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Large deviations in total variation of occupation measures of one-dimensional diffusions

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Abstract

For one-dimensional diffusion processes, we find an explicit necessary and sufficient condition for the large deviation principle of the occupation measures in the total variation and of local times in L^1.

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Keywords: Large deviations; Local times; Uniformly integrable

1. Introduction

Let $\{\Omega, \mathcal{F}, (X_t)_{t\geq 0}, \mathbb{P}_\nu\}$ be a one-dimensional diffusion process taking values in an open interval $I = (x_0, y_0)$ $(-\infty \leq x_0 < y_0 \leq +\infty)$, which is the weak solution of the stochastic differential equation (SDE in short):

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t + b(X_t)\mathrm{d}t \tag{1.1}$$

with initial distribution ν, where (B_t) is a standard Brownian motion, and

(H1) $\sigma(\cdot) > 0$, $b(\cdot) : I \to \mathbb{R}$ are locally bounded, measurable and $\sigma^{-1}(\cdot)$ is locally bounded (the so-called ellipticity).

Under (H1) and the non-explosion assumption, the SDE (1.1) has a unique weak solution (X_t) which is a Markov process with generator $\mathcal{L}$ given by

$$\mathcal{L} := \frac{1}{2}\sigma^2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x}.$$

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We are interested in the large deviation principle (LDP in short) of occupation (or empirical) measures of (X_t),

$$L_t(\mathrm{d}a) := \frac{1}{t}\int_0^t \delta_{X_s}(\mathrm{d}a)\mathrm{d}s$$

in the topology of total variation $\|\cdot\|_{TV}$.

Usually large deviations for occupation measures of Markov processes are studied in the weak convergence topology or the τ-topology, see Donsker–Varadhan [1], Deuschel–Stroock [2], Dembo–Zeitouni [3], Wu [4] etc. In general, the LDP for occupation measures of Markov processes does not hold with respect to (w.r.t. in short) the total variation topology. For instance, for any sequence of i.i.d.r.v. (X_n) valued in E of common law μ, the well known Sanov theorem says that the occupation measure $L_n = (1/n)\sum_{k=1}^n \delta_{X_k}$ satisfies the LDP w.r.t. the weak convergence or the τ-topology. On the other hand, the corresponding LDP w.r.t. the topology of total variation does not remain true once μ is diffuse, for the reason that $\|L_n - \mu\|_{TV} = 2, a.s.$ This observation means that the question raised above is new in the theory of large deviations for Markov processes, and is pertinent only in the case where $\|L_t - \mu\|_{TV} \to 0$ in probability. Two such situations are: either the state space of the Markov process is countable where the LDP w.r.t. $\|\cdot\|_{TV}$ is equivalent to that w.r.t. the weak convergence topology (in this discrete case if $\mu_n \to \mu$ weakly, then $\|\mu_n - \mu\|_{TV} \to 0$, i.e., the weak convergence topology and the $\|\cdot\|_{TV}$-topology on the space $M_1(E)$ of probability measures on the countable space E are the same), or it is one-dimensional: which is exactly the case studied in this paper.

Let l_t^a denote the local time for the diffusion process (X_t) at $a \in I$ until time t, which is given by the Tanaka formula [5]

$$l_t^a := |X_t - a| - |X_0 - a| - \int_0^t \mathrm{sgn}(X_s - a)\mathrm{d}X_s \tag{1.2}$$

where $\mathrm{sgn}(x) = 1_{x>0} - 1_{x\le 0}$. Since $\int_0^t f(X_s)\mathrm{d}[X]_s = \int_{\mathbb{R}} f(a) l_t^a \mathrm{d}a$, $\forall t > 0$, where f is any bounded Borel function and $[X]_t = \int_0^t \sigma^2(X_s)\mathrm{d}s$ is the quadratic variational process of the continuous semimartingale (X_t), we have

$$L_t(\mathrm{d}a) = \frac{l_t^a \mathrm{d}a}{t\sigma^2(a)}. \tag{1.3}$$

Thus the LDP of L_t in the topology of total variation is equivalent to the LDP of the local time $(l_t^a)_{a\in I}$ in $L^1(I, \sigma^{-2}(a)\mathrm{d}a)$. The study of large deviations of local times of the Brownian motion was initiated by Donsker–Varadhan [1,6]. The subject has been recently thoroughly studied by Bass and Chen [7] and Bass, Chen and Rosen [8] (and the references therein). On the other hand, a necessary and sufficient condition for the central limit theorem of functional type for empirical processes of one-dimensional diffusions was established by van der Vaart and van Zanten [9].

The main purpose of this short paper is to present a necessary and sufficient condition for the LDP of L_t in the total variation topology. It is organized as follows. The main result is stated in the next section. In the preparatory Section 3 we explain where our condition (2.9) comes from, by means of Wu's uniform integrability criterion, Muckenhoupt's generalized Hardy inequality and Chen's criterion for the compactness of transition kernel P_t of (X_t), and we present a key tool: Lemma 3.5, on the exponential tightness of l_t^a/t in $L^1(I, \sigma^{-2}(a)\mathrm{d}a)$. Finally we prove the main result in Section 4.

2. Main results

Introduce at first

$$s'(x) := \exp\left(-\int_c^x \frac{2b(y)}{\sigma^2(y)}\mathrm{d}y\right) \tag{2.1}$$

and

$$m'(x) := \frac{2}{\sigma^2(x)}\exp\left(\int_c^x \frac{2b(y)}{\sigma^2(y)}\mathrm{d}y\right), \tag{2.2}$$

the derivatives of the scale function and the speed function of Feller, respectively. Here $c \in I$ is some fixed point.

