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Finite difference schemes for linear stochastic integro-differential equations

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Abstract

We study the rate of convergence of an explicit and an implicit–explicit finite difference scheme for linear stochastic integro-differential equations of parabolic type arising in non-linear filtering of jump–diffusion processes. We show that the rate is of order one in space and order one-half in time.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$, $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$, be a complete filtered probability space such that the filtration is right continuous and $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. Let $\{w^\varrho\}_{\varrho=1}^\infty$ be a sequence of independent real-valued $\mathbf{F}$-adapted Wiener processes. Let $\pi_1(dz)$ and $\pi_2(dz)$ be a Borel sigma-finite measures on $\mathbf{R}^d$ satisfying

$$\int_{\mathbf{R}^d} |z|^2 \wedge 1 \, \pi_r(dz) < \infty, \quad r \in \{1, 2\}.$$

Let $q(dt, dz) = p(dt, dz) - \pi_2(dz)dt$ be a compensated $\mathbf{F}$-adapted Poisson random measure on $\mathbf{R}_+ \times \mathbf{R}^d$. Let $T > 0$ be an arbitrary fixed constant. On $[0, T] \times \mathbf{R}^d$, we consider finite difference

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approximations for the following stochastic integro-differential equation (SIDE)

$$du_t = ((\mathcal{L}_t + I)u_t + f_t)\,dt + \sum_{\varrho=1}^{\infty} \left(\mathcal{N}_t^{\varrho} u_t + g_t^{\varrho}\right) dw_t^{\varrho} + \int_{\mathbf{R}^d} (\mathcal{I}(z)u_{t-} + o_t(z))\, q(dt, dz), \tag{1.1}$$

with initial condition

$$u_0(x) = \varphi(x), \quad x \in \mathbf{R}^d,$$

where the operators are given by

$$\begin{aligned} \mathcal{L}_t\phi(x) &:= \sum_{i,j=0}^{d} a_t^{ij}(x)\partial_{ij}\phi(x), \\ I\phi(x) &:= \int_{\mathbf{R}^d} \left(\phi(x+z) - \phi(x) - \mathbf{1}_{[-1,1]}(|z|)\sum_{j=1}^{d} z_j\partial_j\phi(x)\right)\pi_1(dz), \\ \mathcal{N}_t^{\varrho}\phi(x) &:= \sum_{i=0}^{d} \sigma_t^{i\varrho}(x)\partial_i\phi(x), \quad \mathcal{I}(z)\phi(x) = \phi(x+z) - \phi(x). \end{aligned} \tag{1.2}$$

Here, we denote the identity operator by ∂_0.

Eq. (1.1) arises naturally in non-linear filtering of jump–diffusion processes. We refer the reader to [4,5] for more information about non-linear filtering of jump–diffusions and the derivation of the Zakai equation. Various methods have been proposed to solve stochastic partial differential equations (SPDEs) numerically. For SPDEs driven by continuous martingale noise see, for example, [3,7,8,13,21,16,23] and for SPDEs driven by discontinuous martingale noise, see [15,14,20,1]. Among the various methods considered in the literature is the method of finite differences. For second order linear SPDEs driven by continuous martingale noise it is well-known that the $L^p(\Omega)$-pointwise error of approximation in space is proportional to the parameter h of the finite difference (see, e.g., [24]). In [13], I. Gyöngy and A. Millet consider abstract discretization schemes for stochastic evolution equations driven by continuous martingale noise in the variational framework and, as a particular example, show that the $L^2(\Omega)$-pointwise rate of convergence of an Euler–Maruyama (explicit and implicit) finite difference scheme is of order one in space and one-half in time. More recently, it was shown by I. Gyöngy and N.V. Krylov that under certain regularity conditions, the rate of convergence in space of a semi-discretized finite difference approximation of a linear second order SPDE driven by continuous martingale noise can be accelerated to any order by Richardson's extrapolation method. For the non-degenerate case, we refer to [10,11], and for the degenerate case, we refer to [9]. In [17,18], E. Hall proved that the same method of acceleration can be applied to implicit time-discretized SPDEs driven by continuous martingale noise.

In the literature, finite element, spectral, and, more generally, Galerkin schemes have been studied for SPDEs driven by discontinuous martingale noise. One of the earliest works in this direction is a paper [15] by E. Hausenblas and I. Marchis concerning $L^p(\Omega)$-convergence of Galerkin approximation schemes for abstract stochastic evolution equations in Banach spaces driven by Poisson noise of impulsive-type. As an application of their result, they study a spectral approximation of a linear SPDE in $L^2([0,1])$ with Neumann boundary conditions driven by Poisson noise of impulsive-type and derive $L^p(\Omega)$-error estimates in the $L^2([0,1])$-norm.

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In [14], E. Hausenblas considers finite element approximations of linear SPDEs in polyhedral domains D driven by Poisson noise of impulsive-type and derives $L^p(\Omega)$ error estimates in the $L^p(D)$-norm. In a more recent work [20], A. Lang studied semi-discrete Galerkin approximation schemes for SPDEs of advection diffusion type in bounded domains D driven by cádlág square integrable martingales in a Hilbert space. A. Lang showed that the rate of convergence in the $L^p(\Omega)$ and almost-sure sense in the $L^2(D)$-norm is of order two for a finite-element Galerkin scheme. In [1], A. Lang and A. Barth derive $L^2(\Omega)$ and almost-sure estimates in the $L^2(D)$-norm for the error of a Milstein–Galerkin approximation scheme for the same equation considered in [20] and obtain convergence of order two in space and order one in time.

In the articles [20,1,15,14], the authors make use of the semigroup theory of stochastic evolution equations (mild solution) and only consider stochastic evolution equations in which the principal part of the operator in the drift is non-random. In this paper, since we use the variational framework (L^2-theory) of SPDEs, we are easily able to treat the case of random-coefficients.

The principal part of the operator in the drift of the Zakai equation is, in general, random, and hence numerical schemes that approximate SPDEs or SIDEs with adapted principal part are of importance. The coefficients of the Zakai equation are random if the coefficients of the SDE governing the signal depend on the observation or some observation measurable process-perhaps a control. In this case, the diffusion coefficient $a_t^{ij}(x,\omega)$ in (1.1) will be of the form $a_t^{ij}(x,\omega) = \bar{\sigma}^i(x, y_t(\omega))\bar{\sigma}^j(x, y_t(\omega))$, where $y_t(\omega)$ is an adapted random process and $\bar{\sigma}^i(x,y)$ is a diffusion coefficient in an SDE. Due to the form of the random coefficient in this case, to impose uniform boundedness of $a_t^{ij}(x,\omega)$ in t, x and ω, we need only impose uniform boundedness of $\bar{\sigma}(x,y)$ in x and y, and to impose uniformly ellipticity of $a_t^{ij}(x,\omega)$ in t, x and ω, we need only impose that standard uniform ellipticity of $\bar{\sigma}^i(x,y)\bar{\sigma}^j(x,y)$ in x and y. These assumptions are not uncommon in the SDE literature. Furthermore, since any numerical scheme for (1.1) will be implemented pathwise-note also that in filtering, one only gets to see one path of the observation-the additional computational complexity involved in implementing a numerical scheme for (1.1) with random coefficients of the form $a_t^{ij}(x,\omega) = \bar{\sigma}^i(x, y_t(\omega))\bar{\sigma}^j(x, y_t(\omega))$ compared with $a^{ij}(x) = \bar{\sigma}^i(x)\bar{\sigma}^j(y)$ is simply the time dependence of the coefficient. In the case of an implicit scheme, this does mean that one has to invert an operator at each time step, but this is the case for deterministic PDEs with time-dependent coefficients as well.

The articles [20,1,15,14] do not address the approximation of equations with non-local operators in the drift and noise. There is, however, some work in the literature on deterministic non-local differential equations. In dimension one, a finite difference scheme for degenerate integro-differential equations (deterministic) has been studied by R. Cont and E. Voltchkova in [2]. The authors in [2] first approximate the integral operator near the origin with a second derivative operator. The resulting PDE is then non-degenerate and has an integral operator of order zero. The error of this approximation is studied by means of the probabilistic representation of the solution of both the original equation and the non-degenerate equation. In the second step of their approximation, R. Cont and E. Voltchkova consider an implicit–explicit finite difference scheme and obtain pointwise error estimates of order one in space. As a consequence of the two-step approximation scheme, there are two separate errors for the approximation. We are able to avoid the two-step approximation in our work, when restricted to the non-degenerate diffusion case.

In this paper, we consider the non-degenerate stochastic integro-differential equation (1.1) with random coefficients and apply the method of finite differences in the time and space variables. To the best of our knowledge, this article is the first to use the finite difference method

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to approximate stochastic integro-differential equations. The approximations of the non-local integral operators in the drift and in the noise of (1.1) we choose are natural. In particular, we are able to treat the singularity of the integral operators near the origin directly. We consider a fully-explicit time-discretization scheme and an implicit–explicit time-discretization scheme, where we treat part of the approximation of the integral operator in the drift explicitly. We also provide a numerical verification of our theoretical convergence rates for an equation that has an "analytic" solution.

To obtain error estimates for our approximations, we use the approach in [24], where the discretized equations are first solved as time-discretized SDEs in Sobolev spaces over $\mathbf{R}^d$ and an error estimate is obtained in Sobolev norms. After obtaining $L^2(\Omega)$ error estimates in Sobolev norms, the Sobolev embedding theorem is used to obtain $L^2(\Omega)$-pointwise error estimates. So, in sum, we obtain two types of error estimates: in Sobolev norms and on the grid. Naturally, when using the Sobolev embedding to obtain the pointwise estimates, we do not need the equation to be differentiable to obtain pointwise error estimates, only continuous. Using the approach of first obtaining estimates in Sobolev spaces, we are also easily able to deduce that the more regularity on the coefficients and data we have, the stronger the error estimates we can obtain (see Corollaries 5.3 and 5.4).

The paper is organized as follows. In the next section (Section 2), we introduce the notation that will be used throughout the paper and state the main results. In Section 3, we give a numerical verification of the convergence rates for a simple test problem. In Section 4, we prove auxiliary results that will be used in the proof of the main theorems. In Section 4, we prove the main theorems of the paper.

2. Notation and the main results

For $x \in \mathbf{R}^d$, denote by $|x|$ the Euclidean norm of x. Let $\mathbf{N}_0 = \{0, 1, 2, \dots\}$. For $i \in \{1, \dots, d\}$, let $\partial_{-i} = -\partial_i$, and let ∂_0 be the identity. For a multi-index $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbf{N}_0$ of length $|\gamma| = \gamma_1 + \dots + \gamma_d$, set $\partial^\gamma = \partial_1^{\gamma_1} \dots \partial_d^{\gamma_d}$. Let ℓ_2 be the space of all square-summable real-valued sequences $b = (b^\varrho)_{\varrho=1}^\infty$. For an ℓ_2-valued function f on $\mathbf{R}^d$, the derivative of f with respect to x^i is denoted by $\partial_i f$.

Let $C_c^\infty(\mathbf{R}^d)$ be the space of all smooth real-valued functions on $\mathbf{R}^d$ with compact support. We write $(\cdot, \cdot)_0$ for the inner product and $\|\cdot\|_0$ for the norm in $L_2(\mathbf{R}^d) =: H^0$. For $m \in \mathbf{N}$, denote by H^m the Sobolev space of all functions $u \in L_2(\mathbf{R}^d)$ having distributional derivatives up to order m in $L_2(\mathbf{R}^d)$. We denote by

$$(\cdot, \cdot)_m := \sum_{|\gamma| \le m} (\partial^\gamma \cdot, \partial^\gamma \cdot)_0$$

the inner product in H^m and by $\|\cdot\|_m$ the corresponding norm. Define H^{-1} to be the completion of $C_c^\infty(\mathbf{R}^d)$ with respect to the norm $\|\cdot\|_{-1} = \|(1-\Delta)^{-1/2} \cdot\|_0$, where Δ is the Laplace operator. It is easy to see that for all $u \in H^1$ and $v \in H^0$, $(u, v)_0 \le \|u\|_1 \|v\|_{-1}$. Since H^1 is dense in H^{-1}, we may define the pairing $[\cdot, \cdot]_0 : H^1 \times H^{-1} \to \mathbf{R}$ by $[v, v']_0 = \lim_{n\to\infty} (v, v_n)_0$ for all $v \in H^1$ and $v' \in H^{-1}$, where $(v_n)_{n=1}^\infty \subset H^1$ is such that $\|v_n - v'\|_{-1} \to 0$ as $n \to \infty$. The mapping from H^{-1} to $(H^1)^*$ given by $v' \mapsto [\cdot, v']_0$ is an isometric isomorphism. For more details, see [22]. For an integer $m \ge 0$, we write $H^m(\ell_2)$ for the space of all ℓ_2-valued functions

$g(x) = (g^\varrho(x))_{\varrho=1}^{\infty}$ on $\mathbf{R}^d$ such that for each ϱ, $g^\varrho \in H^m$ and

$$\|g\|_{m,\ell_2}^2 := \sum_{\varrho=1}^{\infty} \|g^\varrho\|_m^2 < \infty.$$

On $[0, T] \times \mathbf{R}^d$, we consider the stochastic integro-differential equation

$$du_t = ((\mathcal{L}_t + I)u_t + f_t)\,dt + \sum_{\varrho=1}^{\infty} \left(\mathcal{N}_t^\varrho u_t + g_t^\varrho\right) dw_t^\varrho + \int_{\mathbf{R}^d} (\mathcal{I}(z)u_{t-} + o_t(z))\, q(dt, dz) \tag{2.3}$$

with initial condition

$$u_0(x) = \varphi(x), \quad x \in \mathbf{R}^d.$$

Denote the predictable sigma-algebra on $\Omega \times [0, T]$ relative to $\mathbf{F}$ by $\mathcal{P}_T$. Let $m \geq 0$ be an integer.

Assumption 2.1. For $i, j \in \{0, \ldots, d\}$, $a_t^{ij} = a_t^{ij}(x)$ are real-valued functions defined on $\Omega \times [0, T] \times \mathbf{R}^d$ that are $\mathcal{P}_T \otimes \mathcal{B}(\mathbf{R}^d)$-measurable and $\sigma_t^i = (\sigma_t^{i\varrho}(x))_{\varrho=1}^{\infty}$ are ℓ_2-valued functions that are $\mathscr{P}_T \otimes \mathcal{B}(\mathbf{R}^d)$-measurable. Moreover,

(i) for each $(\omega, t) \in \Omega \times [0, T]$, the functions a_t^{ij} are $\max(m, 1)$-times continuously differentiable in x for all $i, j \in \{1, \ldots, d\}$, a_t^{i0} and a_t^{0i} are m-times continuously differentiable in x for all $i \in \{0, 1, \ldots, d\}$, and σ_t^i are m-times continuously differentiable in x as ℓ_2-valued functions for all $i \in \{0, \ldots, d\}$. Furthermore, there is a constant $K > 0$ such that for all $(\omega, t, x) \in \Omega \times [0, T] \times \mathbf{R}^d$,

$$|\partial^\gamma a_t^{ij}| \leq K, \quad \forall\, i, j \in \{1, \ldots, d\},\ \forall\, |\gamma| \leq \max(m, 1),$$
$$|\partial^\gamma a_t^{i0}| + |\partial^\gamma a_t^{0i}| + |\partial^\gamma \sigma_t^i|_{\ell_2} \leq K, \quad \forall\, i \in \{0, \ldots, d\},\ \forall |\gamma| \leq m;$$

(ii) there exists a positive constant $\varkappa > 0$ such that for all $(\omega, t, x) \in \Omega \times [0, T] \times \mathbf{R}^d$ and $\eta \in \mathbf{R}^d$

$$\sum_{i,j=1}^{d} \left(2a_t^{ij} - \sum_{\varrho=1}^{\infty} \sigma_t^{i\varrho}\sigma_t^{j\varrho}\right) \eta_i \eta_j \geq \varkappa |\eta|^2.$$

We define the following spaces:

$$\mathbf{H}^m := L_2(\Omega \times [0, T], \mathcal{P}_T; H^m), \qquad \mathbf{H}^m(\ell_2) := L_2(\Omega \times [0, T], \mathcal{P}_T; H^m(\ell_2))$$
$$\mathbf{H}^m(\pi_2) := L_2(\Omega \times [0, T] \times \mathbf{R}^d,\ \mathcal{P}_T \otimes \mathcal{B}(\mathbf{R}^d),\ dP \times dt \times \pi_2(dz); H^m).$$

Assumption 2.2. The initial condition φ is $\mathcal{F}_0$-measurable with values in H^m such that $\mathbf{E}|\varphi|_m^2 < \infty$. Moreover, $f \in \mathbf{H}^{m-1}$, $g \in \mathbf{H}^m(\ell_2)$, and $o \in \mathbf{H}^m(\pi_2)$. Set

$$\kappa_m^2 = \mathbf{E}\|\varphi\|_m^2 + \mathbf{E}\int_{]0,T]} \left(\|f_t\|_{m-1}^2 + \|g_t\|_{m,\ell_2}^2 + \int_{\mathbf{R}^d} \|o_t(z)\|_m^2 \pi_2(dz)\right) dt.$$

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For a real-valued twice continuous differentiable function ϕ on $\mathbf{R}^d$, it is easy to see that for all $x, z \in \mathbf{R}^d$,

$$\phi(x+z) - \phi(x) - \sum_{j=1}^{d} z^j \partial_j \phi(x) = \int_0^1 \sum_{i,j=1}^{d} z^i z^j \partial_{ij}\phi(x+\theta z)(1-\theta)d\theta. \tag{2.4}$$

For each $\delta \in (0, 1]$, let

$$\varsigma_1(\delta) = \int_{|z|\le\delta} |z|^2 \pi_1(dz), \qquad \varsigma_2(\delta) = \int_{|z|\le\delta} |z|^2 \pi_2(dz), \quad \text{and} \quad \varsigma(\delta) = \varsigma_1(\delta) + \varsigma_2(\delta).$$

Fix $\delta \in (0, 1]$ such that

$$\varsigma(\delta) < \varkappa, \tag{2.5}$$

and notice that

$$\sum_{r=1}^{2} \pi_r(\{|z| > \delta\}) < \infty. \tag{2.6}$$

We write $I = I_\delta + I_{\delta^c}$, where

$$I_\delta \phi(x) := \int_{|z|\le\delta} \int_0^1 \sum_{i,j=1}^{d} z^i z^j \partial_{ij}\phi(x+\theta z)(1-\theta)d\theta \pi_1(dz)$$

and I_{δ^c} is defined as in (1.2) with integration over $\{|z| > \delta\}$ instead of $\mathbf{R}^d$.

