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Stochastic Processes and their Applications 119 (2009) 1401–1415
stochastic processes and their applications
www.elsevier.com/locate/spa

On permanental processes

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Received 21 December 2007; received in revised form 27 June 2008; accepted 9 July 2008
Available online 16 July 2008

Abstract

Permanental processes can be viewed as a generalization of squared centered Gaussian processes. We analyze the connections of these processes with the local time process of general Markov processes. The obtained results are related to the notion of infinite divisibility.

MSC: 60J25; 60J55; 60G15; 60G55

Keywords: Gaussian process; Local times; Infinite divisibility; Permanental process

1. Introduction

Permanental processes can be viewed as a generalization of the squared centered Gaussian processes. Their Laplace transform is given by the power $(-\frac{1}{\alpha})$ of a determinant $(\alpha > 0)$ involving a kernel. Squared Gaussian processes correspond to the case of a symmetric kernel and $\alpha = 2$. The value of α is called the index of the permanental process. Permanental processes are so called because any joint moment of a permanental process is equal to a permanent. The problem of the existence of such processes has been solved by Vere-Jones [20]. This paper analyzes the connections between permanental processes and the local time process of general Markov processes.

This subject is based on the natural emergence of the permanental processes in the study of the local times of Markov processes. In the case of a symmetric Markov process, this presence has allowed the writing of the so-called "Isomorphism Theorems" connecting directly the law of

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2008.07.003

the local times to the law of a squared Gaussian process. The most famous one is the identity of Dynkin [3]. In Marcus and Rosen's recent book, these Isomorphism Theorems are the main tool for studying properties of the local time process of symmetric Markov processes. For example, Marcus and Rosen have established the fact that the local time process is jointly continuous in space and time iff the associated (squared) Gaussian process is continuous. For non-symmetric Markov processes, a permanental process is going to replace this squared Gaussian process, and identities similar to Dynkin's Isomorphism Theorem can then be written. We establish here two identities: one for the total accumulated local time of a transient Markov process, and a second one for recurrent Markov processes stopped at inverse local times (extending an identity proven in [10]).

The problem then is that of use of these identities. As an example of use, we show here that the total accumulated local time process of a transient Markov process is continuous in a given distance d almost surely if, and only if, the associated permanental process is continuous, in that distance, almost surely. But in the general case, unlike for squared Gaussian processes, the continuity of permanental processes is still unknown territory. We hope that the finite-dimensional absolute continuity relation between permanental processes and squared Gaussian processes (Proposition 2.6) will provide the key to understanding the pathwise behavior of permanental processes.

In previous works [5,9], we have shown that the property of infinite divisibility characterizes the squared Gaussian processes associated with symmetric Markov processes. Here we extend this characterization to the non-symmetric case. Namely we show that a permanental process is infinitely divisible iff it is associated with a Markov process. Further, we give its Lévy measure in terms of the law of the local times of the associated Markov processes.

The paper is organized as follows. In Section 2 we define permanental processes and treat some of their general properties such as existence, conditioning and absolute continuity. Section 3 deals with permanental processes associated with Markov processes. In Section 4 we establish a characterization of the infinitely divisible permanental processes. Section 5 contains the proof of the main results of Sections 2, 3 and 4. We end the paper by giving a translation of Shirai and Takahashi's conjecture on α-permanents in Section 6.

2. Existence, conditioning and absolute continuity

A permanental process with parameter set E is a positive process whose finite-dimensional Laplace transforms are given by a negative power of a determinant. To make this precise:

Definition 2.1. A real-valued positive process $(\psi_x, x \in E)$ is a permanental process if its finite-dimensional Laplace transforms satisfy, for every $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\mathbb{R}_+^n$ and every $(x_1, x_2, \ldots, x_n)$ in E^n,

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\alpha_i\psi_{x_i}\right\}\right] = |I + \alpha G|^{-1/\beta}, \tag{2.1}$$

where I is the $n \times n$ identity matrix, α is the diagonal matrix $\operatorname{diag}(\alpha_i)_{1\le i\le n}$, $G = (g(x_i, x_j))_{1\le i,j\le n}$ and β is a fixed positive number.

Such a process $(\psi_x, x \in E)$ is called a permanental process with kernel $(g(x, y), x, y \in E)$ and index β.

Let $(\eta_x)_{x\in E}$ be a centered Gaussian process indexed by E; then the process $(\eta_x^2)_{x\in E}$ satisfies Definition 2.1 with $\beta = 2$ and $g(x, y) = \mathbb{E}(\eta_x\eta_y)$.

Vere-Jones has established in [19] the theorem below which gives necessary and sufficient conditions on matrices $G = (g(x_i, x_j))_{1\le i,j\le n}$ and $\beta > 0$ for the existence of a corresponding permanental vector $(\psi_{x_i}, 1 \le i \le n)$. His theorem is based on the following definitions.

Definition 2.2. For any $n \times n$ matrix M

$$\det_\beta M = \sum_{\sigma\in\mathcal{S}_n} \beta^{n-\nu(\sigma)} \prod_{i=1}^{n} M_{i,\sigma(i)},$$

where $\mathcal{S}_n$ is the symmetric group of order n and $\nu(\sigma)$ is the number of cycles of σ.

Note that $\det_{-1} M = |M|$ and $\det_1 M = \mathrm{Per}(M)$. If M is a diagonal matrix, then $\det_\beta M = |M|$ for every β.

Definition 2.3. For $\beta > 0$, an $n \times n$ matrix M is said to be β-positive definite if for every multi-index $k = (k_1, k_2, \ldots, k_n)$ in $\mathbb{N}^n$

$$\det_\beta(M(k)) \ge 0,$$

where $M(k)$ denotes the $|k| \times |k|$ matrix (where $|k| = k_1 + k_2 + \cdots + k_n$) obtained from M by setting

$$M(k)_{i,j} = M_{p_i,p_j},$$

with $p_i = 1$ if $1 \le i \le k_1$, and $p_i = \ell$ if $k_{\ell-1} < i \le k_\ell$ with $\ell \ge 2$.

