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Optimal lower bounds on hitting probabilities for non-linear systems of stochastic fractional heat equations

Robert C. Dalang[1], Fei Pu[*,1]

Institut de Mathématiques, Ecole Polytechnique Fédérale de Lausanne, Station 8, CH-1015 Lausanne, Switzerland

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Abstract

We consider a system of d non-linear stochastic fractional heat equations in spatial dimension 1 driven by multiplicative d-dimensional space–time white noise. We establish a sharp Gaussian-type upper bound on the two-point probability density function of $(u(s, y), u(t, x))$. From this result, we deduce optimal lower bounds on hitting probabilities of the process $\{u(t, x) : (t, x) \in [0, \infty[\times\mathbb{R}\}$ in the non-Gaussian case, in terms of Newtonian capacity, which is as sharp as that in the Gaussian case. This also improves the result in Dalang et al. (2009) for systems of classical stochastic heat equations. We also establish upper bounds on hitting probabilities of the solution in terms of Hausdorff measure.

MSC: primary 60H15; 60J45; secondary 60H07; 60G60

Keywords: Hitting probabilities; Systems of non-linear stochastic fractional heat equations; Malliavin calculus; Gaussian-type upper bound; Space–time white noise

1. Introduction

We consider a system of non-linear stochastic fractional heat equations with vanishing initial conditions on the whole space $\mathbb{R}$, that is,

$$\frac{\partial u_i}{\partial t}(t, x) = {}_xD^\alpha u_i(t, x) + \sum_{j=1}^{d} \sigma_{ij}(u(t, x))\dot{W}^j(t, x) + b_i(u(t, x)), \tag{1.1}$$

* Corresponding author.
E-mail addresses: robert.dalang@epfl.ch (R.C. Dalang), fei.pu@epfl.ch (F. Pu).
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for $1 \leqslant i \leqslant d, t \in [0, T], x \in \mathbb{R}$, where $u := (u_1, \ldots, u_d)$, with initial conditions $u(0, x) = 0$ for all $x \in \mathbb{R}$. Here, $\dot{W} := (\dot{W}^1, \ldots, \dot{W}^d)$ is a vector of d independent space–time white noises on $[0, T] \times \mathbb{R}$ defined on a probability space $(\Omega, \mathscr{F}, \mathrm{P})$. The functions $b_i, \sigma_{ij} : \mathbb{R}^d \to \mathbb{R}$ are globally Lipschitz continuous for all $1 \leqslant i, j \leqslant d$. We set $b = (b_i)$, $\sigma = (\sigma_{ij})$. The fractional differential operator D^α $(1 < \alpha \leqslant 2)$ is given by

$$D^\alpha \varphi(x) = \mathscr{F}^{-1}\{-|\lambda|^\alpha \mathscr{F}\{\varphi(x); \lambda\}; x\},$$

where $\mathscr{F}$ denotes the Fourier transform. The operator D^α coincides with the fractional power $\alpha/2$ of the Laplacian. When $\alpha = 2$, it is the Laplacian itself. For $1 < \alpha < 2$, it can also be represented by

$$D^\alpha \varphi(x) = c_\alpha \int_{\mathbb{R}} \frac{\varphi(x+y) - \varphi(x) - y\varphi'(x)}{|y|^{1+\alpha}} dy$$

with certain positive constant c_α depending only on α; see [16,17,19] and [6]. We refer to [21] for additional equivalent definitions of D^α.

Let $I \subset]0, T]$ and $J \subset \mathbb{R}$ be two fixed compact intervals with positive length. We choose m sufficiently large so that $I \times J \subset [0, m] \times [-m, m]$. We are interested in the hitting probability $\mathrm{P}\{u(I \times J) \cap A \neq \emptyset\}$, where $u(I \times J)$ denotes the range of $I \times J$ under the random map $(t, x) \mapsto u(t, x)$. For systems of stochastic heat equations on the spatial interval [0, 1], in the case where the noise is additive, i.e., $\sigma \equiv \mathrm{Id}$, $b \equiv 0$, Dalang, Khoshnevisan and Nualart [9] have established upper and lower bounds on hitting probabilities for the Gaussian solution. They show that there exists $c > 0$ depending on M, I, J with $M > 0$, such that, for all Borel sets $A \subseteq [-M, M]^d$,

$$c^{-1}\mathrm{Cap}_{d-6}(A) \leqslant \mathrm{P}\{u(I \times J) \cap A \neq \emptyset\} \leqslant c\, \mathscr{H}_{d-6}(A), \tag{1.2}$$

where Cap_β denotes the capacity with respect to the Newtonian β-kernel and $\mathscr{H}_\beta$ denotes the β-dimensional Hausdorff measure (see (1.8), (1.9) for definitions). If the noise is multiplicative, i.e., σ and b are not constants (but are sufficiently regular), then using techniques of Malliavin calculus, Dalang, Khoshnevisan and Nualart [10] have obtained upper and lower bounds on hitting probabilities for the non-Gaussian solution. Indeed, they prove that there exists $c > 0$ depending on M, I, J, η with $M > 0, \eta > 0$, such that, for all Borel sets $A \subseteq [-M, M]^d$,

$$c^{-1}\mathrm{Cap}_{d+\eta-6}(A) \leqslant \mathrm{P}\{u(I \times J) \cap A \neq \emptyset\} \leqslant c\, \mathscr{H}_{d-\eta-6}(A). \tag{1.3}$$

Furthermore, these results have been extended to higher spatial dimensions driven by spatially homogeneous noise in [11]. This type of question has also been studied for systems of stochastic wave equations in [12], and in higher spatial dimensions [14] and [15], and for systems of stochastic Poisson equations [31].

The objective of this paper is to remove the η in the dimension of capacity in (1.3) so that the lower bound on hitting probabilities is consistent with the Gaussian case in (1.2), and to extend these results to systems of stochastic fractional heat equations.

Consider the following three hypotheses on the coefficients of the system (1.1):

- **P1** The functions σ_{ij} and b_i are bounded and infinitely differentiable with bounded partial derivatives of *all* orders, for $1 \leqslant i, j \leqslant d$.
- **P1'** The functions σ_{ij} and b_i are infinitely differentiable with bounded partial derivatives of *all positive* orders, and the σ_{ij} are bounded, for $1 \leqslant i, j \leqslant d$.
- **P2** The matrix σ is uniformly elliptic, that is, $\|\sigma(x)\xi\|^2 \geqslant \rho^2 > 0$ for some $\rho > 0$, for all $x \in \mathbb{R}^d$, $\|\xi\| = 1$.

Notice that hypothesis **P1’** is weaker than hypothesis **P1**, since in **P1’**, the functions b_i, $i = 1, \ldots, d$ are not assumed to be bounded.

Adapting the results from [4] to the case $d \geqslant 1$, the $\mathbb{R}^d$-valued random vector $u(t,x) = (u_1(t,x), \ldots, u_d(t,x))$ admits a smooth probability density function, denoted by $p_{t,x}(\cdot)$ for all $(t,x) \in [0,T] \times \mathbb{R}$: see our Proposition 3.2. For $(s,y) \neq (t,x)$, let $p_{s,y;t,x}(\cdot,\cdot)$ denote the joint density function of the $\mathbb{R}^{2d}$-valued random vector

$$(u(s,y), u(t,x)) = (u_1(s,y), \ldots, u_d(s,y), u_1(t,x), \ldots, u_d(t,x))$$

(the existence of $p_{s,y;t,x}(\cdot,\cdot)$ is a consequence of (2.4), Proposition 4.7 and [10, Theorem 3.1]). Define the fractional parabolic metric

$$\Delta_\alpha((t,x);(s,y)) := |t-s|^{\frac{\alpha-1}{2\alpha}} + |x-y|^{\frac{\alpha-1}{2}}, \quad \text{for } t,\ s \in [0,T] \text{ and } x,\ y \in \mathbb{R}. \tag{1.4}$$

Theorem 1.1. *Assume **P1’** and **P2**. Fix $T > 0$ and let $I \subset]0,T]$ and $J \subset \mathbb{R}$ be two fixed non-trivial compact intervals.*

(a) The density $p_{t,x}(z)$ is a C^∞ function of z and is uniformly bounded over $z \in \mathbb{R}^d$ and $(t,x) \in I \times J$.

(b) There exists $c > 0$ such that for all $s,t \in I, x,y \in J$ with $(s,y) \neq (t,x)$, $z_1, z_2 \in \mathbb{R}^d$ and $p \geqslant 1$,

$$p_{s,y;t,x}(z_1,z_2) \leqslant c(\Delta_\alpha((t,x);(s,y)))^{-d} \left[\frac{(\Delta_\alpha((t,x);(s,y)))^2}{\|z_1 - z_2\|^2} \wedge 1\right]^{p/(4d)}. \tag{1.5}$$

*(c) Assume also **P1**. Then there exists $c > 0$ such that for all $s,t \in I, x,y \in J$ with $(s,y) \neq (t,x)$ and $z_1, z_2 \in \mathbb{R}^d$,*

$$p_{s,y;t,x}(z_1,z_2) \leqslant c(\Delta_\alpha((t,x);(s,y)))^{-d} \exp\left(-\frac{\|z_1 - z_2\|^2}{c(\Delta_\alpha((t,x);(s,y)))^2}\right). \tag{1.6}$$

The right-hand side of (1.5) is larger than the r.h.s. of (1.6) (after adjusting the constant). In fact, the boundedness of the functions b_i, $i = 1, \ldots, d$ in hypothesis **P1** is only used when we derive the exponential factor on the right-hand side of (1.6) by applying Girsanov’s theorem. However, under the hypothesis **P1’**, when b_i is not bounded, Girsanov’s theorem is no longer applicable. We establish (1.5) in Section 4.3 and, following [11,15], show in Section 5.2 that this estimate is also sufficient for our purposes.

We prove the smoothness and uniform boundedness of the one-point density (Theorem 1.1(a)) in Section 3. We present the Gaussian-type upper bound on the two-point density (Theorem 1.1(b)) in Section 4.3.

We will also need the strict positivity of $p_{t,x}(\cdot)$.

Theorem 1.2. *Assume* **P1$'$** *and* **P2**. *For all $(t,x) \in]0,T] \times \mathbb{R}$ and $z \in \mathbb{R}^d$, the density $p_{t,x}(z)$ is strictly positive.*

The proof of the strict positivity of the one-point density (Theorem 1.2) is quite similar to that in [26], using the inverse function theorem and Girsanov’s theorem. We refer to [28, Chapter 2.4] for a complete proof. We mention that Chen, Hu and Nualart [7] have recently studied the strict positivity of the density on the support of the law for the non-linear stochastic fractional heat equation without drift term and with measure-valued initial data and unbounded diffusion coefficient.

Our main contribution is to obtain the upper bounds in Theorem 1.1(b) and (c), which are an improvement over [10, Theorem 1.1(c)]. There, for the stochastic heat equation, the optimal Gaussian-type upper bound was shown to hold when $t = s$, while an extra term η appeared in the exponent when $t \neq s$; see [10, Theorem 1.1]. We manage to remove this η in the Gaussian-type upper bound on the joint density in [10, Theorem 1.1(c)], so that this becomes the best possible upper bound, as in the Gaussian case. This requires a detailed analysis of the small eigenvalues of the Malliavin matrix γ_Z of $Z := (u(s, y), u(t, x) - u(s, y))$; see Proposition 4.8. We prove Proposition 4.8 by giving a better estimate on the Malliavin derivative of the solution; see Lemma A.4, which, for a certain range of parameters, is an improvement of Morien [23, Lemma 4.2]; see also Lemma A.3. This estimate is used in Lemma 4.4 to obtain a bound on the integral terms in the Malliavin derivative of u (compare with [10, Lemma 6.11]), then in Proposition 4.8 to bound negative moments of the smallest eigenvalue of the Malliavin matrix (compare with [10, Proposition 6.9]), and finally in Proposition 4.7 and Theorem 4.11 to bound negative moments of the Malliavin matrix (compare with [10, Proposition 6.6] and [10, Theorem 6.3]). This improves the result of [10, Theorem 1.1(c)], and the method extends to systems of stochastic fractional heat equations (1.1) for $1 < \alpha \leqslant 2$ with a unified proof.

Coming back to potential theory, let us introduce some notation, following [18]. For all Borel sets $F \subseteq \mathbb{R}^d$, we define $\mathscr{P}(F)$ to be the set of all probability measures with compact support contained in F. For all integers $k \geqslant 1$ and $\mu \in \mathscr{P}(\mathbb{R}^k)$, we let $I_\beta(\mu)$ denote the *β-dimensional energy* of μ, that is,

$$I_\beta(\mu) := \iint \mathrm{K}_\beta(\|x - y\|)\mu(dx)\mu(dy),$$

where $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^k$,

$$\mathrm{K}_\beta(r) := r^{-\beta}\, 1_{\{\beta>0\}} + \log_+(1/r)1_{\{\beta=0\}} + 1_{\{\beta<0\}} \tag{1.7}$$

where $\log_+(x) := \log(x \vee e)$.

For all $\beta \in \mathbb{R}$, integers $k \geqslant 1$, and Borel sets $F \subseteq \mathbb{R}^k$, $\mathrm{Cap}_\beta(F)$ denotes the *β-dimensional capacity* of F:

$$\mathrm{Cap}_\beta(F) := \left[\inf_{\mu\in\mathscr{P}(F)} I_\beta(\mu)\right]^{-1}, \tag{1.8}$$

where $1/\infty := 0$. Note that if $\beta < 0$, then $\mathrm{Cap}_\beta(\cdot) \equiv 1$.

Given $\beta \geqslant 0$, the β-dimensional *Hausdorff measure* of F is defined by

$$\mathscr{H}_\beta(F) = \lim_{\epsilon\to 0^+} \inf\left\{\sum_{i=1}^{\infty}(2r_i)^\beta : F \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i), \sup_{i\geqslant 1} r_i \leqslant \epsilon\right\}. \tag{1.9}$$

When $\beta < 0$, we define $\mathscr{H}_\beta(F)$ to be infinite.

Using Theorems 1.1 and 1.2, together with results from Dalang, Khoshnevisan and Nualart [9], we shall prove the following results for the hitting probabilities of the solution (note that the constants depend on the fixed $\alpha \in\,]1, 2]$).

Theorem 1.3. *Assume **P1'** and **P2**. Fix $T > 0$, $M > 0$ and $\eta > 0$. Let $I \subset\,]0, T]$ and $J \subset \mathbb{R}$ be two fixed non-trivial compact intervals.*

(a) There exist $c_1 > 0$ depending on I, J and M, and $c_2 > 0$ depending on I, J and η such that for all compact sets $A \subseteq [-M, M]^d$,

$$c_1\, \mathit{Cap}_{d-\frac{2(\alpha+1)}{\alpha-1}}(A) \leqslant \mathrm{P}\{u(I \times J) \cap A \neq \emptyset\} \leqslant c_2\, \mathscr{H}_{d-\frac{2(\alpha+1)}{\alpha-1}-\eta}(A).$$

(b) *For all* $t \in]0, T]$, *there exist* $c_1 > 0$ *depending on* J *and* M, *and* $c_2 > 0$ *depending on* J *and* η *such that for all compact sets* $A \subseteq [-M, M]^d$,

$$c_1 \, Cap_{d-\frac{2}{\alpha-1}}(A) \leqslant \mathrm{P}\{u(\{t\} \times J) \cap A \neq \emptyset\} \leqslant c_2 \, \mathscr{H}_{d-\frac{2}{\alpha-1}-\eta}(A).$$

(c) *For all* $x \in \mathbb{R}$, *there exist* $c_1 > 0$ *depending on* I *and* M, *and* $c_2 > 0$ *depending on* I *and* η *such that for all compact sets* $A \subseteq [-M, M]^d$,

$$c_1 \, Cap_{d-\frac{2\alpha}{\alpha-1}}(A) \leqslant \mathrm{P}\{u(I \times \{x\}) \cap A \neq \emptyset\} \leqslant c_2 \, \mathscr{H}_{d-\frac{2\alpha}{\alpha-1}-\eta}(A).$$

The optimal lower bounds for the hitting probabilities on the left-hand sides of Theorem 1.3 are mainly the consequence of the sharp upper bound on the two-point density function in (1.5) (or the sharp Gaussian-type upper bound (1.6) under the slightly stronger condition **P1**).

Remark 1.4. For $\alpha = 2$, Theorems 1.1, 1.2, 1.3 (as well as Theorem 1.6) are also valid for stochastic heat equations on a bounded interval with Neumann or Dirichlet boundary conditions; see Remark 4.12. The upper bounds on hitting probabilities on the right-hand sides of Theorem 1.3 are an extension to $1 < \alpha \leqslant 2$ of the corresponding results of [10, Theorem 1.2] for $\alpha = 2$.

Remark 1.5. The main technical improvement in this paper, which yields the sharp upper bound on the two-point density function (and hence optimal lower bounds on hitting probabilities) can also be used for the solution to non-linear stochastic heat equations in higher spatial dimension; see [13].

If $\sigma \equiv \mathrm{Id}$ and $b \equiv 0$, by [35, Theorem 7.6], the upper bounds in Theorem 1.3 can be improved to the best result available for the Gaussian case.

