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Extremes and clustering of nonstationary max-AR(1) sequences

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Abstract

We consider general nonstationary max-autoregressive sequences $\{X_i, i \geqslant 1\}$, with $X_i = Z_i \max(X_{i-1}, Y_i)$ where $\{Y_i, i \geqslant 1\}$ is a sequence of i.i.d. random variables and $\{Z_i, i \geqslant 1\}$ is a sequence of independent random variables $(0 \leqslant Z_i \leqslant 1)$, independent of $\{Y_i\}$. We deal with the limit law of extreme values $M_n = \max\{X_i, i \leqslant n\}$ (as $n \to \infty$) and evaluate the extremal index for the case where the marginal distribution of Y_i is regularly varying at ∞. The limit of the point process of exceedances of a boundary u_n by X_i, $i \leqslant n$, is derived (as $n \to \infty$) by analysing the convergence of the cluster distribution and of the intensity measure.

Keywords: Nonstationary; Extreme values; Point processes; Regular variation; Weak limits; Max-autoregressive sequences

1. Introduction

Let $\{Y_i, i \geqslant 1\}$ and $\{Z_i, i \geqslant 1\}$ be independent sequences of independent random variables, where Y_i are identically distributed with common distribution $G(\cdot)$ and Z_i with distribution $F_i(\cdot)$. We assume that $P\{0 \leqslant Z_i \leqslant 1\} = 1$ for all i's. This is essential for the following analysis of $M_n = \max\{X_i, 0 \leqslant i \leqslant n\}$; otherwise, a rather different behaviour of the extremes of $\{X_i, i \geqslant 0\}$ could be observed. We define the max-AR(1) sequence $\{X_i, i \geqslant 1\}$ by

$$X_i = \begin{cases} X_0 & \text{if } i = 0, \\ Z_i \max\{X_{i-1}, Y_i\} & \text{if } i \geqslant 1, \end{cases} \tag{1}$$

with any random variable X_0. Let $H_i(\cdot)$ denote the marginal distribution of X_i.

Extremal properties of a special case of our model where Z_i is a constant less than one is studied by Alpuim (1989). Alpuim and Athayde (1990) characterized the class of stationary distributions arising from the max-AR(1) sequence as defined in (1),

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especially for Z_i having a beta distribution. The solar thermal energy model

$$X_n = \max(c_2 X_{n-1}, c_1 c_2 X_{n-1} + Y_n),$$

with $c_1 \in [0,1]$, $c_2 \in (0,1)$, introduced by Haslett (1979) and further investigated by Daley and Haslett (1982), Hooghiemstra and Keane (1985) and Hooghiemstra and Scheffer (1986) reduces to a special case of our model if $c_1 = 0$.

In the following we assume that for some $\alpha > 0$,

$$G \in D(\Phi_\alpha), \tag{2}$$

i.e. G belongs to the domain of attraction of the extreme value distribution Φ_α, which is equivalent to

$$G^n(a_n x) \to \exp(-x^{-\alpha}) = \Phi_\alpha(x), \quad \forall x > 0$$

for some sequence of normalizing constants $\{a_n, n \geqslant 0\}$. Since $a_n \to \infty$, $P\{X_0 > a_n x\} \to 0$, which means that X_0 is asymptotically negligible. Moreover, we assume that

$$\sup_{i \geqslant 1} E(Z_i^\beta) < 1 \quad \text{for some } \beta < \alpha \tag{3}$$

implying

$$\sup_{i \geqslant 1} E(Z_i^\alpha) = s < 1. \tag{4}$$

In Section 2 we analyse the behaviour of the extremes of this max-AR(1) sequence. Assuming $Z_i = C < 1$ with probability 1 and that $\{X_i\}$ is a (strictly) stationary sequence, Alpuim (1989) showed that the exceedances do cluster if (2) holds. In this case, the cluster sizes are geometrically distributed with mean value $1/\theta$, where $\theta = 1 - C^\alpha$ is the so-called extremal index (see O'Brien, 1974, 1987; Leadbetter, 1983 or Leadbetter et al., 1983).

We show that this property still holds true for the more general case with random Z_i and nonstationary sequence X_i. Surprisingly, θ depends only on the moments $E(Z_i^\alpha)$ and not on the explicit form of the distributions of Z_i.

In Section 3 we deal with N_n, the point process of exceedances and prove that under additional conditions on the moments $E(Z_i^\alpha)$ N_n converges to a compound Poisson process, where the compounding distribution is geometric.

Instead of (2) we might have assumed that $G \in D(\Psi_\alpha)$ or $G \in D(\Lambda)$, where $\Psi_\alpha(x) = \exp(-(-x)^\alpha)$, for $x \leqslant 0$, and $\Lambda(x) = \exp(-\exp(-x))$. Under the assumption $Z_i = C < 1$ and that $\{X_i\}$ is stationary, the exceedances do not cluster in these cases assuming $x_0 = \sup(x\colon G(x) < 1\} > 0$ (Alpuim, 1989). The same holds true for our general model under slightly different conditions than introduced so far. Assuming that for all i the distributions $\tilde{G}_i$ of $Z_i Y_i$ belong to $D(\Psi_\alpha)$ or $D(\Lambda)$, we have to consider the following cases. Let $x_{i,0}$ denote the right endpoint of $\tilde{G}_i$. Then for the cases $P\{Z_i < 1\} = 1$ for all i, the exceedances do not cluster if

(i) $\tilde{G}_i \in D(\Psi_\alpha)$ and $x_{i,0} > 0$ for all i, $X_0 < \inf_i x_{i,0}$ a.s., or

(ii) $\tilde{G}_i \in D(\Lambda)$ for all i and $X_0 < \inf_i x_{i,0}$ a.s..