Theorem A (*Vere-Jones*). *A permanental vector $(\psi_{x_i}, 1 \le i \le n)$ corresponding to $G = (g(x_i, x_j))_{1\le i,j\le n}$ and index β exists if and only if:*

(1) *All the real, non-zero, eigenvalues of G are positive.*
(2) *For every $r > 0$, set $Q_r = G(I + rG)^{-1}$; then Q_r is β-positive definite.*

Note that (1) is equivalent to $|I + rG| > 0$ for every $r > 0$.

Here is our first result concerning these permanental processes. The proof is provided in Section 5.

Proposition 2.4. *Let $(g(x, y), (x, y) \in E \times E)$ be a real function on $E \times E$ such that there exists a point a in E with $g(x, a) = g(a, x) = 0$ for every x in E. For a fixed $\delta > 0$, assume that there exists a permanental process $(\psi_x, x \in E)$ with a kernel $G + \delta = (g(x, y) + \delta, (x, y) \in E \times E)$ and index $\beta > 0$. We then have*

$$\mathbb{E}\left[\exp\left\{-\frac{t}{2}\psi_a\right\}\right] = (1 + \delta t)^{-1/\beta}, \tag{2.2}$$

and for every n, every $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\mathbb{R}^n_+$, every $(x_1, x_2, \ldots, x_n)$ in E^n and every $r \ge 0$,

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\alpha_i\psi_{x_i}\right\}\middle|\,\psi_a = r\right] = |I + \alpha G|^{-1/\beta} \exp\left\{-\frac{1}{2} r 1^{\mathrm{t}}(I + \alpha G)^{-1}\alpha 1\right\}, \tag{2.3}$$

where I is the $n \times n$ identity matrix, α is the diagonal matrix $\mathrm{diag}(\alpha_i)_{1\le i\le n}$, $G = (g(x_i, x_j))_{1\le i,j\le n}$, 1 is the n-colomn vector of 1 's and 1^{t} is its transpose.

The existence of ψ is equivalent to Vere-Jones conditions (1) and (2) for $(G + \delta)$, for every n and every $x_1, x_2, \ldots, x_n$ in E. As a consequence of Proposition 2.4, we obtain, under the same assumptions, the following result which cannot be easily seen using just (1) and (2).

Corollary 2.5. *If there is a $\delta_0 > 0$ such that there exists a permanental process ψ with kernel $(G + \delta_0)$ and index $\beta > 0$, then for every $\delta \geq 0$ there exists a permanental process with kernel $(G+\delta)$ and index β. In particular, denoting by ϕ a permanental process with kernel G and index β, we obtain*

$$(\psi_x, x \in E \mid \psi_a = 0) \overset{\text{(law)}}{=} (\phi_x, x \in E).$$

In the case when G is positive definite and $\beta = 2$, permanental processes with kernel $G + \delta$ and index 2 exist for every $\delta \geq 0$ (see the remark below). Section 3 deals with a class of kernels G for which permanental processes with kernel $(G + \delta)$ and index β exist for every $\delta \geq 0$ and $\beta \geq 0$.

Remark 2.5.1. Proposition 2.4 implies the following property for ψ:

$$\begin{aligned}&(\psi_x, x \in E|\psi_a = r) + (\tilde{\psi}_x, x \in E|\tilde{\psi}_a = r')\\ &\quad\overset{\text{(law)}}{=} (\psi_x, x \in E|\psi_a = t) + (\tilde{\psi}_x, x \in E|\tilde{\psi}_a = t'),\end{aligned}$$

where $\tilde{\psi}$ an independent copy of ψ, and r, r', t, t' non-negative numbers satisfying $r+r' = t+t'$. This property is well known for when $(g(x, y), (x, y) \in E \times E)$ is symmetric and $\beta = 2$. Indeed, in that case $(\phi_x, x \in E)$ is a squared centered Gaussian process. More precisely there exists a centered Gaussian process $(\eta_x, x \in E)$ with covariance $(g(x, y), x, y \in E)$ such that $(\phi_x, x \in E) = (\eta_x^2, x \in E)$. One can always add a point a by setting $g(a, x) = g(x, a) = 0$; $(\eta_x, x \in E \cup \{a\})$ remains a centered Gaussian process. We then have $\psi \overset{\text{(law)}}{=} (\eta + N)^2$ where N is a centered Gaussian variable with a variance equal to δ, independent of η. This gives

$$(\psi_x, x \in E|\psi_a = r^2) \overset{\text{(law)}}{=} ((\eta_x + N)^2, x \in E|N = r) \overset{\text{(law)}}{=} ((\eta_x + r)^2, x \in E),$$

and for $\tilde{\eta}$, an independent copy of η,

$$(\eta + a)^2 + (\tilde{\eta} + b)^2 \overset{\text{(law)}}{=} (\eta + c)^2 + (\tilde{\eta} + d)^2,$$

for all a, b, c and d such that $a^2 + b^2 = c^2 + d^2$.

Although the Laplace transform of a permanental process looks somewhat like that of a squared Gaussian process, there is no result in the literature on the path behavior of these processes. The following proposition connects some permanental processes of index 2 with squared Gaussian processes and should provide the key to understanding their path behavior.

Proposition 2.6. *Let $(\psi_x, x \in E)$ be a permanental process with kernel $(g(x, y), (x, y) \in E \times E)$ and index 2. Assume that $(\frac{1}{2}(g(x, y) + g(y, x)), (x, y) \in E \times E)$ defines a positive definite kernel. Let $(\eta_x, x \in E)$ be a centered Gaussian process with this covariance. Let $\tilde{\eta}$ be an independent copy of η and $\tilde{\psi}$ an independent copy of ψ. For $x_1, x_2, \ldots, x_n$ in E, we set $G = (g(x_k, x_j))_{1\leq k,j\leq n}$, and let Λ be the complex vector $(\eta_{x_k} + i\tilde{\eta}_{x_k}, 1 \leq k \leq n)$. Then for any functional F on $\mathbb{R}^n$,*

$$\mathbb{E}[F(\psi_{x_k} + \tilde{\psi}_{x_k}, 1 \leq k \leq n)] = \frac{\mathbb{E}[\exp\{\frac{1}{2}\langle A\Lambda, \overline{\Lambda}\rangle\}F(\eta_{x_k}^2 + \tilde{\eta}_{x_k}^2, 1 \leq k \leq n)]}{\mathbb{E}[\exp\{\frac{1}{2}\langle A\Lambda, \overline{\Lambda}\rangle\}]}, \tag{2.4}$$

where the matrix A is defined by $A_{ij} = ((\frac{1}{2}(G + G^{\mathrm{t}}))^{-1} - G^{-1})_{ij}$.