We assume that (X_t) is non-explosive, which, by Feller's criterion (see [10]), is equivalent to

$$\begin{aligned} &\int_c^{y_0} s'(x)\mathrm{d}x \int_0^x m'(y)\mathrm{d}y = +\infty,\\ &\int_{x_0}^c s'(x)\mathrm{d}x \int_x^0 m'(y)\mathrm{d}y = +\infty, \end{aligned} \tag{2.3}$$

and furthermore it is positive recurrent or equivalently (under (2.3))

$$Z := \int_I m'(x)\mathrm{d}x < +\infty. \tag{2.4}$$

In that case, the probability measure

$$\mu(\mathrm{d}x) := \frac{1}{Z}m'(x)\mathrm{d}x = \frac{1}{Z}\frac{2}{\sigma^2(x)}\exp\left(\int_c^x \frac{2b(y)}{\sigma^2(y)}\mathrm{d}y\right)\cdot \mathrm{d}x$$

is the unique invariant measure of (X_t) and is symmetric, i.e., $((X_t), \mathbb{P}_\mu)$ is reversible.

Throughout this paper, we suppose (H1), (2.3) and (2.4).

Let:

- $L^2(\mu) := L^2(I, \mu)$,
- $M_1(I)$: the space of all probability measures on I,
- $\nu(f) := \int f\mathrm{d}\nu$,
- $\mathcal{A}_{\mu,p}(L) := \{\nu \in M_1(I); \nu \ll \mu, \|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\|_{L^p(\mu)} \le L\}$.

Define the Dirichlet form

$$\begin{aligned} &\mathbb{D}(\mathcal{E}) = \left\{f \in \mathcal{AC}(x_0, y_0)\bigcap L^2(\mu); \int_{x_0}^{y_0} \sigma^2(x)(f'(x))^2\mathrm{d}\mu < +\infty\right\},\\ &\mathcal{E}(f, f) = \frac{1}{2}\int_{x_0}^{y_0} \sigma^2(x)(f'(x))^2\mathrm{d}\mu, \quad \forall f \in \mathbb{D}(\mathcal{E}), \end{aligned} \tag{2.5}$$

where $\mathcal{AC}(x_0, y_0)$ is the space of real absolutely continuous functions on (x_0, y_0). By the $L^1(\mu)$-uniqueness in [11] or [12], the space $C_0^\infty(I)$ of infinitely differentiable functions with compact support on I is a form core for $(\mathcal{E}, \mathbb{D}(\mathcal{E}))$, and $(\mathcal{E}, \mathbb{D}(\mathcal{E}))$ is associated with (X_t). More precisely, let $(\mathcal{L}, \mathbb{D}_2(\mathcal{L}))$ be the generator of the transition semigroup (P_t) of (X_t) on $L^2(\mu)$, then $\mathbb{D}(\mathcal{E}) = \mathbb{D}_2(\sqrt{-\mathcal{L}})$ and $\mathcal{E}(f, f) = \langle\sqrt{-\mathcal{L}}f, \sqrt{-\mathcal{L}}\rangle_\mu$.

In the symmetric case, the rate function governing the large deviations of L_t in the τ-topology is given by (see [1] under some absolute continuity and Feller assumption, and [13, Corollary B.11] in full generality)

$$J_{\mathcal{E}}(\nu) := \begin{cases} \mathcal{E}(\sqrt{f}, \sqrt{f}), & \text{if } \nu = f\mu,\ \sqrt{f} \in \mathbb{D}(\mathcal{E}) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.6}$$

Once $\nu = f\mu$ with $f > 0$ and $\sqrt{f} \in \mathbb{D}(\mathcal{E})$, then

$$J_{\mathcal{E}}(\nu) = \frac{1}{8}\int_E \sigma^2(x)\frac{(f'(x))^2}{f(x)}\mathrm{d}\mu(x)$$

which is, up to a factor $1/8$, the Fisher information of ν w.r.t. μ.

The main result of this paper is:

Theorem 2.1. *The following properties are equivalent:*

(a) *The occupation measure $L_t(\cdot)$ satisfies the LDP in $(M_1(I), \|\cdot\|_{TV})$, uniformly over initial measures $\nu \in \mathcal{A}_{\mu,p}(L)$ for any $L \geq 1$ and $p > 1$, with the rate function $J_{\mathcal{E}}$ given in (2.6). More precisely, $J_{\mathcal{E}}$ is inf-compact on $(M_1(I), \|\cdot\|_{TV})$ (i.e. the level-sets $[J_{\mathcal{E}} \leq a], a \geq 0$ are compact in $(M_1(I), \|\cdot\|_{TV})$) and for any measurable subset $A \subset M_1(I)$,*

$$\begin{aligned} -\inf_{\beta\in A^o} J_{\mathcal{E}}(\beta) &\leq \liminf_{t\to\infty}\frac{1}{t}\log \inf_{\nu\in\mathcal{A}_{\mu,p}(L)} \mathbb{P}_\nu(L_t(\cdot)\in A) \\ &\leq \limsup_{t\to\infty}\frac{1}{t}\log \sup_{\nu\in\mathcal{A}_{\mu,p}(L)} \mathbb{P}_\nu(L_t(\cdot)\in A) \leq -\inf_{\beta\in\bar{A}} J_{\mathcal{E}}(\beta), \end{aligned} \tag{2.7}$$

where A^o, $\bar{A}$ denote respectively the interior and the closure of A w.r.t. the total variation norm $\|\cdot\|_{TV}$.