If in (i) $x_{i,0} \leqslant 0$ for all i, then we must have $P\{Z_i > 1\} = 1$ for all i in order to prevent clustering of exceedances. Note that here $P\{Z_i > 1\} = 1$, is a necessary condition since for example $P\{Z_i = 1\} = p > 0$ for all i would lead to clustering of exceedances.

Although the results presented in Sections 2 and 3 are derived under the assumption (2), they are still applicable for instance for the following related models:

(1) Let $G \in D(\Psi_\alpha)$ for some $\alpha > 0$ with $x_0 = 0$ and that $\{X_i\}$ is as in (1). Let $Y_i^* = -1/Y_i$, $Z_i^* = 1/Z_i$ where now $P\{Z_i \geqslant 1\} = 1$ and $\sup_i E(Z_i^{-\beta}) < 1$ for some $\beta > \alpha$ and all i, and $X_0^* = -1/X_0$. Then $X_i^* = -1/X_i$ is as in (1) with $Y_i^* \sim G^* \in D(\Phi_\alpha)$.

(2) Let $G \in D(\Lambda)$ with $x_0 = \infty$ and $X_i = Z_i + \max(X_{i-1}, Y_i)$. Assume that G is such that $G^n(a_n x + b_n) \xrightarrow{w} \Lambda(x)$ with $a_n = 1/\alpha$ and some b_n. Then define $Y_i^* = \exp(Y_i)$, $Z_i^* = \exp(Z_i)$ and $X_i^* = \exp(X_i)$, where $Z_i \leqslant 0$ with probability 1. This transforms the sequence $\{X_i\}$ to a sequence $\{X_i^*\}$ which satisfies the model (1) with $Y_i^* \sim G^* \in D(\Phi_\alpha)$. Thus clustering occurs also in this different model $\{X_i\}$.

Finally, note that the assumption $P\{0 \leqslant Z_i \leqslant 1\} = 1$ in model (1) is an important one, since allowing $Z_i > 1$ would result in a rather different pattern of the extremal behaviour of $\{X_i, i \geqslant 0\}$ such as rapid variation of the sequence around the threshold. In this case a different approach than the one introduced here is needed.

2. Limiting distribution for the maximum

We discuss now the limiting behaviour of the maximum

$$M_n = \max\{X_i, 0 \leqslant i \leqslant n\}.$$

We use the same normalization a_n as for the sequence Y_i, to show that also M_n/a_n has asymptotically a Fréchet-distribution.

Lemma 1. *Assume condition* (2) *holds. Then for any $x > 0$ and $j \geqslant 0$:*

$$nP\{Z_i \cdots Z_{i-j} Y_{i-j} > a_n x\} \to x^{-\alpha} E(Z_i^\alpha) \cdots E(Z_{i-j}^\alpha)$$

as $n \to \infty$, uniformly in i.

Proof. The distribution G of the Y_i is regularly varying with exponent $-\alpha$ (see deHaan, 1970), thus

$$n[1 - G(a_n y)] \to y^{-\alpha}$$

as $n \to \infty$, uniformly for all $y \geqslant y_0$ with any $y_0 > 0$. Hence for any $x > 0$ and v, $0 \leqslant v \leqslant 1$, we have obviously $x/v \geqslant x$ and thus

$$n[1 - G(a_n x/v)] \to x^{-\alpha} v^\alpha$$

uniformly for all v. For any $\varepsilon > 0$ there exists n_0 such that for all $n \geqslant n_0$

$$|nP\{Z_i \cdots Z_{i-j} Y_{i-j} > a_n x\} - x^{-\alpha} E(Z_i^\alpha) \cdots E(Z_{i-j}^\alpha)|$$

$$\leqslant \int \cdots \int |n[1 - G(a_n x/(z_i \cdots z_{i-j}))] - x^{-\alpha} z_i^\alpha \cdots z_{i-j}^\alpha| \, \mathrm{d}F_i(z_i) \cdots \mathrm{d}F_{i-j}(z_{i-j})$$

$$\leqslant \int \cdots \int \varepsilon \, \mathrm{d}F_i(z_i) \cdots \mathrm{d}F_{i-j}(z_{i-j}) = \varepsilon$$

by using the uniform bound for $v = z_i \cdots z_{i-j} \in [0, 1]$. □

In the following we assume that for some $j \geqslant 0$,

$$(1/n) \sum_{i=j+1}^{n} E(Z_i^\alpha) \cdots E(Z_{i-j}^\alpha) \to c_j) \quad \text{as } n \to \infty. \tag{5}$$

Theorem 1. *Suppose that* (2) *and* (5) *for $j = 0$ hold. Then*

$$P\{M_n \leqslant a_n x\} \to \exp(-c_0 x^{-\alpha})$$

as $n \to \infty$ for any $x > 0$, where $\{a_n\}$ are the normalizing constants in (2).

