

Explicit solutions to quadratic BSDEs and applications to utility maximization in multivariate affine stochastic volatility models

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Abstract

Over the past few years quadratic Backward Stochastic Differential Equations (BSDEs) have been a popular field of research. However there are only very few examples where explicit solutions for these equations are known. In this paper we consider a class of quadratic BSDEs involving affine processes and show that their solution can be reduced to solving a system of generalized Riccati ordinary differential equations. In other words we introduce a rich and flexible class of quadratic BSDEs which are analytically tractable, i.e. explicit up to the solution of an ODE. Our results also provide analytically tractable solutions to the problem of utility maximization and indifference pricing in multivariate affine stochastic volatility models. This generalizes univariate results of Kallsen and Muhle-Karbe (2010) and some results in the multivariate setting of Leippold and Trojani (2010) by establishing the full picture in the multivariate affine jump-diffusion setting. In particular we calculate the interesting quantity of the power utility indifference value of change of numeraire. Explicit examples in the Heston, Barndorff-Nielsen–Shephard and multivariate Heston setting are calculated.

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1. Introduction

Since Bismut [9] introduced linear BSDEs in the context of Pontryagin's maximum principle, they have been intensively studied. Their popularity stems from the fact that they can be applied to many different areas, e.g. in the study of properties of partial differential equations (PDEs), see Briand and Confortola [11] and N'Zi et al. [46]. BSDEs also appear in many fields of mathematical finance, see El Karoui et al. [27] or more recently El Karoui and Hamadène [26] for an overview. Problems such as pricing and hedging of European options (compare [27]), stochastic recursive utility (see [24]), utility maximization problems (e.g. [34]) and risk measures (e.g. [5,48]) have been tackled using BSDE techniques.

For linear BSDEs it is possible under certain integrability and boundedness conditions to describe the first component of a solution as conditional expectation, compare [27, Section 2]. This is already not possible anymore in the Lipschitz case and therefore solutions and their properties can mostly be studied numerically. In the meantime there is a huge literature on numerics for Lipschitz BSDEs, see [7,10,28] amongst many others. The present work focuses on BSDEs with drivers of quadratic growth which were first investigated by Kobylanski [39] in a Brownian setting and later extended to a continuous martingale setting by Morlais in [43]. Imposing certain growth and Lipschitz conditions on the generator and assuming bounded terminal conditions existence and uniqueness of quadratic growth BSDEs are guaranteed. However there exists much less research on numerics for quadratic BSDEs (see [35,49]) and only very few examples where an explicit solution is known.

Motivated by this lack of examples and by the most important applications of this theory we analyze conditions under which one can find explicit solutions to a class of quadratic growth BSDEs. Our setting is as follows. We consider a forward affine process valued in S_d^+ , the cone of positive semidefinite $d \times d$ matrices, and analyze BSDEs whose terminal condition and generator depend on this forward process. The main question from a BSDE-point of view addressed here is: Which structural conditions on the terminal condition and the generator are needed (e.g. linear, affine, quadratic) to allow us to solve the BSDE explicitly?

We have chosen the forward process to be an affine process X on S_d^+ . As Kallsen and Muhle-Karbe [38] in the univariate case we can relate systems with S_d^+ -valued forward processes to multivariate, realistically modeled utility optimization problems. Notice that affine processes have found a growing interest in the literature due to their analytic tractability which stems from the affine transform formula

$$\mathbb{E}[\exp(-\text{Tr}(X_t u))] = \exp(-\phi(t, u) - \text{Tr}(\psi(t, u)X_0)),$$

for all $t \in [0, T]$ and $u \in S_d^+$. The functions ϕ and ψ solve a system of generalized Riccati ordinary differential equations (ODEs), which are specified via the model parameters. Affine processes have been applied in various fields in mathematical finance such as the theory of term structure of interest rates, option pricing in stochastic volatility models and credit risk, see e.g. [2, 16,22,23] and the references therein. Note that it is not necessary to look at matrix-valued affine processes as we do, but we could have equally chosen $\mathbb{R}_+^n \times \mathbb{R}^m$, which was characterized by Duffie et al. [25], or $\mathbb{R}^n \times S_d^+$. For the sake of presentation we chose S_d^+ since this state space is complicated enough to make important pitfalls visible (e.g. no infinite divisibility, no polyhedral property, etc., see [18]), but still allows for simple notation. We emphasize that most of our results carry over to the general state space case.

The forward–backward system we consider consists of an affine process X on S_d^+ and a BSDE whose terminal condition is an affine function of this process. Moreover the generator is allowed

to have a more involved structure including a dependence on X and a quadratic dependence in the control process Z . In [Theorem 3.5](#) we carry out which analytic form the generator and the terminal value of the BSDE need for the solution to be determined by a matrix ODE. This ODE is a generalized Riccati ODE which may explode in finite time because of its quadratic term. Therefore it is necessary to find conditions such that the ODE possesses a unique finite solution on the whole time interval $[0, T]$.

We apply our results to the problem of maximizing expected utility of terminal wealth in multivariate affine stochastic volatility models. This problem is typically approached either by stochastic control methods leading to Hamilton–Jacobi–Bellman equations or martingale methods which we will use here. The martingale method has also been used in [\[38\]](#), where the authors were able to solve the power utility optimization problem in several univariate affine models. They obtain the solution using semimartingale characteristics and represent the optimal strategies in terms of an opportunity process. Using a combination of martingale methods and our results on explicit solutions of BSDEs we extend these ideas to higher dimensions. In particular we derive explicit results for power and exponential utility in multivariate extensions of the model of Heston [\[32\]](#) and the model of Barndorff-Nielsen and Shephard [\[2\]](#). The multivariate results presented here are mostly new and provide a thorough extension of Fonseca et al. [\[19\]](#), where the power utility case in the multivariate Heston model is treated. We also want to mention the work of Leippold and Trojani [\[40\]](#) where in a multivariate setting the optimal strategy and the value function for power utility maximization is given. Their considerations are justified in our general affine setting. Note that Baeuerle and Li [\[1\]](#) solve the portfolio problem in a Wishart process setting for power and logarithmic utility via HJB equations and their verification.

A particularly interesting application of our findings is the following: in the case of exponential utility we are able to provide analytic expressions for the indifference prices of variance swaps, which is well-known and can also be found in the literature. In the case of power utility – due to the additive structure of indifference prices – one cannot find tractable expressions of those prices. However, the equally interesting concept of *indifference value of change of numeraire* is again analytically tractable. The indifference value of change of numeraire is the price one is willing to pay to swap one numeraire with another one. This can have two applications: one is the case where an institution actually bases their portfolio optimization, e.g., on fixed interest rates, even though interest rates are floating. The indifference value of change of numeraire is consequently the value of a swap contract particularly designed for compensating this model misspecification. The second one is a foreign exchange situation where replacing one numeraire by another one influences the optimal portfolio problem and therefore leads to an indifference value. We can provide fully tractable formulas in all these cases, see [Section 4.2](#).

In [Section 2](#) we introduce necessary notation and collect several results by Cuchiero et al. [\[18\]](#) who give a complete characterization of affine processes on S_d^+ . [Section 3](#) studies explicit solutions to BSDEs and the connection with generalized matrix-valued Riccati ODEs. The results are then applied to expected utility maximization in [Section 4](#).

2. Notation and characterization of affine processes

We start with the notation we use subsequently. The space M_d stands for $d \times d$ matrices with real entries and I_d is the $d \times d$ -identity matrix. We denote by A^\top the transpose of the matrix A , by S_d the space of symmetric $d \times d$ -matrices equipped with the scalar product $\text{Tr}(xy)$. This inner product naturally induces a norm $\|x\| = \sqrt{\text{Tr}(xx)}$. We write S_d^+ (or S_d^-) for the closed cone of symmetric $d \times d$ positive (or negative) semidefinite matrices and S_d^{++} (or S_d^{--}) for the

open cone of $d \times d$ positive (or negative) definite matrices. By $\partial S_d^+ = S_d^+ \setminus S_d^{++}$ we denote the boundary of S_d^+ . The cones S_d^+ and S_d^{++} induce a strict partial order relation on S_d . We write $x \leq y$ if $y - x \in S_d^+$ and $x < y$ if $y - x \in S_d^{++}$. For $i, j = 1, \dots, d$, we also introduce the matrices $e^{ij} \in M_d$ with $e_{ij}^{ij} = 1$ and the remaining entries being 0.

The Borel σ -algebra on a space $U \subseteq S_d$ is denoted by $\mathcal{B}(U)$ and bS_d^+ refers to the set of bounded real-valued measurable functions f on S_d . The vector space \mathbb{R}^d is equipped with the Euclidean norm $|\cdot|$. We work on the finite time interval $[0, T]$, where $T > 0$ is fixed.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ denote a complete stochastic basis, where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$, and let \mathcal{P} be the σ -field of predictable sets on $[0, T] \times \Omega$. We consider time-homogeneous Markov processes X with state space S_d^+ and semigroup $(P_t)_{t \in [0, T]}$, where

$$P_t f(x) = \int_{S_d^+} f(\xi) p_t(x, d\xi),$$

and $x \in S_d^+$, $f \in bS_d^+$ and p_t a probability transition function. We refer to [51] for further details. Let us now define an affine process on S_d^+ .

Definition 2.1. A time-homogeneous Markov process X is called affine if it satisfies the following conditions.

- (i) X is stochastically continuous.
- (ii) The Laplace transform of X depends in an exponential affine way on the initial state. More precisely, there exist functions $\phi : [0, T] \times S_d^+ \rightarrow \mathbb{R}_+$ and $\psi : [0, T] \times S_d^+ \rightarrow S_d^+$ such that

$$P_t \exp(-\text{Tr}(xu)) = \int_{S_d^+} \exp(-\text{Tr}(\xi u)) p_t(x, d\xi) = \exp(-\phi(t, u) - \text{Tr}(\psi(t, u)x)),$$

for all $t \in [0, T]$ and $u, x \in S_d^+$.

As we will see later, the functions ϕ and ψ can be expressed in terms of ODEs involving an admissible parameter set. Note that this set is always given w.r.t. a truncation function $\chi : S_d \rightarrow S_d$ which is a continuous bounded function such that $\chi(\xi) = \xi$ for ξ in a neighborhood of 0.

Definition 2.2. We call $(\alpha, b, \beta^{ij}, m, \mu, \iota, \gamma)$ an admissible parameter set associated to a truncation function χ if it satisfies the following conditions.

- (i) The linear diffusion coefficient α belongs to the cone S_d^+ .
- (ii) The constant drift term b is such that $b \geq (d-1)\alpha$.
- (iii) The constant jump term m is a Borel measure on $\mathcal{B}(S_d^+ \setminus \{0\})$ satisfying

$$\int_{S_d^+ \setminus \{0\}} (\|\xi\| \wedge 1) m(d\xi) < \infty. \quad (2.1)$$

- (iv) The linear jump term consists of a $d \times d$ -matrix $\mu = (\mu_{ij})$ of finite signed measures on $\mathcal{B}(S_d^+ \setminus \{0\})$ such that $\mu(E) \in S_d^+$ for all $E \in \mathcal{B}(S_d^+ \setminus \{0\})$. The kernel

$$M(x, d\xi) = \frac{\text{Tr}(x\mu(d\xi))}{\|\xi\|^2 \wedge 1}, \quad (2.2)$$

satisfies

$$\int_{S_d^+ \setminus \{0\}} \text{Tr}(\chi(\xi)u) M(x, d\xi) < \infty, \quad \text{for all } x, u \in S_d^+ \text{ with } \text{Tr}(xu) = 0. \quad (2.3)$$

- (v) The linear drift coefficient is composed of a family $(\beta^{ij})_{i,j=1,\dots,d}$ of symmetric matrices with $\beta^{ij} = \beta^{ji} \in S_d$ for all $i, j = 1, \dots, d$, and such that the linear map $B : S_d \rightarrow S_d$ with

$$B(x) = \sum_{i,j=1}^d \beta^{ij} x_{ij}, \quad (2.4)$$

fulfills

$$\text{Tr}(B(x)u) - \int_{S_d^+ \setminus \{0\}} \text{Tr}(\chi(\xi)u) M(x, d\xi) \geq 0, \quad (2.5)$$

for all $x, u \in S_d^+$ with $\text{Tr}(xu) = 0$.

- (vi) The constant killing rate coefficient ι has values in \mathbb{R}_+ .

- (vii) The linear killing rate coefficient γ has values in S_d^+ .

A discussion on the conditions of the parameters can be found in Section 2.1 in [18]. The authors of [18] also give a full characterization of affine processes on S_d^+ .

Theorem 2.3 ([18, Theorem 2.4]). *Let X be an affine process on S_d^+ . Then there exists an admissible parameter set $(\alpha, b, \beta^{ij}, m, \mu, \iota, \gamma)$ w.r.t. a truncation function χ such that the functions ϕ and ψ from Definition 2.1(ii) solve the generalized Riccati ODE*

$$\frac{\partial \phi(t, u)}{\partial t} = \mathcal{F}(\psi(t, u)), \quad \phi(0, u) = 0, \quad (2.6)$$

$$\frac{\partial \psi(t, u)}{\partial t} = \mathcal{R}(\psi(t, u)), \quad \psi(0, u) = u \in S_d^+, \quad (2.7)$$

with

$$\begin{aligned} \mathcal{F}(u) &= \text{Tr}(bu) + \iota - \int_{S_d^+ \setminus \{0\}} (e^{-\text{Tr}(u\xi)} - 1) m(d\xi), \\ \mathcal{R}(u) &= -2u\alpha u + B^*(u) + \gamma - \int_{S_d^+ \setminus \{0\}} \frac{e^{-\text{Tr}(u\xi)} - 1 + \text{Tr}(\chi(\xi)u)}{\|\xi\|^2 \wedge 1} \mu(d\xi), \end{aligned}$$

where $B_{ij}^*(u) = \text{Tr}(\beta^{ij}u)$ for $i, j = 1, \dots, d$.

Conversely, let $(\alpha, b, \beta^{ij}, m, \mu, \iota, \gamma)$ be an admissible parameter set associated to a truncation function χ . Then there exists a unique affine process on S_d^+ and the condition of Definition 2.1(ii) holds for all $(t, u) \in [0, T] \times S_d^+$, where ϕ and ψ are given by (2.6) and (2.7).

Every conservative affine process on S_d^+ with killing rate coefficients $\iota = \gamma = 0$ is a semimartingale.

