\, \mathrm{d}^n\tau = \infty. \tag{2.9}$$

Otherwise it is short range dependent.

Proposition 2.8. *The GFGCC $X_{\alpha,\beta}(t)$ is a long range dependent random field if and only if $0 < \alpha\beta \le n$.*

Proof. In order to obtain the condition for the Gaussian random field with covariance (2.1) to be LRD, we make use of the following integral identity given in Gradshteyn and Ryzhik [22], 3.251, no. 11:

$$\int_0^\infty x^{\mu-1}\left(1+x^\rho\right)^{-\nu}\mathrm{d}x = \frac{1}{\rho}B\left(\frac{\mu}{\rho}, \nu-\frac{\mu}{\rho}\right),$$

where $\rho > 0$, $0 < \mu < \rho v$ and $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ is the beta function. Using polar coordinates, one gets

$$\begin{aligned}\int_{\mathbb{R}^n_+} \left|C_{\alpha,\beta}(\tau)\right| \mathrm{d}^n\tau &= \int_{\mathbb{R}^n_+} \left(1+\|\tau\|^\alpha\right)^{-\beta}\mathrm{d}^n\tau \\ &= \frac{2\pi^{\frac{n}{2}}}{2^n\Gamma\left(\frac{n}{2}\right)}\int_0^\infty r^{n-1}(1+r^\alpha)^{-\beta}\mathrm{d}r.\end{aligned} \tag{2.10}$$

For large r,

$$r^{n-1}(1+r^\alpha)^{-\beta} \sim r^{n-1-\alpha\beta}.$$

Therefore, the integral (2.10) is divergent for all $\beta > 0$, $0 < \alpha\beta \le n$. For $\beta > 0$ and $\alpha\beta > n$, we have

$$\frac{2\pi^{\frac{n}{2}}}{2^n\Gamma\left(\frac{n}{2}\right)}\int_0^\infty r^{n-1}(1+r^\alpha)^{-\beta}\mathrm{d}r = \frac{\pi^{\frac{n}{2}}}{2^{n-1}\alpha\Gamma\left(\frac{n}{2}\right)}B\left(\frac{n}{\alpha}, \beta-\frac{n}{\alpha}\right) < \infty.$$

Therefore the condition for $X_{\alpha,\beta}(t)$ to be a Gaussian field with LRD is $0 < \alpha\beta \le n$. $\square$

The discussion above shows that it is possible to characterize the fractal dimension D and the LRD property separately. If the covariance is re-expressed as $(1 + \|\tau\|^\alpha)^{-\gamma/\alpha}$, which behaves like $\|\tau\|^{-\gamma}$ in the large-$\|\tau\|$ limit, then $X_{\alpha,\gamma/\alpha}(t)$ is LRD if and only if $0 < \gamma \le n$. Thus α ($0 < \alpha \le 2$) and γ ($\gamma > 0$), respectively, provide separate characterization of fractal dimension and LRD/SRD. The separate characterization of the fractal dimension (local property) and LRD (global property) for GFGCC appears to offer a more natural and flexible model than that based on a single parameter such as in Lévy fractional Brownian field. We note that this feature of separate characterization of local self-similarity (hence fractal dimension) and long range dependence is present in any stationary Gaussian field with covariance $C(\tau)$ satisfying the asymptotic behaviors $C(\tau) \sim A - B\|\tau\|^{\alpha}$ as $\|\tau\| \to 0^+$, and $C(\tau) \to \|\tau\|^{-\gamma}$ as $\|\tau\| \to \infty$, with $\alpha \in (0, 2]$, $\gamma > 0$. Similarly, one can also have a Gaussian stationary process which has separate parametrization of fractal dimension and short range dependence [38]. The ability to have separate characterization of fractal dimension and Hurst effect is a desirable property in the modeling of physical and geological phenomena.

3. Asymptotic properties of spectral density of GFGCC

In this section, we consider the spectral density of $X_{\alpha,\beta}(t)$ and its asymptotic properties. Though the covariance of GFGCC is given by a relatively simple expression, the analytic simplicity of the covariance function is not inherited by the corresponding spectral density. This is similar to the case of the stationary Gaussian field $\Xi_{\alpha,\beta}(t)$ with powered exponential covariance (2.4) which have simple form, but its spectral density in general does not have closed analytic expression. In the case of GFGCC in $\mathbb{R}$, a detailed study of spectral densities (in terms of probability distributions correspond to the characteristic functions of generalized Linnik distributions) have been carried out by Kotz et al. [29] for $0 < \alpha < 2$, $\beta = 1$, $n = 1$; by Ostrovskii [41] for $0 < \alpha < 2$, $\beta = 1$, $n \in \mathbb{N}$; and by Erdogan and Ostrovskii [13] for $0 < \alpha < 2$, $\beta > 0$, $n = 1$. They employed the contour integration representations and series expansions of the generalized Linnik distributions. However the techniques used in these works are less accessible to practitioners. In this section, we derive the asymptotic properties of the spectral densities of GFGCC for $0 < \alpha \leq 2$ and $\beta > 0$, which can be regarded as an extension to the results on generalized multivariate Linnik distributions. The techniques used in our derivations are mathematically more tractable.

Recall that the spectral density $S(\omega)$ of a stationary field $X(t)$ is defined as the Fourier transform of its covariance function $C(t) = \langle X(t)X(0)\rangle$:

$$S(\omega) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\omega.t} C(t) \mathrm{d}^n t,$$

if the integral is convergent. If the integral does not converge, we consider $C(t)$ as a generalized function and define $S(\omega)$ as the Fourier transform of $C(t)$ in the Schwartz space of test functions [17]. Namely, for any test function $\psi(\omega)$ in the Schwartz class of $\mathbb{R}^n$, we require

$$\langle S(\omega), \psi(\omega)\rangle = \langle C(t), \hat{\psi}(t)\rangle,$$

where

$$\hat{\psi}(t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\omega.t} \psi(\omega) \mathrm{d}^n \omega.$$

Alternatively, the spectral density can also be defined to be the function satisfying

$$C(t) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\omega.t} S(\omega) \mathrm{d}^n \omega.$$

For GFGCC, since

$$J_\nu(z) \sim \sqrt{\frac{\pi}{2z}} \cos\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \tag{3.1}$$

as $z \to \infty$ ([3], page 209), we find that when $\alpha\beta > \frac{n-1}{2}$, it's spectral density is

$$S_{\alpha,\beta}(\omega) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i\omega.t}}{(1+\|t\|^\alpha)^\beta} d^n t = \frac{\|\omega\|^{\frac{2-n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_0^\infty \frac{J_{\frac{n-2}{2}}(\|\omega\|t)}{(1+t^\alpha)^\beta} t^{\frac{n}{2}} dt, \tag{3.2}$$

where $J_\nu(z)$ is the Bessel function. When $\alpha = 2$, using no. 4 of 6.565 in [22], we have the explicit formula

$$S_{2,\beta}(\omega) = \frac{\|\omega\|^{\beta-\frac{n}{2}}}{2^{\frac{n}{2}+\beta-1}\pi^{\frac{n}{2}}\Gamma(\beta)} K_{\frac{n}{2}-\beta}(\|\omega\|), \tag{3.3}$$

if $\beta > (n-1)/4$. Here $K_\nu(z)$ is the modified Bessel function. The formula no. 7 of 6.576 in [22] shows that if $\beta \in (0, n)$, then

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{i\omega.t} \left(\frac{\|\omega\|^{\beta-\frac{n}{2}}}{2^{\frac{n}{2}+\beta-1}\pi^{\frac{n}{2}}\Gamma(\beta)} K_{\frac{n}{2}-\beta}(\|\omega\|) \right) d^n\omega \\
&\quad = (2\pi)^{\frac{n}{2}} \int_0^\infty \frac{J_{\frac{n-2}{2}}(\omega\|t\|)}{(\omega\|t\|)^{\frac{n-2}{2}}} \left(\frac{\omega^{\beta-\frac{n}{2}}}{2^{\frac{n}{2}+\beta-1}\pi^{\frac{n}{2}}\Gamma(\beta)} K_{\frac{n}{2}-\beta}(\omega) \right) \omega^{n-1} d\omega = \left(1+\|t\|^2\right)^{-\beta}.
\end{aligned}$$

Therefore, (3.3) is still the spectral density when $\beta \in (0, (n-1)/4]$. For general $\alpha < 2$, no explicit formula such as (3.3) can be found for $S_{\alpha,\beta}(\omega)$. When $n = 1$, the formula (3.2) gives the spectral density of $X_{\alpha,\beta}(t)$ for all values of $\alpha \in (0, 2]$ and $\beta > 0$. For $n \geq 2$, we would also like to find a formula for the spectral density that is valid for all $\alpha \in (0, 2)$ and $\beta > 0$. For this purpose, it would be beneficial to investigate the case $n = 1$ first. When $n = 1$, we can rewrite (3.2) as

$$S_{\alpha,\beta}(\omega) = \frac{1}{\pi} \mathrm{Re} \int_0^\infty \frac{e^{i|\omega|t}}{(1+t^\alpha)^\beta} dt.$$

Let

$$f(\zeta) = \frac{e^{i|\omega|\zeta}}{(1+\zeta^\alpha)^\beta}, \quad -\pi < \arg\zeta \leq \pi,$$

and consider the region $\mathfrak{D}_r$ in the complex plane defined by

$$\mathfrak{D}_r = \{z \in \mathbb{C} : |z| \leq r,\ \mathrm{Re}\, z > 0,\ \mathrm{Im}\, z > 0\}.$$

When $\alpha \in (0, 2)$, the function f is an analytic function on the domain $\mathfrak{D}_r$. Therefore, by Cauchy integral formula,

$$\oint_{\partial\mathfrak{D}_r} f(\zeta) d\zeta = 0. \tag{3.4}$$

Notice that the boundary of $\mathfrak{D}_r$, $\partial\mathfrak{D}_r$, consists of three components: the line segment $l_{r,1}$ along the real axis from 0 to r, the arc C_r of the circle $|z| = r$ from r to ir, and the line segment $l_{r,2}$ along the imaginary axis from ir to 0. On the arc C_r, if $r > 1$, then

$$|f(\zeta)| \leq \frac{e^{-|\omega| \mathrm{Im}\,\zeta}}{(r^\alpha - 1)^\beta}.$$

Therefore,

$$\lim_{r\to\infty}\int_{C_r} f(\zeta)\mathrm{d}\zeta = 0,$$

and (3.4) implies that

$$\lim_{r\to\infty}\int_{l_{r,1}} f(\zeta)\mathrm{d}\zeta = -\lim_{r\to\infty}\int_{l_{r,2}} f(\zeta)\mathrm{d}\zeta.$$

This gives us:

$$S_{\alpha,\beta}(\omega) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}|\omega|t}\mathrm{d}t}{(1+t^\alpha)^\beta} = -\frac{1}{\pi}\mathrm{Im}\int_0^\infty \frac{\mathrm{e}^{-|\omega|u}}{\left(1+\mathrm{e}^{\frac{\mathrm{i}\pi\alpha}{2}}u^\alpha\right)^\beta}\mathrm{d}u. \tag{3.5}$$

For $n \geq 2$, we can derive a formula similar to (3.5). Recall that the Hankel's function of the first kind $H_\nu^{(1)}(z)$ is defined as

$$H_\nu^{(1)}(z) = J_\nu(z) + \mathrm{i}N_\nu(z),$$

where $N_\nu(z)$ is the modified Bessel function of the second kind or called the Neumann function. Using Hankel's function, we can rewrite (3.2) as

$$S_{\alpha,\beta}(\omega) = \frac{\|\omega\|^{\frac{2-n}{2}}}{(2\pi)^{\frac{n}{2}}}\mathrm{Re}\int_0^\infty \frac{H^{(1)}_{\frac{n-2}{2}}(\|\omega\|t)}{(1+t^\alpha)^\beta}t^{\frac{n}{2}}\mathrm{d}t.$$

For $z \to \infty$, we have ([22], no. 3 of 8.451)

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\exp\left\{\mathrm{i}\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\right\}.$$

Therefore, we can show as in the $n = 1$ case that

$$S_{\alpha,\beta}(\omega) = -\frac{\|\omega\|^{\frac{2-n}{2}}}{(2\pi)^{\frac{n}{2}}}\mathrm{Im}\int_0^\infty \frac{H^{(1)}_{\frac{n-2}{2}}(\mathrm{i}\|\omega\|u)}{\left(1+\mathrm{e}^{\frac{\mathrm{i}\pi\alpha}{2}}u^\alpha\right)^\beta}(\mathrm{i}u)^{\frac{n}{2}}\mathrm{d}u.$$

Using the formula ([22], no. 1 of 8.407)

$$K_\nu(z) = \frac{\mathrm{i}\pi}{2}\mathrm{e}^{\frac{\mathrm{i}\nu\pi}{2}}H_\nu^{(1)}(\mathrm{i}z),$$

we have finally

$$S_{\alpha,\beta}(\omega) = -\frac{\|\omega\|^{\frac{2-n}{2}}}{2^{\frac{n-2}{2}}\pi^{\frac{n+2}{2}}}\mathrm{Im}\int_0^\infty \frac{K_{\frac{n-2}{2}}(\|\omega\|u)}{\left(1+\mathrm{e}^{\frac{\mathrm{i}\pi\alpha}{2}}u^\alpha\right)^\beta}u^{\frac{n}{2}}\mathrm{d}u. \tag{3.6}$$