Definition 2.1. An H^0-valued càdlàg adapted process u is called a solution of (2.3) if

(i) $u_t \in H^1$ for $dP \times dt$-almost-every $(\omega, t) \in \Omega \times [0, T]$;
(ii) $\mathbf{E}\int_{]0,T]} \|u_t\|_1^2 dt < \infty$;
(iii) there exists a set $\tilde{\Omega} \subset \Omega$ of probability one such that for all $(\omega, t) \in [0, T] \times \tilde{\Omega}$ and $\phi \in C_c^\infty(\mathbf{R}^d)$,

$$\begin{aligned}
(u_t, \phi)_0 = (\varphi, \phi)_0 &+ \int_{]0,t]} \left(\sum_{i,j=1}^{d} \left(\partial_j u_s, \partial_{-i}(a_s^{ij}\phi)\right)_0 + [\phi, f_s]_0 \right) ds \\
&+ \int_{]0,t]} \int_{|z|\le\delta} \int_0^1 \sum_{i,j=1}^{d} \left(z^j \partial_j u_s(\cdot + \theta z), z^i \partial_{-i}\phi\right)_0 (1-\theta) d\theta \pi_1(dz) ds \\
&+ \int_{]0,t]} \int_{|z|>\delta} \left(u_s(\cdot + z) - u_s - \mathbf{1}_{[-1,1]}(|z|) \sum_{j=1}^{d} z^j \partial_j u_s, \phi \right)_0 \pi_1(dz) ds \\
&+ \sum_{\varrho=1}^{\infty} \int_{]0,t]} \sum_{i=0}^{d} \left(\sigma_s^{i\varrho} \partial_i u_s + g_s^\varrho, \phi\right)_0 dw_s^\varrho \\
&+ \int_{]0,t]} \int_{\mathbf{R}^d} (u_{s-}(\cdot + z) - u_{s-} + o_t(z), \phi)_0 \, q(dz, ds).
\end{aligned}$$

Remark 2.1. In the above definition, instead of δ we may choose any other positive constant.

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The following existence theorem is a consequence of Theorems 2.9, 2.10, and 4.1 in [6] and will be verified in Section 4. The notation $N = N(\cdot, \dots, \cdot)$ is used to denote a positive constant depending only on the quantities appearing in the parentheses. In a given context, the same letter is repeatedly used to denote different constants depending on the same parameter.

Theorem 2.1. *If Assumptions 2.1 and 2.2 hold with $m \geq 0$, then there exist a unique solution u of (2.3). Furthermore, u is a cádlág H^m-valued process with probability one and there is a constant $N = N(d, m, \varkappa, K, T)$ such that*

$$\mathbf{E} \sup_{t \leq T} \|u_t\|_m^2 + \mathbf{E} \int_{]0,T]} \|u_s\|_{m+1}^2 ds \leq N \kappa_m^2. \tag{2.7}$$

Remark 2.2. We have used the standard definition of solution for the variational (or L^2) theory of stochastic partial differential equations. In what follows below, we will always assume $m \geq 2$ (though for our schemes, we assume $m \geq 3$), and so we have enough regularity to formulate the solution in the weak sense in (H^1, H^0, H^{-1}) without integrating by parts.

The following proposition is needed to establish the rate of convergence in time of our approximation scheme and is proved in Section 4.

Proposition 2.2. *Let Assumptions 2.1 and 2.2 hold with $m \geq 1$ and u be the solution of (2.3). Moreover, assume that*

$$\sup_{t \leq T} \mathbf{E} \|g_t\|_{m-1,\ell_2}^2 + \sup_{t \leq T} \mathbf{E} \int_{\mathbf{R}^d} \|o_t(z)\|_{m-1}^2 \pi_2(dz) \leq K.$$

Then there is a constant $\lambda = \lambda(d, m, K, T, \varkappa, \kappa_m^2)$ such that for all $s, t \in [0, T]$,

$$\mathbf{E} \|u_t - u_s\|_{m-1}^2 \leq \lambda |t - s|.$$

Assumption 2.3. For $m \geq 3$, in addition to Assumption 2.2, there exists a random variable ξ with $\mathbf{E}\xi < K$ such that for all $\omega \in \Omega,\ t, s \in [0, T]$,

$$\|g_t\|_{m-1,\ell_2}^2 + \int_{\mathbf{R}^d} \|o_t(z)\|_{m-1}^2 \pi_2(dz) \leq \xi$$

$$\|f_t - f_s\|_{m-2}^2 + \|g_t - g_s\|_{m-2,\ell_2}^2 + \int_{\mathbf{R}^d} \|o_t(z) - o_s(z)\|_{m-1}^2 \pi_2(dz) \leq \xi |t - s|.$$

Assumption 2.4. For $m \geq 3$, in addition to Assumption 2.1(i), there is a constant $C > 0$ such that for all $(\omega, x) \in \Omega \times \mathbf{R}^d,\ s, t \in [0, T],\ i, j \in \{0, 1, \dots, d\}$,

$$\left|\partial^\gamma \left(a_t^{ij} - a_s^{ij}\right)\right|^2 + \left|\partial^\gamma \left(\sigma_t^i - \sigma_s^i\right)\right|_{\ell_2}^2 \leq C|t - s|, \quad \forall |\gamma| \leq m - 2.$$

We turn our attention to the discretization of Eq. (2.3). For each $h \in \mathbf{R} - \{0\}$ and standard basis vector $e_i,\ i \in \{1, \dots, d\}$, of $\mathbf{R}^d$ we define the first-order difference operator $\delta_{h,i}$ by

$$\delta_{h,i}\phi(x) := \frac{\phi(x + he_i) - \phi(x)}{h},$$

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for all real-valued functions ϕ on $\mathbf{R}^d$. We define $\delta_{h,0}$ to be the identity operator. Notice that for all $\psi, \phi \in H^0$, we have

$$(\phi, \delta_{-h,i}\psi)_0 = -(\delta_{h,i}\phi, \psi)_0. \tag{2.8}$$

Set

$$\delta_i^h := \frac{1}{2}(\delta_{h,i} + \delta_{-h,i})$$

and observe that for all $\phi \in H^0$,

$$(\phi, \delta_i^h \phi)_0 = 0. \tag{2.9}$$

For each $h \neq 0$, we introduce the grid $\mathbf{G}_h := \{hz_k : z_k \in \mathbf{Z}^d, k \in \mathbf{N}_0, z_0 = 0\}$ with step size $|h|$. Let $\ell_2(\mathbf{G}_h)$ be the Hilbert space of real-valued functions ϕ on $\mathbf{G}_h$ such that

$$\|\phi\|^2_{\ell_2(\mathbf{G}_h)} := |h|^d \sum_{x \in \mathbf{G}_h} |\phi(x)|^2 < \infty.$$

We approximate the operators $\mathcal{L}$ and $\mathcal{N}^\varrho$ by

$$\mathcal{L}_t^h \phi(x) := \sum_{i,j=0}^d a_t^{ij}(x)\delta_{h,i}\delta_{-h,j}\phi(x) \quad \text{and} \quad \mathcal{N}_t^{\varrho;h}\phi(x) := \sum_{i=0}^d \sigma_t^{i\varrho}(x)\delta_{h,i}\phi(x),$$

respectively. In order to approximate I, we approximate I_δ and I_{δ^c} separately. For each $k \in \mathbf{N} \cup \{0\}$ and $h \neq 0$, define the rectangles in $\mathbf{R}^d$

$$A_k^h := \left(z_k^1|h| - \frac{|h|}{2}, z_k^1|h| + \frac{|h|}{2}\right] \times \cdots \times \left(z_k^d|h| - \frac{|h|}{2}, z_k^d|h| + \frac{|h|}{2}\right],$$

where z_k^i, $i \in \{1, \ldots, d\}$, are the coordinates of $z_k \in \mathbf{Z}^d$, and set

$$B_k^h := A_k^h \cap \{|z| \leq \delta\}, \qquad \bar{B}_k^h := A_k^h \cap \{|z| > \delta\}.$$

We approximate I_{δ^c} by

$$I_{\delta^c}^h \phi(x) := \sum_{k=0}^\infty \left((\phi(x + hz_k) - \phi(x))\,\bar{\zeta}_{h,k} - \sum_{i=1}^d \bar{\xi}_{h,k}^i \delta_i^h \phi(x) \right),$$

where

$$\bar{\zeta}_{h,k} := \pi_1(\bar{B}_k^h) \quad \text{and} \quad \bar{\xi}_{h,k}^i := \int_{\bar{B}_k^h \cap \{|z| \leq 1\}} z^i \pi_1(dz).$$

We continue with the approximation of the operator I_δ. By (2.4), for all $x \in \mathbf{G}_h$,

$$I_\delta \phi(x) = \sum_{k=0}^\infty \int_{B_k^h} \int_0^1 \sum_{i,j=1}^d z^i z^j \partial_{ij}\phi(x + \theta z)(1-\theta)d\theta \pi_1(dz),$$

where there are only a finite number of non-zero terms in the infinite sum over k. The closest point in $\mathbf{G}_h$ to any point $z \in B_k^h$ is clearly hz_k. This simple observation leads us to the following

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(intermediate) approximation of $I_\delta\phi(x)$:

$$\sum_{k=0}^{\infty}\int_0^1\sum_{i,j=1}^{d}\int_{B_k^h} z^i z^j \pi_1(dz)\partial_{ij}\phi(x+\theta h z_k)(1-\theta)d\theta.$$

However, in order to ensure that our approximation is well-defined for functions $\phi \in \ell_2(\mathbf{G}_h)$, we need to approximate the integral over $\theta \in [0,1]$. Fix $k \in \mathbf{N}_0$ and $h \neq 0$. Consider the directed line segment $\{\theta h z_k : \theta \in [0,1]\}$ extending from the origin to the point $hz_k \in \mathbf{R}^d$. It is clear that this line segment intersects a unique finite sequence of rectangles from the set $\{A_{\bar{k}}^h\}_{\bar{k}\in\mathbf{N}_0}$. Denote the number of rectangles by $\chi(h,k)$. Since the line's start point is the origin, the first rectangle it intersects is A_0^h, and since the line's endpoint is hz_k, the last rectangle it intersects is A_k^h, the center of which is the point hz_k. If $\chi(h,k) > 2$, then in between these two rectangles, the line segment intersects $\chi(h,k)-2$ additional rectangles from the set $\{A_{\bar{k}}^h\}_{\bar{k}\in\mathbf{N}_0} - \{A_0^h, A_k^h\}$. Denote the indices of these rectangles by $r_l^{h,k}$, $l \in \{2,\dots,\chi(h,k)-1\}$, and set $r_1^{h,k} = 0$ and $r_{\chi(h,k)}^{h,k} = k$; that is, $\{\theta h z_k ; \theta \in [0,1]\} \subseteq \cup_{l=1}^{\chi(h,k)} A_{r_l^{h,k}}^h$. Corresponding to the set of rectangles $\{A_{r_l^{h,k}}^h\}_{l=1}^{\chi(h,k)}$ is a partition $0 = \theta_0^{h,k} \le \cdots \le \theta_{\chi(h,k)}^{h,k} = 1$ of the interval $[0,1]$ such that for each $l \in \{1,\dots,\chi(h,k)\}$ and $\theta \in (\theta_{l-1}^{h,k}, \theta_l^{h,k})$, $\theta h z_k \in A_{r_l^{h,k}}^h$. Since the diagonal of a d-dimensional hypercube with side length $|h|$ has length $\sqrt{d}h$, for each $k \in \mathbf{N}_0$, $z \in B_k^h$, and $l \in \{1,\dots,\chi(h,k)\}$,

$$|\theta z - h z_{r_l^{h,k}}| \le |\theta z - \theta h z_k| + |\theta h z_k - h z_{r_l^{h,k}}| \le \sqrt{d}|h|, \tag{2.10}$$

for all $\theta \in (\theta_{l-1}^{h,k}, \theta_l^{h,k})$. Set

$$\zeta_{h,k}^{ij} = \int_{B_k^h} z^i z^j \pi_1(dz), \qquad \bar{\theta}_l^{h,k} = \int_{\theta_{l-1}^{h,k}}^{\theta_l^{h,k}} (1-\theta)d\theta$$

and define the operator

$$I_\delta^h\phi(x) =: \sum_{k=0}^{\infty}\sum_{l=1}^{\chi(h,k)} \bar{\theta}_l^{h,k} \sum_{i,j=1}^{d} \zeta_{h,k}^{ij}\delta_{h,i}\delta_{-h,j}\phi(x+hz_{r_l^{h,k}}),$$

where there are only a finite number of non-zero terms in the infinite sum over k. Set $I^h = I_\delta^h + I_{\delta^c}^h$ and introduce the martingales

$$p_t^{h,k,i} = \int_{]0,t]}\int_{B_k^h} z^i q(dt,dz), \qquad \bar{p}_t^{h,k} = q(\bar{B}_k^h,]0,t]).$$

