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Statistics on crossings of discretized diffusions and local time

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Let $\bar{X}_\Delta$ be the process obtained by linear interpolation from discrete observations of a diffusion X. In the first part of this paper we study the statistical properties of the observation sgn $\bar{X}_\Delta$. In the second part we prove that the number of zero-crossings of $\bar{X}_\Delta$, suitably normalized, converges in (L^2-norm) to the zero local time of X.

diffusion * local time * crossings * estimation

1. Introduction

Let X be a diffusion on $\mathbb{R}$ which is a solution of the stochastic differential equation

$$\mathrm{d}X_t = b(X_t, \theta)\,\mathrm{d}t + \mathrm{d}W_t, \quad X_0 = x,$$

where θ is an unknown parameter.

Let $\bar{X}_\Delta$ be the process obtained from X by linear interpolation from the sequence $\{X_{k\Delta}, k \geqslant 0\}$.

We are interested in the observation $\{Z_k = \operatorname{sgn} X_{k\Delta}, k \geqslant 0\}$. From a statistical point of view, this observation is convenient and has some robustness, but it is not easy to compute its likelihood. In what follows, marginal likelihood functions are considered (working as if Z_k were independent or a Markov chain), giving asymptotically normal estimators. The variances of these estimators are functions of Δ and go to infinity when Δ goes to zero, as usual for discrete observations. The asymptotic is, of course, linked to stationarity and to positive recurrent models.

Let us now consider a finite interval of observation, for instance $[0, 1]$. The observations are $Z_k^{(n)} = \operatorname{sgn} X_{k/n}$. If we are able to prove that the number $N_{1/n}$ of zero-crossings of the process $\bar{X}_{1/n}$ goes to the local time L, we have an interesting statistical interpretation of L.

Thus, the last part of this paper is dedicated to proving that $\sqrt{\tfrac{1}{2}\pi}\,\sqrt{\Delta}\; N_\Delta(t) \to L(t)$ as $\Delta \to 0$ (in L^2-norm).

In Azais and Florens-Zmirou (1987) we prove a similar result for Gaussian processes and the two parts of this work are related to our work (Florens-Zmirou,

1988) where we studied the random discretization obtained from the excursions of X longer than Δ.

2. Statistical model

We consider the statistical model

$$\mathrm{d}X_t = b(X_t, \theta)\,\mathrm{d}t + \mathrm{d}W_t, \quad X_0 = x, \ \theta \in \Theta,$$

where Θ is a compact subset of $\mathbb{R}$.

For each θ, $((X_t)_{t\in\mathbb{R}^+}, \mathbb{P}_\theta)$ is supposed to be a recurrent positive diffusion. The density of the transition probability Π_t is denoted by $\pi_t(x, y)$ and that of the invariant probability is denoted by $\mu_\theta(x)$ (we abuse the notation by setting $\mu_\theta(f) = \int f(x)\mu_\theta(x)\,\mathrm{d}x$).

The true value of the parameter is denoted by θ_0. We denote by a comma the derivative with respect to x and by a dot the one with respect to θ.

The following regularity hypotheses are used:

(H.1) $(x, \theta) \to b(x, \theta) \in C^2$.

(H.2) There exist ρ and K such that for all θ,

$$\liminf_{|x|\to\infty} \frac{b(x, \theta)}{x} > -\rho,$$

$$\inf_x [b^2(x, \theta) + b'(x, \theta)] > -K, \quad K > 0,$$

$$b^2(x, \theta) + b'(x, \theta) = \mathrm{O}(x^2), \quad x \to \infty.$$

(H.3) $(x, \theta) \to \mu_\theta(x)$ is uniformly bounded.

(H.4) There exists λ such that for all (θ, x),

$$|\dot{b}(x, \theta)| \leq \lambda \exp \lambda|x|, \qquad |\dot{b}'(x, \theta)| \leq \lambda \exp \lambda|x|.$$

Let

$$Z_k = \begin{cases} 1 & \text{if } X_{k\Delta} \geq 0, \\ -1 & \text{if } X_{k\Delta} < 0. \end{cases}$$

We set

$$M_\theta(x) = \int_0^x \mu_\theta(y)\,\mathrm{d}y, \qquad \mu_\theta^+ = M_\theta(\infty), \quad \mu_\theta^- = -M_\theta(-\infty),$$

$$p_{\theta,\Delta} = P^\mu[X_0 \geq 0, X_\Delta < 0], \qquad q_{\theta,\Delta} = P^\mu[X_0 \leq 0, X_\Delta > 0].$$

From now, we choose $\mu_\theta(x)\,\mathrm{d}x$ as the initial distribution. Then the process is stationary, and therefore $p_\Delta = q_\Delta$. Let us remark that from the exponential convergence of the transition probability to μ, proved in Florens-Zmirou (1984), it is not too difficult to derive the case of any initial distribution.

We shall omit θ when there is no ambiguity and we denote in an evident way

$$n^+ = \sum_{k=0}^{n-1} \mathbf{1}_{(Z_k=1)}, \qquad n^- = n - n^+,$$

$$n^{+-} = \sum_{k=0}^{n-1} \mathbf{1}_{(Z_k=1, Z_{k+1}=-1)}, \qquad n^{++} = \sum_{k=0}^{n-1} \mathbf{1}_{(Z_k=1, Z_{k+1}=1)},$$

and n^{-+} and n^{--} by an evident analogy.