Theorem 1.6. *Denote by* v *the solution of* (1.1) *with* $\sigma \equiv Id$ *and* $b \equiv 0$. *Fix* $T > 0$. *Let* $I \subset]0, T]$ *and* $J \subset \mathbb{R}$ *be two fixed non-trivial compact intervals. The upper bounds in Theorem* 1.3*(a), (b) and (c) hold when* u *is replaced by* v *and* η *is set to* 0 *in the Hausdorff measure on the right-hand sides.*

Theorems 1.3 and 1.6 will be proved in Section 5. We conclude this introduction by giving a rigorous formulation of Eq. (1.1), following Walsh [32]. For $t \geqslant 0$, let $\mathscr{F}_t = \sigma\{W(s, x), s \in [0, t], x \in \mathbb{R}\} \vee \mathscr{N}$, where $\mathscr{N}$ is the σ-field generated by P-null sets. A *mild solution* of (1.1) is a jointly measurable $\mathbb{R}^d$-valued process $u = \{u(t, x), t \geqslant 0, x \in \mathbb{R}\}$, adapted to the filtration $(\mathscr{F}_t)_{t \geqslant 0}$, such that for $i \in \{1, \ldots, d\}$,

$$\begin{aligned} u_i(t, x) = & \int_0^t \int_{\mathbb{R}} G_\alpha(t - r, x - v) \sum_{j=1}^d \sigma_{ij}(u(r, v)) W^j(dr, dv) \\ & + \int_0^t \int_{\mathbb{R}} G_\alpha(t - r, x - v) b_i(u(r, v)) dr dv, \end{aligned} \tag{1.10}$$

where the stochastic integral is interpreted as in [32] and $G_\alpha(t, x)$ denotes the Green kernel for the (fractional) heat equation. If $\alpha = 2$, the Green kernel $G_2(t, x)$ (denoted by $G(t, x)$) for the heat equation without boundary is given by $G(t, x) = (4\pi t)^{-1/2} \exp(-x^2/(4t))$. The Green kernel for the fractional heat equation ($1 < \alpha < 2$) is given via Fourier transform:

$$G_\alpha(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-i\lambda x - t|\lambda|^\alpha) d\lambda.$$

We refer to [2,6,17,34] for the properties of the Green kernel. In fact, to make sense of the stochastic integral in (1.10), the function $(r, v) \mapsto 1_{\{r<t\}} G_\alpha(t-r, x-v)$ must belong to $L^2([0,T] \times \mathbb{R})$. This explains the requirement $1 < \alpha \leqslant 2$; see also [6,17].

The problems of existence, uniqueness and Hölder continuity of the solution to non-linear stochastic fractional heat equations have been studied by many authors; see, e.g., [1,4,6,17] and the references therein. Adapting these results to the case $d \geqslant 1$, one can show that there exists a unique process $u = \{u(t,x), t \geqslant 0, x \in \mathbb{R}\}$ that is a mild solution of (1.1), such that for any $T > 0$ and $p \geqslant 1$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[|u_i(t,x)|^p\right] < \infty, \quad i \in \{1,\dots,d\}. \tag{1.11}$$

Moreover, the following estimate holds for the moments of increments of the solution (see [1, Theorem 3.1]): for all $s, t \in [0,T]$, $x, y \in \mathbb{R}$ and $p > 1$,

$$\mathrm{E}[\|u(t,x) - u(s,y)\|^p] \leqslant C_{T,p}(\Delta_\alpha((t,x);(s,y)))^p, \tag{1.12}$$

where Δ_α is defined in (1.4).

Denote by $K_m = [0,m] \times [-m,m]$ and $\beta_p = 1 - \frac{2(\alpha+1)}{p(\alpha-1)}$ with $p > \frac{2(\alpha+1)}{\alpha-1}$. By (1.12), Kolmogorov's continuity theorem (see [20, Theorem 1.4.1, p. 31] and [5, Proposition 4.2]), the solution u has a continuous modification which we continue to denote by u that satisfies, for all integers m and $0 \leqslant \beta < \beta_p$,

$$\mathrm{E}\left[\left(\sup_{\substack{(t,x),(s,y)\in K_m\\(t,x)\neq(s,y)}} \frac{\|u(t,x)-u(s,y)\|}{[\Delta_\alpha((t,x);(s,y))]^\beta}\right)^p\right] < \infty. \tag{1.13}$$

2. Elements of Malliavin calculus

For the basic notions of Malliavin calculus, we refer to Nualart [25] (see also [30]). Let $W = \{W(h),\ h \in \mathscr{H}\}$ denote the isonormal Gaussian process (see [25, Definition 1.1.1]) associated with our space–time white noise $\dot{W}$, where $\mathscr{H}$ is the Hilbert space $L^2([0,T]\times\mathbb{R},\mathbb{R}^d)$. We then have the notion of Malliavin derivative $DG = (D_{t,x}G = (D^{(1)}_{t,x}G,\dots,D^{(d)}_{t,x}G), (t,x) \in [0,T]\times\mathbb{R})$ of a smooth random variable G, and for $p, k \geqslant 1$, the Sobolev space $\mathbb{D}^{k,p}$ with the seminorm $\|\cdot\|_{k,p}$ defined by

$$\|G\|^p_{k,p} = \mathrm{E}[|G|^p] + \sum_{j=1}^k \mathrm{E}\left[\|D^j G\|^p_{\mathscr{H}^{\otimes j}}\right],$$

where

$$\|D^j G\|^2_{\mathscr{H}^{\otimes j}} = \sum_{i_1,\dots,i_j=1}^d \int_0^T dt_1 \int_{\mathbb{R}} dx_1 \cdots \int_0^T dt_j \int_{\mathbb{R}} dx_j \left(D^{(i_1)}_{t_1,x_1} \cdots D^{(i_j)}_{t_j,x_j} G\right)^2.$$

We set $\mathbb{D}^\infty = \cap_{p\geqslant1} \cap_{k\geqslant1} \mathbb{D}^{k,p}$.

The adjoint of the derivative operator D on $L^2(\Omega)$ is the Skorohod integral, denoted by δ and characterized by the duality relation

$$\mathrm{E}[G\,\delta(u)] = \mathrm{E}\left[\sum_{j=1}^d \int_0^T \int_{\mathbb{R}} D^{(j)}_{t,x} G\ u_j(t,x)dtdx\right], \quad \text{for all } \ G \in \mathbb{D}^{1,2}$$

(see also [10, Section 3]).

In [4], the Malliavin differentiability and smoothness of the density of the solution to fractional SPDEs driven by spatially correlated noise were established when $d = 1$. These can also be applied to SPDEs driven by space–time white noise and the extension to $d > 1$ under **P1'** and **P2** can easily be done by working coordinate by coordinate. In particular, for any $(t, x) \in [0, T] \times \mathbb{R}$, $i, k \in \{1, \ldots, d\}$, the derivative of $u_i(t, x)$ satisfies the system of equations

$$D^{(k)}_{r,v}(u_i(t, x)) = G_\alpha(t - r, x - v)\sigma_{ik}(u(r, v)) + a_i(k, r, v, t, x), \tag{2.1}$$

where

$$\begin{aligned} a_i(k, r, v, t, x) = &\sum_{j=1}^{d} \int_r^t \int_{\mathbb{R}} G_\alpha(t - \theta, x - \eta) D^{(k)}_{r,v}(\sigma_{ij}(u(\theta, \eta)))W^j(d\theta, d\eta) \\ &+ \int_r^t \int_{\mathbb{R}} G_\alpha(t - \theta, x - \eta) D^{(k)}_{r,v}(b_i(u(\theta, \eta)))d\theta d\eta, \end{aligned} \tag{2.2}$$

if $r < t$ and $D^{(k)}_{r,v}(u_i(t, x)) = 0$ when $r > t$. By iterating the calculation which leads to (2.1), we see that the order m derivative $D^m u_i(t, x)$ also satisfies a system of stochastic partial differential equations which are analogous to the equations in Proposition 4.1 of [10]; see also [27, (6.29)]. Moreover, for any $p > 1, m \geqslant 1$ and $i \in \{1, \ldots, d\}$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\left\| D^m(u_i(t, x))\right\|^p_{\mathscr{H}^{\otimes m}}\right] < \infty. \tag{2.3}$$

Furthermore, for all $(t, x) \in [0, T] \times \mathbb{R}$,

$$u(t, x) \in (\mathbb{D}^\infty)^d. \tag{2.4}$$

3. Existence, smoothness and uniform boundedness of the one-point density

Our objective in this section is to prove Theorem 1.1(a) by using [10, Proposition 3.4]. Let $\gamma_{u(t,x)}$ be the Malliavin matrix of $u(t, x)$. The next result proves property (a) in [10, Proposition 3.4] when F is replaced by $u(t, x)$.

Proposition 3.1. *Fix $T > 0$ and assume hypotheses **P1'** and **P2**. Then, for any $p \geqslant 1$, $\mathrm{E}[(\det \gamma_{u(t,x)})^{-p}]$ is uniformly bounded over (t, x) in any closed non-trivial rectangle $I \times J \subset]0, T] \times \mathbb{R}$.*

Proof. The proof follows along the same lines as [10, Proposition 4.2]; see also [11, Proposition 4.1]. The main differences are the exponents appearing in the estimate. Let $(t, x) \in I \times J$ be fixed. We write

$$\det \gamma_{u(t,x)} \geqslant \left(\inf_{\xi\in\mathbb{R}^d:\|\xi\|=1} \xi^T \gamma_{u(t,x)}\xi \right)^d .$$

Let $\xi \in \mathbb{R}^d$ with $\|\xi\| = 1$ and fix $\epsilon \in]0, 1[$. Using (2.1) and the inequality

$$(a + b)^2 \geqslant \frac{2}{3}a^2 - 2b^2, \tag{3.1}$$

valid for all $a, b \in \mathbb{R}$, we see that

$$\xi^T \gamma_{u(t,x)}\xi = \int_0^t dr \int_{\mathbb{R}} dv \left\| \sum_{i=1}^d D_{r,v}(u_i(t,x))\xi_i \right\|^2$$
$$\geqslant \int_{t(1-\epsilon)}^t dr \int_{\mathbb{R}} dv \left\| \sum_{i=1}^d D_{r,v}(u_i(t,x))\xi_i \right\|^2 \geqslant \frac{2}{3} I_1 - 2I_2,$$

where

$$I_1 = \int_{t(1-\epsilon)}^t dr \int_{\mathbb{R}} dv \sum_{k=1}^d \left(\sum_{i=1}^d G_\alpha(t-r, x-v)\sigma_{ik}(u(r,v))\xi_i \right)^2,$$
$$I_2 = \int_{t(1-\epsilon)}^t dr \int_{\mathbb{R}} dv \sum_{k=1}^d \left(\sum_{i=1}^d a_i(k,r,v,t,x)\xi_i \right)^2,$$

and $a_i(k, r, v, t, x)$ is defined in (2.2). By hypothesis **P2** and semi-group property of the Green kernel [6, Lemma 4.1(iii)],

$$I_1 \geqslant c \int_{t(1-\epsilon)}^t \int_{\mathbb{R}} G_\alpha^2(t-r, x-v) dv dr = c \int_{t(1-\epsilon)}^t G_\alpha(2(t-r), 0) dr$$
$$= \frac{c}{2} \int_0^{2t\epsilon} G_\alpha(r, 0) dr = c'(2t\epsilon)^{\frac{\alpha-1}{\alpha}} \geqslant c'' \epsilon^{\frac{\alpha-1}{\alpha}}, \tag{3.2}$$

where in the third equality we use the scaling property of the Green kernel [6, Lemma 4.1(iv)], and the constants c, c' and c'' are uniform over $(t, x) \in I \times J$.

Next we apply the Cauchy–Schwarz inequality to find that, for any $q \geqslant 1$,

$$\mathrm{E}\left[\sup_{\xi \in \mathbb{R}^d : \|\xi\|=1} |I_2|^q \right] \leqslant c(\mathrm{E}[|I_{21}|^q] + \mathrm{E}[|I_{22}|^q]),$$

where

$$I_{21} = \sum_{i,j,k=1}^d \int_{t(1-\epsilon)}^t dr \int_{\mathbb{R}} dv \left(\int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) D_{r,v}^{(k)}(\sigma_{ij}(u(\theta,\eta))) W^j(d\theta, d\eta) \right)^2,$$
$$I_{22} = \sum_{i,k=1}^d \int_{t(1-\epsilon)}^t dr \int_{\mathbb{R}} dv \left(\int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) D_{r,v}^{(k)}(b_i(u(\theta,\eta))) d\theta d\eta \right)^2.$$

The term I_{21} is bounded in the same way as A_1 in [10, (4.5)], with G there replaced by our G_α. Instead of using their Lemmas 7.6, 7.3 and 7.5, we use Lemma A.2, (4.1) and Lemma A.3. This leads to $\mathrm{E}[|I_{21}|^q] \leqslant C_T \epsilon^{2(\alpha-1)q/\alpha}$, where the constant C_T is uniform over $(t, x) \in I \times J$. For details, see [28, Proof of Prop. 2.3.1].

We next derive a similar bound for I_{22}. First, we use the Cauchy–Schwarz inequality with respect to the measure $G_\alpha(t-\theta, x-\eta) d\theta d\eta$ to see that

$$I_{22} \leqslant \sum_{i,k=1}^d \int_{t(1-\epsilon)}^t (t-r) dr \int_{\mathbb{R}} dv \int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) \left(D_{r,v}^{(k)}(b_i(u(\theta,\eta))) \right)^2 d\theta d\eta$$
$$\leqslant \sum_{i,k=1}^d t\epsilon \int_{t(1-\epsilon)}^t dr \int_{\mathbb{R}} dv \int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) \left(D_{r,v}^{(k)}(b_i(u(\theta,\eta))) \right)^2 d\theta d\eta.$$

Since the partial derivatives of b_i are bounded, by Fubini's theorem,

$$\begin{aligned}
\mathrm{E}[|I_{22}|^q] &\leqslant c \sum_{l,k=1}^{d} (t\epsilon)^q \mathrm{E}\left[\left| \int_{t(1-\epsilon)}^{t} dr \int_{\mathbb{R}} dv \int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta)\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2 d\theta d\eta \right|^q\right] \\
&= c \sum_{l,k=1}^{d} (t\epsilon)^q \mathrm{E}\left[\left| \int_{t(1-\epsilon)}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \int_{t(1-\epsilon)}^{t\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2 \right|^q\right].
\end{aligned}$$

Applying Hölder's inequality with respect to the measure $G_\alpha(t-\theta, x-\eta)d\theta d\eta$,

$$\begin{aligned}
\mathrm{E}[|I_{22}|^q] &\leqslant c \sum_{l,k=1}^{d} (t\epsilon)^q \left| \int_{t(1-\epsilon)}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \right|^{q-1} \\
&\quad \times \int_{t(1-\epsilon)}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \mathrm{E}\left[\left| \int_{t(1-\epsilon)}^{t\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2 \right|^q\right].
\end{aligned}$$

Using Lemma A.3, this yields $\mathrm{E}[|I_{22}|^q] \leqslant C_T (t\epsilon)^q (t\epsilon)^q (t\epsilon)^{(\alpha-1)q/\alpha} = C_T (t\epsilon)^{(3-1/\alpha)q}$.

Thus, we have proved that

$$\mathrm{E}\left[\sup_{\xi\in\mathbb{R}^d:\|\xi\|=1} |I_2|^q\right] \leqslant C_T \epsilon^{2(\alpha-1)q/\alpha}, \tag{3.3}$$

where the constant C_T is clearly uniform over $(t,x) \in I \times J$.

Finally, we apply [10, Prop. 3.5] with $Z := \inf_{\|\xi\|=1}(\xi^T \gamma_{u(t,x)}\xi)$, $Y_{1,\epsilon} = Y_{2,\epsilon} = \sup_{\|\xi\|=1} I_2$, $\epsilon_0 = 1$, $\alpha_1 = \alpha_2 = (\alpha-1)/\alpha$ and $\beta_1 = \beta_2 = 2(\alpha-1)/\alpha$, to get $\mathrm{E}\left[(\det \gamma_{u(t,x)})^{-p}\right] \leqslant C_T$, where all the constants are independent of $(t,x) \in I \times J$. □

In [4], the authors established the existence and smoothness of the density of the solution of one single stochastic fractional partial differential equation driven by spatially correlated noise. For a system of d equations driven by space–time white noise, we have the following.

Proposition 3.2. *Assume **P1'** and **P2**. Fix $T > 0$ and let I and J be compact intervals as in Theorem 1.1. Then for any $(t,x) \in]0,T] \times \mathbb{R}$, $u(t,x) \in (\mathbb{D}^\infty)^d$, $\det \gamma_{u(t,x)}^{-1} \in L^p(\Omega)$ for all $p \geqslant 1$, and density function of $u(t,x)$ is infinitely differentiable and uniformly bounded over $z \in \mathbb{R}^d$ and $(t,x) \in I \times J$.*

Proof. The conclusions follow from Proposition 3.1 and (2.4) together with [10, Theorem 3.1], (2.3) and [10, Proposition 3.4]. □

Proof of Theorem 1.1(a). This is an immediate consequence of Proposition 3.2. □

4. Gaussian-type upper bound on the two-point density

The aim of this section is to prove Theorem 1.1(b) and (c). We will follow the general approach in [10, Section 6]; see also [11, Section 5].