Proof. Note that the maximum M_n can be written as

$$M_n = \max\{X_0, X_1, \ldots, X_n\} = \max\{X_0, Z_1 Y_1, Z_2 Y_2, \ldots, Z_n Y_n\}$$

since the Z_i are concentrated on $[0, 1]$. Consequently,

$$P\{M_n \leqslant a_n x\} = P\{X_0 \leqslant a_n x\} \prod_{i=1}^{n} P\{Z_i Y_i \leqslant a_n x\}.$$

The convergence of this product to $\exp(-c_0 x^{-\alpha})$ is equivalent to

$$\sum_{i=1}^{n} P\{Z_i Y_i > a_n x\} \to c_0 x^{-\alpha}.$$

Since by Lemma 1 each term of the sum can be approximzated uniformly by $x^{-\alpha} E(Z_i^\alpha)/n$, we get immediately that

$$\lim_{n\to\infty} \sum_{i \leqslant n} P\{Z_i Y_i > a_n x\} = x^{-\alpha} \lim_{n\to\infty} (1/n) \sum_{i \leqslant n} E(Z_i^\alpha) = x^{-\alpha} c_0$$

by (5). □

For a stationary sequence it is well-known that if $n[1 - F(u_n(\tau))] \to \tau$ for some normalization $u_n(\tau)$ and if weak mixing conditions hold, then $P\{M_n \leqslant u_n(\tau)\}$ converges to $\exp(-\theta\tau)\}$, where θ is a constant ($\leqslant 1$) not depending on τ. θ is called the extremal index and is related to the clustering of exceedances of the sequence. If $\theta = 1$ then the exceedances do not cluster, i.e. the cluster sizes are asymptotically equal to 1 with probability 1. For the max-AR(1) sequence this would be the case if $G \in D(\Lambda)$ or $G \in D(\Psi_\alpha)$. Therefore these cases are of less interest for our purposes.

For nonstationary sequences the extremal index can be defined in a similar way (see Hüsler, 1986)

$$\theta = \lim_{n\to\infty} \frac{-\log P\{M_n \leqslant u_u(\tau)\}}{\sum_{i\leqslant n}[1 - H_i(u_n(\tau))]},$$

where the H_i's are the marginal distributions of the nonstationary sequence X_i. Here θ may depend on τ. However, for many nonstationary sequences θ does not depend on τ and one can use the same interpretation of θ as in the stationary case. We show that the extremal index exists for the max-AR(1) sequence $\{X_i\}$ and that it does not depend on τ.

We use the following bounds for the regularly varying function G.

Lemma 2. *Assume condition* (2) *holds. Then for any fixed* $x > 0$ *and* $\varepsilon > 0$, *there exists* n_0 *such that for all* $n > n_0$ *and all* $0 \leqslant z \leqslant 1$

$$(1-\varepsilon)z^{\alpha+\varepsilon}x^{-\alpha} \leqslant n[1 - G(a_n x/z)] \leqslant (1+\varepsilon)z^{\alpha-\varepsilon}x^{-\alpha}.$$

This follows straightforward from the representation of regularly varying functions and Potter bounds (de Haan, 1970; Bingham et al., 1987). Using this lemma we now prove the central approximations needed for the main result of this section.

Lemma 3. *Let* $\{X_i\}$ *be defined by* (1). *Assume that* (2), (3) *and* (5) *for all* $j \geqslant 0$ *hold. Then for any* $x > 0$

(i) *as* $n \to \infty$

$$\left| \sum_{i\leqslant n} [1 - H_i(a_n x)] - \sum_{i\leqslant n}\sum_{j=0}^{i-1} P\{Z_i \cdots Z_{i-j}Y_{i-j} > a_n x\} \right| \to 0$$

(ii)

$$\lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=0}^{i-1} P\{Z_i \cdots Z_{i-j}Y_{i-j} > a_n x\} = x^{-\alpha}\sum_{j\geqslant 0} c_j < \infty$$

and consequently

$$\lim_{n\to\infty} \sum_{i\leqslant n} [1 - H_i(a_n x)] = x^{-\alpha}\sum_{j\geqslant 0} c_j.$$

Proof. From the definition of the sequence X_i we have

$$X_i = \max\{Z_i \cdots Z_1 X_0, Z_i \cdots Z_1 Y_1, Z_i \cdots Z_2 Y_2, \ldots, Z_i Y_i\}.$$

Let $Z^*_{i,j} := \prod_{k=i}^{j} Z_k$. Then

$$1 - H_i(a_n x) = P(\textstyle\bigcup_{j=0}^{i-1}\{Z^*_{i-j,i}Y_{i-j} > a_n x\})(1 + o(1))$$

uniformly in i, and using the inequality of Bonferroni we get

$$\sum_{j=0}^{i-1} P\{Z_{i-j,i}^{*}Y_{i-j} > a_n x\} - \sum_{j=1}^{i-1}\sum_{k=0}^{j-1} P\{Z_{i-j,i}^{*}Y_{i-j} > a_n x, Z_{i-k,i}^{*}Y_{i-k} > a_n x\}$$
$$\leqslant 1 - H_i(a_n x) \leqslant \sum_{j=0}^{i-1} P\{Z_{i-j,i}^{*}Y_{i-j} > a_n x\}.$$

The double sum is approximated first. In the same way as in Lemma 1, using Lemma 2 and letting $\varepsilon = \alpha - \beta > 0$, we find that each term of the sum is bounded from above by

$$x^{-2\alpha}n^{-2}(1+\varepsilon)^2 E(Z_i^{2\beta}) \cdots E(Z_{i-k}^{2\beta})E(Z_{i-k-1}^{\beta}) \cdots E(Z_{i-j}^{\beta}).$$