This formula agrees with the formula for multivariate Linnik distribution proved in [41] for $\alpha \in (0, 2)$, $\beta = 1$ and $n \in \mathbb{N}$. Notice that the right hand side of (3.6) is well-defined for all $\alpha, \beta > 0$. Using the formula (no. 2 of 6.521 in [22]),

$$\int_0^\infty xK_\nu(ax)J_\nu(bx)\mathrm{d}x = \frac{b^\nu}{a^\nu(a^2+b^2)}, \quad \nu > -1,$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{i\omega.t}\left\{-\frac{\|\omega\|^{\frac{2-n}{2}}}{2^{\frac{n-2}{2}}\pi^{\frac{n+2}{2}}}\mathrm{Im}\int_0^\infty \frac{K_{\frac{n-2}{2}}(\|\omega\|u)}{\left(1+e^{\frac{i\pi\alpha}{2}}u^\alpha\right)^\beta}u^{\frac{n}{2}}du\right\}d^n\omega\\
&=(2\pi)^{\frac{n}{2}}\|t\|^{\frac{2-n}{2}}\int_0^\infty J_{\frac{n-2}{2}}(\omega\|t\|)\omega^{\frac{n}{2}}\left\{-\frac{\omega^{\frac{2-n}{2}}}{2^{\frac{n-2}{2}}\pi^{\frac{n+2}{2}}}\mathrm{Im}\int_0^\infty \frac{K_{\frac{n-2}{2}}(\omega u)}{\left(1+e^{\frac{i\pi\alpha}{2}}u^\alpha\right)^\beta}u^{\frac{n}{2}}du\right\}d\omega\\
&=-\frac{2\|t\|^{\frac{2-n}{2}}}{\pi}\mathrm{Im}\int_0^\infty \frac{u^{\frac{n}{2}}}{\left(1+e^{\frac{i\pi\alpha}{2}}u^\alpha\right)^\beta}\int_0^\infty \omega K_{\frac{n-2}{2}}(\omega u)J_{\frac{n-2}{2}}(\omega\|t\|)d\omega du\\
&=-\frac{2}{\pi}\mathrm{Im}\int_0^\infty \frac{u}{\left(1+e^{\frac{i\pi\alpha}{2}}u^\alpha\right)^\beta(u^2+\|t\|^2)}du\\
&=-\frac{1}{\pi i}\int_{-\infty}^\infty \frac{u}{\left(1+e^{\frac{i\pi\alpha}{2}}u^\alpha\right)^\beta(u^2+\|t\|^2)}du.
\end{aligned}$$

When $\alpha\in(0,2)$, residue calculus implies that this last integral is equal to

$$2\mathrm{Re\,s}_{u=-i\|t\|}\frac{u}{(u-i\|t\|)\left(1+e^{\frac{i\pi\alpha}{2}}u^\alpha\right)^\beta}=\frac{1}{(1+\|t\|^\alpha)^\beta}.$$

This shows that (3.6) is indeed the spectral density of GFGCC for all $\alpha\in(0,2)$ and $\beta>0$. We would also like to remark that although the formula (3.6) is derived under the assumption $n\geq 2$, but since

$$K_{-1/2}(z)=\sqrt{\frac{\pi}{2z}}e^{-z},$$

therefore when $n=1$, the formula (3.6) reduces to the formula (3.5). We summarize the result as follows:

Proposition 3.1. *If $\alpha\in(0,2)$ and $\beta>0$, the spectral density of the GFGCC $X_{\alpha,\beta}(t)$ is given by*

$$S_{\alpha,\beta}(\omega)=-\frac{\|\omega\|^{\frac{2-n}{2}}}{2^{\frac{n-2}{2}}\pi^{\frac{n+2}{2}}}\mathrm{Im}\int_0^\infty \frac{K_{\frac{n-2}{2}}(\|\omega\|u)}{\left(1+e^{\frac{i\pi\alpha}{2}}u^\alpha\right)^\beta}u^{\frac{n}{2}}du. \tag{3.7}$$

If $\alpha=2$ and $\beta>0$, the spectral density of the GFGCC $X_{2,\beta}(t)$ is given by

$$S_{2,\beta}(\omega)=\frac{\|\omega\|^{\beta-\frac{n}{2}}}{2^{\frac{n}{2}+\beta-1}\pi^{\frac{n}{2}}\Gamma(\beta)}K_{\frac{n}{2}-\beta}(\|\omega\|). \tag{3.8}$$

The spectral density for different values of α and β are plotted in Figs. 1–3. To find the high frequency behavior of the spectral density, we first consider the case where $\alpha=2$. Using the fact that ([22], no. 6 of 8.451) as $z\to\infty$,

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} \sum_{j=0}^\infty \frac{1}{(2z)^j} \frac{\Gamma\left(\nu + j + \frac{1}{2}\right)}{j!\Gamma\left(\nu - j + \frac{1}{2}\right)}.$$

This implies that as $\|\omega\| \to \infty$,

$$S_{2,\beta}(\omega) \sim \frac{\|\omega\|^{\beta - \frac{n+1}{2}}}{2^{\frac{n-1}{2}+\beta} \pi^{\frac{n-1}{2}} \Gamma(\beta)} \mathrm{e}^{-\|\omega\|} \sum_{j=0}^\infty \frac{1}{(2\|\omega\|)^j} \frac{\Gamma\left(j - \beta + \frac{n+1}{2}\right)}{j!\Gamma\left(\frac{n+1}{2} - j - \beta\right)}, \tag{3.9}$$

with leading term

$$S_{2,\beta}(\omega) \sim \frac{\|\omega\|^{\beta - \frac{n+1}{2}}}{2^{\frac{n-1}{2}+\beta} \pi^{\frac{n-1}{2}} \Gamma(\beta)} \mathrm{e}^{-\|\omega\|}.$$

For general $\alpha \in (0, 2)$, to find the high frequency behavior of $S_{\alpha,\beta}(\omega)$, we make use of Eq. (3.7). Making a change of variable and using

$$\frac{1}{\left(1 + \mathrm{e}^{\frac{\mathrm{i}\pi\alpha}{2}} \frac{u^\alpha}{\|\omega\|^\alpha}\right)^\beta} = \sum_{j=0}^m \frac{(-1)^j}{j!} \frac{\Gamma(\beta + j)}{\Gamma(\beta)} \mathrm{e}^{\frac{\mathrm{i}\pi\alpha j}{2}} \frac{u^{\alpha j}}{\|\omega\|^{\alpha j}} + \mathrm{O}(\|\omega\|^{-\alpha(m+1)}), \tag{3.10}$$

as $\|\omega\| \to \infty$, we find that

$$\begin{aligned} S_{\alpha,\beta}(\omega) &= -\frac{\|\omega\|^{-n}}{2^{\frac{n-2}{2}} \pi^{\frac{n+2}{2}}} \mathrm{Im} \int_0^\infty K_{\frac{n-2}{2}}(u) \frac{u^{\frac{n}{2}}}{\left(1 + \mathrm{e}^{\frac{\mathrm{i}\pi\alpha}{2}} \frac{u^\alpha}{\|\omega\|^\alpha}\right)^\beta} \mathrm{d}u \\ &= -\frac{\|\omega\|^{-n}}{2^{\frac{n-2}{2}} \pi^{\frac{n+2}{2}}} \frac{1}{\Gamma(\beta)} \mathrm{Im} \int_0^\infty K_{\frac{n-2}{2}}(u) \sum_{j=0}^m \frac{(-1)^j}{j!} \Gamma(\beta + j) \mathrm{e}^{\frac{\mathrm{i}\pi\alpha j}{2}} \frac{u^{\alpha j}}{\|\omega\|^{\alpha j}} u^{\frac{n}{2}} \mathrm{d}u \\ &\quad + \mathrm{O}\left(\|\omega\|^{-\alpha(m+1)-n}\right) \quad \text{as } \|\omega\| \to \infty. \end{aligned}$$

Using the formula ([22], no. 16 of 6.561)

$$\int_0^\infty x^\mu K_\nu(x) \mathrm{d}x = 2^{\mu-1} \Gamma\left(\frac{1 + \mu + \nu}{2}\right) \Gamma\left(\frac{1 + \mu - \nu}{2}\right), \quad \mathrm{Re}\,(\mu + 1 - |\nu|) > 0,$$

and the definition of asymptotic expansion ([3], page 611), we conclude that as $\|\omega\| \to \infty$, $S_{\alpha,\beta}(\omega)$ behaves asymptotically as

$$\frac{1}{\pi^{\frac{n+2}{2}}} \frac{1}{\Gamma(\beta)} \sum_{j=1}^\infty \frac{(-1)^{j-1} 2^{\alpha j}}{j!} \Gamma(\beta + j) \Gamma\left(\frac{\alpha j + n}{2}\right) \Gamma\left(\frac{\alpha j + 2}{2}\right) \sin \frac{\pi \alpha j}{2} \|\omega\|^{-\alpha j - n}. \tag{3.11}$$

Notice that there is a drastic change of high frequency limit of $S_{\alpha,\beta}(\omega)$ when $\alpha < 2$ and $\alpha = 2$. In fact, naively putting $\alpha = 2$ in (3.11) give identically zero terms. This is a hint that as $\|\omega\| \to \infty$, $S_{2,\beta}(\omega)$ does not have polynomial decay, instead it decays exponentially as is verified by (3.9). We summarize the results as follows:

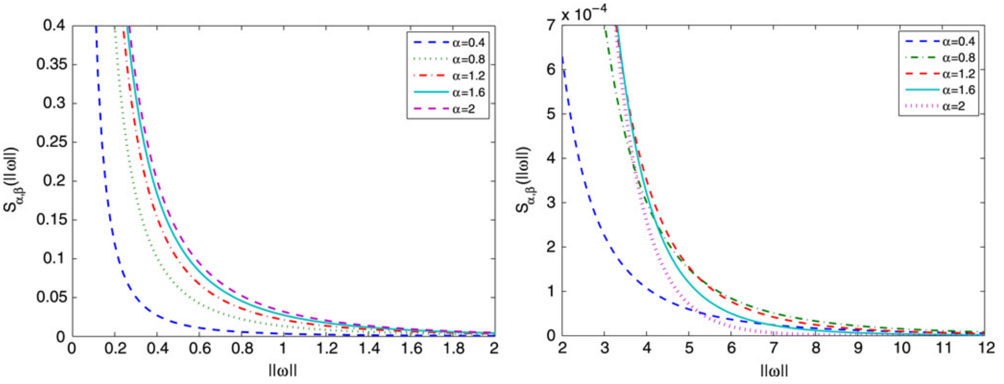

Fig. 1. The spectral density $S_{\alpha,\beta}(\|\omega\|)$ as a function of $\|\omega\|$ when $n = 3$ and $\alpha\beta = 1.5$.

Proposition 3.2. *If $\alpha \in (0, 2)$ and $\beta > 0$, the high frequency limit of the spectral density $S_{\alpha,\beta}(\omega)$ is given by the following asymptotic series:*

$$S_{\alpha,\beta}(\omega) \sim \frac{1}{\pi^{\frac{n+2}{2}}} \frac{1}{\Gamma(\beta)} \sum_{j=1}^{\infty} \frac{(-1)^{j-1} 2^{\alpha j}}{j!} \Gamma(\beta + j) \Gamma\left(\frac{\alpha j + n}{2}\right) \times \Gamma\left(\frac{\alpha j + 2}{2}\right) \sin \frac{\pi \alpha j}{2} \|\omega\|^{-\alpha j - n}. \quad (3.12)$$

If $\alpha = 2$ and $\beta > 0$, the high frequency limit of the spectral density $S_{2,\beta}(\omega)$ is given by the following asymptotic series:

$$S_{2,\beta}(\omega) \sim \frac{\|\omega\|^{\beta - \frac{n+1}{2}}}{2^{\frac{n-1}{2}+\beta} \pi^{\frac{n-1}{2}} \Gamma(\beta)} \mathrm{e}^{-\|\omega\|} \sum_{j=0}^{\infty} \frac{1}{(2\|\omega\|)^j} \frac{\Gamma\left(j - \beta + \frac{n+1}{2}\right)}{j! \Gamma\left(\frac{n+1}{2} - j - \beta\right)}.$$

When $\alpha \in (0, 2)$, $\beta = 1$, $n \in \mathbb{N}$ and $\alpha \in (0, 2)$, $\beta > 0$, $n = 1$, (3.12) agrees with the results given in [41,13] respectively. In particular, we observe that when $\alpha \in (0, 2)$, the high frequency behavior of the spectral density of GFGCC is

$$S_{\alpha,\beta}(\omega) \sim \frac{2^\alpha \beta}{\pi^{\frac{n+2}{2}}} \Gamma\left(\frac{\alpha + n}{2}\right) \Gamma\left(\frac{\alpha + 2}{2}\right) \sin \frac{\pi\alpha}{2} \|\omega\|^{-\alpha - n} \to 0^+, \quad \|\omega\| \to \infty, \quad (3.13)$$

which is independent of β. Kent and Wood [27] have shown that if a random field has spectral density satisfying (3.13), then its covariance satisfies (2.2) with locally self-similar property. However, the converse is not true.