4.1. Technical lemmas and propositions

In this subsection, we present several technical lemmas and propositions which will be used for the analysis of the Malliavin matrix.

Lemma 4.1 (*[6, Proposition 4.4]*). *For any $s, t \in [0, T], s \leqslant t$, and $x, y \in \mathbb{R}$, there exists a constant $C_T > 0$ such that*

$$\int_0^T \int_{\mathbb{R}} (g_\alpha(r, v))^2 dr dv \leqslant C_T(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}),$$

where $g_\alpha(r, v) := g^\alpha_{t,x,s,y}(r, v) = 1_{\{r<t\}} G_\alpha(t-r, x-v) - 1_{\{r<s\}} G_\alpha(s-r, y-v)$.

The following identity, which follows from a simple calculation by using the semigroup property and scaling property of Green kernel [6, Lemma 4.1(iii), (iv)], will be used several times later on:

$$\int_a^b \int_{\mathbb{R}} G_\alpha^2(t-r, x-v) dv dr = c_\alpha \left((t-a)^{\frac{\alpha-1}{\alpha}} - (t-b)^{\frac{\alpha-1}{\alpha}} \right), \quad a \leqslant b \leqslant t, \tag{4.1}$$

where c_α is a positive constant depending on α.

We next give an estimate on the L^p-modulus of continuity of the derivative of the increment, analogous to [10, Proposition 6.2], which is comparable to (1.12).

Proposition 4.2. *For any $p \geqslant 2, m \geqslant 1$, there exists a constant $C_{p,T}$ such that for all $s, t \in [0, T], s \leqslant t, x, y \in \mathbb{R}$,*

$$\mathrm{E}\left[\left\| D^m(u_i(t,x) - u_i(s,y)) \right\|^p_{\mathscr{H}^{\otimes m}} \right] \leqslant C_{p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{p/2}, \quad i = 1, \dots, d. \tag{4.2}$$

Proof. The proof is different from that of [10, Proposition 6.2], in particular regarding the estimate for I_3 in [10, Proposition 6.2].

Assume $m = 1$. Using (2.1), we see that, for any $p \geqslant 2$,

$$\mathrm{E}\left[\|D(u_i(t,x) - u_i(s,y))\|^p_{\mathscr{H}} \right] \leqslant c \left(\mathrm{E}\left[|I_1|^{p/2}\right] + \mathrm{E}\left[|I_2|^{p/2}\right] + \mathrm{E}\left[|I_3|^{p/2}\right] + \mathrm{E}\left[|I_4|^{p/2}\right] \right), \tag{4.3}$$

where

$$I_1 = \sum_{k=1}^d \int_0^T dr \int_{\mathbb{R}} dv\, (g_\alpha(r,v)\sigma_{ik}(u(r,v)))^2,$$

$$I_2 = \sum_{j,k=1}^d \int_0^T dr \int_{\mathbb{R}} dv \left(\int_0^T \int_{\mathbb{R}} g_\alpha(\theta,\eta) D^{(k)}_{r,v}(\sigma_{ij}(u(\theta,\eta))) W^j(d\theta, d\eta) \right)^2,$$

$$I_3 = \sum_{k=1}^d \int_0^T dr \int_{\mathbb{R}} dv \left(\int_s^t \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) D^{(k)}_{r,v}(b_i(u(\theta,\eta))) d\theta d\eta \right)^2,$$

$$I_4 = \sum_{k=1}^d \int_0^T dr \int_{\mathbb{R}} dv \left(\int_0^s \int_{\mathbb{R}} G_\alpha(s-\theta, y-\eta) \right.$$
$$\left. \times\ D^{(k)}_{r,v}(b_i(u(t-s+\theta, x-y+\eta)) - b_i(u(\theta,\eta))) d\theta d\eta \right)^2.$$

By hypothesis **P1'** and Lemma 4.1,

$$\mathrm{E}\left[|I_1|^{p/2}\right] \leqslant C_{p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{p/2}. \tag{4.4}$$

For the term I_2, we proceed as in [10, Proof of Prop. 6.2], using Lemma A.2, (2.3) and Lemma 4.1 instead of their Lemma 7.6, (4.1) and Lemma 6.1, and we obtain

$$\mathrm{E}\left[|I_2|^{p/2}\right] \leqslant C_{p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{p/2}. \tag{4.5}$$

To estimate I_3, denoting $\Theta_{k,l} := D^{(k)}_{r,v}(u_l(\theta,\eta))$, we use the Cauchy–Schwartz inequality and the Minkowski inequality with respect to the measure $G_\alpha(t-\theta, x-\eta)d\theta d\eta$ to get that

$$\begin{aligned}
\mathrm{E}\left[|I_3|^{p/2}\right] &\leqslant C_{p,T} \sum_{k,l=1}^{d} (t-s)^{p/2}\mathrm{E}\Big[\Big(\int_0^{t-s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \int_0^T dr \int_{\mathbb{R}} dv\, \Theta_{k,l}^2\Big)^{p/2}\Big] \\
&\leqslant C_{p,T} \sum_{k,l=1}^{d} (t-s)^{p/2}\Big(\int_0^{t-s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \\
&\qquad \times \sup_{(\theta,\eta)\in[0,T]\times\mathbb{R}} \Big(\mathrm{E}\Big[\Big[\int_0^T dr \int_{\mathbb{R}} dv\, \Theta_{k,l}^2\Big]^{p/2}\Big]\Big)^{2/p}\Big)^{\frac{p}{2}} \\
&\leqslant C_{p,T}(t-s)^p,
\end{aligned} \tag{4.6}$$

where in the last inequality we use (2.3). Using Hölder's inequality with respect to the measure $G_\alpha(t-\theta, x-\eta)d\theta d\eta$,

$$I_4 \leqslant c \sum_{k=1}^{d} \int_0^T dr \int_{\mathbb{R}} dv \int_0^s d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(s-\theta, y-\eta) \left(D^{(k)}_{r,v}\left(b_i(u(t-s+\theta, x-y+\eta)) - b_i(u(\theta,\eta))\right)\right)^2.$$

We apply the chain rule to compute $D^{(k)}_{r,v} b_i(u(t-s+\theta, x-y+\eta)) - D^{(k)}_{r,v} b_i(u(\theta,\eta))$, subtract and add the term $\sum_{l=1}^{d} \frac{\partial b_i}{\partial x_l}(u(t-s+\theta, x-y+\eta)) D^{(k)}_{r,v} u_l(\theta,\eta)$. Then by hypothesis **P1'**, this is bounded above by

$$\begin{aligned}
&c \sum_{k,l=1}^{d} \int_0^T dr \int_{\mathbb{R}} dv \int_0^s d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(s-\theta, y-\eta) \left(D^{(k)}_{r,v}(u_l(t-s+\theta, x-y+\eta) - u_l(\theta,\eta))\right)^2 \\
&\quad + c \sum_{k,l=1}^{d} \int_0^T dr \int_{\mathbb{R}} dv \int_0^s d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(s-\theta, y-\eta)\, (u_l(t-s+\theta, x-y+\eta) - u_l(\theta,\eta))^2\, \Theta_{k,l}^2 \\
&:= I_{41} + I_{42}.
\end{aligned}$$

Using the Minkowski inequality with respect to the measure $G_\alpha(t-\theta, x-\eta)d\theta d\eta$, we have

$$\begin{aligned}
\mathrm{E}\left[|I_{42}|^{p/2}\right] &\leqslant c \sum_{k,l=1}^{d} \Big(\int_0^s d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(s-\theta, y-\eta) \\
&\qquad \times \Big(\mathrm{E}\Big[|u_l(t-s+\theta, x-y+\eta) - u_l(\theta,\eta)|^p \Big(\int_0^T dr \int_{\mathbb{R}} dv\, \Theta_{k,l}^2\Big)^{p/2}\Big]\Big)^{2/p}\Big)^{p/2}.
\end{aligned}$$

By the Cauchy–Schwartz inequality, this is bounded above by

$$\begin{aligned}
&c s^{p/2} \sum_{k,l=1}^{d} \sup_{(\theta,\eta)\in[0,T]\times\mathbb{R}} \mathrm{E}\Big[\Big(\int_0^T dr \int_{\mathbb{R}} dv\, \Theta_{k,l}^2\Big)^p\Big]^{1/2} \mathrm{E}\Big[|u_l(t-s+\theta, x-y+\eta) - u_l(\theta,\eta)|^{2p}\Big]^{1/2} \\
&\qquad \leqslant C_{p,T} s^{p/2}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{p/2}
\end{aligned} \tag{4.7}$$

where we use (2.3) and (1.12).

Denote

$$\varphi(h, z, \theta) := \sup_{\eta\in\mathbb{R}} \sum_{k,l=1}^{d} \mathrm{E}\Big[\Big(\int_0^T \int_{\mathbb{R}} \left(D^{(k)}_{r,v}(u_l(h+\theta, z+\eta) - u_l(\theta,\eta))\right)^2 dr dv\Big)^{\frac{p}{2}}\Big].$$

By Hölder's inequality,

$$\mathrm{E}\left[|I_{41}|^{p/2}\right] \leqslant c \sum_{k,l=1}^{d} \left(\int_0^s \int_{\mathbb{R}} G_\alpha(s-\theta, y-\eta) d\theta d\eta\right)^{\frac{p}{2}-1} \int_0^s d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(s-\theta, y-\eta)$$
$$\times \mathrm{E}\Big[\Big(\int_0^T \int_{\mathbb{R}} (D_{r,v}^{(k)}(u_l(t-s+\theta, x-y+\eta) - u_l(\theta,\eta)))^2 dr dv\Big)^{\frac{p}{2}}\Big]$$
$$\leqslant C_{p,T} \int_0^s \varphi(t-s, x-y, \theta) d\theta. \tag{4.8}$$

Denote $h = t - s$ and $z = x - y$. From (4.3)–(4.8), we conclude that for all $h \geqslant 0$, $z \in \mathbb{R}$, $s \in [0, T]$, $y \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\mathrm{E}\left[\|D(u_i(h+s, z+y) - u_i(s, y))\|_{\mathscr{H}}^p\right] \leqslant C_{p,T}(|h|^{\frac{\alpha-1}{\alpha}} + |z|^{\alpha-1})^{p/2} + C_{p,T} \int_0^s \varphi(h, z, \theta) d\theta.$$

Taking the supremum over $y \in \mathbb{R}$ on the left-hand side of the above inequality, we obtain that for all $h \geqslant 0$, $z \in \mathbb{R}$ and $s \in [0, T]$,

$$\varphi(h, z, s) \leqslant C_{p,T}(|h|^{\frac{\alpha-1}{\alpha}} + |z|^{\alpha-1})^{p/2} + C_{p,T} \int_0^s \varphi(h, z, \theta) d\theta.$$

By Gronwall's lemma (see [29, p. 543]), we obtain that

$$\sup_{s\in[0,T]} \varphi(h, z, s) \leqslant C_{p,T}(|h|^{\frac{\alpha-1}{\alpha}} + |z|^{\alpha-1})^{p/2},$$

which implies (4.2) with $m = 1$.

The case $m > 1$ follows along the same lines by using (2.3) and the stochastic partial differential equations satisfied by the iterated derivatives (see [10, Proposition 4.1]). □

The following lemma is another version of [10, Lemma 6.11].

Lemma 4.3. *Assume **P1'**. Fix $T > 0, q \geqslant 1$. There exists a constant $c = c(q, T) \in]0, \infty[$ such that for every $0 < 2\epsilon \leqslant s \leqslant t \leqslant T$ and $x \in \mathbb{R}$,*

$$\mathrm{E}\Big[\Big(\sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} a_i^2(k, r, v, t, x)\Big)^q\Big] \leqslant c(t-s+\epsilon)^{(\alpha-1)q/\alpha} \epsilon^{(\alpha-1)q/\alpha}.$$

Proof. The proof follows the same lines as [10, Lemma 6.11]. Define

$$A := \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} a_i^2(k, r, v, t, x).$$

From (2.2), we write $\mathrm{E}\left[|A|^q\right] \leqslant c\left(\mathrm{E}\left[|A_1|^q\right] + \mathrm{E}\left[|A_2|^q\right]\right)$, where

$$A_1 := \sum_{i,j,k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left|\int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) D_{r,v}^{(k)}(\sigma_{ij}(u(\theta,\eta))) W^j(d\theta, d\eta)\right|^2, \tag{4.9}$$

$$A_2 := \sum_{i,k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left|\int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) D_{r,v}^{(k)}(b_i(u(\theta,\eta))) d\theta d\eta\right|^2. \tag{4.10}$$

We bound the qth moment of A_1 and A_2 separately. As regards A_1, we follow the calculation in [10, pp. 416–417], with their G replaced by our G_α, and we use (4.1) instead of their Lemma 7.3 and our Lemma A.3 instead of their Lemma 7.5. This replaces their exponent $\frac{1}{2}$ with $\frac{\alpha-1}{\alpha}$, and we obtain

$$\mathrm{E}\left[|A_1|^q\right] \leqslant c(t-s+\epsilon)^{\frac{\alpha-1}{\alpha}q}\epsilon^{\frac{\alpha-1}{\alpha}q}. \tag{4.11}$$

Next we derive a similar bound for A_2. By the Cauchy–Schwartz inequality with respect to the measure $G_\alpha(t-\theta, x-\eta)d\theta d\eta$,

$$\begin{aligned}A_2 &\leqslant \sum_{i,k=1}^{d}\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv\,(t-r)\int_r^t\int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta)\left(D_{r,v}^{(k)}(b_i(u(\theta,\eta)))\right)^2 d\theta d\eta\\ &\leqslant \sum_{i,k=1}^{d}(t-s+\epsilon)\int_{s-\epsilon}^{s} dr\int_{\mathbb{R}} dv\int_r^t\int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta)\left(D_{r,v}^{(k)}(b_i(u(\theta,\eta)))\right)^2 d\theta d\eta.\end{aligned}$$

By hypothesis **P1’** and Fubini’s theorem,

$$\begin{aligned}\mathrm{E}\left[|A_2|^q\right] &\leqslant c(t-s+\epsilon)^q\sum_{k,l=1}^{d}\mathrm{E}\left[\left|\int_{s-\epsilon}^{s} dr\int_{\mathbb{R}} dv\int_r^t d\theta\int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)(D_{r,v}^{(k)}(u_l(\theta,\eta)))^2\right|^q\right]\\ &= c(t-s+\epsilon)^q\sum_{k,l=1}^{d}\mathrm{E}\left[\left|\int_{s-\epsilon}^{t} d\theta\int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)\int_{s-\epsilon}^{s\wedge\theta} dr\int_{\mathbb{R}} dv\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right].\end{aligned} \tag{4.12}$$

We apply Hölder’s inequality with respect to the measure $G_\alpha(t-\theta, x-\eta)d\theta d\eta$ to find that

$$\begin{aligned}\mathrm{E}\left[|A_2|^q\right] &\leqslant c(t-s+\epsilon)^q\sum_{k,l=1}^{d}\left|\int_{s-\epsilon}^{t} d\theta\int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)\right|^{q-1}\\ &\quad\times\int_{s-\epsilon}^{t} d\theta\int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)\mathrm{E}\left[\left|\int_{s-\epsilon}^{s\wedge\theta} dr\int_{\mathbb{R}} dv\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right]\\ &\leqslant c(t-s+\epsilon)^q\left|\int_{s-\epsilon}^{t} d\theta\int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)\right|^q\epsilon^{\frac{\alpha-1}{\alpha}q}\\ &= c(t-s+\epsilon)^{2q}\epsilon^{\frac{\alpha-1}{\alpha}q},\end{aligned} \tag{4.13}$$

where in the second inequality we use Lemma A.3. Hence (4.11) and (4.13) prove the lemma. □

The following lemma improves Lemma 4.3 by using Lemma A.4. As we mentioned in Section 1, this is a key ingredient in our improvement of the lower bound in (1.3), which has also been adapted in [13] to prove the optimal lower bounds on hitting probabilities for stochastic heat equations in higher spatial dimension (see [13, Lemma 5.3]).

Lemma 4.4. *Assume **P1’**. Fix $T > 0, c_0 > 1$ and $0 < \gamma_0 < 1$. For all $q \geqslant 1$, there exists a constant $c = c(c_0, q, T) \in]0,\infty[$ such that for every $0 < 2\epsilon \leqslant s \leqslant t \leqslant T$ with $t-s > c_0\epsilon^{\gamma_0}$ and $x \in \mathbb{R}$,*

$$\mathrm{E}\left[\left(\sum_{k=1}^{d}\int_{s-\epsilon}^{s} dr\int_{\mathbb{R}} dv\sum_{i=1}^{d} a_i^2(k,r,v,t,x)\right)^q\right] \leqslant c\epsilon^{\min((1+\gamma_0)\frac{\alpha-1}{\alpha},1-\frac{\gamma_0}{\alpha})q}.$$

Proof. We use again the notations from the proof of Lemma 4.3. From (4.9) and Burkholder's inequality for Hilbert-space-valued martingales (Lemma A.2), we have

$$\begin{aligned}\mathrm{E}\left[|A_1|^q\right] &\leqslant c \sum_{k,l=1}^{d} \mathrm{E}\Big[\Big|\int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \big(D_{r,v}^{(k)}(u_l(\theta,\eta))\big)^2\Big|^q\Big] \\ &\leqslant A_{11} + A_{12} + A_{13},\end{aligned}$$

where for $i = 1, 2, 3$,

$$A_{1i} := c \sum_{k,l=1}^{d} \mathrm{E}\Big[\Big|\int_{a_i}^{b_i} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\Big|^q\Big],$$

with

$$a_1 = s-\epsilon,\ b_1 = s,\ a_2 = s,\ b_2 = s + c_0\epsilon^{\gamma_0},\ a_3 = s + c_0\epsilon^{\gamma_0},\ b_3 = t, \tag{4.14}$$

and from (4.12),

$$\begin{aligned}\mathrm{E}\left[|A_2|^q\right] &\leqslant c \sum_{k,l=1}^{d} \mathrm{E}\Big[\Big|\int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \big(D_{r,v}^{(k)}(u_l(\theta,\eta))\big)^2\Big|^q\Big] \\ &\leqslant A_{21} + A_{22} + A_{23},\end{aligned}$$

where A_{2i} is defined in the same way as A_{1i}, but with G_α^2 replaced by G_α, $i = 1, 2, 3$.

