

Generalization of Itô's formula for smooth nondegenerate martingales

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Abstract

In this paper we prove the existence of the quadratic covariation $[(\partial F/\partial x_k)(X), X^k]$ for all $1 \leq k \leq d$, where F belongs locally to the Sobolev space $\mathcal{W}^{1,p}(\mathbb{R}^d)$ for some $p > d$ and X is a d -dimensional smooth nondegenerate martingale adapted to a d -dimensional Brownian motion. This result is based on some moment estimates for Riemann sums which are established by means of the techniques of the Malliavin calculus. As a consequence we obtain an extension of Itô's formula where the complementary term is one-half the sum of the quadratic covariations above. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $W = \{W_t, t \in [0, T]\}$ be a d -dimensional Brownian motion, with $d > 1$. Consider a d -dimensional square integrable martingale $X = \{X_t, t \in [0, T]\}$. It is well known that X has a representation of the form $X_t^k = \sum_{i=1}^d \int_0^t u_s^{k,i} dW_s^i$, where for all $k, i = 1, \dots, d$, $u^{k,i}$ are continuous and adapted stochastic processes satisfying $\int_0^T E|u_s^{k,i}|^2 ds < \infty$.

Let F be a function which belongs locally to the Sobolev space $\mathcal{W}^{1,p}(\mathbb{R}^d)$ for some $p > d$. The purpose of this paper is to prove the existence of the quadratic covariation of the processes X^k and $(\partial F/\partial x_k)(X)$ for all $k = 1, \dots, d$, defined as the following limit in probability:

$$\left[\frac{\partial F}{\partial x_k}(X), X^k \right]_t = \lim_n \sum_{t_i \in D_n, t_i < t} (X_{t_{i+1}}^k - X_{t_i}^k) \left(\frac{\partial F}{\partial x_k}(X_{t_{i+1}}) - \frac{\partial F}{\partial x_k}(X_{t_i}) \right), \quad (1.1)$$

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where D_n is a sequence of partitions of $[0, T]$ such that

$$\lim_n \sup_{t_i \in D_n} (t_{i+1} - t_i) = 0, \quad \sup_n \sup_{t_i \in D_n} \frac{t_{i+1}}{t_i} < \infty.$$

The existence of this limit will allow us to prove the following extension of the Itô's formula:

$$F(X_t) = F(0) + \sum_{k=1}^d \int_0^t \frac{\partial F}{\partial x_k}(X_s) dX_s^k + \frac{1}{2} \sum_{k=1}^d \left[\frac{\partial F}{\partial x_k}(X), X^k \right]_t. \quad (1.2)$$

Notice that for a smooth function F on \mathbb{R}^d we have

$$\sum_{k=1}^d \left[\frac{\partial F}{\partial x_k}(X), X^k \right]_t = \int_0^t \Delta F(X_s) ds.$$

The result (existence of the quadratic covariation and Itô's formula) in the one-dimensional case holds for any absolutely continuous function F such that its derivative f belongs to $L^2_{\text{loc}}(\mathbb{R})$ (Moret and Nualart, 2000), assuming suitable nondegeneracy and regularity properties on the martingale X . The proof is based on the estimate

$$E(f(X_t)^2 Z) \leq \frac{c}{\sqrt{t}} \|f\|_2^2 \quad (1.3)$$

for any nice random variable Z , and for any function $f \in L^2(\mathbb{R})$, which is derived using the techniques of Malliavin calculus. Clearly, inequality (1.3) implies

$$\int_0^T E(f(X_t)^2 Z) dt \leq 2\sqrt{T}c \|f\|_2^2.$$

When $d > 1$, (1.3) is replaced by $E(f(X_t)^2 Z) \leq c' t^{-d/2} \|f\|_2^2$, and the right-hand side of this inequality is not integrable. However, using exponential estimates for the law of X_t and applying Hölder's inequality for some $p > d$ we can show, for some constant M ,

$$E(f(X_t)^2 Z) \leq c' \int_{\mathbb{R}^d} f(x)^2 t^{-d/2} e^{-|x|^2/2tM} dx \leq c'' t^{-d/p} \|f\|_p^2$$

and hence, if $f \in L^p(\mathbb{R}^d)$, the right-hand side of this inequality is integrable. In this paper we will make use of this argument and for this reason we are forced to assume that the partial derivatives of our function F are locally in $L^p(\mathbb{R}^d)$ for some $p > d$.