Let $\tilde{s} = \sup_{i \geqslant 1} E(Z_i^{\beta}) < 1$ by (3). Hence also $E(Z_i^{2\beta}) \leqslant \tilde{s}$ and

$$\sum_{j=1}^{i-1}\sum_{k=0}^{j-1} P\{Z_{i-j,i}^{*}Y_{i-j} > a_n x, Z_{i-k,i}^{*}Y_{i-k} > a_n x\} \leqslant n^{-2}x^{-2\alpha}(1+\varepsilon)^2 \sum_{j=1}^{i-1} j\tilde{s}^{j+1}.$$

By taking the sum on i we get

$$\sum_{i=1}^{n}\sum_{j=1}^{i-1}\sum_{k=0}^{j-1} P\{Z_{i-j,i}^{*}Y_{i-j} > a_n x, Z_{i-k,i}^{*}Y_{i-k} > a_n x\}$$
$$\leqslant n^{-2}x^{-2\alpha}(1+\varepsilon)^2 \sum_{i=1}^{n}\sum_{j=1}^{i-1} j\tilde{s}^{j+1}$$
$$= n^{-2}x^{-2\alpha}(1+\varepsilon)^2 \mathrm{O}(n).$$

Therefore, taking the sum of the bounds of $1 - H_i(a_n x)$ and letting $n \to \infty$, statement (i) follows.

To prove the second statement, note that

$$\sum_{i=1}^{n}\sum_{j=0}^{i-1} P\{Z_{i-j,i}^{*}Y_{i-j} > a_n x\} = \sum_{j=0}^{n} \frac{1}{n} \sum_{i=j+1}^{n} nP\{Z_{i-j,i}^{*}Y_{i-j} > a_n x\}.$$

By Lemma 1 each term $nP\{Z_{i-j,i}^{*}Y_{i-j} > a_n x\}$ converges uniformly (in i) to $x^{-\alpha}E(Z_i^{\alpha}) \cdots E(Z_{i-j}^{\alpha})$, so that with (5) we get for fixed $j \geqslant 0$

$$\frac{1}{n} \sum_{i=j+1}^{n} nP\{Z_{i-j,i}^{*}Y_{i-j} > a_n x\} \to x^{-\alpha}c_j \tag{6}$$

as $n \to \infty$. Because of (3), the sum of c_j is finite, since c_j is bounded by $s^{j+1} < \tilde{s}^{j+1}$.

Therefore, for any $\delta > 0$ there exists j_0 such that

$$\tilde{s}^{j_0}/(1-\tilde{s}) < x^{\alpha}\delta.$$

Thus

$$x^{-\alpha}\sum_{j=j_0}^{\infty} c_j \leqslant x^{-\alpha}\sum_{j=j_0}^{\infty} \tilde{s}^j < \delta. \tag{7}$$

Again by Lemma 2, there exists n_0 such that for all $n \geqslant n_0$, all $0 \leqslant z_i, \ldots, z_{i-j} \leqslant 1$ and any $j \geqslant 0$ and $\varepsilon \leqslant \min(\alpha - \beta, 1)$

$$n[1 - G(a_n x / z_i \cdots z_{i-j})] \leqslant (1 + \varepsilon)(z_i \cdots z_{i-j})^{\beta} x^{-\alpha}$$

and consequently

$$\begin{aligned} nP\{Z^*_{i-j,i} Y_{i-j} > a_n x\} &\leqslant (1 + \varepsilon) E(Z_i^{\beta}) \cdots E(Z_{i-j}^{\beta}) x^{-\alpha} \\ &\leqslant (1 + \varepsilon) x^{-\alpha} \tilde{s}^{j+1}. \end{aligned} \tag{8}$$

For j_0 such that (7) holds, we can select by (6) $n_1 \geqslant n_0$ such that for all $j < j_0$ and $n \geqslant n_1$

$$\left| \frac{1}{n} \sum_{i=j+1}^{n} nP\{Z^*_{i-j,i} Y_{i-j} > a_n x\} - x^{-\alpha} c_j \right| < \delta / j_0.$$

Then using these bounds for $n \geqslant n_1$

$$\begin{aligned} &\left| \sum_{j=0}^{n} \frac{1}{n} \sum_{i=j+1}^{n} nP\{Z^*_{i-j,i} Y_{i-j} > a_n x\} - x^{-\alpha} \sum_{j=0}^{n} c_j \right| \\ &\leqslant \left| \sum_{j=0}^{j_0 - 1} \left(\frac{1}{n} \sum_{i=j+1}^{n} nP\{Z^*_{i-j,i} Y_{i-j} > a_n x\} - x^{-\alpha} c_j \right) \right| \\ &\quad + x^{-\alpha} \sum_{j=j_0}^{n} c_j + \sum_{j=j_0}^{n} \frac{1}{n} \sum_{i=j+1}^{n} nP\{Z^*_{i-j,i} Y_{i-j} > a_n x\} \\ &= |A_n| + B_n + C_n. \end{aligned} \tag{9}$$

Thus $|A_n| < \delta$ and also $B_n < \delta$ by (7) and the choice of j_0. Finally by (8), for $n \geqslant n_1$

$$\begin{aligned} C_n &\leqslant \sum_{j=j_0}^{n} \frac{x^{-\alpha}}{n} \sum_{i=j+1}^{n} (1 + \varepsilon) E(Z_i^{\beta}) \cdots E(Z_{i-j}^{\beta}) \\ &\leqslant (1 + \varepsilon) x^{-\alpha} \sum_{j=j_0}^{n} \frac{n-j}{n} \tilde{s}^{j} \\ &\leqslant (1 + \varepsilon) x^{-\alpha} \tilde{s}^{j_0} / (1 - \tilde{s}) \leqslant 2\delta \end{aligned}$$

by (7). Therefore, (9) is bounded by 4δ, which proves statement (ii). □

Remark. Note that the proof shows also that $n(1 - H_i(a_n x)) = \mathrm{O}(x^{-\alpha})$, uniformly in i.