We shall use as contrast functions the marginal likelihoods of Z_k. The first one, denoted by $\mathcal{M}_1(\theta)$, is the binomial distribution $b(n, \mu^+(\theta))$ and the second one, denoted by $\mathcal{M}_2(\theta)$, is the multinomial distribution $M[n, p_\Delta(\theta), \mu_\theta^+ - p_\Delta(\theta), \mu_\theta^- - p_\Delta(\theta)]$. The two contrasts are respectively,

$$L_n^{(1)}(\theta) = n^+ \log \mu^+(\theta) + n^- \log \mu^-(\theta),$$

$$\begin{aligned} L_n^{(2)}(\theta) = {} & n^{+-} \log p_\Delta(\theta) + n^{-+} \log p_\Delta(\theta) \\ & + n^{++} \log(\mu^+(\theta) - p_\Delta(\theta)) \\ & + n^{--} \log(\mu^-(\theta) - p_\Delta(\theta)). \end{aligned}$$

The law of large numbers proves that as $n \to \infty$,

$$\frac{1}{n}[L_n^{(1)}(\theta_0) - L_n^{(1)}(\theta)] \to K[\mathcal{M}_1(\theta_0), \mathcal{M}_1(\theta)] = K_1(\theta_0, \theta)$$

and

$$\frac{1}{n}[L_n^{(2)}(\theta_0) - L_n^{(2)}(\theta)] \to K[\mathcal{M}_2(\theta_0), \mathcal{M}_2(\theta)] = K_2(\theta_0, \theta),$$

where K is the Kullback distance.

$\mathcal{M}_1$ is a marginal model of $\mathcal{M}_2$ and therefore

$$K_1(\theta_0, \theta) \leq K_2(\theta_0, \theta).$$

Let $\hat{\theta}_n^{(i)} = \arg\max L_n^{(i)}(\theta)$. (If there are several maxima, $\hat{\theta}_n^{(i)}$ is the argument of one of them.) We make now the additional assumptions:

(H.5) $\theta \to \mu^+(\theta), \theta \to \mu^-(\theta) \in C^2$.

(H.6) $\theta \to p_\Delta(\theta) \in C^2$.

We need these assumptions for the proof of the consistency and normality of $\hat{\theta}_n^{(i)}$. Of course the hypotheses (H1)-(H6) are not independent. It follows from (H.5) and (H.6) that $\theta \to (\mu^+(\theta), \mu^-(\theta), p_\Delta(\theta))$ are continuous and Theorem 3.2.8 of Dacunha-Castelle and Duflo (1986) can be applied to models $\mathcal{M}_i(\theta)$ in order to prove the consistency of $\hat{\theta}_n^{(i)}$.

The main result gives the expansion of the asymptotic variance with respect to Δ.

Theorem 1. *Under hypothesis* (H):

(1) *Let* $I_1(\theta_0)$ *be the Fisher information of* $\mathcal{M}_1(\theta_0)$. *If* $I_1(\theta_0)\neq 0$ *then*

$$\sqrt{n\Delta}\,(\hat{\theta}_n^{(1)}-\theta_0)\xrightarrow{\mathcal{L}(P_{\theta_0})}\mathrm{N}(0,\sigma_1^2(\theta_0))$$

where $\sigma_1^2(\theta_0)=V^0(\theta_0)/I_1^2(\theta_0)+\mathrm{O}(\Delta)$,

$$V^0=4\left[A^+\left(\frac{\dot\mu^+}{\mu^+}\right)^2+A^-\left(\frac{\dot\mu^-}{\mu^-}\right)^2\right],$$

$$A^+=\int_0^\infty\frac{M^2(x)}{\mu(x)}\,\mathrm{d}x,\qquad A^-=\int_{-\infty}^0\frac{M^2(x)}{\mu(x)}\,\mathrm{d}x$$

and $M(x)=\int_0^x\mu(u)\,\mathrm{d}u$.

(2) *If* $I_1(\theta_0)=0$ (μ^+ *and* μ^- *do not depend on* θ), *then*

$$\sqrt{n}\,\Delta^{3/4}(\hat{\theta}_n^{(2)}-\theta_0)\xrightarrow{\mathcal{L}(P_{\theta_0})}\mathrm{N}(0,\sigma_2^2)$$

with, for every $\varepsilon>0$, $\sigma_2^2(\theta_0,\Delta)\leqslant(1+\varepsilon)\sqrt{2\pi}/I_0(\theta_0)s$ (*as soon as* $\Delta<\Delta(\varepsilon)$), *where* $I_0(\theta_0)=[\dot\mu_{\theta_0}(0)]^2/\mu_{\theta_0}(0)$, *and s is the lower strictly positive bound of the spectrum of the diffusion.*

Proof. By the law of large numbers, we have

$$-\frac{1}{n}\ddot{L}_n(\theta_0)\to I_1(\theta_0)=\left[\frac{(\dot\mu^+)^2}{\mu^+}+\frac{(\dot\mu^-)^2}{\mu^-}\right](\theta_0).$$

In order to prove a central limit theorem for $\hat{\theta}_n^{(1)}$, we need to know the behaviour of

$$\frac{1}{\sqrt{n}}\dot{L}_n^{(1)}(\theta_0)=\frac{1}{\sqrt{n}}\left[\left(\frac{\dot\mu^+}{\mu^+}\right)\sum_{i=1}^n\mathbf{1}_{(X_{i\Delta}>0)}+\left(\frac{\dot\mu^-}{\mu^-}\right)\sum_{i=1}^n\mathbf{1}_{(X_{i\Delta}\leqslant 0)}\right].$$