In this connection we remark that for the Gaussian stationary field $\Xi_{\alpha,\beta}(t)$ with powered exponential covariance which is lss with

$$\left\langle \Xi_{\alpha,\beta}(t + \tau) \Xi_{\alpha,\beta}(t) \right\rangle = \mathrm{e}^{-\beta\|\tau\|^\alpha} = 1 - \beta\|\tau\|^\alpha \left[1 + \mathrm{O}(\|\tau\|^\alpha)\right], \quad \|\tau\| \to 0^+, \quad (3.14)$$

its small $\|\tau\|$ behavior has a similar form as that of GFGCC (2.3). Thus it is not surprising that this two random fields have the same tail behavior for their spectral densities at high frequencies as given by (3.13). The detailed calculation carried out by Garoni and Frankel [16] for the

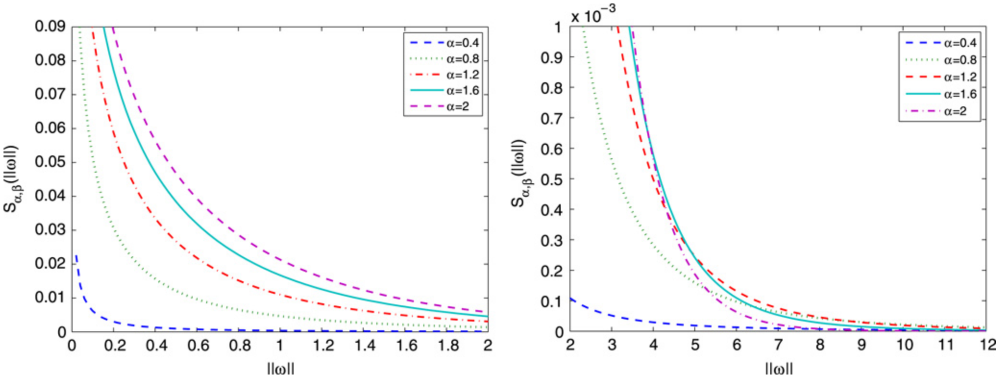

Fig. 2. The spectral density $S_{\alpha,\beta}(\|\omega\|)$ as a function of $\|\omega\|$ when $n = 3$ and $\alpha\beta = 3$.

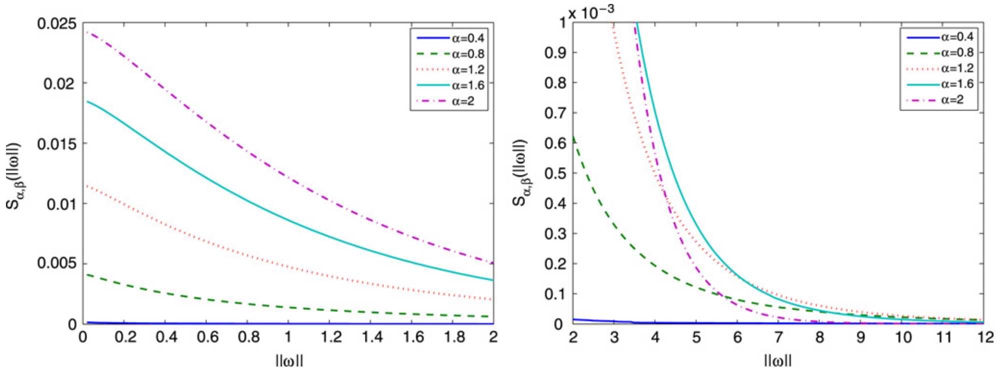

Fig. 3. The spectral density $S_{\alpha,\beta}(\|\omega\|)$ as a function of $\|\omega\|$ when $n = 3$ and $\alpha\beta = 4.5$. $\alpha = 0.4, 0.8, 1.2, 1.6, 2$ for S_1, S_2, S_3, S_4, S_5 respectively.

probability distribution of the multivariate Lévy stable distribution with characteristic function given by (3.14) confirms this. We also note that (3.12) can be used to verify that the tangent field at any point t_0 has spectral density which varies as $\|\omega\|^{-\alpha-n}$ for $\|\omega\| \to \infty$. If we let $\alpha = 2H$, then the tangent field is just the Lévy fractional Brownian field in $\mathbb{R}^n$. Such a relationship can be viewed as a consequence of the Tauberian–Abelian theorem (see e.g. [28]).

For the low frequency behavior of the spectral density $S_{\alpha,\beta}(\omega)$, we first consider the case $\alpha = 2$. Using 8.485, 8.445, 8.446 of [22], we find that if $\nu \notin \mathbb{Z}$,

$$K_\nu(z) = K_{-\nu}(z) = \frac{\pi}{2\sin(\pi\nu)}\left\{\sum_{j=0}^{\infty}\frac{(z/2)^{2j-\nu}}{j!\Gamma(j+1-\nu)} - \sum_{j=0}^{\infty}\frac{(z/2)^{2j+\nu}}{j!\Gamma(j+1+\nu)}\right\}; \tag{3.15}$$

whereas when $\nu = \pm m$, where m is a nonnegative integer,

$$K_\nu(z) = \frac{1}{2}\sum_{j=0}^{m-1}\frac{(-1)^j(m-j-1)!}{j!}\left(\frac{z}{2}\right)^{2j-m} + (-1)^{m+1}\sum_{j=0}^{\infty}\frac{(z/2)^{m+2j}}{j!(m+j)!}\left\{\ln\frac{z}{2} - \frac{1}{2}\psi(j+1) - \frac{1}{2}\psi(j+1+m)\right\}. \tag{3.16}$$

Here $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithm derivative of the Gamma function. Therefore from (3.8), we find that:

- if $\beta > n/2$, then as $\|\omega\| \to 0^+$,

$$S_{2,\beta}(\omega) \sim \frac{\Gamma\left(\beta - \frac{n}{2}\right)}{2^n \pi^{\frac{n}{2}} \Gamma(\beta)} \tag{3.17}$$

- if $\beta = n/2$, then as $\|\omega\| \to 0^+$,

$$\begin{aligned} S_{2,\beta}(\omega) &\sim \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\left\{-\ln\|\omega\| + \ln 2 + \psi(1)\right\} \\ &= \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\left\{-\ln\|\omega\| + \ln 2 - \gamma\right\}, \end{aligned} \tag{3.18}$$

  where γ is the Euler constant
- if $\beta < n/2$, then as $\|\omega\| \to 0^+$,

$$S_{2,\beta}(\omega) \sim \frac{\Gamma\left(\frac{n}{2} - \beta\right)}{2^{2\beta}\pi^{\frac{n}{2}}\Gamma(\beta)} \|\omega\|^{2\beta - n}. \tag{3.19}$$

In fact, by considering the cases $\beta - \frac{n}{2} \in \mathbb{Z}$ and $\beta - \frac{n}{2} \neq \mathbb{Z}$ separately and substituting the series (3.15) and (3.16) into (3.8), we can express the spectral density $S_{2,\beta}(\omega)$ in terms of convergent power series in $\|\omega\|$.

For general $\alpha \in (0, 2)$, the low frequency behavior of $S_{\alpha,\beta}(\omega)$ depends on the arithmetic nature of α and β. The method we are going to employ does not allow the derivation of the whole asymptotic series as obtained by Kotz et al. [29], Erdogan and Ostrovskii [41,13]. We will only derive the leading behavior of the spectral density $S_{\alpha,\beta}(\omega)$, which only depends on the algebraic conditions $\alpha\beta > n$, $\alpha\beta = n$ or $\alpha\beta < n$. These conditions are less stringent than the arithmetic conditions considered in [29,41,13]. However, the simpler method employed here provides the necessary $S_{\alpha\beta}(\omega)$, $\|\omega\| \to 0^+$ asymptotic behaviors which are sufficient for most practical purposes.

When $\alpha\beta > n$, since ([22], Eq. 8.402)

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} + \mathrm{O}\left(z^{\nu+2}\right) \quad \text{as } z \to 0, \tag{3.20}$$

we find from (3.2) that as $\|\omega\| \to 0^+$,

$$S_{\alpha,\beta}(\omega) \sim \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{t^{n-1}}{(1+t^\alpha)^\beta}\mathrm{d}t = \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{n}{\alpha}\right)\Gamma\left(\beta - \frac{n}{\alpha}\right)}{\alpha\Gamma(\beta)}, \tag{3.21}$$

([22], no. 11 of 3.251). When $\alpha\beta = n$, we make a change of variable on (3.2) to get

$$S_{\alpha,\beta}(\omega) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^\infty J_{\frac{n-2}{2}}(t) \frac{t^{\frac{n}{2}}}{(\|\omega\|^\alpha + t^\alpha)^\beta}\mathrm{d}t.$$

In view of the leading behavior of the Bessel function $J_\nu(z)$ as $z \to 0$ (3.20), we write $S_{\alpha,\beta}(\omega)$ as the sum of two terms $S^1_{\alpha,\beta}(\omega)$ and $S^2_{\alpha,\beta}(\omega)$ where

$$\begin{aligned}S^1_{\alpha,\beta}(\omega) = &\frac{1}{(2\pi)^{\frac{n}{2}}}\int_0^1 \left(J_{\frac{n-2}{2}}(t) - \frac{t^{\frac{n-2}{2}}}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}\right)\frac{t^{\frac{n}{2}}}{(\|\omega\|^\alpha + t^\alpha)^\beta}\mathrm{d}t\\ &+\frac{1}{(2\pi)^{\frac{n}{2}}}\int_1^\infty J_{\frac{n-2}{2}}(t)\frac{t^{\frac{n}{2}}}{(\|\omega\|^\alpha + t^\alpha)^\beta}\mathrm{d}t,\end{aligned}$$

and

$$S^2_{\alpha,\beta}(\omega) = \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\int_0^1 \frac{t^{n-1}}{(\|\omega\|^\alpha + t^\alpha)^\beta}\mathrm{d}t.$$

As $\|\omega\| \to 0^+$, (3.1) and (3.20) show that $S^1_{\alpha,\beta}(\omega)$ has a finite limit. By Lebesgue's dominated convergence theorem, the limit is given by $S^1_{\alpha,\beta}(0)$. Namely

$$\begin{aligned}&S^1_{\alpha,\beta}(\omega) \xrightarrow{\|\omega\|\to 0^+} S^1_{\alpha,\beta}(0)\\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_0^1 \left(J_{\frac{n-2}{2}}(t) - \frac{t^{\frac{n-2}{2}}}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}\right)t^{-\frac{n}{2}}\mathrm{d}t + \frac{1}{(2\pi)^{\frac{n}{2}}}\int_1^\infty J_{\frac{n-2}{2}}(t)t^{-\frac{n}{2}}\mathrm{d}t.\end{aligned}$$

This expression can be evaluated using regularization method. More precisely, using Lebesgue's dominated convergence theorem again, we find that

$$\begin{aligned}S^1_{\alpha,\beta}(0) &= \lim_{\varepsilon\to 0^+}\left\{\frac{1}{(2\pi)^{\frac{n}{2}}}\int_0^1 \left(J_{\frac{n-2}{2}}(t) - \frac{t^{\frac{n-2}{2}}}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}\right)t^{-\frac{n}{2}+\varepsilon}\mathrm{d}t\right.\\ &\quad\left.+\ \frac{1}{(2\pi)^{\frac{n}{2}}}\int_1^\infty J_{\frac{n-2}{2}}(t)t^{-\frac{n}{2}+\varepsilon}\mathrm{d}t\right\}\\ &= \lim_{\varepsilon\to 0^+}\left\{\frac{1}{(2\pi)^{\frac{n}{2}}}\int_0^\infty J_{\frac{n-2}{2}}(t)t^{-\frac{n}{2}+\varepsilon}\mathrm{d}t - \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\int_0^1 t^{-1+\varepsilon}\mathrm{d}t\right\}.\end{aligned}$$

The formula no. 14 of 6.561 in [22] then gives

$$\begin{aligned}S^1_{\alpha,\beta}(0) &= \lim_{\varepsilon\to 0^+}\left\{\frac{2^\varepsilon\Gamma\left(\frac{\varepsilon}{2}\right)}{2^n\pi^{\frac{n}{2}}\Gamma\left(\frac{n-\varepsilon}{2}\right)} - \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\frac{1}{\varepsilon}\right\}\\ &= \lim_{\varepsilon\to 0^+}\frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\frac{1}{\varepsilon}\left\{(1+\varepsilon\ln 2)\left(1+\frac{\varepsilon}{2}\psi(1)\right)\left(1+\frac{\varepsilon}{2}\psi\left(\frac{n}{2}\right)\right)-1\right\}\\ &= \frac{1}{2^n\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\left\{2\ln 2 + \psi(1) + \psi\left(\frac{n}{2}\right)\right\}.\end{aligned}$$