Combining the results of Theorem 1, Lemma 3(ii) and Hüsler (1986), we get immediately.

Theorem 2. *Suppose that* (2), (3) *and* (5), *for each* $j \geqslant 0$, *hold for the max-AR*(1)*-random sequence* $\{X_i\}$ *defined in* (1). *Then* $\{X_i\}$ *has extremal index* $\theta = c_0 / \sum_{j=0}^{\infty} c_j$.

Remarks. Note that the extremal index does not depend on x as mentioned above. In particular, if $Z_i = C$ with probability 1, for all i, then $EZ_i^\alpha = C^\alpha = c_0$ and $c_j = C^{\alpha(j+1)}$. This implies that $\theta = C^\alpha/(\sum_{j\geqslant 1} C^{\alpha j}) = 1 - C^\alpha$, which is the result obtained for the stationary case. But this particular result still holds, if only for instance $EZ_i^\alpha = C^\alpha$ for all $i \geqslant 1$. Even weaker assumptions on $\{Z_i\}$ would lead to the same result.

3. Point process of exceedances

In this section we discuss the point process N_n of exceedances of the boundary $u_n = a_n x$ by $\{X_i\}$. We define N_n on $[0, 1]$ by

$$N_n = \sum_{i\leqslant n} \delta_{i/n} 1(X_i > u_n).$$

Since the exceedances do cluster, as is shown in Section 2, we expect that N_n converges asymptotically to a compound Poisson process with a certain distribution for the multiplicities representing the cluster sizes. To derive such a result, we assume in this section in addition to the previous assumptions that

$$EZ_i^\alpha \to c \quad \text{as } i \to \infty. \tag{10}$$

This implies that (5) holds for all $j \geqslant 0$ with $c_j = c^{j+1}$. In the stationary case with fixed nonrandom Z_i it was shown by Alpuim (1988) that the cluster size distribution is asymptotically a geometric distribution. To derive a similar result for the nonstationary model, we use a general result for the convergence of point processes of exceedances given in Nandagopalan (1990) and Nandagopalan et al. (1992), which holds under certain mixing and smoothness conditions.

The mixing condition Δ is the following: Let

$$\mathscr{B}_{k_1}^{k_2}(u_n) = \sigma\{\{X_i > u_n\}, k_1 \leqslant i \leqslant k_2\}$$

and for $1/n < l < (n-1)/n$ define

$$\alpha_{n,l} = \sup\{|P(A\cap B) - P(A)P(B)|, A \in \mathscr{B}_0^m(u_n), B \in \mathscr{B}_{m+[nl]}^n(u_n),$$
$$0 \leqslant m < m + [nl] \leqslant n\}.$$

The condition Δ is said to hold if $\alpha_{n,l_n} \to 0$ for some sequence $l_n \to 0$ as $n \to \infty$.

Then there exists a sequence $\{k_n, n \geqslant 1\}$ such that

$$k_n \to \infty,\ k_n/n \to 0,\ k_n(\alpha_{n,l_n} + l_n) \to 0 \quad \text{as } n \to \infty. \tag{11}$$

Take for instance $k_n = \min(\sqrt{n}, \alpha_{n,l_n}^{-1/2}, l_n^{-1/2})$.

In order that the limiting point process N is infinitely divisible and has independent increments, i.e. $N(B)$ and $N(C)$ are independent whenever B and C are disjoint subsets of $[0, 1]$, we need in addition that the exceedances in small intervals are asymptotically negligible, more precisely

$$\sup_{J: m(J) \leqslant l_n} P\{N_n(J) \neq 0\} \to 0 \quad \text{as } n \to \infty.$$

where $m(\cdot)$ denotes Lebesgue measure. This will follow from the verification of the following condition (12).

We assume that for each $n > 0$ there exists an interval partition $\{J_i = J_i(n), 1 \leqslant i \leqslant k_n\}$ of $[0, 1]$ such that

$$\gamma_n = \max_{1 \leqslant i \leqslant k_n} P\{N_n(J_i) \neq 0\} \to 0 \quad \text{as } n \to \infty. \tag{12}$$

With respect to this partition we define the (conditional) cluster size distribution

$$\pi_{n,i}(y) = P\{N_n(J_i) \leqslant y \mid N_n(J_i) \neq 0\}, \quad y > 0,\ i \leqslant k_n,$$

and set $\pi_{n,x} = \pi_{n,i}$ whenever $x \in J_i$.