We apply our theorem (Florens-Zmirou, 1984) for functionals of the chain $X_{k\Delta}$:

$$\frac{1}{\sqrt{n}}\dot{L}_n^{(1)}(\theta_0)\xrightarrow{\mathcal{L}(P_{\theta_0})}\mathrm{N}\left(0,\frac{V_\Delta^{(1)}}{\Delta}\right)$$

with $V_\Delta^{(1)}=\Delta\mu(f^2)+2\mu(f\Pi_\Delta F_\Delta)$ and $F_\Delta=\Delta\sum_{n=0}^\infty\Pi_{n\Delta}f$, where

$$f(x)=\frac{\dot\mu^+}{\mu^+}\mathbf{1}_{(x>0)}+\frac{\dot\mu^-}{\mu^-}\mathbf{1}_{(x\leqslant 0)}.$$

In Florens-Zmirou (1984), we also proved that

$$V_\Delta^{(1)}=V^0+\mathrm{O}(\Delta)$$

with

$$V^0=4\int_{-\infty}^{+\infty}\mu(x)(\mathrm{d}x)\left[\frac{1}{\mu(x)}\int_0^x f(u)\mu(u)\,\mathrm{d}u\right]^2$$
$$=4\left[A^+\left(\frac{\dot\mu^+}{\mu^+}\right)^2+A^-\left(\frac{\dot\mu^-}{\mu^-}\right)^2\right].$$

Then, if $I_1(\theta_0) \neq 0$,

$$\sqrt{n\Delta}\,\frac{\dot{L}_n^{(1)}(\theta_0)}{\ddot{L}_n^{(1)}(\theta_0)} \to \mathrm{N}(0, \sigma_1^2(\theta_0))$$

with $\sigma_1^2(\theta_0) = V^0(\theta_0)/I_1^2(\theta_0) + \mathrm{O}(\Delta)$.

In order to prove that

$$\sqrt{n\Delta}\,(\hat{\theta}_n^{(1)} - \theta_0) \to \mathrm{N}(0, \sigma_1^2(\theta_0)),$$

we apply Theorem 3.3.15 of Dacunha-Castelle and Duflo (1986). We have to check that

$$\lim_{h\to 0} \lim_{n\to\infty} \sup_{|\theta-\theta'|<h} \frac{1}{n} |\ddot{L}_n^{(1)}(\theta) - \ddot{L}_n^{(1)}(\theta')| = 0;$$

this is a consequence of hypothesis (H.5) and of the law of large numbers.

Let us study the case of $\mathcal{M}_2(\theta)$. We have

$$\frac{1}{n}\,\dot{L}_n^{(2)} = \dot{p}_\Delta \left[\left(\frac{n^{+-}}{p_\Delta} - \frac{n^{++}}{\mu^+ - p_\Delta}\right) + \left(\frac{n^{-+}}{p_\Delta} - \frac{n^{--}}{\mu^- - p_\Delta}\right)\right],$$

and we suppose that $\theta \to \mu^+(\theta)$ and $\theta \to \mu^-(\theta)$ are constant.

First,

$$-\frac{1}{n}\,\ddot{L}_n^{(2)}(\theta_0) \to I_2(\theta_0) \quad \text{a.s.}$$

where $I_2(\theta_0)$ is the Fisher information of $\mathcal{M}_2(\theta_0)$, given by

$$I_2(\theta_0) = \frac{\dot{p}_\Delta^2}{p_\Delta}\left[\frac{\mu^+}{(\mu^+ - p_\Delta)} + \frac{\mu^-}{(\mu^- - p_\Delta)}\right](\theta_0).$$

In the same way as for the first result, Theorem 1 of Florens-Zmirou (1984) proves that

$$\frac{1}{\sqrt{n}}\,\dot{L}_n^{(2)}(\theta_0) \xrightarrow{\mathcal{L}(P_{\theta_0})} \mathrm{N}\left(0, \frac{V_\Delta^{(2)}(\theta_0)}{\Delta}\right)$$

with $V_\Delta^{(2)} = (\Pi_\Delta \otimes \mu)[\Delta f^2 + 2fF_\Delta]$ and $F_\Delta = \Delta \sum_{n=1}^{\infty} \Pi_{n\Delta} f$, where

$$f(x, y) = \frac{\dot{p}_\Delta}{p_\Delta}\mathbf{1}_{(y>0, x\leq 0)} - \frac{\dot{p}_\Delta}{\mu^+ - p_\Delta}\mathbf{1}_{(y>0,x>0)} + \frac{\dot{p}_\Delta}{p_\Delta}\mathbf{1}_{(y<0,x\geq 0)} - \frac{\dot{p}_\Delta}{\mu^- - p_\Delta}\mathbf{1}_{(y<0,x<0)}.$$

Then, using 3.3.15 of Dacunha-Castelle and Duflo (1986) and (H6), we obtain

$$\sqrt{n\Delta}\,(\hat{\theta}_n^{(2)} - \theta_0) \xrightarrow{\mathcal{L}(P_{\theta_0})} \mathrm{N}\left(0, \frac{V_\Delta^{(2)}(\theta_0)}{[I_2(\theta_0)]^2}\right).$$

It remains to get Δ-estimates of $V_\Delta^{(2)}$ and I_2.