For the term $S^2_{\alpha,\beta}(\omega)$, we make a change of variable $u = t^\alpha$ or equivalently $t = u^{\frac{\beta}{n}}$, to get

$$S^2_{\alpha,\beta}(\omega) = \frac{\beta}{2^{n-1}\pi^{\frac{n}{2}}n\Gamma\left(\frac{n}{2}\right)}\int_0^1 \frac{u^{\beta-1}\mathrm{d}u}{(\|\omega\|^\alpha+u)^\beta}.$$

We split $S^2_{\alpha,\beta}(\omega)$ again into a sum of two terms $S^3_{\alpha,\beta}(\omega)$ and $S^4_{\alpha,\beta}(\omega)$, where

$$\begin{aligned}S^3_{\alpha,\beta}(\omega) &= \frac{\beta}{2^{n-1}\pi^{\frac{n}{2}} n\Gamma\left(\frac{n}{2}\right)}\int_0^1 \frac{1}{(\|\omega\|^\alpha+u)}\mathrm{d}u\\ &= \frac{\beta}{2^{n-1}\pi^{\frac{n}{2}} n\Gamma\left(\frac{n}{2}\right)}\ln\frac{1+\|\omega\|^\alpha}{\|\omega\|^\alpha} \sim \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\ln\frac{1}{\|\omega\|}+\mathrm{O}(\|\omega\|^\alpha),\end{aligned}$$

and

$$\begin{aligned}S^4_{\alpha,\beta}(\omega) &= \frac{\beta}{2^{n-1}\pi^{\frac{n}{2}} n\Gamma\left(\frac{n}{2}\right)}\int_0^1 \frac{u^{\beta-1}-(\|\omega\|^\alpha+u)^{\beta-1}}{(\|\omega\|^\alpha+u)^\beta}\mathrm{d}u\\ &= \frac{\beta}{2^{n-1}\pi^{\frac{n}{2}} n\Gamma\left(\frac{n}{2}\right)}\int_0^{\frac{1}{\|\omega\|^\alpha}} \frac{u^{\beta-1}-(1+u)^{\beta-1}}{(1+u)^\beta}\mathrm{d}u.\end{aligned}$$

When $\|\omega\|\to 0^+$,

$$S^4_{\alpha,\beta}(\omega) \sim \frac{\beta}{2^{n-1}\pi^{\frac{n}{2}} n\Gamma\left(\frac{n}{2}\right)}\int_0^\infty \frac{u^{\beta-1}-(1+u)^{\beta-1}}{(1+u)^\beta}\mathrm{d}u.$$

The integral is a convergent integral with value given by ([22], 3.219 page 316)

$$\int_0^\infty \frac{u^{\beta-1}-(1+u)^{\beta-1}}{(1+u)^\beta}\mathrm{d}u = -\psi(\beta)-\gamma.$$

Putting everything together, we find that

$$S^2_{\alpha,\beta}(\omega) \sim \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\left\{\ln\frac{1}{\|\omega\|}-\frac{\beta}{n}\left(\psi(\beta)+\gamma\right)\right\} \quad \text{as } \|\omega\|\to 0^+.$$

Therefore, as $\|\omega\|\to 0^+$,

$$S_{\alpha,\beta}(\omega) \sim \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\left\{\ln\frac{1}{\|\omega\|}-\frac{\beta}{n}\left(\psi(\beta)+\gamma\right)+\ln 2-\frac{1}{2}\gamma+\frac{1}{2}\psi\left(\frac{n}{2}\right)\right\}. \tag{3.22}$$

When $\alpha\beta<n$, care has to be taken since (3.2) is defined only for $\alpha\beta>\frac{n-1}{2}$. Making a change of variable, one finds that

$$S_{\alpha,\beta}(\omega) = \frac{\|\omega\|^{\alpha\beta-n}}{(2\pi)^{\frac{n}{2}}}\int_0^\infty \frac{J_{\frac{n-2}{2}}(u)u^{\frac{n}{2}}}{(\|\omega\|^\alpha+u^\alpha)^\beta}\mathrm{d}u.$$

It is easy to verify that for any $\alpha>0$ and $\beta>0$, when $\|\omega\|\to 0^+$,

$$S_{\alpha,\beta}(\omega) \sim \frac{\|\omega\|^{\alpha\beta-n}}{(2\pi)^{\frac{n}{2}}} \lim_{\varepsilon\to 0^+} \lim_{\|\omega\|\to 0} \int_0^\infty \frac{J_{\frac{n-2}{2}}(u)\mathrm{e}^{-\varepsilon u} u^{\frac{n}{2}}}{(\|\omega\|^\alpha + u^\alpha)^\beta} \mathrm{d}u.$$

Now using no. 1 of 6.621 in [22], we have

$$\begin{aligned}
&\lim_{\|\omega\|\to 0} \int_0^\infty \frac{J_{\frac{n-2}{2}}(u)\mathrm{e}^{-\varepsilon u} u^{\frac{n}{2}}}{(\|\omega\|^\alpha + u^\alpha)^\beta} \mathrm{d}u = \int_0^\infty J_{\frac{n-2}{2}}(u)\mathrm{e}^{-\varepsilon u} u^{\frac{n}{2}-\alpha\beta} \\
&\quad = \frac{1}{2^{\frac{n-2}{2}}} \frac{\Gamma(n-\alpha\beta)}{\sqrt{(\varepsilon^2+1)^{n-\alpha\beta}}\,\Gamma\left(\frac{n}{2}\right)} {}_2F_1\left(\frac{n-\alpha\beta}{2}, \frac{\alpha\beta-1}{2}; \frac{n}{2}; \frac{1}{1+\varepsilon^2}\right),
\end{aligned}$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function

$${}_2F_1(a,b;c;z) = \sum_{j=0}^\infty \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!}, \qquad (x)_j := x(x+1)\dots(x+j-1) = \frac{\Gamma(x+j)}{\Gamma(x)}.$$

Using the formula no. 2 of 9.131 in [22], we find that

$$\begin{aligned}
&{}_2F_1\left(\frac{n-\alpha\beta}{2}, \frac{\alpha\beta-1}{2}; \frac{n}{2}; \frac{1}{1+\varepsilon^2}\right) \\
&\quad = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha\beta}{2}\right)\Gamma\left(\frac{n+1-\alpha\beta}{2}\right)} {}_2F_1\left(\frac{n-\alpha\beta}{2}, \frac{\alpha\beta-1}{2}; \frac{1}{2}; \frac{\varepsilon^2}{1+\varepsilon^2}\right) \\
&\qquad + \left(\frac{\varepsilon^2}{1+\varepsilon^2}\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{n-\alpha\beta}{2}\right)\Gamma\left(\frac{\alpha\beta-1}{2}\right)} {}_2F_1\left(\frac{\alpha\beta}{2}, \frac{n-\alpha\beta+1}{2}; \frac{3}{2}; \frac{\varepsilon^2}{1+\varepsilon^2}\right).
\end{aligned}$$

Since

$$\lim_{z\to 0} {}_2F_1(a,b;c;z) = 1,$$

therefore,

$$\lim_{\varepsilon\to 0^+} \lim_{\|\omega\|\to 0} \int_0^\infty \frac{J_{\frac{n-2}{2}}(u)\mathrm{e}^{-\varepsilon u} u^{\frac{n}{2}}}{(\|\omega\|^\alpha + u^\alpha)^\beta} \mathrm{d}u = \frac{\sqrt{\pi}}{2^{\frac{n-2}{2}}} \frac{\Gamma(n-\alpha\beta)}{\Gamma\left(\frac{\alpha\beta}{2}\right)\Gamma\left(\frac{n+1-\alpha\beta}{2}\right)} = 2^{\frac{n}{2}-\alpha\beta} \frac{\Gamma\left(\frac{n-\alpha\beta}{2}\right)}{\Gamma\left(\frac{\alpha\beta}{2}\right)},$$

where we have used the formula $\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+(1/2))$. This gives for $0 < \alpha\beta < n$,

$$S_{\alpha,\beta}(\omega) \sim \frac{\|\omega\|^{\alpha\beta-n}}{2^{\alpha\beta}\pi^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n-\alpha\beta}{2}\right)}{\Gamma\left(\frac{\alpha\beta}{2}\right)} \quad \text{as } \|\omega\| \to 0^+. \tag{3.23}$$

It is easy to verify that by putting $\alpha = 2$ in (3.21), (3.22) and (3.23), we get back (3.17), (3.18) and (3.19) respectively. The low frequency behaviors of $S_{\alpha,\beta}(\omega)$ are summarized below.

Proposition 3.3. *For all $\alpha \in (0, 2]$ and $\beta > 0$, the low frequency limit of the spectral density $S_{\alpha,\beta}(\omega)$ is given by*

$$S_{\alpha,\beta}(\omega) \sim \frac{\|\omega\|^{\alpha\beta-n}}{2^{\alpha\beta}\pi^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n-\alpha\beta}{2}\right)}{\Gamma\left(\frac{\alpha\beta}{2}\right)}, \quad \text{if } \alpha\beta < n;$$

$$S_{\alpha,\beta}(\omega) \sim \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \left\{\ln\frac{1}{\|\omega\|} - \frac{\beta}{n}\left(\psi(\beta)+\gamma\right) + \ln 2 - \frac{1}{2}\gamma + \frac{1}{2}\psi\left(\frac{n}{2}\right)\right\}, \quad \text{if } \alpha\beta = n;$$

$$S_{\alpha,\beta}(\omega) \sim \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{n}{\alpha}\right)\Gamma\left(\beta-\frac{n}{\alpha}\right)}{\alpha\Gamma(\beta)}, \quad \text{if } \alpha\beta > n.$$

We see that if $0 < \alpha\beta \le n$, $S_{\alpha,\beta}(\omega)$ is divergent at the origin and if $0 < \alpha\beta < n$, $S_{\alpha,\beta}(\omega) \sim \|\omega\|^{\alpha\beta-n}$ as $\|\omega\| \to 0^+$. Note that the condition $0 < \alpha\beta \le n$ agrees with the LRD condition. In fact, it is a basic fact that for a stationary field with positive covariance, it is LRD if and only if its spectral density diverges at the origin. We would also like to point out that the low frequency limit of the spectral density of the Gaussian field with powered exponent covariance $\Xi_{\alpha,\beta}(t)$ (2.4) is finite as shown by Garoni and Frankel [16], in agreement with the fact that $\Xi_{\alpha,\beta}(t)$ is SRD.

We remark that by considering

$$S_{\alpha,\beta}(\omega) - \text{“leading order term as } \|\omega\| \to 0^+\text{”},$$

we can find the next order term in the asymptotic expansion of $S_{\alpha,\beta}(\omega)$ when $\|\omega\| \to 0^+$ using the same methods we employed above. The results depend on more complicated conditions on α and β.

We also briefly remark that the asymptotic behavior of the spectral density at low frequency is connected to the large time behavior of the covariance function. For the covariance function $C_{\alpha,\beta}(\tau)$ which satisfies

$$C_{\alpha,\beta}(\tau) \sim L(\tau)\|\tau\|^{-\alpha\beta}, \quad \|\tau\| \to \infty, \tag{3.24}$$

where $L(\tau)$ is a slowly varying function for large $\|\tau\|$, i.e. $L(c\tau)/L(\tau) \to 1$ as $\|\tau\| \to \infty$ for all positive constant c, Hardy–Littlewood–Karamata–Tauberian theorem (see e.g. [34,35]) implies that the spectral density $S_{\alpha,\beta}(\omega)$ has the following asymptotic behavior

$$S_{\alpha,\beta}(\omega) \sim c_{n,\alpha\beta}\|\omega\|^{\alpha\beta-n} L\left(\frac{1}{\omega}\right), \quad \|\omega\| \to 0^+, \text{ if } \alpha\beta < n,$$

where

$$c_{n,\alpha\beta} = \frac{1}{2^{\alpha\beta}\pi^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n-\alpha\beta}{2}\right)}{\Gamma\left(\frac{\alpha\beta}{2}\right)}.$$

4. Gaussian sheet with generalized Cauchy covariance

We can also introduce another type of n-dimensional process $X^{\#}_{\alpha,\beta}(t)$, called Gaussian sheet with generalized Cauchy covariance (GSGCC).

Definition 4.1. The GSGCC is a centered Gaussian random field $X^{\#}_{\alpha,\beta}(t)$ indexed by two multidimensional parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, with $\alpha_i \in (0, 2]$, $\beta_i > 0$ for all $i = 1, \ldots, n$, and with covariance given by

$$\begin{aligned} C^{\#}_{\alpha,\beta}(\tau) &= \left\langle X^{\#}_{\alpha,\beta}(t+\tau) X^{\#}_{\alpha,\beta}(t) \right\rangle \\ &= \prod_{i=1}^{n} C_{\alpha_i,\beta_i}(\tau_i) = \prod_{i=1}^{n} \left(1 + |\tau_i|^{\alpha_i}\right)^{-\beta_i}, \end{aligned} \tag{4.1}$$

where $C_{\alpha_i,\beta_i}(\tau_i)$ is the covariance of one-dimensional GFGCC indexed by α_i and β_i.