Theorem 2.4 ([18, Theorem 2.6]). *Let X be a conservative affine process on S_d^+ and $(\alpha, b, \beta^{ij}, m, \mu, 0, 0)$ the related admissible parameter set associated to a truncation function χ . Then X is a semimartingale whose characteristics (D, A, ν) with respect to χ are given by*

$$\begin{aligned} D_t &= \int_0^t \left(b + \int_{S_d^+ \setminus \{0\}} \chi(\xi) m(d\xi) + B(X_s) \right) ds, \\ A_{t,ijkl} &= \int_0^t A_{ijkl}(X_s) ds, \\ \nu([0, t], G) &= \int_0^t (m(G) + M(X_s, G)) ds, \end{aligned}$$

for $i, j, k, l \in \{1, \dots, d\}$, $t \in [0, T]$ and $G \in \mathcal{B}(S_d^+ \setminus \{0\})$. The matrix B is given by (2.4), M by (2.2) and A_{ijkl} by

$$A_{ijkl}(x) = x_{ik}\alpha_{jl} + x_{il}\alpha_{jk} + x_{jk}\alpha_{il} + x_{jl}\alpha_{ik}, \quad (2.8)$$

for all $i, j, k, l \in \{1, \dots, d\}$, and $x \in S_d^+$. Moreover there exists a $d \times d$ matrix of standard Brownian motions W such that X has the following canonical representation

$$\begin{aligned} X_t = & x + \int_0^t \sqrt{X_s} dW_s \Sigma + \int_0^t \Sigma^\top dW_s^\top \sqrt{X_s} \\ & + \int_0^t \left(b + B(X_s) + \int_{S_d^+ \setminus \{0\}} \chi(\xi) m(d\xi) \right) ds \\ & + \int_0^t \int_{S_d^+ \setminus \{0\}} \chi(\xi) \left(\mu^X(ds, d\xi) - \nu(ds, d\xi) \right) \\ & + \int_0^t \int_{S_d^+ \setminus \{0\}} (\xi - \chi(\xi)) \mu^X(ds, d\xi), \end{aligned} \quad (2.9)$$

where $\Sigma \in M_d$ satisfies $\Sigma \Sigma^\top = \alpha$ and μ^X denotes the random measure associated with the jumps of X .

Remark 2.5. The canonical representation (2.9) follows via the canonical semimartingale representation (see [36, Theorem II.2.34]) and the construction of a matrix-valued Brownian motion. For the latter one has to find a matrix which coincides with the covariation of the affine process.

Note that the constant drift term of an affine semimartingale is independent of the truncation function χ while χ influences the linear drift coefficient B .

From now on we fix a truncation function χ , and then write “admissible parameter set” for “admissible parameter set associated to truncation function χ ”. The affine process X with admissible parameter set $(\alpha, b, \beta^{ij}, m, \mu, \iota, \gamma)$ is continuous if and only if m and μ vanish, i.e. $(\alpha, b, \beta^{ij}, 0, 0, \iota, \gamma)$. Since we only consider affine semimartingales we write $(\alpha, b, \beta^{ij}, m, \mu)$ for $(\alpha, b, \beta^{ij}, m, \mu, 0, 0)$.

To fix ideas let us give an example of a matrix-valued affine processes, the Wishart processes. These processes were first rigorously studied in Bru [13] extending squares of matrix Ornstein–Uhlenbeck processes. The dynamics of a Wishart process satisfies

$$dX_t = (b + HX_t + X_t H^\top) dt + \sqrt{X_t} dW_t \Sigma + \Sigma^\top dW_t^\top \sqrt{X_t}, \quad (2.10)$$

where $b, H, \Sigma \in M_d$ and W is a $d \times d$ matrix Brownian motion. These processes have been widely used to model stochastic covariation, see e.g. [14,20,21]. In order to obtain a well defined matrix volatility process Bru [13] required the constant drift part $b = k \Sigma^\top \Sigma$ for some $k > d - 1$. Then X has a Wishart distribution. In the above notation the admissible parameter set for the Wishart process is $(\Sigma^\top \Sigma, b, \beta^{ij}, 0, 0)$ with $B(x) = Hx + x H^\top$.

In contrast to affine processes on the state space $\mathbb{R}_+^m \times \mathbb{R}^n$, which were fully characterized in [25], and where the diffusion term consists of a constant and linear part, the diffusion term of an affine process on S_d^+ with admissible parameter set $(\alpha, b, \beta^{ij}, m, \mu)$ only allows for a linear

part of the specific form

$$\sum_{i,j,k,l=1}^d u_{ij} A_{ijkl}(x) u_{kl} = 4 \operatorname{Tr}(x u \alpha u), \quad x, u \in S_d^+. \quad (2.11)$$

Note that the necessity and sufficiency of conditions (ii) and (v) in Definition 2.2 was first shown in [18]. In particular formula (2.4) allows for a more general form than $B(x) = Hx + xH^\top$, $x \in S_d^+$, compare also [18, Chapter 2.1.2].

3. Explicit solutions of quadratic FBSDEs

In this section we examine how the solutions for a class of quadratic BSDEs can be reduced to solving ODEs. In contrast to many existence results in the literature, e.g. [6,39,44,47], where the generator f is usually required to satisfy certain Lipschitz and growth conditions, we suggest an analytic expression for f which gives the problem extra structure. Consider the following motivating example where such a form appears naturally.

Take the one-dimensional Heston model (compare [32]) for the dynamics of an asset H . The stochastic logarithm N of H satisfies

$$\begin{aligned} dN_t &= \eta R_t dt + \sqrt{R_t} dQ_t, \\ dR_t &= (b + \lambda R_t) dt + \sigma \sqrt{R_t} dW_t, \quad t \in [0, T]. \end{aligned} \quad (3.1)$$

Here R is the stochastic volatility process, $b, \sigma > 0$, $\eta, \lambda \in \mathbb{R}$ are constants and Q, W are two Brownian motions with constant correlation $\rho \in [-1, 1]$. The volatility process R satisfies our definition of an affine process on \mathbb{R}_+ with admissible parameter set $(\frac{1}{4}\sigma^2, b, \lambda, 0, 0)$. We study an investor who is interested in maximizing their expected utility from terminal wealth. The investor's initial capital is denoted by $x \in \mathbb{R}$ and their trading strategies are deterministic functions π of time, where $\pi(t)$ describes the amount of money invested in stock H at time $t \in [0, T]$. The wealth process $X^{x,\pi}$ for initial endowment x and strategy π is given by

$$X_t^{x,\pi} = x + \int_0^t \frac{\pi(s)}{H_s} dH_s = x + \int_0^t \pi(s) dN_s,$$

for $t \in [0, T]$. We can solve the exponential utility maximization problem

$$V(x) = \sup_{\pi} \mathbb{E} \left[-\exp(-\gamma X_T^{x,\pi}) \right], \quad x \in \mathbb{R}, \quad \gamma > 0,$$

by finding the generator f of the BSDE

$$Y_t = 0 - \int_t^T Z_s dW_s + \int_t^T f(R_s, Z_s) ds, \quad (3.2)$$

such that the process $L_t^\pi = -\exp(-\gamma(X_t^{x,\pi} + Y_t))$, $t \in [0, T]$, is a supermartingale for all strategies π and a martingale for a particular strategy π^{opt} . The required generator can be shown (see Lemma 4.12) to be

$$f(r, z) = \frac{\gamma}{2}(\rho^2 - 1)z^2 + \frac{1}{2\gamma^3}\eta^2 r - \frac{1}{\gamma}\eta\rho z\sqrt{r}, \quad (r, z) \in \mathbb{R}^2, \quad (3.3)$$

similarly to [34, Theorem 7]. Notice that the generator is quadratic in the z -component. In order to solve this BSDE we apply the Itô formula to an affine function of R . More precisely we make

an affine ansatz for $Y_t = \Gamma(t)R_t + w(t)$, $t \in [0, T]$, where $\Gamma, w : [0, T] \rightarrow \mathbb{R}$ are differentiable functions. This leads to

$$\begin{aligned} \Gamma(t)R_t + w(t) &= \Gamma(T)R_T + w(T) - \int_t^T \Gamma(s)\sigma\sqrt{R_s}dW_s \\ &\quad - \int_t^T \left(\Gamma(s)(b + \lambda R_s) + \frac{d\Gamma(s)}{ds}R_s + \frac{dw(s)}{ds} \right) ds. \end{aligned} \quad (3.4)$$

It can be immediately read off the equation that Γ and w must satisfy

$$\Gamma(T) = w(T) = 0 \quad \text{and} \quad Z_s = \Gamma(s)\sigma\sqrt{R_s}, \quad s \in [0, T]. \quad (3.5)$$

The finite variation parts of (3.2) and (3.4) coincide if for all $s \in [0, T]$

$$\begin{aligned} 0 &= f(R_s, Z_s) + \Gamma(s)(b + \lambda R_s) + \frac{d\Gamma(s)}{ds}R_s + \frac{dw(s)}{ds} \\ &= \frac{\gamma}{2}(\rho^2 - 1)\Gamma^2(s)\sigma^2R_s + \frac{1}{2\gamma^3}\eta^2R_s - \frac{1}{\gamma}\eta\rho\Gamma(s)\sigma R_s + \Gamma(s)(b + \lambda R_s) \\ &\quad + \frac{d\Gamma(s)}{ds}R_s + \frac{dw(s)}{ds}, \end{aligned}$$

where we have used the equation for the generator f and formula (3.5) for Z . Equating coefficients this leads to an ODE of Riccati type

$$-\frac{d\Gamma(t)}{dt} = q\Gamma^2(t) + l\Gamma(t) + c, \quad \Gamma(T) = 0, \quad t \in [0, T],$$

with constants

$$q = \frac{\gamma}{2}\sigma^2(\rho^2 - 1), \quad l = \lambda - \frac{1}{\gamma}\sigma\rho\eta, \quad c = \frac{1}{2\gamma^3}\eta^2,$$

and an ODE of the simpler form

$$-\frac{dw(t)}{dt} = \Gamma(t)b, \quad w(T) = 0, \quad t \in [0, T].$$

Hence the solution of (3.2) is

$$\begin{aligned} Y_t &= \Gamma(t)R_t + \int_t^T b\Gamma(s)ds \\ Z_t &= \Gamma(t)\sigma\sqrt{R_t}, \quad t \in [0, T]. \end{aligned}$$

Generally, Riccati ODEs have the property that their solution can blow up in finite time, however our model parameter choices admit a non-explosive solution. In the one-dimensional case considered here we can even give an fully explicit solution, compare [12, Section 21.5.1.2]. More specifically we distinguish two different cases depending on the value of

$$d = l^2 - 4qc = \left(\lambda - \frac{1}{\gamma}\sigma\eta\rho \right)^2 + \frac{1}{\gamma^2}\sigma^2\eta^2(1 - \rho^2) \geq 0.$$

If $d > 0$, then

$$\Gamma(t) = -2c \frac{e^{\sqrt{d}(T-t)} - 1}{e^{\sqrt{d}(T-t)}(l + \sqrt{d}) - l + \sqrt{d}}, \quad t \in [0, T].$$

If $d = 0$, then $\rho = 1$, $\lambda = \frac{1}{\gamma}\sigma\eta$ and hence

$$\Gamma(t) = \frac{1}{2\gamma^3}\eta^2(T-t), \quad t \in [0, T].$$

In both cases the martingale property of $-\exp(-\gamma(X^{x,\pi^{\text{opt}}} + Y))$ then gives the value function and the optimal strategy

$$V(x) = -\exp\left(-\gamma\left(x + \Gamma(0)R_0 + \int_0^T b\Gamma(s)ds\right)\right),$$

$$\pi^{\text{opt}}(t) = \frac{1}{\gamma^2}\eta - \Gamma(t)\sigma\rho,$$

for $x \in \mathbb{R}$, $t \in [0, T]$.

In the previous example the ansatz $Y_t = \Gamma(t)R_t + w(t)$ and the method of equating coefficients enabled us to reduce the solution of a BSDE to solving ODEs. Now that we have seen how we exploit the affine structure in an one-dimensional example, we generalize this to BSDEs depending on affine processes on S_d^+ and even an additional process which has affine semimartingale characteristics with respect to the affine process. The question is how general we are allowed to choose the generator and the terminal condition in order to still be able to apply the above method.

We associate the affine process X with admissible parameter set $(\alpha, b, \beta^{ij}, m, \mu)$ to a BSDE. To allow for a more flexible financial modeling, especially in view of pricing of variance swaps, the BSDE will moreover depend on the matrix-valued process

$$dO_t = \sigma(t)\sqrt{X_t}d\hat{Q}_t + (o_1(t) + o_2(t)X_t)dt, \quad t \in [0, T], \quad (3.6)$$

where $o_1, o_2 : [0, T] \rightarrow M_d$ and $\sigma : [0, T] \rightarrow M_d$ are continuous functions. Here the process \hat{Q} denotes a $d \times d$ matrix-valued Brownian motion which is independent of the Brownian motion W . This process will enable us to calculate indifference prices and delta hedges for variance swaps, see e.g. Section 4.2.2.

Our (nonstandard) real-valued BSDE will have the following form

$$\begin{aligned} Y_t = & F(X_T, O_T) - \int_t^T \text{Tr}(Z_s^\top dW_s) - \int_t^T \text{Tr}(\hat{Z}_s^\top d\hat{Q}_s) \\ & - \int_0^t \int_{S_d^+ \setminus \{0\}} K_s(\xi) \left(\mu^X(ds, d\xi) - \nu(ds, d\xi) \right) \\ & + \int_t^T f(s, X_s, Y_s, Z_s, \hat{Z}_s, K_s)ds, \end{aligned} \quad (3.7)$$

for $t \in [0, T]$, where the terminal condition F is allowed to depend on the affine process X and on the process O . Recall that W is the Brownian motion of the underlying affine process X and the generator is a deterministic Borel measurable function $f : [0, T] \times S_d^+ \times \mathbb{R} \times M_d \times M_d \times \mathbb{R} \rightarrow \mathbb{R}$.

Definition 3.1. A solution to BSDE (3.7) is a family of adapted processes (Y, Z, \hat{Z}, K) with values in $\mathbb{R} \times M_d \times M_d \times \mathbb{R}$ such that:

- (i) The equation (3.7) is a.s. satisfied.
- (ii) The processes Z and \hat{Z} are predictable processes such that $\int_0^T |Z_s^\top|^2 ds < \infty$ and respectively $\int_0^T |\hat{Z}_s^\top|^2 ds < \infty$.