Moreover, set

$$\tilde{\theta}_l^{h,k} := \theta_{l+1}^{h,k} - \theta_l^{h,k}.$$

Let $\mathcal{T} \ge 1$ be an integer and set $\tau = T/\mathcal{T}$ and $t_n = n\tau$ for $i \in \{0,1,\dots,\mathcal{T}\}$. For any $\mathbf{F}$-martingale $(p_t)_{t\le T}$, we use the notation $\Delta p_{n+1} := p_{t_{n+1}} - p_{t_n}$. Define recursively the $\ell_2(\mathbf{G}_h)$-valued random variables $(\hat{u}_n^{h,\tau})_{n=0}^{\mathcal{T}}$ by

$$\hat{u}_n^{h,\tau}(x) = \hat{u}_{n-1}^{h,\tau}(x) + \left((\mathcal{L}_{t_{n-1}}^h + I^h)\hat{u}_{n-1}^{h,\tau}(x) + f_{t_{n-1}}(x)\right)\tau$$

$$+\sum_{\varrho=1}^{\infty}(\mathcal{N}_{t_{n-1}}^{\varrho;h}\hat{u}_{n-1}^{h,\tau}(x)+g_{t_{n-1}}^{\varrho}(x))\Delta w_n^{\varrho}$$
$$+\sum_{k=0}^{\infty}\sum_{i=1}^{d}\left(\sum_{l=1}^{\chi(h,k)}\tilde{\theta}_l^{h,k}\delta_{h,i}\hat{u}_{n-1}^{h,\tau}(x+hz_{r_l^{h,k}})\right)\Delta p_n^{h,k,i}$$
$$+\int_{\mathbf{R}^d}o_{t_{n-1}}(x,z)q(]t_{n-1},t_n],dz)$$
$$+\sum_{k=0}^{\infty}\left(\hat{u}_{n-1}^{h,\tau}(x+hz_k)-\hat{u}_{n-1}^{h,\tau}(x)\right)\Delta\bar{p}_n^{h,k},\quad n\in\{1,\dots,\mathcal{T}\},\tag{2.11}$$

with initial condition

$$\hat{u}_0^{h,\tau}(x)=\varphi(x),\quad x\in\mathbf{G}_h.$$

It is clear that $\hat{u}_n^{h,\tau}$ is $\mathcal{F}_{t_n}$-measurable for every $n\in\{0,1,\dots,\mathcal{T}\}$. Define the operators

$$\tilde{\mathcal{L}}_t^h\phi=\sum_{i,j=0}^{d}a_t^{ij}\delta_{h,i}\delta_{-h,j}\phi-\pi_1(\{|z|>\delta\})\phi-\sum_{i=1}^{d}\int_{\delta<|z|\le 1}z^i\pi_1(dz)\delta_i^h\phi$$

and

$$\tilde{I}_{\delta^c}^h\phi=\sum_{k=0}^{\infty}\phi(x+hz_k)\bar{\zeta}_{h,k}$$

and note that $\tilde{\mathcal{L}}^h+\tilde{I}_{\delta^c}^h+I_\delta=\mathcal{L}^h+I^h$. On $\mathbf{G}_h$, we also consider the following implicit–explicit discretization scheme of (2.3):

$$\hat{v}_n^{h,\tau}(x)=\hat{v}_{n-1}^{h,\tau}(x)+\left((\tilde{\mathcal{L}}_{t_n}^h+I_\delta^h)\hat{v}_n^{h,\tau}(x)+\tilde{I}_{\delta^c}^h v_{n-1}^{h,\tau}(x)+f_{t_n}(x)\right)\tau$$
$$+\mathbf{1}_{n>1}\sum_{\varrho=1}^{\infty}(\mathcal{N}_{t_{n-1}}^{\varrho;h}\hat{v}_{n-1}^{h,\tau}(x)+g_{t_{n-1}}^{\varrho}(x))\Delta w_n^{\varrho}$$
$$+\mathbf{1}_{n>1}\sum_{k=0}^{\infty}\sum_{i=1}^{d}\left(\sum_{l=1}^{\chi(h,k)}\tilde{\theta}_l^{h,k}\delta_{h,i}\hat{v}_{n-1}^{h,\tau}(x+hz_{r_l^{h,k}})\right)\Delta p_n^{h,k,i}$$
$$+\int_{\mathbf{R}^d}o_{t_{n-1}}(x,z)q(]t_{n-1},t_n],dz)$$
$$+\mathbf{1}_{n>1}\sum_{k=0}^{\infty}\left(\hat{v}_{n-1}^{h,\tau}(x+hz_k)-\hat{v}_{n-1}^{h,\tau}(x)\right)\Delta\bar{p}_n^{h,k},\quad n\in\{1,\dots,\mathcal{T}\},\tag{2.12}$$

with initial condition

$$\hat{v}_0^{h,\tau}(x)=\varphi(x),\quad x\in\mathbf{G}_h,$$

where $\mathbf{1}_{n>1}=0$ if $n=1$ and $\mathbf{1}_{n>1}=1$ if $n\ge 2$. A solution $(\hat{v}_n^{h,\tau})_{n=0}^{M}$ of (2.12) is understood as a sequence of $\ell_2(\mathbf{G}_h)$-valued random variables such that $\hat{v}_n^{h,\tau}$ is $\mathcal{F}_{t_n}$-measurable for every $n\in\{0,1,\dots,M\}$ and satisfies (2.3).

Remark 2.3. Under Assumptions 2.2 and 2.3, for $m>2+d/2$, by virtue of the embedding $H^{m-2}\hookrightarrow\ell_2(\mathbf{G}^h)$, the free-terms f, g, and $o(z)$ are continuous $\ell_2(\mathbf{G}^h)$ valued processes, and consequently the above schemes make sense. Moreover, for $0 < |h| < 1$, there is a constant N independent of h such that—

$$\|\phi\|_{\ell_2(\mathbf{G}^h)} \leq N\|\phi\|_{m-2}. \tag{2.13}$$

Assumption 2.5. The parameters $h \neq 0$ and $\mathcal{T}$ are such that

$$d\frac{\tau}{h^2} < \frac{\varkappa - \varsigma(\delta)}{(2\Gamma + \varsigma_1(\delta))^2}, \tag{2.14}$$

where $\Gamma := \left(\sup_{t,x,\omega}\sum_{i,j=1}^d |a^{ij}(x)|^2\right)^{1/2}$.

The following are our main theorems.

Theorem 2.3. *Let Assumptions 2.1 through 2.4 hold with $m > 2+\frac{d}{2}$ and let Assumption 2.5 hold. Let u be the solution of (2.3) and let $(\hat{u}_n^{h,\tau})_{n=0}^{\mathcal{T}}$ be defined by (2.11). Then there is a constant $N = N(d, m, \varkappa, K, T, C, \lambda, \kappa_m^2, \delta)$ such that for any real number h with $0 < |h| < 1$,*

$$\mathbf{E}\max_{0\leq n\leq\mathcal{T}}\sup_{x\in\mathbf{G}_h}|u_{t_n}(x) - \hat{u}_n^{h,\tau}(x)|^2 + \mathbf{E}\max_{0\leq n\leq\mathcal{T}}\|u_{t_n} - \hat{u}_n^{h,\tau}\|^2_{\ell_2(\mathbf{G}_h)} \leq N\left(|h|^2 + \tau\right).$$

Theorem 2.4. *Let Assumptions 2.1 through 2.4 hold with $m > 2 + \frac{d}{2}$ and let u be a solution of (2.3). There exists a constant $R = R(d, m, \varkappa, K, \delta)$ such that if $\mathcal{T} > R$, then there exists a unique solution $(\hat{v}_n^{h,\tau})_{n=0}^{\mathcal{T}}$ of (2.12) and a constant $N = N(d, m, \varkappa, K, T, C, \lambda, \kappa_m^2, \delta)$ such that for any real number h with $0 < |h| < 1$,*

$$\mathbf{E}\max_{0\leq n\leq\mathcal{T}}\sup_{x\in\mathbf{G}_h}|u_{t_n}(x) - \hat{v}_n^{h,\tau}(x)|^2 + \mathbf{E}\max_{0\leq n\leq\mathcal{T}}\|u_{t_n} - \hat{v}_n^{h,\tau}\|^2_{\ell_2(\mathbf{G}_h)} \leq N\left(|h|^2 + \tau\right).$$

3. Numerical simulation

Let us consider finite difference approximations of the following SIDE on $[0, T] \times \mathbf{R}$:

$$\begin{aligned} du_t(x) &= \left(\left(\frac{\bar{\sigma}_1^2}{2} + \frac{\bar{\sigma}_2^2}{2}\right)\partial_1^2 u_t(x) + \int_{\mathbf{R}}(u_t(x+z) - u_t(x) - \partial_1 u_t(x)z)\pi(dz)\right)dt \\ &\quad + \bar{\sigma}_2\partial_1 u_t(x)dw_t + \int_{\mathbf{R}}(u(x+z) - u(x))\,q(dt, dz), \\ u_0(x) &= \frac{1}{\sqrt{2\pi}\bar{\sigma}_0}\exp\left(-\frac{x^2}{\bar{\sigma}_1^2\bar{\sigma}_0^2}\right), \end{aligned} \tag{3.15}$$

where $\pi(dz) = c_-\exp(-\beta_- z)\frac{dz}{|z|^{1+\alpha_-}}\mathbf{1}_{(-\infty,0)}(z) + c_+\exp(-\beta_+ z)\frac{dz}{|z|^{1+\alpha_+}}\mathbf{1}_{(0,\infty)}(z)$. It is easily verified that for $(t, x) \in [0, T] \times \mathbf{R}$,

$$v_t(x) = \frac{1}{\sqrt{\pi(2\bar{\sigma}_0^2 + 4t)}}\exp\left(\frac{x^2}{\bar{\sigma}_1^2(\bar{\sigma}_0^2 + 2t)}\right)$$

solves

$$dv_t(x) = \frac{\bar{\sigma}_1^2}{2}\partial_1^2 v_t(x)dt, \qquad v_0(x) = \frac{1}{\sqrt{2\pi}\bar{\sigma}_0}\exp\left(-\frac{x^2}{\bar{\sigma}_1^2\bar{\sigma}_0^2}\right).$$

Moreover, applying Itô's formula, we find that

$$u_t(x) = v_t\left(x + \bar{\sigma}_2 w_t + \int_{\mathbf{R}} zq(dt, dz)\right) \tag{3.16}$$

solves (3.15). Thus, we can compare our finite difference approximations with (3.16).

In our numerical simulations, we used MATLAB 2013a and made the following parameter specification:

$$\begin{aligned}&\bar{\sigma}_1 = \frac{1}{2}, \qquad \bar{\sigma}_2 = \frac{1}{4}, \qquad \bar{\sigma}_0 = \frac{1}{2}, \qquad c_- = c_+ = 1, \qquad \beta_- = \beta_+ = 1,\\ &\alpha_- = \alpha_+ = 1.1, \qquad T = 1.\end{aligned}$$

We also made a few practical simplifications. Both the explicit and implicit–explicit approximations were assumed to take the value zero on $(-\infty, 8] \cup [8, \infty)$. We also restricted the support of $\pi(dz)$ to $[-3, 3]$. We would like to investigate the associated error with these reductions in the future. We also mention that a good heuristic is to choose the size of domain and terminal time T according to the exit time of the diffusion associated with the drift of the SIDE. In fact, it is more than a heuristic and we aim to address this in a future work.

In our simulation, we took $\delta = \frac{1}{100}$. It follows that $\kappa = \bar{\sigma}_1^2 = \frac{1}{2}$ and

$$\begin{aligned}\varsigma(\delta) &= c_- \int_0^\delta \exp(-\beta_- z) z^{1-\alpha_-} dz + c_+ \int_0^\delta \exp(-\beta_+ z) z^{1-\alpha_+} dz + z\\ &= c_- \beta_-^{\alpha_- - 2}\gamma(2-\alpha_-, \beta_-\delta) + c_+\beta_+^{\alpha_+ - 2}\gamma(2-\alpha_+, \beta_+\delta) \approx 0.0082,\end{aligned}$$

where $\gamma(\eta, z)$ denotes the lower incomplete gamma function. Thus, the right-hand-side of (2.14) is approximately 1.0559, and hence we can always set $\tau = h^2$. The quantities $\zeta_{h,k}^{11}$, $\bar{\zeta}_{h,k}$, and $\xi_{h,k}^1$ can all be calculated using MATLAB's built-in upper and lower incomplete gamma functions, or by implementing an appropriate numerical integration procedure. The calculation of $\theta_l^{h,k}$, $\bar{\theta}_l^{h,k}$, and $\tilde{\theta}_l^{h,k}$ are all straightforward in one-dimension. Some more thought would need to spent on how to calculate these quantities in higher dimensions. Of course as an alternative, one could set $\delta = \frac{h}{2}$, but then the schemes are not guaranteed to converge as h tends to zero. This is the drawback of taking $\delta = \frac{h}{2}$ and not including the additional terms in I_δ (see the paragraph at the bottom of page 1620 in [2]). It does seem that the method we propose to discretize I_δ is novel in this respect. In our error analysis, we have considered $h \in \{2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}\}$ and $\tau = h^2$.

The term

$$\int_{|z|>\delta} (u_t(x+z) - u_t(x))\,\pi(dz)$$

in the drift of (3.15) can be canceled with the compensator of the compensated Poisson random measure term. We get a similar cancellation in the corresponding finite difference equations, and thus we can replace $\bar{p}_t^{h,k} = q(\bar{B}_k^h,]t_n, t_{n+1}])$ with $\hat{p}_t^{h,k} = p(\bar{B}_k^h,]t_n, t_{n+1}])$ in the explicit (2.11) and implicit–explicit (2.12) scheme.

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In order to simulate

$$\Delta p_n^{h,k} = \int_{]t_n,t_{n+1}]} \int_{B_k^h} zq(dt,dz), \qquad \hat{p}_t^{h,k} = p(\bar{B}_k^h,]t_n, t_{n+1}]),$$

for the finest time step size $\tau = 2^{-14}$, we used the algorithm discussed in Section 4 of [19]. In this algorithm, a parameter ϵ is chosen for which the process $\Delta p_n^{h,0} = \int_{]0,t]} \int_{|z|<\epsilon} zq(dt,dz)$ is approximated by a Wiener process with infinitesimal variance $\int_{|z|<\epsilon} z^2 \pi(dz)$. We chose the parameter $\epsilon = 2^{-8}$, which is one-half times the smallest step size h under consideration in our error analysis. The process $\int_{]0,t]} \int_{|z|>\epsilon} zq(dt,dz) = \int_{]0,t]} \int_{|z|>\epsilon} zp(dt,dz)$ (we have used symmetry of the measure $\pi(dz)$) is a compound Poisson process with jump intensity

$$\lambda := 2 \int_\epsilon^3 \pi(dz) \approx 68.9676$$

and jump-size density

$$\bar{f}(z) = \frac{1}{\lambda}\left(c_- \exp(-\beta_- z) \frac{dz}{|z|^{1+\alpha_-}} \mathbf{1}_{(-3,2^{-8})}(z) + c_+ \exp(-\beta_+ z) \frac{dz}{|z|^{1+\alpha_+}} \mathbf{1}_{(2^{-8},3)}(z) \right).$$

The underlying Poisson process was simulated using MATLAB's built-in Poisson random variable generator; of course there are other simple methods that one can use as an alternative (e.g. exponential times or uniform times for fixed number of jumps). We sampled random variables from the density $\bar{f}$ by sampling the positive and negative parts separately and using an acceptance–rejection algorithm with a Pareto random variable. We refer to [19] for more details. Once we simulated the point process on $[0,T] \times [-3,-\epsilon] \cup [\epsilon, 3]$, we then computed $\int_{]0,t]} \int_{|z|>\epsilon} zp(dt,dz)$. In order to compute $\hat{p}_t^{h,k} = p(\bar{B}_k^h,]t_n, t_{n+1}])$, we ran a histogram with the intervals $\bar{B}_k^h$.

The quantity $\Delta p_n^{h,k} = \int_{]t_n,t_{n+1}]} \int_{B_k^h} zq(dt,dz)$ is zero for $k \neq 0$ when $h < \frac{\delta}{2}$ (for $h \in \{2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}\}$) since $B_k^h = \emptyset$ for $k \neq 0$ when $h < \frac{\delta}{2}$. For $h \in \{2^{-6}, 2^{-7}\}$, $\Delta p_n^{h,k}$ is non-zero for $k \in \{-1,0,1\}$. A similar analysis holds for the quantity $\zeta_{h,k}^{11}$. As mentioned above, we set $\hat{p}_t^{h,0}$ equal to the Wiener process approximating the small jumps. To compute $\int_{]t_n,t_{n+1}]} \int_{B_k^h} zq(dt,dz)$ for $k \in \{-1,1\}$ in the case $h \in \{2^{-6}, 2^{-7}\}$, we summed the jump sizes in their respective bins and compensated. To obtain the above quantities for coarser time step sizes, we cumulatively summed the finer increments and took the union of jump sizes.