(b) *The local time $l_t^{\cdot} = (a \to l_t^a)$ satisfies the LDP in $L^1(I, \sigma^{-2}(a)\mathrm{d}a)$, uniformly over initial measures $\nu \in \mathcal{A}_{\mu,p}(L)$ for any $L \geq 1$ and $p > 1$, with the rate function $I(\cdot|\mu) : L^1(I, \sigma^{-2}(a)\mathrm{d}a) \to [0, +\infty]$ given by*

$$I(f|\mu) := \begin{cases} \mathcal{E}\left(\sqrt{f}/\sigma, \sqrt{f}/\sigma\right), & \text{if } f \geq 0, \int_I \frac{f}{\sigma^2(a)}\mathrm{d}a = 1, \sqrt{f}/\sigma \in \mathbb{D}(\mathcal{E}) \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.8}$$

(c)

$$\begin{aligned} &\limsup_{x\to y_0}\sup_{y\geq x}\mu[y, y_0)\int_x^y s'(z)\mathrm{d}z = 0, \\ &\limsup_{x\to x_0}\sup_{y\leq x}\mu(x_0, y]\int_y^x s'(z)\mathrm{d}z = 0. \end{aligned} \tag{2.9}$$

Condition (2.9) comes from Muckenhoupt's generalized Hardy inequality and Chen's characterization of empty essential spectrum for $\mathcal{L}$ in $L^2(I, \mu)$, see Lemma 3.2. Intuitively it says that the diffusion comes back to a compact sub-interval J of I with an exponential rate which grows to infinity when J increases to the whole interval I (see Section 3 about the Dirichlet eigenvalue which is just the exponential rate).

For example, when $I = \mathbb{R}, \sigma(x) = 1$ and $b(x) = -\mathrm{sgn}(x)a|x|^{a-1}$ $(a > 0)$, the invariant measure is given by $\mu = \mathrm{e}^{-2|x|^a}\mathrm{d}x$, condition (2.9) is satisfied iff $a > 1$.

We now present two applications of this theorem.

Let $\mathcal{G}$ be any non-empty family of bounded and measurable functions on I such that

$$\sup_{g\in\mathcal{G}}\sup_{x\in I}|g(x)| \le C < +\infty \tag{2.10}$$

and $l^\infty(\mathcal{G})$ the space of bounded functions $F:\mathcal{G}\to\mathbb{R}$ on $\mathcal{G}$ equipped with the uniform norm $\|F\| := \sup_{g\in\mathcal{G}}|F(g)|$. For any $\nu\in M_1(E)$, $\nu^{\mathcal{G}}: g\to\nu(g)$ is an element of $l^\infty(\mathcal{G})$. The mapping $\nu\to\nu^{\mathcal{G}}$ is continuous from $(M_1(E),\|\cdot\|_{TV})$ to $(l^\infty(\mathcal{G}),\|\cdot\|_{\mathcal{G}})$. Then by the contraction principle we obtain the LDP for the empirical process $L_t^{\mathcal{G}} = (g\to L_t(g))_{g\in\mathcal{G}}$:

Corollary 2.2. *Assume (2.9). Then for any non-empty family $\mathcal{G}$ of bounded and measurable functions on I satisfying (2.10), $\mathbb{P}_\nu\left(L_t^{\mathcal{G}}\in\cdot\right)$ satisfies the LDP on $(l^\infty(\mathcal{G}),\|\cdot\|_{\mathcal{G}})$ uniformly over $\nu\in\mathcal{A}_{\mu,p}(L)$ (for any $p, L>1$) with the rate function given by*

$$J(F) = \inf\{J_{\mathcal{E}}(\nu);\ \nu\in M_1(I), \nu^{\mathcal{G}} = F\},\quad F\in l^\infty(\mathcal{G})$$

($\inf\emptyset := +\infty$).

Recall that the central limit theorem for $L_t^{\mathcal{G}}$ in $(l^\infty(\mathcal{G}),\|\cdot\|_{\mathcal{G}})$ was known under a much weaker condition than (2.9) but with some extra condition on $\mathcal{G}$, see van der Vaart and van Zanten [9].

Another application is about the kernel density estimator $f^*_{n,\varepsilon}(x)$ of $\mathrm{d}\mu(a)/\mathrm{d}a = m'(a)/Z$ given by

$$f^*_{n,\varepsilon}(x) := \frac{1}{n}\int_0^n \psi_\varepsilon(x-X_s)\mathrm{d}s = \int_{\mathbb{R}}\psi_\varepsilon(x-y)L_n(\mathrm{d}y)$$

where ψ is some fixed probability density function on $\mathbb{R}$, and $\psi_\varepsilon(x) := (1/\varepsilon)\psi(x/\varepsilon)$. Here $\varepsilon=\varepsilon_n\to 0$ is the bandwidth. This is a typical statistical problem when the sample path $(X_t)_{0\le t\le n}$ is observed but $a(x), b(x)$ are unknown.

Assume (2.9). Since the mapping $F_\varepsilon: g\to\psi_\varepsilon * g$ from $L^1(\mathbb{R},\mathrm{d}a)$ to $L^1(\mathbb{R},\mathrm{d}a)$ converges, as $\varepsilon\to 0+$, to the identity operator Id uniformly over any compact subsets of $L^1(\mathbb{R},\mathrm{d}a)$,

$$f^*_{n,\varepsilon} = F_\varepsilon\left(1_I(a)\frac{L_n(\mathrm{d}a)}{\mathrm{d}a}\right)$$

and $1_I(a)L_n(\mathrm{d}a)/\mathrm{d}a$ satisfies the LDP on $L^1(\mathbb{R},\mathrm{d}a)$ (by Theorem 2.1) and then it is exponentially tight, thus we have for any $\delta>0$ fixed and $\varepsilon=\varepsilon_n\to 0$,

$$\lim_{n\to\infty}\frac{1}{n}\log\sup_{\nu\in\mathcal{A}_{\mu,p}(L)}\mathbb{P}_\nu\left(\int_{\mathbb{R}}\left|f^*_{n,\varepsilon}(a) - 1_I(a)\frac{L_n(\mathrm{d}a)}{\mathrm{d}a}\right|\mathrm{d}a>\delta\right) = -\infty.$$