- (iii) The integrability condition $\int_0^T |f(s, X_s, Y_s, Z_s, \hat{Z}_s, K_s)| ds < \infty$ holds true.
- (iv) The mapping $t \mapsto Y_t$ is càdlàg.
- (v) The process K is predictable and satisfies $\int_0^T \int_{S_d^+ \setminus \{0\}} |K_t(\xi)|^2 (m(d\xi) + M(X_t, d\xi)) dt < \infty$.

For a certain class of generators and terminal conditions we can give the solution to the above BSDE in terms of matrix ODEs. Suppose the terminal condition $F : S_d^+ \times M_d \rightarrow \mathbb{R}$ is affine, more precisely

$$F(x, o) = \text{Tr}(ux) + \text{Tr}(ao) + v, \quad x \in S_d^+, o \in M_d, \quad (3.8)$$

where $u \in S_d$, $a \in M_d$ and $v \in \mathbb{R}$. Let us define the set L^0 as the space which contains all functions $k : S_d^+ \rightarrow \mathbb{R}$. The class of generators f is more involved, more precisely the generator $f : [0, T] \times S_d^+ \times \mathbb{R} \times M_d \times M_d \times L^0 \rightarrow \mathbb{R}$ is allowed to have the following form

$$\begin{aligned} f(t, x, y, z, \hat{z}, k) = & \text{Tr}(zc_{zz}(t)z^\top) + \text{Tr}(zc_{z\sqrt{x}}(t)\sqrt{x}) + \text{Tr}(c_x(t)x) + c_y(t)y + c_t(t) \\ & + \text{Tr}(\hat{z}c_{\hat{z}\hat{z}}(t)\hat{z}^\top) + \text{Tr}(\hat{z}c_{\hat{z}z}(t)z^\top) + \text{Tr}(\hat{z}c_{\hat{z}\sqrt{x}}(t)\sqrt{x}) \\ & + \int_{S_d^+ \setminus \{0\}} g_M(t, k(\xi))M(x, d\xi) + \int_{S_d^+ \setminus \{0\}} \left(\text{Tr}(zg_{z\sqrt{x}}(t, k(\xi))\sqrt{x}) \right. \\ & + \text{Tr}(xg_x(t, k(\xi))) + g_t(t, k(\xi)) + yg_y(t, k(\xi)) \Big) m(d\xi) \\ & + \int_{S_d^+ \setminus \{0\}} \left(\text{Tr}(\hat{z}g_{\hat{z}\hat{z}}(t, k(\xi))\hat{z}^\top) + \text{Tr}(\hat{z}g_{\hat{z}z}(t, k(\xi))z^\top) \right. \\ & + \left. \text{Tr}(\hat{z}g_{\hat{z}\sqrt{x}}(t, k(\xi))\sqrt{x}) \right) m(d\xi), \end{aligned} \quad (3.9)$$

for all $(t, x, y, z, \hat{z}, k) \in [0, T] \times S_d^+ \times \mathbb{R} \times M_d \times M_d \times L^0$. In the above,

$$\begin{aligned} c_{zz}, c_{z\sqrt{x}}, c_x, c_{\hat{z}\hat{z}}, c_{\hat{z}z}, c_{\hat{z}\sqrt{x}} : [0, T] &\rightarrow M_d, \\ c_t, c_y : [0, T] &\rightarrow \mathbb{R}, \end{aligned}$$

are continuous functions and $g_M : [0, T] \times L^0 \rightarrow \mathbb{R}$ is an $M(x, d\xi)$ -integrable function, $x \in S_d^+$, which is continuous in time. Finally

$$\begin{aligned} g_{z\sqrt{x}}, g_x, g_{\hat{z}\hat{z}}, g_{\hat{z}z}, g_{\hat{z}\sqrt{x}} : [0, T] \times L^0 &\rightarrow M_d, \\ g_t, g_y : [0, T] \times L^0 &\rightarrow \mathbb{R}, \end{aligned}$$

are $m(d\xi)$ -integrable functions which are also assumed continuous in time.

Remark 3.2. As we have seen in the introductory example an affine ansatz for the solution process Y leads to an explicit form of Z in terms of ODEs and the forward process X . The affine form for Y and the application of Itô's formula then implies the affine form of the terminal condition F and already determines the processes Z , \hat{Z} and K in terms of X and up to the solvability of a matrix ODE. The form of the generators f looks more complicated but is governed by the affine ansatz for Y and Itô's formula as well. The finite variation part of Itô's formula applied to an affine function necessarily needs to coincide with the generator of the BSDE. Additionally, the generator written exclusively in terms of X (instead of Z , \hat{Z} and K) needs to be affine so that we can use the method of equating coefficients. This leads to the described class of generators.

Remark 3.3. We restrict ourselves to the case where f does not depend explicitly on O since the structure of f is already quite involved. If this were not the case we would derive further coupled ODEs. However the terminal value F depends affine on O . We could allow for jumps in O provided those jumps have affine characteristics in X , but do not for reasons of brevity. It is possible to only consider functional forms of time $\sigma(\cdot)$ rather than $\sigma(\cdot)\sqrt{X}$ in (3.6).

Remark 3.4. With applications in utility maximization in mind, we have chosen to examine a class of quadratic BSDEs with an affine terminal condition. However the approach can be transferred to (exponentially) affine or polynomial terminal conditions. For illustration consider a quadratic BSDE without jumps and with a terminal condition depending quadratically on the affine process R from (3.1) with parameters $2b > \sigma^2$. Then a quadratic ansatz for the solution process Y , i.e. $Y_t = \Xi(t)R_t^2 + \Gamma(t)R_t + w(t)$ with differentiable functions of time Ξ , Γ and w , leads to $Z_t = (2\Xi(t)R_t + \Gamma(t))\sigma\sqrt{R_t}$. The BSDE (Y, Z) is solvable if its generator $f(r, z) = f(r, (2\Xi r + \Gamma)\sigma\sqrt{r})$ is quadratic in r and the system of 3 ODEs is solvable. This is the case for example for the BSDE with quadratic generator $f(r, z) = z^2/r + z\sqrt{r}$, because the corresponding ODE for Ξ turns out to be a Riccati ODE, Γ is a linear ODE depending on the solution of Ξ and w can be found as an integral involving the solution of Γ .

We can now give the main theorem which describes the explicit form of the solution processes (Y, Z, \hat{Z}, K) in terms of the solution to a system of generalized Riccati equations.

Theorem 3.5. Let X be an affine semimartingale on S_d^+ associated to the admissible parameter set $(\alpha, b, \beta^{ij}, m, \mu)$ such that

$$\int_{S_d^+ \setminus \{0\}} (\xi - \chi(\xi))(m(d\xi) + M(x, d\xi)) < \infty, \quad x \in S_d^+. \quad (3.10)$$

Furthermore suppose that there exists a unique solution $\Gamma(\cdot, u) : [0, T] \rightarrow S_d$ to the generalized Riccati ODE

$$-\frac{\partial \Gamma(t, u)}{\partial t} = \theta(t, \Gamma(t, u)), \quad \Gamma(T, u) = u, \quad (3.11)$$

with

$$\begin{aligned} \theta(t, u) = & 4u\Sigma^\top c_{zz}(t)\Sigma u + \mathcal{L}(t)u + u\mathcal{L}^\top(t) + B^*(u) + \mathcal{C}(t) \\ & + \int_{S_d^+ \setminus \{0\}} \frac{\text{Tr}(u(\xi - \chi(\xi))) + g_M(t, \text{Tr}(u\xi))}{\|\xi\|^2 \wedge 1} \mu(d\xi) \\ & + \int_{S_d^+ \setminus \{0\}} \left(u\Sigma^\top g_{z\sqrt{x}}(t, \text{Tr}(u\xi)) + g_{z\sqrt{x}}^\top(t, \text{Tr}(u\xi))\Sigma u + ug_y(t, \text{Tr}(u\xi)) \right. \\ & + g_x(t, \text{Tr}(u\xi)) + \sigma^\top(t)ag_{\hat{z}\hat{z}}(t, \text{Tr}(u\xi))a^\top\sigma(t) + \sigma^\top(t)ag_{\hat{z}\hat{z}}(t, \text{Tr}(u\xi))\Sigma u \\ & \left. + u\Sigma^\top g_{\hat{z}\hat{z}}^\top(t, \text{Tr}(u\xi))a^\top\sigma(t) + \sigma^\top(t)ag_{\hat{z}\sqrt{x}}(t, \text{Tr}(u\xi)) \right) m(d\xi), \end{aligned} \quad (3.12)$$

for $(t, u) \in [0, T] \times S_d$. The functions $\mathcal{L}(t)$ and $\mathcal{C}(t)$ are given by

$$\begin{aligned} \mathcal{L}(t) &= \frac{1}{2}c_y(t) + c_{z\sqrt{x}}^\top(t)\Sigma + \sigma(t)^\top ac_{\hat{z}\hat{z}}(t)\Sigma \\ \mathcal{C}(t) &= c_x(t) + \sigma(t)^\top ac_{\hat{z}\hat{z}}(t)a^\top\sigma(t) + \sigma^\top(t)ac_{\hat{z}\sqrt{x}}(t) + a\sigma_2(t), \end{aligned}$$

for all $t \in [0, T]$. Let $w(\cdot, u, v) : [0, T] \rightarrow \mathbb{R}$ be the solution of

$$-\frac{\partial w(t, u, v)}{\partial t} = \varpi(t, \Gamma(t, u), w(t, u, v)), \quad w(T, u, v) = v, \quad (3.13)$$

with

$$\begin{aligned} \varpi(t, u, v) = & c_y(t)v + c_t(t) + \text{Tr}(ao_1(t)) + \text{Tr}(ub) \\ & + \int_{S_d^+ \setminus \{0\}} (\text{Tr}(u\xi) + g_y(t, \text{Tr}(u\xi))v + g_t(t, \text{Tr}(u\xi))) m(d\xi), \end{aligned}$$

for $(t, u, v) \in [0, T] \times S_d \times \mathbb{R}$. Then the above BSDE has the unique solution

$$\begin{cases} Y_t = \text{Tr}(\Gamma(t, u)X_t) + \text{Tr}(aO_t) + w(t, u, v), \\ Z_t = 2\sqrt{X_t}\Gamma(t, u)\Sigma^\top, \\ \hat{Z}_t = \sqrt{X_t}\sigma^\top(t)a, \\ K_t(\xi) = \text{Tr}(\Gamma(t, u)\xi), \end{cases} \quad (3.14)$$

for all $\xi \in S_d^+, t \in [0, T]$.

Proof. Let Γ and w be the unique solutions of (3.11) and (3.13). We apply Itô's formula for semimartingales to the function $[0, T] \times S_d^+ \times M_d \ni (t, x, o) \mapsto \text{Tr}(\Gamma(t, u)x) + \text{Tr}(ao) + w(t, u, v)$. Using representation (2.9) and (3.6) this gives

$$\begin{aligned} & \text{Tr}(\Gamma(t, u)X_t) + \text{Tr}(aO_t) + w(t, u, v) \\ &= \text{Tr}(\Gamma(T, u)X_T) + w(T, u, v) + \text{Tr}(aO_T) - \int_t^T 2 \text{Tr} \left(\Sigma \Gamma(s, u) \sqrt{X_s} dW_s \right) \\ & \quad - \int_t^T \left(\text{Tr}(\Gamma(s, u)b) + \text{Tr}(\Gamma(s, u)B(X_s)) + \text{Tr} \left(\Gamma(s, u) \int_{S_d^+ \setminus \{0\}} \chi(\xi) m(d\xi) \right) \right) ds \\ & \quad - \int_t^T \int_{S_d^+ \setminus \{0\}} \text{Tr}(\Gamma(s, u)\chi(\xi)) (\mu^X(ds, d\xi) - \nu(ds, d\xi)) \\ & \quad - \int_t^T \int_{S_d^+ \setminus \{0\}} \text{Tr}(\Gamma(s, u)(\xi - \chi(\xi))) \mu^X(ds, d\xi) \\ & \quad - \int_t^T \left(\text{Tr} \left(\frac{\partial \Gamma(s, u)}{\partial s} X_s \right) + \frac{\partial w(s, u, v)}{\partial s} \right) ds \\ & \quad - \int_t^T \text{Tr} \left(a\sigma(s) \sqrt{X_s} d\hat{Q}_s \right) - \int_t^T \text{Tr} (a(o_1(s) + o_2(s)X_s)) ds, \end{aligned}$$

where we have used basic properties of the trace to derive

$$\text{Tr} \left(\Gamma(t, u) (\sqrt{X_t} dW_t \Sigma + \Sigma^\top dW_t^\top \sqrt{X_t}) \right) = 2 \text{Tr} \left(\Sigma \Gamma(t, u) \sqrt{X_t} dW_t \right).$$

The integral

$$\int_t^T \int_{S_d^+ \setminus \{0\}} \text{Tr}(\Gamma(s, u)\xi) (\mu^X(ds, d\xi) - \nu(ds, d\xi)), \quad t \in [0, T],$$

is well defined. Indeed, recall the form of the compensator ν from [Theorem 2.4](#) and the admissibility conditions [\(2.1\)](#) and [\(2.3\)](#). This yields

$$\int_t^T \int_{S_d^+ \setminus \{0\}} \text{Tr}(\Gamma(s, u) \chi(\xi)) (\mu^X(ds, d\xi) - \nu(ds, d\xi)) < \infty, \quad t \in [0, T],$$

and with the integrability condition [\(3.10\)](#) on the measures m and M , we ensure

$$\int_t^T \int_{S_d^+ \setminus \{0\}} \text{Tr}(\Gamma(s, u) (\xi - \chi(\xi))) (\mu^X(ds, d\xi) - \nu(ds, d\xi)) < \infty, \quad t \in [0, T].$$

Hence we may write the above equation in the following form

$$\begin{aligned} & \text{Tr}(\Gamma(t, u) X_t) + \text{Tr}(a O_t) + w(t, u, v) \\ &= \text{Tr}(\Gamma(T, u) X_T) + w(T, u, v) + \text{Tr}(a O_T) - \int_t^T 2 \text{Tr} \left(\Sigma \Gamma(s, u) \sqrt{X_s} dW_s \right) \\ & \quad - \int_t^T \left(\text{Tr}(\Gamma(s, u) b) + \text{Tr}(\Gamma(s, u) B(X_s)) + \text{Tr} \left(\Gamma(s, u) \int_{S_d^+ \setminus \{0\}} \chi(\xi) m(d\xi) \right) \right) ds \\ & \quad - \int_t^T \int_{S_d^+ \setminus \{0\}} \text{Tr}(\Gamma(s, u) \xi) (\mu^X(ds, d\xi) - \nu(ds, d\xi)) \\ & \quad - \int_t^T \int_{S_d^+ \setminus \{0\}} \text{Tr}(\Gamma(s, u) (\xi - \chi(\xi)) \nu(ds, d\xi)) \\ & \quad - \int_t^T \left(\text{Tr} \left(\frac{\partial \Gamma(s, u)}{\partial s} X_s \right) + \frac{\partial w(s, u, v)}{\partial s} \right) ds \\ & \quad - \int_t^T \text{Tr} \left(a \sigma(s) \sqrt{X_s} d\hat{Q} \right) - \int_t^T \text{Tr} (a o_1(s) + o_2(s) X_s) ds. \end{aligned} \quad (3.15)$$