We first bound $\mathrm{E}\left[|A_1|^q\right]$. We apply Hölder's inequality with respect to the measure $G_\alpha^2(t-\theta, x-\eta)d\theta d\eta$ to find that

$$\begin{aligned}A_{1i} \leqslant c \sum_{k,l=1}^{d} &\Big(\int_{a_i}^{b_i} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta)\Big)^{q-1} \\ &\times \int_{a_i}^{b_i} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta) \mathrm{E}\Big[\Big|\int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\Big|^q\Big].\end{aligned} \tag{4.15}$$

In the case $i = 1$, for $\theta \in [s-\epsilon, s]$, we have $s - \epsilon \geqslant \theta - \epsilon \geqslant 0$. Hence by Lemma A.3,

$$\mathrm{E}\Big[\Big|\int_{s-\epsilon}^{\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\Big|^q\Big] \leqslant \mathrm{E}\Big[\Big|\int_{\theta-\epsilon}^{\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\Big|^q\Big] \leqslant c\epsilon^{\frac{\alpha-1}{\alpha}q}, \tag{4.16}$$

where $c \in]0, \infty[$ does not depend on $(\theta, \eta, s, t, \epsilon, x)$. Therefore, by (4.1),

$$\begin{aligned}A_{11} &\leqslant c\epsilon^{\frac{\alpha-1}{\alpha}q} \left(\int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta)\right)^q \\ &= c\epsilon^{\frac{\alpha-1}{\alpha}q} \left((t-s+\epsilon)^{(\alpha-1)/\alpha} - (t-s)^{(\alpha-1)/\alpha}\right)^q \leqslant c'\epsilon^{\frac{\alpha-1}{\alpha}q}\epsilon^{(1-\frac{\gamma_0}{\alpha})q},\end{aligned} \tag{4.17}$$

where, in the last inequality, we perform the same calculation as in (A.12) under the assumption $t - s > c_0\epsilon^{\gamma_0}$.

Similarly, by (4.15) and Lemma A.3,

$$\begin{aligned}A_{12} &\leqslant c\Big(\int_{s}^{s+c_0\epsilon^{\gamma_0}} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta)\Big)^q \epsilon^{\frac{\alpha-1}{\alpha}q} \\ &= c\left((t-s)^{\frac{\alpha-1}{\alpha}} - (t-s-c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}}\right)^q \epsilon^{\frac{\alpha-1}{\alpha}q}\end{aligned}$$

$$\leqslant c\left((c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}} - (c_0\epsilon^{\gamma_0} - c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}}\right)^q \epsilon^{\frac{\alpha-1}{\alpha}q}$$
$$= c(c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}q}\epsilon^{\frac{\alpha-1}{\alpha}q} = c'\epsilon^{(1+\gamma_0)\frac{\alpha-1}{\alpha}q}, \tag{4.18}$$

where the second inequality holds by Lemma A.1(a).

To estimate A_{13}, we see that Lemma A.4 implies that for any $\theta \in]s + c_0\epsilon^{\gamma_0}, t[$,

$$\sum_{k,l=1}^{d} \mathrm{E}\left[\left|\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right] \leqslant c\epsilon^{(1-\frac{\gamma_0}{\alpha})q},$$

where $c \in]0, \infty[$ does not depend on $(\theta, \eta, s, t, \epsilon, x)$. Thus, by (4.15) and (4.1),

$$A_{13} \leqslant c\left(\int_{s+c_0\epsilon^{\gamma_0}}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta, x-\eta)\right)^q \epsilon^{(1-\frac{\gamma_0}{\alpha})q} = c(t-s-c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}q}\epsilon^{(1-\frac{\gamma_0}{\alpha})q} \leqslant c'\epsilon^{(1-\frac{\gamma_0}{\alpha})q}, \tag{4.19}$$

where in the last inequality, we bound $t - s - c_0\epsilon^{\gamma_0}$ by T.

We proceed to derive a similar bound for $\mathrm{E}\left[|A_2|^q\right]$. We apply Hölder's inequality with respect to the measure $G_\alpha(t-\theta, x-\eta)d\theta d\eta$ to find that

$$A_{2i} \leqslant c\sum_{k,l=1}^{d}\left|\int_{a_i}^{b_i} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)\right|^{q-1}$$
$$\times \int_{a_i}^{b_i} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)\mathrm{E}\left[\left|\int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right|^q\right]. \tag{4.20}$$

In the case $i = 1$, by (4.16),

$$A_{21} \leqslant c\left|\int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)\right|^q \epsilon^{\frac{\alpha-1}{\alpha}q} = c\,\epsilon^{(\frac{\alpha-1}{\alpha}+1)q}. \tag{4.21}$$

Similarly, by (4.20),

$$A_{22} \leqslant c\left|\int_{s}^{s+c_0\epsilon^{\gamma_0}} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)\right|^q \epsilon^{\frac{\alpha-1}{\alpha}q} = c(c_0\epsilon^{\gamma_0})^q\epsilon^{\frac{\alpha-1}{\alpha}q} = c'\epsilon^{(\frac{\alpha-1}{\alpha}+\gamma_0)q}. \tag{4.22}$$

It remains to estimate A_{23}. By (4.20) and Lemma A.4,

$$A_{23} \leqslant c\left|\int_{s+c_0\epsilon^{\gamma_0}}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)\right|^q \epsilon^{(1-\frac{\gamma_0}{\alpha})q} = c(t-s-c_0\epsilon^{\gamma_0})^q\epsilon^{(1-\frac{\gamma_0}{\alpha})q} \leqslant c'\epsilon^{(1-\frac{\gamma_0}{\alpha})q}, \tag{4.23}$$

where, in the last inequality, we bound $t - s - c_0\epsilon^{\gamma_0}$ by T.

Finally, from (4.17), (4.18), (4.19), (4.21), (4.22) and (4.23), together with the choice of γ_0, we obtain the desired result. □

4.2. Study of the Malliavin matrix

Fix $T > 0$. For $s, t \in [0, T]$, $s \leqslant t$, and $x, y \in \mathbb{R}$, consider the $2d$-dimensional random vector

$$Z := (u(s, y), u(t, x) - u(s, y)). \tag{4.24}$$

Let γ_Z be the Malliavin matrix of Z. Note that $\gamma_Z = ((\gamma_Z)_{m,l})_{m,l=1,\dots,2d}$ is a symmetric $2d \times 2d$ random matrix with four $d \times d$ blocs of the form

$$\gamma_Z = \begin{pmatrix} \gamma_Z^{(1)} & \vdots & \gamma_Z^{(2)} \\ \dots & \vdots & \dots \\ \gamma_Z^{(3)} & \vdots & \gamma_Z^{(4)} \end{pmatrix}$$

where

$$\begin{aligned}
\gamma_Z^{(1)} &= \left(\left\langle D(u_i(s,y)), D(u_j(s,y))\right\rangle_{\mathscr{H}}\right)_{i,j=1,\dots,d},\\
\gamma_Z^{(2)} &= \left(\left\langle D(u_i(s,y)), D(u_j(t,x)-u_j(s,y))\right\rangle_{\mathscr{H}}\right)_{i,j=1,\dots,d},\\
\gamma_Z^{(3)} &= \left(\left\langle D(u_i(t,x)-u_i(s,y)), D(u_j(s,y))\right\rangle_{\mathscr{H}}\right)_{i,j=1,\dots,d},\\
\gamma_Z^{(4)} &= \left(\left\langle D(u_i(t,x)-u_i(s,y)), D(u_j(t,x)-u_j(s,y))\right\rangle_{\mathscr{H}}\right)_{i,j=1,\dots,d}.
\end{aligned}$$

We let (**1**) denote the couples of $\{1,\dots,d\}\times\{1,\dots,d\}$, (**2**) denote the couples of $\{1,\dots,d\}\times\{d+1,\dots,2d\}$, (**3**) denote the couples of $\{d+1,\dots,2d\}\times\{1,\dots,d\}$ and (**4**) denote the couples of $\{d+1,\dots,2d\}\times\{d+1,\dots,2d\}$.

The next two results follow exactly along the same lines as [10, Propositions 6.5 and 6.7] using (2.3) and Proposition 4.2, with Δ there replaced by Δ_α^2. We omit the proofs.

Proposition 4.5. *Fix $T > 0$ and let I and J be compact intervals as in Theorem 1.1. Let A_Z denote the cofactor matrix of γ_Z. Assuming **P1'**, for any $(s,y),(t,x) \in I \times J$, $(s,y) \neq (t,x)$, $p > 1$,*

$$\mathrm{E}\left[|(A_Z)_{m,l}|^p\right]^{1/p} \leqslant \begin{cases} c_{p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^d & \text{if } (m,l) \in (\mathbf{1}),\\ c_{p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{d-\frac{1}{2}} & \text{if } (m,l) \in (\mathbf{2}) \text{ or } (\mathbf{3}),\\ c_{p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{d-1} & \text{if } (m,l) \in (\mathbf{4}). \end{cases}$$

Proposition 4.6. *Fix $T > 0$ and let I and J be compact intervals as in Theorem 1.1. Assuming **P1'**, for any $(s,y),(t,x) \in I \times J$, $(s,y) \neq (t,x)$, $p > 1$,*

$$\mathrm{E}\left[\|D^k(\gamma_Z)_{m,l}\|^p_{\mathscr{H}^{\otimes k}}\right]^{1/p} \leqslant \begin{cases} c_{k,p,T} & \text{if } (m,l) \in (\mathbf{1}),\\ c_{k,p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{\frac{1}{2}} & \text{if } (m,l) \in (\mathbf{2}) \text{ or } (\mathbf{3}),\\ c_{k,p,T}(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}) & \text{if } (m,l) \in (\mathbf{4}). \end{cases}$$

The main technical effort in this subsection is the proof of the following proposition, which improves [10, Proposition 6.6(a)] and is why the η can be removed in the lower bound on hitting probabilities.

Proposition 4.7. *Fix $T > 0$ and let I and J be compact intervals as in Theorem 1.1. Assume **P1'** and **P2**. There exists C depending on T such that for any $(s,y),(t,x) \in I \times J$, $(s,y) \neq (t,x)$, $p > 1$,*

$$\mathrm{E}\left[(\det\gamma_Z)^{-p}\right]^{1/p} \leqslant C(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{-d}. \tag{4.25}$$

Proof. The proof has the same structure as that of [10, Proposition 6.6]; see also [11, Proposition 5.5]. We write

$$\det \gamma_Z = \prod_{i=1}^{2d} (\xi^i)^T \gamma_Z \xi^i, \tag{4.26}$$

where $\xi = \{\xi^1, \ldots, \xi^{2d}\}$ is an orthogonal basis of $\mathbb{R}^{2d}$ consisting of eigenvectors of γ_Z.

We use the perturbation argument of [10, Proposition 6.6]. Let $\mathbf{0} \in \mathbb{R}^d$. Consider the spaces $E_1 = \{(\lambda, \mathbf{0}) : \lambda \in \mathbb{R}^d\}$ and $E_2 = \{(\mathbf{0}, \mu) : \mu \in \mathbb{R}^d\}$. Each ξ^i can be written

$$\xi^i = (\lambda^i, \mu^i) = \beta_i(\tilde{\lambda}^i, \mathbf{0}) + \sqrt{1-\beta_i^2}(\mathbf{0}, \tilde{\mu}^i), \tag{4.27}$$

where $\lambda^i, \mu^i \in \mathbf{R}^d$, $(\tilde{\lambda}^i, \mathbf{0}) \in E_1$, $(\mathbf{0}, \tilde{\mu}^i) \in E_2$, with $\|\tilde{\lambda}^i\| = \|\tilde{\mu}^i\| = 1$ and $0 \leqslant \beta_i \leqslant 1$. In particular, $\|\xi^i\|^2 = \|\lambda^i\|^2 + \|\mu^i\|^2 = 1$.

By the argument between (6.10) and (6.11) in [10], Propositions 4.8 and 4.9 conclude the proof of Proposition 4.7. □

Proposition 4.8. *Fix $T > 0$. Assume **P1'** and **P2**. There exists C depending on T such that for all $s, t \in I$, $0 \leqslant t - s < 1$, $x, y \in J$, $(s, y) \neq (t, x)$, and $p > 1$,*

$$\mathrm{E}\left[\left(\inf_{\substack{\xi = (\lambda, \mu) \in \mathbb{R}^{2d} : \\ \|\lambda\|^2 + \|\mu\|^2 = 1}} \xi^T \gamma_Z \xi\right)^{-2dp}\right] \leqslant C(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{-2dp}. \tag{4.28}$$

We are going to apply Lemma 4.4 to prove this proposition. This is an improvement over the proof of [10, Proposition 6.9] in which an extra exponent η appears. Notice that a similar improvement has been obtained in [13, Theorem 1.3] for stochastic heat equations in higher spatial dimensions.

Proposition 4.9. *Assume **P1'** and **P2**. Fix $T > 0$, $p > 1$ and $\beta_0 > 0$ sufficiently small. Let K be a subset of $\{1, \ldots, 2d\}$ with $|K| = d$. Then there exists $C = C(p, T)$ such that for all $s, t \in I$ with $t \geqslant s$, $x, y \in J$, $(s, y) \neq (t, x)$,*

$$\mathrm{E}\left[\mathbf{1}_{A_K}\left(\prod_{i \in K} (\xi^i)^T \gamma_Z \xi^i\right)^{-p}\right] \leqslant C, \tag{4.29}$$

where $A_K = \cap_{i \in K}\{\beta_i \geqslant \beta_0\}$.

Proof of Proposition 4.8. Since γ_Z is a matrix of inner products, we can write

$$\xi^T \gamma_Z \xi = \sum_{k=1}^d \int_0^T dr \int_{\mathbb{R}} dv \Big(\sum_{i=1}^d \big(\lambda_i D_{r,v}^{(k)}(u_i(s,y)) + \mu_i(D_{r,v}^{(k)}(u_i(t,x)) - D_{r,v}^{(k)}(u_i(s,y)))\big)\Big)^2.$$

From here on, the proof is divided into two cases.

Case 1. In the first case, we assume that $t - s > 0$ and $|x-y|^\alpha \leqslant t - s$. Choose and fix an $\epsilon \in]0, \delta(t-s)[$, where $0 < \delta < 1$ is small but fixed; its specific value will be decided later on (see the line above (4.35)). Then we may write

$$\xi^T \gamma_Z \xi \geqslant J_1 + J_2,$$

where

$$J_1 := \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} (\lambda_i - \mu_i)[G_\alpha(s-r, y-v)\sigma_{ik}(u(r,v)) + a_i(k,r,v,s,y)] + W \right)^2,$$

$$J_2 := \sum_{k=1}^{d} \int_{t-\epsilon}^{t} dr \int_{\mathbb{R}} dv\, W^2,$$

$a_i(k, r, v, s, y)$ is defined in (2.2) and

$$W := \sum_{i=1}^{d} [\mu_i G_\alpha(t-r, x-v)\sigma_{ik}(u(r,v)) + \mu_i a_i(k,r,v,t,x)].$$

Sub-case A: $\epsilon \leqslant \delta(t-s)^{1/\gamma_0}$ with $0 < \gamma_0 < 1$. In this sub-case, by the elementary inequality (3.1),

$$J_2 \geqslant \hat{Y}_{1,\epsilon} - Y_{1,\epsilon},$$

where

$$\hat{Y}_{1,\epsilon} := \frac{2}{3} \sum_{k=1}^{d} \int_{t-\epsilon}^{t} dr \int_{\mathbb{R}} dv \Big(\sum_{i=1}^{d} \mu_i \sigma_{ik}(u(r,v)) \Big)^2 G_\alpha^2(t-r, x-v),$$

$$Y_{1,\epsilon} := 2 \sup_{\|\mu\| \leqslant 1} \sum_{k=1}^{d} \int_{t-\epsilon}^{t} dr \int_{\mathbb{R}} dv \Big(\sum_{i=1}^{d} \mu_i a_i(k,r,v,t,x) \Big)^2.$$

In agreement with hypothesis **P2** and by (4.1),

$$\hat{Y}_{1,\epsilon} \geqslant c\|\mu\|^2 \int_{t-\epsilon}^{t} dr \int_{\mathbb{R}} dv\, G_\alpha^2(t-r, x-v) = c'\|\mu\|^2 \epsilon^{\frac{\alpha-1}{\alpha}}.$$

Next we apply Lemma 4.3 [with $s := t$] to find that $\mathrm{E}\Big[|Y_{1,\epsilon}|^q\Big] \leqslant c\epsilon^{\frac{2\alpha-2}{\alpha}q}$, for any $q \geqslant 1$.