Consequently by the approximation lemma [3, Theorem 4.2.13], we get:

Corollary 2.3. *Assume (2.9). When the bandwidth $\varepsilon=\varepsilon_n\to 0$, then $\mathbb{P}_\nu(f^*_{n,\varepsilon}\in\cdot)$ satisfies the LDP on $L^1(\mathbb{R},\mathrm{d}a)$ uniformly over $\nu\in\mathcal{A}_{\mu,p}(L)$ (for any $p, L>1$), with the rate function*

$$J(f) = \begin{cases}\mathcal{E}(\sqrt{g},\sqrt{g}), & \text{if } f\ge 0, \int_{\mathbb{R}} f\mathrm{d}a = 1,\ f1_{\mathbb{R}\setminus I} = 0,\ \sqrt{g} := \sqrt{Zf/m'}\in\mathbb{D}(\mathcal{E})\\ +\infty, & \text{otherwise.}\end{cases}$$

This is completely different from the discrete time case: even for the $\mathbb{R}$-valued i.i.d. sample sequence, a necessary and sufficient condition for the consistency of the kernel density estimator $f^*_{n,\varepsilon}$ is that $\varepsilon_n n \to +\infty$ (due to Devroye [14]), i.e., $\varepsilon_n \gg 1/n$. Under the last condition on the bandwidth, we have obtained the weak LDP of $f^*_{n,\varepsilon}$ in $L^1(\mathbb{R}, \mathrm{d}a)$ for discrete time Markov chains which are either uniformly ergodic [15] or μ-symmetric with transition kernel uniformly integrable on $L^2(\mu)$ [16], but the LDP of $f^*_{n,\varepsilon}$ in $L^1(\mathbb{R}, \mathrm{d}a)$ is false in general even for the i.i.d. sequence [17]! In the point of view of the consistency of $f^*_{n,\varepsilon}$, this type of difference between discrete time and continuous time samples was already observed by Bosq et al. [18].

3. Several lemmas

3.1. Discussions around the necessary and sufficient condition (2.9)

We begin with Wu's uniform integrability criterion for the LDP of L_t in the τ-topology (for general reversible and ergodic Markov process (X_t)). A bounded operator P on $L^2(\mu)$ is said to be uniformly integrable if $\{(Pf)^2;\ \|f\|_2 \le 1\}$ is μ-uniformly integrable. The τ-topology on $M_1(I)$ is the weakest topology in which $\nu \to \nu(f)$ is continuous for all real Borel measurable and bounded functions f (written as $f \in b\mathcal{B}$).

Lemma 3.1 *([4, Corollary 5.5]). The following conditions are equivalent:*

(i) $\mathbb{P}_\nu(L_t \in \cdot)$ *satisfies the LDP w.r.t. the* τ*-topology* $\sigma(M_1(I), b\mathcal{B})$ *on* $M_1(I)$*, uniformly over* $\mathcal{A}_{\mu,p}(L)$ *for each* L, $p > 1$.
(ii) $\{f^2;\ \mu(f^2) + \mathcal{E}(f, f) \le 1\}$ *is* μ*-uniformly integrable.*
(iii) P_t $(t > 0)$ *is uniformly integrable on* $L^2(\mu)$.
(iv) P_t *is compact on* $L^2(\mu)$ *for each* $t > 0$.

Here the equivalence between (i), (ii) and (iii) holds without the absolute continuity, and their equivalence with (iv) holds since for $t > 0$, $P_t(x, \mathrm{d}y) \ll \mathrm{d}y \sim \mu(\mathrm{d}y)$ by our ellipticity condition in (H1).

Next let us see where condition (2.9) comes from. For every interval $I_0 \subset (x_0, y_0) = I$, consider the smallest Dirichlet eigenvalue on I_0,

$$\lambda_D(I_0) = \inf\{\mathcal{E}(f, f);\ f \in \mathbb{D}(\mathcal{E});\ f(x) = 0,\ \forall x \notin I_0\}, \tag{3.1}$$

and set $\lambda_D(x+) := \lambda_D([x, y_0))$ and $\lambda_D(x-) := \lambda_D((x_0, x])$ for every $x \in E$. Consider the Muckenhoupt constants

$$\begin{aligned} B(x+) &:= \sup_{y \ge x} \int_y^{y_0} m'(y)\mathrm{d}y \int_x^y s'(z)\mathrm{d}z, \\ B(x-) &:= \sup_{y \le x} \int_{x_0}^{x} m'(y)\mathrm{d}y \int_y^x s'(z)\mathrm{d}z. \end{aligned} \tag{3.2}$$

Muckenhoupt's lemma [19] says that

$$B(x\pm) \le \frac{1}{\lambda_D(x\pm)} \le 4B(x\pm). \tag{3.3}$$

Thus our condition (2.9) means that $\lim_{x\to y_0} B(x+) = \lim_{x\to x_0} B(x-) = 0$ or equivalently

$$\lim_{x\to y_0} \lambda_D(x+) = \lim_{x\to x_0} \lambda_D(x-) = +\infty. \tag{3.4}$$

This last property turns out to be equivalent to the compactness of the resolvent $(1-\mathcal{L})^{-1}$ or of $P_t, t>0$ in $L^2(\mu)$, as given in Chen [20]:

Lemma 3.2 (*[20]*). *The following properties are equivalent:*

(iv) *P_t is compact on $L^2(\mu)$ for each $t>0$ or $(1-\mathcal{L})^{-1}$ is compact in $L^2(\mu)$.*
(v) $\lim_{x\to y_0}\lambda_D(x+)=\lim_{x\to x_0}\lambda_D(x-)=+\infty$.
(vi) $\lim_{x\to y_0}B(x+)=\lim_{x\to x_0}B(x-)=0$, *i.e.*, (2.9).