Hence the BSDE [\(3.7\)](#) is solved by [\(3.14\)](#) provided the finite variation parts of [\(3.15\)](#) and the BSDE coincide, i.e. if

$$\begin{aligned} & f(t, X_t, Y_t, Z_t, \hat{Z}, K_t) \\ &= -\text{Tr}(\Gamma(t, u) b) - \text{Tr}(\Gamma(t, u) B(X_t)) - \text{Tr} \left(\Gamma(t, u) \int_{S_d^+ \setminus \{0\}} \chi(\xi) m(d\xi) \right) \\ & \quad - \int_{S_d^+ \setminus \{0\}} \text{Tr}(\Gamma(t, u) (\xi - \chi(\xi))) (m(d\xi) + M(X_t, d\xi)) \\ & \quad - \text{Tr} \left(\frac{\partial \Gamma(t, u)}{\partial t} X_t \right) - \frac{\partial w(t, u, v)}{\partial t} - \text{Tr}(a o_1(t)) - \text{Tr}(a o_2(t) X_t), \end{aligned}$$

for $t \in [0, T]$. Using the special form [\(3.9\)](#) of the generator f and formulas [\(3.14\)](#) we calculate

$$\begin{aligned} & f(t, X_t, Y_t, Z_t, \hat{Z}, K_t) \\ &= 4 \text{Tr}(\sqrt{X_t} \Gamma(t, u) \Sigma^\top c_{zz}(t) \Sigma \Gamma(t, u) \sqrt{X_t}) + 2 \text{Tr}(\sqrt{X_t} \Gamma(t, u) \Sigma^\top c_{z\sqrt{x}}(t) \sqrt{X_t}) \\ & \quad + \text{Tr}(c_x(t) X_t) + c_y(t) \text{Tr}(\Gamma(t, u) X_t) + c_y(t) w(t, u, v) + c_t(t) \\ & \quad + \text{Tr}(\sqrt{X_t} \sigma^\top(t) a c_{\hat{z}\hat{z}}(t) a^\top \sigma(t) \sqrt{X_t}) + \text{Tr}(\sqrt{X_t} \sigma^\top(t) a c_{\hat{z}\hat{z}}(t) 2 \Sigma \Gamma(t, u) \sqrt{X_t}) \end{aligned}$$

$$\begin{aligned}
& + \operatorname{Tr}(\sqrt{X_t} \sigma^\top(t) a c_{\hat{z}\sqrt{x}}(t) \sqrt{X_t}) + \int_{S_d^+ \setminus \{0\}} g_M(t, \operatorname{Tr}(\Gamma(t, u) \xi)) M(X_t, d\xi) \\
& + \int_{S_d^+ \setminus \{0\}} (\operatorname{Tr}(2X_t \Gamma(t, u) \Sigma^\top g_{z\sqrt{x}}(t, \operatorname{Tr}(\Gamma(t, u) \xi))) + \operatorname{Tr}(g_x(t, \operatorname{Tr}(\Gamma(t, u) \xi)) X_t) \\
& + \operatorname{Tr}(g_y(t, \operatorname{Tr}(\Gamma(t, u) \xi)) \Gamma(t, u) X_t) + g_y(t, \operatorname{Tr}(\Gamma(t, u) \xi)) w(t, u, v) \\
& + g_t(t) + \operatorname{Tr}(\sqrt{X_t} \sigma^\top(t) a g_{\hat{z}\hat{z}}(t, \operatorname{Tr}(\Gamma(t, u) \xi)) a^\top \sigma(t) \sqrt{X_t}) \\
& + \operatorname{Tr}(\sqrt{X_t} \sigma^\top(t) a g_{\hat{z}z}(t, \operatorname{Tr}(\Gamma(t, u) \xi)) 2 \Sigma \Gamma(t, u) \sqrt{X_t}) \\
& + \operatorname{Tr}(\sqrt{X_t} \sigma^\top(t) a g_{z\sqrt{x}}(t, \operatorname{Tr}(\Gamma(t, u) \xi)) \sqrt{X_t}) m(d\xi) \\
& = -\operatorname{Tr}(\Gamma(t, u) b) - \operatorname{Tr}(\Gamma(t, u) B(X_t)) - \int_{S_d^+ \setminus \{0\}} \operatorname{Tr}(\Gamma(t, u) \chi(\xi)) m(d\xi) \\
& - \int_{S_d^+ \setminus \{0\}} \operatorname{Tr}(\Gamma(t, u) (\xi - \chi(\xi))) (m(d\xi) + M(X_t, d\xi)) \\
& - \operatorname{Tr} \left(\frac{\partial \Gamma(t, u)}{\partial t} X_t \right) - \frac{\partial w(t, u, v)}{\partial t} - \operatorname{Tr}(a o_1(t)) - \operatorname{Tr}(a o_2(t) X_t),
\end{aligned}$$

where the last equality is obtained from (3.11) and (3.13), the definition of the adjoint operator B^* and basic properties of the trace. \square

Obviously our results also apply to ‘standard’ FBSDEs, where the BSDE is only allowed to depend the affine process X itself. For more details and a study on the existence and uniqueness of solutions to the generalized matrix Riccati ODE (3.11) we refer the reader to [50, Chapter 8].

4. Application in multivariate affine stochastic volatility models

In this chapter we apply the results of the previous chapter to the classical problem of utility maximization in a multivariate stochastic volatility setting. Stochastic volatility models are an extension of the Black–Scholes model, where the previously constant assumed volatility is now modeled as a stochastic process. The key feature of affine stochastic volatility models is that their joint Fourier–Laplace transform has an exponentially affine form. For a multivariate model consider the d -dimensional logarithmic price process N whose stochastic volatility is given by an affine process R on S_d^+ , then the following formula holds

$$\mathbb{E} \left[e^{\operatorname{Tr}(u R_t) + v^\top N_t} \right] = e^{\operatorname{Tr}(\Psi(t, u, v) R_0) + v^\top N_0 + \Phi(t, u, v)},$$

for suitable arguments $t \in [0, T]$, $u \in S_d + i S_d$ and $v \in \mathbb{C}^d$. The functions Φ and Ψ solve a system of generalized Riccati ODEs which are specified by the model parameters. This formula is the main reason for the analytic tractability of affine stochastic volatility models. In the multivariate stochastic volatility models mainly used in the literature, the dynamics of R follow

$$\begin{aligned}
dR_t &= (b + \hat{B} R_t + R_t \hat{B}^\top) dt + \sqrt{R_t} dW_t \Sigma + \Sigma^\top dW_t^\top \sqrt{R_t} + dJ_t, \\
R_0 &= r \in S_d^+,
\end{aligned}$$

where W is a matrix-valued Brownian motion possibly correlated with the Brownian motion driving N . Moreover b is a suitably chosen matrix in S_d^+ , Σ , \hat{B} are some invertible matrices and J is a pure jump process with a compensator that is affine in R . Without jumps this process is a Wishart-process (see also (2.10) and thereafter). They were introduced by [13] and have

been applied to many different fields such as term structure modeling and derivative pricing in [20,21,30,31]. The authors of [3,4] consider multivariate stochastic volatility models for a class of matrix-valued Ornstein–Uhlenbeck processes driven by a Lévy process of finite variation. Closer to our subject is the work of [19]. There the authors investigate the power utility maximization problem in a multivariate Heston model where the covariation process follows a Wishart process. They obtain the optimal portfolio and utility via a duality approach.

We start with the martingale property of the stochastic exponential of some process which will allow us to prove optimality in the utility maximization problem. More precisely we want to maximize expected utility of terminal wealth. The wealth process $X^{x,\pi}$ is composed of the initial capital $x \in \mathbb{R}$ and gains from trading with strategy π in the market. We want to solve the problem in presence of random revenues F which are paid at terminal time T , i.e.

$$V(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E} [U(X_T^{x,\pi} + F)], \quad x \in \mathbb{R},$$

where U is an exponential utility function. Once a notion of admissibility is fixed we call any $\pi \in \mathcal{A}$ an admissible (trading) strategy. Our aim is to explicitly describe the value function V and the corresponding optimal strategy π^{opt} . Similarly we examine the problem

$$\tilde{V}(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E} [U(X_T^{x,\pi} \exp(F))], \quad x \in \mathbb{R},$$

where U is now a power utility function. It is known that the logarithmic utility maximization problem with $F = 0$ can be solved explicitly for almost all semimartingale models, see e.g. [29] and the references therein. This is why we do not consider logarithmic utility in this work.

4.1. The martingale property

In the following sections it will play an important role under which conditions the stochastic exponential of a process involving an affine process becomes a true martingale. This problem has applications in fields including absolute continuity of distributions of stochastic processes (see [15]) and the verification of optimality in utility maximization as we use it here. On the state space $\mathbb{R}_+^n \times \mathbb{R}^m$ and in a time-homogeneous setting the problem has been addressed already in [37,42], which has then been extended to the state space $S_d^+ \times \mathbb{R}^d$ in [17].

Suppose R is an affine process with admissible parameter set $(\alpha, b, \beta^{ij}, m, 0)$ associated with truncation function χ^R . Let for all $s \in [0, T]$

$$\int_{\{|\text{Tr}(\sigma_\mu(s)\xi)| > 1\}} e^{\text{Tr}(\sigma_\mu(s)\xi)} m(d\xi) < \infty, \quad (4.1)$$

and consider the process

$$\begin{aligned} P_t &= \int_0^t \sigma_Q^\top(s) \sqrt{R_s} dQ_s + \int_0^t \text{Tr}(\sigma_W(s) \sqrt{R_s} dW_s) + \int_0^t \text{Tr}(\sigma_{\hat{Q}}(s) \sqrt{R_s} d\hat{Q}_s) \\ &\quad + \int_0^t \int_{S_d^+ \setminus \{0\}} (e^{\text{Tr}(\sigma_\mu(s)\xi)} - 1) d(\mu^R(ds, d\xi) - m(d\xi)ds), \quad t \in [0, T], \end{aligned} \quad (4.2)$$

where $\sigma_Q : [0, T] \rightarrow \mathbb{R}^d$ and $\sigma_W, \sigma_{\hat{Q}}, \sigma_\mu : [0, T] \rightarrow M^d$ are continuous functions of time. The d -dimensional Brownian motion Q is correlated to the matrix Brownian motion W by

$$dQ_t = dW_t \rho + \sqrt{1 - \rho^\top \rho} dD_t.$$

Here D is a d -dimensional Brownian motion independent of W and ρ a d -dimensional vector with entries $\rho_i \in [-1, 1], i = 1, \dots, d$, satisfying $\rho^\top \rho \leq 1$. The process \hat{Q} is another independent $d \times d$ -matrix Brownian motion. Finally μ^R denotes the random measure associated to the jumps of R . It will be crucial that the conditional Fourier–Laplace transform of (R, \hat{P}) with

$$\hat{P} = \ln(\mathcal{E}(P)),$$

is exponentially affine which is made precise in Lemma 4.1. From this one can determine when the stochastic exponential of P is a true martingale. The proofs of the following lemma and the proposition rely on the careful application of Itô's formula and σ -martingales and can be found in [50, Chapter 9].

Lemma 4.1. *The conditional Fourier–Laplace transform of (R, \hat{P}) has an exponentially affine form. More precisely, there exist functions $(s, t, u, v) \mapsto \Psi(s, t, u, v)$ and $(s, t, u, v) \mapsto \Phi(s, t, u, v)$ such that*

$$\mathbb{E} \left[\exp(\text{Tr}(u R_t) + v \hat{P}_t) | (R_s, \hat{P}_s) \right] = \exp \left(\text{Tr}(\Psi(s, t, u, v) R_s) + v \hat{P}_s + \Phi(s, t, u, v) \right),$$

for all $(s, t, u, v) \in \mathcal{I}$, where

$$\begin{aligned} \mathcal{I} = & \left\{ (s, t, u, v) \in [0, T] \times [0, T] \times S_d \right. \\ & \left. + i S_d \times \mathbb{C} : s \leq t, \mathbb{E} \left[e^{\text{Tr}(u R_t) + v \hat{P}_t} | (R_s, \hat{P}_s) \right] < \infty \right\}. \end{aligned}$$

The functions Φ and Ψ have the following form

$$\begin{aligned} -\frac{\partial \Phi(s, t, u, v)}{\partial s} &= \mathcal{F}(s, \Psi(s, t, u, v), v), & \Phi(t, t, u, v) &= 0, \\ -\frac{\partial \Psi(s, t, u, v)}{\partial s} &= \mathcal{R}(s, \Psi(s, t, u, v), v), & \Psi(t, t, u, v) &= u, \end{aligned}$$

where

$$\begin{aligned} \mathcal{F}(s, u, v) &= \text{Tr}(bu) + \int_{S_d^+ \setminus \{0\}} \left(e^{\text{Tr}(u\xi) - v \text{Tr}(\sigma_\mu(s)\xi)} - v e^{\text{Tr}(\sigma_\mu(s)\xi)} \right. \\ &\quad \left. + v - 1 - \text{Tr}(u\xi) \right) m(d\xi), \\ \mathcal{R}(s, u, v) &= 2u\alpha u + B^*(u) + \frac{1}{2}v(v-1)(2\sigma_Q(s)\rho^\top \sigma_W(s) \\ &\quad + \sigma_W^\top(s)\sigma_W(s) + \sigma_{\hat{Q}}^\top(s)\sigma_{\hat{Q}}(s) \\ &\quad + \sigma_Q(s)\sigma_Q^\top(s)) + v u(\sigma_Q(s)\rho^\top + \sigma_W^\top(s))\Sigma + v \Sigma^\top(\sigma_W(s) + \rho\sigma_Q^\top(s))u, \end{aligned}$$

for all $(s, t, u, v) \in \mathcal{I}$.

Proposition 4.2. *Assume (4.1), then the process $\mathcal{E}(P)$ is a martingale.*

4.2. Solution in a continuous multivariate affine stochastic volatility model

In this section we introduce a continuous affine stochastic volatility model which is a natural multivariate extension of the Heston model. We then formulate the utility maximization problem in this model and solve it for power and exponential utility. This allows to describe the optimal strategy and maximal expected utility in terms of the model parameters and a Riccati ODE.