For J_1, we find that

$$J_1 \geqslant \hat{Y}_{2,\epsilon} - Y_{2,\epsilon},$$

where

$$\hat{Y}_{2,\epsilon} := \frac{2}{3} \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \Big(\sum_{i=1}^{d} (\lambda_i - \mu_i) \sigma_{ik}(u(r,v)) \Big)^2 G_\alpha^2(s-r, y-v),$$

and

$$Y_{2,\epsilon} := 6(W_1 + W_2 + W_3),$$

where

$$W_1 := \sup_{\|\xi\|=1} \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} \mu_i G_\alpha(t-r, x-v)\sigma_{ik}(u(r,v)) \right)^2,$$

$$W_2 := \sup_{\|\xi\|=1} \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} (\lambda_i - \mu_i) a_i(k,r,v,s,y) \right)^2, \tag{4.30}$$

$$W_3 := \sup_{\|\xi\|=1} \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d} \mu_i a_i(k,r,v,t,x) \right)^2. \tag{4.31}$$

Hypothesis **P2** implies that $\hat{Y}_{2,\epsilon} \geqslant c\|\lambda-\mu\|\epsilon^{\frac{\alpha-1}{\alpha}}$. We next give an estimate on the qth moment of W_1, which is better than in [10]. We apply the Cauchy–Schwartz inequality to find that, for any $q \geqslant 1$,

$$\mathrm{E}\left[|W_1|^q\right] \leqslant \sup_{\|\xi\|=1} \|\mu\|^{2q} \times \mathrm{E}\left[\left|\sum_{k=1}^d \int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \sum_{i=1}^d (\sigma_{ik}(u(r,v)))^2 G_\alpha^2(t-r,x-v)\right|^q\right].$$

Thanks to hypothesis **P1'** and (4.1), this is bounded above by

$$c\left|\int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv\, G_\alpha^2(t-r,x-v)\right|^q = c((t-s+\epsilon)^{\frac{\alpha-1}{\alpha}} - (t-s)^{\frac{\alpha-1}{\alpha}})^q \leqslant c'\epsilon^{(1-\frac{\gamma_0}{\alpha})q},$$

where, in the inequality, we perform the same calculation as in (A.12) under the assumption $t-s > c_0\epsilon^{\gamma_0}$ of the Sub-case A.

We bound the qth moment of W_2 similarly as in [10]: By the Cauchy–Schwarz inequality,

$$\begin{aligned}\mathrm{E}\left[|W_2|^q\right] &\leqslant \sup_{\|\xi\|=1} \|\lambda-\mu\|^{2q} \times \mathrm{E}\left[\left|\sum_{k=1}^d \int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \sum_{i=1}^d a_i^2(k,r,v,s,y)\right|^q\right] \\ &\leqslant c\,\mathrm{E}\left[\left|\sum_{k=1}^d \int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \sum_{i=1}^d a_i^2(k,r,v,s,y)\right|^q\right].\end{aligned}$$

We apply Lemma 4.3 [with $t := s$] to find that $\mathrm{E}\left[|W_2|^q\right] \leqslant c\epsilon^{\frac{2\alpha-2}{\alpha}q}$.

Furthermore, different from the estimate of the qth moment of W_3 in [10], under the assumption of the Sub-case A, by Lemma 4.4 we find that, for any $q \geqslant 1$,

$$\mathrm{E}\left[|W_3|^q\right] \leqslant \sup_{\|\xi\|=1} \|\mu\|^{2q} \times \mathrm{E}\left[\left|\sum_{k=1}^d \int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \sum_{i=1}^d a_i^2(k,r,v,t,x)\right|^q\right] \leqslant c\epsilon^{\min((1+\gamma_0)\frac{\alpha-1}{\alpha},1-\frac{\gamma_0}{\alpha})q}.$$

The preceding bounds for W_1, W_2 and W_3 prove, in conjunction, that

$$\mathrm{E}\left[|Y_{2,\epsilon}|^q\right] \leqslant c\epsilon^{\min((1+\gamma_0)\frac{\alpha-1}{\alpha},1-\frac{\gamma_0}{\alpha})q}.$$

Thus we have

$$\begin{aligned}J_1 + J_2 &\geqslant \hat{Y}_{1,\epsilon} + \hat{Y}_{2,\epsilon} - Y_{1,\epsilon} - Y_{2,\epsilon} \geqslant c(\|\mu\|^2 + \|\lambda-\mu\|^2)\epsilon^{\frac{\alpha-1}{\alpha}} - Y_{1,\epsilon} - Y_{2,\epsilon} \\ &\geqslant c\epsilon^{\frac{\alpha-1}{\alpha}} - Y_\epsilon,\end{aligned} \tag{4.32}$$

where $Y_\epsilon := Y_{1,\epsilon} + Y_{2,\epsilon}$ satisfies

$$\mathrm{E}\left[|Y_\epsilon|^q\right] \leqslant c\epsilon^{\min((1+\gamma_0)\frac{\alpha-1}{\alpha},1-\frac{\gamma_0}{\alpha})q}. \tag{4.33}$$

Sub-case B: $\delta(t-s)^{1/\gamma_0} < \epsilon < \delta(t-s)$. In this sub-case, we are going to give a different estimate on J_1:

$$J_1 \geqslant \tilde{Y}_\epsilon - 4(W_2 + W_3),$$

where

$$\tilde{Y}_\epsilon := \frac{2}{3}\sum_{k=1}^d \int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^d [(\lambda_i-\mu_i)G_\alpha(s-r,y-v)+\mu_i G_\alpha(t-r,x-v)]\sigma_{ik}(u(r,v))\right)^2$$

and W_2 and W_3 are defined in (4.30) and (4.31). Using the inequality $(a+b)^2 \geqslant a^2 - 2|ab|$, we see that

$$\tilde{Y}_\epsilon \geqslant \hat{Y}_{2,\epsilon} - \frac{4}{3} B_1^{(3)},$$

where as above, $\hat{Y}_{2,\epsilon} \geqslant c\|\lambda - \mu\| \epsilon^{\frac{\alpha-1}{\alpha}}$, and

$$B_1^{(3)} := \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left| \sum_{i=1}^{d} (\lambda_i - \mu_i) G_\alpha(s-r, y-v) \sigma_{ik}(u(r,v)) \right| \times \left| \sum_{i=1}^{d} \mu_i G_\alpha(t-r, x-v) \sigma_{ik}(u(r,v)) \right|. \tag{4.34}$$

Hypothesis **P1'** assures us that

$$\begin{aligned} \left| B_1^{(3)} \right| &\leqslant c \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv\, G_\alpha(s-r, y-v) G_\alpha(t-r, x-v) \\ &= c \int_{s-\epsilon}^{s} dr\, G_\alpha(t+s-2r, x-y) = c \int_0^{\epsilon} dr\, G_\alpha(t-s+2r, x-y), \end{aligned}$$

where, in the first equality, we use the semi-group property of the Green kernel [6, Lemma 4.1(iii)]. Since for any $t > 0$, the function $x \mapsto G_\alpha(t,x)$ attains its maximum at 0, this is bounded above by

$$\begin{aligned} c \int_0^{\epsilon} dr\, G_\alpha(t-s+2r, 0) = \tilde{c} \int_0^{\epsilon} dr (t-s+2r)^{-\frac{1}{\alpha}} &= c'((t-s+2\epsilon)^{\frac{\alpha-1}{\alpha}} - (t-s)^{\frac{\alpha-1}{\alpha}}) \\ &= c' \epsilon^{\frac{\alpha-1}{\alpha}} \left((\frac{t-s}{\epsilon} + 2)^{\frac{\alpha-1}{\alpha}} - (\frac{t-s}{\epsilon})^{\frac{\alpha-1}{\alpha}} \right) \\ &\leqslant c' \epsilon^{\frac{\alpha-1}{\alpha}} ((1/\delta + 2)^{\frac{\alpha-1}{\alpha}} - (1/\delta)^{\frac{\alpha-1}{\alpha}}), \end{aligned}$$

where the first equality is due to the scaling property of Green kernel [6, Lemma 4.1(iv)] and in the inequality we use the assumption $\epsilon < \delta(t-s)$ and Lemma A.1(a). Hence we have

$$\begin{aligned} J_1 + J_2 &\geqslant \hat{Y}_{1,\epsilon} + \hat{Y}_{2,\epsilon} - \frac{4}{3} B_1^{(3)} - 4W_2 - 4W_3 - Y_{1,\epsilon} \\ &\geqslant c(\|\mu\|^2 + \|\lambda - \mu\|^2) \epsilon^{\frac{\alpha-1}{\alpha}} - c' \epsilon^{\frac{\alpha-1}{\alpha}} ((1/\delta + 2)^{\frac{\alpha-1}{\alpha}} - (1/\delta)^{\frac{\alpha-1}{\alpha}}) - 4W_2 - 4W_3 - Y_{1,\epsilon} \\ &\geqslant c_0 \epsilon^{\frac{\alpha-1}{\alpha}} - c' \epsilon^{\frac{\alpha-1}{\alpha}} ((1/\delta + 2)^{\frac{\alpha-1}{\alpha}} - (1/\delta)^{\frac{\alpha-1}{\alpha}}) - 4W_2 - 4W_3 - Y_{1,\epsilon} \end{aligned}$$

We can choose δ small so that $c_0 > c'((1/\delta + 2)^{\frac{\alpha-1}{\alpha}} - (1/\delta)^{\frac{\alpha-1}{\alpha}})$ and therefore,

$$J_1 + J_2 \geqslant c \epsilon^{\frac{\alpha-1}{\alpha}} - 4W_2 - 4W_3 - Y_{1,\epsilon}. \tag{4.35}$$

In this sub-case,

$$\begin{aligned} \mathrm{E}\left[|W_2|^q \right] &\leqslant \sup_{\|\xi\|=1} \|\lambda - \mu\|^{2q} \times \mathrm{E}\left[\left| \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} a_i^2(k,r,v,s,y) \right|^q \right] \\ &\leqslant c\, \mathrm{E}\left[\left| \sum_{k=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} a_i^2(k,r,v,s,y) \right|^q \right]. \end{aligned}$$

We apply Lemma 4.3 to find that $\mathrm{E}\left[|W_2|^q\right] \leqslant c\epsilon^{\frac{2\alpha-2}{\alpha}q}$. Similarly, we find using Lemma 4.3 and the assumption $\delta(t-s)^{1/\gamma_0} < \epsilon$ that

$$\begin{aligned}\mathrm{E}\left[|W_3|^q\right] &\leqslant \sup_{\|\xi\|=1} \|\mu\|^{2q} \times \mathrm{E}\left[\left|\sum_{k=1}^d \int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \sum_{i=1}^d a_i^2(k,r,v,t,x)\right|^q\right] \\ &\leqslant c(t-s+\epsilon)^{\frac{\alpha-1}{\alpha}q}\epsilon^{\frac{\alpha-1}{\alpha}q} \leqslant c(\delta^{-\gamma_0}\epsilon^{\gamma_0}+\epsilon)^{\frac{\alpha-1}{\alpha}q}\epsilon^{\frac{\alpha-1}{\alpha}q} \leqslant c'\epsilon^{(1+\gamma_0)\frac{\alpha-1}{\alpha}q}.\end{aligned}$$

Combining (4.32) and (4.35), we have for $\epsilon \in \,]0, \delta(t-s)[$,

$$\inf_{\|\xi\|=1} \xi^T \gamma_Z \xi \geqslant c\epsilon^{\frac{\alpha-1}{\alpha}} - \tilde{Z}_\epsilon, \tag{4.36}$$

where

$$\tilde{Z}_\epsilon := Y_\epsilon 1_{\{\epsilon\leqslant\delta(t-s)^{1/\gamma_0}\}} + 4(W_2+W_3+Y_{1,\epsilon})1_{\{\delta(t-s)^{1/\gamma_0}<\epsilon<\delta(t-s)\}}$$

and for all $q \geqslant 1$,

$$\mathrm{E}\left[|Y_\epsilon 1_{\{\epsilon\leqslant\delta(t-s)^{1/\gamma_0}\}}|^q\right] \leqslant c\epsilon^{\min((1+\gamma_0)\frac{\alpha-1}{\alpha},1-\frac{\gamma_0}{\alpha})q}, \tag{4.37}$$

and

$$\mathrm{E}\left[|4(W_2+W_3+Y_{1,\epsilon})1_{\{\delta(t-s)^{1/\gamma_0}<\epsilon<\delta(t-s)\}}|^q\right] \leqslant c\epsilon^{(1+\gamma_0)\frac{\alpha-1}{\alpha}q}. \tag{4.38}$$

We use [10, Proposition 3.5] to find that

$$\mathrm{E}\left[\left(\inf_{\|\xi\|=1}\xi^T\gamma_Z\xi\right)^{-2pd}\right] \leqslant c\,(\delta(t-s))^{-2pd\frac{\alpha-1}{\alpha}} = c'(t-s)^{-2pd\frac{\alpha-1}{\alpha}} \leqslant \tilde{c}\left[|t-s|^{\frac{\alpha-1}{\alpha}}+|x-y|^{\alpha-1}\right]^{-2pd},$$

whence follows the result in the case that $|x-y|^\alpha \leqslant t-s < 1$.

Case 2. Now we work on the second case where $|x-y| > 0$ and $|x-y|^\alpha \geqslant t-s \geqslant 0$. Let $\epsilon > 0$ be such that $(1+\beta)\epsilon^{1/\alpha} < \frac{1}{2}|x-y|$, where $\beta > 0$ is large but fixed; its specific value will be decided on later (see the explanation for (4.49) and (4.50)). Then

$$\xi^T\gamma_Z\xi \geqslant I_1 + I_2,$$

where

$$I_1 := \sum_{k=1}^d \int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv\,(\varphi_1+\varphi_2)^2, \quad I_2 := \sum_{k=1}^d \int_{(t-\epsilon)\vee s}^t dr \int_{\mathbb{R}} dv\,\varphi_2^2,$$

and

$$\varphi_1 := \sum_{i=1}^d (\lambda_i-\mu_i)[G_\alpha(s-r,y-v)\sigma_{ik}(u(r,v)) + a_i(k,r,v,s,y)],$$

$$\varphi_2 := \sum_{i=1}^d [\mu_i G_\alpha(t-r,x-v)\sigma_{ik}(u(r,v)) + \mu_i a_i(k,r,v,t,x)].$$

From here on, Case 2 is divided into two further sub-cases.