Note that when the kernel G is symmetric, then the law of η^2 is equal to that of ψ and $A = 0$.

3. Permanental processes associated with Markov processes

We work with a transient Markov process $(X_t)_{t\geq 0}$ with a state space E. Denote by $(L_t^x, x \in E, t \geq 0)$ its local time process and by $(g(x, y), (x, y) \in E \times E)$ its potential density. We shall normalize the local time so that it satisfies $g(x, y) = \mathbb{E}_x(L_\infty^y)$. Let a be a point in E for which $g(a, a) > 0$. We define the probability $\tilde{\mathbb{P}}_a$ by

$$\tilde{\mathbb{P}}_a|_{\mathcal{F}_t} = \frac{g(X_t, a)}{g(a, a)} \mathbb{P}_a|_{\mathcal{F}_t},$$

where $\mathcal{F}_t$ is the σ-field generated by $(X_s, 0 \leq s \leq t)$, augmented as usual, and $\mathbb{P}_a$ the probability under which X starts at a. Under $\tilde{\mathbb{P}}_a$, the process X starts at a and is killed at its last visit to a. The expectation with respect to $\tilde{\mathbb{P}}_a$ is denoted by $\tilde{\mathbb{E}}_a$.

Theorem 3.1. *For every $\beta > 0$, there exists a positive process $(\psi_x, x \in E)$ such that for every $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\mathbb{R}_+^n$ and every $(x_1, x_2, \ldots, x_n)$ in E^n,*

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{i=1}^{n} \alpha_i \psi_{x_i}\right\}\right] = |I + \alpha G|^{-1/\beta}, \tag{3.1}$$

where I is the $n \times n$ identity matrix, α is the diagonal matrix $\mathrm{diag}(\alpha_i)_{1\leq i\leq n}$ and $G = (g(x_i, x_j))_{1\leq i,j\leq n}$.

In the case $\beta = 2$, we note that for every fixed $x \in E$, $\psi(x)$ has the law of a squared centered Normal variable with variance equal to $g(x, x)$. If moreover the potential density is symmetric, ψ is the square of a centered Gaussian process with covariance equal to $(g(x, y), x, y \in E)$. This has been already noted and exploited by many authors (Dynkin [3,4], Marcus and Rosen [15], Eisenbaum [5], Eisenbaum et al. [10], …). This Gaussian process is named the "Gaussian process associated with X".

The process ψ defined in Theorem 3.1 will be called the *permanental process with index β, associated with X*.

We will see in Section 4 that even when the potential density g is not symmetric, it might happen that the associated permanental process ψ with index 2 is the square of a centered Gaussian process.

From now on $(\psi_x, x \in E)$ will denote the permanental process with index 2 associated with the Markov process X. We assume that ψ is independent of X; this is always possible by defining ψ on a probability space unrelated to that of X. On this probability space, the expectation will be denoted by $\langle\ ;\ \rangle$. The existence of an associated permanental process with index β for every $\beta > 0$ implies immediately the infinite divisibility of ψ. Lemma 3.1 of [7] characterizes the property of infinite divisibility of positive processes as follows: for every $a \in E$ such that $g(a, a) > 0$, there exists a process $(l_x^{(a)}, x \in E)$ independent of ψ such that

$$\psi^{(a)} \overset{(\text{law})}{=} \psi + l^{(a)}, \tag{3.2}$$

where $(\psi_x^{(a)}, x \in E)$ denotes the process $(\psi_x, x \in E)$ under the probability $\frac{1}{\langle\psi_a\rangle}\langle\psi_a, .\rangle$.

The theorem below shows that for every a, $l^{(a)} \overset{(\text{law})}{=} L_\infty$ under $\tilde{\mathbb{P}}_a$. It provides the connection between the law of ψ, the permanental process with index $\beta = 2$ associated with X, and the law of $(L_\infty^x, x \in E)$.

Theorem 3.2. *For every $a \in E$ such that $g(a,a) > 0$, for every functional F on the space of measurable functions from E into $\mathbb{R}$, we have*

$$\tilde{\mathbb{P}}_a\left\langle F\left(L^x_\infty + \frac{1}{2}\psi_x; x \in E\right)\right\rangle = \left\langle \frac{\psi_a}{g(a,a)} F\left(\frac{1}{2}\psi_x; x \in E\right)\right\rangle. \tag{3.3}$$

We mention that a paper by Le Jan [14] posted on ArXiv refers to this identity.

The following corollary gives the Lévy measure of ψ.

Corollary 3.3. *The process ψ is an infinitely divisible process with a Lévy measure ν characterized by the following marginals:*

$$\begin{aligned}&\nu_{(\psi(a)/2,\psi(x_2)/2,\ldots\psi(x_n)/2)}(\mathrm{d}y_1, \mathrm{d}y_2, \ldots, \mathrm{d}y_n)\\ &\quad = \frac{g(a,a)}{2y_1}\tilde{\mathbb{P}}_a(L^a_\infty \in \mathrm{d}y_1, L^{x_i}_\infty \in \mathrm{d}y_i, 2 \le i \le n).\end{aligned}$$

Equivalently the Lévy measure of $\psi/2$ is equal to the law of $(L^x_\infty, x \in E)$ under $\frac{g(a,a)}{2L^a_\infty}\tilde{\mathbb{P}}_a$ for every $a \in E$ with $g(a,a) > 0$.

Theorem 3.2 and Corollary 3.3 provide interesting connections between the path properties of the process $(\psi_x, x \in E)$ and the path properties of the local time process. To state some of them, assume that (E,d) is a locally compact metric space. We see immediately, thanks to (3.3), that if $(\psi_x, x \in E)$ is d-continuous then $(L^x_\infty, x \in E)$ is d-continuous.

The next theorem follows directly from this last remark and from Corollary 3.3.