Assume that there exists a financial market with one riskless bond with zero interest rate and d risky assets $H = (H_1, \dots, H_d)$. The process H is modeled as stochastic exponential $H = H_0 \mathcal{E}(N)$, where $N = (N_1, \dots, N_d)$ is given by

$$dN_t = R_t \eta dt + \sqrt{R_t} dQ_t, \quad t \in [0, T],$$

with Q being a vector Brownian motion with values in \mathbb{R}^d and $\eta \in \mathbb{R}^d$ being a constant vector. The stochastic volatility process is continuous and affine with admissible parameter set $(\alpha, b, \rho^{ij}, 0, 0)$, i.e. it satisfies the SDE

$$\begin{aligned} dR_t &= (b + B(R_t))dt + \sqrt{R_t} dW_t \Sigma + \Sigma^\top dW_t^\top \sqrt{R_t}, \\ R_0 &= r \in S_d^+, \quad t \in [0, T], \end{aligned} \quad (4.3)$$

where W is a $d \times d$ matrix-valued Brownian motion. The Brownian motion Q driving the assets returns and the Brownian motion W of the stochastic covariation matrix are allowed to be correlated in a certain way. Let $\rho = (\rho_1, \dots, \rho_d)^\top$ be a vector with entries $\rho_i \in [-1, 1]$, $i = 1, \dots, d$, and such that $\rho^\top \rho \leq 1$. With a d -dimensional Brownian motion D independent of W we can write

$$dQ_t = dW_t \rho + \sqrt{1 - \rho^\top \rho} dD_t.$$

Hence the correlation between the scalar Brownian motions Q^i and W^{mn} is given by ρ_n if $i = m$ and else it is 0. The structure of the correlation between Q and W has been chosen in this way in order to ensure the model to be affine:

Proposition 4.3. *The pair of processes (R, N) is a multivariate stochastic volatility model with functions Φ and Ψ solving*

$$\begin{aligned} \frac{\partial \Phi(t, u, v)}{\partial t} &= b \Psi(t, u, v), \\ \Phi(0, u, v) &= 0, \\ \frac{\partial \Psi(t, u, v)}{\partial t} &= 2 \Psi(t, u, v) \alpha \Psi(t, u, v) + B^*(\Psi(t, u, v)) + (v - \eta) \rho^\top \Sigma \Psi(t, u, v) \\ &\quad + \Psi(t, u, v) \Sigma^\top \rho (v - \eta)^\top + \frac{1}{2} v v^\top, \\ \Psi(0, u, v) &= u, \end{aligned} \quad (4.4)$$

for all $(t, u, v) \in \mathcal{Q}$.

The proof has been omitted here and can be found in [50, Chapter 9].

Remark 4.4. We have chosen to model the asset price process H as stochastic exponential of N . It is also possible to model H as ordinary exponential, i.e. $H = H_0 e^N$. In this case we have $H = H_0 \mathcal{E}(\tilde{N})$ with $d\tilde{N}_t = \sqrt{R_t} dQ_t + R_t(\eta + \frac{1}{2})dt$, $t \in [0, T]$. Hence we are back in the setting considered above.

4.2.1. Power utility

We set $F = 0$ and assume that the investor's preferences are described by the power utility function

$$U(x) = \frac{1}{\gamma} x^\gamma, \quad x \geq 0, \quad \gamma \in (0, 1).$$

Let \mathcal{A} be the set of all d -dimensional predictable processes π that satisfy $\int_0^T \pi_s^\top \pi_s ds < \infty$ a.s. Note that π_i denotes the fraction of the wealth invested in stock i , where $i = 1, \dots, d$. Any process $\pi \in \mathcal{A}$ is called an admissible (trading) strategy. Under these assumptions the wealth process evolves as follows

$$X_t^{x,\pi} = x + \int_0^t X_s^{x,\pi} \pi_s^\top dN_s = x + \int_0^t X_s^{x,\pi} \pi_s^\top R_s \eta ds + \int_0^t X_s^{x,\pi} \pi_s^\top \sqrt{R_s} dQ_s,$$

for $t \in [0, T]$. It can also be written as stochastic exponential

$$X_t^{x,\pi} = x \mathcal{E} \left(\int_0^t \pi_s^\top R_s \eta ds + \int_0^t \pi_s^\top \sqrt{R_s} dQ_s \right), \quad t \in [0, T].$$

As described earlier the investor wants to maximize their expected utility of terminal wealth. In order to model interest and exchange rates later, we want to take the function

$$F(O_T) = \text{Tr}(a O_T),$$

into account, where a is a $d \times d$ -matrix and O_T the final value of the process

$$O_t = \int_0^t \sigma \sqrt{R_s} d\hat{Q}_s + \int_0^t (o_1 + o_2 R_s) ds, \quad t \in [0, T]. \quad (4.5)$$

Here $o_1, o_2, \sigma \in M_d$ and \hat{Q} is a $d \times d$ -dimensional Brownian motion independent of the Brownian motions W and Q . From now on we will write F^{a,σ,o_1,o_2} for $F(O_T)$ with O_T given by (4.5) to depict the structure of F in more detail. We want to solve the maximization problem

$$V^{a,\sigma,o_1,o_2}(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E} \left[\frac{1}{\gamma} \left(X_T^{x,\pi} \exp(F^{a,\sigma,o_1,o_2}) \right)^\gamma \right], \quad x \geq 0.$$

Our main result describes the value function and the optimal strategy explicitly in terms of the model parameters.

Theorem 4.5. *Let the linear diffusion term α belong to S_d^{++} and suppose the linear drift term B in (4.3) is of the form $B(r) = r\hat{B} + \hat{B}^\top r$, $r \in S_d^+$, with $\hat{B} \in M_d$. Define the matrix-valued functions $A : [0, T] \rightarrow M_{2d}$ and $A_{ij} : [0, T] \rightarrow M_d$, $i, j = 1, \dots, d$, by*

$$\begin{aligned} A(t) &= \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} \\ &= \exp \left((T-t) \begin{pmatrix} \frac{\gamma}{1-\gamma} \Sigma^\top \rho \eta^\top + \hat{B}^\top & -2\alpha - \frac{2\gamma}{1-\gamma} \Sigma^\top \rho \rho^\top \Sigma \\ \frac{1}{2} \sigma^\top a a^\top \sigma + \frac{\gamma}{2(1-\gamma)} \eta \eta^\top + a o_2 & -\frac{\gamma}{1-\gamma} \eta \rho^\top \Sigma - \hat{B} \end{pmatrix} \right). \end{aligned} \quad (4.6)$$

Then the value function and the corresponding optimal strategy are given by

$$\begin{aligned} V^{a,\sigma,o_1,o_2}(x) &= \frac{1}{\gamma} x^\gamma \exp \left(\text{Tr}(A_{22}^{-1}(0) A_{21}(0) r) + \int_0^T \text{Tr} \left(A_{22}^{-1}(s) A_{21}(s) b + a o_1 \right) ds \right), \\ \pi_t^{\text{opt}} &= \frac{1}{1-\gamma} \left(\eta + 2 A_{22}^{-1}(t) A_{21}(t) \Sigma^\top \rho \right), \quad x \geq 0, \quad t \in [0, T]. \end{aligned}$$

In contrast to the classical Merton problem in the Black–Scholes setting, the strategy is now not a constant proportion of wealth anymore but is a deterministic function of time. Note that the strategy consists of the Merton ratio $\eta/(1 - \gamma)$ and a hedging component $1/(1 - \gamma)2A_{22}^{-1}(t)A_{21}(t)\Sigma^\top\rho$, the impact of which is discussed in a numerical example in [1, Chapter 7].

Before we prove the above theorem we motivate the present approach which can also be found in [34]. Note that in contrast to [34] the coefficients in the evolution of N are not bounded. We solve this problem by using the martingale optimality principle, in particular we aim to construct processes L^π as well as a strategy π^{opt} such that

- $L_T^\pi = U(X_T^{x,\pi} \exp(F^{a,\sigma,o_1,o_2}))$ for all $\pi \in \mathcal{A}$,
- L^π is a supermartingale for all $\pi \in \mathcal{A}$ and there is a particular strategy $\pi^{\text{opt}} \in \mathcal{A}$ such that $L^{\pi^{\text{opt}}}$ is a martingale.

Note that our assumptions on the filtration then imply that $L_0^\pi = C$ for all $\pi \in \mathcal{A}$ and a constant $C > 0$. Applying the utility function to $X_T^{x,\pi} \exp(F^{a,\sigma,o_1,o_2})$ we get

$$\begin{aligned} \frac{1}{\gamma} \left(X_T^{x,\pi} \exp(F^{a,\sigma,o_1,o_2}) \right)^\gamma &= \frac{1}{\gamma} x^\gamma \exp \left(\int_0^T \gamma \pi_s^\top R_s \eta ds + \int_0^T \gamma \pi_s^\top \sqrt{R_s} dQ_s \right. \\ &\quad \left. - \frac{1}{2} \int_0^T \gamma \pi_s^\top R_s \pi_s ds + \gamma F^{a,\sigma,o_1,o_2} \right). \end{aligned}$$

This suggests the following choice of L^π

$$L_t^\pi = x^\gamma \exp \left(\int_0^t \gamma \pi_s^\top R_s \eta ds + \int_0^t \gamma \pi_s^\top \sqrt{R_s} dQ_s - \frac{1}{2} \int_0^t \gamma \pi_s^\top R_s \pi_s ds + Y_t \right),$$

where Y is the first component of the solution of a BSDE with terminal condition $\gamma F^{a,\sigma,o_1,o_2}$. More precisely we want to find a generator f for the BSDE

$$\begin{aligned} Y_t &= \gamma F^{a,\sigma,o_1,o_2} - \int_t^T \text{Tr}(Z_s^\top dW_s) - \int_t^T \text{Tr}(\hat{Z}_s^\top d\hat{Q}_s) \\ &\quad + \int_t^T f(R_s, Z_s, \hat{Z}_s) ds, \quad t \in [0, T], \end{aligned} \quad (4.7)$$

such that its solution (Y, Z, \hat{Z}) implies that L^π meets the above requirements.

Lemma 4.6. Let $\alpha \in S_d^{++}$, $B(r) = r\hat{B} + \hat{B}^\top r$ with $\hat{B} \in M_d$ and recall (4.6). If the generator $f : S_d^+ \times M_d \rightarrow \mathbb{R}$ is of the form

$$\begin{aligned} f(r, z, \hat{z}) &= \frac{1}{2} \text{Tr}(zz^\top) + \frac{1}{2} \text{Tr}(\hat{z}\hat{z}^\top) + \frac{\gamma}{2(1-\gamma)} |\sqrt{r}\eta + z\rho|^2, \\ (r, z, \hat{z}) &\in S_d^+ \times M_d \times M_d, \end{aligned} \quad (4.8)$$

then (4.7) is solved by

$$\begin{aligned} Y_t &= \text{Tr}(A_{22}^{-1}(t)A_{21}(t)R_t) + \text{Tr}(aO_t) + \int_t^T \text{Tr} \left(A_{22}^{-1}(s)A_{21}(s)b + a o_1 \right) ds, \\ Z_t &= 2\sqrt{R_t}A_{22}^{-1}(t)A_{21}(t)\Sigma^\top, \\ \hat{Z}_t &= \sqrt{R_t}\sigma^\top a, \quad t \in [0, T]. \end{aligned} \quad (4.9)$$

Moreover L^π is a supermartingale for every strategy $\pi \in \mathcal{A}$ and for

$$\pi_t^{\text{opt}} = \frac{1}{1-\gamma} \left(\eta + 2A_{22}^{-1}(t)A_{21}(t)\Sigma^\top \rho \right), \quad t \in [0, T], \quad (4.10)$$

the process $L^{\pi^{\text{opt}}}$ is a martingale.

Proof. Let us define the constants

$$\begin{aligned} c_{zz} &= \frac{1}{2}I_d + \frac{\gamma}{2(1-\gamma)}\rho\rho^\top, & c_{\hat{z}\hat{z}} &= \frac{1}{2}I_d, \\ c_{z\sqrt{x}} &= \frac{\gamma}{2(1-\gamma)}\rho\eta^\top, & c_x &= \frac{\gamma}{2(1-\gamma)}\eta\eta^\top. \end{aligned}$$

Note that c_{zz} is positive definite. By [50, Proposition 8.4] we know that the ODE

$$\begin{aligned} -\frac{d\Gamma(t)}{dt} &= \Gamma(t)\Sigma^\top c_{zz}\Sigma\Gamma(t) + B^*(\Gamma(t)) + 2\Gamma(t)\Sigma^\top c_{z\sqrt{x}} + 2c_{z\sqrt{x}}^\top\Sigma\Gamma(t) \\ &\quad + \frac{1}{2}\sigma^\top a a^\top \sigma + c_x + a o_2, \quad \Gamma(T) = 0, \end{aligned}$$

has the solution $\Gamma(t) = A_{22}^{-1}(t)A_{21}(t)$, $t \in [0, T]$. We then obtain from Theorem 3.5 that the BSDE (4.7) with generator (4.8) is solved by (4.9).