Lastly, we made use of the Fast Fourier Transform to compute terms of the form

$$\sum_{k=0}^{\infty} \phi(x + hz_k) \Delta \hat{p}_n^{h,k},$$

which would be quite computationally expensive otherwise. In our error analysis, we ran 3000 simulations of the explicit and implicit–explicit schemes on 30 CPUs and computed the following errors:

$$\sqrt{\frac{1}{3000} \sum_{m=1}^{3000} \max_{0 \le n \le \mathcal{T}} \sup_{x \in \mathbf{G}_h} |u_{t_n}(x) - \hat{u}_n^{h,\tau}(x)|^2}, \qquad \sqrt{\frac{1}{3000} \sum_{m=1}^{3000} \max_{0 \le n \le \mathcal{T}} \|u_{t_n} - \hat{u}_n^{h,\tau}\|^2_{\ell_2(\mathbf{G}_h)}}$$

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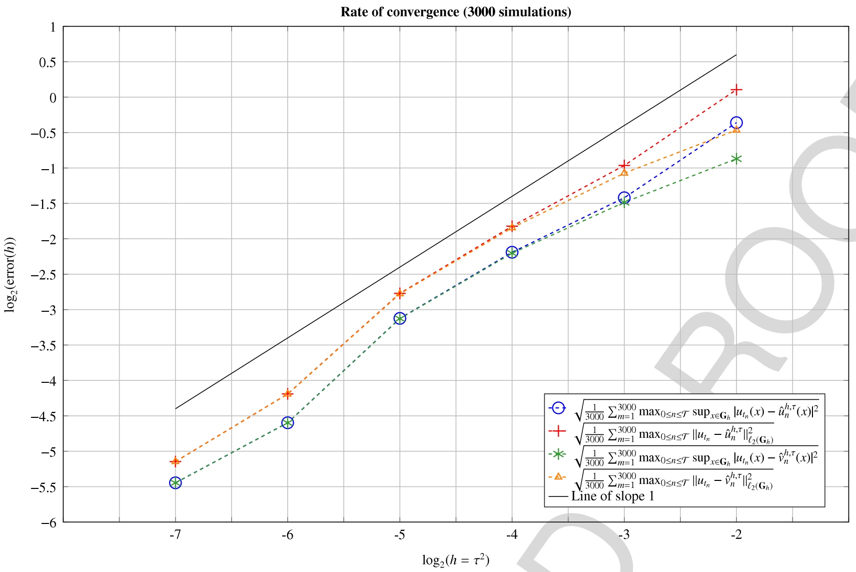

Fig. 1. Simulated errors with respect to the space discretization and a line as reference slope on a $\log_2$ scale.

$$\sqrt{\frac{1}{3000}\sum_{m=1}^{3000}\max_{0\le n\le\mathcal{T}}\sup_{x\in\mathbf{G}_h}|u_{t_n}(x)-\hat{v}_n^{h,\tau}(x)|^2},\qquad \sqrt{\frac{1}{3000}\sum_{m=1}^{3000}\max_{0\le n\le\mathcal{T}}\|u_{t_n}-\hat{v}_n^{h,\tau}\|^2_{\ell_2(\mathbf{G}_h)}}.$$

By our main theorems and the relation $\tau = h^2$, these errors should proportional to h (i.e. $O(h)$). This is precisely what we observe in Fig. 1. The slight bump down at the finest two spatial step-sizes $h \in \{2^{-6}, 2^{-7}\}$ is most likely due to the increase in the number of terms in the approximation of I^h_δ (three to be precise) and the analogous small jump term in the noise.

4. Auxiliary results

In this section, we present some results that will be needed for the proof of Theorems 2.3 and 2.4. Introduce the operators

$$\begin{aligned}
\mathcal{I}^{\delta;h}(z)\phi(x) &:= \sum_{k=0}^{\infty}\mathbf{1}_{B_k^h}(z)\sum_{l=1}^{\chi(h,k)}\sum_{i=1}^{d}\tilde{\theta}_l^{h,k}z^i\delta_{h,i}\phi(x+hz_{r_l^{h,k}}),\\
\mathcal{I}^{\delta^c;h}(z)\phi(x) &:= \sum_{k=0}^{\infty}\mathbf{1}_{\bar{B}_k^h}(z)(\phi(x+hz_k)-\phi(x)),\\
\mathcal{I}^h(z)\phi(x) &:= \mathcal{I}^{\delta;h}(z)\phi(x)+\mathcal{I}^{\delta^c;h}(z)\phi(x).
\end{aligned}$$

Consider the following explicit and implicit–explicit schemes in H^0:

$$\begin{aligned}
u_n^{h,\tau} &= u_{n-1}^{h,\tau}+\left((\mathcal{L}^h_{t_{n-1}}+I^h)u_{n-1}^{h,\tau}+f_{t_{n-1}}\right)\tau+\sum_{\varrho=1}^{\infty}(\mathcal{N}^{\varrho;h}_{t_{n-1}}u_{n-1}^{h,\tau}+g^{\varrho}_{t_{n-1}})\Delta w_n^{\varrho}\\
&\quad+\int_{\mathbf{R}^d}\left(\mathcal{I}^h(z)u_{n-1}^{h,\tau}+o_{t_{n-1}}(z)\right)q(dz,]t_{n-1},t_n]),\qquad n\in\{1,\ldots,\mathcal{T}\},
\end{aligned}\tag{4.17}$$

and

$$v_n^{h,\tau} = v_{n-1}^{h,\tau} + \left((\tilde{\mathcal{L}}_{t_n}^h + I_\delta^h) v_n^{h,\tau} + \tilde{I}_{\delta^c}^h v_{n-1}^{h,\tau} + f_{t_n} \right) \tau + \mathbf{1}_{n>1} \sum_{\varrho=1}^{\infty} (\mathcal{N}_{t_{n-1}}^{\varrho;h} v_{n-1}^{h,\tau} + g_{t_{n-1}}^{\varrho}) \Delta w_n^{\varrho}$$
$$+ \mathbf{1}_{n>1} \int_{\mathbf{R}^d} \left(\mathcal{I}^h(z) v_{n-1}^{h,\tau} + o_{t_{n-1}}(z) \right) q(dz,]t_{n-1}, t_n]), \quad n \in \{1, \dots, \mathcal{T}\}, \tag{4.18}$$

with initial condition

$$u_0^{h,\tau}(x) = v_0^{h,\tau}(x) = \varphi(x), \quad x \in \mathbf{R}^d.$$

We now prove some lemmas that will help us to establish the consistency of our approximations. The following lemma is well-known and we omit the proof (see, e.g., [10]).

Lemma 4.1. *For each integer $m \geq 0$, there is a constant $N = N(d, m)$ such that for all $u \in H^{m+2}$ and $v \in H^{m+3}$,*

$$\|\delta_{h,i} u - \partial_i u\|_m \leq \frac{1}{2}|h| \, \|u\|_{m+2},$$
$$\|\delta_{h,i} \delta_{-h,j} v - \partial_{ij} v\|_m \leq N|h| \, \|v\|_{m+3}.$$

Lemma 4.2. *For each integer $m \geq 0$, there is a constant $N = N(d, m, \delta)$ such that for all $u \in H^{m+3}$, we have*

$$\|Iu - I^h u\|_m \leq N|h| \, \|u\|_{m+3}. \tag{4.19}$$

Proof. It suffices to show (4.19) for $u \in C_c^\infty(\mathbf{R}^d)$. We begin with $m = 0$. A simple calculation shows that

$$I_{\delta^c} u(x) - I_{\delta^c}^h u(x) = \sum_{k=0}^{\infty} \int_{\bar{B}_k^h} \int_0^1 \sum_{i=1}^d (z^i - hz_k^i) \partial_i u(x + hz_k + \theta(z - hz_k)) d\theta \pi_1(dz)$$
$$- \sum_{k=0}^{\infty} \int_{\bar{B}_k^h \cap \{|z| \leq 1\}} \sum_{i=1}^d z^i (\partial_i u(x) - \delta_i^h u(x)) \pi_1(dz).$$

By Minkowski's inequality, we get

$$\|I_{\delta^c} u - I_{\delta^c}^h u\|_0 \leq \sum_{k=0}^{\infty} \int_{\bar{B}_k^h} \sum_{i=1}^d |z^i - hz_k^i| \, \|\partial_i u\|_0 \pi_1(dz)$$
$$+ \sum_{k=0}^{\infty} \int_{\bar{B}_k^h \cap \{|z| \leq 1\}} \sum_{i=1}^d |z^i| \, \|\partial_i u(x) - \delta_i^h u(x)\|_0 \pi_1(dz)$$
$$\leq N|h| \, \|u\|_3 + N \sum_{i=1}^d \|\partial_i u(x) - \delta_i^h u(x)\|_0,$$

since $|z - hz_k| \leq |h|\sqrt{d}/2$ and (2.6) holds. Thus, by Lemma 4.1, we have

$$\|I_{\delta^c} u - I_{\delta^c}^h u\|_0 \leq N|h| \, \|u\|_3. \tag{4.20}$$

We also have

$$I_\delta u(x) - I^h_\delta u(x) = \sum_{k=0}^{\infty} \int_{B^h_k} \sum_{l=1}^{\chi(h,k)} \int_{\theta^{h,k}_{l-1}}^{\theta^{h,k}_l} \sum_{i,j=1}^{d} z^i z^j \left(\partial_{ij} u(x+\theta z) - \delta_{h,i}\delta_{-h,j} u(x + h z_{r^{h,k}_l}) \right) \times (1-\theta) d\theta \pi_1(dz). \tag{4.21}$$

Note that

$$\begin{aligned} &\partial_{ij} u(x+\theta z) - \delta_{h,i}\delta_{-h,j} u(x + h z_{r^{h,k}_l}) \\ &\quad = \partial_{ij} u(x+\theta z) - \partial_{ij} u(x + h z_{r^{h,k}_l}) + \partial_{ij} u(x + h z_{r^{h,k}_l}) - \delta_{h,i}\delta_{-h,j} u(x + h z_{r^{h,k}_l}) \\ &\quad = \int_0^1 \sum_{q=1}^{d} \left(\theta z^q - h z^q_{r^{h,k}_l} \right) \partial_q \partial_{ij} u \left(x + h z_{r^{h,k}_l} + \rho(\theta z - h z_{r^{h,k}_l}) \right) d\rho \\ &\qquad + \partial_{ij} u(x + h z_{r^{h,k}_l}) - \delta_{h,i}\delta_{-h,j} u(x + h z_{r^{h,k}_l}). \end{aligned}$$

By (2.10), we have $|\theta z^q - h z^q_{r^{h,k}_l}| \le N|h|$. Hence, substituting the above relation in (4.21), using Minkowski's inequality, (2.5), and Lemma 4.1, we obtain

$$\|I_\delta u - I^h_\delta u\|_0 \le |h| N \|u\|_3. \tag{4.22}$$

Combining (4.20) and (4.22), we have (4.19) for $m = 0$. The case $m > 0$ follows from the case $m = 0$, since for a multi-index γ, we have

$$\partial^\gamma (Iu - I^h u) = I \partial^\gamma u - I^h \partial^\gamma u. \quad \square$$

Lemma 4.3. *For each integer $m \ge 0$, there is a constant $N = N(d, m, \delta)$, such that for all $u \in H^{m+2}$, we have*

$$\int_{\mathbf{R}^d} \|\mathcal{I}^h(z) u - \mathcal{I}(z) u\|^2_m \pi_2(dz) \le N |h|^2 \|u\|^2_{m+2}. \tag{4.23}$$

Proof. It suffices to prove the lemma for $u \in C^\infty_c(\mathbf{R}^d)$ and $m = 0$. We have

$$\begin{aligned} &\mathcal{I}^\delta(z) u(x) - \mathcal{I}^{\delta;h}(z) u(x) \\ &\quad = \sum_{k=0}^{\infty} \mathbf{1}_{B^h_k}(z) \sum_{l=1}^{\chi(h,k)} \int_{\theta^{h,k}_{l-1}}^{\theta^{h,k}_l} \sum_{i=1}^{d} z^i (\partial_i u(x+\theta z) - \delta_{h,i} u(x + h z_{r^{h,k}_l})) d\theta. \end{aligned}$$

Notice that

$$\begin{aligned} \partial_i u(x+\theta z) - \delta_{h,i} u(x + h z_{r^{h,k}_l}) = &\int_0^1 \sum_{i,j=1}^{d} \partial_{ij} u(x + \rho(\theta z - h z_{r^{h,k}_l}))(\theta z^j - h z^j_{r^{h,k}_l}) d\rho \\ &+ \partial_i u(x + h z_{r^{h,k}_l}) - \delta_{h,i} u(x + h z_{r^{h,k}_l}). \end{aligned}$$

Q6 Thus, by Remark (2.10) and Lemma 4.1, we get

$$\|\mathcal{I}^{\delta;h}(z) u - \mathcal{I}^\delta(z) u\|^2_0 \le \mathbf{1}_{|z|\le\delta} |z|^2 N |h|^2 \|u\|^2_2,$$

and hence by (2.5), we obtain

$$\int_{\mathbf{R}^d} \|\mathcal{I}^{\delta;h}(z) u - \mathcal{I}^\delta(z) u\|^2_0 \pi_2(dz) \le N |h|^2 \|u\|^2_2. \tag{4.24}$$

We also have

$$|\mathcal{I}^{\delta^c}(z)u(x) - \mathcal{I}^{\delta^c;h}(z)u(x)| = \sum_{k=0}^{\infty} \mathbf{1}_{\bar{B}_k^h}(z)|u(x+z) - u(x+hz_k)|$$

$$\leq \sum_{k=0}^{\infty} \mathbf{1}_{\bar{B}_k^h}(z) \int_0^1 \sum_{i=1}^{d} |\partial_i u(x+hz_k+\rho(z-hz_k))|\,|z^i - hz_k^i| d\rho.$$

Consequently,

$$\|\mathcal{I}^{\delta^c;h}(z)u - \mathcal{I}^{\delta^c}(z)u\|_0^2 \leq \mathbf{1}_{|z|>\delta} N|h|^2\|u\|_1^2,$$

which implies by (2.6) that

$$\int_{\mathbf{R}^d} \|\mathcal{I}^{\delta^c;h}(z)u - \mathcal{I}^{\delta^c}(z)u\|_0^2 \pi_2(dz) \leq N|h|^2\|u\|_1^2. \tag{4.25}$$

Combining (4.25) and (4.24), we have (4.23) for $m = 0$. The case $m > 0$ follows from the case $m = 0$, since for a multi-index γ, we have

$$\partial^\gamma(\mathcal{I}u - \mathcal{I}^h u) = \mathcal{I}\partial^\gamma u - \mathcal{I}^h \partial^\gamma u. \quad \square$$

Lemma 4.4. *If Assumption 2.1 holds for some $m \geq 0$, then for any $\epsilon \in (0,1)$ there exist constants $N_1 = N_1(d, m, \varkappa, K, \delta, \epsilon)$ and $N_2 = N_2(d, m, \varkappa, K, \delta, \epsilon)$ such that for any $u \in H^m$,*

$$\mathbf{G}_t^{(m)}(u) := 2(u, \mathcal{L}_t^h u)_m + \|\mathcal{N}_t^h u\|_{m,\ell_2}^2 + 2(u, I^h u)_m + \int_{\mathbf{R}^d} \|\mathcal{I}^h(z)u\|_m^2 \pi_2(dz)$$

$$\leq -(\varkappa - \varsigma(\delta) - \epsilon) \sum_{i=1}^{d} \|\delta_{h,i} u\|_m^2 + N_1\|u\|_m^2, \tag{4.26}$$

and

$$(u, \tilde{\mathcal{L}}_t^h u)_m + (u, I_\delta^h u)_m \leq -(\varkappa - \varsigma_1(\delta) - \epsilon) \sum_{i=1}^{d} \|\delta_{h,i} u\|_m^2 + N_2\|u\|_n^m. \tag{4.27}$$

Proof. By virtue of Lemma 3.1 and Theorem 3.2 in [10], under Assumption 2.1, there is a constant $N = N(d, m, \varkappa)$ such that for any $u \in H^m$ and $\epsilon > 0$,

$$2(u, \mathcal{L}_t^h u)_m + \|\mathcal{N}_t^h u\|_{m,\ell_2}^2 \leq -(\varkappa - \epsilon) \sum_{i=1}^{d} \|\delta_{h,i} u\|_m^2 + N\|u\|_m^2.$$

Therefore, it suffices to show that there is a constant $N = N(\delta)$ such that for all $u \in C_c^\infty(\mathbf{R}^d)$,

$$2(u, I^h u)_m + \int_{\mathbf{R}^d} \|\mathcal{I}^h(z)u\|_m^2 \pi_2(dz) \leq \varsigma(\delta) \sum_{i=1}^{d} \|\delta_{h,i} u\|_m^2 + N\|u\|_m^2. \tag{4.28}$$

We start with $m = 0$. Since

$$(u, I_\delta^h u)_0 = \sum_{k=0}^{\infty} \int_{B_k^h} \sum_{l=1}^{\chi(h,k)} \sum_{i,j=1}^{d} \bar{\theta}_l^{k,h} z^i z^j \int_{\mathbf{R}^d} \delta_{h,i}\delta_{-h,j} u(x + hz_{r_l^{h,k}}) u(x) dx \pi_1(dz)$$

and

$$\int_{\mathbf{R}^d} \delta_{h,i}\delta_{-h,j}u(x+hz_{r_l^{h,k}})u(x)dx = -\int_{\mathbf{R}^d} \delta_{h,i}u(x+hz_{r_l^{h,k}})\delta_{h,j}u(x)dx,$$

by Hölder's inequality, we get

$$2(u, I_\delta^h u)_0 \le \int_{|z|\le\delta} |z|^2\pi_1(dz)\sum_{i=1}^d \|\delta_{h,i}u\|_0^2 = \varsigma_1(\delta)\sum_{i=1}^d \|\delta_{h,i}u\|_0^2.$$