The approach we use in this paper was introduced by Föllmer et al. (1995) to treat the case $F(B_t)$, where F is an absolutely continuous function with locally square integrable derivative and B is a one-dimensional Brownian motion. The results of Föllmer et al. (1995) have been extended to elliptic diffusions by Bardina and Jolis (1997) and to nondegenerate diffusion processes with nonsmooth coefficients in a recent work of Flandoli et al. (2000).

In the d -dimensional case Föllmer and Protter (2000), obtained an Itô's formula for functions $F \in \mathcal{W}_{\text{loc}}^{1,2}$ of a Brownian motion starting at x_0 , where x_0 must be outside of some polar set. There are also results for multidimensional diffusion processes when $F \in \mathcal{W}_{\text{loc}}^{1,p}$ with $p > 2 \vee d$ (Rozkosz, 1996) and for càdlàg processes, when $F \in \mathcal{C}^1$ and has a locally Hölder continuous derivative (Errami et al., 1999).

Using integration with respect to the local time Wolf (1997) established an extension of Itô's formula for semimartingales and absolutely continuous functions with derivative in L^1_{loc} satisfying some technical assumptions. Eisenbaum (1997) has proved a generalization of Itô's formula to time-dependent functions of a Brownian motion, where the complementary term is a two-parameter integral with respect to the local time.

By means of a regularization approach, Russo and Vallois (1995) obtained an Itô's formula for \mathcal{C}^1 transformations of time reversible continuous semimartingales. In the framework of Dirichlet forms, an extension of Itô's formula has been established by Lyons and Zhang (1994).

The paper is organized as follows. Section 2 contains some basic material on Malliavin calculus. In Section 3 we show a general result on the existence of the quadratic covariation and Itô's formula for d -dimensional martingales (Theorem 5). Section 4 is devoted to prove basic estimates for stochastic integrals and its derivatives, and on the Sobolev norm of the inverse of the Malliavin matrix. Finally, in Section 5 we apply these results to estimate the Riemann sums and deduce the main result of the paper.

2. Preliminaries

Let $W = \{W_t, t \in [0, T]\}$ be a d -dimensional Brownian motion defined on the canonical probability space (Ω, \mathcal{F}, P) . That is, Ω is the space of continuous functions from $[0, T]$ to \mathbb{R}^d which vanish at zero, \mathcal{F} is the Borel σ -field on Ω completed with respect to P , and P is the Wiener measure. For every $t \in [0, T]$ we denote by \mathcal{F}_t the σ -algebra generated by the random variables $\{W_s, s \leq t\}$ and the P -null sets. Let $H = L^2([0, T]; \mathbb{R}^d)$. For any $h \in H$ we denote the Wiener integral of h by $W(h) = \sum_{i=1}^d \int_0^T h_t^i dW_t^i$.

We will use the notation $|x|$ (resp. $|t|$) for the Euclidian norm of a vector x in \mathbb{R}^d (resp. a tensor t in $\mathbb{R}^d \otimes \dots \otimes \mathbb{R}^d$). That is $|t|^2 = \sum_{i_1, \dots, i_j=1}^d (t^{i_1, \dots, i_j})^2$. We will also make use of the notation $\langle x, y \rangle$ for the scalar product in \mathbb{R}^d .

Let us first introduce the derivative operator D . We denote by $C_b^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ such that f and all of its partial derivatives are bounded.

Let \mathcal{S} denote the class of smooth cylindrical random variables of the form

$$F = f(W(h_1), \dots, W(h_n)), \quad (2.1)$$

where f belongs to $C_b^\infty(\mathbb{R}^n)$, and $h_1, \dots, h_n \in H$. If F has form (2.1) we define its derivative DF as the d -dimensional stochastic process given by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_n)) h_i(t). \quad (2.2)$$

We will denote $D^{(k)}F$ the k th component of DF .