We show the local (super)martingale property for L^π . With Itô's formula applied to L^π we have for all $\pi \in \mathcal{A}$

$$\begin{aligned} dL_t^\pi &= L_t^\pi \left(\gamma\pi_t^\top \sqrt{R_t} dQ_t + \text{Tr}(Z_t^\top dW_t) + \text{Tr}(\hat{Z}_t^\top d\hat{Q}_t) \right) \\ &\quad + L_t^\pi \left(\gamma\pi_t^\top R_t \eta - \frac{1}{2}\gamma\pi_t^\top R_t \pi_t - f(R_t, Z_t, \hat{Z}_t) \right) dt \\ &\quad + \frac{1}{2}L_t^\pi \left(\left| \gamma\sqrt{R_t}\pi_t\rho^\top + Z_t \right|^2 + \left| \gamma\sqrt{1-\rho^\top\rho}\pi_t^\top\sqrt{R_t} \right|^2 + \text{Tr}(\hat{Z}_t\hat{Z}_t^\top) \right) dt, \end{aligned}$$

where we have used $dQ_t = dW_t\rho + \sqrt{1-\rho^\top\rho}dD_t$. If the finite variation part satisfies $dt \otimes \mathbb{P}$ -a.e.

$$\begin{aligned} L_t^\pi &\left(\gamma\pi_t^\top R_t \eta - \frac{1}{2}\gamma\pi_t^\top R_t \pi_t - f(R_t, Z_t, \hat{Z}_t) + \frac{1}{2} \left| \gamma\sqrt{R_t}\pi_t\rho^\top + Z_t \right|^2 \right. \\ &\quad \left. + \frac{1}{2} \left| \gamma\sqrt{1-\rho^\top\rho}\pi_t^\top\sqrt{R_t} \right|^2 + \frac{1}{2}|\hat{Z}_t|^2 \right) \leq 0, \end{aligned}$$

then we know that L^π is a local supermartingale. Indeed, since $L_t^\pi > 0$ for all $t \in [0, T]$ we only need to check whether

$$\begin{aligned} -f(R_t, Z_t, \hat{Z}_t) &\leq -\gamma\pi_t^\top R_t \eta + \frac{1}{2}\gamma\pi_t^\top R_t \pi_t - \frac{1}{2} \left| \gamma\sqrt{R_t}\pi_t\rho^\top + Z_t \right|^2 \\ &\quad - \frac{1}{2} \left| \gamma\sqrt{1-\rho^\top\rho}\pi_t^\top\sqrt{R_t} \right|^2 - \frac{1}{2}|\hat{Z}_t|^2. \end{aligned}$$

This is equivalent to

$$\begin{aligned} -f(R_t, Z_t, \hat{Z}_t) &\leq -\gamma\pi_t^\top R_t \eta + \frac{1}{2}\gamma\pi_t^\top R_t \pi_t - \frac{1}{2}|Z_t|^2 - \gamma \text{Tr}(\sqrt{R_t}\pi_t\rho^\top Z_t^\top) \\ &\quad - \frac{1}{2}\gamma^2\pi_t^\top R_t \pi_t - \frac{1}{2}|\hat{Z}_t|^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \gamma (1 - \gamma) \left| \sqrt{R_t} \pi_t - \frac{1}{1 - \gamma} (\sqrt{R_t} \eta + Z_t \rho) \right|^2 \\
&\quad - \frac{\gamma}{2(1 - \gamma)} \left| \sqrt{R_t} \eta + Z_t \rho \right|^2 - \frac{1}{2} |Z_t|^2 - \frac{1}{2} |\hat{Z}_t|^2.
\end{aligned}$$

If we use (4.8), we see that this inequality is true for every $\pi \in \mathcal{A}$. For π^{opt} from (4.10) and applying the particular form of Z , the above inequality turns out to be an equality and hence the process $L^{\pi^{\text{opt}}}$ is a local martingale. Note that $\pi^{\text{opt}} \in \mathcal{A}$.

We proceed showing that L^π is a true supermartingale for all $\pi \in \mathcal{A}$. By definition there exists a sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ converging to T such that $L^{\pi_{\cdot \wedge \tau_n}}$ is a supermartingale. Since L^π is bounded below by zero we may use Fatou's lemma to pass to the limit:

$$\begin{aligned}
\mathbb{E}[L_t^\pi | \mathcal{F}_s] &= \mathbb{E}\left[\lim_{n \rightarrow \infty} L_{t \wedge \tau_n}^\pi | \mathcal{F}_s\right] \leq \lim_{n \rightarrow \infty} \mathbb{E}[L_{t \wedge \tau_n}^\pi | \mathcal{F}_s] \leq \lim_{n \rightarrow \infty} L_{s \wedge \tau_n}^\pi = L_s^\pi, \\
s &\leq t \in [0, T].
\end{aligned}$$

Note that from (4.9) and (4.10) we have

$$\begin{aligned}
L_t^{\pi^{\text{opt}}} &= x^\gamma \mathcal{E}\left(\gamma \int_0^t (\pi^{\text{opt}})_s^\top \sqrt{R_s} dQ_s + \int_0^t \text{Tr}\left(2\Sigma(A_{22}^{-1}(s)A_{21}(s))^\top \sqrt{R_s} dW_s\right) \right. \\
&\quad \left. + \int_0^t \text{Tr}\left(a^\top \sigma \sqrt{R_s} d\hat{Q}_s\right)\right).
\end{aligned}$$

By choosing $\sigma_Q(s) = \gamma \pi_s^{\text{opt}}$, $\sigma_W(s) = 2\Sigma(A_{22}^{-1}(s)A_{21}(s))^\top$, $\sigma_{\hat{Q}}(s) \equiv a^\top \sigma$ and $\sigma_\mu \equiv 0$, $s \in [0, T]$, we derive from Proposition 4.2 that $L^{\pi^{\text{opt}}}$ is a true martingale. \square

Proof of Theorem 4.5. Note that we derive from Lemma 4.6 for all $\pi \in \mathcal{A}$

$$\mathbb{E}[U(X_T^{x, \pi} \exp(F^{a, \sigma, o_1, o_2}))] = \mathbb{E}\left[\frac{1}{\gamma} L_T^\pi\right] \leq \mathbb{E}\left[\frac{1}{\gamma} L_0^\pi\right] = \frac{1}{\gamma} x^\gamma \exp(Y_0).$$

The strategy π^{opt} is indeed optimal since we have that $L^{\pi^{\text{opt}}}$ is a martingale and hence

$$\mathbb{E}\left[U(X_T^{x, \pi^{\text{opt}}} \exp(F^{a, \sigma, o_1, o_2}))\right] = \mathbb{E}\left[\frac{1}{\gamma} L_0^{\pi^{\text{opt}}}\right].$$

This immediately gives the value function. \square

Remark 4.7. In dimension $d = 1$, for the case $F(O_T) = 0$ and with a slightly different choice of parameters, this result was derived by the authors of [38]. They represent the optimal strategy in terms of an opportunity process and use semimartingale characteristics. In our setting the opportunity process is e^Y , see also [45] and in particular [33] for a survey on the relationship between BSDEs and duality methods in utility maximization. On a heuristic level the result for $d = 1$ and $F(O_T) = 0$ appears in [41]. Also using duality methods [19] derive a result similar to Theorem 4.5.

Finally we are able to give the indifference value of change of numeraire in two examples. Let us first look at the special situation where

$$\begin{aligned}
F^{-I_d, 0, o_3, 0} &= -\text{Tr}(o_3)T \quad \text{or} \quad F^{-I_d, 0, o_1, o_2} = -\int_0^T \text{Tr}(o_1 + o_2 R_s) ds, \\
o_1, \dots, o_3 &\in M_d,
\end{aligned}$$

and understand this as the possibly stochastic discounting of the investors terminal wealth. The indifference value p of changing between those two numeraire is then defined by

$$V^{-I_d,0,o_1,o_2}(x - p(x)) = V^{-I_d,0,o_3,0}(x).$$

Proposition 4.8. *The indifference value of changing from a fixed interest rate $F^{-I_d,0,o_3,0}$ to the floating one $F^{-I_d,0,o_1,o_2}$ is*

$$p(x) = x - x \exp \left(\frac{1}{\gamma} \left(\text{Tr}(B_{22}^{-1}(0)B_{21}(0)r) + \int_0^T \text{Tr} \left(B_{22}^{-1}(s)B_{21}(s)b - o_3 \right) ds \right. \right. \\ \left. \left. - \text{Tr}(A_{22}^{-1}(0)A_{21}(0)r) - \int_0^T \text{Tr} \left(A_{22}^{-1}(s)A_{21}(s)b - o_1 \right) ds \right) \right),$$

with

$$\begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} \\ = \exp \left((T-t) \begin{pmatrix} \frac{\gamma}{1-\gamma} \Sigma^\top \rho \eta^\top + \hat{B}^\top & -2\alpha - \frac{2\gamma}{1-\gamma} \Sigma^\top \rho \rho^\top \Sigma \\ \frac{\gamma}{2(1-\gamma)} \eta \eta^\top - o_2 & -\frac{\gamma}{1-\gamma} \eta \rho^\top \Sigma - \hat{B} \end{pmatrix} \right) \\ \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix} \\ = \exp \left((T-t) \begin{pmatrix} \frac{\gamma}{1-\gamma} \Sigma^\top \rho \eta^\top + \hat{B}^\top & -2\alpha - \frac{2\gamma}{1-\gamma} \Sigma^\top \rho \rho^\top \Sigma \\ \frac{\gamma}{2(1-\gamma)} \eta \eta^\top & -\frac{\gamma}{1-\gamma} \eta \rho^\top \Sigma - \hat{B} \end{pmatrix} \right).$$

Proof. By Theorem 4.5 we have that

$$V^{-I_d,0,o_1,o_2}(x - p(x)) = \frac{1}{\gamma}(x - p(x))^\gamma \exp \left(\text{Tr}(A_{22}^{-1}(0)A_{21}(0)r) \right. \\ \left. + \int_0^T \text{Tr} \left(A_{22}^{-1}(s)A_{21}(s)b - o_1 \right) ds \right) \\ V^{-I_d,0,o_3,0}(x) = \frac{1}{\gamma}x^\gamma \exp \left(\text{Tr}(B_{22}^{-1}(0)B_{21}(0)r) + \int_0^T \text{Tr} \left(B_{22}^{-1}(s)B_{21}(s)b - o_3 \right) ds \right),$$

which gives the result. \square

In a similar way we can describe the indifference value of change of numeraire from a fixed exchange rate

$$F^{I_d,0,o_3,0} = \text{Tr}(o_3)T,$$

with $o_3 \in M_d$ to a random valued exchange rate

$$F^{a,\sigma,o_1,o_2} = \text{Tr}(aO_T),$$

with $a \in M_d$ and O_T from (4.5).

Proposition 4.9. *The indifference value of changing from a fixed exchange rate $F^{-I_d,0,o_3,0}$ to the floating one F^{a,σ,o_1,o_2} is*

$$p(x) = x - x \exp \left(\frac{1}{\gamma} \left(\text{Tr}(B_{22}^{-1}(0)B_{21}(0)r) + \int_0^T \text{Tr} \left(B_{22}^{-1}(s)B_{21}(s)b + o_3 \right) ds \right. \right. \\ \left. \left. - \text{Tr}(A_{22}^{-1}(0)A_{21}(0)r) - \int_0^T \text{Tr} \left(A_{22}^{-1}(s)A_{21}(s)b + ao_1 \right) ds \right) \right),$$

with

$$\begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} \\ = \exp \left((T-t) \begin{pmatrix} \frac{\gamma}{1-\gamma} \Sigma^\top \rho \eta^\top + \hat{B}^\top & -2\alpha - \frac{2\gamma}{1-\gamma} \Sigma^\top \rho \rho^\top \Sigma \\ \frac{1}{2} \sigma^\top a a^\top \sigma + \frac{\gamma}{2(1-\gamma)} \eta \eta^\top + ao_2 & -\frac{\gamma}{1-\gamma} \eta \rho^\top \Sigma - \hat{B} \end{pmatrix} \right) \\ \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix} \\ = \exp \left((T-t) \begin{pmatrix} \frac{\gamma}{1-\gamma} \Sigma^\top \rho \eta^\top + \hat{B}^\top & -2\alpha - \frac{2\gamma}{1-\gamma} \Sigma^\top \rho \rho^\top \Sigma \\ \frac{\gamma}{2(1-\gamma)} \eta \eta^\top & -\frac{\gamma}{1-\gamma} \eta \rho^\top \Sigma - \hat{B} \end{pmatrix} \right).$$

Since the proof is very similar to the proof of Proposition 4.8 we omit it here.

4.2.2. Exponential utility

In this section we want to solve the utility maximization problem for the exponential utility function

$$U(x) = -\exp(-\gamma x), \quad x \in \mathbb{R},$$

where $\gamma > 0$ denotes the risk aversion. Note that we have already discussed this problem in the one-dimensional case in the beginning of Section 3 and now study it in a multivariate setting in detail. We are also interested in pricing variance swaps which depend on the realized variance via utility indifference pricing. We consider the case where the variance swaps are not available in the market and the initial capital x is invested in the (incomplete) financial market H . For $i = 1, \dots, d$, a variance swap on the i th asset of maturity T is a contract which pays

$$\frac{1}{T} \int_0^T (R_{ii})_s ds$$

at terminal time T in exchange for a previously fixed amount K_i . That is to say the payoff of a variance swap on H_i is a function of $O_T = \int_0^T R_s ds$, more precisely

$$F^i(O_T) = \text{Tr}(a^{ii} O_T) - K_i,$$

where $a^{ii} = \frac{1}{T} e^{ii}$. If we are only interested in the utility maximization problem without a random endowment, i.e. $F^i = 0$, we define $a^{ii} = 0$, $K_i = 0$ for $i = 0$.

In this section we also need a notion of admissibility. For $\mathbb{R}_+^m \times \mathbb{R}^n$ -valued affine stochastic volatility models Vierthauer [52] shows in Theorem 3.17 that the optimal strategy in the

exponential utility maximization problem is a deterministic function of time. Motivated by this we introduce the set \mathcal{A} of admissible trading strategies as the set of d -dimensional deterministic functions of time $\pi = (\pi(t))_{t \in [0, T]}$. This time, the trading strategy π describes the amount of money invested in the stocks H so that the number of shares is π_j/H_j for $j = 1, \dots, d$. The wealth process $X^{x, \pi}$ corresponding to strategy π and initial capital x is then given by

$$X_t^{x, \pi} = x + \sum_{i=1}^d \int_0^t \frac{\pi_i(s)}{H_{i,s}} dH_{i,s} = x + \int_0^t \pi^\top(s) R_s \eta ds + \int_0^t \pi^\top(s) \sqrt{R_s} dQ_s.$$

Remark 4.10. Note that we measure the trading strategies π in different units than in the power utility case. This then leads to a similar exponential structure in the process L . This was also deployed in [34] for example.

In the following theorem we characterize the maximal expected utility from trading in the financial market in presence of a variance swap on the i th asset

$$V^{F^i}(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E} \left[-\exp \left(-\gamma \left(X_T^{x, \pi} + F^i(O_T) \right) \right) \right], \quad x \in \mathbb{R}, \quad (4.11)$$

and the optimal strategy π^{F^i} for $i = 0, 1, \dots, d$.