In addition, owing to Holder's inequality and (2.9), we have

$$\begin{aligned}2(u, I_{\delta^c}^h u)_0 &= \sum_{k=0}^\infty \int_{\bar{B}_k^h}\int_{\mathbf{R}^d}\left(u(x+hz_k)-u(x)-\mathbf{1}_{[-1,1]}(|z|)\sum_{i=1}^d z^i\delta_i^h u(x)\right)u(x)dx\pi_1(dz)\\ &\le 0.\end{aligned}$$

By Minkowski's inequality, we have

$$\|\mathcal{I}^{\delta;h}(z)u\|^2 \le \sum_{k=0}^\infty \mathbf{1}_{B_k^h}(z)|z|^2\sum_{i=1}^d\|\delta_{h,i}u\|_0^2 \quad \text{and} \quad \|\mathcal{I}^{\delta^c;h}(z)u\|_0^2 \le 4\sum_{k=0}^\infty \mathbf{1}_{\bar{B}_k^h}(z)\|u\|_0^2$$

and hence

$$\int_{\mathbf{R}^d}\|\mathcal{I}^h(z)u\|_0^2\pi_2(dz) \le \varsigma_2(\delta)\sum_{i=1}^d\|\delta_{h,i}u\|_0^2 + 4\pi_1(\{|z|>\delta\})\|u\|_0^2,$$

which proves (4.28) for $m=0$. The case $m>0$ follows by replacing u with $\partial^\gamma u$ for $|\gamma|\le m$. This proves (4.26), which implies (4.27). □

Remark 4.1. It follows that for $m\ge 0$, there is a constant $N_5 = N_5(d,m,K,\delta)$ such that for any $u\in H^m$,

$$\|\mathcal{N}_t^h u\|_{m,\ell_2}^2 + \int_{\mathbf{R}^d}\|\mathcal{I}^h(z)u\|_m^2\pi_2(dz) \le N_5\sum_{i=0}^d\|\delta_{h,i}u\|_m^2 \tag{4.29}$$

$$\le N_5\left(1+\frac{4d}{h^2}\right)\|u\|_m^2. \tag{4.30}$$

Lemma 4.5. *For any $m\ge 0$ and $u\in H^m$,*

$$\|\tilde{I}_{\delta^c}^h u\|_m^2 \le \pi_1(\{|z|>\delta\})^2\|u\|_m^2. \tag{4.31}$$

Moreover, if Assumption 2.1 holds for some $m\ge 0$, then for any $\epsilon>0$ and $u\in H^m$,

$$\|(\mathcal{L}_t^h + I^h)u\|_m^2 \le (1+\varepsilon)\frac{N_3 d}{h^2}\sum_{i=1}^d\|\delta_{h,i}u\|_m^2 + N_4\left(1+\frac{1}{h^2}\right)\|u\|_m^2 \tag{4.32}$$

where

$$N_3 := \left(2\left(\sup_{t,x,\omega}\sum_{i,j=1}^d|a^{ij}(x)|^2\right)^{1/2} + \varsigma_1(\delta)\right)^2$$

and N_4 is a constant depending only on d, m, K, δ, and ϵ.

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Proof. It suffices to prove the lemma for $u \in C_c^\infty(\mathbf{R}^d)$. It follows that

$$
\begin{aligned}
(\mathcal{L}_t^h + I_\delta^h)u(x) = &\sum_{k=0}^{\infty} \sum_{l=1}^{\chi(h,k)} \bar{\theta}_l^{h,k} \sum_{i,j=1}^{d} \hat{\zeta}_{t,h,k}^{ij}(x)\delta_{h,i}\delta_{-h,j}u(x + hz_{r_l^{h,k}}) \\
&+ \sum_{\substack{i,j=0 \\ i \text{ or } j=0}}^{d} a_t^{ij}\delta_{h,i}\delta_{-h,j}u(x)
\end{aligned}
$$

where $\hat{\zeta}_{t,h,k}^{ij}(x) := \zeta_{h,k}^{ij}$ for $k \neq 0$ and $\hat{\zeta}_{t,h,0}^{ij}(x) := \zeta_{h,0}^{ij} + 2a_t^{ij}(x)$ (recall that $\bar{\theta}_1^{h,0} = \frac{1}{2}$ and $\chi(h,0) = 1$). Moreover, for each multi-index γ with $1 \leq |\gamma| \leq m$,

$$
\begin{aligned}
\partial^\gamma(\mathcal{L}_t^h + I_\delta^h)u(x) \; = \; &\sum_{k=0}^{\infty} \sum_{l=1}^{\chi(h,k)} \bar{\theta}_l^{h,k} \sum_{i,j=1}^{d} \hat{\zeta}_{h,k}^{ij}(x)\delta_{h,i}\delta_{-h,j}\partial^\gamma u(x + hz_{r_l^{h,k}}) \\
&+ \sum_{\{\beta:\, \beta<\gamma\}} N(\beta,\gamma) \sum_{i,j=1}^{d} \left(\partial^{\gamma-\beta} a_t^{ij}(x)\right) \delta_{h,i}\delta_{-h,j}\partial^\beta u(x) \\
&+ \sum_{\{\beta:\, \beta\leq\gamma\}} N(\beta,\gamma) \sum_{\substack{i,j=0 \\ i \text{ or } j=0}}^{d} \left(\left(\partial^{\gamma-\beta} a_t^{ij}(x)\right) \delta_{h,i}\delta_{-h,j}\partial^\beta u(x)\right) \\
=: &\; (A_1(\gamma) + A_2(\gamma) + A_3(\gamma))u(x),
\end{aligned}
$$

where $N(\beta,\gamma)$ are constants depending only on β and γ. By Young's inequality and Jensen's inequality, for any $\epsilon \in (0,1)$, we have

$$
\begin{aligned}
\|(\mathcal{L}_t^h + I^h)u\|_m^2 \leq &\; (1+\epsilon) \sum_{|\gamma|\leq m} \|A_1(\gamma)u\|_0^2 \\
&+ 3\left(1 + \frac{1}{\epsilon}\right)\left[\sum_{|\gamma|\leq m}\left(\|A_2(\gamma)u\|_0^2 + \|A_3(\gamma)u\|_0^2\right) + \|I_{\delta^c}^h u\|_m^2\right].
\end{aligned}
$$

Applying Minkowski's inequality and the Cauchy–Bunyakovsky–Schwarz inequality and noting that $\sum_{l=1}^{\chi(h,k)} \bar{\theta}_l^{h,k} = \frac{1}{2}$ and

$$
\|\delta_{h,i}\partial^\beta u\|_0 \leq \frac{2}{h}\|\partial^\beta u\|_0; \quad \forall i \in \{0, 1 \ldots, d\},\ \forall |\beta| = m,
$$

we obtain

$$
\begin{aligned}
&\|A_1(\gamma)u\|_0 \\
&\leq \sum_{k=0}^{\infty} \sum_{l=1}^{\chi(h,k)} \bar{\theta}_l^{h,k} \left(\sup_{t,x,\omega} \sum_{i,j=1}^{d} \left|\hat{\zeta}_{h,k}^{ij}(x)\right|^2\right)^{1/2} \left(\sum_{i,j=1}^{d} \left\|\delta_{h,i}\delta_{-h,j}u(\cdot + hz_{r_l^{h,k}})\right\|_m^2\right)^{1/2} \\
&\leq \frac{\sqrt{d}}{h} \sum_{k=0}^{\infty} \left(\sup_{t,x,\omega} \sum_{i,j=1}^{d} |\hat{\zeta}_{h,k}^{ij}(x)|^2\right)^{1/2} \left(\sum_{i=1}^{d} \|\delta_{h,i}\partial^\gamma u\|_0^2\right)^{1/2}
\end{aligned}
$$

and

$$
\sum_{k=0}^{\infty} \left(\sup_{t,x,\omega} \sum_{i,j=1}^{d} |\hat{\zeta}_{h,k}^{ij}(x)|^2\right)^{1/2} = \left(\sup_{t,x,\omega} \sum_{i,j=1}^{d} \left|\int_{B_0^h} z^i z^j \pi_1(dz) + 2a_t^{ij}(x)\right|^2\right)^{1/2}
$$

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$$+\sum_{k=1}^{\infty}\left(\sum_{i,j=1}^{d}\left|\int_{B_k^h} z^i z^j \pi_1(dz)\right|^2\right)^{1/2}$$

$$\leq 2\left(\sup_{t,x,\omega}\sum_{i,j=1}^{d}|a_t^{ij}(x)|^2\right)^{1/2}+\varsigma(\delta).$$

Thus,

$$\sum_{|\gamma|\leq m}\|A_1(\gamma)u\|_0^2 \leq \frac{N_3 d}{h^2}\sum_{i=1}^{d}\|\partial_{h,i}u\|_m^2.$$

Another application of the Cauchy–Bunyakovsky–Schwarz inequality and Minkowski's inequality, combined with the inequalities

$$\|\delta_{h,i}\partial^\beta u\|_0 \leq \|\partial_i\partial^\beta u\|_0 \quad \forall i\in\{0,1\ldots,d\},\ \forall|\beta|\leq m-1,$$
$$\|\delta_{h,i}\delta_{-h,j}\partial^\beta u\|_0 \leq \|\partial_{ij}\partial^\beta u\|_0, \quad \forall; i,j\in\{1,\ldots,d\},\ \forall|\beta|\leq m-2,$$

and

$$\|\delta_{h,i}\delta_{-h,j}\partial^\beta u\|_0 \leq \frac{2}{h}\|\delta_{h,i}u\|_m, \quad \forall; i,j\in\{1,\ldots,d\},\ \forall|\beta|=m-1,$$

yields

$$\sum_{|\gamma|\leq m}\left(\|A_2(\gamma)u\|_0^2+\|A_3(\gamma)\|_0^2\right)\leq N\left(1+\frac{1}{h^2}\right)\|u\|_m^2.$$

By Minkowski's integral inequality, we have

$$\|I_{\delta^c}^h u\|_m \leq \int_{\mathbf{R}^d}\sum_{k=0}^{\infty}\mathbf{1}_{\overline{B}_k^h}\|u(\cdot+hz_k)-u-\mathbf{1}_{[-1,1]}(z)\sum_{i=1}^{d}z^i\delta_{h,i}u\|_m\pi_1(dz)$$
$$\leq 3\left(\pi_1(\{|z|>\delta\})+\frac{2d\int_{\delta<|z|\leq 1}|z|\pi_1(dz)}{h}\right)\|\partial^\gamma u\|_0.$$

It is also easy to see that (4.31) holds. Combining above inequalities, we obtain (4.32). □

The following theorem establishes the stability of the explicit approximate scheme (4.17).

Theorem 4.6. *Let Assumption 2.1 hold with $m \geq 0$ and Assumption 2.5 hold. Let $F^i \in \mathbf{H}^m$ for $i\in\{0,\ldots,d\}$, $G\in\mathbf{H}^m(\ell_2)$, and $R\in\mathbf{H}^m(\pi_2)$. Consider the following scheme in H^m:*

$$u_n^{h,\tau}=u_{n-1}^{h,\tau}+\int_{]t_{n-1},t_n]}\left((\mathcal{L}_{t_{n-1}}^h+I^h)u_{n-1}^{h,\tau}+\sum_{i=0}^{d}\delta_{h,i}F_t^i\right)dt$$
$$+\int_{]t_{n-1},t_n]}\left(\mathcal{N}_{t_{n-1}}^{\varrho;h}u_{n-1}^{h,\tau}+G_t^\varrho\right)dw_t^\varrho$$
$$+\int_{]t_{n-1},t_n]}\int_{\mathbf{R}^d}\left(\mathcal{I}^h(z)u_{n-1}^{h,\tau}+R_t(z)\right)q(dt,dz), \quad n\in\{1,\ldots,\mathcal{T}\},$$

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for any H^m-valued $\mathscr{F}_0$-measurable initial condition φ. If $\mathbf{E}\|\varphi\|_m^2 < \infty$, then there is a constant $N = N(d, m, \varkappa, K, T, \delta)$ such that

$$\mathbf{E}\max_{0\le n\le \mathcal{T}} \|u_n^{h,\tau}\|_m^2 + \mathbf{E}\sum_{n=0}^{\mathcal{T}} \tau \sum_{i=0}^{d} \|\delta_{h,i} u_n^{h,\tau}\|_m^2$$
$$\le N\mathbf{E}\|\varphi\|_m^2 + N\mathbf{E}\int_0^T \Big(\sum_{i=0}^d \|F_t^i\|_m^2 + \|G_t\|_m^2 + \int_{\mathbf{R}^d} \|R_t(z)\|_m^2 \pi_2(dz)\Big) dt. \tag{4.33}$$

Proof. If $\mathbf{E}\|\varphi\|_m^2 < \infty$, then proceeding by induction on n and using Young's and Jensen's inequality, Itô's isometry, (4.32), and (4.30), we get that for all $n \in \{0, 1, \dots, \mathcal{T}\}$, $\mathbf{E}\|u_n^{h,\tau}\|_m^2 < \infty$. Applying the identity $\|y\|_m^2 - \|x\|_m^2 = 2(x, y - x)_m + \|y - x\|_m^2$, $x, y \in H^m$, for each $n \in \{1, \dots, \mathcal{T}\}$, we obtain

$$\|u_n^{h,\tau}\|_m^2 = \|u_{n-1}^{h,\tau}\|_m^2 + \sum_{i=1}^{6} I_i(t_n), \tag{4.34}$$

where

$$I_1(t_n) := 2\tau \left(u_{n-1}^{h,\tau}, \left(\mathcal{L}_{t_{n-1}}^h + I^h\right) u_{n-1}^{h,\tau}\right)_m + \|\eta(t_n)\|_m^2,$$

$$I_2(t_n) := 2\int_{]t_{n-1},t_n]} \sum_{i=0}^{d} (u_{n-1}^{h,\tau}, \delta_{h,i} F_t^i)_m dt,$$

$$I_3(t_n) := \left\| \tau \left(\mathcal{L}_{t_{n-1}}^h + I^h\right) u_{n-1}^{h,\tau} + \int_{]t_{n-1},t_n]} \sum_{i=0}^{d} \delta_{h,i} F_t^i dt \right\|_m^2,$$

$$I_4(t_n) := 2\int_{]t_{n-1},t_n]} \left(u_{n-1}^{h,\tau}, \mathcal{N}_{t_{n-1}}^{\varrho;h} u_{n-1}^{h,\tau} + G_t^\varrho\right)_m dw_t^\varrho,$$

$$I_5(t_n) := 2\int_{]t_{n-1},t_n]} \int_{\mathbf{R}^d} \left(u_{n-1}^{h,\tau}, \mathcal{I}^h(z) u_{n-1}^{h,\tau} + R_t(z)\right)_m q(dt, dz),$$

$$I_6(t_n) := 2\left(\tau(\mathcal{L}_{t_{n-1}}^h + I^h) u_{n-1}^{h,\tau}, \eta(t_n)\right)_m + 2\left(\int_{]t_{n-1},t_n]} \sum_{i=0}^{d} \delta_{h,i} F_t^i dt, \eta(t_n)\right)_m,$$

and where

$$\eta(t_n) := \int_{]t_{n-1},t_n]} \left(\mathcal{N}_{t_{n-1}}^{\varrho;h} u_{n-1}^{h,\tau} + G_t^\varrho\right) dw_t^\varrho + \int_{]t_{n-1},t_n]} \int_{\mathbf{R}^d} \left(\mathcal{I}^h(z) u_{n-1}^{h,\tau} + R_t(z)\right) q(dt, dz).$$

By virtue of Assumption 2.5, we fix $\tilde{q} > 0$ and $\epsilon > 0$ small enough such that

$$\overline{q} := \varkappa - \varsigma(\delta) - \epsilon - (1+\epsilon)(1+\tilde{q}) N_3 d \frac{\tau}{h^2} - \tilde{q} > 0,$$

where N_3 is the constant in 4.5. Since the two stochastic integrals that define η are orthogonal square-integrable martingales, by Young's inequality and (4.29), for all $\mathfrak{q} > 0$, Q7

$$\mathbf{E}\|\eta(t_n)\|_m^2 \le \mathbf{E}\tau\|\mathcal{N}_{t_{n-1}}^h u_{n-1}^{h,\tau}\|_{m,\ell_2}^2 + \mathbf{E}\tau \int_{\mathbf{R}^d} \|\mathcal{I}^h(z) u_{n-1}^{h,\tau}\|_m^2 \pi_2(dz) + \mathfrak{q}\mathbf{E}\tau \sum_{i=0}^{d} \|\delta_{h,i} u_{n-1}^{h,\tau}\|_m^2$$