Moreover, we define for $n > 0$ the measures ν_n for the occurrence of cluster of exceedances by

$$\nu_n(B) = \sum_{i=1}^{k_n} P\{N_n(J_i) \neq 0\} \frac{m(B \cap J_i)}{m(J_i)}, \quad B \subset [0, 1].$$

A smoothness condition is also required. For each $n > 0$ and $a \in G$ where G is a nonempty open subset of $\mathbb{R}_+ \backslash \{0\}$, define the family of functions $g_n(\cdot, a)$

$$g_n(x, a) = \int_{y > 0} (1 - \exp(-ay))\, \mathrm{d}\pi_{n,x}(y).$$

We suppose that for each a the family $\{g_n(\cdot, a), n \geqslant 1\}$ is equicontinuous, i.e. for each $x \in [0, 1]$ and $\varepsilon > 0$, there exists $\tilde{N}(x) > 0$ and $\delta(x) > 0$ such that $|g_n(x, a) - g_n(x', a)| < \varepsilon$ whenever $n > \tilde{N}(x)$ and $|x - x'| < \delta(x)$.

Finally we need that

$$\limsup_{n \geqslant 1} \nu_n([0, 1]) < \infty. \tag{13}$$

If these mixing and smoothness conditions together with (12) and (13) hold and if in addition $\nu_n \xrightarrow{w} \nu$ and also $\pi_{n,x} \xrightarrow{w} \pi_x$, $\nu - a.e.\ x$, then Corollary 5.3 of Nandagopalan et al. (1992) implies that $N_n \xrightarrow{d} N$, where N is a point process with Laplace transform

$$-\log L_N(f) = \int_{x \in [0, 1]} \int_{y > 0} [1 - \exp(-y f(x))]\, \mathrm{d}\pi_x(y)\, \mathrm{d}\nu(x).$$

We shall show that the cluster size distribution π_x does not depend on x, which together with the representation of the Laplace transform above implies that the resulting limiting point process N is a compound Poisson process.

(1) We begin by verifying condition (12). Let $J_i = [(i-1)/k_n, i/k_n)$, $1 \leqslant i \leqslant k_n - 1$ and $J_{k_n} = [(k_n - 1)/k_n, 1]$ be an interval partition of $[0, 1]$, with $k_n \to \infty$ such that (11)

holds. Obviously, $m(J_i) = 1/k_n$ and $1/k_n > l_n$ because of (11). Now,

$$P\{N_n(J_i) \neq 0\} \leqslant \sum_{1 \leqslant j \in nJ_i} (1 - H_j(u_n)) = \mathrm{O}(1/k_n) \to 0,$$

since by the remark following Lemma 3, $P\{X_i > u_n\} = 1 - H_i(u_n) \leqslant \mathrm{O}(1)x^{-\alpha}/n = \mathrm{O}(1/n)$, for all $n > n_0$, uniformly in i.

(2) To verify (13) note that by the definition of ν_n, we have

$$\nu_n([0,1]) = \sum_{i=1}^{k_n} P\{N_n(J_i) \neq 0\} < \sum_{1 \leqslant j \leqslant n} P\{X_j > u_n\} = \mathrm{O}(1)$$

using (1).

(3) We consider now the weak convergence of ν_n. A similar calculation as in the proof of Theorem 1 gives for any subinterval J of $[0,1]$ where $m(J) \to 0$ and $nm(J) \to \infty$ as $n \to \infty$

$$\begin{aligned} P\{N_n(J) \neq 0 &\sim P\left\{\max_{j \in nJ} Z_j Y_j > u_n\right\} \\ &\sim \sum_{j \in nJ} P\{Z_j Y_j > u_n\} \\ &\sim (x^{-\alpha}/n) \sum_{j \in nJ} EZ_j^{\alpha}. \end{aligned} \tag{14}$$

The convergence of ν_n is now implied by (10) since

$$\nu_n(B) \sim \sum_{i \leqslant k_n} \frac{x^{-\alpha}}{n}\left(\sum_{j \in nJ_i} EZ_j^{\alpha}\right)\frac{m(B \cap J_i)}{m(J_i)} \to cx^{-\alpha}m(B) = \nu(B). \tag{15}$$

(4) We continue by analysing the convergence of the cluster size distribution $\pi_{n,x}$. Let k be fixed, $k \geqslant 1$. Again, we use first any subinterval J of $[0,1]$ introduced in (3). It follows easily that with

$$B_j = \{X_j \leqslant u_n,\ X_{j+1} > u_n,\ \ldots, X_{j+k} > u_n\},$$

$j \geqslant 0$,

$$P\{N_n(J) \geqslant k\} - P(\textstyle\bigcup_{j \in nJ} B_j) = \mathrm{O}((P\{N_n(J) \neq 0\})^2) + \mathrm{O}(1/n).$$

In the same way with

$$\begin{aligned} A_j = \{&Z_{j+1}Y_{j+1} > Y_{j+2}, Z_{j+2}Z_{j+1}Y_{j+1} > Y_{j+3}, \ldots, \\ &Z^*_{j+1,j+k-1}Y_{j+1} > Y_{j+k}, Z^*_{j+1,j+k}Y_{j+1} > u_n\}, \end{aligned}$$

denoting the event that the weighted 'input' Y_{j+1} dominates the following $k-1$ 'inputs' $Y_{j+2}, \ldots, Y_{j+k}$ and remains above u_n for the next k time points, we get

$$P\left\{\bigcup_{j \in nJ} B_j\right\} - P\left\{\bigcup_{j \in nJ} (B_j \cap A_j)\right\} = o(P\{N_n(J) \neq 0\}).$$