Heuristically, one can regard $X^{\#}_{\alpha,\beta}(t)$ as the product of n independent one-dimensional GFGCC processes $X_{\alpha_i,\beta_i}(t_i)$. However, since the product of independent normal random variables is not a normal random variable, we shall not write $X^{\#}_{\alpha,\beta}(t) = \prod_{i=1}^{n} X_{\alpha_i,\beta_i}(t_i)$. Nevertheless, it is true that for any integer $m \geq 1$,

$$E\left(\left[X^{\#}_{\alpha,\beta}(t)\right]^m\right) = \prod_{i=1}^{n} E\left(\left[X_{\alpha_i,\beta_i}(t_i)\right]^m\right).$$

The GSGCC $X^{\#}_{\alpha,\beta}(t)$ is an anisotropic Gaussian random field and it does not satisfy the definition of local self-similarity given in Definition 2.2. However, we can show that it is lass and has tangent field according to Definition 2.3. Assume that $\alpha_1 = \cdots = \alpha_{m_\alpha} < \alpha_{m_\alpha+1} \leq \cdots \leq \alpha_n$. Namely, for some $m_\alpha \in \{1, \ldots, n\}$, $\alpha_1 = \alpha_2 = \cdots = \alpha_{m_\alpha} = \min\alpha$, and for all $i \geq m_\alpha + 1$, $\alpha_i \gneq \min\alpha$. Here $\min\alpha = \min\{\alpha_1, \ldots, \alpha_n\}$.

Proposition 4.2. *GSGCC is lass with tangent field $T_{\alpha,\beta}(t)$, which is a stationary centered Gaussian field with covariance*

$$\left\langle T_{\alpha,\beta}(u) T_{\alpha,\beta}(v) \right\rangle = \sum_{i=1}^{m_\alpha} \beta_i \left(|u_i|^{\min\alpha} + |v_i|^{\min\alpha} - |u_i - v_i|^{\min\alpha}\right).$$

Proof. From definition, it is easy to see that

$$C^{\#}_{\alpha,\beta}(\tau) = \prod_{i=1}^{n} \left\{1 - \beta_i |\tau_i|^{\alpha_i} \left[1 + \mathrm{O}|\tau_i|^{\alpha_i}\right]\right\} \quad \text{as } |\tau_i| \to 0^+,\ i = 1, \ldots, n. \tag{4.2}$$

Therefore,

$$C^{\#}_{\alpha,\beta}(\tau) = 1 - \sum_{i=1}^{m_\alpha} \beta_i |\tau_i|^{\min\alpha} + \mathrm{O}\left(\sum_{i=m_\alpha+1}^{n} |\tau_i|^{\alpha_i} + \sum_{i=1}^{m_\alpha} |\tau_i|^{2\min\alpha}\right). \tag{4.3}$$

Consequently,

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \left\langle \frac{\Delta_{\varepsilon u} X^{\#}_{\alpha,\beta}(t)}{\varepsilon^{\min\alpha/2}} \frac{\Delta_{\varepsilon v} X^{\#}_{\alpha,\beta}(t)}{\varepsilon^{\min\alpha/2}} \right\rangle \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{\min\alpha}} \left\{C^{\#}_{\alpha,\beta}(\varepsilon(u-v)) - C^{\#}_{\alpha,\beta}(\varepsilon u) - C^{\#}_{\alpha,\beta}(\varepsilon v) + 1\right\} \\ &= \sum_{i=1}^{m_\alpha} \beta_i \left(|u_i|^{\min\alpha} + |v_i|^{\min\alpha} - |u_i - v_i|^{\min\alpha}\right). \quad \square \end{aligned}$$

From this proposition we see that in general, the tangent field $T_{\alpha,\beta}(u)$ of the GSGCC $X^{\#}_{\alpha,\beta}(t)$ defined by Definition 2.3 are uncorrelated in some of the directions of $\mathbb{R}^n$. It does not fully capture the locally self-similar property of $X^{\#}_{\alpha,\beta}(t)$. There is another measure of local self-similarity, called local multiple self-similarity that better describe this kind of process. Recall that [18] a stochastic process $X(t)$ is called multi-self-similar (mss) of index $H \in \mathbb{R}^n_+$ if and only if for any $c \in \mathbb{R}^n_+$,

$$X(c_1 t_1, \dots, c_n t_n) =_d c_1^{H_1} \dots c_n^{H_n} X(t_1, \dots, t_n).$$

It is obvious that a H-mss field is a $\left[\sum_{i=1}^n H_i\right]$-ss field, but in general a self-similar field is not multi-self-similar. The notion of lss can be generalized accordingly.

Definition 4.3. A centered stationary Gaussian field is n-ple locally self-similar of order $\alpha/2$ if its covariance function $C(\tau)$ satisfies for $\|\tau\| \to 0^+$,

$$C(\tau) = \prod_{i=1}^n \left(A_i - B_i |\tau_i|^{\alpha_i} \left[1 + \mathrm{O}\left(|\tau_i|^{\delta_i}\right)\right]\right),$$

for some $A_i, B_i, \delta_i > 0$, $1 \le i \le n$.

In view of (4.2), it is easy to see that $X^{\#}_{\alpha,\beta}(t)$ is n-ple locally self-similar with $A_i = 1$, $B_i = \beta_i$ and $\delta_i = \alpha_i$. We also note that the local multi-self-similarity is equivalent to

$$X^{\#}_{\alpha,\beta}(t + c\tau_i e_i) - X^{\#}_{\alpha,\beta}(t) =_d c^{\alpha_i} \left[X^{\#}_{\alpha,\beta}(t + \tau_i e_i) - X^{\#}_{\alpha,\beta}(t)\right] \quad \text{as } \tau_i \to 0^+, \tag{4.4}$$

for any $c \in \mathbb{R}^+$. Here e_i is the unit vector in the t_i direction. This can also be rephrased as n-ple lass.

Proposition 4.4. *$X^{\#}_{\alpha,\beta}(t)$ is n-ple lass. More precisely, for every $i = 1, \dots, n$,*

$$\lim_{\varepsilon \to 0^+} \left\{ \frac{X^{\#}_{\alpha,\beta}(t + \varepsilon u_i e_i) - X^{\#}_{\alpha,\beta}(t)}{\varepsilon_i^{\alpha_i/2}} \right\}_{u_i \in \mathbb{R}} = \sqrt{2\beta} B_{\alpha_i/2}(u_i), \tag{4.5}$$

where $B_{\alpha/2}(u)$ is the one dimensional fractional Brownian motion of index $\alpha/2$.

The proof is the same as Proposition 2.4.

One should observe that the n-ple lass here is different from the lass defined in Definition 2.3. We take the limit process in each of the t_i-direction separately and the limit process can be regarded as the partial derivative process of $X^{\#}_{\alpha,\beta}(t)$ in the t_i-direction. A disadvantage of this definition is that the limit process of an n-dimensional field is a one-dimensional process. Therefore, we will define another limit process that will better reflect local multi-self-similarity, as were considered in [23,5]. Given a stochastic process $X(t)$, $t \in \mathbb{R}^n$, we define its total increment by $u \in \mathbb{R}^n$ at the point $t \in \mathbb{R}^n$ as

$$\square_u X(t) = \sum_{\delta \in \{0,1\}^n} (-1)^{n - \sum_{i=1}^n \delta_i} X\left(t + \sum_{i=1}^n \delta_i u_i e_i\right).$$

When $n = 1$, this is the same as the increment, i.e. $\square_u X(t) = \Delta_u X(t)$. When $n = 2$, we have

$$\square_{(u_1,u_2)} X(t_1, t_2) = X(t_1 + u_1, t_2 + u_2) - X(t_1 + u_1, t_2) - X(t_1, t_2 + u_2) + X(t_1, t_2),$$

which is also known as the rectangular increment. For general n, given $t, u \in \mathbb{R}^n$, the set of points

$$\left\{ t + \sum_{i=1}^n \delta_i u_i e_i \right\}_{\delta \in \{0,1\}^n}$$

are the vertices of the hyperrectangle with t and $t + u$ as one of the main diagonals. By giving the vertex $t + u$ a weight $+1$, the other vertices of the hyperrectangle can be given a weight $+1$ and -1 alternatingly so that adjacent vertices has different weights. $\square_u X(t)$ is then the weighted sum of the field $X(t)$ at the vertices of the hyperrectangle.

Recall that the fractional Brownian sheet $B^{\#}_{\alpha/2}(t)$, $t \in \mathbb{R}^n$ of index $\alpha/2 \in (0, 1)^n$ is a centered Gaussian process with covariance

$$\left\langle B^{\#}_{\alpha/2}(t) B^{\#}_{\alpha/2}(s) \right\rangle = \frac{1}{2^n} \prod_{i=1}^n \left(|t_i|^{\alpha_i} + |s_i|^{\alpha_i} - |t_i - s_i|^{\alpha_i} \right).$$

It is well known that $B^{\#}_{\alpha/2}(t)$ is a self-similar field of order $\sum_{i=1}^n \alpha_i/2$, and multi-self-similar of order $\alpha/2$. However, unlike the Lévy fractional Brownian field (2.6), the fractional Brownian sheet is not a process with stationary increment. Nevertheless, it is a process with stationary total increment, i.e.,

$$\left\{ \square_u B^{\#}_{\alpha/2}(t), u \in \mathbb{R}^n \right\} =_d \left\{ \square_u B^{\#}_{\alpha/2}(s), u \in \mathbb{R}^n \right\}, \quad \forall\, t, s \in \mathbb{R}^n.$$

Now returning to the local asymptotic multi-self-similarity of GSGCC, we can show the following:

Proposition 4.5.

$$\lim_{\varepsilon \to 0^+} \left\langle \frac{\square_{\varepsilon.u} X^{\#}_{\alpha,\beta}(t)}{\prod_{i=1}^n \varepsilon_i^{\alpha_i/2}} \right\rangle_{u \in \mathbb{R}^n} =_d \left[\prod_{i=1}^n \sqrt{2\beta_i} \right] B^{\#}_{\alpha/2}(u).$$

Here $\varepsilon.u = \sum_{i=1}^n \varepsilon_i u_i e_i$.

Proof. If we heuristically write $X^{\#}_{\alpha,\beta}(t)$ as $\prod_{i=1}^n X_{\alpha_i,\beta_i}(t_i)$, then heuristically the total increment $\square_u X^{\#}_{\alpha,\beta}(t)$ can be written as the product of increments $\prod_{i=1}^n \Delta_{u_i} X_{\alpha_i,\beta_i}(t_i)$, and the result follows from Proposition 2.4.

For a more rigorous proof, notice that

$$\left\langle \square_{\varepsilon.u} X^{\#}_{\alpha,\beta}(t) \square_{\varepsilon.v} X^{\#}_{\alpha,\beta}(t) \right\rangle = \sum_{\delta \in \{0,1\}^n} \sum_{\eta \in \{0,1\}^n} (-1)^{\sum_{i=1}^n \delta_i + \sum_{i=1}^n \eta_i} C^{\#}_{\alpha,\beta} \left(\sum_{i=1}^n \varepsilon_i \left[\delta_i u_i - \eta_i v_i \right] e_i \right)$$

$$= \sum_{\delta \in \{0,1\}^n} \sum_{\eta \in \{0,1\}^n} (-1)^{\sum_{i=1}^n \delta_i + \sum_{i=1}^n \eta_i} \prod_{i=1}^n C_{\alpha_i,\beta_i} \left(\varepsilon_i \left[\delta_i u_i - \eta_i v_i \right] \right)$$

$$= \prod_{i=1}^{n} \left\{ C_{\alpha_i,\beta_i}(0) - C_{\alpha_i,\beta_i}(\varepsilon_i u_i) - C_{\alpha_i,\beta_i}(\varepsilon_i v_i) + C_{\alpha_i,\beta_i}(\varepsilon_i [u_i - v_i]) \right\}$$

$$= \prod_{i=1}^{n} \left\langle \Delta_{\varepsilon_i u_i} X_{\alpha_i,\beta_i}(t_i) \Delta_{\varepsilon_i v_i} X_{\alpha_i,\beta_i}(t_i) \right\rangle.$$

The result follows. □

From this proposition, we see that the total increment process can capture the locally multi-self-similar property of the GSGCC better. It also shows that locally, GSGCC behaves similarly as the fractional Brownian sheet. One would tend to use Propositions 4.2 and 2.6 to conclude that the Hausdorff dimension of the graph of GSGCC over a hyperrectangle is

$$d_{n,\alpha} := n + 1 - \frac{1}{2} \min\{\alpha_1, \ldots, \alpha_n\}.$$

However, Proposition 2.6 cannot be applied here since Proposition 4.2 does not imply that the fractal index of GSGCC is $\min \alpha$. For the fractional Brownian sheet $B^{\#}_{\alpha/2}$, Kamont [26] showed that the Hausdorff dimension of its graph is bounded above by $d_{n,\alpha}$. In [4], Ayache used wavelet method to show that the Hausdorff dimension of the graph of the fractional Brownian sheet $B^{\#}_{\alpha/2}$ is indeed equal to $d_{n,\alpha}$. This fact was proved again by Ayache and Xiao [6] as a special case of a more general result on Hausdorff dimension of fractional Brownian sheets from $\mathbb{R}^n$ to $\mathbb{R}^d$, using a different method. Therefore, it is natural for us to conjecture that the Hausdorff dimension of the graph of GSGCC is also $d_{n,\alpha}$. In fact, (4.3) implies that

$$\sigma^2_{X^{\#}_{\alpha,\beta}}(\tau) = \left\langle \left[X^{\#}_{\alpha,\beta}(t+\tau) - X^{\#}_{\alpha,\beta}(t) \right]^2 \right\rangle = \mathrm{O}\left(\|\tau\|^{\min \alpha} \right) \quad \text{as } \|\tau\| \to 0^+.$$

By a well-known theorem (see e.g. [1]), this implies that the Hausdorff dimension of the graph of GSGCC is bounded above by $d_{n,\alpha}$. On the other hand, it is easy to show that

Lemma 4.6. *Let $\mathcal{C} = \prod_{i=1}^{n} [a_i, b_i]$ be a hyperrectangle in $\mathbb{R}^n$. There exist constants c_1 and c_2 such that*

$$c_1 \sum_{i=1}^{n} |t_i - s_i|^{\alpha_i} \le \left\langle \left[X^{\#}_{\alpha,\beta}(t) - X^{\#}_{\alpha,\beta}(s) \right]^2 \right\rangle \le c_2 \sum_{i=1}^{n} |t_i - s_i|^{\alpha_i}, \tag{4.6}$$

for all $t, s \in \mathcal{C}$.