3.2. Exponential tightness in $L^1(I,\nu)$

Hence under (2.9), L_t satisfies the LDP in $M_1(E)$ w.r.t. the τ-topology. For the passage from the LDP in the τ-topology to that in the $\|\cdot\|_{TV}$ topology, we shall require a general observation from large deviations.

Lemma 3.3. *Let (E,d) be a complete separable metric space and σ a regular Hausdorff topology weaker than d, and $(\mu_n)_{n\geq1}$ a family of probability measures on E. Assume that as $n\to\infty$, (μ_n) satisfies the LDP on E w.r.t. the σ-topology with the rate function I. If (μ_n) is exponentially tight on (E,d), i.e., for any $L>0$ there is some compact subset $K=K_L\subset(E,d)$,*

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_n\left(K^c\right)<-L$$

then (μ_n) satisfies the LDP with the rate function I on (E,d).

Proof. At first, we prove the lower bound, which means that for any non-empty open O of (E,d), we have to prove that

$$l(O):=\liminf_{n\to\infty}\frac{1}{n}\log\mu_n(O)\geq-\inf_O I.$$

We may assume that $\inf_O I<+\infty$. Let $L>\inf_O I$ and $K=K_L$ be the compact set in (E,d) specified by the exponential tightness condition. As K is also σ-compact, K^c is σ-open, then

$$-\inf_{K^c}I\leq l(K^c)<-L.$$

Since for every $F\subset K$, if F is d-closed, it is σ-compact then σ-closed, then every $G\subset K$, open in (K,d) is also open in (K,σ). Thus $O\bigcap K$ is open in (K,σ), i.e., there exists a σ-open $\tilde{O}$ such that $\tilde{O}\bigcap K=O\bigcap K$. Consequently by the lower bound in the σ-topology,

$$-\inf_{O\bigcap K}I\leq-\inf_{\tilde{O}}I\leq l(\tilde{O})\leq\max\left\{l\left(O\bigcap K\right),-L\right\}\leq\max\{l(O),-L\}.$$

Since $\inf_{K^c}I>L>\inf_O I$, we have $\inf_{O\bigcap K}I=\inf_O I$, so we get

$$-\inf_O I\leq\max\{l(O),-L\},$$

where the desired result follows by letting $L\to+\infty$.

Having the lower bound in the d-topology and the upper bound of large deviation for compacts of (E,d), the desired LDP follows from the exponential tightness. $\square$

In other words the key to Theorem 2.1 (c) $\Longrightarrow$ (a) is to prove the exponential tightness of $\mathbb{P}_\nu(l_t^\cdot/t\in\cdot)$ in $L^1(I,\sigma^{-2}(a)\mathrm{d}a)$. We begin with a lemma of independent interest:

Lemma 3.4. *Let (μ_n) be a sequence of probability measures on the complete separable metric space (E, d). If for any $L, \delta > 0$, there exists some compact $K = K_{L,\delta}$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(B(K,\delta)^c) < -L,$$

where $B(K,\delta) := \{y \in E; d(y, K) := \inf_{x\in E} \mathrm{d}(y, x) \le \delta\}$, then (μ_n) is exponentially tight.

Proof. Let $L > 0$ be fixed. For any integer $m \ge 1$, let $K_m = K_{mL,1/m}$ be the compact given in the lemma. Thus there exists some $N_m \ge 1$ such that for all $n \ge N(m)$,

$$\mu_n(B(K_m, 1/m)^c) \le \mathrm{e}^{-nmL}.$$

Since $(\mu_n, 1 \le n \le N_m)$ is tight, we can find compact $\tilde{K}_m \supset K_m$ so that the relation above with $\tilde{K}_m$ in place of K_m holds for all $n \ge 1$. Let

$$K = \bigcap_{m\ge1} B(\tilde{K}_m, 1/m).$$

It is closed and totally bounded, then compact in (E, d). Also for any $n \ge 1$,

$$\mu_n(K^c) \le \sum_{m=1}^{\infty} \mathrm{e}^{-nmL} = \frac{\mathrm{e}^{-nL}}{1 - \mathrm{e}^{-nL}},$$

where the desired exponential tightness follows. □

Let ψ be a C^∞ even probability density function on $\mathbb{R}$ with support contained in $[-1, 1]$ and $\psi_\varepsilon(x) := \frac{1}{\varepsilon}\psi(x/\varepsilon)$. Define

$$\mathcal{M}_\varepsilon f(x) := \int_{\mathbb{R}} f(y)\psi_\varepsilon(x-y)\mathrm{d}y = \psi_\varepsilon * f(x). \tag{3.5}$$

Note that $1_{[l,r]}\mathcal{M}_\varepsilon : L^1([l-\varepsilon, r+\varepsilon], \mathrm{d}x) \to L^1([l, r], \mathrm{d}x)$ is a compact operator: this follows from

$$\sup_{x\in[l,r]} |(\mathcal{M}_\varepsilon f)'(x)| = \sup_{x\in[l,r]} |\psi'_\varepsilon * f(x)| \le \sup_{x\in\mathbb{R}} |\psi'_\varepsilon(x)| \cdot \int |f|\mathrm{d}x$$

and the Arzelà–Ascoli theorem.