Lemma 1. *Under hypothesis* (H), *if* $\mu^+(\theta)$ *and* $\mu^-(\theta)$ *are constant*:
(a) $q_{\theta,\Delta}=p_{\theta,\Delta}=\sqrt{\Delta/2\pi}\;\mu_\theta(0)+\mathrm{O}(\Delta)$ *uniformly in* θ,
(b) $\dot q_{\theta,\Delta}=\dot p_{\theta,\Delta}=\sqrt{\Delta/2\pi}\;\dot\mu_\theta(0)+\mathrm{O}(\Delta)$ *uniformly in* θ.

Proof. (a) We have

$$\pi_\Delta(x,y)=\frac{1}{\sqrt{2\pi\Delta}}\exp-\frac{(y-x)^2}{2\Delta}\left[\frac{\mu(y)}{\mu(x)}\right]^{1/2}E[\exp-\Delta H(x,y)]$$

where

$$H(x,y)=\int_0^1(b^2+b')((1-v)x+vy+\sqrt{\Delta}\;B_v)\,\mathrm{d}v$$

and B is a standard Brownian bridge (Dacunha-Castelle and Florens-Zmirou, 198(
Let $u=(y-x)/\sqrt{\Delta}$ and

$$\varphi_\Delta(u)=\frac{1}{u\sqrt{\Delta}}\int_0^{-u\sqrt{\Delta}}[\mu(x)\mu(x+u\sqrt{\Delta})]^{1/2}\times E[\exp-\Delta H(x,x+u\sqrt{\Delta})]\,\mathrm{d}x;$$

then

$$p_\Delta=\int_0^\infty\mu(x)\,\mathrm{d}x\int_{-\infty}^0\pi_\Delta(x,y)\,\mathrm{d}y=\sqrt{\Delta/2\pi}\int_{-\infty}^0u\,\mathrm{e}^{-u^2/2}\varphi_\Delta(u)\,\mathrm{d}u.$$

- If $u\leqslant 0$, set $J(u,\Delta)=\{x;|x|\leqslant-u\sqrt{\Delta}\}$; we have

$$\sup_{0\leqslant x\leqslant-u\sqrt{\Delta}}|(\mu(x)\mu(x+\sqrt{\Delta}\;u))^{1/2}-\mu(0)|\leqslant\sup_{x\in J(u,\Delta)}|\mu(x)-\mu(0)|.$$

One has

$$\begin{aligned}\sup_{x\in J(u,\Delta)}|\mu(x)-\mu(0)|&\leqslant|u|\sqrt{\Delta}\sup_{x\in J(u,\Delta)}|\mu'(x)|\\&=|u|\sqrt{\Delta}\sup_{x\in J(u,\Delta)}2|b(x)\mu(x)|\\&\leqslant 2|u|\sqrt{\Delta}\;\|\mu\|_\infty(C+\rho|u|\sqrt{\Delta})\quad\text{by (H2)}\\&\leqslant C_1|u|\sqrt{\Delta}+C_2u^2\Delta,\end{aligned}$$

where here (and later) C_i denote constants independent of Δ (and of θ).
- On the other hand, by hypothesis (H2), $H\geqslant-K$, and hence one has

$$|E\,\mathrm{e}^{-\Delta H}-1|\leqslant\Delta E|H|\,\mathrm{e}^{\Delta K}.$$

Since

$$|H(x,x+u\sqrt{\Delta})|=\int_0^1|b^2+b'|(x+vu\sqrt{\Delta}+\sqrt{\Delta}\;B_v)\,\mathrm{d}v$$

and from (H2) it is clear that

$$\sup_{x\in J(u,\Delta)} E|H(x, x+u\sqrt{\Delta}\,| \leq C_3 + C_4 u^2 \Delta,$$

then $\varphi_\Delta(u) = \mu(0) + S_2(u, \Delta)$ with $S_2(u, \Delta) \leq C_5|u|\sqrt{\Delta} + C_6 u^2 \Delta$, u^2 is integrable with respect to $\mathrm{e}^{-u^2/2}\,\mathrm{d}u$ and therefore the first statement of lemma is proved.

(b) We need an analogous result for $\dot{p}_\Delta$. Now $\dot{\varphi}_\Delta$ is sum of two terms,

$$\dot{\varphi}_\Delta^{(1)} = \frac{1}{u\sqrt{\Delta}} \int_0^{-u\sqrt{\Delta}} \frac{1}{2}\left(\frac{\dot{\mu}(x)}{\mu(x)} + \frac{\dot{\mu}(x+\sqrt{\Delta}\,u)}{\mu(x+\sqrt{\Delta}\,u)}\right)(\mu(x)\mu(x+u\sqrt{\Delta}))^{1/2}$$
$$\cdot\, E[\exp - \Delta H(x, x+u\sqrt{\Delta})]\,\mathrm{d}x,$$

and

$$\dot{\varphi}_\Delta^{(2)} = \frac{1}{u\sqrt{\Delta}} \int_0^{-\sqrt{\Delta}u} (\mu(x)\mu(x+\sqrt{\Delta}\,u))^{1/2}$$
$$\cdot\, E[-\Delta \dot{H}(x, x+u\sqrt{\Delta}) \exp - \Delta H(x, x+u\sqrt{\Delta})]\,\mathrm{d}x.$$