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$$+\left(1+\frac{N_5}{\mathfrak{q}}\right)\mathbf{E}\int_{]t_{n-1},t_n]}\left(\|G_t\|^2_{m,\ell_2}+\int_{\mathbf{R}^d}\|R_t(z)\|^2_m\pi_2(dz)\right)dt. \tag{4.35}$$

Thus, taking $\mathfrak{q}=\frac{\tilde{q}}{3}$ in (4.35), we have

$$EI_1(t_n)\le \mathbf{E}\tau\mathbf{G}^{(m)}_{t_{l-1}}(u^{h,\tau}_{l-1})+\frac{\tilde{q}}{3}\mathbf{E}\tau\sum_{i=0}^d\|\delta_{h,i}u^{h,\tau}_{n-1}\|^2_m$$
$$+\left(1+\frac{3N_5}{\tilde{q}}\right)\mathbf{E}\int_{]t_{n-1},t_n]}\left(\|G_t\|^2_{m,\ell_2}+\int_{\mathbf{R}^d}\|R_t(z)\|^2_m\pi_2(dz)\right)dt.$$

Using (2.8) and Young's inequality, we obtain

$$EI_2(t_n)\le\frac{\tilde{q}}{3}\mathbf{E}\tau\sum_{i=0}^d\|\delta_{h,i}u^{h,\tau}_{n-1}\|^2_m+\frac{3}{\tilde{q}}\mathbf{E}\int_{]t_{n-1},t_n]}\sum_{i=0}^d\|F^i_t\|^2_m dt.$$

An application of Young's inequality and (4.32) yields

$$EI_3(t_n)\le(1+\epsilon)(1+\tilde{q})N_3d\frac{\tau}{h^2}\mathbf{E}\tau\sum_{i=1}^d\|\delta_{h,i}u^{h,\tau}_{n-1}\|^2_m+(1+\tilde{q})N_4\left(\tau+\frac{\tau}{h^2}\right)\mathbf{E}\tau\|u^{h,\tau}_{n-1}\|^2_m$$
$$+(d+1)\left(1+\frac{1}{\tilde{q}}\right)\mathbf{E}\int_{]t_{n-1},t_n]}\left(\tau\|F^0_t\|^2_m+\frac{4d\tau}{h^2}\sum_{i=1}^d\|F^i_t\|^2_m\right)dt.$$

Making use of the estimate (4.30) and noting that $\mathbf{E}\|u^{h,\tau}_n\|^2_m<\infty$, $G\in\mathbf{H}^m(\ell_2)$, and $R\in\mathbf{H}^m(\pi_2)$, we obtain $EI_4(t_n)=EI_5(t_n)=0$. Moreover, as $(\mathcal{L}^h_{t_{n-1}}+I^h)u^{h,\tau}_{n-1}$ is $\mathcal{F}_{t_{n-1}}$-measurable and $E(\eta(t_n)|\mathcal{F}_{t_{n-1}})=0$, the expectation of first term in $I_6(t_n)$ is zero, and hence by Young's inequality, for any $\mathfrak{q}_1>0$,

$$EI_6(t_n)\le\mathfrak{q}_1\mathbf{E}\|\eta(t_n)\|^2_m+\frac{1}{\mathfrak{q}_1}E\left\|\int_{]t_{n-1},t_n]}\sum_{i=0}^d\delta_{h,i}F^i_t dt\right\|^2_m.$$

Moreover, by Jensen's inequality, (4.35), and (4.29), for any $\mathfrak{q}_1>0$ and $\mathfrak{q}>0$,

$$EI_6(t_n)\le(\mathfrak{q}_1\mathfrak{q}+\mathfrak{q}_1N_5)\mathbf{E}\tau\sum_{i=0}^d\|\delta_{h,i}u^{h,\tau}_{n-1}\|^2_m$$
$$+\mathbf{E}\int_{]t_{n-1},t_n]}\left(\frac{(d+1)\tau}{\mathfrak{q}_1}\|F^0_t\|^2_m+\frac{4d(d+1)\tau}{\mathfrak{q}_1h^2}\sum_{i=1}^d\|F^i_t\|^2_m\right)dt$$
$$+\mathfrak{q}_1\left(1+\frac{N_5}{\mathfrak{q}}\right)\mathbf{E}\int_{]t_{n-1},t_n]}\left(\|G_t\|^2_{m,\ell_2}+\int_{\mathbf{R}^d}\|R_t(z)\|^2_m\pi_2(dz)\right)dt.$$

We choose $\mathfrak{q}$ and $\mathfrak{q}_1$ such that $\mathfrak{q}_1\mathfrak{q}+\mathfrak{q}_1N_5\le\tilde{q}/3$. Thus, owing to (4.26), we have

$$E\mathbf{G}^{(m)}_{t_{n-1}}(u^{h,\tau}_{n-1})+\left(\tilde{q}+(1+\epsilon)(1+\tilde{q})N_3d\frac{\tau}{h^2}\right)\mathbf{E}\tau\sum_{i=1}^d\|\delta_{h,i}u^{h,\tau}_{n-1}\|^2_m$$
$$\le-\overline{q}\mathbf{E}\tau\sum_{i=1}^d\|\delta_{h,i}u^{h,\tau}_{n-1}\|^2_m+N_1\mathbf{E}\tau\|u^{h,\tau}_{n-1}\|^2_m.$$

Taking the expectation of both sides of (4.34), summing-up, and combining the above inequalities and identities, we find that there is a constant $N = N(d, m, \varkappa, K, \delta)$ such that for all $n \in \{0, 1, \dots, \mathcal{T}\}$,

$$\begin{aligned} \mathbf{E}\|u_n^{h,\tau}\|_m^2 &\le \mathbf{E}\|\varphi\|_m^2 - \overline{q}\mathbf{E}\sum_{l=1}^n \tau \sum_{i=1}^d \|\delta_{h,i}u_{l-1}^{h,\tau}\|_m^2 \\ &\quad + \left(N_1 + \tilde{q} + (1+\tilde{q})N_4\left(\tau + \frac{\tau}{h^2}\right)\right)\mathbf{E}\sum_{l=1}^n \tau\|u_{l-1}^{h,\tau}\|_m^2 \\ &\quad + N\left(\tau + \frac{\tau}{h^2}\right)\mathbf{E}\int_{]0,t_n]}\sum_{i=0}^d \|F_t^i\|_m^2 dt + N\mathbf{E} \\ &\quad \times \int_{]0,t_n]}\left(\|G_t\|_{m,\ell_2}^2 + \int_{\mathbf{R}^d}\|R_t(z)\|_m^2 \pi_2(dz)\right)dt. \end{aligned}$$

Therefore, by discrete Gronwall's inequality, there is a constant $N = N(d, m, \varkappa, K, T, \delta)$ such that

$$\begin{aligned} &\mathbf{E}\|u_n^{h,\tau}\|_m^2 + \mathbf{E}\sum_{l=0}^n \tau \sum_{i=0}^d \|\delta_{h,i}u_l^{h,\tau}\|_m^2 \\ &\le N\mathbf{E}\|\varphi\|_m^2 + N\mathbf{E}\int_{]0,T]}\left(\sum_{i=0}^d \|F_t^i\|_m^2 + \|G_t\|_{m,\ell_2}^2 + \int_{\mathbf{R}^d}\|R_t(z)\|_m^2\pi_2(dz)\right)dt. \end{aligned} \tag{4.36}$$

Now that we have proved (4.36), we will show (4.33). Estimating as we did above, we get that there is a constant N such that

$$\begin{aligned} &\mathbf{E}\max_{0\le n\le\mathcal{T}}\sum_{l=1}^n (I_1(t_l) + I_2(t_l) + I_3(t_l) + I_6(t_l)) \le N\mathbf{E}\sum_{l=0}^{\mathcal{T}-1}\tau\sum_{i=0}^d \|\delta_{h,i}u_l^{h,\tau}\|_m^2 \\ &+ N\mathbf{E}\int_{]0,T]}\left(\sum_{i=0}^d \|F_t^i\|_m^2 + \|G_t\|_{m,\ell_2}^2 + \int_{\mathbf{R}^d}\|R_t(z)\|_m^2\pi_2(dz)\right)dt. \end{aligned}$$

Applying the Burkholder–Davis–Gundy inequality and Young's inequality, we obtain

$$\begin{aligned} \mathbf{E}\max_{0\le n\le\mathcal{T}}\sum_{l=1}^n I_5(t_l) &\le 6\mathbf{E}\left|\sum_{l=1}^n \int_{]t_{n-1},t_n]}\int_{\mathbf{R}^d}\left(u_{n-1}^{h,\tau}, \mathcal{I}^h(z)u_{n-1}^{h,\tau} + R_t(z)\right)_m^2 \pi_2(dz)dt\right|^{1/2} \\ &\le \frac{1}{4}\mathbf{E}\max_{0\le n\le\mathcal{T}}\|u_n^{h,\tau}\|_m^2 + N\left(\mathbf{E}\sum_{l=0}^{\mathcal{T}-1}\tau\mathbf{E}\|\delta_{h,i}u_l^{h,\tau}\|_m^2 + \mathbf{E}\sum_{l=0}^{\mathcal{T}-1}\tau\mathbf{E}\|u_l^{h,\tau}\|_m^2\right) \\ &\quad + N\mathbf{E}\int_{]0,T]}\int_{\mathbf{R}^d}\|R_t(z)\|_m^2\pi_2(dz)dt. \end{aligned}$$

We can estimate $\mathbf{E}\max_{0\le n\le\mathcal{T}}\sum_{l=1}^n I_4(t_l)$ in similar way. Combining the above $\mathbf{E}\max_{0\le n\le\mathcal{T}}$-estimates and (4.36), we obtain (4.33). □

The following theorem establishes the existence and uniqueness of a solution to (4.18) and the stability of the implicit–explicit approximation scheme.

Theorem 4.7. *Let Assumption 2.1 hold with $m \ge 0$. Let $F^i \in \mathbf{H}^m$ for $i \in \{0, \dots, d\}$, $G \in \mathbf{H}^m(\ell_2)$ and $R \in \mathbf{H}^m(\pi_2)$. Then there exists a constant $R = R(d, m, \varkappa, K, \delta)$ such that* *if $\mathcal{T} > R$, then for any $h \neq 0$, there exists a unique H^m-valued solution $(v_n^{h,\tau})_{n=0}^{\mathcal{T}}$ of*

$$\begin{aligned} v_n^{h,\tau} &= v_{n-1}^{h,\tau} + \int_{]t_{n-1},t_n]} \left((\tilde{\mathcal{L}}^h_{t_n} + I^h_\delta) v_n^{h,\tau} + \tilde{I}^h_{\delta^c} v_{n-1}^{h,\tau} + \sum_{i=0}^d \delta_{h,i} F_t^i \right) dt \\ &\quad + \int_{]t_{n-1},t_n]} \left(\mathbf{1}_{n>1} \mathcal{N}^{\varrho;h}_{t_{n-1}} v_{n-1}^{h,\tau} + G_t^\varrho \right) dw_t^\varrho \\ &\quad + \int_{]t_{n-1},t_n]} \int_{\mathbf{R}^d} \left(\mathbf{1}_{n>1} \mathcal{I}^h(z) v_{n-1}^{h,\tau} + R_t(z) \right) q(dt,dz), \end{aligned} \tag{4.37}$$

Q8 *for $n \in \{1,\ldots,\mathcal{T}\}$, for any H^m-valued $\mathscr{F}_0$-measurable initial condition φ. Moreover, if $\mathbf{E}\|\varphi\|_m^2 < \infty$, then there is a constant $N = N(d,m,\varkappa,K,T,\delta)$ such that*

$$\begin{aligned} &\mathbf{E} \max_{0\le n\le \mathcal{T}} \|v_n^{h,\tau}\|_m^2 + \mathbf{E} \sum_{n=0}^{\mathcal{T}} \tau \sum_{i=0}^d \|\delta_{h,i} v_n^{h,\tau}\|_m^2 \\ &\quad \le N\mathbf{E}\|\varphi\|_m^2 + N\mathbf{E} \int_0^T \left(\sum_{i=0}^d \|F_t^i\|_m^2 + \|G_t\|_m^2 + \int_{\mathbf{R}^d} \|R_t(z)\|_m^2 \pi_2(dz) \right) dt. \end{aligned} \tag{4.38}$$

Proof. For each $n \in \{1,\ldots,\mathcal{T}\}$, we write (4.37) as

$$D_n v_n^{h,\tau} = y_{n-1},$$

where D_n is the operator defined by

$$D_n \phi := \phi - \tau \left(\tilde{\mathcal{L}}^h_{t_n} + I^h_\delta \right) \phi$$

Q9 and

$$\begin{aligned} y_{n-1} &:= v_{n-1}^{h,\tau} + \int_{]t_{n-1},t_n]} \left(\tilde{I}^h_{\delta^c} v_{n-1}^{h,\tau} + \sum_{i=0}^d \delta_{h,i} F_t^i \right) dt \\ &\quad + \int_{]t_{n-1},t_n]} \left(\mathbf{1}_{n>1} \mathcal{N}^{\varrho;h}_{t_{n-1}} v_{n-1}^{h,\tau} + G_t^\varrho \right) dw_t^\varrho \\ &\quad + \int_{]t_{n-1},t_n]} \int_{\mathbf{R}^d} \left(\mathbf{1}_{n>1} \mathcal{I}^h(z) v_{n-1}^{h,\tau} + R_t(z) \right) q(dt,dz). \end{aligned}$$

Fix ϵ_1 and ϵ_2 in $(0,1)$ such that

$$\overline{q}_1 := \varkappa - \varsigma_1(\delta) - \epsilon_1 > 0$$

and

$$\overline{q}_2 := \varkappa - \varsigma(\delta) - \epsilon_2 > 0.$$

Owing to Lemma 4.5, there is a constant $N = N(d,m,K,\delta)$ such that for all $\phi \in H^m$,

$$\|D_n \phi\|_m^2 \le N \left(1 + \tau^2 \left(\frac{1}{h^2} + \frac{1}{h^4} \right) \right) \|\phi\|_m^2. \tag{4.39}$$

Assume $\mathcal{T} > TN_2$. By (4.27), for all $\phi \in H^m$, we have

$$(\phi, D_n\phi)_m \ge (1-\tau N_2)\|\phi\|_m^2 + \overline{q}_1 \tau \sum_{i=1}^d \|\delta_{h,i}\phi\|_m^2 \ge (1-\tau N_2)\|\phi\|_m^2. \tag{4.40}$$

Using Jensen's inequality and (4.31), we get

$$\|y_0\|_m^2 \leq 5\left(1+\pi_1(\{|z|>\delta\})^2\tau^2\right)\|\phi\|_m^2 + \frac{20\tau}{h^2}\int_{]0,t_1]}\sum_{i=0}^d \|F_t^i\|_m^2 dt + 5\left\|\int_{]0,t_1]} G_t^\varrho dw_t^\varrho\right\|_m^2 + 5\left\|\int_{]0,t_1]}\int_{\mathbf{R}^d} R_t(z)q(dt,dz)\right\|_m^2. \tag{4.41}$$

Since $\varphi \in H^m$, $F^i \in \mathbf{H}^m$, $i \in \{0,1,\dots,d\}$, $G \in \mathbf{H}^m(\ell_2)$, and $R \in \mathbf{H}^m(\pi_2)$, it follows that $y_0 \in H^m$. By (4.39), and (4.40), owing to Proposition 3.4 in [12] ($p=2$), there exists a unique $v_1^{h,\tau}$ in H^m such that $D_1 v_1^{h,\tau} = y_0$, and moreover

$$\|v_1^{h,\tau}\|_m^2 \leq 1 + \frac{\|y_0\|_m^2}{(1-\tau N_2)^2} < \infty. \tag{4.42}$$

Proceeding by induction on $n \in \{1,\dots,\mathcal{T}\}$, one can show that there exists a unique $v_n^{h,\tau}$ in H^m such that $D_n v_n^{h,\tau} = y_{n-1}$, and moreover

$$\|v_n^{h,\tau}\|_m^2 \leq 1 + \frac{\|y_{n-1}\|_m^2}{(1-\tau N_2)^2} < \infty. \tag{4.43}$$