The operator D is closable from $\mathcal{S} \subset L^p(\Omega)$ into $L^p(\Omega; H)$ for each $p \geq 1$. We will denote by $\mathbb{D}^{1,p}$ the closure of the class of smooth random variables \mathcal{S} with respect to

the norm

$$\|F\|_{1,p}^p = E(|F|^p) + E(\|DF\|_H^p).$$

We can define the iteration of the operator D in such a way that for a smooth random variable F , the derivative $D_{t_1, \dots, t_k}^k F$ is a k -parameter process. Then, for every $p \geq 1$ and any natural number k we introduce the space $\mathbb{D}^{k,p}$ as the completion of the family of smooth random variables \mathcal{S} with respect to the norm

$$\|F\|_{k,p}^p = E(|F|^p) + \sum_{j=1}^k E(\|D^j F\|_{H^{\otimes j}}^p). \quad (2.3)$$

Let V be a real separable Hilbert space. We can also introduce the corresponding Sobolev spaces $\mathbb{D}^{k,p}(V)$ of V -valued random variables. More precisely, if \mathcal{S}_V denotes the family of V -valued random variables of the form

$$F = \sum_{j=1}^n F_j v_j, \quad v_j \in V, \quad F_j \in \mathcal{S},$$

we define $D^k F = \sum_{j=1}^n D^k F_j \otimes v_j$, $k \geq 1$. Then D^k is a closable operator from $\mathcal{S}_V \subset L^p(\Omega; V)$ into $L^p(\Omega; H^{\otimes k} \otimes V)$ for any $p \geq 1$. For any integer $k \geq 1$ and any real number $p \geq 1$ we can define a seminorm on \mathcal{S}_V by

$$\|F\|_{k,p,V}^p = E(\|F\|_V^p) + \sum_{j=1}^k E(\|D^j F\|_{H^{\otimes j} \otimes V}^p).$$

We denote by $\mathbb{D}^{k,p}(V)$ the completion of \mathcal{S}_V with respect the seminorm $\|\cdot\|_{k,p,V}$.

We will denote by δ the adjoint of the operator D as an unbounded operator from $L^2(\Omega)$ into $L^2(\Omega; H)$. That is, the domain of δ , denoted by $\text{Dom } \delta$, is the set of H -valued square integrable random variables u such that there exists a square integrable random variable $\delta(u)$ verifying

$$E(F\delta(u)) = E(\langle DF, u \rangle_H) \quad (2.4)$$

for any $F \in \mathcal{S}$. We will make use of the notation $\delta(u) = \int_0^T u_s dW_s$. We refer to Nualart, 1995a,b for a detailed account of the basic properties of the operators D and δ .

The following integration by parts formula will be one of the main ingredients in the proof of our results.

Proposition 1. Fix $m \geq 1$ and $0 \leq a < b \leq T$. Let $Y = (Y^1, \dots, Y^d)$ be a random vector in the space $(\mathbb{D}^{m+1,p})^d$, for all $p > 1$. Define the matrix

$$\gamma_Y^{a,b} = \left(\sum_{k=1}^d \int_a^b D_t^{(k)} Y^i D_t^{(k)} Y^j dt \right)_{1 \leq i, j \leq d}. \quad (2.5)$$

Suppose that $\gamma_Y^{a,b}$ is invertible a.s and $(\det \gamma_Y^{a,b})^{-1} \in \bigcap_{p \geq 1} L^p(\Omega)$. Let $Z \in \mathbb{D}^{m,p}$, for all $p > 1$. Then, for any function $f \in C_b^1(\mathbb{R}^d)$ and for any multi-index $\alpha \in \{1, \dots, d\}^m$ we have

$$E((\partial_\alpha f)(Y)Z) = E(f(Y)H_\alpha^{a,b}(Y, Z)), \quad (2.6)$$

where $H_{\alpha}^{a,b}(Y, Z)$ is recursively given by

$$H_{(i)}^{a,b}(Y, Z) = \sum_{j=1}^d \int_a^b Z(\gamma_Y^{a,b})_{ij}^{-1} D_s Y^j \, dW_s, \quad (2.7)$$

$$H_{\alpha}^{a,b}(Y, Z) = H_{\alpha_k}^{a,b}(Y, H_{(\alpha_1, \dots, \alpha_{k-1})}^{a,b}(Y, Z)).$$