Let us study the first term:

$$\sup_{x\in J(u,\Delta)} \left|\frac{\dot{\mu}(x)}{\mu(x)} - \frac{\dot{\mu}(0)}{\mu(0)}\right| = 2 \sup_{x\in J(u,\Delta)} \left|\int_0^x \dot{b}(v)\,\mathrm{d}v\right|$$
$$\leq 2\lambda\sqrt{\Delta}\,|u|\,\mathrm{e}^{\lambda\sqrt{\Delta}|u|} \quad \text{by (H4).}$$

Using (a) we get

$$|\dot{\varphi}_\Delta^{(1)} - \dot{\mu}(0)| \leq \text{const.}\sqrt{\Delta}\,|u|\,\mathrm{e}^{\lambda\sqrt{\Delta}|u|}.$$

Next, $\dot{\varphi}^{(2)}$ has an expression analogous to φ_Δ and then using (H4) and (a) we get

$$\dot{\varphi}_\Delta^{(2)}(u) \leq \text{const.}\,\Delta\,\mathrm{e}^{\lambda\sqrt{\Delta}|u|}.$$

Because of the integrability with respect to $\mathrm{e}^{u^2/2}\,\mathrm{d}u$, we get $\dot{p}_\Delta = \sqrt{\Delta/2\pi}\,\dot{\mu}(0) + \mathrm{O}(\Delta)$ and the lemma is proved.

Now, we are able to compute $V_\Delta^{(2)}$. Using Lemma 1 one has

$$I_2 = \sqrt{\frac{2}{\pi}}\sqrt{\Delta}\,\frac{[\dot{\mu}(0)]^2}{\mu(0)} + \mathrm{o}(\sqrt{\Delta}) = \sqrt{\frac{2}{\pi}}\,I_0\sqrt{\Delta} + \mathrm{o}(\sqrt{\Delta}).$$

It is not possible to get an equivalent of $V_\Delta^{(2)}$;

$$V_\Delta^{(2)} = \mu \otimes \Pi_\Delta(f^2\Delta + 2F_\Delta f)$$
$$= \Delta \int f^2(x,y)\pi_\Delta(x,y)\mu(x)\,\mathrm{d}x\,\mathrm{d}y$$
$$+ 2\int f(x,y)F_\Delta(y)\pi_\Delta(x,y)\mu(x)\,\mathrm{d}x\,\mathrm{d}y.$$

If $\| \ \|$ is the norm in $L^2(\mu \otimes \Pi_\Delta)$, one has $\|f\|^2 = I_2$ and $\|F_\Delta\| \leqslant \Delta\|f\|/(1-\mathrm{e}^{-s\Delta})$ where $s>0$ is the lower strictly positive bound of the spectrum of the diffusion. Then for $\Delta \leqslant \Delta(\varepsilon)$,

$$V_\Delta^{(2)} \leqslant \frac{2}{s}\|f\|^2 + \Delta\|f\|^2.$$

We get finally

$$V_\Delta^{(2)} \leqslant \sqrt{\frac{2}{\pi}}\, I_0 \frac{2}{s}\sqrt{\Delta} + R(\Delta) \quad \text{with } R(\Delta) = \mathrm{O}(\Delta)$$

and the asymptotic variance of $\sqrt{n}(\hat{\theta}_n^{(2)} - \theta_0)$ is equal to $\sigma_2^2(\theta_0, \Delta)/\Delta^{3/2}$ and if $\Delta < \Delta(\varepsilon)$,

$$\sigma_2^2(\theta_0, \Delta) \leqslant [(1+\varepsilon)]\frac{\sqrt{2\pi}}{I_0 s}.$$

We studied in this first part the behaviour of the number of crossings of $\bar{X}_\Delta$, when the interval of observation goes to infinity and Δ is fixed. In order to study, in a subsequent paper, the same problem for a discretization $(X_{k\Delta})$ with $k = 1, \ldots, n$, $\Delta_n \to 0$, $n\Delta_n \to \infty$, we study now the convergence of the number of crossings when Δ goes to zero and the interval of observation is fixed.

3. Zero-crossings of X and zero local time of X

Let $L(t)$ be the zero local time of the diffusion (see Rice (1944) for references on local time), that we define here by

$$L(t) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_0^t \mathbf{1}_{(-\delta, +\delta)}(X_s)\, \mathrm{d}s.$$

If $I = [a, b]$, $0 < a < b$, $L(I) = L(b) - L(a)$.

Let $N_\Delta(I)$ be the number of zero-crossings of $\bar{X}_\Delta$ between I.

Theorem 2. *If hypotheses* (H.1) *to* (H.4) *are satisfied* (*for a fixed* θ), *we have*

$$\sqrt{\frac{\pi}{2}}\sqrt{\Delta}\, N_\Delta(I) \xrightarrow[\Delta \to 0]{L^2(P)} L(I).$$

We shall use the framework of Azema and Yor (1978) and Azais (1990).