Theorem 3.4. *The local time process $(L^x_\infty, x \in E)$ is d-continuous iff $(\psi_x, x \in E)$ is d-continuous.*

When the Markov process X is symmetric, Theorem 3.4 implies that the associated Gaussian process is continuous, a result which has already been established by Marcus and Rosen [15]. Their proof is based on the Isomorphism Theorems but does not make use of the infinite divisibility of ψ.

We end this section by giving a version in the non-symmetric case of a theorem established in [10]. Assume that X is a recurrent Markov process with state space E and having a local time at each $x \in E$. For $a \in E$, define $T_a = \inf\{t \ge 0 : X_t = a\}$ and $\tau_r = \inf\{t \ge 0 : L^a_t > r\}$. Let S_θ be an exponential time with parameter θ, independent of X. Then X killed at T_a and X killed at τ_{S_θ} are both transient Markov processes. We denote by ϕ and ψ their respective associated permanental processes with index 2. We have the following identity for the process $(L^x_{\tau_r}, x \in E)$.

Corollary 3.5. *Let X be a recurrent Markov process. For $a \in E$ and every functional F on the space of measurable functions from E into $\mathbb{R}$,*

$$\mathbb{P}_a\left\langle F\left(L^x_{\tau_r} + \frac{1}{2}\phi_x; x \in E\right)\right\rangle = \left\langle F\left(\frac{1}{2}\psi_x; x \in E\right) | \psi_a = r\right\rangle. \tag{3.4}$$

Further, $((\psi_x; x \in E)|\psi_a = 0) \overset{(\mathrm{law})}{=} (\phi_x; x \in E)$.

4. Characterization of the infinitely divisible permanental processes

Similarly to what has been done in the symmetric case (see [9]), one might ask whether the property of infinite divisibility characterizes the associated permanental processes. The answer is affirmative. A permanental process is infinitely divisible if, and only if, it is associated with a Markov process. In particular, a squared Gaussian process is infinitely divisible if and only if it is a permanental process associated with a Markov process. This does not imply necessarily that the Gaussian process itself is associated with a Markov process; indeed, we have proved in [6] that this is a stronger property.

If a permanental process with index $\beta > 0$ is infinitely divisible then the permanental process with the same kernel and index 2 is infinitely divisible too. Hence from now on in this section we will take $\beta = 2$.

Towards proving the above result for the non-symmetric case, we need to extend the main tool, namely Bapat's criterion, to that case. This is the content of Lemma 4.2.

Definition 4.1. An $n \times n$ matrix A is an M-matrix if:
(i) $A_{ij} \leq 0$ for $i \neq j$;
(ii) A is non-singular and $A^{-1} \geq 0$ (i.e. $A^{-1}_{ij} \geq 0$ for every i, j).

Lemma 4.2. *Let $(G_{i,j}, 1 \leq i, j \leq n)$ be a real non-singular $n \times n$ matrix. There exists a positive infinitely divisible random vector $(\psi_1, \psi_2, \ldots, \psi_n)$ such that for every $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n_+$,*

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\alpha_i\psi_i\right\}\right] = |I + \alpha G|^{-1/2} \tag{4.1}$$

if, and only if, there exists a signature matrix S such that $SG^{-1}S$ is an M-matrix.

It follows from Lemma 4.2 that the real eigenvalues of a matrix G satisfying (4.1) must be positive.

Theorem 4.3. *Let $(G_{i,j}, 1 \leq i, j \leq n)$ be a real non-singular $n \times n$ matrix. There exists a positive infinitely divisible random vector $(\psi_1, \psi_2, \ldots, \psi_n)$ such that (4.1) holds if, and only if, for every (i, j),*

$$G(i, j) = d(i)g(i, j)d(j), \tag{4.2}$$

where d is a function on $\{1, 2, \ldots, n\}$ and g the potential density of a Markov process.

Remark 4.3.1. As has been noticed in [9], the property (4.2) is equivalent to the following property:

$$G(i, j) = D^{-1}(i)\tilde{g}(i, j)D(j) \tag{4.3}$$

where D is a positive function and $\tilde{g}$ the potential density of a Markov process. But then $|I + \alpha G| = |I + \alpha\tilde{g}|$, which means precisely that the vector ψ is a permanental vector associated with the Markov process with potential density $\tilde{g}$.

Under an assumption of continuity, the following theorem extends Theorem 4.3 from vectors to processes.

Theorem 4.4. *Let $(k(x, y), x, y \in E)$ be a jointly continuous function on $E \times E$ such that $k(x, x) > 0$ for every $x \in E$. There exists a positive infinitely divisible process $(\psi_x, x \in E)$ such that for every $(\alpha_1, \alpha_2, \dots, \alpha_n)$ in $\mathbb{R}_+^n$ and every $(x_1, x_2, \dots, x_n)$ in E^n,*

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\alpha_i\psi_{x_i}\right\}\right] = |I + \alpha K|^{-1/2},$$

where $K = (k(x_i, x_j))_{1\le i,j\le n}$, if, and only if,

$$k(x, y) = d(x)g(x, y)d(y), \tag{4.4}$$

where d is a positive function and g the potential density of a Markov process.

Similarly to the case of vectors, Remark 4.3.1 leads to the following corollary.

Corollary 4.5. *Let $(k(x, y), x, y \in E)$ be a jointly continuous function on $E \times E$ such that $k(x, x) > 0$ for every $x \in E$. Let $(\psi_x, x \in E)$ be a process such that for every $(\alpha_1, \alpha_2, \dots, \alpha_n)$ in $\mathbb{R}_+^n$ and every $(x_1, x_2, \dots, x_n)$ in E^n,*

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\alpha_i\psi_{x_i}\right\}\right] = |I + \alpha K|^{-1/2},$$

where $K = (k(x_i, x_j))_{1\le i,j\le n}$.

Then $(\psi_x, x \in E)$ is infinitely divisible if, and only if, it is associated with a Markov process.