Sub-Case A. Suppose, in addition, that $\epsilon \geqslant \delta(t-s)$, where δ is chosen as in Case 1. In this sub-case, we are going to prove that

$$\inf_{\|\xi\|=1}\xi^T\gamma_Z\xi \geqslant c\epsilon^{\frac{\alpha-1}{\alpha}} - Z_{1,\epsilon}, \tag{4.39}$$

where for all $q \geqslant 1$,

$$\mathrm{E}\left[|Z_{1,\epsilon}|^q\right] \leqslant c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q}. \tag{4.40}$$

Indeed, by the elementary inequality (3.1) we find that

$$I_1 \geqslant \frac{2}{3}\tilde{A}_1 - B_1^{(1)} - B_1^{(2)},$$

where

$$\tilde{A}_1 := \sum_{k=1}^{d}\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \Big(\sum_{i=1}^{d}[(\lambda_i-\mu_i)G_\alpha(s-r,y-v)+\mu_i G_\alpha(t-r,x-v)]\sigma_{ik}(u(r,v))\Big)^2,$$

$$B_1^{(1)} := 4\|\lambda-\mu\|^2\sum_{k=1}^{d}\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} a_i^2(k,r,v,s,y), \tag{4.41}$$

$$B_1^{(2)} := 4\|\mu\|^2\sum_{k=1}^{d}\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \sum_{i=1}^{d} a_i^2(k,r,v,t,x). \tag{4.42}$$

Using the inequality $(a+b)^2 \geqslant a^2+b^2-2|ab|$, we see that $\tilde{A}_1 \geqslant A_1 + A_2 - 2B_1^{(3)}$, where

$$A_1 := \sum_{k=1}^{d}\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \Big(\sum_{i=1}^{d}(\lambda_i-\mu_i)G_\alpha(s-r,y-v)\sigma_{ik}(u(r,v))\Big)^2,$$

$$A_2 := \sum_{k=1}^{d}\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \Big(\sum_{i=1}^{d}\mu_i G_\alpha(t-r,x-v)\sigma_{ik}(u(r,v))\Big)^2$$

and $B_1^{(3)}$ has the same expression as in (4.34). We can combine terms to find that

$$I_1 \geqslant \frac{2}{3}(A_1+A_2) - (B_1^{(1)} + B_1^{(2)} + 2B_1^{(3)}).$$

Moreover, we appeal to the elementary inequality (3.1) to find that $I_2 \geqslant \frac{2}{3}A_3 - B_2$, where

$$A_3 := \sum_{k=1}^{d}\int_{(t-\epsilon)\vee s}^{t} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d}\mu_i G_\alpha(t-r,x-v)\sigma_{ik}(u(r,v))\right)^2,$$

$$B_2 := 2\sum_{k=1}^{d}\int_{(t-\epsilon)\vee s}^{t} dr \int_{\mathbb{R}} dv \left(\sum_{i=1}^{d}\mu_i a_i(k,r,v,t,x)\right)^2. \tag{4.43}$$

By hypothesis **P2** and using (4.1) three times,

$$\begin{aligned}
&A_1+A_2+A_3 \\
&\quad\geqslant \rho^2\Big(\|\lambda-\mu\|^2\int_{s-\epsilon}^{s} dr\int_{\mathbb{R}} dv\, G_\alpha^2(s-r,y-v) + \|\mu\|^2\int_{s-\epsilon}^{s} dr\int_{\mathbb{R}} dv\, G_\alpha^2(t-r,x-v) \\
&\qquad\qquad + \|\mu\|^2\int_{(t-\epsilon)\vee s}^{t} dr\int_{\mathbb{R}} dv\, G_\alpha^2(t-r,x-v)\Big) \\
&\quad= c\rho^2\Big(\|\lambda-\mu\|^2\epsilon^{\frac{\alpha-1}{\alpha}} + \|\mu\|^2\left((t-s+\epsilon)^{\frac{\alpha-1}{\alpha}} - (t-s)^{\frac{\alpha-1}{\alpha}} + (t-((t-\epsilon)\vee s))^{\frac{\alpha-1}{\alpha}}\right)\Big) \\
&\quad= c\rho^2\Big(\|\lambda-\mu\|^2\epsilon^{\frac{\alpha-1}{\alpha}} + \|\mu\|^2\left((t-s+\epsilon)^{\frac{\alpha-1}{\alpha}} - (t-s)^{\frac{\alpha-1}{\alpha}} + ((t-s)\wedge\epsilon)^{\frac{\alpha-1}{\alpha}}\right)\Big) \\
&\quad= c\rho^2\epsilon^{\frac{\alpha-1}{\alpha}}\left(\|\lambda-\mu\|^2 + \|\mu\|^2\left((\frac{t-s}{\epsilon}+1)^{\frac{\alpha-1}{\alpha}} - (\frac{t-s}{\epsilon})^{\frac{\alpha-1}{\alpha}} + ((\frac{t-s}{\epsilon})\wedge 1)^{\frac{\alpha-1}{\alpha}}\right)\right).
\end{aligned}$$

Denote $\zeta(x) := (x+1)^{\frac{\alpha-1}{\alpha}} - x^{\frac{\alpha-1}{\alpha}} + (x\wedge 1)^{\frac{\alpha-1}{\alpha}}$, $x \in [0,\infty[$. Then it is clear that

$$\hat{c}_0 := \min_{0\leqslant x<\infty} \zeta(x) > 0. \tag{4.44}$$

Thus we have

$$A_1 + A_2 + A_3 \geqslant c\rho^2 \epsilon^{\frac{\alpha-1}{\alpha}} \left(\|\lambda-\mu\|^2 + \hat{c}_0\|\mu\|^2 \right) \geqslant c'\epsilon^{\frac{\alpha-1}{\alpha}}.$$

We are aiming for (4.39), and will bound the absolute moments of $B_1^{(i)}, i = 1, 2, 3$ and B_2, separately. According to Lemma 4.3 with $s = t$,

$$\mathrm{E}\left[\sup_{\|\xi\|=1} |B_2|^q\right] \leqslant c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q} \quad \text{and} \quad \mathrm{E}\left[\sup_{\|\xi\|=1} |B_1^{(1)}|^q\right] \leqslant c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q}. \tag{4.45}$$

In the same way, we see that

$$\mathrm{E}\left[\sup_{\|\xi\|=1} |B_1^{(2)}|^q\right] \leqslant c(t-s+\epsilon)^{\frac{\alpha-1}{\alpha}q}\epsilon^{\frac{\alpha-1}{\alpha}q}. \tag{4.46}$$

Since we are in the Sub-case A where $t - s \leqslant \delta^{-1}\epsilon$, we obtain

$$\mathrm{E}\left[\sup_{\|\xi\|=1} |B_1^{(2)}|^q\right] \leqslant c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q}. \tag{4.47}$$

We can combine (4.45) and (4.47) as follows:

$$\mathrm{E}\left[\sup_{\|\xi\|=1} \left(B_1^{(1)} + B_1^{(2)}\right)^q\right] \leqslant c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q}. \tag{4.48}$$

Finally, we turn to bounding the absolute moments of $B_1^{(3)}$. Hypothesis **P1'** assures us that

$$\begin{aligned}
|B_1^{(3)}| &\leqslant c\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv\, G_\alpha(s-r, y-v)G_\alpha(t-r, x-v) \\
&= c\int_{s-\epsilon}^{s} dr\, G_\alpha(t+s-2r, x-y) = c\int_0^{\epsilon} dr\, G_\alpha(t-s+2r, x-y),
\end{aligned}$$

thanks to the semi-group property.

When $\alpha = 2$, we can follow the arguments of [10, p. 414] to find that

$$\inf_{\|\xi\|=1} \xi^T \gamma_Z \xi \geqslant c\epsilon^{1/2} - Z_{1,\epsilon}, \tag{4.49}$$

where $Z_{1,\epsilon} := B_1^{(1)} + B_1^{(2)} + B_2$ satisfies $\mathrm{E}[|Z_{1,\epsilon}|^q] \leqslant c_1(q)\epsilon^q$.

When $1 < \alpha < 2$, by the scaling property of the Green kernel [6, Lemma 4.1(iv)], and the bound in [6, Lemma 4.1(vi)], we have

$$\begin{aligned}
\left|B_1^{(3)}\right| &\leqslant c\int_0^{\epsilon} dr (t-s+2r)^{-1/\alpha} G_\alpha(1, (x-y)(t-s+2r)^{-1/\alpha}) \\
&\leqslant cK_\alpha \int_0^{\epsilon} \frac{(t-s+2r)^{-1/\alpha}}{1+\left|(x-y)(t-s+2r)^{-1/\alpha}\right|^{1+\alpha}} dr \\
&\leqslant cK_\alpha \int_0^{\epsilon} \frac{(t-s+2r)^{-1/\alpha}}{\left|(x-y)(t-s+2r)^{-1/\alpha}\right|^{1+\alpha}} dr \\
&= cK_\alpha |x-y|^{-1-\alpha} \int_0^{\epsilon} (t-s+2r)dr = cK_\alpha |x-y|^{-1-\alpha}[(t-s)\epsilon + \epsilon^2].
\end{aligned}$$

Since $t-s \leqslant |x-y|^\alpha$ and $(1+\beta)\epsilon^{1/\alpha} < \frac{1}{2}|x-y|$ (since we are in Case 2), this is bounded above by

$$cK_\alpha(|x-y|^{-1}\epsilon + |x-y|^{-1-\alpha}\epsilon^2) \leqslant cK_\alpha\left(\frac{1}{(1+\beta)}\epsilon^{\frac{\alpha-1}{\alpha}} + \frac{1}{(1+\beta)^{1+\alpha}}\epsilon^{2-(1+\alpha)/\alpha}\right)$$
$$= cK_\alpha\left(\frac{1}{(1+\beta)} + \frac{1}{(1+\beta)^{1+\alpha}}\right)\epsilon^{\frac{\alpha-1}{\alpha}}.$$

Therefore, for $1 < \alpha \leqslant 2$, we can choose and fix β large enough so that

$$\inf_{\|\xi\|=1} \xi^T \gamma_Z \xi \geqslant c\epsilon^{\frac{\alpha-1}{\alpha}} - Z_{1,\epsilon}, \tag{4.50}$$

where for all $q \geqslant 1$, $\mathrm{E}\left[|Z_{1,\epsilon}|^q\right] \leqslant c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q}$, as in (4.39) and (4.40).

Sub-case B. In this final (sub-) case we suppose that $\epsilon < \delta(t-s) \leqslant \delta|x-y|^\alpha$. Choose and fix $0 < \epsilon < \delta(t-s)$. During the course of our proof of Case 1, we established the following:

$$\inf_{\|\xi\|=1} \xi^T \gamma_Z \xi \geqslant c\epsilon^{\frac{\alpha-1}{\alpha}} - \tilde{Z}_\epsilon, \tag{4.51}$$

where, for all $q \geqslant 1$,

$$\mathrm{E}\left[|\tilde{Z}_\epsilon|^q\right] \leqslant c\epsilon^{\min((1+\gamma_0)\frac{\alpha-1}{\alpha}, 1-\frac{\gamma_0}{\alpha})q}$$

(see (4.37) and (4.38)). This inequality remains valid in this Sub-case B.

Combine Sub-Cases A and B, and, in particular, (4.39) and (4.51), to find that for all $0 < \epsilon < 2^{-\alpha}(1+\beta)^{-\alpha}|x-y|^\alpha$,

$$\inf_{\|\xi\|=1} \xi^T \gamma_Z \xi \geqslant c\epsilon^{\frac{\alpha-1}{\alpha}} - (\tilde{Z}_\epsilon \mathbf{1}_{\{\epsilon<\delta(t-s)\}} + Z_{1,\epsilon}\mathbf{1}_{\{t-s\leqslant\delta^{-1}\epsilon\}}).$$

Because of this and (4.40), by [10, Proposition 3.5], this implies that

$$\mathrm{E}\left[\left(\inf_{\|\xi\|=1} \xi^T \gamma_Z \xi\right)^{-2pd}\right] \leqslant c|x-y|^{\alpha(-2dp)(\frac{\alpha-1}{\alpha})} \leqslant c'\left(|x-y|^\alpha + |t-s|\right)^{(\frac{\alpha-1}{\alpha})(-2dp)}$$
$$\leqslant c\left(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}\right)^{-2dp}.$$

This completes the proof of Proposition 4.8. □

Remark 4.10. From the proof of Proposition 4.8, we see that Propositions 4.7 and 4.8 are also valid for the solutions of stochastic heat equations on a bounded interval with Neumann or Dirichlet boundary conditions. This is because in this case, Lemma A.4 is still valid (see Remark A.5) and implies Lemma 4.4, which is used to prove Propositions 4.8 and 4.7.

Proof of Proposition 4.9. The proof follows along the same lines as those of [10, Proposition 6.13]. Let $0 < \epsilon < s \leqslant t$. We fix $i_0 \in \{1,\dots,2d\}$ and write $\tilde{\lambda}^{i_0} = (\tilde{\lambda}_1^{i_0},\dots,\tilde{\lambda}_d^{i_0})$ and $\tilde{\mu}^{i_0} = (\tilde{\mu}_1^{i_0},\dots,\tilde{\mu}_d^{i_0})$. We look at $(\xi^{i_0})^T\gamma_Z\xi^{i_0}$ on the event $\{\beta_{i_0} \geqslant \beta_0\}$. As in the proof of Proposition 4.8 and using the notation from (4.27), this is bounded below by

$$\sum_{k=1}^d \int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \Bigg(\sum_{i=1}^d \Bigg[\Big(\beta_{i_0}\tilde{\lambda}_i^{i_0} G_\alpha(s-r, y-v)$$
$$+ \tilde{\mu}_i^{i_0}\sqrt{1-\beta_{i_0}^2}(G_\alpha(t-r,x-v) - G_\alpha(s-r,y-v))\Big)\sigma_{ik}(u(r,v))$$

$$+ \beta_{i_0} \tilde{\lambda}_i^{i_0} a_i(k,r,v,s,y)$$
$$+ \tilde{\mu}_i^{i_0} \sqrt{1-\beta_{i_0}^2}(a_i(k,r,v,t,x) - a_i(k,r,v,s,y))\Big]\Big)^2$$
$$+ \sum_{k=1}^d \int_{s\vee(t-\epsilon)}^t dr \int_{\mathbb{R}} dv \Big(\sum_{i=1}^d \Big[\tilde{\mu}_i^{i_0} \sqrt{1-\beta_{i_0}^2} G_\alpha(t-r, x-v)\sigma_{ik}(u(r,v))$$
$$+ \tilde{\mu}_i^{i_0} \sqrt{1-\beta_{i_0}^2} a_i(k,r,v,t,x)\Big]\Big)^2. \tag{4.52}$$

We seek lower bounds for this expression for $0 < \epsilon < \epsilon_0$ where $\epsilon_0 \in]0, \frac{1}{2}[$ is fixed. Notice that this expression corresponds to [10, (6.35)], with α there replaced by β, G by G_α, and $\int_0^1 dv$ by $\int_{\mathbb{R}} dv$. By following essentially the same proof as in their Cases 1 and 2 [10, pp. 419–425], we find that for $0 < \epsilon \leqslant \epsilon_0$,

$$1_{\{\beta_{i_0} \geqslant \beta_0\}} (\xi^{i_0})^T \gamma_Z \xi^{i_0} \geqslant 1_{\{\beta_{i_0} \geqslant \beta_0\}} Z,$$

where

$$Z = \min\Big(c\rho^2 \epsilon^{\frac{\alpha-1}{\alpha}+\eta} - 2I_{3,\epsilon}, c\epsilon^{\frac{\alpha-1}{\alpha}} - 2I_\epsilon \mathbf{1}_{\{\epsilon \geqslant t-s\}} - \tilde{J}_\epsilon \mathbf{1}_{\{\epsilon < t-s\}}\Big)$$

where $I_{3,\epsilon}$, I_ϵ and $\tilde{J}_\epsilon$ are random variables such that $\mathrm{E}[|I_{3,\epsilon}|^q] \leqslant c(q)\epsilon^{2\frac{\alpha-1}{\alpha}q}$, $\mathrm{E}[|I_\epsilon|^q] \leqslant c(q)\epsilon^{\frac{2\alpha-2}{\alpha}q}$, and $\mathrm{E}[|\tilde{J}_\epsilon|^q] \leqslant c(q)\epsilon^{(\frac{\alpha-1}{\alpha}+\eta)q}$ ($\eta > 0$). Note that all the constants are independent of i_0. Then, using [10, Proposition 3.5], we deduce that for all $p \geqslant 1$, there is $C > 0$ such that

$$\mathrm{E}\Big[1_{\{\beta_{i_0} \geqslant \beta_0\}} \big((\xi^{i_0})^T \gamma_Z \xi^{i_0}\big)^{-p}\Big] \leqslant \mathrm{E}\Big[1_{\{\beta_{i_0} \geqslant \beta_0\}} Z^{-p}\Big] \leqslant \mathrm{E}[Z^{-p}] \leqslant C.$$

Since this applies to any $p \geqslant 1$, we can use Hölder's inequality to deduce (4.29). Details can be found in [28, Proof of Proposition 2.5.11]. This proves Proposition 4.9. □

4.3. Proof of Theorem 1.1(b) and (c)

Fix two compact intervals I and J as in Theorem 1.1. Let $(s,y),(t,x) \in I \times J$, $s \leqslant t$, $(s,y) \neq (t,x)$, and $z_1, z_2 \in \mathbb{R}^d$. Let Z be as in (4.24) and let p_Z be the density of Z. Then

$$p_{s,y;t,x}(z_1,z_2) = p_Z(z_1, z_2 - z_1).$$

Use [10, Corollary 3.3] with $\sigma = \{i \in \{d+1,\dots,2d\} : z_2^{i-d} - z_1^{i-d} \geqslant 0\}$ and Hölder's inequality to see that

$$p_Z(z_1, z_2 - z_1) \leqslant \|H_{(1,\dots,2d)}(Z,1)\|_{0,2} \prod_{i=1}^d \left(\mathrm{P}\left\{|u_i(t,x) - u_i(s,y)| > |z_1^i - z_2^i|\right\}\right)^{\frac{1}{2d}}. \tag{4.53}$$

Therefore, in order to prove the desired results of Theorem 1.1(b) and (c), it suffices to prove that:

$$\|H_{(1,\dots,2d)}(Z,1)\|_{0,2} \leqslant c_T(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})^{-d/2}, \tag{4.54}$$

and

$$\prod_{i=1}^d \left(\mathrm{P}\left\{|u_i(t,x) - u_i(s,y)| > |z_1^i - z_2^i|\right\}\right)^{\frac{1}{2d}} \leqslant c \exp\left[-\frac{\|z_1 - z_2\|^2}{c_T(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1})}\right] \tag{4.55}$$

under the hypothesis **P1**, and

$$\prod_{i=1}^{d}\left(\mathrm{P}\left\{|u_i(t,x)-u_i(s,y)|>|z_1^i-z_2^i|\right\}\right)^{\frac{1}{2d}}\leqslant c\left[\frac{|t-s|^{\frac{\alpha-1}{\alpha}}+|x-y|^{\alpha-1}}{\|z_1-z_2\|^2}\wedge 1\right]^{p/(4d)} \tag{4.56}$$

under the hypothesis **P1'**.

The proof of (4.55) under the hypothesis **P1** is essentially the same as that of [10, (6.2)], with Δ there replaced by Δ_α^2, by using Lemma 4.1, the exponential martingale inequality [25, (A.5)] and Girsanov's theorem. As for the proof of (4.56) under the hypothesis **P1'**, it is analogous to that of [11, Theorem 1.6(b)], with $\frac{\gamma}{2}$ there replaced by $\frac{\alpha-1}{\alpha}$ and γ there replaced by $\alpha-1$. Details can be found in [28, Section 2.5.3].