We need the following Lemma 2. First let us denote by $f_{X,Y,Z,\ldots}$ a continuous version of the density of random variables $X, Y, Z, \ldots$ with respect to Lebesgue measure (when its exits) and let

$$f_{t_1,t_2} = f_{X(t_1),X(t_2)}$$

and if $t_i \neq k\Delta$, $i = 1, 2$,

$$f_{t_1,t_2}^{\Delta,\Delta} = f_{\bar{X}_\Delta(t_1),\bar{X}_\Delta(t_2),\sqrt{\Delta}(\bar{X}_\Delta)'(t_1),\sqrt{\Delta}(\bar{X}_\Delta)'(t_2)} \quad \text{and} \quad f_{t_1,t_2}^{\Delta} = f_{X(t_1),\bar{X}_\Delta(t_2),\sqrt{\Delta}(\bar{X}_\Delta)'(t_2)}.$$

Lemma 2. *Under hypotheses* (H.1) *to* (H.4) (*for a fixed* θ), *we have*

$$(1)\qquad f_{t_1,t_2}(0,0) \leq \frac{C_1}{\sqrt{t_2-t_1}}, \quad t_1<t_2,$$

$$(2a)\qquad f^{\Delta,\Delta}_{t_1,t_2}(x_1,x_2,z_1,z_2) \leq \frac{C_2}{\sqrt{t_2-t_1}} \exp -\frac{z_1^2+z_2^2}{2}, \quad t_1<t_2,$$

$$(2b)\qquad \lim_{\Delta\to 0} f^{\Delta,\Delta}_{t_1,t_2}(0,0,z_1,z_2) = f_{t_1,t_2}(0,0)\frac{1}{2\pi}\exp -\frac{z_1^2+z_2^2}{2}.$$

$$(3a)\qquad f^{\Delta}_{t_1,t_2}(x_1,x_2,z) \leq \frac{C_3}{\sqrt{t_2-t_1}} e^{-z^2/2}, \quad t_1<t_2,$$

$$(3b)\qquad \lim_{\Delta\to 0} f^{\Delta}_{t_1,t_2}(0,0,z) = f_{t_1,t_2}(0,0)\frac{e^{-z^2/2}}{\sqrt{2\pi}},$$

with C_1, C_2, C_3 *suitable constants.*

Suppose that we have proved Lemma 2. To prove Theorem 2 from Lemma 2, we proceed as follows. Let

$$\zeta^{\Delta}(I) = \sqrt{\pi\Delta/2}\, N_{\Delta}(I) \quad \text{and} \quad \eta^{\delta}(I) = \frac{1}{2\delta}\int_I \mathbf{1}_{]-\delta,\delta[}(X_s)\,ds.$$

It is known (Rice, 1944) that $\eta^{\delta}(I)$ goes to $L(I)$ (in L^2-norm) when δ goes to zero. Then we have to prove that

$$\lim_{(\Delta,\delta)\to 0} E[\zeta^{\Delta}(I)-\eta^{\delta}(I)]^2 = 0.$$

Let $n = |I|/\Delta + 1$ and $(I_1,\ldots,I_n)$ be a partition of I in n intervals of length Δ (except the last one).

Let $\zeta_i^{\Delta} = \zeta^{\Delta}(I_i)$ and $\eta_i^{\delta} = \eta^{\delta}(I_i)$. Set $J_{\Delta} = \{(i,j); |j-i|>1\}$ and $J_{\Delta}^{c} = \{(i,j); |j-i|\leq 1\}$.

First note that since the diffusion is stationary, if $(i,j)\in J_{\Delta}^{c}$,

$$\sum_{(i,j)\in J_{\Delta}^{c}} E[\zeta_i^{\Delta}\zeta_j^{\Delta}] \leq 3nE[\zeta_i^{\Delta}]^2$$

with $E[\zeta_i^{\Delta}]^2 = \frac{1}{2}\pi\Delta(p_{\Delta}+q_{\Delta}) = O(\Delta^{3/2})$ from Lemma 1. Hence

$$\sum_{(i,j)\in J_{\Delta}^{c}} E[\zeta_i^{\Delta}\zeta_j^{\Delta}] \to 0 \quad \text{as } \Delta\to 0.$$

It remains only to prove that if $(i,j)\in J_{\Delta}$,

$$\lim_{\Delta\to 0} E\left[\sum_{(i,j)\in J_{\Delta}} \zeta_i^{\Delta}\zeta_j^{\Delta}\right] = \lim_{\delta\to 0} E[\eta^{\delta}(I)]^2$$

$$= \lim_{(\Delta,\delta)\to 0} E\left[\sum_{(i,j)\in J_{\Delta}} \zeta_j^{\Delta}\eta_i^{\delta}\right].$$

Let us remark that the trajectories of $\bar{X}_\Delta$ are absolutely continuous and its derivative $\bar{X}'_\Delta(t)$ is defined except at the points $k\Delta$. The process $\bar{X}_\Delta$ has finitely many local extrema on each bounded interval then we can apply Kac's version of Rice's formulas (Azais, 1990; see also Rice, 1944).