5. Proofs of Sections 2, 3 and 4

Proof of Proposition 2.4. We note first that for any $\alpha = \mathrm{Diag}(\alpha_i, 1 \le i \le n)$,

$$(I + \alpha(G + \delta)) = (I + \alpha G)(I + \delta(I + \alpha G)^{-1}\alpha\mathbb{1})$$

where $\mathbb{1}$ denotes the $n \times n$ matrix with all its entries equal to 1. Hence

$$|I + \alpha(G + \delta)| = |I + \alpha G||I + D\mathbb{1}|$$

where D is the diagonal matrix such that $D_{ii} = \sum_{j=1}^{n}(\delta(I + \alpha G)^{-1}\alpha)_{ij}$. Now note that $|I + D\mathbb{1}| = 1 + \mathrm{Tr}(D) = 1 + 1^{\mathrm{t}}D1$, where 1^{t} is the vector $(1, 1, \dots 1)$ of $\mathbb{R}^n$ (see for example [20] identity (4) in Section 2). Consequently

$$|I + \alpha(G + \delta)| = |I + \alpha G|(1 + \delta 1^{\mathrm{t}}(I + \alpha G)^{-1}\alpha 1). \tag{5.1}$$

By assumption the term $|I + \alpha(G + \delta)|$ is positive for every α. Consider now the terms $|I + \alpha G|$ and $(1 + \delta 1^{\mathrm{t}}(I + \alpha G)^{-1}\alpha 1)$; they are both continuous in α and for $\inf_{1\le i\le n}|\alpha_i|$ small enough they are both positive. Thanks to (5.1), their product is always positive. Consequently they are both always positive. It follows that

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\alpha_i\psi_{x_i}\right\}\right] = |I + \alpha G|^{-1/\beta}(1 + \delta 1^{\mathrm{t}}(I + \alpha G)^{-1}\alpha 1)^{-1/\beta}. \tag{5.2}$$

Let N be a centered Gaussian variable with variance δ. The variable N^2 is always infinitely divisible; hence there exists a positive random variable Z with the Laplace transform given by

$$\mathbb{E}(\mathrm{e}^{-\frac{t}{2}Z}) = (1 + \delta t)^{-1/\beta} \quad \text{for every } t \ge 0. \tag{5.3}$$

Actually (5.3) extends to negative t such that $(1+\delta t)>0$. In particular, since $(1+\delta 1^{\mathrm{t}}(I+\alpha G)^{-1}\alpha 1)$ is positive, we have

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}(1^{\mathrm{t}}(I+\alpha G)^{-1}\alpha 1)Z\right\}\right]=(1+\delta 1^{\mathrm{t}}(I+\alpha G)^{-1}\alpha 1)^{-1/\beta}.$$

Hence (5.2) can be rewritten as follows:

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\alpha_i\psi_{x_i}\right\}\right]=|I+\alpha G|^{-1/\beta}\mathbb{E}\left[\exp\left\{-\frac{1}{2}(1^{\mathrm{t}}(I+\alpha G)^{-1}\alpha 1)Z\right\}\right].$$

Since this identity is true for every n, we can write it for $n+1$ with $x_{n+1}=a$. Set $\tilde{G}=(G(x_i,x_j))_{1\le i,j\le n+1}$ and $\tilde{\alpha}=\mathrm{diag}(\alpha_i)_{1\le i\le n+1}$. We then have $1^{\mathrm{t}}(I+\tilde{\alpha}\tilde{G})^{-1}\tilde{\alpha}1=\alpha_{n+1}+1^{\mathrm{t}}(I+\alpha G)^{-1}\alpha 1$, where 1^{t} and 1 denote without ambiguity vectors of $\mathbb{R}^{n+1}$ on the left hand side and of $\mathbb{R}^n$ on the right hand side. Hence we obtain

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{i=1}^{n+1}\alpha_i\psi_{x_i}\right\}\right]\\&\quad=|I+\alpha G|^{-1/\beta}\mathbb{E}\left[\exp\left\{-\frac{1}{2}(1^{\mathrm{t}}(I+\alpha G)^{-1}\alpha 1+\alpha_{n+1})Z\right\}\right].\end{aligned}\tag{5.4}$$

Note that $\mathbb{E}[\exp\{-\frac{1}{2}\alpha_{n+1}\psi_a\}]=\mathbb{E}[\exp\{-\frac{1}{2}\alpha_{n+1}Z\}]$. Developing then both sides of (5.4), we obtain

$$\begin{aligned}&\int_0^{+\infty}\mathbb{P}(\psi_a\in \mathrm{d}r)\mathrm{e}^{-\frac{1}{2}\alpha_{n+1}r}\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\alpha_i\psi_{x_i}\right\}|\psi_a=r\right]\\&\quad=|I+\alpha G|^{-1/\beta}\int_0^{+\infty}\mathbb{P}(\psi_a\in \mathrm{d}r)\mathrm{e}^{-\frac{1}{2}\alpha_{n+1}r}\exp\left\{-\frac{r}{2}(1^{\mathrm{t}}(I+\alpha G)^{-1}\alpha 1)\right\},\end{aligned}$$

which leads to (2.3) dr a.e. We now choose to define the conditional Laplace transform of $(\psi_{x_1},\ldots,\psi_{x_n})$ given $\psi_a=r$ for every $r\ge 0$ by the right hand side of (2.3). $\square$

Proof of Proposition 2.6. First note that by assumption $\tilde{G}=\frac{1}{2}(G+G^{\mathrm{t}})$ is positive definite and hence G is invertible. We have for any diagonal matrix α with non-negative entries

$$\begin{aligned}|I+(\alpha-A)\tilde{G}|&=|\tilde{G}||\tilde{G}^{-1}-A+\alpha|\\&=|\tilde{G}||\tilde{G}^{-1}-A||I+\alpha(\tilde{G}^{-1}-A)^{-1}|,\end{aligned}$$

which leads to

$$|I+(\alpha-A)\tilde{G}|=|\tilde{G}||G|^{-1}|I+\alpha G|.\tag{5.5}$$

Now since $\tilde{G}$ is positive definite one checks that $(I-\tilde{G}^{1/2}(\frac{A+A^{\mathrm{t}}}{2})\tilde{G}^{1/2})$ is also positive definite. Hence with standard arguments we obtain

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\langle(\alpha-A)\Lambda,\overline{\Lambda}\rangle\right\}\right]=|I+(\alpha-A)\tilde{G}|^{-1},$$

and in particular for $\alpha=0:\mathbb{E}[\exp\{\frac{1}{2}\langle A\Lambda,\overline{\Lambda}\rangle\}]=\frac{|G|}{|\tilde{G}|}$. Identity (2.4) now follows from (5.5).