Proof. By the chain rule we have

$$D_s(f(Y)) = \sum_{i=1}^d \partial_i f(Y) D_s Y^i$$

and as a consequence we obtain,

$$\int_a^b \langle D_s Y^j, D_s(f(Y)) \rangle \, ds = \sum_{i=1}^d \partial_i f(Y) (\gamma_Y^{a,b})_{ij}.$$

Hence, using the duality relationship (2.4) for the operator δ yields

$$\begin{aligned} E(\partial_i f(Y) Z) &= E \left(\sum_{j=1}^d (\gamma_Y^{a,b})_{ij}^{-1} Z \int_a^b \langle D_s Y^j, D_s(f(Y)) \rangle \, ds \right) \\ &= E(f(Y) H_{(i)}^{a,b}(Y, Z)). \end{aligned}$$

We complete the proof by means of a recurrence argument. \square

Notice that by the Bouleau and Hirsch criterion (Bouleau and Hirsch, 1986) the condition on $\gamma_Y^{a,b}$ implies that the law of Y is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^d . Moreover, if the assumptions of Proposition 1 hold with $m = d$, then the density of Y is given by

$$p(x) = E(\mathbf{1}_{\{Y > x\}} H_{(1, \dots, d)}^{a,b}(Y, 1)). \quad (2.8)$$

Corollary 2. Let $Y = (Y^1, \dots, Y^d)$ be a random vector and Z a random variable satisfying the assumptions of Proposition 1 with $m = d$. Suppose also $E|Zf(Y)^2| < \infty$, where $f \in L^2(\mathbb{R}^d)$. Then, we have

$$E(f(Y)^2 Z) = \int_{\mathbb{R}^d} f(x)^2 E(\mathbf{1}_{\{Y > x\}} H_{(1, \dots, d)}^{a,b}(Y, Z)) \, dx. \quad (2.9)$$

Proof. We can assume that f is bounded by replacing f^2 by $f^2 \wedge M$ and letting M tend to infinity. By the Lebesgue differentiation theorem and using that Y has an absolutely continuous probability distribution we obtain

$$\left(\frac{n}{2}\right)^d \int_{Y^1 - (1/n)}^{Y^1 + (1/n)} \cdots \int_{Y^d - (1/n)}^{Y^d + (1/n)} f(x_1, \dots, x_d)^2 \, dx_1 \cdots dx_d \rightarrow f(Y^1, \dots, Y^d)^2,$$

a.s. as n tends to infinity. For any fixed $x \in \mathbb{R}^d$ set

$$\begin{aligned} f_n(x, y) &= \left(\frac{n}{2}\right)^d \prod_{i=1}^d \mathbf{1}_{[x_i-1/n, x_i+1/n]}(y_i), \\ g_n(x, y) &= \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_d} f_n(x, \eta_1, \dots, \eta_d) d\eta_1 \dots d\eta_d. \end{aligned} \quad (2.10)$$

Then, by the dominated convergence theorem and applying Proposition 1 with $\alpha = (1, \dots, d)$ to the function $g_n(x, \cdot)$ we get

$$\begin{aligned} E(f(Y)^2 Z) &= \lim_n \left(\frac{n}{2}\right)^d \int_{\mathbb{R}^d} f(x)^2 E \left(Z \prod_{i=1}^d \mathbf{1}_{[x_i-1/n, x_i+1/n]}(Y^i) \right) dx \\ &= \lim_n \int_{\mathbb{R}^d} f(x)^2 E(f_n(x, Y) Z) dx \\ &= \lim_n \int_{\mathbb{R}^d} f(x)^2 E((\partial_\alpha g_n)(x, Y) Z) dx \\ &= \lim_n \int_{\mathbb{R}^d} f(x)^2 E(g_n(x, Y) H_{(1, \dots, d)}^{a, b}(Y, Z)) dx \\ &= \int_{\mathbb{R}^d} f(x)^2 E(\mathbf{1}_{\{Y > x\}} H_{(1, \dots, d)}^{a, b}(Y, Z)) dx, \end{aligned}$$

which completes the proof. \square

We will make use of the following estimate for the $\|\cdot\|_{k, p}$ -norm of the divergence operator (Nualart, 1995a,b).