Then by adapting the proof in [51] for the lower bound on the Hausdorff dimension of general anisotropic Gaussian fields with stationary increments, it can be verified that the graph of GSGCC over a hyperrectangle has Hausdorff dimension equal to $d_{n,\alpha}$.

The condition for LRD can be generalized to random sheet and it becomes

$$\int_{\mathbb{R}^n_+} \left| C^{\#}_{\alpha,\beta}(\tau) \right| \mathrm{d}^n \tau = \infty. \tag{4.7}$$

Since

$$C^{\#}_{\alpha,\beta}(\tau) = \prod_{i=1}^{n} C_{\alpha_i,\beta_i}(\tau_i),$$

and Proposition 2.8 for $n = 1$ gives

$$\int_{\mathbb{R}^+} \left| C_{\alpha_i,\beta_i}(\tau_i) \right| d\tau_i = \infty \Longleftrightarrow 0 < \alpha_i \beta_i \leq 1,$$

we have

Proposition 4.7. *GSGCC is LRD if and only if for some $1 \leq i \leq n$, $0 < \alpha_i \beta_i \leq 1$; it is SRD if and only if $\alpha_i \beta_i > 1$ for all $1 \leq i \leq n$.*

In order to determine the asymptotic behavior for the spectral density of the generalized Cauchy sheet, we observe that the spectral density of GSGCC, which we denote by $S^{\#}_{\alpha,\beta}(\omega)$ can be expressed as products of spectral densities of one-dimensional isotropic GFGCC considered in Section 3. Namely,

$$S^{\#}_{\alpha,\beta}(\omega) = \prod_{i=1}^{n} S_{\alpha_i,\beta_i}(\omega_i). \tag{4.8}$$

Therefore, the high frequency and low frequency behaviors of the spectral density $S^{\#}_{\alpha,\beta}(\omega)$ can be obtained from Propositions 3.2 and 3.3 respectively.

Proposition 4.8. *The high frequency limit of the spectral density $S^{\#}_{\alpha,\beta}(\omega)$ is given by*

$$S^{\#}_{\alpha,\beta}(\omega) \sim \prod_{i=1}^{n} \mathcal{H}_i(\alpha_i, \beta_i; \omega_i), \quad \|\omega\| \to \infty,$$

where

$$\mathcal{H}_i(\alpha_i, \beta_i; \omega_i) = \frac{\beta_i}{\pi} \Gamma(\alpha_i + 1) \sin \frac{\pi \alpha_i}{2} |\omega_i|^{-\alpha_i - 1}, \quad \text{if } \alpha_i \in (0, 2);$$

and

$$\mathcal{H}_i(\alpha_i, \beta_i; \omega_i) = \frac{|\omega_i|^{\beta_i - 1}}{2^{\beta_i} \Gamma(\beta_i)} e^{-|\omega_i|}, \quad \text{if } \alpha_i = 2.$$

Proposition 4.9. *The low frequency limit of the spectral density $S^{\#}_{\alpha,\beta}(\omega)$ is given by*

$$S^{\#}_{\alpha,\beta}(\omega) \sim \prod_{i=1}^{n} \mathcal{L}_i(\alpha_i, \beta_i; \omega_i), \quad \|\omega\| \to 0^+,$$

where

$$\mathcal{L}_i(\alpha_i, \beta_i; \omega_i) = \frac{\Gamma(1 - \alpha_i \beta_i)}{\pi} \sin \frac{\pi \alpha_i \beta_i}{2} |\omega|^{\alpha_i \beta_i - 1}, \quad \text{if } \alpha_i \beta_1 < 1;$$

$$\mathcal{L}_i(\alpha_i, \beta_i; \omega_i) = \frac{1}{\pi} \left\{ \ln \frac{1}{|\omega_i|} - \beta_i (\psi(\beta_i) + \gamma) - \gamma \right\}, \quad \text{if } \alpha_i \beta_i = 1;$$

$$\mathcal{L}_i(\alpha_i, \beta_i; \omega_i) = \frac{1}{\pi} \frac{\Gamma\left(\frac{1}{\alpha_i}\right) \Gamma\left(\beta_i - \frac{1}{\alpha_i}\right)}{\alpha_i \Gamma(\beta_i)}, \quad \text{if } \alpha_i \beta_i > 1.$$

Here we have used the fact that $\Gamma(2z) = 2^{2z-1} \pi^{-\frac{1}{2}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right)$ ([22], no. 1 of 8.335), $\Gamma(z)\Gamma(1 - z) = \pi / \sin(\pi z)$ and $\psi(1/2) = -\gamma - 2 \ln 2$ ([22], no. 2 of 8.366).

5. Lamperti transformation of GFGCC and GSGCC

In his seminal paper [32], Lamperti introduced a transformation which provides a one to one correspondence between a self-similar process and a stationary process. For a stationary process $X(t)$, $t \in \mathbb{R}$, we let

$$Y(t) = t^H X(\ln t), \tag{5.1}$$

for $t \in \mathbb{R}^+$, $H > 0$, and $Y(0) = 0$ be its H-Lamperti transfrom. Then $Y(t)$ is an H-self-similar (H-ss) process. Conversely, if $\{Y(t), t \geq 0\}$ is H-ss with zero mean, then the inverse Lamperti transformation of $Y(t)$ defined by

$$X(t) = \mathrm{e}^{-Ht} Y(\mathrm{e}^t), \quad t \in \mathbb{R}, \tag{5.2}$$

is a stationary process.

There are two ways to define extensions of Lamperti transformation to $\mathbb{R}^n$ linking stationary random field to a self-similar random field. Given a stationary random field $X(t)$, $t \in \mathbb{R}^n$, the first way to define its Lamperti transformation is, given $H \in \mathbb{R}^+$, defined by

$$Y(t) = \|t\|^H X(\ln t_1, \ldots, \ln t_n), \tag{5.3}$$

for $t \in \mathbb{R}^n_+$. It is easy to show that

Proposition 5.1. *Let $Y(t)$, $t \in \mathbb{R}^n_+$ be the H-Lamperti transform of a stationary field $X(t)$, $t \in \mathbb{R}^n$ defined by (5.3). Then $Y(t)$ is a H-ss field.*

Proof. For any $c \in \mathbb{R}^+$, we have

$$\begin{aligned} Y(ct) &=_d \|ct\|^H X(\ln[ct_1], \ldots, \ln[ct_n]) \\ &=_d c^H \|t\|^H X(\ln t_1 + \ln c, \ldots, \ln t_n + \ln c) \\ &=_d c^H \|t\|^H X(\ln t_1, \ldots, \ln t_n) \\ &=_d c^H Y(t), \end{aligned}$$

i.e. $Y(t)$ is H-ss. □

The inverse to this Lamperti transformation transforms a random field $Y(t)$ to

$$X(t) = (\mathrm{e}^{2t_1} + \cdots + \mathrm{e}^{2t_n})^{-H/2} Y(\mathrm{e}^{t_1}, \ldots, \mathrm{e}^{t_n}) \tag{5.4}$$

for $t \in \mathbb{R}^n$. For a field $X(t)$, $t \in \mathbb{R}^n$ to be stationary, it must satisfy $X(t+u) =_d X(t)$ for all $u \in \mathbb{R}^n$. This is an n-parameter family of conditions. However, in proving that $Y(t)$ is H-ss, we only use one parameter family of these conditions, i.e. we only look at those u of the form $u_1 = \cdots = u_n$. Therefore we cannot expect that the inverse Lamperti transform (5.4) of a H-ss field is a stationary field. As an example, consider the Lévy fractional Brownian field with index H, $B_H(t)$, which is a H-ss field. Define its inverse H-Lamperti transform $Z(t)$ by (5.4). We find that its covariance is

$$\langle Z(t+\tau) Z(t) \rangle = \frac{1}{2} \left(\mathrm{e}^{2(t_1+\tau_1)} + \cdots + \mathrm{e}^{2(t_n+\tau_n)} \right)^{-H/2} \left(\mathrm{e}^{2t_1} + \cdots + \mathrm{e}^{2t_n} \right)^{-H/2}$$

$$\times\left\{\left(e^{2(t_1+\tau_1)}+\cdots+e^{2(t_n+\tau_n)}\right)^H+\left(e^{2t_1}+\cdots+e^{2t_n}\right)^H\right.$$
$$\left.-\left([e^{t_1+\tau_1}-e^{t_1}]^2+\cdots+[e^{t_n+\tau_n}-e^{t_n}]^2\right)^H\right\}.$$

It is easy to verify that for $n \geq 2$, this expression is not independent of $t \in \mathbb{R}^n$. Therefore, $Z(t)$ is not a stationary process.

To make full use of the n-parameter families of symmetry in a stationary process, there is another definition of Lamperti transformation introduced by [18]. Given a stationary process $X(t)$, $t \in \mathbb{R}^n$, and a multi-index $H \in \mathbb{R}^n_+$, the second H-Lamperti transform of $X(t)$, denoted by $\mathbb{Y}(t)$, is defined as

$$\mathbb{Y}(t) = t_1^{H_1} \dots t_n^{H_n} X(\ln t_1, \dots, \ln t_n) \tag{5.5}$$

for $t \in \mathbb{R}^n_+$. It was shown in [18] that

Proposition 5.2. *Let $\mathbb{Y}(t)$, $t \in \mathbb{R}^n_+$ be the H-Lamperti transform of a stationary field $X(t)$, $t \in \mathbb{R}^n$ defined by (5.5). Then $\mathbb{Y}(t)$ is a H-mss field.*

The inverse of this second Lamperti transformation transform a field $\mathbb{Y}(t)$, $t \in \mathbb{R}^n_+$ to $X(t)$, $t \in \mathbb{R}^n$, where

$$X(t) = e^{-\sum_{i=1}^n t_i H_i}\, \mathbb{Y}(e^{t_1}, \dots, e^{t_n}). \tag{5.6}$$

It has a nice property, namely, as was shown in [18]:

Proposition 5.3. *Let $X(t)$, $t \in \mathbb{R}^n$ be the inverse second H-Lamperti transform of a H-mss field $\mathbb{Y}(t)$, $t \in \mathbb{R}^n_+$ defined by (5.6). Then $X(t)$ is a stationary field.*

As a side remark, it can be shown that (5.5) is essentially the unique (up to some multiplicative constants) transformation that takes a stationary process to a H-mss. However, if for $c \in \mathbb{R}^n_+$, $\beta \in \mathbb{R}^+$, $f_{c,\beta}(t)$, $t \in \mathbb{R}^n_+$ is a function of the form

$$f_{c,\beta}(t) = \sum_{i=1}^n c_i t_i^{\beta},$$

then if $c[1], \dots, c[m] \in \mathbb{R}^n_+$, $\beta_1, \dots, \beta_m \in \mathbb{R}^+$ and $\gamma_1, \dots, \gamma_m \in \mathbb{R}^+$ are such that $\beta_1\gamma_1 + \cdots + \beta_m\gamma_m = H$, it is easy to verify that the transform

$$X(t) \mapsto \prod_{i=1}^m f_{c[i],\beta_i}(t)^{\gamma_i} X(\ln t_1, \dots, \ln t_n)$$

takes a stationary process to a H-ss process, but its inverse in general does not take a H-ss process to a stationary process. The first Lamperti transformation (5.3) corresponds to $m = 1$, $c[1]_1 = \cdots = c[1]_n = 1$, $\beta_1 = 2$ and $\gamma_1 = H/2$.