So, for any $(i,j)\in J_\Delta$,

$$E[\zeta_i^\Delta\zeta_j^\Delta]=\tfrac{1}{2}\pi\int_{I_i\times I_j}\mathrm{d}t_1\,\mathrm{d}t_2\int_{\mathbb{R}^2}|z_1z_2|f_{t_1,t_2}^{\Delta,\Delta}(0,0,z_1,z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2,$$

where (from Lemma 2),

$$f_{t_1,t_2}^{\Delta,\Delta}(0,0,z_1,z_2)\leq\frac{C_1}{\sqrt{t_2-t_1}}\exp-\frac{z_1^2+z_2^2}{2}$$

and

$$\lim_{\Delta\to0}f_{t_1,t_2}^{\Delta,\Delta}(0,0,z_1,z_2)=\frac{1}{2\pi}\exp-\frac{z_1^2+z_2^2}{2}f_{t_1,t_2}(0,0).$$

So, Lebesgue's theorem implies that

$$\lim_{\Delta\to0}\sum_{(i,j)\in J_\Delta}E[\zeta_i^\Delta\zeta_j^\Delta]=\int_{I\times I}f_{t_1,t_2}(0,0)\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

On the other hand,

$$E[\eta^\delta(I)]^2=\frac{1}{4\delta^2}\int_{I\times I}\mathrm{d}t_1\,\mathrm{d}t_2\int_{\mathbb{R}^2}f_{t_1,t_2}(u,v)\mathbf{1}_{]-\delta,+\delta]^2}(u,v)\,\mathrm{d}u\,\mathrm{d}v,$$

f_{t_1,t_2} is continuous in (u,v) and by Lemma 2,

$$f_{t_1,t_2}(u,v)\leq\frac{C_1}{\sqrt{t_2-t_1}}.$$

So by Lebesgue's theorem

$$\lim_{\delta\to0}E[\eta^\delta(I)]^2=\int_{I\times I}f_{t_1,t_2}(0,0)\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

We now want to compute

$$\sum_{(i,j)\in J_\Delta^c}E[\zeta_i^\Delta\eta_j^\delta]\quad\text{and}\quad\sum_{(i,j)\in J_\Delta}E[\zeta_i^\Delta\eta_j^\delta].$$

First, note that if $(i,j)\in J_\Delta^c$, then

$$\sum_{(i,j)\in J_\Delta^c}E[\eta_j^\delta\zeta_i^\Delta]\leq\sum_{(i,j)\in J_\Delta^c}[E[\eta_j^\delta]^2E[\zeta_i^\Delta]^2]^{1/2},$$

where

$$E[\eta_j^\delta]^2\leq\int_{I_i\times I_j}\frac{C}{\sqrt{t_2-t_1}}\mathrm{d}t_2\,\mathrm{d}t_1\leq2C\Delta^{3/2}$$

and $E[\zeta_i^\Delta]^2 = O(\Delta^{3/2})$ by Lemma 1, so we get

$$\lim_{(\Delta,\delta)\to 0} E \sum_{(i,j)\in J_\Delta^c} (\eta_j^\delta \zeta_i^\Delta) = 0.$$

Next note that for $(i,j)\in J_\Delta$ we get by Rice's formulae,

$$E[\eta_j^\delta \zeta_i^\Delta] = \sqrt{\tfrac{1}{2}\pi}\,\frac{1}{2\delta}\int_{I_i\times I_j} \mathrm{d}t_1\,\mathrm{d}t_2 \int_{-\delta}^{+\delta} \mathrm{d}x \int_{\mathbb{R}} |z_2| f_{t_1,t_2}^\Delta(x,0,z_2)\,\mathrm{d}z_2.$$

By Lemma 2 we know that

$$f_{t_1,t_2}^\Delta(x,0,z_2) \le \frac{C}{\sqrt{t_2-t_1}}\,\mathrm{e}^{-z^2/2},$$

and

$$\frac{1}{2\delta}\int_{-\delta}^{+\delta} \mathrm{d}x \int_{\mathbb{R}} |z_2| f_{t_1,t_2}^\Delta(x,0,z_2)\,\mathrm{d}z_2$$
$$\to \sqrt{2/\pi}\, f_{t_1,t_2}(0,0) \quad \text{as } (\delta,\Delta)\to 0.$$

Finally, using Lebesgue's theorem, this proves that as $(\delta,\Delta)\to 0$,

$$\sum_{(i,j)\in J_\Delta} E[\eta_i^\delta \zeta_j^\Delta] \to \int_{I\times I} f_{t_1,t_2}(0,0)\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

Our theorem is an immediate consequence of the equality

$$E[\zeta^\Delta(I)-\eta^\delta(I)]^2 = E[\zeta^\Delta(I)]^2 - 2E[\zeta^\Delta(I)\eta^\delta(I)] + E[\eta^\delta(I)]^2.$$

To complete this section it remains only to prove Lemma 2.

Proof of Lemma 2. We have

$$f_{t_1,t_2}(x,y) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{t_2-t_1}}\exp -\frac{(y-x)^2}{2(t_2-t_1)}[\mu(x)\mu(y)]^{1/2}\bar{H}(x,y,t_2-t_1)$$

with $\bar{H}(x,y,t) = E\exp -tH(x,y)$ (H was defined in part 1).

So

$$f_{t_1,t_2}(x,y) \le \frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{t_2-t_1}}\,\mathrm{e}^{n\Delta K}\|\mu\|_\infty$$

by (H.2), so (1) is proved with $C_1 = \sqrt{\tfrac{1}{2}\pi}\,\mathrm{e}^{n\Delta K}\|\mu\|_\infty$.