We turn to proving (4.54), which requires the following estimate on the inverse of the matrix γ_Z.

Theorem 4.11. *Fix $T>0$. Assume **P1'** and **P2**. Let I and J be compact intervals as in Theorem 1.1. For any $(s,y),(t,x)\in I\times J$, $s\leqslant t$, $(s,y)\neq(t,x)$, $k\geqslant 0$ and $p>1$,*

$$E\left[\|(\gamma_Z)^{-1}_{m,l}\|_{k,p}\right]\leqslant\begin{cases} c_{k,p,T} & \text{if } (m,l)\in(\mathbf{1}),\\ c_{k,p,T}(|t-s|^{\frac{\alpha-1}{\alpha}}+|x-y|^{\alpha-1})^{-\frac{1}{2}} & \text{if } (m,l)\in(\mathbf{2}) \text{ or } (\mathbf{3}),\\ c_{k,p,T}(|t-s|^{\frac{\alpha-1}{\alpha}}+|x-y|^{\alpha-1})^{-1} & \text{if } (m,l)\in(\mathbf{4}).\end{cases} \tag{4.57}$$

Proof. As in the proof of [10, Theorem 6.3], we shall use Propositions 4.5–4.7.

When $k=0$, the result is a consequence of the estimates of Propositions 4.5 and 4.7, using the fact that the inverse of a matrix is the inverse of its determinant multiplied by its cofactor matrix. Comparing to the proof of [10, Theorem 6.3(a)], the extra exponent η does not appear due to the optimal estimate of negative moments of $\det\gamma_Z$ in Proposition 4.7.

For $k\geqslant 1$, we proceed recursively as in the proof of [10, Theorem 6.3], using Proposition 4.6 instead of Proposition 4.5. □

Proof of (4.54). The proof is similar to that of [10, (6.3)] by using the continuity of the Skorohod integral δ (see [25, Proposition 3.2.1] and [24, (1.11) and p. 131]) and Hölder's inequality for Malliavin norms (see [33, Proposition 1.10, p. 50]); the main difference is that Δ there is replaced by Δ_α^2. Comparing with the estimate in [10, (6.3)], we are able to remove the extra exponent η because of the correct estimate on the inverse of the matrix γ_Z in Theorem 4.11. □

Remark 4.12. We conclude this section by remarking that (4.54), and therefore Theorem 1.1, is also valid for the solutions of stochastic heat equations with Neumann or Dirichlet boundary conditions, since the result of Theorem 4.11 is true in that case by applying Proposition 4.7; see Remark 4.10. Because the proofs in the next section of Theorems 1.3 and 1.6 are based on Theorem 1.1, these two theorems are also valid for the solutions of these stochastic heat equations.

5. Proof of Theorems 1.3 and 1.6

In this section, we give the proof of Theorems 1.3 and 1.6. The organization of the proof is similar to [11, Section 2.3].

5.1. Proof of Theorem 1.3: upper bounds

For all positive integers n, set $t_k^n := k2^{-\frac{2n\alpha}{\alpha-1}}$, $x_l^n := l2^{-\frac{2n}{\alpha-1}}$ and $I_k^n = [t_k^n, t_{k+1}^n]$, $J_l^n = [x_l^n, x_{l+1}^n]$, $R_{k,l}^n = I_k^n \times J_l^n$. By (1.13), we have

$$\mathrm{E}\Big[\sup_{(t,x)\in R_{k,l}^n} \|u(t,x) - u(t_k^n, x_l^n)\|^p\Big] \leqslant C2^{-n\beta p}, \tag{5.1}$$

where β is chosen as in (1.13).

Lemma 5.1. *Fix $\eta > 0$. There exists $c > 0$ such that for all $z \in \mathbb{R}^d$, n large and $R_{k,l}^n \subset I \times J$,*

$$\mathrm{P}\left\{u(R_{k,l}^n) \cap B(z, 2^{-n}) \neq \emptyset\right\} \leqslant c2^{-n(d-\eta)}. \tag{5.2}$$

Proof. The proof is similar to that of [9, Theorem 3.3], using Theorem 1.1(a) and (5.1); see also [11, Lemma 2.2]. The details are left to the reader. □

Proof of Theorem 1.3: upper bounds. We start by proving the upper bound on hitting probability in Theorem 1.3(a). Fix $\epsilon \in]0, 1[$ and $n \in \mathbb{N}$ such that $2^{-n-1} < \epsilon \leqslant 2^{-n}$, and write

$$\mathrm{P}\{u(I \times J) \cap B(z,\epsilon) \neq \emptyset\} \leqslant \sum_{(k,l): R_{k,l}^n \cap I \times J \neq \emptyset} \mathrm{P}\left\{u(R_{k,l}^n) \cap B(z, 2^{-n}) \neq \emptyset\right\}.$$

The number of pairs (k,l) involved in the sum is at most $2^{2n(\alpha+1)/(\alpha-1)}$ times a constant. Lemma 5.1 implies that for all $z \in A$, $\eta > 0$ and large n,

$$\mathrm{P}\{u(I \times J) \cap B(z,\epsilon) \neq \emptyset\} \leqslant \tilde{C}2^{-n(d-\eta)}2^{\frac{2n(\alpha+1)}{\alpha-1}} \leqslant C\epsilon^{d-\frac{2(\alpha+1)}{\alpha-1}-\eta}. \tag{5.3}$$

Note that C does not depend on (n,ϵ). Therefore, (5.3) is valid for all $\epsilon \in]0, 1[$.

Now we use a covering argument (see also the end of the proof of Theorem 1.2(a) in [11, p. 104]): Choose $\tilde{\epsilon} \in]0, 1[$ and let $\{B_i\}_{i=1}^\infty$ be a sequence of open balls in $\mathbb{R}^d$ with respective radii $r_i \in [0, \tilde{\epsilon}[$ such that

$$A \subseteq \cup_{i=1}^\infty B_i \quad \text{and} \quad \sum_{i=1}^\infty (2r_i)^{d-\frac{2(\alpha+1)}{\alpha-1}-\eta} \leqslant \mathscr{H}_{d-\frac{2(\alpha+1)}{\alpha-1}-\eta}(A) + \tilde{\epsilon}. \tag{5.4}$$

Because $\mathrm{P}\{u(I \times J) \neq \emptyset\}$ is at most $\sum_{i=1}^\infty \mathrm{P}\{u(I \times J) \cap B_i \neq \emptyset\}$, the bounds (5.3) and (5.4) imply that

$$\mathrm{P}\{u(I \times J) \neq \emptyset\} \leqslant C\left(\mathscr{H}_{d-\frac{2(\alpha+1)}{\alpha-1}-\eta}(A) + \tilde{\epsilon}\right).$$

Let $\tilde{\epsilon} \to 0$ to conclude that the upper bound in Theorem 1.3(a) holds.

The proof of the upper bounds on hitting probabilities in Theorem 1.3(b) and (c) is similar; see also [9, Theorem 3.1(2), (3)]. □

5.2. Proof of Theorem 1.3: lower bounds

The proof is similar to that of [9, Theorem 2.1]; see also [11, Section 2.4], which requires the following lemma analogous to [11, Lemma 2.3].

Lemma 5.2. *Fix $N > 0$ and $\beta > 0$.*

(a) For $p > 4d(\frac{d}{2} - \frac{2}{\alpha-1} - 1)$, there exists a finite and positive constant $C = C(I, J, d, N, p, \alpha)$ such that for all $a \in [0, N]$,

$$\int_I dt \int_I ds \int_J dx \int_J dy (\Delta_\alpha((t,x);(s,y)))^{-d} \left[\frac{(\Delta_\alpha((t,x);(s,y)))^2}{a^2} \wedge 1\right]^{p/(4d)} \leqslant C\, K_{d-\frac{2(\alpha+1)}{\alpha-1}}(a). \tag{5.5}$$

(b) For $p > 4d(\frac{d}{2} - \frac{1}{\beta})$, there exists a finite and positive constant $C = C(I, d, N, p, \beta)$ such that for all $a \in [0, N]$,

$$\int_I dt \int_I ds\, |t-s|^{-\frac{d\beta}{2}} \left[\frac{|t-s|^\beta}{a^2} \wedge 1\right]^{p/(4d)} \leqslant C\, K_{d-\frac{2}{\beta}}(a). \tag{5.6}$$

Proof. We start by proving (a). Using the change of variables $\tilde{u} = t - s$ (t fixed), $\tilde{v} = x - y$ (x fixed), we see that the integral on the left-hand side of (5.5) is bounded above by

$$4|I\,\|\,J| \int_0^{|I|} d\tilde{u} \int_0^{|J|} d\tilde{v}\, (\tilde{u}^{\frac{\alpha-1}{2\alpha}} + \tilde{v}^{\frac{\alpha-1}{2}})^{-d} \left[\frac{(\tilde{u}^{\frac{\alpha-1}{2\alpha}} + \tilde{v}^{\frac{\alpha-1}{2}})^2}{a^2} \wedge 1\right]^{p/(4d)}.$$

Another change of variables $[\tilde{u} = (ua^2)^{\alpha/(\alpha-1)}, \tilde{v} = (va^2)^{1/(\alpha-1)}]$ implies that this is less than

$$Ca^{\frac{2\alpha+2}{\alpha-1}-d} \int_0^{|I|^{(\alpha-1)/\alpha} a^{-2}} du \int_0^{|J|^{\alpha-1} a^{-2}} dv\, \frac{u^{1/(\alpha-1)} v^{(2-\alpha)/(\alpha-1)}}{(u+v)^{d/2}} [(u+v) \wedge 1]^{p/(4d)}.$$

Passing to the polar coordinates, this is bounded above by

$$Ca^{\frac{2\alpha+2}{\alpha-1}-d}(I_1 + I_2(a)), \tag{5.7}$$

where

$$I_1 = \int_0^{\bar{K}N^{-2}} d\rho\, \rho^{\frac{2}{\alpha-1}-\frac{d}{2}} \rho^{p/(4d)} \quad \text{and} \quad I_2(a) = \int_{\bar{K}N^{-2}}^{\bar{K}a^{-2}} d\rho\, \rho^{\frac{2}{\alpha-1}-\frac{d}{2}}$$

with $\bar{K} = (|I|^{2(\alpha-1)/\alpha} + |J|^{2(\alpha-1)})^{1/2}$. Clearly, $I_1 \leqslant C < \infty$ since $\frac{2}{\alpha-1} - \frac{d}{2} + \frac{p}{4d} > -1$ by the hypothesis on p. Moreover, if $\frac{2}{\alpha-1} - \frac{d}{2} + 1 \neq 0$, i.e., $\frac{2(\alpha+1)}{\alpha-1} \neq d$, then

$$I_2(a) = \bar{K}^{(\alpha+1)/(\alpha-1)-d/2} \frac{a^{d-2(\alpha+1)/(\alpha-1)} - N^{d-2(\alpha+1)/(\alpha-1)}}{(\alpha+1)/(\alpha-1) - d/2}. \tag{5.8}$$

There are three separate cases to consider. (i) If $\frac{2(\alpha+1)}{\alpha-1} < d$, then $I_2(a) \leqslant C < \infty$ for all $a \in [0, N]$. (ii) If $\frac{2(\alpha+1)}{\alpha-1} > d$, then $I_2(a) \leqslant ca^{d-2(\alpha+1)/(\alpha-1)}$. (iii) If $\frac{2(\alpha+1)}{\alpha-1} = d$, then

$$I_2(a) = 2(\log\frac{1}{a} + \log N) \leqslant (2\log N + 2)\log_+(\frac{1}{a}). \tag{5.9}$$

We combine these observations to conclude that the expression in (5.7) is bounded above by $C\, K_{d-\frac{2(\alpha+1)}{\alpha-1}}(a)$.

Next we prove (b). Fix t and change variables $[u = t - s]$ to see that

$$\int_I dt \int_I ds\, |t-s|^{-\frac{d\beta}{2}} \left[\frac{|t-s|^\beta}{a^2} \wedge 1\right]^{p/(4d)} \leqslant 2\int_0^{|I|} du\, u^{-\frac{d\beta}{2}} \left[\frac{u^\beta}{a^2} \wedge 1\right]^{p/(4d)}. \tag{5.10}$$

Another change of variables $[u = a^{2/\beta} v^{1/\beta}]$ simplifies this expression to

$$C\, a^{\frac{2}{\beta}-d} \int_0^{|I|^\beta a^{-2}} dv\, v^{\frac{1}{\beta}-\frac{d}{2}-1} [v \wedge 1]^{p/(4d)}.$$

Observe that

$$\int_0^{|I|^\beta a^{-2}} dv\, v^{\frac{1}{\beta}-\frac{d}{2}-1} [v \wedge 1]^{p/(4d)} \leqslant I_1 + I_2(a),$$

where

$$I_1 := \int_0^{|I|^\beta N^{-2}} dv\, v^{\frac{1}{\beta}-\frac{d}{2}-1+\frac{p}{4d}} \quad \text{and} \quad I_2(a) := \int_{|I|^\beta N^{-2}}^{|I|^\beta a^{-2}} dv\, v^{\frac{1}{\beta}-\frac{d}{2}-1}.$$

Clearly, $I_1 \leqslant C < \infty$ provided that $p > 4d(\frac{d}{2} - \frac{1}{\beta})$. The remainder of the proof is the same as that of (a). □

Proof of Theorem 1.3: lower bounds. We start by proving the lower bound on hitting probabilities in (a). The proof follows along the same lines as the proof of [9, Theorem 2.1(1)], therefore we will only sketch the steps that differ; see also the proof of [11, Theorem 1.2(b)]. We need to replace their $\beta - 6$ by $d - \frac{2(\alpha+1)}{\alpha-1}$.

We first note that our Theorems 1.1(a) and 1.2 imply that

$$\inf_{\|z\| \leqslant M} \int_I dt \int_J dx\, p_{t,x}(z) \geqslant C > 0, \tag{5.11}$$

which proves hypothesis **A1’** of [9, Theorem 2.1(1)] (see [9, Remark 2.5(a)]).

Let us now follow the proof of [9, Theorem 2.1(1)]. Define, for all $z \in \mathbb{R}^d$ and $\epsilon > 0$, $\tilde{B}(z, \epsilon) := \{y \in \mathbb{R}^d : |y - z| < \epsilon\}$, where $|z| := \max_{1 \leqslant j \leqslant d} |z_j|$, and

$$J_\epsilon(z) = \frac{1}{(2\epsilon)^d} \int_I dt \int_J dx\, \mathbf{1}_{\tilde{B}(z,\epsilon)}(u(t, x)), \tag{5.12}$$

as in [9, (2.28)].

Assume first that $d < \frac{2(\alpha+1)}{\alpha-1}$. Using Theorem 1.1(b), we find, instead of [9, (2.30)],

$$\mathrm{E}[(J_\epsilon(z))^2] \leqslant c \int_I dt \int_I ds \int_J dx \int_J dy\, [\Delta_\alpha((t, x); (s, y))]^{-d}.$$

The change of variables $u = t - s$ (t fixed), $v = x - y$ (x fixed), implies that the above integral is bounded above by

$$C \int_0^{|I|} du \int_0^{|J|} dv \left(u^{\frac{\alpha-1}{2\alpha}} + v^{\frac{\alpha-1}{2}}\right)^{-d} \leqslant C' \int_0^{|I|} du\, \Psi_{|J|,(\alpha-1)d/2}(u^{(\alpha-1)d/(2\alpha)}), \tag{5.13}$$

where Ψ is defined by $\Psi_{a,\nu}(\rho) := \int_0^a \frac{dx}{\rho + x^\nu}$, for all $a, \nu, \rho > 0$, as in (2.23) of [9]. Hence, by Lemma 2.3 of [9], for all $\epsilon > 0$,

$$\mathrm{E}\left[(J_\epsilon(z))^2\right] \leqslant C \int_0^{|I|} du\, \mathrm{K}_{1-\frac{2}{(\alpha-1)d}}(u^{(\alpha-1)d/(2\alpha)}).$$

In order to bound the above integral, we consider three different cases: (i) If $0 < d < \frac{2}{\alpha-1}$, then $1 - \frac{2}{(\alpha-1)d} < 0$ and the integral equals $|I|$. (ii) If $\frac{2}{\alpha-1} < d < \frac{2(\alpha+1)}{\alpha-1}$, then $\mathrm{K}_{1-\frac{2}{(\alpha-1)d}}(u^{(\alpha-1)d/(2\alpha)}) = u^{1/\alpha-(\alpha-1)d/(2\alpha)}$ and the integral is finite. (iii) If $d = \frac{2}{\alpha-1}$, then

$K_0(u^{1/\alpha}) = \log_+(u^{-1/\alpha})$ and the integral is also finite. The remainder of the proof of Theorem 1.3(a) when $d < \frac{2(\alpha+1)}{\alpha-1}$ follows exactly as in [9, Theorem 2.1(1) Case 1].