Returning to the GFGCC $X_{\alpha,\beta}(t)$ and the GSGCC $X^{\#}_{\alpha,\beta}(t)$, Proposition 5.1 shows that their first Lamperti transforms, $Y_{\alpha,\beta}(t)$ and $Y^{\#}_{\alpha,\beta}(t)$, are H-ss fields with covariances

$$\begin{aligned}\left\langle Y_{\alpha,\beta}(t)Y_{\alpha,\beta}(s)\right\rangle &= \|t\|^H\|s\|^H\left\langle X_{\alpha,\beta}(\ln t_1,\ldots,\ln t_n)X_{\alpha,\beta}(\ln s_1,\ldots,\ln s_n)\right\rangle\\ &= \|t\|^H\|s\|^H\left[1+\left(\sum_{i=1}^n(\ln t_i-\ln s_i)^2\right)^{\alpha/2}\right]^{-\beta},\end{aligned}$$

and

$$\left\langle Y^{\#}_{\alpha,\beta}(t)Y^{\#}_{\alpha,\beta}(s)\right\rangle = \|t\|^H\|s\|^H\prod_{i=1}^n\left[1+(\ln t_i-\ln s_i)^{\alpha_i}\right]^{-\beta_i}$$

respectively. On the other hand, Proposition 5.2 shows that their second Lamperti transforms, $\mathbb{Y}_{\alpha,\beta}(t)$ and $\mathbb{Y}^{\#}_{\alpha,\beta}(t)$, are H-mss fields with covariances

$$\left\langle \mathbb{Y}_{\alpha,\beta}(t)\mathbb{Y}_{\alpha,\beta}(s)\right\rangle = \prod_{i=1}^n t_i^{H_i}\prod_{i=1}^n s_i^{H_i}\left[1+\left(\sum_{i=1}^n(\ln t_i-\ln s_i)^2\right)^{\alpha/2}\right]^{-\beta},$$

and

$$\left\langle \mathbb{Y}^{\#}_{\alpha,\beta}(t)\mathbb{Y}^{\#}_{\alpha,\beta}(s)\right\rangle = \prod_{i=1}^n t_i^{H_i}s_i^{H_i}\left[1+(\ln t_i-\ln s_i)^{\alpha_i}\right]^{-\beta_i}$$

respectively.

Next we consider whether the LRD property of GFGCC and GSGCC is preserved under the Lamperti transformations. First we note that the LRD condition (2.9) can be extended to non-stationary field in the following way.

Definition 5.4. Let

$$R_X(t,t+\tau) = \frac{C(t,t+\tau)}{\sqrt{C(t+\tau,t+\tau)C(t,t)}} \tag{5.7}$$

be the correlation function of a non-stationary Gaussian field $X(t)$. Then the condition of LRD for the non-stationary Gaussian field $X(t)$ is given by

$$\int_{\mathbb{R}^n_+}|R_X(t,t+\tau)|\,\mathrm{d}^n\tau = \infty. \tag{5.8}$$

Proposition 5.5. A. *The H-ss Gaussian field $Y_{\alpha,\beta}(t)$ is LRD for all $\alpha\in(0,2]$ and $\beta>0$.*
B. *The H-ss Gaussian field $Y^{\#}_{\alpha,\beta}(t)$ is LRD for all $\alpha,\beta\in\mathbb{R}^n_+$ satisfying $\alpha_i\in(0,2]$ and $\beta_i>0$, $1\le i\le n$.*
C. *The H-mss Gaussian field $\mathbb{Y}_{\alpha,\beta}(t)$ is LRD for all $\alpha\in(0,2]$ and $\beta>0$.*
D. *The H-mss Gaussian field $\mathbb{Y}^{\#}_{\alpha,\beta}(t)$ is LRD for all $\alpha,\beta\in\mathbb{R}^n_+$ satisfying $\alpha_i\in(0,2]$ and $\beta_i>0$, $1\le i\le n$.*

Proof. It is easy to verify that both the processes $Y_{\alpha,\beta}(t)$ and $\mathbb{Y}_{\alpha,\beta}(t)$ have the same correlation function

$$R_{Y_{\alpha,\beta}}(t+\tau,t) = R_{\mathbb{Y}_{\alpha,\beta}}(t+\tau,t)$$
$$= \left[1+\left(\sum_{i=1}^{n}\left(\ln\left[1+\frac{\tau_i}{t_i}\right]\right)^2\right)^{\alpha/2}\right]^{-\beta}, \quad \tau\in\mathbb{R}_+^n, \tag{5.9}$$

which is independent of H. In order to show that these processes are LRD, we have by condition (5.8),

$$\begin{aligned}\int_{\mathbb{R}_+^n}\left|R_{Y_{\alpha,\beta}}(t+\tau,t)\right|\mathrm{d}^n\tau &= \int_{\mathbb{R}_+^n}\left[1+\left(\sum_{i=1}^{n}\left(\ln\left[1+\frac{\tau_i}{t_i}\right]\right)^2\right)^{\alpha/2}\right]^{-\beta}\mathrm{d}^n\tau\\ &= \left[\prod_{i=1}^{n}t_i\right]\int_{\mathbb{R}_+^n}\left[1+\left(\sum_{i=1}^{n}(\ln[1+u_i])^2\right)^{\alpha/2}\right]^{-\beta}\mathrm{d}^n u\\ &= \left[\prod_{i=1}^{n}t_i\right]\int_{\mathbb{R}_+^n}\left[1+\left(\sum_{i=1}^{n}v_i^2\right)^{\alpha/2}\right]^{-\beta}\left[\prod_{i=1}^{n}\mathrm{e}^{v_i}\right]\mathrm{d}^n v,\end{aligned}$$

where we have used the substitutions $u_i = \tau_i/t_i$ and $v_i = \ln(1+u_i)$. Clearly, when $\|v\|\to\infty$, the integrand also approaches ∞. Therefore, the last integral diverges for all α, β, which is the condition for $Y_{\alpha,\beta}(t)$ and $\mathbb{Y}_{\alpha,\beta}(t)$ to be LRD. The statements for $Y^{\#}_{\alpha,\beta}(t)$ and $\mathbb{Y}^{\#}_{\alpha,\beta}(t)$ are proved analogously. □

Note that the LRD property of GFGCC and GSGCC is preserved under the Lamperti transformations, but the SRD property is not preserved. Here we have an example that the application of Lamperti transformation to a LRD stationary process (in this case the GFGCC and GSGCC) gives a (multi)-self-similar process with LRD. Examples of Lamperti transformation encountered so far relate either two short memory processes (for example, Ornstein-Uhlenbeck process and Brownian motion), or between a SRD process and a LRD process (in the case of fBm and its inverse Lamperti transformed process). For examples, one can show that the inverse Lamperti transformation of fractional Brownian sheet with LRD property gives rise to a stationary random sheet with short range dependence, and the stationary field associated with the Lévy fractional Brownian field is a stationary field with SRD.

Recall that Lévy fractional Brownian field is essentially the only self-similar Gaussian field with stationary increments [46]. Similarly, one can show that fractional Brownian sheet is essentially the only multi-self-similar Gaussian random field that has stationary total increments. Hence $Y_{\alpha,\beta}(t)$, $Y^{\#}_{\alpha,\beta}(t)$, $\mathbb{Y}_{\alpha,\beta}(t)$ and $\mathbb{Y}^{\#}_{\alpha,\beta}(t)$, the self-similar and multi-self-similar fields associated with GFGCC and GSGCC, do not have stationary increments or total increments. There exists a weaker stationary property known as asymptotically locally stationarity, which requires the field to be stationary in the limit $\|\tau\|\to 0^+$.

Definition 5.6. A centered Gaussian random field $X(t)$ is said to have asymptotically locally stationary increment if and only if as $\|\tau\|\to 0^+$, the variance of its increment $\sigma_t^2(\tau) = \left\langle[\Delta_\tau X(t)]^2\right\rangle$ is independent of t. More precisely,

$$\sigma_t^2(\tau) = f(\tau) + g(t,\tau), \tag{5.10}$$

where $f(\tau)$ is independent of t, and for any fixed t, $g(t,\tau) = \mathrm{o}(f(\tau))$ as functions of

τ. Similarly, a centered Gaussian random field $X(t)$ is said to have asymptotically locally stationary total increment if and only if as $\|\tau\| \to 0^+$, the variance of its total increment $\hat{\sigma}_t^2(\tau) = \langle [\Box_\tau X(t)]^2 \rangle$ is independent of t.

For the random field $Y_{\alpha,\beta}(t)$, we have

$$\begin{aligned}\left\langle \left[\Delta_\tau Y_{\alpha,\beta}(t)\right]^2 \right\rangle &= \left\langle \left[Y_{\alpha,\beta}(t+\tau)\right]^2 \right\rangle + \left\langle \left[Y_{\alpha,\beta}(t)\right]^2 \right\rangle - 2\left\langle Y_{\alpha,\beta}(t+\tau) Y_{\alpha,\beta}(t)\right\rangle \\ &= \|t+\tau\|^{2H} + \|t\|^{2H} - 2\|t+\tau\|^H \|t\|^H \left[1 + \left(\sum_{i=1}^n \left(\ln\left[1 + \frac{\tau_i}{t_i}\right]\right)^2\right)^{\alpha/2}\right]^{-\beta}.\end{aligned}$$

Using

$$\begin{aligned}\|t+\tau\| = \sqrt{(t_1+\tau_1)^2 + \cdots + (t_n+\tau_n)^2} &= \|t\| \left(1 + \frac{2\sum_{i=1}^n t_i\tau_i}{\|t\|^2} + \mathrm{O}(\|\tau\|^2)\right)^{1/2} \\ &= \|t\| \left(1 + \frac{\sum_{i=1}^n t_i\tau_i}{\|t\|^2} + \mathrm{O}(\|\tau\|^2)\right)\end{aligned}$$

and

$$\begin{aligned}\left[1 + \left(\sum_{i=1}^n \left(\ln\left[1 + \frac{\tau_i}{t_i}\right]\right)^2\right)^{\alpha/2}\right]^{-\beta} &= \left[1 + \left(\sum_{i=1}^n \left(\frac{\tau_i}{t_i} + \mathrm{O}(\tau_i^2)\right)^2\right)^{\alpha/2}\right]^{-\beta} \\ &= \left[1 + \left(\sum_{i=1}^n \left[\frac{\tau_i}{t_i}\right]^2 + \mathrm{O}(\|\tau\|^3)\right)^{\alpha/2}\right]^{-\beta} \\ &= \left[1 + \left(\sum_{i=1}^n \left[\frac{\tau_i}{t_i}\right]^2\right)^{\alpha/2} + \mathrm{O}(\|\tau\|^{\alpha+1})\right]^{-\beta} \\ &= 1 - \beta\left(\sum_{i=1}^n \left[\frac{\tau_i}{t_i}\right]^2\right)^{\alpha/2} + \mathrm{O}\left(\|\tau\|^{\min\{2\alpha,\alpha+1\}}\right),\end{aligned}$$

we find that as $\|\tau\| \to 0^+$,

$$\begin{aligned}\left\langle \left[\Delta_\tau Y_{\alpha,\beta}(t)\right]^2 \right\rangle = \|t\|^{2H} &\left(1 + 2H\frac{\sum_{i=1}^n t_i\tau_i}{\|t\|^2}\right) + \|t\|^{2H} \\ &- 2\|t\|^{2H}\left(1 + H\frac{\sum_{i=1}^n t_i\tau_i}{\|t\|^2}\right)\left(1 - \beta\left(\sum_{i=1}^n \left[\frac{\tau_i}{t_i}\right]^2\right)^{\alpha/2}\right)\end{aligned}$$

$$+\mathrm{O}\left(\|\tau\|^{\min\{2,2\alpha,\alpha+1\}}\right)$$
$$= 2\beta\|t\|^{2H}\left(\sum_{i=1}^{n}\left[\frac{\tau_i}{t_i}\right]^2\right)^{\alpha/2}+\mathrm{O}\left(\|\tau\|^{\min\{2,2\alpha,\alpha+1\}}\right). \tag{5.11}$$