Finally, we approximate the last term

$$P\left\{\bigcup_{j\in nJ}(B_j\cap A_j)\right\} \sim P\left\{\bigcup_{j\in nJ}(X_j \leqslant u_n, Z^*_{j+1,j+k}Y_{j+1} > u_n)\right\}$$

$$\sim \sum_{j\in nJ} P\{Z^*_{j+1,j+k}Y_{j+1} > u_n\}$$

$$\sim \sum_{j\in nJ} \frac{x^{-\alpha}}{n}\int\cdots\int (z_{j+1}\cdots z_{j+k})^\alpha \mathrm{d}F_{j+1}\cdots \mathrm{d}F_{j+k}$$

$$\sim \sum_{j\in nJ} \frac{x^{-\alpha}}{n} EZ^\alpha_{j+1}\cdots EZ^\alpha_{j+k}.$$

Using (14) we get now

$$P\left\{\bigcup_{j\in nJ} B_j \,|\, N_n(J)\neq 0\right\} = (1+\mathrm{o}(1))\frac{\sum_{j\in nJ} EZ^\alpha_{j+1}\cdots EZ^\alpha_{j+k}}{\sum_{1\leqslant j\in nJ} EZ^\alpha_j}. \tag{16}$$

Now taking the intervals of the partition and using (10), it follows that for any $k \geqslant 1$ and any $i \leqslant k_n$

$$1-\pi_{n,i}(k-1) = P\{N_n(J_i) \geqslant k \,|\, N_n(J_i) \neq 0\}$$

$$\sim \frac{\sum_{j\in nJ_i} EZ^\alpha_{j+1}\cdots EZ^\alpha_{j+k}}{\sum_{1\leqslant j\in nJ_i} EZ^\alpha_j} \sim c^{k-1}.$$

Hence for all $x\in[0,1]$, $\pi_x(k) = 1-c^k$ is a geometric distribution, independent of x. The above approximation holds uniformly for $i \leqslant k_n$ which implies that $\{g_n\}$ is equicontinuous and the corresponding smoothness condition is obviously satisfied.

(5) Left to verify is the mixing condition Δ. If two events A and B are conditionally independent given E with $P(E) > 0$, it follows by a straightforward calculation that

$$|P(A\cap B) - P(A)P(B)| \leqslant P(A\cap B|E) - P(A|E)P(B|E)| + \mathrm{O}(P(E^c))$$

We consider first two special events $A^* \in \mathscr{B}^m_0$ and $B^* \in \mathscr{B}^n_{m+[nl_n]}$ where $0 \leqslant m < m + [nl_n] \leqslant n$ with $l_n = \mathrm{o}(1)$, $nl_n \to \infty$ (as $n \to \infty$):

$$A^* = \left\{\bigcap_{j=0}^{m} (X_j \in I_j)\right\}.$$

$$B^* = \left\{\bigcap_{j=m+[nl_n]}^{n} (X_j \in I_j)\right\},$$

where $I_j \in S_n := \{\emptyset, (-\infty, u_n], (u_n, \infty), \mathbb{R}\}$.

Furthermore, let

$$E = \bigcup_{j=1}^{[nl_n]-1} \{Z^*_{m+1,m+j}X_m < Z_{m+j}Y_{m+j}\} \cap \bigcap_{k=1}^{[nl_n]-1} \{Z_{m+k}Y_{m+k} \leqslant u_n\}.$$

E denotes the event that in the index set$\{m+1, \ldots, m+[nl_n]-1\}$ the sequence X_j is at least once exceeded by an input Z_jY_j whereas all the inputs fail to exceed u_n.

Observe that

$$\{X_{m+[nl_n]} \in I_{m+[nl_n]}\} = \{Z_{m+[nl_n]} \max(Z^*_{m+1,m+[nl_n]-1} X_m, Z^*_{m+1,m+[nl_n]-1}$$

$$Y_{m+1}, \ldots, Z_{m+[nl_n]-1} Y_{m+[nl_n]-1}, Y_{m+[nl_n]}) \in I_{m+[nl_n]}\}.$$

Taking the intersection with E we get

$$\{X_{m+[nl_n]} \in I_{m+[nl_n]}\} \cap E = \{Z_{m+[nl_n]} Y_{m+[nl_n]} \in I_{m+[nl_n]}\} \cap E.$$

Therefore

$$P(B^* \cap E) = P\{Z_{m+[nl_n]} Y_{m+[nl_n]} \in I_{m+[nl_n]}, \hat{X}_k \in I_k, m+[nl_n]+1 \leqslant k \leqslant n\} P(E)$$

$$= P(\hat{B}) P(E)$$

where $\hat{X}_k = Z_k \max(\hat{X}_{k-1}, Y_k)$ for $k > m + [nl_n], \hat{X}_{m+[nl_n]} = Z_{m+[nl_n]} Y_{m+[nl_n]}$, and $\hat{B} = \{\bigcap_{j=m+[nl_n]}^{n} (\hat{X}_j \in I_j)\}$; also

$$P(A^* \cap B^* \cap E) = P(A^* \cap E) P(\hat{B})$$

which implies

$$P(A^* \cap B^* | E) = P(A^* | E) P(\hat{B}) = P(A^* | E) P(B^* | E).$$

Next we show that $P(E) \to 1$ as $n \to \infty$. We rewrite $E = E_1 \cap E_2$, where