Note that for the Gaussian vector $\eta=(\eta_{x_j})_{1\le j\le n}$, we have

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\langle(\alpha-A)\eta,\eta\rangle\right\}\right]=\left|I+\left(\alpha-\frac{1}{2}(A+A^{\mathrm{t}})\right)\tilde{G}\right|^{-1/2},$$

and hence (2.4), which connects $\psi + \tilde{\psi}$ to the norm of $(\eta, \tilde{\eta})$, cannot be reduced to a one-dimensional identity that would connect ψ to η^2. $\square$

Proof of Theorem 3.1. We use Theorem A of Vere-Jones recalled in Section 2. First note that, as in the symmetric case, G^{-1} is an M-matrix (see Definition 4.1 and [8]). By Assertion (D_{16}) in Chap. 6, p. 135 of Berman and Plemmons's book [2], all the real eigenvalues of G^{-1} are hence positive. This implies that the real eigenvalues of G are positive. Then, since the resolvent matrices $Q_\sigma = \sigma G(I + \sigma G)^{-1}$, $\sigma > 0$, have only non-negative entries, they are all β-positive definite for every $\beta > 0$. Hence ψ is well defined. $\square$

Proof of Theorem 3.2. Thanks to Theorem 3.1, we have for $(x_1, x_2, \dots, x_n) \in E^n$

$$\left\langle \exp\left\{-\frac{1}{2}\sum_{i=1}^n \alpha_i \psi_{x_i}\right\}\right\rangle = |I + \alpha G|^{-1/2}. \tag{5.6}$$

Differentiating (5.6) with respect to α_1 and setting $x_1 = a$, we obtain

$$\left\langle \psi_a \exp\left\{-\frac{1}{2}\sum_{i=1}^n \alpha_i \psi_{x_i}\right\}\right\rangle = \frac{\partial}{\partial \alpha_1}(|I + \alpha G|)|I + \alpha G|^{-3/2}. \tag{5.7}$$

Developing $|I + \alpha G|$ with respect to its first column gives

$$\begin{aligned} |I + \alpha G| &= (1 + \alpha_1 g(a, a))(I + \alpha G)^{11} - \alpha_1 g(x_2, a)(I + \alpha G)^{21} \\ &\quad + \alpha_1 g(x_3, a)(I + \alpha G)^{31} + \cdots + (-1)^{n+1} g(x_n, a)\alpha_1 (I + \alpha G)^{n1} \end{aligned}$$

and hence

$$\begin{aligned} \frac{\partial}{\partial \alpha_1}(|I + \alpha G|) &= g(a, a)(I + \alpha G)^{11} - g(x_2, a)(I + \alpha G)^{21} \\ &\quad + g(x_3, a)(I + \alpha G)^{31} + \cdots + (-1)^{n+1} g(x_n, a)(I + \alpha G)^{n1} \\ &= |V|, \end{aligned}$$

where the matrix $V = (V_{ij})_{1 \le i, j \le n}$ is defined by $V_{ij} = (I + \alpha G)_{ij}$ if $i \ne 1$ and $V_{1j} = g(x_j, a)$. Consequently (5.7) becomes

$$\left\langle \psi_a \exp\left\{-\frac{1}{2}\sum_{i=1}^n \alpha_i \psi_{x_i}\right\}\right\rangle = |V||I + \alpha G|^{-3/2}. \tag{5.8}$$

But as is well known (see for example Marcus and Rosen [16] Lemma 2.6.2)

$$\tilde{\mathbb{P}}_a\left(\exp\left\{-\sum_{i=1}^n \alpha_i L_\infty^{x_i}\right\}\right) = \frac{|V|}{g(a, a)|I + \alpha G|},$$

which, together with (5.6) and (5.8), gives Theorem 3.2. $\square$

Proof of Corollary 3.3. Set $\mu = \nu_{(\psi(a)/2, \psi(x_2)/2, \dots \psi(x_n)/2)}$; then

$$\left\langle \exp\left\{-\frac{1}{2}\sum_{i=1}^n \alpha_i \psi_{x_i}\right\}\right\rangle = \exp\left\{-\int_{\mathbb{R}^n}\left(1 - \mathrm{e}^{-\sum_{i=1}^n \alpha_i y_i}\right)\mu(\mathrm{d}y)\right\}.$$

For $x_1 = a$ we obtain, after taking the derivative with respect to α_1,

$$\left\langle \psi(a) \exp\left\{-\frac{1}{2}\sum_{i=1}^n \alpha_i \psi_{x_i}\right\}\right\rangle = \left\langle \exp\left\{-\frac{1}{2}\sum_{i=1}^n \alpha_i \psi_{x_i}\right\}\right\rangle \int_{\mathbb{R}^n} \left(2y_1 \mathrm{e}^{-\sum_{i=1}^n \alpha_i y_i}\right) \mu(\mathrm{d}y).$$

Consequently, by Theorem 3.2,

$$g(a,a)\tilde{\mathbb{P}}_a\left(\exp\left\{-\sum_{i=1}^n \alpha_i L_\infty^{x_i}\right\}\right) = \int_{\mathbb{R}^n} \left(2y_1 \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^n \alpha_i y_i}\right) \mu(\mathrm{d}y),$$

which leads to

$$\nu_{(\psi(a)/2,\psi(x_2)/2,\ldots,\psi(x_n)/2)}(\mathrm{d}y_1, \mathrm{d}y_2, \ldots, \mathrm{d}y_n) = \frac{g(a,a)}{2y_1}\tilde{\mathbb{P}}_a(L_\infty^a \in \mathrm{d}y_1, L_\infty^{x_i} \in \mathrm{d}y_i, 2 \le i \le n). \quad \square$$