Theorem 4.11. For $i \in \{0, 1, \dots, d\}$ let Γ^i be the solution of the ODE

$$\begin{aligned} -\frac{\partial \Gamma^i(t)}{\partial t} &= \Gamma^i(t) \left(-2\gamma\alpha + 2\gamma \Sigma^\top \rho \rho^\top \Sigma \right) \Gamma^i(t) + B^*(\Gamma^i(t)) \\ &\quad - \frac{1}{\gamma} \Gamma^i(t) \Sigma^\top \rho \eta^\top - \frac{1}{\gamma} \eta \rho^\top \Sigma \Gamma^i(t) + \frac{1}{2\gamma^3} \eta \eta^\top + a^{ii}, \\ \Gamma^i(T) &= 0, \end{aligned} \quad (4.12)$$

for all $t \in [0, T]$. Then the value function has the form

$$V^{F^i}(x) = -\exp \left(-\gamma \left(x - K_i + \text{Tr}(\Gamma^i(0)r) + \int_0^T \text{Tr}(\Gamma^i(s)b) ds \right) \right), \quad x \in \mathbb{R},$$

and the optimal strategy π^{F^i} is given by

$$\pi^{F^i}(t) = \frac{1}{\gamma^2} \eta - 2\Gamma^i(t) \Sigma^\top \rho, \quad (4.13)$$

for all $t \in [0, T]$.

Note that the optimal strategy is only then a constant cash amount (equivalent to the classical Merton problem) if the Brownian motions driving the risky assets and the covariation process are uncorrelated. As before we want to use the martingale optimality principle in order to establish this theorem. Therefore we dynamize the problem. For all $i = 0, 1, \dots, d$, we define

$$L_t^{\pi, i} = -\exp(-\gamma(X_t^{x, \pi} + Y_t^i)), \quad t \in [0, T], \pi \in \mathcal{A},$$

where (Y^i, Z^i) is the solution to

$$Y_t^i = F^i(O_T) - \int_t^T \text{Tr}((Z_s^i)^\top dW_s) + \int_t^T f(R_s, Z_s^i) ds, \quad t \in [0, T]. \quad (4.14)$$

The generator f needs to be selected in a way such that $L^{\pi,i}$ possesses the desired properties. This is done in the following lemma.

Lemma 4.12. For $i \in \{0, 1, \dots, d\}$, let the generator $f : S_d^+ \times M_d \rightarrow \mathbb{R}$ have the form

$$f(r, z^i) = -\frac{\gamma}{2} \text{Tr}(z^i (z^i)^\top) + \frac{1}{2\gamma} \left| \frac{1}{\gamma} \sqrt{r} \eta - \gamma z^i \rho \right|^2. \quad (4.15)$$

Then the solution to (4.14) is given by

$$\begin{aligned} Y_t^i &= \text{Tr}(\Gamma^i(t) R_t) + \text{Tr}(a^{ii} O_t) - K_i + \int_t^T \text{Tr}(\Gamma^i(s) b) ds, \\ Z_t^i &= 2\sqrt{R_t} \Gamma^i(t) \Sigma^\top, \end{aligned} \quad (4.16)$$

where $\Gamma^i \in S_d^+$ is the solution to (4.12) and $O_t = \int_0^t R_s ds$. Furthermore $L^{\pi,i}$ is a supermartingale for every strategy $\pi \in \mathcal{A}$ and for

$$\pi^{F^i}(t) = \frac{1}{\gamma^2} \eta - 2\Gamma^i(t) \Sigma^\top \rho, \quad (4.17)$$

the process $L^{\pi^{F^i},i}$ is a martingale.

Proof. Fix $i \in \{0, 1, \dots, d\}$ and define

$$c_{zz} = \frac{\gamma}{2} (\rho \rho^\top - I_d), \quad c_{z\sqrt{x}} = -\frac{1}{2\gamma} \rho \eta^\top, \quad c_x = \frac{1}{2\gamma^3} \eta \eta^\top + a^{ii}.$$

Note that c_{zz} is negative definite. Indeed, if $\rho = 0$, we have $c_{zz} = -\frac{\gamma}{2} I_d \in S_d^{--}$. If $\rho \neq 0$, we know that $I_d - \rho \rho^\top$ is the inverse of the positive definite matrix $I_d + \frac{1}{1-\rho^\top \rho} \rho \rho^\top$ and hence is itself positive definite. The conclusion is that $c_{zz} \in S_d^{--}$. Then by [50, Proposition 8.6] there exists a unique solution $\Gamma^i \in S_d^+$. This allows us to find solution (4.16) via Theorem 3.5.

Fix $\pi \in \mathcal{A}$. By Itô's formula we see that $L^{\pi,i}$ can be described by the product of the local martingale

$$M_t^{\pi,i} = -L_0^{\pi,i} \mathcal{E} \left(-\gamma \left(\int_0^t \pi^\top(s) \sqrt{R_s} dQ_s - \int_0^t \text{Tr}((Z_s^i)^\top dW_s) \right) \right),$$

and the bounded variation process

$$\begin{aligned} A_t^{\pi,i} &= -\exp \left(\int_0^t \left(-\gamma \pi^\top(s) R_s \eta + \gamma f(R_s, Z_s^i) + \frac{1}{2} \gamma^2 |\sqrt{R_s} \pi(s) \rho^\top + (Z_s^i)^\top|^2 \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \gamma^2 |\sqrt{1 - \rho^\top \rho} \pi^\top(s) \sqrt{R_s}|^2 \right) ds \right). \end{aligned}$$

Proposition 4.2 implies that $M^{\pi,i}$ is a true martingale. The process $A^{\pi,i}$ is non-increasing, if

$$\begin{aligned} &-\gamma \pi^\top(s) R_s \eta + \gamma f(R_s, Z_s^i) + \frac{1}{2} \gamma^2 |\sqrt{R_s} \pi(s) \rho^\top + (Z_s^i)^\top|^2 \\ &+ \frac{1}{2} \gamma^2 |\sqrt{1 - \rho^\top \rho} \pi^\top(s) \sqrt{R_s}|^2 \geq 0 \end{aligned}$$

for all $s \in [0, T]$. This is equivalent to

$$\begin{aligned} -f(R_t, Z_t^i) &\leq \frac{\gamma}{2} \text{Tr}(Z_t^i (Z_t^i)^\top) + \gamma \text{Tr}(\sqrt{R_t} \pi(t) \rho^\top (Z_t^i)^\top) \\ &\quad - \frac{1}{\gamma} \pi(t)^\top R_t \eta + \frac{\gamma}{2} |\pi^\top \sqrt{R_t}|^2 \\ &= \frac{\gamma}{2} \left| \sqrt{R_t} \pi(t) - \left(\frac{1}{\gamma^2} \sqrt{R_t} \eta - Z_t^i \rho \right) \right|^2 - \frac{1}{2\gamma} \left(\frac{1}{\gamma} \sqrt{R_t} \eta - \gamma Z_t^i \rho \right)^2 \\ &\quad + \frac{\gamma}{2} \text{Tr}(Z_t^i (Z_t^i)^\top), \end{aligned} \quad (4.18)$$

which holds true by formula (4.15). Hence $A^{\pi,i}$ is non-increasing and $L^{\pi,i} = M^{\pi,i} A^{\pi,i}$ is a supermartingale. From (4.18) we see in particular that $A^{\pi^{F^i},i} = -1$ is constant and thus $L^{\pi^{F^i},i} = -M^{\pi^{F^i},i}$ is a true martingale. \square

Proof of Theorem 4.11. Follows the same reasoning as Theorem 4.5. \square

For all $i \in \{1, \dots, d\}$, the indifference price of the variance swap

$$F^i(O_T) = \text{Tr}(a^{ii} O_T) - K_i$$

on the i th asset is defined as the value p^i for which the investor is indifferent between buying F^i for the amount p^i and receiving a random income F^i at terminal time T or not having it, i.e.

$$V^{F^i}(x - p^i) = V^0(x),$$

for all $x \in \mathbb{R}$. The optimal strategy π^{F^i} which attains the maximal expected utility in the presence of F^i can be decomposed into a sum of a pure investment part π^0 and a hedging component Δ^i , i.e.

$$\pi^{F^i}(t) = \pi^0(t) + \Delta^i(t), \quad t \in [0, T]. \quad (4.19)$$

We therefore call Δ^i the optimal hedge.

Proposition 4.13. For $i \in \{1, \dots, d\}$ the indifference price p^i and the optimal hedge Δ^i of $F^i(O_T)$ are explicitly given by

$$\begin{aligned} p^i &= -K_i + \text{Tr}((\Gamma^i(0) - \Gamma^0(0))r) + \int_0^T \text{Tr}((\Gamma^i(s) - \Gamma^0(s))b)ds, \\ \Delta^i(t) &= 2(\Gamma^i(t) - \Gamma^0(t))\Sigma^\top \rho, \quad t \in [0, T], \end{aligned}$$

where Γ^i and Γ^0 are the solutions of (4.12).

Proof. Fix $i \in \{1, \dots, d\}$ and recall the value functions

$$\begin{aligned} V^{F^i}(x - p^i) &= -\exp\left(-\gamma\left(x - p^i - K_i + \text{Tr}(\Gamma^i(0)r) + \int_0^T \text{Tr}(\Gamma^i(s)b)ds\right)\right), \\ V^0(x) &= -\exp\left(-\gamma\left(x + \text{Tr}(\Gamma^0(0)r) + \int_0^T \text{Tr}(\Gamma^0(s)b)ds\right)\right), \end{aligned}$$

from Theorem 4.11. Equating them immediately gives the first part of the result. The second part then follows from (4.13) and (4.19). \square

4.3. Solution in a multivariate affine stochastic volatility model with jumps

We now consider a model with jumps which is a natural multivariate extension of the model of [2] and has been applied e.g. in optimal portfolio selection, see [8] and the references therein. As before the asset price process H is modeled as stochastic exponential $H = H_0 \mathcal{E}(N)$ with

$$dN_t = R_t \eta dt + \sqrt{R_t} dQ_t, \quad t \in [0, T], \quad (4.20)$$

where Q is a d -dimensional vector Brownian motion and η a constant parameter. By R we denote the Ornstein–Uhlenbeck-type stochastic process with dynamics

$$dR_t = (\lambda + \Lambda(R_t))dt + dJ_t, \quad (4.21)$$

and a starting value $R_0 = r$. Here $\lambda \in S_d^+$ and $\Lambda : S_d \rightarrow S_d$ is the linear map $\Lambda(r) = \sum_{i,j} \beta^{ij} r_{ij}$ with $\beta^{ij} = \beta^{ji} \in S_d$ and such that $\text{Tr}(\Lambda(r)x) \geq 0$ for all $r, x \in S_d^+$ with $\text{Tr}(rx) = 0$. We denote its adjoint operator by Λ^* . The process J is an independent affine process with admissible parameter set $(0, b^J, 0, m^J, 0)$, starting at 0. Our goal is again to maximize the expected terminal wealth from trading in the market.

Proposition 4.14. *The process (R, N) is a multivariate stochastic volatility model with functions Φ and Ψ solving*

$$\begin{aligned} \frac{\partial \Phi(t, u, v)}{\partial t} &= (\lambda + b^J) \Psi(t, u, v) - \int_{S_d^+ \setminus \{0\}} (e^{\text{Tr}(\xi \Psi(t, u, v))} - 1) m^J(d\xi), \\ \Phi(0, u, v) &= 0, \\ \frac{\partial \Psi(t, u, v)}{\partial t} &= \Lambda^*(\Psi(t, u, v)) + \frac{1}{2} v v^\top, \quad \Psi(0, u, v) = u, \end{aligned} \quad (4.22)$$

for all $(t, u, v) \in \mathcal{Q}$.

For brevity reasons the proof is omitted.

4.3.1. Power utility

The investor wants to maximize their expected utility of terminal wealth, i.e. we search for the value function

$$V(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E} \left[\frac{1}{\gamma} (X_T^{x, \pi})^\gamma \right], \quad x \geq 0, \quad \gamma \in (0, 1).$$

Here the strategies π and the wealth process $X^{x, \pi}$ are defined as in Section 4.3.1. We describe the value function and the optimal strategy of the maximization problem in terms of an ODE.

Theorem 4.15. *Suppose the jump measure m^J satisfies*

$$\int_{|\text{Tr}(\Gamma(t)\xi)| > 1} e^{-\text{Tr}(\Gamma(t)\xi)} m^J(d\xi) < \infty, \quad t \in [0, T],$$

where Γ is the solution of the ODE

$$-\frac{d\Gamma(t)}{dt} = \Lambda^*(\Gamma(t)) - \frac{\gamma}{2(1-\gamma)} \eta \eta^\top, \quad \Gamma(T) = 0. \quad (4.23)$$

Then the value function is given by

$$V(x) = \frac{1}{\gamma} x^\gamma \exp \left(-\text{Tr}(\Gamma(0)r) - \int_0^T \text{Tr}(\Gamma(s)(b^J + \lambda))ds \right. \\ \left. - \int_0^T \int_{S_d^+ \setminus \{0\}} (e^{-\text{Tr}(\Gamma(s)\xi)} - 1)m^J(d\xi)ds \right),$$

for $x \geq 0$, and the optimal strategy π^{opt} is

$$\pi_t^{\text{opt}} \equiv \frac{1}{1-\gamma} \eta, \quad t \in [0, T].$$

As we have chosen a stochastic covariation process without a diffusion part, the obtained optimal strategy becomes a constant just as in the classical Merton problem. Note that the jumps in R influence the maximal utility but not the strategy leading to this utility.

We choose the process L^π as we did in Section 4.3.1, only that this time the involved BSDE is the following

$$Y_t = 0 - \int_t^T \int_{S_d^+ \setminus \{0\}} K_s(\xi)(\mu^J(ds, d\xi) - m^J(d\xi)ds) + \int_t^T f(R_s, K_s)ds, \quad (4.24)$$

$t \in [0, T]$.

Lemma 4.16. Let the jump measure m^J satisfy

$$\int_{|\text{Tr}(\Gamma(t)\xi)| > 1} e^{-\text{Tr}(\Gamma(t)\xi)} m^J(d\xi) < \infty, \quad t \in [0, T], \quad (4.25)$$

where Γ is the solution of (4.23). Suppose the generator in (4.7) is of the following form

$$f(r, k) = -\frac{\gamma}{2(1-\gamma)} \eta^\top r \eta - \int_{S_d^+ \setminus \{0\}} \left(e^{-k(\xi)} - 1 + k(\xi) \right) m^J(d\xi), \quad (4.26)$$

for all $r \in S_d^+$ and $k : S_d^+ \rightarrow \mathbb{R}$. Then BSDE (4.24) is solved by

$$Y_t = \text{Tr}(\Gamma(t)R_t) + \int_t^T \text{Tr}(\Gamma(s)(b^J + \lambda))ds + \int_t^T \int_{S_d^+ \setminus \{0\}} (1 - e^{-\text{Tr}(\Gamma(s)\xi)})m^J(d\xi)ds \\ K_t(\xi) = \text{Tr}(\Gamma(t)\xi), \quad t \in [0, T], \quad \xi \in S_d^+.$$

Moreover L^π is a supermartingale for every strategy $\pi \in \mathcal{A}$ and if π^{opt} satisfies

$$\pi_t^{\text{opt}} = \frac{1}{1-\gamma} \eta, \quad t \in [0, T], \quad (4.27)$$

then $L^{\pi^{\text{opt}}}$ is a martingale.