Lemma 3.5. *Let $h > 0$ be locally bounded and measurable on I such that $1/h$ is locally bounded on I. Let (μ_n) be a sequence of probability measures on $L^1(I, h\mathrm{d}a)$. Assume*

(i)

$$\lim_{N\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log \mu_n\left(f; \int_I |f|h\mathrm{d}a > N\right) = -\infty.$$

(ii) *for any $\delta > 0$*

$$\lim_{l\to x_0+, r\to y_0-} \limsup_{n\to\infty} \frac{1}{n} \log \mu_n\left(f; \int_{[l,r]^c} |f|h\mathrm{d}a > \delta\right) = -\infty.$$

(iii) *for any $\delta > 0$ and compact interval $[l, r] \subset (x_0, y_0)$,*

$$\lim_{\varepsilon\to0} \limsup_{n\to\infty} \frac{1}{n} \log \mu_n\left(f; \int_l^r |\mathcal{M}_\varepsilon f(a) - f(a)|\mathrm{d}a > \delta\right) = -\infty$$

where $\mathcal{M}_\varepsilon f(a)$, $a \in [l, r]$ is given in (3.5).

Then (μ_n) is exponentially tight in $L^1(I, h\mathrm{d}a)$.

Proof. For any fixed $L > 0$ and any $\delta > 0$, we use condition (i) to find some natural number N such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(B(0,N)^c) < -L,\ \ B(0,N) := \left\{ f \in L^1(I, h\mathrm{d}a);\ \int_I |f| h\mathrm{d}a \le N \right\};$$

using condition (ii) we choose $[l, r] \subset I$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(A^c) < -L, \quad A = \left\{ f \in L^1(I, h\mathrm{d}a);\ \int_{[l,r]^c} |f| h\mathrm{d}a \le \delta/2 \right\};$$

and finally using condition (iii) we can take $\varepsilon > 0$ such that $[l-\varepsilon, r+\varepsilon] \subset I$ and

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(C^c) < -L,$$

where $C := \left\{ f \in L^1(I, h\mathrm{d}a);\ \int_l^r |\mathcal{M}_\varepsilon f(a) - f(a)|\mathrm{d}a \le \frac{\delta}{2\|h 1_{[l,r]}\|_\infty} \right\}$. Consider

$$K := \text{the closure of } \{1_{[l,r]}\mathcal{M}_\varepsilon f;\ \ f \in B(0,N)\} \text{ in } L^1(I, h\mathrm{d}a).$$

Since the linear mapping $1_{[l,r]}\mathcal{M}_\varepsilon : L^1([l-\varepsilon, r+\varepsilon], \mathrm{d}a) \to L^1([l,r], \mathrm{d}a)$ is compact, using the local boundedness of h and h^{-1}, we see that K is compact in $L^1([l,r], \mathrm{d}a)$, and hence in $L^1(I, h\mathrm{d}a)$. For every $f \in B(0,N) \bigcap A \bigcap C$, $1_{[l,r]}\mathcal{M}_\varepsilon f \in K$, and

$$\begin{aligned}\int_I |1_{[l,r]}\mathcal{M}_\varepsilon f - f| h\mathrm{d}a &\le \int_{[l,r]^c} |f| h\mathrm{d}a + \int_l^r |\mathcal{M}_\varepsilon f(a) - f(a)| h(a)\mathrm{d}a \\ &\le \frac{\delta}{2} + \|h 1_{[l,r]}\|_\infty \frac{\delta}{2\|h 1_{[l,r]}\|_\infty} = \delta.\end{aligned}$$

Thus $B(0,N) \bigcap A \bigcap C \subset B(K, \delta)$, and therefore

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(B(K,\delta)^c) \le \limsup_{n\to\infty} \frac{1}{n} \log \left[\mu_n(B(0,N)^c) + \mu_n(A^c) + \mu_n(C^c)\right] < -L.$$

This implies the desired exponential tightness by Lemma 3.4. □

4. Proof of Theorem 2.1

(a) $\Longleftrightarrow$ (b). This has been already noticed in the Introduction; it follows at once by (1.3).
(a) $\Longrightarrow$ (c). As (a) is stronger than the LDP of L_t in the τ-topology in Lemma 3.1(i), this implication follows from Lemmas 3.1 and 3.2.

We now turn to the crucial part (c) $\Longrightarrow$ (a). The proof is divided into three steps.

Step 1. By Lemmas 3.1 and 3.2, under condition (2.9), the LDP of L_t in the τ-topology as stated in Lemma 3.1(i) holds true. Since the uniform LDP in (a) is equivalent to saying that $\mathbb{P}_{\nu_n}(L_n \in \cdot)$ satisfies the LDP on $(M_1(I), \|\cdot\|_{TV})$ for any sequence $(\nu_n) \subset \mathcal{A}_{\mu,p}(L)$ (here the passage from continuous time t to discrete time $t = n$ exists because $\|L_t - L_{[t]}\|_{TV} \le t^{-1} + (1 - [t]/t) \to 0$ as $t \to \infty$), by Lemma 3.3, it is enough to show that $\mathbb{P}_{\nu_n}(L_n \in \cdot)$ is exponentially tight in $(M_1(I), \|\cdot\|_{TV})$ or equivalently $\mathbb{P}_n(l_n^{\cdot}/n \in \cdot)$ is exponentially tight in $L^1(I, \sigma^{-2}(a)\mathrm{d}a)$ by (1.3). Since $\int_I (l_t^a/t)\sigma^{-2}(a)\mathrm{d}a = L_t(I) = 1$ and σ^2, σ^{-2} are locally bounded, by Lemma 3.5, it is enough to establish

$$\lim_{l\to x_0, r\to y_0} \limsup_{t\to\infty} \frac{1}{t} \sup_{\nu\in\mathcal{A}_{\mu,p}(L)} \log \mathbb{P}_\nu \left(\int_{[l,r]^c} l_t^a \sigma^{-2}(a)\mathrm{d}a > \delta \right) = -\infty, \quad \forall \delta > 0 \qquad (4.1)$$

and for any $x_0 < l < r < y_0$,

$$\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log\sup_{\nu\in\mathcal{A}_{\mu,p}(L)}\mathbb{P}_\nu\left(\frac{1}{t}\int_l^r |l_t^x - \mathcal{M}_\varepsilon l_t^{\cdot}(x)|\mathrm{d}x > \delta\right) = -\infty, \quad \forall\delta > 0, \tag{4.2}$$

where $\mathcal{M}_\varepsilon$ is the operator defined in (3.5).