Proof of Corollary 3.5. We denote by $g_{\tau_{S_\theta}}$ the potential densities of X killed at τ_{S_θ}, and by g_{T_a} the Green function of X killed at T_a. It has been proved in [8] that for every $x, y \in E$, $g_{\tau_{S_\theta}}(x,a) = g_{\tau_{S_\theta}}(a,x) = 1/\theta$ and $g_{\tau_{S_\theta}}(x,y) = g_{T_a}(x,y) + 1/\theta$. Hence for the process X killed at τ_{S_θ} : $\tilde{\mathbb{P}}_a = \mathbb{P}_a$, and by Theorem 3.2,

$$\mathbb{P}_a\left\langle F\left(L_{\tau_{S_\theta}}^x + \frac{1}{2}\psi_x; x \in E\right)\right\rangle = \left\langle \frac{\psi_a}{g_{\tau_{S_\theta}}(a,a)} F\left(\frac{1}{2}\psi_x; x \in E\right)\right\rangle, \tag{5.9}$$

with $\langle \exp\{-\frac{1}{2}\sum_{i=1}^n \alpha_i \psi_{x_i}\}\rangle = |I + \alpha G_{\tau_{S_\theta}}|^{-1/2}$ where $G_{\tau_{S_\theta}} = (g_{T_a}(x_i,x_j) + 1/\theta)_{1\le i,j\le n}$. We set $G_{T_a} = (g_{T_a}(x_i,x_j))_{1\le i,j\le n}$.

By the proof of Proposition 2.4 we can define a measurable function f on $\mathbb{R}_+^n \times E^n$ such that for any α and any $x = (x_1, x_2, \ldots, x_n) \in E^n$,

$$|I + \alpha G_{\tau_{S_\theta}}|^{-1/2} = |I + \alpha G_{T_a}|^{-1/2}\left(1 + \frac{1}{\theta} f(\alpha,x)\right)^{-1/2}. \tag{5.10}$$

Assuming that $x_1 = a$ and taking derivatives with respect to α_1, we obtain then

$$-1/2\left\langle \psi_a \exp\left\{-\frac{1}{2}\sum_{i=1}^n \alpha_i \psi_{x_i}\right\}\right\rangle = \frac{\partial}{\partial \alpha_1}\left(|I + \alpha G_{T_a}|^{-1/2}\left(1 + \frac{1}{\theta} f(\alpha,x)\right)^{-1/2}\right)$$
$$= -\frac{1}{2\theta}|I + \alpha G_{T_a}|^{-1/2} \frac{\partial f(\alpha,x)}{\partial \alpha_1}\left(1 + \frac{1}{\theta} f(\alpha,x)\right)^{-3/2},$$

where we have used the fact that $|I + \alpha G_{T_a}|$ does not depend on α_1. Since under $\mathbb{P}_a$ the process $(L_{\tau_r}^{\cdot}, r > 0)$ is a Lévy process (with values in function space), there exists a measurable function h on $\mathbb{R}_+^n \times E^n$ such that for every (α, x)

$$\mathbb{P}_a\left(\exp\left\{-\frac{1}{2}\sum_{i=1}^n \alpha_i L_{\tau_r}^{x_i}\right\}\right) = \mathrm{e}^{-h(\alpha,x)r}.$$

Hence (5.9) can be expressed as

$$\mathbb{E}\left(\mathrm{e}^{-h(\alpha,x)S_\theta}\right)|I+\alpha G_{T_a}|^{-1/2}\left(1+\frac{1}{\theta}f(\alpha,x)\right)^{-1/2}$$
$$=|I+\alpha G_{T_a}|^{-1/2}\frac{\partial f(\alpha,x)}{\partial\alpha_1}\left(1+\frac{1}{\theta}f(\alpha,x)\right)^{-3/2},$$

which is equivalent to

$$\left(1+\frac{1}{\theta}h(\alpha,x)\right)^{-1}=\frac{\partial f(\alpha,x)}{\partial\alpha_1}\left(1+\frac{1}{\theta}f(\alpha,x)\right)^{-1}.$$

Consequently, for every $\theta>0$, we have $\frac{\partial f(\alpha,x)}{\partial\alpha_1}=\frac{\theta+f(\alpha,x)}{\theta+h(\alpha,x)}$. Letting θ tend to ∞, we finally obtain $\frac{\partial f(\alpha,x)}{\partial\alpha_1}=1$ and $f(\alpha,x)=h(\alpha,x)$. We can now rewrite (5.10) as follows:

$$\left\langle\exp\left\{-\frac{1}{2}\sum_{i=1}^n\alpha_i\psi_{x_i}\right\}\right\rangle=\mathbb{P}_a\left\langle\exp\left\{-\sum_{i=1}^n\alpha_i\left(L^{x_i}_{\tau_{S_\theta}}+\frac{1}{2}\phi_{x_i}\right)\right\}\right\rangle.$$

Conditioning on both sides with the respective value at a of the processes, and recalling that, by its definition, $\phi_a=0$, we obtain

$$\left\langle\exp\left\{-\frac{1}{2}\sum_{i=1}^n\alpha_i\psi_{x_i}\right\}\middle|\psi_a=r\right\rangle=\mathbb{P}_a\left\langle\exp\left\{-\sum_{i=1}^n\alpha_i\left(L^{x_i}_{\tau_{S_\theta}}+\frac{1}{2}\phi_{x_i}\right)\right\}\middle|L^a_{\tau_{S_\theta}}=r\right\rangle,$$

which is equivalent to

$$\left\langle\exp\left\{-\frac{1}{2}\sum_{i=1}^n\alpha_i\psi_{x_i}\right\}\middle|\psi_a=r\right\rangle=\mathbb{P}_a\left\langle\exp\left\{-\sum_{i=1}^n\alpha_i\left(L^{x_i}_{\tau_r}+\frac{1}{2}\phi_{x_i}\right)\right\}\right\rangle.\quad\square$$

Proof of Lemma 4.2. First assume (4.1). For $a>0$, let $Q_a=aG(I+aG)^{-1}$ and for $U=\mathrm{diag}(u_i)_{1\le i\le n}$ with $|u_i|\le 1$, $1\le i\le n$, define $P_a(U)=|I-Q_a||I-UQ_a|^{-1}$.

For $\alpha=\mathrm{diag}(\alpha_1,\alpha_2,\dots,\alpha_n)$, we set $F(\alpha)=|I+\alpha G|^{-1}$.