It is obvious that the leading term is not independent of t for any H and α, unless when $n = 1$ and $\alpha = 2H$. Therefore for $n \geq 2$, $Y_{\alpha,\beta}(t)$ does not have asymptotically locally stationary increments. Similarly, for the fields $\mathbb{Y}_{\alpha,\beta}(t)$, $Y^{\#}_{\alpha,\beta}(t)$ and $\mathbb{Y}^{\#}_{\alpha,\beta}(t)$, one can show similarly that as $\|\tau\| \to 0^+$,

$$\left\langle\left[\Delta_\tau \mathbb{Y}_{\alpha,\beta}(t)\right]^2\right\rangle = 2\beta t_1^{2H_1}\dots t_n^{2H_n}\left(\sum_{i=1}^{n}\left[\frac{\tau_i}{t_i}\right]^2\right)^{\alpha/2}+\mathrm{O}\left(\|\tau\|^{\min\{2,2\alpha,\alpha+1\}}\right), \tag{5.12}$$

$$\left\langle\left[\Delta_\tau Y^{\#}_{\alpha,\beta}(t)\right]^2\right\rangle = 2\|t\|^{2H}\sum_{i=1}^{n}\beta_i\left|\frac{\tau_i}{t_i}\right|^{\alpha_i}+\mathrm{O}\left(\|\tau\|^{\min\{2,2\alpha_i,\alpha_i+1\}}\right), \tag{5.13}$$

$$\left\langle\left[\Delta_\tau \mathbb{Y}^{\#}_{\alpha,\beta}(t)\right]^2\right\rangle = 2t_1^{2H_1}\dots t_n^{2H_n}\sum_{i=1}^{n}\beta_i\left|\frac{\tau_i}{t_i}\right|^{\alpha_i}+\mathrm{O}\left(\|\tau\|^{\min\{2,2\alpha_i,\alpha_i+1\}}\right). \tag{5.14}$$

Therefore for $n \geq 2$, none of the fields $\mathbb{Y}_{\alpha,\beta}(t)$, $Y^{\#}_{\alpha,\beta}(t)$ and $\mathbb{Y}^{\#}_{\alpha,\beta}(t)$ have asymptotically locally stationary increments. When $n = 1$, all the fields $Y_{\alpha,\beta}(t)$, $\mathbb{Y}_{\alpha,\beta}(t)$, $Y^{\#}_{\alpha,\beta}(t)$ and $\mathbb{Y}^{\#}_{\alpha,\beta}(t)$ are actually the same, and they have asymptotically locally stationary increment if and only if $\alpha = 2H$, in which case the variance of the increment $\sigma_t^2(\tau)$ behaves like

$$\sigma_t^2(\tau) \sim 2\beta|\tau|^{\alpha} \quad |\tau| \to 0.$$

Next we consider the total increments. Since increment is the same as total increment when $n = 1$, we only need to consider $n \geq 2$. Using similar computations as given above, one can verify that for $n \geq 2$, the fields $Y_{\alpha,\beta}(t)$, $\mathbb{Y}_{\alpha,\beta}(t)$ and $Y^{\#}_{\alpha,\beta}(t)$ do not have asymptotically locally stationary increments, but $\mathbb{Y}^{\#}_{\alpha,\beta}(t)$ have if $\alpha = 2H$. We show the computation of the latter case here. By definition,

$$\begin{aligned}
\left\langle\left[\Box_\tau \mathbb{Y}^{\#}_{\alpha,\beta}(t)\right]^2\right\rangle &= \sum_{\delta\in\{0,1\}^n}\sum_{\eta\in\{0,1\}^n}(-1)^{\sum_{i=1}^{n}\delta_i+\sum_{i=1}^{n}\eta_i} \\
&\quad\times\left\langle \mathbb{Y}^{\#}_{\alpha,\beta}\left(t+\sum_{i=1}^{n}\delta_i\tau_i e_i\right)\mathbb{Y}^{\#}_{\alpha,\beta}\left(t+\sum_{i=1}^{n}\eta_i\tau_i e_i\right)\right\rangle \\
&= \prod_{i=1}^{n}\sum_{(\delta_i,\eta_i)\in\{0,1\}^2}(-1)^{\delta_i+\eta_i}(t_i+\delta_i\tau_i)^{H_i}(t_i+\eta_i\tau_i)^{H_i}C_{\alpha_i,\beta_i}\left(\ln[t_i+\delta_i\tau_i],\ln[t_i+\eta_i\tau_i]\right) \\
&= \prod_{i=1}^{n}\left\{t_i^{2H_i}-2(t_i+\tau_i)^{H_i}t_i^{H_i}\left(1+\left|\ln\left[1+\frac{\tau_i}{t_i}\right]\right|^{\alpha_i}\right)^{-\beta_i}+(t_i+\tau_i)^{2H_i}\right\}
\end{aligned}$$

$$= \prod_{i=1}^{n} \left\{ 2\beta_i t_i^{2H_i} \left| \frac{\tau_i}{t_i} \right|^{\alpha_i} + \mathrm{O}\left(|\tau_i|^{\min\{2, 2\alpha_i, \alpha_i+1\}} \right) \right\}$$

$$= 2^n \prod_{i=1}^{n} \left\{ \beta_i t_i^{2H_i} \left| \frac{\tau_i}{t_i} \right|^{\alpha_i} \right\} + \mathrm{O}\left(\|\tau\|^{\sum_{i=1}^{n} \alpha_i + \delta} \right),$$

where $\delta = \min\{2 - \alpha_i, \alpha_i, 1\}_{i=1}^{n}$. If $\alpha = 2H$, then as $\|\tau\| \to 0^+$,

$$\left\langle \left[\Box_\tau \mathbb{Y}^{\#}_{\alpha,\beta}(t) \right]^2 \right\rangle \sim 2^n \prod_{i=1}^{n} [\beta_i |\tau_i|^{\alpha_i}].$$

Therefore, we have verified the following:

Proposition 5.7. *The total increments of $\mathbb{Y}^{\#}_{\alpha,\beta}(t)$ are asymptotically locally stationary if $\alpha = 2H$.*

Next we consider the tangent fields (Definition 2.3) of the Lamperti transforms of GFGCC $Y_{\alpha,\beta}(t)$ and $\mathbb{Y}_{\alpha,\beta}(t)$. We have

Proposition 5.8. A. *The field $Y_{\alpha,\beta}(t)$ is lass of order $\alpha/2$, with tangent field at $t \in \mathbb{R}^n_+$ being*

$$\sqrt{2\beta} \|t\|^H B_{\alpha/2} \left(\frac{u_1}{t_1}, \ldots, \frac{u_n}{t_n} \right).$$

B. *The field $\mathbb{Y}_{\alpha,\beta}(t)$ is lass of order $\alpha/2$, with tangent field at $t \in \mathbb{R}^n_+$ being*

$$\sqrt{2\beta} \left[\prod_{i=1}^{n} t_i^{H_i} \right] B_{\alpha/2} \left(\frac{u_1}{t_1}, \ldots, \frac{u_n}{t_n} \right).$$

Proof. Using the formula

$$\begin{aligned} &\left\langle \Delta_{\varepsilon u} Y_{\alpha,\beta}(t) \Delta_{\varepsilon v} Y_{\alpha,\beta}(t) \right\rangle \\ &\quad = \frac{1}{2} \left\{ \left\langle \left[\Delta_{\varepsilon u} Y_{\alpha,\beta}(t) \right]^2 \right\rangle + \left\langle \left[\Delta_{\varepsilon v} Y_{\alpha,\beta}(t) \right]^2 \right\rangle - \left\langle \left[\Delta_{\varepsilon(u-v)} Y_{\alpha,\beta}(t + \varepsilon v) \right]^2 \right\rangle \right\}, \end{aligned}$$

we obtain immediately from (5.11) that as $\varepsilon \to 0$,

$$\begin{aligned} &\left\langle \frac{\Delta_{\varepsilon u} Y_{\alpha,\beta}(t)}{\varepsilon^{\alpha/2}} \frac{\Delta_{\varepsilon v} Y_{\alpha,\beta}(t)}{\varepsilon^{\alpha/2}} \right\rangle \\ &\quad \sim \beta \|t\|^{2H} \left\{ \left(\sum_{i=1}^{n} \left[\frac{u_i}{t_i} \right]^2 \right)^{\alpha/2} + \left(\sum_{i=1}^{n} \left[\frac{v_i}{t_i} \right]^2 \right)^{\alpha/2} - \left(\sum_{i=1}^{n} \left[\frac{u_i - v_i}{t_i} \right]^2 \right)^{\alpha/2} \right\}. \end{aligned}$$

Notice that this is up to the factor $2\beta\|t\|^{2H}$, the covariance of the scaled Lévy Brownian field

$$\left\{ B_{\alpha/2} \left(\frac{u_1}{t_1}, \ldots, \frac{u_n}{t_n} \right) : u \in \mathbb{R}^n \right\}.$$

The statement for $\mathbb{Y}_{\alpha,\beta}(t)$ is proved similarly. $\Box$

In fact, one can also deduce from (5.11) and (5.12) that the local fractal index of the fields $Y_{\alpha,\beta}(t)$ and $\mathbb{Y}_{\alpha,\beta}(t)$ are both equal to $\alpha/2$, and the Hausdorff dimension of their graphs over a hyperrectangle is $n+1-\alpha/2$.

For the tangent fields of the Lamperti transforms of GSGCC $Y^{\#}_{\alpha,\beta}(t)$ and $\mathbb{Y}^{\#}_{\alpha,\beta}(t)$, we recall that given $\alpha \in (0,2]^n$, $\min\alpha = \min\{\alpha_i\}_{i=1}^n$, and we assume WLOG that $\min\alpha = \alpha_1 = \cdots = \alpha_{m_\alpha} < \alpha_{m_\alpha+1} \leq \cdots \leq \alpha_n$ for some $1 \leq m_\alpha \leq n$. Then as in Proposition 5.8, we can use (5.13) and (5.14) to show that

Proposition 5.9. A. *The field* $Y^{\#}_{\alpha,\beta}(t)$ *is lass of order* $\min\alpha/2$, *with tangent field at* $t \in \mathbb{R}^n_+$ *being*

$$\|t\|^H T_{\alpha,\beta}\left(\frac{u_1}{t_1},\ldots,\frac{u_n}{t_n}\right),$$

where the field $T_{\alpha,\beta}(u)$, $u \in \mathbb{R}^n$ *is defined in Proposition 4.2.*
B. *The field* $\mathbb{Y}^{\#}_{\alpha,\beta}(t)$ *is lass of order* $\min\alpha/2$, *with tangent field at* $t \in \mathbb{R}^n_+$ *being*

$$\left[\prod_{i=1}^n t_i^{H_i}\right] T_{\alpha,\beta}\left(\frac{u_1}{t_1},\ldots,\frac{u_n}{t_n}\right).$$

From Propositions 2.4, 5.8, 4.2 and 5.9, we find that the tangent fields of GFGCC and GSGCC are related to the tangent fields of their Lamperti transforms by some change of variable formulas. Namely, if $X(t)$ has tangent field $T(u)$ at $t \in \mathbb{R}^n$, then its first Lamperti transform $Y(s)$ has tangent field

$$\|s\|^H T\left(\frac{u_1}{s_1},\ldots,\frac{u_n}{s_n}\right)$$

at $s \in \mathbb{R}^n_+$, and its second Lamperti transform $\mathbb{Y}(s)$ has tangent field

$$\left[\prod_{i=1}^n s_i^{H_i}\right] T\left(\frac{u_1}{s_1},\ldots,\frac{u_n}{s_n}\right)$$

at $s \in \mathbb{R}^n_+$. We also notice that the order of self-similarity of the fields $Y_{\alpha,\beta}(t)$ and $Y^{\#}_{\alpha,\beta}(t)$, H, is in general different from their order of local asymptotic self-similarity, being $\alpha/2$ and $\min\alpha/2$ respectively.

Since the stochastic process $\mathbb{Y}^{\#}_{\alpha,\beta}(t)$ resembles the GSGCC $X^{\#}_{\alpha,\beta}(t)$ in the sense that both their covariances can be written as products of covariance of one-dimensional processes, we can consider the limit of the total increments of $\mathbb{Y}^{\#}_{\alpha,\beta}(t)$. It is easy to obtain as in Proposition 4.5, using the $n=1$ case of Proposition 5.8 that

Proposition 5.10.

$$\lim_{\varepsilon\to 0^+}\left\langle \frac{\Box_{\varepsilon.u}\mathbb{Y}^{\#}_{\alpha,\beta}(t)}{\prod_{i=1}^n \varepsilon_i^{\alpha_i/2}}\right\rangle_{u\in\mathbb{R}^n} =_d \left[\prod_{i=1}^n \sqrt{2\beta_i}\, t_i^{H_i-(\alpha_i/2)}\right] B^{\#}_{\alpha/2}(u).$$

Here $\varepsilon.u = \sum_{i=1}^n \varepsilon_i u_i e_i$.

6. Concluding remarks

We have studied some of the basic properties of GFGCC and GSGCC, and their associated Lamperti transforms. The asymptotic properties of the spectral densities of GFGCC and GSGCC are considered. We expect the separate characterization of fractal dimension and long range dependence for GFGCC and GSGCC will provide more flexibility in their applications to modeling various surfaces and images. In the one-dimensional case, GFGCC has been applied to model the Havriliak–Negami relaxation law [37]. The estimations of parameters for stationary Gaussian processes have been widely studied. Some of these estimations can be adapted for GFGCC and GSGCC, and they are crucial to the applications. Further applications in spatial-temporal processes are possible if GFGCC is extended and modified to a space-time field to include non-stationarity and anisotropy [40,31,9,19,47,20,12]. We hope to apply results obtained in this paper to model various physical systems such as thin film surfaces in semiconductors [42, 45], surface ocean waves [8], geological morphology [12], etc, in a future work.

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