Assume now that $d > \frac{2(\alpha+1)}{\alpha-1}$. Define, for all $\mu \in \mathscr{P}(A)$ and $\epsilon > 0$,

$$J_\epsilon(\mu) = \frac{1}{(2\epsilon)^d} \int_{\mathbb{R}^d} \mu(dz) \int_I dt \int_J dx\, \mathbf{1}_{\tilde{B}(z,\epsilon)}(u(t,x)), \tag{5.14}$$

as [9, (2.35)]. Fix $\mu \in \mathscr{P}(A)$ such that

$$I_{d-\frac{2(\alpha+1)}{\alpha-1}}(\mu) \leqslant \frac{2}{\mathrm{Cap}_{d-\frac{2(\alpha+1)}{\alpha-1}}(A)}.$$

Analogous to the proof of [9, (2.41)], we use Theorem 1.1(b) and Lemma 5.2(a), to see that for all $\epsilon > 0$

$$\mathrm{E}\left[(J_\epsilon(\mu))^2\right] \leqslant C_2 I_{d-\frac{2(\alpha+1)}{\alpha-1}}(\mu) \leqslant \frac{2C_2}{\mathrm{Cap}_{d-\frac{2(\alpha+1)}{\alpha-1}}(A)}.$$

The remainder of the proof of Theorem 1.3(a) when $d > \frac{2(\alpha+1)}{\alpha-1}$ follows as in [9, Theorem 2.1(1) Case 2].

The case $d = \frac{2(\alpha+1)}{\alpha-1}$ is proved exactly along the same lines as the proof of [9, Theorem 2.1(1) Case 3], appealing to (5.11), Theorem 1.1(b) and Lemma 5.2(a).

The proof of lower bounds on hitting probabilities in (b) and (c) follows similarly by using Theorem 1.1(b) and Lemma 5.2(b). □

5.3. *Proof of* Theorem 1.6

In the case $b \equiv 1$ and $\sigma \equiv I_d$, the components of $v = (v_1, \ldots, v_d)$ are independent and identically distributed.

Proposition 5.3. *For any $0 < t_0 < T$, $p > 1$ and K a compact set, there exists $c_1 = c_1(p, t_0, K) > 0$ such that for any $t_0 \leqslant s \leqslant t \leqslant T$, $x, y \in K$,*

$$\mathrm{E}\left[|v_1(t,x) - v_1(s,y)|^p\right] \geqslant c_1 \left(|t-s|^{\frac{\alpha-1}{\alpha}} + |x-y|^{\alpha-1}\right)^{p/2}. \tag{5.15}$$

Proof. The proof is similar to that of Proposition 2.1 of [11]. Details can be found in [28, Proposition 2.2.2] □

Proof of Theorem 1.6. As in [11, Theorem 1.5], we first apply [35, Theorem 7.6] to prove that the upper bound holds in Theorem 1.3(a) when u is replaced by v and η is set to 0 in the Hausdorff measure on the right-hand side. For this, it suffices to verify Conditions (C1) and (C2) of [35, Section 2.4, p. 158] with $N = 2$, $H_1 = \frac{\alpha-1}{2\alpha}$, $H_2 = \frac{\alpha-1}{2}$.

First, we observe that $\mathrm{E}[v_1(t,x)^2] = c_\alpha t^{\frac{\alpha-1}{\alpha}}$ (see (4.1)), which implies that there are positive constants c_1, c_2 such that for all $(t,x), (s,y) \in I \times J$,

$$c_1 \leqslant \mathrm{E}[v_1(t,x)^2] \leqslant c_2. \tag{5.16}$$

By (5.15) and (1.12), there exist positive constants c_3, c_4 such that for all $(t,x), (s,y) \in I \times J$,

$$c_3(\Delta_\alpha((t,x);(s,y)))^2 \leqslant \mathrm{E}\left[|v_1(t,x) - v_1(s,y)|^2\right] \leqslant c_4(\Delta_\alpha((t,x);(s,y)))^2. \tag{5.17}$$

Hence condition (C1) is satisfied by (5.16) and (5.17). Condition (C2) holds by applying the fourth point of Remark 2.2 in [35], since $(t, x) \mapsto \mathrm{E}[v_1(t, x)] = c_\alpha t^{\frac{\alpha-1}{\alpha}}$ is continuous in $I \times J$ with continuous partial derivatives.

The rest of the proof of Theorem 1.6 follows the same lines by using (5.16), (5.17) and the fact that $(t, x) \mapsto \mathrm{E}[v_1(t, x)] = c_\alpha t^{\frac{\alpha-1}{\alpha}}$ is continuous in $I \times J$ with continuous partial derivatives.

Therefore we have finished the proof of Theorem 1.6. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

We first state an elementary fact that is used several times.

Lemma A.1. *Fix $\gamma \in]0, 1[$ and $\mu > 0$.*

(a) The function $x \mapsto (x + \mu)^\gamma - x^\gamma$ is nonincreasing on $[0, \infty[$ and the function $x \mapsto x^\gamma - (x - \mu)^\gamma$ is nonincreasing on $[\mu, \infty[$.

(b) $(1 + x)^\gamma - 1 \leqslant \gamma x$ for all $x \geqslant 0$.

We recall Burkholder's inequality for Hilbert-space-valued martingales; see also [3, Eq.(4.18)] and [10, Lemma 7.6].

Lemma A.2 *([22, E.2. p. 212])***.** *Let $H_{s,y}$ be a predictable $L^2(([0, t]\times\mathbb{R})^m, d\alpha)$-valued process, where $m \geqslant 1$ and $d\alpha$ denotes Lebesgue measure. Then, for any $p \geqslant 2$, there exists $C > 0$ such that*

$$\mathrm{E}\left[\left|\int_{([0,t]\times\mathbb{R})^m}\left|\int_0^t\int_{\mathbb{R}} H_{s,y}(\alpha)W(ds, dy)\right|^2 d\alpha\right|^p\right] \leqslant C\mathrm{E}\left[\left|\int_0^t\int_{\mathbb{R}}\left|\int_{([0,t]\times\mathbb{R})^m} H^2_{s,y}(\alpha)d\alpha\right|dyds\right|^p\right].$$

The next result is an extension of Morien [23, Lemma 4.2] for the solution of the fractional stochastic heat equation (1.1).

Lemma A.3. *Assume **P1**. For all $p \geqslant 1, T > 0$ there exists $C > 0$ such that for all $T \geqslant t \geqslant s \geqslant \epsilon > 0$ and $x \in \mathbb{R}$,*

$$\sum_{k,i=1}^d \mathrm{E}\left[\left(\int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \left|D^{(k)}_{r,v}(u_i(t, x))\right|^2\right)^p\right] \leqslant C\epsilon^{(\alpha-1)p/\alpha}.$$

Proof. The proof follows the same lines as [23, Lemma 4.2]. We include it because the ingredients will be needed for Lemma A.4. We define

$$H_i(t, x) := \mathrm{E}\left[\left(\int_{s-\epsilon}^s dr \int_{\mathbb{R}} dv \left|D^{(k)}_{r,v}(u_i(t, x))\right|^2\right)^p\right], \tag{A.1}$$

and

$$K_s(t) := \sum_{i=1}^{d} \sup_{s \leqslant \lambda \leqslant t} \sup_{y \in \mathbb{R}} H_i(\lambda, y) \tag{A.2}$$

which are finite by (2.3). Thanks to formula (2.1), we have

$$\begin{aligned} H_i(t,x) &\leqslant c\left(\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv\, G_\alpha^2(t-r,x-v)\right)^p \\ &\quad + c\sum_{j=1}^{d} \mathrm{E}\Big[\Big[\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \Big(\int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta,x-\eta) D_{r,v}^{(k)}(\sigma_{ij}(u(\theta,\eta))) W^j(d\theta,d\eta)\Big)^2\Big]^p\Big] \\ &\quad + c\,\mathrm{E}\Big[\Big[\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \Big(\int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta,x-\eta) D_{r,v}^{(k)}(b_i(u(\theta,\eta))) d\theta d\eta\Big)^2\Big]^p\Big] \\ &:= A_1 + A_2 + A_3. \end{aligned} \tag{A.3}$$

By (4.1), we see that

$$\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv\, G_\alpha^2(t-r,x-v) = c((t-s+\epsilon)^{\frac{\alpha-1}{\alpha}} - (t-s)^{\frac{\alpha-1}{\alpha}}) \leqslant c'\epsilon^{\frac{\alpha-1}{\alpha}}, \tag{A.4}$$

by Lemma A.1(b). This implies that

$$A_1 \leqslant c_p \epsilon^{(\alpha-1)p/\alpha}. \tag{A.5}$$

Using Burkholder's inequality for Hilbert-space-valued martingales (Lemma A.2) first, and then the Cauchy–Schwarz inequality together with the fact that the partial derivatives of σ_{ij} are bounded, we obtain

$$\begin{aligned} A_2 &\leqslant c\sum_{l=1}^{d} \mathrm{E}\Big[\Big[\int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta \int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv\, G_\alpha^2(t-\theta,x-\eta)\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\Big]^p\Big] \\ &\quad + c\sum_{l=1}^{d} \mathrm{E}\Big[\Big[\int_{s}^{t} d\theta \int_{\mathbb{R}} d\eta \int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv\, G_\alpha^2(t-\theta,x-\eta)\left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\Big]^p\Big] \\ &:= A_{21} + A_{22}. \end{aligned} \tag{A.6}$$

We now use Hölder's inequality with respect to the measure $G_\alpha^2(t-\theta,x-\eta)d\theta d\eta$ to find that

$$\begin{aligned} A_{21} &\leqslant c\left|\int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta,x-\eta)\right|^p \sup_{(\theta,\eta)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\left(\int_0^T dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right)^p\right] \\ &\leqslant c\,\epsilon^{(\alpha-1)p/\alpha}, \end{aligned} \tag{A.7}$$

where the last inequality follows from (2.3) and (A.4). Again, applying Hölder's inequality with respect to the measure $G_\alpha^2(t-\theta,x-\eta)d\theta d\eta$, we see that

$$\begin{aligned} A_{22} &\leqslant c\left|\int_s^t d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta,x-\eta)\right|^{p-1} \\ &\qquad \times \int_s^t d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta,x-\eta) \sum_{l=1}^{d} \mathrm{E}\left[\left(\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right)^p\right] \\ &\leqslant c(t-s)^{\frac{\alpha-1}{\alpha}(p-1)} \int_s^t d\theta \int_{\mathbb{R}} d\eta\, G_\alpha^2(t-\theta,x-\eta) K_s(\theta) \\ &\leqslant c\int_s^t (t-\theta)^{-\frac{1}{\alpha}} K_s(\theta) d\theta. \end{aligned} \tag{A.8}$$

We handle the third term in (A.3) in a similar way. First, by the Cauchy–Schwarz inequality with respect to the measure $G_\alpha(t-\theta, x-\eta)d\theta d\eta$, we have

$$\begin{aligned}
A_3 &\leqslant c\,\mathrm{E}\left[\left[\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \int_r^t \int_{\mathbb{R}} G_\alpha(t-\theta, x-\eta) \sum_{l=1}^{d} \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2 d\theta d\eta\right]^p\right] \\
&= c\,\mathrm{E}\left[\left[\int_{s-\epsilon}^{t} d\theta \int_{\mathbb{R}} d\eta \int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv\, G_\alpha(t-\theta, x-\eta) \sum_{l=1}^{d} \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right]^p\right] \\
&\leqslant c\,\mathrm{E}\left[\left[\int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \sum_{l=1}^{d} \int_{s-\epsilon}^{s\wedge\theta} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right]^p\right] \\
&\quad + c\,\mathrm{E}\left[\left[\int_{s}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \sum_{l=1}^{d} \int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right]^p\right] \\
&:= A_{31} + A_{32}.
\end{aligned} \tag{A.9}$$

By Hölder's inequality with respect to the measure $G_\alpha(t-\theta, x-\eta)d\theta d\eta$,

$$\begin{aligned}
A_{31} &\leqslant c\left|\int_{s-\epsilon}^{s} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)\right|^p \sum_{l=1}^{d} \sup_{(\theta,\eta)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[\left(\int_0^T dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right)^p\right] \\
&\leqslant c\,\epsilon^p \leqslant c\,\epsilon^{(\alpha-1)p/\alpha},
\end{aligned} \tag{A.10}$$

where in the third inequality we use [6, Lemma 4.1(i)] and (2.3). Similarly,

$$\begin{aligned}
A_{32} &\leqslant c\left|\int_{s}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)\right|^{p-1} \\
&\quad \times \int_s^t d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) \sum_{l=1}^{d} E\left[\left(\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_l(\theta,\eta))\right)^2\right)^p\right] \\
&\leqslant c\left|\int_{s}^{t} d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta)\right|^{p-1} \int_s^t d\theta \int_{\mathbb{R}} d\eta\, G_\alpha(t-\theta, x-\eta) K_s(\theta) \\
&\leqslant c\int_s^t K_s(\theta)d\theta.
\end{aligned} \tag{A.11}$$

Finally, we put (A.3) and (A.5)–(A.11) together and obtain that

$$K_s(t) \leqslant c\,\epsilon^{(\alpha-1)p/\alpha} + c\int_s^t (1+(t-\theta)^{-\frac{1}{\alpha}})K_s(\theta)d\theta \leqslant c\,\epsilon^{(\alpha-1)p/\alpha} + \overline{c}\int_s^t (t-\theta)^{-\frac{1}{\alpha}} K_s(\theta)d\theta.$$

Define $\overline{K}_s(\lambda) := K_s(\lambda + s)$. From the above inequality we have

$$\overline{K}_s(t-s) \leqslant c\,\epsilon^{(\alpha-1)p/\alpha} + \overline{c}\int_0^{t-s} (t-s-\theta)^{-\frac{1}{\alpha}} \overline{K}_s(\theta)d\theta.$$

By Gronwall's lemma [8, Lemma 15], we have

$$K_s(t) = \overline{K}_s(t-s) \leqslant c\,\epsilon^{(\alpha-1)p/\alpha}, \quad \text{for all } s \leqslant t. \quad \square$$

The following lemma is an improvement of Lemma A.3.

Lemma A.4. *Fix $T > 0, c_0 > 1$ and $0 < \gamma_0 < 1$. For all $p \geqslant 1$ there exists $C > 0$ such that for all $T \geqslant t \geqslant s \geqslant \epsilon > 0$ with $t - s > c_0\epsilon^{\gamma_0}$ and $x \in \mathbb{R}$,*

$$\sum_{k,i=1}^{d} \mathrm{E}\left[\left(\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv \left(D_{r,v}^{(k)}(u_i(t,x))\right)^2\right)^p\right] \leqslant C\epsilon^{(1-\frac{\gamma_0}{\alpha})p}.$$

Proof. We use the same notations as in the proof of Lemma A.3. First, under the condition $t - s > c_0\epsilon^{\gamma_0}$, using (4.1), we have

$$\begin{aligned}\int_{s-\epsilon}^{s} dr \int_{\mathbb{R}} dv\, G_\alpha^2(t-r, x-v) &= c((t-s+\epsilon)^{\frac{\alpha-1}{\alpha}} - (t-s)^{\frac{\alpha-1}{\alpha}})\\ &\leqslant c((c_0\epsilon^{\gamma_0}+\epsilon)^{\frac{\alpha-1}{\alpha}} - (c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}}) = c(c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}}((1+c_0^{-1}\epsilon^{1-\gamma_0})^{\frac{\alpha-1}{\alpha}} - 1)\\ &\leqslant c(c_0\epsilon^{\gamma_0})^{\frac{\alpha-1}{\alpha}} c_0^{-1}\epsilon^{1-\gamma_0}(\alpha-1)/\alpha = c\, c_0^{-1/\alpha}\epsilon^{1-\frac{\gamma_0}{\alpha}}(\alpha-1)/\alpha, \end{aligned} \tag{A.12}$$

where the first inequality follows from Lemma A.1(a) because $t - s > c_0\epsilon^{\gamma_0}$, and the second inequality is due to Lemma A.1(b). Therefore, $A_1 \leqslant c\,\epsilon^{(1-\frac{\gamma_0}{\alpha})p}$. Using (A.12) instead of (A.4), we see that $A_{21} \leqslant c\,\epsilon^{(1-\frac{\gamma_0}{\alpha})p}$. Due to the choice of γ_0 and by (A.10), we have $A_{31} \leqslant c\,\epsilon^p \leqslant c'\,\epsilon^{(1-\frac{\gamma_0}{\alpha})p}$. The estimates for other terms remain the same as in the proof of Lemma A.3. Therefore, we have obtained that

$$K_s(t) \leqslant c\,\epsilon^{(1-\frac{\gamma_0}{\alpha})p} + c\int_s^t (1+(t-\theta)^{-\frac{1}{\alpha}})K_s(\theta)d\theta \leqslant c\,\epsilon^{(1-\frac{\gamma_0}{\alpha})p} + \bar{c}\int_s^t (t-\theta)^{-\frac{1}{\alpha}} K_s(\theta)d\theta.$$

Applying Gronwall's lemma ([8, Lemma 15]), we have $K_s(t) \leqslant c\,\epsilon^{(1-\frac{\gamma_0}{\alpha})p}$, for all $s \leqslant t$. □

Remark A.5. The result of Lemma A.4 is also valid for the solutions of stochastic heat equations with Neumann or Dirichlet boundary conditions in which case $\alpha = 2$. This is because the Green kernel of the heat equation with Neumann or Dirichlet boundary conditions shares similar properties with the Green kernel of heat equation, which enables us to derive the same estimates as in (A.12) and the lines that follow, for the solutions of stochastic heat equations with Neumann or Dirichlet boundary conditions.

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