If the function $F(\alpha)$ is the Laplace transform of an infinitely divisible vector $(\phi_1,\phi_2,\dots,\phi_n)$, then for every $a>0$, the function $P_a(U)$ is the probability generating function of an infinitely divisible vector. This has been used in the symmetric case by Griffiths [11] and Griffiths and Milne [12], but it is still true without the assumption of symmetry. Indeed, we have $P_a(U)=F(a(I-U))$, which can be rewritten as

$$P_a(U)=\mathbb{E}\left[\prod_{i=1}^n u_i^{N_i}\right],$$

where conditionally on $(\phi_1,\phi_2,\dots,\phi_n)$, $N_1,N_2,\dots,N_n$ are n independent Poisson variables with respective parameters $a\phi_1,a\phi_2,\dots,a\phi_n$.

Towards proving the necessity we use the following criterion that is due to Griffiths and Milne [12].

Theorem B. *Let Q be a $n\times n$ real matrix. The function $|I-Q||I-QU|^{-1}$ is an infinitely divisible probability generating function if and only if:*

(i) *the eigenvalues of Q are strictly bounded in modulus by* 1;
(ii) $Q_{ii}\ge 0$ *and* $Q_{ij}Q_{ji}\ge 0$, $i\ne j$, $i,j\in\{1,2,\dots,n\}$;

(iii) *for every $k \leq n$, for every subset $\{i_1, i_2, \ldots, i_k\}$ of k distinct indices from $\{1, 2, \ldots, n\}$,*

$$T_{i_1,i_2} T_{i_2,i_3} T_{i_{k-1},i_k} T_{i_k,i_1} \geq 0,$$

where $T = Q + Q^{\mathrm{t}}$ (Q^{t} denotes the transpose of Q).

We note that $P_a(U) = |I - Q_a||I - UQ_a|^{-1} = |I - Q_a^{\mathrm{t}}||I - Q_a^{\mathrm{t}} U|^{-1}$. We are going to use Theorem B for the matrix Q_a^{t}. We have

$$I - Q_a = \frac{1}{a}(a^{-1} I + G)^{-1}. \tag{5.11}$$

We can choose a large enough in order that for every (i, j), if $G_{ij}^{-1} \neq 0$, then G_{ij}^{-1} and $(a^{-1} I + G)_{ij}^{-1}$ have the same sign; if $G_{ij}^{-1} \neq 0$ and $G_{ji}^{-1} = 0$, then $(G_{ij}^{-1} + G_{ji}^{-1})$ and $(a^{-1} I + G)_{ij}^{-1} + (a^{-1} I + G^{\mathrm{t}})_{ji}^{-1}$ have the same sign.

Now thanks to Theorem B(ii), and (5.11),

$$G_{ij}^{-1} G_{ji}^{-1} \geq 0. \tag{5.12}$$

Making use of the argument of Bapat to prove Theorem 1 [1], which does not use symmetry, we know that there exists a signature matrix S such that the off-diagonal terms of the matrix $S(-T)S$ are all negative. Now

$$2I - T = \frac{1}{a}\{(a^{-1} I + G)^{-1} + (a^{-1} I + G^{\mathrm{t}})^{-1}\}. \tag{5.13}$$

With S as above, we have $(S(2I - T)S)_{ij} \leq 0$ for $i \neq j$ which, by (5.13), leads to

$$S(i)S(j)\{(a^{-1} I + G)_{ij}^{-1} + (a^{-1} I + G)_{ji}^{-1}\} \leq 0.$$

If $G_{ij}^{-1} G_{ji}^{-1} \neq 0$, then $S(i)S(j)(G_{ij}^{-1} + G_{ji}^{-1}) \leq 0$ and hence by (5.12), we obtain $S(i)S(j)G_{ij}^{-1} \leq 0$. If $G_{ij}^{-1} \neq 0$ and $G_{ji}^{-1} = 0$, then it follows that $S(i)S(j)G_{ij}^{-1} \leq 0$.

Consequently for every (i, j), $S(i)S(j)G_{ij}^{-1} \leq 0$. By (N_{38}) of Berman and Plimmons in [2], p.137, chap. 6, we finally obtain that $SG^{-1}S$ is an M-matrix.

Conversely, assume that there exists a signature matrix S such that $SG^{-1}S$ is an M-matrix. We can then reproduce the proof of Theorem 3.2 [9] to show that

$$S(i)G(i, j)S(j) = d(i)g(i, j)d(j),$$

where d is a non-negative deterministic function on $\{1, 2, \ldots, n\}$ and g is the potential density of a transient Markov process with a state space equal to $\{1, 2, \ldots, n\}$.

Using then Theorem 3.1 and Corollary 3.3, we conclude that (4.1) is satisfied. □

Proof of Theorem 4.3. By Theorem 3.1 and Corollary 3.3, the sufficiency of this condition is immediate. The necessity follows from the argument developed in [9] to establish Theorem 3.2. This argument is based on Bapat's criterion, which is valid for the non-symmetric case as well. □

Proof of Theorem 4.4. The sufficiency of condition (4.4) follows from Theorem 3.1 and Corollary 3.3. To prove the necessity, we can, thanks to Lemma 4.2, make use of the proof of Theorem 3.4 [9] which works similarly, since there was no use of symmetry in the proof. □

6. A conjecture on α-permanents

Here is Shirai and Takahashi's conjecture:

Let α *be in* $[0, 2]$. *Then* $\det_\alpha A$ *is non-negative for every non-negative definite square matrix* A.

Shirai and Takahashi have made substantial progress in the direction of proving the conjecture. We send the interested reader to the last section of their paper [17]. Shirai has communicated to us his last paper [18] on this conjecture, where he goes a little further. We also mention that in [13], the authors give an example of a 3×3 positive definite matrix K such that $\det_\alpha(K) < 0$ for $\alpha > 4$.

Actually we have checked that the conjecture is true for 3×3 positive definite matrices. We would like just to point out the fact that in view of the results of Vere-Jones, this conjecture has the following form, more appealing for probabilists:

For every centered Gaussian vector $(\eta_1, \eta_2, \ldots, \eta_n)$ *and for every* $\delta \geq 1$,

$$\left(\mathbb{E}\left[\exp\left\{-\sum_{i=1}^{n} z_i \eta_i^2\right\}\right]\right)^{\delta}$$

is a Laplace transform in $(z_1, z_2, \ldots, z_n)$.

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