Let us now prove (2). Let $t_i = h_i\Delta + u_i\Delta$, $i=1,2$, with $h_i = [t_i/\Delta]$, $0\le u_i<1$, and suppose $t_1<t_2$ such that $h_2-h_1 = k>1$. Then we have

$$\bar{X}_\Delta(t_i) = (1-u_i)X(h_i\Delta) + u_iX((h_i+1)\Delta).$$

Thus, the density $f_{t_1,t_2}^{\Delta,\Delta}$ can be written as

$$\begin{aligned} f_{t_1,t_2}^{\Delta,\Delta}(x_1,x_2,z_1,z_2) = {} & \Delta\mu(x_1-u_1\sqrt{\Delta}\,z_1)\pi_\Delta(x_1-u_1\sqrt{\Delta}\,z_1, x_1+(1-u_1)\sqrt{\Delta}\,z_1)\\ & \cdot \pi_{(k-1)\Delta}(x_1+(1-u_1)\sqrt{\Delta}\,z_1, x_2-u_2\sqrt{\Delta}\,z_2)\\ & \cdot \pi_\Delta(x_2-u_2\sqrt{\Delta}\,z_2, x_2+(1-u_2)\sqrt{\Delta}\,z_2).\end{aligned}$$

and using the expression of $\pi_t(x, y)$, we obtain

$$= \frac{1}{2\pi} \exp - \frac{z_1^2 + z_2^2}{2} \frac{1}{\sqrt{2\pi(k-1)\Delta}} \exp - \frac{(x_2 - u_2\sqrt{\Delta}\, z_2 - x_1 - (1-u_1)\sqrt{\Delta}\, z_1)^2}{2(k-1)\Delta}$$

$$\cdot [\mu(x_1 - u_1\sqrt{\Delta}\, z_1)\mu(x_2 + (1-u_2)\sqrt{\Delta}\, z_2)]^{1/2}$$

$$\cdot \bar{H}(x_1 - u_1\sqrt{\Delta}\, z_1, x_1 + (1-u_1)\sqrt{\Delta}\, z_1, \Delta)$$

$$\cdot \bar{H}(x_1 + (1-u_1)\sqrt{\Delta}\, z_1, x_2 - u_2\sqrt{\Delta}\, z_2, (k-1)\Delta)$$

$$\cdot \bar{H}(x_2 - u_2\sqrt{\Delta}\, z_2, x_2 + (1-u_2)\sqrt{\Delta}\, z_2, \Delta).$$

We can apply to $\bar{H}$ the dominated convergence theorem in order to prove that

$$\lim_{\Delta \to 0} \bar{H}(x_1 - u_1\sqrt{\Delta}\, z_1, x_1 + (1-u_1)\sqrt{\Delta}\, z_1, \Delta) = 1.$$

Now $\exp -tH(x, y) \leq \exp tK$ by hypothesis (H.2), and using the same trick for the remaining two other terms in $\bar{H}$ one has

$$\lim_{\Delta \to 0} f_{t_1, t_2}^{\Delta, \Delta} = \frac{1}{2\pi} \exp - \frac{z_1^2 + z_2^2}{2} \mu(x_1) \left[\frac{\mu(x_2)}{\mu(x_1)} \right]^{1/2}$$

$$\cdot \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp -(x_2 - x_1)^2 / 2(t_2 - t_1) H(x_1, x_2, t_2 - t_1).$$

Further we have, for $\Delta < \Delta_0$,

$$f_{t_1, t_2}^{\Delta, \Delta}(x_1, x_2, z_1, z_2) \leq \frac{1}{2\pi} \exp - \frac{z_1^2 + z_2^2}{2} \|\mu\|_\infty \frac{2}{\sqrt{2\pi(t_2 - t_1)}} e^{2[\Delta_0 + (t_2 - t_1)]K}.$$

The dominated convergence theorem implies the convergence (2b) of the lemma. The third part can be proved, exactly in the same way as the second one. So Lemma 2 and Theorem 2 are proved.

References

J. Azema and M. Yor, Temps local, Astérisque 52-53 (1978) 3-16.

J.M. Azais, Conditions for convergence of number of crossings to the local time (1990), to appear.

J.M. Azais and D. Florens-Zmirou, Approximation du temps local des processus gaussiens stationnaires par régularisation des trajectoires, Probab. Theory Rel. Fields 76 (1987) 121-132.

S.M. Berman, Local nondeterminism and local time of Gaussian processes, Indiana Univ. Math. J. 23 (1973) 69-74.

D. Dacunha-Castelle and M. Duflo, Probability and Statistics (Springer, New York, 1986).

D. Dacunha-Castelle and D. Florens-Zmirou, Estimation of the coefficients of a diffusion from discrete observations, Stochastics 19 (1986) 263-284.

D. Florens-Zmirou, Approximate discrete time schemes for statistics on diffusion processes, Stochastics (1990), to appear.

D. Florens-Zmirou, Statistique des diffusions et temps local, Ann. Inst. Henri Poincaré 24 (1988) 99-130.

D. Florens-Zmirou, Théorème de limite centrale pour une diffusion et sa discrétisée, C. R. Acad. Sci. Paris Ser. (1) 19 (1984) 995-998.

M. Kac, On the average number of real roots of a random algebraic equation, Bull. Amer. Soc. 44 (1943) 314–320.

M.B. Marcus, Level crossings of a stochastic process with absolutely continuous sample paths, Ann. Probab. 5 (1977) 52–71.

S.O. Rice, The mathematical analysis of random noise, Bell. Systems Tech. J. 23 (1944) 282–332.

M. Wschebor, Surfaces aléatoires, Mesures géométriques des ensembles de niveau, Lecture Notes in Math. No. 1147 (Springer, New York, 1985).