Step 2. In this step we show (4.1). For any sequence of compact sub-intervals $[x_n, y_n]$ increasing to $I = (x_0, y_0)$, let $F_{n,\delta} = \{\nu \in M_1(E);\ \nu(I \setminus (x_n, y_n)) \geq \delta\}$, which is closed in $M_1(E)$ w.r.t. the weak convergence topology (weaker than the τ-topology). By the LDP of L_t in the weak convergence topology (a consequence of Lemma 3.1(i)) and (1.3) we have

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\sup_{\nu\in\mathcal{A}_{\mu,p}(L)}\log\mathbb{P}_\nu\left(\int_{[x,y]^c} l_t^a\sigma^{-2}(a)\mathrm{d}a > \delta\right)\\
&\quad\leq \limsup_{t\to\infty}\frac{1}{t}\sup_{\nu\in\mathcal{A}_{\mu,p}(L)}\log\mathbb{P}_\nu\left(L_t \in F_{n,\delta}\right)\\
&\quad\leq -\inf_{\nu\in F_{n,\delta}} J_{\mathcal{E}}(\nu).
\end{aligned}$$

Since $F_{n,\delta}$ decreases to the empty set as n goes to infinity, by the inf-compactness of $J_{\mathcal{E}}$ in the weak convergence topology (since it is so in the τ-topology by Lemma 3.1(i)),

$$\lim_{n\to\infty}\inf_{\nu\in F_{n,\delta}} J_{\mathcal{E}}(\nu) = +\infty,$$

where (4.1) follows.

Step 3. It remains to show (4.2). For any $z \in (-\varepsilon, \varepsilon)$ where $0 < \varepsilon < \varepsilon_0$, $[l - \varepsilon_0, r + \varepsilon_0] \subset (x_0, y_0)$ and $x \in [l, r]$, by the Tanaka formula we have,

$$\begin{aligned}
|l_t^x - \mathcal{M}_\varepsilon l_t^{\cdot}(x)| &= \Bigg|\left(|X_t - x| - |X_0 - x| - \int_0^t \mathrm{sgn}(X_s - x)\mathrm{d}X_s\right) - \int \psi_\varepsilon(z)\\
&\qquad\times\left(|X_t - (x+z)| - |X_0 - (x+z)| - \int_0^t \mathrm{sgn}(X_s - x - z)\mathrm{d}X_s\right)\mathrm{d}z\Bigg|\\
&\leq 2\varepsilon + 2\left|\int \psi_\varepsilon(z)\mathrm{d}z\int_0^t 1_{x-z^-\leq X_s\leq x+z^+}\mathrm{d}X_s\right|\\
&\leq 2\varepsilon + 2\left|\int \psi_\varepsilon(z)\mathrm{d}z\int_0^t 1_{x-z^-\leq X_s\leq x+z^+}\sigma(X_s)\mathrm{d}B_s\right|\\
&\qquad + 2\left|\int \psi_\varepsilon(z)\mathrm{d}z\int_0^t 1_{x-z^-\leq X_s\leq x+z^+}b(X_s)\mathrm{d}s\right|.
\end{aligned}\tag{4.3}$$

For the last term, since for $|z| < \varepsilon$, we have

$$\begin{aligned}
&\frac{1}{t}\int_l^r \mathrm{d}x\left|\int_0^t 1_{x-z^-\leq X_s\leq x+z^+}b(X_s)\mathrm{d}s\right|\\
&\quad\leq \frac{1}{t}\int_0^t |b(X_s)|1_{l-\varepsilon\leq X_s\leq r+\varepsilon}\mathrm{d}s\int_l^r 1_{X_s-z^+\leq x\leq X_s+z^-}\mathrm{d}x\\
&\quad\leq \frac{\varepsilon}{t}\int_0^t |b(X_s)|1_{l-\varepsilon\leq X_s\leq r+\varepsilon}\mathrm{d}s\\
&\quad\leq \varepsilon\sup_{x\in[l-\varepsilon_0, r+\varepsilon_0]}|b(x)|,
\end{aligned}$$

then for (4.2) it is enough to show (by (4.3)) that for all $\delta > 0$,

$$\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log\sup_{\nu\in\mathcal{A}_{\mu,p}(L)}\mathbb{P}_\nu\left(\frac{1}{t}\int_l^r \mathrm{d}x\int \psi_\varepsilon(z)|M_t(x,z)|\mathrm{d}z > \delta\right) = -\infty, \tag{4.4}$$

where

$$M_t(x,z) := \int_0^t 1_{x-z^-\le X_s\le x+z^+}\sigma(X_s)\mathrm{d}B_s.$$

Set

$$K_t(\varepsilon) = \frac{1}{t}\int_l^r \mathrm{d}x\int \psi_\varepsilon(z)|M_t(x,z)|\mathrm{d}z.$$

By Cheybechev's inequality, for all $\delta > 0$, $\lambda > 0$,

$$\mathbb{P}_\nu(K_t(\varepsilon) > \delta) \le \mathrm{e}^{-\lambda t\delta}\mathbb{E}^\nu\exp(\lambda t K_t(\varepsilon))$$

thus,

$$\frac{1}{t}\log\mathbb{P}_\nu(K_t(\varepsilon) > \delta) \le -\lambda\delta + \frac{1}{t}\log\mathbb{E}^\nu\exp(\lambda t K_t(\varepsilon)).$$