Proof. For all $y \in \mathbb{R}$ we define

$$c_x = -\frac{\gamma}{2(1-\gamma)} \eta \eta^\top, \quad g_t(y) = -e^{-y} + 1 - y.$$

Then we can see that by [50, Corollary 8.7] there exists a unique solution Γ with values in S_d^- to (4.23). As a result we find the above solution of (4.24) with Theorem 3.5.

We apply Itô's formula which gives that for all $\pi \in \mathcal{A}$

$$\begin{aligned} dL_t^\pi &= L_t^\pi \left(\gamma \pi_t^\top \sqrt{R_t} dQ_t \right) \\ &+ L_t^\pi \left(\gamma \pi_t^\top R_t \eta - \frac{1}{2} \gamma \pi_t^\top R_t \pi_t + f(R_t, K_t) + \frac{1}{2} \gamma^2 \pi_t^\top R_t \pi_t \right) dt \\ &+ L_t^\pi \int_{S_d^+ \setminus \{0\}} (e^{-K_t(\xi)} - 1) \left(\mu^J(dt, d\xi) - m^J(d\xi) dt \right) \\ &+ L_t^\pi \int_{S_d^+ \setminus \{0\}} \left(e^{-K_t(\xi)} - 1 + K_t(\xi) \right) m^J(d\xi) dt, \end{aligned}$$

where we have used integrability condition (4.25). This means that L^π is a local supermartingale for all $\pi \in \mathcal{A}$, if the finite variation part $dt \otimes \mathbb{P}$ -a.e. satisfies

$$\begin{aligned} L_t^\pi &\left(\gamma \pi_t^\top R_t \eta - \frac{1}{2} \gamma \pi_t^\top R_t \pi_t + f(R_t, K_t) + \frac{1}{2} \gamma^2 \pi_t^\top R_t \pi_t \right. \\ &\left. + \int_{S_d^+ \setminus \{0\}} \left(e^{-K_t(\xi)} - 1 + K_t(\xi) \right) m^J(d\xi) \right) \leq 0. \end{aligned}$$

Since $L^\pi > 0$, the generator f needs to fulfill

$$f(R_t, K_t) \leq -\gamma \pi_t^\top R_t \eta + \frac{1}{2} \gamma (1 - \gamma) \pi_t^\top R_t \pi_t - \int_{S_d^+ \setminus \{0\}} \left(e^{-K_t(\xi)} - 1 + K_t(\xi) \right) m^J(d\xi),$$

which is equivalent to

$$\begin{aligned} f(R_t, K_t) &\leq \frac{1}{2} \gamma (1 - \gamma) |\sqrt{R_t} \pi_t - \frac{1}{1 - \gamma} \sqrt{R_t} \eta|^2 - \frac{\gamma}{2(1 - \gamma)} |\sqrt{R_t} \eta|^2 \\ &- \int_{S_d^+ \setminus \{0\}} \left(e^{-K_t(\xi)} - 1 + K_t(\xi) \right) m^J(d\xi). \end{aligned}$$

With (4.26) this inequality is true for all $\pi \in \mathcal{A}$ and hence L^π a local supermartingale. Obviously the inequality is an equality for π^{opt} , for which $L^{\pi^{\text{opt}}}$ is then a local martingale.

Since L^π is bounded below by 0, the fact that L^π is a supermartingale for all $\pi \in \mathcal{A}$ follows as in the proof of Lemma 4.6 by Fatou's lemma. Note that the process $L^{\pi^{\text{opt}}}$ is given by

$$\begin{aligned} L_t^{\pi^{\text{opt}}} &= x^\gamma \mathcal{E} \left(\int_0^t \gamma (\pi_s^{\text{opt}})^\top \sqrt{R_s} dQ_s \right. \\ &\left. + \int_0^t \int_{S_d^+ \setminus \{0\}} (e^{-K_s(\xi)} - 1) (\mu^J(ds, d\xi) - m^J(d\xi) ds) \right), \end{aligned}$$

and hence, setting $\sigma_Q(s) \equiv \frac{\gamma}{1-\gamma} \eta$, $\sigma_W(s) \equiv 0$ and $\sigma_\mu(s) = -\Gamma(s)$, $s \in [0, T]$, we obtain the martingale property of $L^{\pi^{\text{opt}}}$ by Proposition 4.2. \square

Proof of Theorem 4.15. It is omitted since it follows the same idea as the proof of Theorem 4.5. \square

4.3.2. Exponential utility

Using the same notation as in Section 4.2.2, we will compute the utility indifference prices for variance swaps. This means that we need to solve the problem

$$V^{F^i}(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E} \left[U \left(X_t^{x, \pi} + F^i \right) \right], \quad x \geq 0,$$

for $i \in \{0, 1, \dots, d\}$, which is done in the following theorem.

Theorem 4.17. *Let $i \in \{0, 1, \dots, d\}$,*

$$\int_{|\gamma \operatorname{Tr}(\Gamma^i(s)\xi)| > 1} e^{\gamma \operatorname{Tr}(\Gamma^i(s)\xi)} m^J(d\xi) < \infty, \quad s \in [0, T],$$

where Γ^i is the solution of the ODE

$$-\frac{d\Gamma^i(t)}{dt} = \Lambda^*(\Gamma^i(t)) + \frac{1}{2\gamma} \eta \eta^\top + a^{ii}, \quad \Gamma^i(T) = 0. \quad (4.28)$$

Then the value function satisfies

$$V^{F^i}(x) = -\exp \left(-\gamma \left(x - K_i + \operatorname{Tr}(\Gamma^i(0)r) + \int_0^T \operatorname{Tr}(\Gamma^i(s)(b^J + \lambda)) ds \right. \right. \\ \left. \left. - \frac{1}{\gamma} \int_0^T \int_{S_d^+ \setminus \{0\}} \left(e^{\gamma \operatorname{Tr}(\Gamma^i(s)\xi)} - 1 \right) m^J(d\xi) ds \right) \right),$$

and the optimal strategy π^{F^i} is given by

$$\pi^{F^i}(t) \equiv \frac{1}{\gamma} \eta, \quad t \in [0, T].$$

We will use the martingale optimality principle again and construct for every $i \in \{0, 1, \dots, d\}$, a process

$$L_t^{\pi, i} = -\exp(-\gamma(X_t^{x, \pi} + Y_t^i)), \quad t \in [0, T], \quad \pi \in \mathcal{A},$$

where (Y^i, K^i) is the solution of

$$Y_t^i = F^i - \int_t^T \int_{S_d^+ \setminus \{0\}} K_s^i(\xi) (\mu^J(ds, d\xi) - m^J(d\xi)ds) + \int_t^T f(R_s, K_s^i) ds. \quad (4.29)$$

The generator of BSDE (4.29) needs to be chosen in the following way.

Lemma 4.18. *Let $i \in \{0, 1, \dots, d\}$,*

$$\int_{|\gamma \operatorname{Tr}(\Gamma^i(t)\xi)| > 1} e^{\gamma \operatorname{Tr}(\Gamma^i(t)\xi)} m^J(d\xi) < \infty,$$

for all $t \in [0, T]$ and with Γ^i being the solution of (4.28). Let the generator f in (4.29) have the form

$$f(r, k^i) = \frac{1}{2\gamma} \eta^\top r \eta - \frac{1}{\gamma} \int_{S_d^+ \setminus \{0\}} \left(e^{\gamma k^i(\xi)} - 1 + \gamma k^i(\xi) \right) m^J(d\xi), \quad (4.30)$$

for all $r \in S_d^+$ and $k^i : S_d^+ \rightarrow \mathbb{R}$. Then the solution of BSDE (4.29) is given by

$$\begin{aligned} Y_t^i &= \text{Tr}(\Gamma^i(t)R_t) + \text{Tr}\left(a^{ii} \int_0^t R_s ds\right) - K_i + \int_t^T \text{Tr}(\Gamma^i(s)(b^J + \lambda))ds \\ &\quad - \frac{1}{\gamma} \int_t^T \int_{S_d^+ \setminus \{0\}} \left(e^{\gamma \text{Tr}(\Gamma^i(s)\xi)} - 1\right) m^J(d\xi) ds, \\ K_t^i(\xi) &= \text{Tr}(\Gamma^i(t)\xi), \quad t \in [0, T], \quad \xi \in S_d^+. \end{aligned} \quad (4.31)$$

Moreover for all $\pi \in \mathcal{A}$ the process $L^{\pi,i}$ is a supermartingale and $L^{\pi^{F^i},i}$ is a martingale.

Proof. Fix $i \in \{0, 1, \dots, d\}$ and define for all $y \in \mathbb{R}$

$$g_x = \frac{1}{2\gamma} \eta \eta^\top + a^{ii}, \quad g_t(y) = -\frac{1}{\gamma} (e^{\gamma y} - 1 + \gamma y).$$

Using [50, Proposition 8.6] we see that there exists a unique solution $\Gamma^i \in S_d^+$ of (4.28). This implies (4.31) by Theorem 3.5.

Fix $\pi \in \mathcal{A}$. Note that $L^{\pi,i}$ can be written as a product $M^{\pi,i} V^{\pi,i}$ of the two processes

$$\begin{aligned} M_t^{\pi,i} &= -L_0^{\pi,i} \mathcal{E}\left(-\gamma \int_0^t \pi^\top(s) \sqrt{R_s} dQ_s \right. \\ &\quad \left. + \int_0^t \int_{S_d^+ \setminus \{0\}} \left(e^{\gamma K_s^i(\xi)} - 1\right) (\mu^R(ds, d\xi) - m^J(d\xi) ds)\right), \\ V_t^{\pi,i} &= -\exp\left(\int_0^t \left(-\gamma \pi^\top(s) R_s \eta + \gamma f(R_s, K_s^i) + \frac{1}{2} \gamma^2 \pi^\top(s) R_s \pi_s \right. \right. \\ &\quad \left. \left. + \int_{S_d^+ \setminus \{0\}} \left(e^{\gamma K_s^i(\xi)} - 1 + \gamma K_s^i(\xi)\right) m^J(d\xi)\right) ds\right). \end{aligned}$$

Setting $\sigma_Q(s) = -\gamma \pi_s$, $\sigma_W(s) \equiv 0$ and $\sigma_{mu}(s) = \gamma \Gamma(s)$, we have from Proposition 4.2 that $M^{\pi,i}$ is a true martingale. In order for $V^{\pi,i}$ to be decreasing, it needs to be ensured that

$$\begin{aligned} &-\gamma \pi^\top(s) R_s \eta + \gamma f(R_s, K_s^i) + \frac{1}{2} \gamma^2 \pi^\top(s) R_s \pi_s \\ &+ \int_{S_d^+ \setminus \{0\}} \left(e^{\gamma K_s^i(\xi)} - 1 + \gamma K_s^i(\xi)\right) m^J(d\xi) \geq 0, \end{aligned} \quad (4.32)$$

$ds \otimes \mathbb{P}$ -a.e. Taking formulas (4.30) and (4.31) into account this is indeed true, since (4.32) is equivalent to

$$\begin{aligned} -f(R_t, K_t^i) &\leq \frac{1}{2} \gamma \left| \pi(t)^\top \sqrt{R_t} - \frac{1}{\gamma} \eta^\top \sqrt{R_t} \right|^2 - \frac{1}{2\gamma} \eta^\top R_t \eta \\ &\quad + \int_{S_d^+ \setminus \{0\}} \frac{1}{\gamma} \left(e^{\gamma K_t^i(\xi)} - 1 + \gamma K_t^i(\xi)\right) m^J(d\xi). \end{aligned}$$

Since $M^{\pi,i}$ is a martingale and $V^{\pi,i}$ is non-increasing, $L^{\pi,i} = M^{\pi,i} V^{\pi,i}$ is a supermartingale.

It is straightforward that $V_s^{\pi^{F^i},i} = -1$ for $s \in [0, T]$ and thus $L^{\pi^{F^i},i} = -M^{\pi^{F^i},i}$ is a true martingale. \square

Proof of Theorem 4.17. The proof now follows the same reasoning as the proof of Theorem 4.5. \square

Recall that for $i \in \{1, \dots, d\}$ the indifference price of the variance swap F^i on the i th asset is the value p^i such that for all $x \in \mathbb{R}$ the value $V^{F^i}(x - p^i)$ equals $V^0(x)$.

Proposition 4.19. For $i \in \{1, \dots, d\}$ the indifference price p^i is explicitly given by

$$p^i = -K_i + \text{Tr}((\Gamma^i(0) - \Gamma^0(0))r) + \int_0^T \text{Tr}((\Gamma^i(s) - \Gamma^0(s))(b^J + \lambda))ds \\ - \frac{1}{\gamma} \int_0^T \int_{S_d^+ \setminus \{0\}} \left(e^{\gamma \text{Tr}(\Gamma^i(s)\xi)} - e^{\gamma \text{Tr}(\Gamma^0(s)\xi)} \right) m^J(d\xi),$$

where Γ^i and Γ^0 are the respective solutions of

$$\frac{\partial \Gamma^i(t)}{\partial t} = \Lambda^*(\Gamma^i(t)) + \frac{1}{2\gamma} \eta \eta^\top + a^{ii}, \quad \Gamma^i(T) = 0, \\ \frac{\partial \Gamma^0(t)}{\partial t} = \Lambda^*(\Gamma^0(t)) + \frac{1}{2\gamma} \eta \eta^\top, \quad \Gamma^0(T) = 0.$$

Proof. For $i \in \{0, 1, \dots, d\}$ it follows from Theorem 4.17 that the value functions have the form

$$V^{F^i}(x - p^i) = -\exp \left(-\gamma \left(x - p^i - K_i + \text{Tr}(\Gamma^i(0)r) + \int_0^T \text{Tr}(\Gamma^i(s)(b^J + \lambda))ds \right. \right. \\ \left. \left. - \frac{1}{\gamma} \int_0^T \int_{S_d^+ \setminus \{0\}} \left(e^{\gamma \text{Tr}(\Gamma^i(s)\xi)} - 1 \right) m^J(d\xi) \right) \right).$$

Equating $V^{F^i}(x - p^i)$ and $V^0(x)$ for $i = 1, \dots, d$, immediately gives the result. \square

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