Assume that $\mathbf{E}\|\varphi\|_m^2 < \infty$. By (4.41) and (4.42) and the fact that $f^i \in \mathbf{H}^m$, $i \in \{0,1,\dots,d\}$, $g \in \mathbf{H}^m(\ell_2)$, and $r \in \mathbf{H}^m(\nu)$, it follows that $\mathbf{E}\|v_1^{h,\tau}\|_m^2 < \infty$. By Jensen's inequality, (4.31), and (4.30), we have

$$\mathbf{E}\|y_{n-1}\|_m^2 \leq 7N\left(1+\pi_1(\{|z|>\delta\})^2\tau^2 + \mathbf{1}_{n>1}\tau\left(1+\frac{1}{h^2}\right)\right)\mathbf{E}\|v_{n-1}^{h,\tau}\|_m^2 + \frac{28\tau}{h^2}\mathbf{E}\int_{]0,t_1]}\sum_{i=0}^d\|F_t^i\|_m^2 dt + 7\mathbf{E}\int_{]0,t_1]}\|G_t\|_{m,\ell_2}^2 dt + 7\mathbf{E}\int_{]0,t_1]}\int_{\mathbf{R}^d}\|R_t(z)\|_m^2\pi_2(dz)dt. \tag{4.44}$$

Proceeding by induction on n and combining (4.43) and (4.44), we obtain

$$\mathbf{E}\|v_n^{h,\tau}\|_m^2 < \infty, \quad \forall n \in \{0,1,\dots,\mathcal{T}\}. \tag{4.45}$$

Applying the identity $\|y\|_m^2 - \|x\|_m^2 = 2(x,y-x)_m + \|y-x\|_m^2$, $x,y \in H^m$, for any $n \in \{1,\dots,\mathcal{T}\}$, we have

$$\|v_n^{h,\tau}\|_m^2 = \|v_{n-1}^{h,\tau}\|_m^2 + \sum_{i=1}^6 I_i(t_n),$$

where

$$I_1(t_n) := 2\tau\left(v_n^{h,\tau}, \left(\tilde{\mathcal{L}}_{t_n}^h + I_\delta^h\right)v_n^{h,\tau}\right)_m + 2\tau(v_{n-1}^{h,\tau}, \tilde{I}_{\delta^c}^h v_{n-1}^{h,\tau})_m + \|\eta(t_n)\|_m^2,$$

$$I_2(t_n) := 2\int_{]t_{n-1},t_n]}\sum_{i=0}^d (u_n^{h,\tau}, \delta_{h,i}F_t^i)_m dt,$$

$$I_3(t_n) := -\left\|\tau\left(\tilde{\mathcal{L}}_{t_n}^h + I_\delta^h\right)v_n^{h,\tau} + \sum_{i=0}^d\int_{[t_{n-1},t_n]}\delta_{h,i}F_t^i dt\right\|_m^2 + \left\|\tilde{I}_{\delta^c}^h v_{n-1}^{h,\tau}\right\|_m^2\tau^2,$$

$$I_4(t_n) := 2\int_{]t_{n-1},t_n]} \left(v^{h,\tau}_{n-1}, \mathbf{1}_{n>1}\mathcal{N}^{\varrho;h}_{t_{n-1}} v^{h,\tau}_{n-1} + G^\varrho_t\right)_m dw^\varrho_t,$$
$$I_5(t_n) := 2\int_{]t_{n-1},t_n]} \int_{\mathbf{R}^d} \left(v^{h,\tau}_{n-1}, \mathbf{1}_{n>1}\mathcal{I}^h(z) v^{h,\tau}_{n-1} + R_t(z)\right)_m q(dt,dz),$$
$$I_6(t_n) := \left(\tau \tilde{I}^h_{\delta^c} v^{h,\tau}_{n-1}, \eta(t_n)\right)_m,$$

and where

$$\eta(t_n) := \int_{]t_{n-1},t_n]} \left(\mathbf{1}_{n>1}\mathcal{N}^{\varrho;h}_{t_{n-1}} v^{h,\tau}_{n-1} + G^\varrho_t\right) dw^\varrho_t$$
$$+ \int_{]t_{n-1},t_n]} \int_{\mathbf{R}^d} \left(\mathbf{1}_{n>1}\mathcal{I}^h(z) v^{h,\tau}_{n-1} + R_t(z)\right) q(dt,dz).$$

As in the proof Theorem 4.6, by Young's inequality, (4.26), and (4.31), we have

$$\mathbf{E}\|v^{h,\tau}_n\|^2_m \le (1 + 2\pi_1(\{|z| > \delta\}))\, \mathbf{E}\|\varphi\|^2_m - \bar{q}_2 \mathbf{E}\sum_{l=1}^n \tau \sum_{i=1}^d \|\delta_{h,i} v^{h,\tau}_l\|^2_m$$
$$+ \quad \mathbf{E}\sum_{l=1}^n \tau \left(N_2 + 2\pi_1(\{|z| > \delta\}) + \tau\pi_1(\{|z| > \delta\})^2\right) \|v^{h,\tau}_l\|^2_m$$
$$+ N\mathbf{E}\int_{]0,t_n]} \left(\sum_{i=0}^d \|F^i_t\|^2_m dt + \|G_t\|^2_{m,\ell_2} + \int_{\mathbf{R}^d} \|R_t(z)\|^2_m \pi_2(dz)\right) dt.$$

Q10 Set

$$Z := N_2 + 2\pi_1(\{|z| > \delta\}),$$
$$R := \max\left(\frac{2\pi_1(\{|z| > \delta\})^2}{\sqrt{Z^2 + 4\pi_1(\{|z| > \delta\})^2} - Z}, N_2\right) T.$$

Assume $\mathcal{T} > R$. Making use of (4.45) and applying discrete Gronwall's lemma, we get that there exists a constant $N(d, m, K, \varkappa, T, \delta)$ such that

$$\mathbf{E}\|v^{h,\tau}_n\|^2_m + \mathbf{E}\sum_{l=1}^n \tau \sum_{i=0}^d \|\delta_{h,i} v^{h,\tau}_l\|^2_m$$
$$\le N\mathbf{E}\|\varphi\|^2_m + N\mathbf{E}\int_{]0,T]} \left(\sum_{i=0}^d \|F^i_t\|^2_m + \|G_t\|^2_{m,\ell_2} + \int_{\mathbf{R}^d} \|R_t(z)\|^2_m \pi_2(dz)\right) dt. \quad (4.46)$$

Using (4.31) instead of (4.32), we obtain (4.38) from (4.46) in the same manner as Theorem 4.6. Note that no bound on τ/h^2 is needed in this case. □

5. Proof of the main results

Proof of Theorem 2.1. By virtue of Theorems 2.9, 2.10, and 4.1 in [6], in order to obtain the existence, uniqueness, regularity, and the estimate (2.7), we only need to show that (2.3) may be realized as an abstract stochastic evolution equation in a Gelfand triple and that the growth condition and coercivity condition are satisfied. Indeed, since (2.3) is a linear equation, the hemicontinuity condition is immediate and monotonicity follows directly from the coercivity condition. By Holder's inequality and Assumption 2.1(i), for $u, v \in H^1$, we have

$$\sum_{i,j=0}^{d} \left(\partial_j u, (v\partial_{-i} a_t^{ij} + a^{ij}\partial_{-i} v)\right)_0$$

$$+ \int_{|z|>\delta} \left(u(\cdot + z) - u - \mathbf{1}_{[-1,1]}(|z|) \sum_{j=1}^{d} z^j \partial_j u, v\right)_0 \pi_1(dz)$$

$$+ \int_{|z|\le\delta} \int_0^1 \sum_{i,j=1}^{d} \left(z^j \partial_j u(\cdot + \theta z), z^i \partial_{-i} v\right)_0 (1-\theta) d\theta \pi_1(dz) \le N\|u\|_1 \|v\|_1.$$

Therefore, since the pairing $[\cdot,\cdot]_0$ brings $(H^1)^*$ and H^{-1} into isomorphism, for each $(\omega, t) \in [0, T] \times \Omega$, there exists a linear operator $\tilde{A}_t : H^1 \to H^{-1}$ such that $[v, \tilde{A}_t u]_0$ agrees with the left-hand-side of the above inequality and for $u, v \in H^1$, $\|\tilde{A}_t u\|_{-1} \le N\|u\|_1$. By Assumption 2.2, the operator A defined by $A(u) = \tilde{A}u + f$, maps H^1 to H^{-1} and for $u \in H^1$, $\|A_t(u)\|_{-1} \le N(\|u\|_1 + \|f\|_{-1})$.

For an integer $m \ge 1$, with abuse of notation, we write

$$(\cdot,\cdot)_m = ((1-\Delta)^{m/2}\cdot, (1-\Delta)^{m/2}\cdot)_0$$

and $\|\cdot\|_m$ for the corresponding norm in H^m. It is well known that the above inner product and norm are equivalent to the ones introduced in Section 1. For each $m \ge 1$ and for all $u \in H^{m+1}$ and $v \in H^m$, we have $(u, v)_m \le \|u\|_{m+1}\|v\|_{m-1}$. Since H^{m+1} is dense in H^{m-1}, we may define the pairing $[\cdot,\cdot]_m : H^{m+1} \times H^{m-1} \to \mathbf{R}$ by $[v, v']_m = \lim_{n\to\infty}(v, v_n)_m$ for all $v \in H^{m+1}$ and $v' \in H^{m-1}$, where $(v_n)_{n=1}^{\infty} \subset H^{m+1}$ is such that $\|v_n - v'\|_{m-1} \to 0$ as $n \to \infty$. It can be shown that the mapping from H^{m-1} to $(H^{m+1})^*$ given by $v' \mapsto [\cdot, v']_m$ is an isometric isomorphism. For more details, see [22]. Therefore, for all $m \ge 0$, (H^{m+1}, H^m, H^{m-1}) forms a Gelfand triple with the pairing $[\cdot,\cdot]_m$.

For $m \ge 1$ and all $u \in H^{m+1}$ and $v \in H^m$, using integration by parts, we get $[v, A_t(u)]_0 = ((\mathcal{L}_t + I_t)u + f, v)_0 = [v, (\mathcal{L}_t + I_t)u + f]_0$. Since this is true for all $v \in H^m$, which is dense in H^1, the restriction of A to H^{m+1} coincides with $L + I + f$. Moreover, it can easily be shown under Assumptions 2.1(i) and 2.2 that for all $m \ge 1$ and $u, v \in H^{m+1}$, $\|A_t(u)\|_{m-1} \le N\|u\|_{m+1} + \|f\|_{m-1}$, where N is a constant depending only on m, d, K, and ν, which shows that A satisfies the growth condition. For $u \in H^m$, $m \ge 1$, define $B_t^{\varrho}(u) = b_t^{i\varrho}\partial_i u + g_t^{\varrho}$, $B_t = (B_t^{\varrho})_{\varrho=1}^{\infty}$, and $\mathcal{C}_z(u) = u(\cdot + z) - u + o_t(z)$, $z \in \mathbf{R}^d$. Owing to Assumption 2.1(i), B_t is an operator from H^{m+1} to $H^m(\ell_2)$. Furthermore, $\mathcal{C}$ is an operator from H^{m+1} to $L_2(\mathbf{R}^d, \pi_2(dz); H^m)$ (see (5.48)). It is also clear that A, B, and $\mathcal{C}$ are appropriately measurable. Thus, (2.3) may be realized as the following stochastic evolution equation in the Gelfand triple (H^{m+1}, H^m, H^{m-1}):

$$u_t = u_0 + \int_{]0,t]} A_s(u_s) ds + \int_{]0,t]} B_s^{\varrho}(u_s) dw_s^{\varrho} + \int_{]0,t]} \mathcal{C}_z(u_{s-}) q(dz, ds), \tag{5.47}$$

for $t \in [0, T]$. Let $u \in C_c^{\infty}(\mathbf{R}^d)$. A simple calculation shows that there is a constant $N = N(\delta)$ such that

$$\int_{\mathbf{R}^d} \|u(\cdot + z) - u\|_m^2 \pi_2(dz) \le \varsigma_2(\delta)\|u\|_{m+1}^2 + N\|u\|_m^2. \tag{5.48}$$

Applying Holder's inequality and the identity $(u, \partial_j u) = 0$, we obtain

$$\int_{|z|>\delta'} \left(u(\cdot + z) - u - \mathbf{1}_{[-1,1]}(|z|) \sum_{j=1}^{d} z^j \partial_j u, u \right)_m \pi_1(dz) \leq 0.$$

By Holder's inequality and the Cauchy–Bunyakovsky–Schwarz inequality, we have

$$2 \int_{|z|\leq\delta'} \int_0^1 \sum_{i,j=1}^{d} \left(z^j \partial_j u(\cdot + \theta z), z^i \partial_{-i} u \right)_m (1-\theta) d\theta \pi_1(dz) \leq \varsigma_1(\delta) \|u\|_{m+1}^2.$$

There exists a constant $\epsilon = \epsilon(\varkappa, \delta)$ such that

$$\overline{q} := \varkappa - \varsigma(\delta) - \epsilon > 0.$$

As in Theorem 4.1.2 in [22] and Lemma 4.4, using Holder's and Young's inequalities, the above estimates, and Assumption 2.1, we find that for each $\epsilon > 0$, there is a constant $N = N(d, m, \kappa, K, T, \delta)$ such that for all $(\omega, t) \in \Omega \times [0, T]$,

$$\begin{aligned} &2[u, A_t(u)]_m + \|B_t(u)\|_{m,l^2}^2 + \int_{\mathbf{R}^d} \|\mathcal{C}_z(u)\|_m^2 \pi_2(dz) + \overline{q}\|u\|_{m+1}^2 \\ &\leq N \left(\|u\|_m^2 + \|f_t\|_{m-1} + \|g_t\|_{m,\ell_2} + \int_{\mathbf{R}^d} \|o_t(z)\|_m^2 \pi_2(dz) \right). \end{aligned}$$

Using the self-adjointness of $(1-\Delta)^{1/2}$, the properties of the CBF $[\cdot,\cdot]_m$, and Assumption 2.2, for all $v \in C_c^\infty(\mathbf{R}^d)$ and $u \in H^{m+1}$, $m \geq 1$, we have

$$[v, A(u)]_m = ((L+I)u, (1-\Delta)^m v)_0 + (f, (1-\Delta)^m v)_0. \tag{5.49}$$

Owing to (5.49) and the denseness of $(1-\Delta)^{-m} C_c^\infty(\mathbf{R}^d)$ in H^1, from Theorems 2.9, 2.10, and 4.1 in [6], we obtain the existence and uniqueness of a solution u of (2.3), such that u is a càdlàg H^m-valued process satisfying (2.7). □

Proof of Proposition 2.2. Let A, B, and $\mathcal{C}$ be as in (5.47). Owing to Assumption 2.1, the boundedness of the $m-1$-norm of g in expectation, and estimate (2.7), using Jensen's inequality and Itô's isometry, for $s, t \in [0, T]$, we get

$$\begin{aligned} \mathbf{E}\left\| \int_{]s,t]} A_r(u_r) ds \right\|_{m-1}^2 &\leq |t-s| \left(N\mathbf{E} \int_{]0,T]} \|u_t\|_{m+1}^2 dt + \mathbf{E} \int_{]0,T]} \|f_r\|_{m-1}^2 dr \right) \\ &\leq N|t-s|, \\ \mathbf{E}\left\| \int_{]s,t]} B_r^\varrho(u_r) dw_r^\rho \right\|_{m-1}^2 &= \mathbf{E} \int_{]s,t]} \|B_r(u_r)\|_{m-1,\ell_2}^2 dr \\ &\leq N|t-s| \left(\sup_{t\leq T} \mathbf{E}\|u_t\|_m^2 + \sup_{t\leq T} \mathbf{E}\|g_t\|_{m-1,\ell_2} \right) \leq N|t-s|, \end{aligned}$$

and

$$\begin{aligned} &\mathbf{E}\left\| \int_{]s,t]} \int_{\mathbf{R}^d} \mathcal{C}_z(u_{r-}) q(dr, dz) \right\|_{m-1}^2 = \mathbf{E} \int_{]s,t]} \int_{\mathbf{R}^d} \|\mathcal{C}_z(u_r)\|_{m-1}^2 \pi_2(dz) ds \\ &\leq N|t-s| \left(\sup_{t\leq T} \mathbf{E}\|u_t\|_m^2 + \sup_{t\leq T} \mathbf{E} \int_{\mathbf{R}^d} \|o_t(z)\|_{m-1}^2 \pi_2(dz) \right) \end{aligned}$$