For the control of the last term, using the convexity of $\log\mathbb{E}\mathrm{e}^X$ in X and $\mathrm{e}^{|x|} \le \mathrm{e}^x + \mathrm{e}^{-x}$, we have,

$$\begin{aligned}
\frac{1}{t}\log\mathbb{E}^\nu\exp\{\lambda t K_t(\varepsilon)\} &= \frac{1}{t}\log\mathbb{E}^\nu\exp\left\{\frac{\lambda}{r-l}\int_l^r\mathrm{d}x\int\psi_\varepsilon(z)(r-l)|M_t(x,z)|\mathrm{d}z\right\}\\
&\le \frac{1}{r-l}\int_l^r\mathrm{d}x\int\psi_\varepsilon(z)\mathrm{d}z\left(\frac{1}{t}\log\mathbb{E}^\nu\exp\{\lambda(r-l)\,|M_t(x,z)|\}\right)\\
&\le \frac{1}{r-l}\int_l^r\mathrm{d}x\int\psi_\varepsilon(z)\mathrm{d}z\left(\frac{1}{t}\log\mathbb{E}^\nu\left(\mathrm{e}^{(r-l)\lambda M_t(x,z)} + \mathrm{e}^{-(r-l)\lambda M_t(x,z)}\right)\right). \qquad (4.5)
\end{aligned}$$

Noting that for any continuous local martingale (M_t), $\exp(2M_t - 2[M]_t)$ is a local martingale then a supermartingale, then by Cauchy–Schwartz,

$$\mathbb{E}\mathrm{e}^{M_t} \le \sqrt{\mathbb{E}\mathrm{e}^{2M_t-2[M]_t}}\sqrt{\mathbb{E}\mathrm{e}^{2[M]_t}} \le \sqrt{\mathbb{E}\mathrm{e}^{2[M]_t}} \le \mathbb{E}\mathrm{e}^{2[M]_t}.$$

As

$$[M(x,z)]_t = \int_0^t 1_{x-z^-\le X_s\le x+z^+}\sigma^2(X_s)\mathrm{d}s \le \|\sigma^2 1_{[l-\varepsilon_0,r+\varepsilon_0]}\|_\infty t,$$

we have for all $|z| < \varepsilon < \varepsilon_0$,

$$\begin{aligned}
&\mathbb{E}^\nu\left(\mathrm{e}^{(r-l)\lambda M_t(x,z)} + \mathrm{e}^{-(r-l)\lambda M_t(x,z)}\right)\\
&\quad\le 2\mathbb{E}^\nu\exp\left(2(r-l)^2\lambda^2\int_0^t 1_{x-\varepsilon\le X_s\le x+\varepsilon}\sigma^2(X_s)\mathrm{d}s\right)\\
&\quad\le 2\exp\left(2(r-l)^2\lambda^2\|\sigma^2 1_{[l-\varepsilon_0,r+\varepsilon_0]}\|_\infty t\right).
\end{aligned}$$

Substituting it into (4.5) we obtain by dominated convergence that for all $\delta, \lambda > 0$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{\nu\in\mathcal{A}_{\mu,p}(L)} \mathbb{P}_\nu(K_t(\varepsilon) > \delta) \le -\lambda\delta + \frac{1}{r-l}\int_l^r \mathrm{d}x\, \Lambda(V_{x,\varepsilon}),$$

where $V_{x,\varepsilon}(y) = 2(r-l)^2\lambda^2\sigma^2(y)1_{x-\varepsilon\le y\le x+\varepsilon}$ and

$$\Lambda(V_{x,\varepsilon}) := \limsup_{t\to\infty} \frac{1}{t} \log \sup_{\nu\in\mathcal{A}_{\mu,p}(L)} \mathbb{E}^\nu \exp\left(\int_0^t V_{x,\varepsilon}(X_s)\mathrm{d}s\right).$$

Since $\Lambda(V_{x,\varepsilon}) \le 2(r-l)^2\lambda^2\|\sigma^2 1_{[l-\varepsilon_0,r+\varepsilon_0]}\|_\infty$, by dominated convergence, for (4.4) we only have to show that for any $x \in [l, r]$ fixed and for any $\lambda > 0$,

$$\lim_{\varepsilon\to 0} \Lambda(V_{x,\varepsilon}) = 0. \tag{4.6}$$

By the Laplace principle and the LDP in the τ-topology in Lemma 3.1,

$$\Lambda(V_{x,\varepsilon}) = \sup_{\nu\in M_1(E)} \big(\nu(V_{x,\varepsilon}) - J_{\mathcal{E}}(\nu)\big) = -\inf_{\nu\in M_1(E)} f_\varepsilon(\nu),$$

where $f_\varepsilon(\nu) := J_{\mathcal{E}}(\nu) - \nu(V_{x,\varepsilon})$. Since $f_\varepsilon(\nu) \uparrow J_{\mathcal{E}}(\nu)$ as ε decreases to zero and f_ε is inf-compact on $(M_1(I), \tau)$, we have by an elementary analytic lemma

$$\lim_{\varepsilon\to 0} \inf_{\nu\in M_1(E)} f_\varepsilon(\nu) = \inf_{\nu\in M_1(E)} \lim_{\varepsilon\to 0} f_\varepsilon(\nu) = \inf_{\nu\in M_1(E)} J_{\mathcal{E}}(\nu) = 0$$

(for $J_{\mathcal{E}}(\mu) = 0$) which yields the desired (4.6).

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