$$E_1 = \bigcup_{j=1}^{[nl_n]-1} \{Z_{m+j} Y_{m+j} > Z^*_{m+1,m+j} X_m\}$$

and

$$E_2 = \bigcap_{j=1}^{[nl_n]-1} \{Z_{m+j} Y_{m+j} \leqslant u_n\}.$$

We get

$$P(E_1^c) = \int P\{(Y_{m+1} \leqslant x) \bigcap_{j=2}^{[nl_n]-1} \{Y_{m+j} \leqslant Z^*_{m+1,m+j-1} x)\} H_m(\mathrm{d}x)$$

$$\leqslant \int P\{Y_{m+j} \leqslant x, 1 \leqslant j \leqslant [nl_n]-1\} H_m(\mathrm{d}x)$$

$$= \int [G(x)]^{[nl_n]-1} H_m(\mathrm{d}x).$$

To see that this bound tends to 0, define the sequence n^* by $n^* = [nl_n] - 1$. We have $n^* \to \infty$. Split the integral into two parts with the point $x_0 a_{n^*}$ with $x_0 > 0$, small. For all $x \leqslant x_0 a_{n^*}$

$$\begin{aligned}[G(x)]^{n^*} &\leqslant [G(x_0 a_{n^*})]^{n^*} \\ &\to \exp(-x_0^{-\alpha}) \\ &< \varepsilon\end{aligned}$$

for x_0 sufficiently small. The second part of the integral is bounded by $1 - H_m(x_0 a_{n^*}) = O(x_0^{-\alpha}/n^*)$, uniformly for m, by the remark after Lemma 3. Hence the upper bound of $P(E_1^c)$ tends to zero as $n \to \infty$. By Lemma 2 we get that

$$\begin{aligned}P(E_2) &= \prod_{j=1}^{[nl_n]-1} P\{Z_{m+j} Y_{m+j} \leqslant u_n\} \\ &\geqslant \prod_{j=1}^{[nl_n]-1} (1 - (1+\varepsilon) E(Z_{m+j}^{\alpha-\varepsilon}) x^{-\alpha}/n) \\ &\geqslant (1 - (1+\varepsilon) x^{-\alpha}/n)^{[nl_n]-1} \to 1,\end{aligned}$$

since $l_n \to 0$.

Combining these results we notice that $P(E^c) \leqslant P(E_1^c) + P(E_2^c) \to 0$ as $n \to \infty$. This implies that the mixing property holds for the special events A^* and B^*.

It remains to show that this implies also the mixing property for any events $A \in \sigma\{\{X_j > u_n\},\ j \leqslant m\}$ and $B \in \sigma\{\{X_j > u_n\},\ m + [nl_n] \leqslant j \leqslant n\}$. Observing that $\mathscr{B}_0^m(u_n) = \sigma\{\{X_j \leqslant u_n\},\ 0 \leqslant j \leqslant m\} = \sigma\{(X_0, \ldots, X_m) \in I_0 \times \cdots \times I_m;\ I_j \in S_n,\ 0 \leqslant j \leqslant m\}$ define

$$\mathscr{C}_0^m(u_n) = \{(X_0, \ldots, X_m) \in I_0 \times \cdots \times I_m;\ I_j \in S_n,\ 0 \leqslant j \leqslant m\}$$

and $\mathscr{C}_{m+[nl_n]}^n(u_n)$ similarly. Let

$$\mathscr{D}_1 = \{A \in \mathscr{B}_0^m(u_n):\ P(A \cap B | E) = P(A|E) P(B|E), B \in \mathscr{C}_{m+[nl_n]}^n(u_n)\}.$$

$\mathscr{D}_1$ is obviously a Dynkin system and $\mathscr{D}_1 \supset \mathscr{C}_0^m(u_n)$. Since $\mathscr{C}_0^m(u_n)$ is $\cap$-stable, we have $\mathscr{D}_1 \supset \sigma(\mathscr{C}_0^m(u_n)) = \mathscr{B}_0^m(u_n)$. Now define

$$\mathscr{D}_2 = \{B \in \mathscr{B}_{m+[nl_n]}^n(u_n):\ P(A \cap B | E) = P(A|E) P(B|E), A \in \mathscr{B}_0^m(u_n)\}.$$

Again, $\mathscr{D}_2$ is a Dynkin system and $\mathscr{D}_2 \supset \mathscr{C}_{m+[nl_n]}^n(u_n)$. Since $\mathscr{C}_{m+[nl_n]}^n(u_n)$ is $\cap$-stable, we have $\mathscr{D}_2 \supset \sigma(\mathscr{C}_{m+[nl_n]}^n(u_n)) = \mathscr{B}_{m+[nl_n]}^n(u_n)$. Therefore, we conclude that any two events $A \in \mathscr{B}_0^m(u_n)$ and $B \in \mathscr{B}_{m+[nl_n]}^n(u_n)$ are conditionally independent given E. This together with $P(E^c) \to 0$ implies the Δ mixing condition.

Hence we proved

Theorem 3. *Suppose that* (2), (3) *and* (5), *for each* $j \geqslant 0$, *and* (10) *hold for the max-AR*(1)*-random sequence defined in* (1). *Then*

$$N_n \xrightarrow{d} N \quad \text{as } n \to \infty$$

where N is a compound Poisson process with a geometric cluster size distribution $\pi(k) = 1 - c^k$, $k \geqslant 1$ and intensity $cx^{-\alpha}$.

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