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$$\leq N|t-s|,$$

which completes the proof of the proposition. □

Theorem 5.1. *Let Assumptions 2.1 through 2.5 hold for some $m \geq 2$. Let u be the solution of (2.3) and $(u_n^{h,\tau})_{n=0}^{\mathcal{T}}$ be defined by (4.17). Then there is a constant $N = N(d,m,\varkappa,K,T,C,\lambda,\kappa_m^2,\delta)$ such that*

$$\mathbf{E}\max_{0\leq n\leq \mathcal{T}}\|u_{t_n}-u_n^{h,\tau}\|_{m-2}^2+\mathbf{E}\sum_{n=0}^{\mathcal{T}}\tau\sum_{i=0}^{d}\|\delta_{h,i}u_{t_n}-\delta_{h,i}u_n^{h,\tau}\|_{m-2}^2 ds$$
$$\leq N(|h|^2+\tau). \tag{5.50}$$

Proof. For $t\in[0,T]$, let $\kappa_1(t):=t_{n-1}$ for $t\in]t_{n-1},t_n]$, and set $e_n^{h,\tau}:=u_n^{h,\tau}-u_{t_n}$. One can easily verify that $e_n^{h,\tau}$ satisfies in H^{m-2},

$$e_n^{h,\tau}=e_{n-1}^{h,\tau}+\int_{]t_{n-1},t_n]}\left((\mathcal{L}_{t_n-1}^h+I^h)e_{n-1}^{h,\tau}+\sum_{i=0}^d\delta_{h,i}F_t^i\right)dt$$
$$+\int_{]t_{n-1},t_n]}\left(\mathcal{N}_{t_{n-1}}^{\varrho;h}e_{n-1}^{h,\tau}+G_t^{\varrho}\right)dw_t^{\varrho}+\int_{]t_{n-1},t_n]}\int_{\mathbf{R}^d}\left(\mathcal{I}^h(z)e_{n-1}^{h,\tau}+R_t(z)\right)q(dt,dz),$$

where

$$\begin{aligned}F_t^0:=&\ (\mathcal{L}_{\kappa_1(t)}^h-\mathcal{L}_{\kappa_1(t)})u_t+(\mathcal{L}_{\kappa_1(t)}-\mathcal{L}_t)u_t+(I^h-I)u_t\\&+(f_{\kappa_1(t)}-f_t)+I_{\delta^c}^h(u_{\kappa_1(t)}-u_t)\\&+\sum_{j=1}^d a_{\kappa_1(t)}^{0j}\delta_{-h,j}(u_{\kappa_1(t)}-u_t)+\sum_{i=0}^d a_{\kappa_1(t)}^{i0}\delta_{h,i}(u_{\kappa_1(t)}-u_t)\\&-\sum_{i,j=1}^d\delta_{-h,j}(u_{\kappa_1(t)}-u_t)(\cdot+h)\delta_{h,i}a_{\kappa_1(t)}^{ij},\end{aligned}$$

$$F_t^i:=\sum_{j=1}^d a_{\kappa_1(t)}^{ij}\delta_{-h,j}(u_{\kappa_1(t)}-u_t)+\sum_{k=0}^{\infty}\sum_{l=1}^{\chi(h,k)}\bar{\theta}_l^{k,h}\zeta_{k,h}^{ij}\delta_{-h,j}(u_{\kappa_1(t)}-u_t)(\cdot+hz_{r_l^{h,k}})$$

$$G_t^{\varrho}:=(\mathcal{N}_{\kappa_1(t)}^{\varrho}-\mathcal{N}_t^{\varrho})u_t+(\mathcal{N}_{\kappa_1(t)}^{\varrho;h}-\mathcal{N}_{\kappa_1(t)}^{\varrho})u_t+\mathcal{N}_{\kappa_1(t)}^{\varrho}(u_{\kappa_1(t)}-u_t)+(g_{\kappa_1(t)}^{\varrho}-g_t^{\varrho})$$

$$R_t^h(z):=\left(\mathcal{I}^h(z)-\mathcal{I}(z)\right)u_{t-}+\mathcal{I}^h(z)(u_{\kappa_1(t)}-u_{t-})+\left(o_{\kappa_1(t)}(z)-o_t(z)\right).$$

By Theorem 4.6, we have

$$\mathbf{E}\max_{0\leq n\leq\mathcal{T}}\|e_n^{h,\tau}\|_{m-2}^2+\mathbf{E}\sum_{n=0}^{\mathcal{T}}\tau\sum_{i=0}^d\|\delta_{h,i}e_n^{h,\tau}\|_{m-2}^2$$

$$\leq N\mathbf{E}\int_{]0,T]}\Big(\sum_{i=0}^d\|F_t^i\|_{m-2}^2+\|G_t\|_{m-2,\ell_2}^2+\int_{\mathbf{R}^d}\|R_t(z)\|_{m-2}^2\pi_2(dz)\Big)dt.$$

Using Lemmas 4.1, 4.2, and 4.3 and Assumptions 2.1(i) and 2.4, the right-hand-side of the above relation can be estimated by

$$N\mathbf{E}\int_{]0,T]}\left(|h|^2\|u_t\|_{m+1}^2+|\kappa_1(t)-t|\|u_t\|_m^2+\|u_{\kappa_1(t)}-u_t\|_{m-1}^2\right)dt$$

$$+ N\mathbf{E}\int_{]0,T]}\Big(\|f_{\kappa_1(t)} - f_t\|^2_{m-2} + \|g_{\kappa_1(t)} - g_t\|_{m-2,\ell_2}$$
$$+ \int_{\mathbf{R}^d} \|o_{\kappa_1(t)}(z) - o_t(z)\|_{m-2}\pi_2(dz)\Big)\,dt$$

where N depends only on $d, m, \varkappa, K, C, \lambda, T, \delta$ and ν. By virtue of (2.7), Proposition 2.2, and Assumption 2.3, we obtain (5.50), which completes the proof. □

Theorem 5.2. *Let Assumptions 2.1 through 2.4 hold with $m \geq 2$ and let u be the solution of (2.3). There exists a constant $R = R(d, m, \varkappa, K, \delta)$ such that if $\mathcal{T} > R$, then there exists a unique solution $(v^{h,\tau})_{n=0}^{\mathcal{T}}$ of (4.18) in H^{m-2}. Moreover, there is a constant $N = N(d, m, \varkappa, K, T, C, \lambda, \kappa_m^2, \delta)$ such that*

$$\mathbf{E}\max_{0\leq n\leq\mathcal{T}}\|u_{t_n} - v_n^{h,\tau}\|^2_{m-2} + \mathbf{E}\sum_{n=0}^{\mathcal{T}}\tau\sum_{i=0}^{d}\|\delta_{h,i}u_{t_n} - \delta_{h,i}v_n^{h,\tau}\|^2_{m-2}ds$$
$$\leq N(|h|^2 + \tau). \tag{5.51}$$

Proof. The existence and uniqueness follow directly from Theorem 4.7. Let $\kappa_1(t)$ be as in the previous proof and set $\kappa_2(t) = t_n$ for $t \in]t_{n-1}, t_n]$. Let G and R be defined as in Theorem 5.1 and define $\bar{F}^i$ to be F^i with $\kappa_1(t)$ replaced with $\kappa_2(t)$. Set $e_n^{h,\tau} = v_n^{h,\tau} - u_{t_n}$. As in the proof of Theorem 5.1, we have

$$e_n^{h,\tau} = e_{n-1}^{h,\tau} + \int_{]t_{n-1},t_n]}\Big((\tilde{\mathcal{L}}^h_{t_n} + I^h_\delta)e_n^{h,\tau} + \tilde{I}^h_{\delta^c}e_{n-1}^{h,\tau} + \sum_{i=0}^{d}\delta_{h,i}\tilde{F}^i_t\Big)\,dt$$
$$+ \int_{]t_{n-1},t_n]}\Big(\mathbf{1}_{n>1}\mathcal{N}^{\varrho;h}_{t_{n-1}}e_{n-1}^{h,\tau} + \tilde{G}^\varrho_t\Big)\,dw^\varrho_t$$
$$+ \int_{]t_{n-1},t_n]}\int_{\mathbf{R}^d}\Big(\mathbf{1}_{n>1}\mathcal{I}^h(z)e_{n-1}^{h,\tau} + \tilde{R}_t(z)\Big)\,q(dt,dz),$$

where

$$\tilde{F}^i = \bar{F}^i, \quad \text{for } i \neq 0, \qquad \tilde{F}^0 = \bar{F}^0 + \tilde{I}^h_{\delta^c}(u_{\kappa_1(t)} - u_{\kappa_2(t)}),$$
$$\tilde{G}^\varrho_t = \mathbf{1}_{t\leq t_1}(\mathcal{N}^\varrho_t u_t + g^\varrho_t) + \mathbf{1}_{t>t_1}G^\varrho_t, \qquad \tilde{R}_t(z) = \mathbf{1}_{t\leq t_1}\mathcal{I}(z)u_{t-} + \mathbf{1}_{t>t_1}R_t(z).$$

By Theorem 4.7, we have

$$\mathbf{E}\max_{0\leq n\leq\mathcal{T}}\|e_n^{h,\tau}\|^2_{m-2} + \mathbf{E}\sum_{n=0}^{\mathcal{T}}\tau\sum_{i=0}^{d}\|\delta_{h,i}e_n^{h,\tau}\|^2_{m-2} \leq N(A_1 + A_2 + A_3),$$

where

$$A_1 := \mathbf{E}\int_{]0,T]}\sum_{i=0}^{d}\|\bar{F}^i_t\|^2_{m-2}dt + \int_{]t_1,T]}\Big(\|G_t\|^2_{m-2,\ell_2} + \int_{\mathbf{R}^d}\|R_t(z)\|^2_{m-2}\pi_2(dz)\Big)\,dt,$$
$$A_2 := \mathbf{E}\int_{]0,T]}\|\tilde{I}^h_{\delta^c}(u_{\kappa_1(t)} - u_{\kappa_2(t)})\|^2_{m-2}dt$$
$$A_3 := \mathbf{E}\int_{]0,t_1]}\Big(\|M_t u_t + g_t\|^2_{m-2,\ell_2} + \int_{\mathbf{R}^d}\|\mathcal{I}(z)u_t + o_t(z)\|^2_{m-2}\pi_2(dz)\Big)\,dt.$$

As in the proof of Theorem 5.1, we have $A_1 \le N(|h|^2 + \tau)$. By Proposition 2.2, we get

$$A_2 \le N\mathbf{E}\int_0^T \|u_{\kappa_1(t)} - u_{\kappa_2(t)}\|^2_{m-1} dt \le N\tau.$$

Owing to 2.3, we have Q11

$$A_3 \le N\mathbf{E}\int_0^{t_1} \left(\|u_t\|^2_{m-1} + \|g_t\|^2_{m-2,\ell_2} + \int_{\mathbf{R}^d} \|o_t(z)\|^2_{m-2}\pi_2(dz)\right) dt$$

$$\le N\tau\mathbf{E}\int_0^{t_1} \left(\sup_{t\le T}\|u_t\|^2_{m-1} + \xi\right) dt \le N\tau.$$

Combining the above estimates yields (5.51). □

By virtue of Sobolev's embedding theorem and (2.13), as in [10], we obtain the following corollaries of Theorems 5.1 and 5.2.

Corollary 5.3. *Let $l \ge 0$ be an integer. Suppose the assumptions of Theorem 5.1 hold with $m > l + 2 + d/2$. Then for all $\lambda = (\lambda^1, \dots, \lambda^l) \in \{1\dots, d\}^l$ and $\delta_{h,\lambda} = \delta_{h,\lambda^1}\cdots\delta_{h,\lambda^l}$, there is a constant $N = N(d, m, l, \varkappa, K, T, C, \lambda, \kappa^2_m, \delta)$ such that*

$$\mathbf{E}\max_{0\le n\le \mathcal{T}}\sup_{x\in\mathbf{R}^d}|\delta_{h,\lambda}u_{t_n}(x) - \delta_{h,\lambda}u^{h,\tau}_n(x)|^2 + \mathbf{E}\max_{0\le n\le\mathcal{T}}\|\delta_{h,\lambda}u_{t_n} - \delta_{h,\lambda}u^{h,\tau}_n\|^2_{\ell_2(\mathbf{G}_h)}$$
$$\le N(|h|^2 + \tau).$$

Corollary 5.4. *Let $l \ge 0$ be an integer. Suppose the assumptions of Theorem 5.2 hold with $m > l + 2 + d/2$. Then for all $\lambda = (\lambda^1, \dots, \lambda^l) \in \{1\dots, d\}^l$ and $\delta_{h,\lambda} = \delta_{h,\lambda^1}\cdots\delta_{h,\lambda^l}$, there is a constant $N = N(d, m, l, \varkappa, K, T, C, \lambda, \kappa^2_m, \delta)$ such that*

$$\mathbf{E}\max_{0\le n\le \mathcal{T}}\sup_{x\in\mathbf{R}^d}|\delta_{h,\lambda}u_{t_n}(x) - \delta_{h,\lambda}v^{h,\tau}_n(x)|^2 + \mathbf{E}\max_{0\le n\le\mathcal{T}}\|\delta_{h,\lambda}u_{t_n} - \delta_{h,\lambda}v^{h,\tau}_n\|^2_{\ell_2(\mathbf{G}_h)}$$
$$\le N(|h|^2 + \tau).$$

Proof of Theorems 2.3 and 2.4. Let $(\hat{u}^{h,\tau}_n)^M_{n=0}$ be defined by (2.11). Denote by $(\cdot,\cdot)_{\ell_2(\mathbf{G}_h)}$ the inner product of $\ell_2(\mathbf{G}_h)$. There exists a constant $\epsilon = \epsilon(\varkappa, \delta)$ such that

$$\overline{q} := \varkappa - \varsigma_1(\delta) - \epsilon > 0.$$

As in (4.27), there is a constant $N_6 = N_6(d, \varkappa, K, \delta)$ such that for all $\phi \in \ell_2(\mathbf{G}_h)$,

$$(\phi, \tilde{\mathcal{L}}^h_t\phi)_{\ell_2(\mathbf{G}_h)} + (\phi, I^h_\delta\phi)_{\ell_2(\mathbf{G}_h)} \le -\overline{q}\sum_{i=1}^d \|\delta_{h,i}\phi\|^2_{\ell_2(\mathbf{G}_h)} + N_6\|\phi\|^2_{\ell_2(\mathbf{G}_h)}.$$

Following the arguments in the beginning of the proof of Theorem 4.7, we conclude that if $\mathcal{T} > N_6 T$, then there exists a unique solution $(\hat{v}^{h,\tau}_n)^M_{n=0}$ in $\ell_2(\mathbf{G}_h)$ of (2.12). It is easy to see that $N_6 < N_2$ (for the same choice of ϵ) for all $m > 0$, where N_2 is the constant appearing on the right-hand-side of (4.27), and hence $N_6 < R$, where R is as in Theorem 4.7. Let $(u^{h,\tau}_n)^M_{n=1}$ be defined by (4.17). By Theorem 5.2, there exists a unique solution $(v^{h,\tau}_n)^M_{n=1}$ of (4.18). It suffices to show that almost surely,

$$u^{h,\tau}_n(x) = \hat{u}^{h,\tau}_n(x) \qquad (5.52)$$

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and

$$v_n^{h,\tau}(x) = \hat{v}_n^{h,\tau}(x), \tag{5.53}$$

for all $n \in \{0, \dots, M\}$ and $x \in \mathbf{G}^h$. Let $\mathscr{S} : H^{m-2} \to \ell_2(\mathbf{G}^h)$ denote the embedding from Remark 2.3. Applying $\mathscr{S}$ to both sides of (4.17), one can see that $\mathscr{S}u^{h,\tau}$ and $\hat{u}^{h,\tau}$ satisfy the same recursive relation in $\ell_2(\mathbf{G}^h)$ with common initial condition φ, and hence (5.52) follows. Similarly, $\mathscr{S}v^{h,\tau}$ and $\hat{v}^{h,\tau}$ satisfy the same equation in $\ell_2(\mathbf{G}^h)$ and (5.53) follows from the uniqueness of the $\ell_2(\mathbf{G}^h)$ solution of (2.12). □

Remark 5.1. It follows from Corollaries 5.3, 5.4, and relations (5.52) and (5.53) that if more regularity is assumed of the coefficients and the data of Eq. (2.3), then better estimates can be obtained than the ones presented in Theorems 2.3 and 2.4.

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