Proposition 3. *The operator δ is continuous from $\mathbb{D}^{k+1, p}(V \otimes H)$ into $\mathbb{D}^{k, p}(V)$ for all $p > 1, k \geq 0$. Hence, for any $u \in \mathbb{D}^{k+1, p}(V \otimes H)$ we have*

$$\|\delta(u)\|_{k, p, V} \leq c_{k, p} \|u\|_{k+1, p, V \otimes H} \quad (2.11)$$

for some constant $c_{k, p}$.

For any fixed $0 \leq a < T$ the following conditional version of the duality relationship between the derivative and divergence operators holds

$$E \left(F \int_a^T u_r dW_r | \mathcal{F}_a \right) = E \left(\int_a^T \langle D_r F, u_r \rangle dr | \mathcal{F}_a \right) \quad (2.12)$$

for all $F \in \mathbb{D}^{1, 2}$ and u such that $u \mathbf{1}_{[a, T]} \in \text{Dom } \delta$. Using this duality formula we can formulate the following conditional version of equality (2.9).

Proposition 4. *Let $Y = (Y^1, \dots, Y^d)$ be a random vector and Z a random variable satisfying the assumptions of Proposition 1 with $m = d$. Let A be an \mathcal{F}_a -measurable*

random variable. Suppose also $E|Zf(Y)^2| < \infty$, where $f \in L^2(\mathbb{R}^d)$. Then, we have

$$\begin{aligned} E(f(Y)^2 Z | \mathcal{F}_a) &= \sum_{\sigma \subset \{1, \dots, d\}} (-1)^{d-|\sigma|} \int_{Q_\sigma(A)} f(x)^2 \\ &\quad \times E(\mathbf{1}_{\{Y^i > x^i, i \in \sigma, Y^i < x^i, i \notin \sigma\}} H_{(1, \dots, d)}^{a,b}(Y, Z) | \mathcal{F}_a) dx, \end{aligned} \quad (2.13)$$

where $Q_\sigma(A) = \{x \in \mathbb{R}^d : A^i < x^i, i \in \sigma, A^i > x^i, i \notin \sigma\}$ and $|\sigma|$ is the cardinal of σ .

As a consequence, taking $\sigma = \{1, \dots, d\}$, the conditional density of Y given \mathcal{F}_a has the following expression:

$$p_a(x) = E(\mathbf{1}_{\{Y > x\}} H_{(1, \dots, d)}^{a,b}(Y, 1) | \mathcal{F}_a).$$

Proof. As in Corollary 2 we have

$$\begin{aligned} E(f(Y)^2 Z | \mathcal{F}_a) &= \lim_n \int_{\mathbb{R}^d} f(x)^2 E(f_n(x, Y) Z | \mathcal{F}_a) dx \\ &= \lim_n \sum_{\sigma \subset \{1, \dots, d\}} \int_{Q_\sigma(A)} f(x)^2 E(f_n(x, Y) Z | \mathcal{F}_a) dx, \end{aligned} \quad (2.14)$$

where f_n is defined by (2.10). For any $\sigma = \{i_1, \dots, i_j\} \subset \{1, \dots, d\}$ consider the function

$$g_n^\sigma(x, y) = \int_{-\infty}^{y_{i_1}} \dots \int_{-\infty}^{y_{i_j}} \int_{y_{i_{j+1}}}^\infty \dots \int_{y_{i_d}}^\infty f_n(x, \eta_1, \dots, \eta_d) d\eta_1 \dots d\eta_d.$$

We have the following relationship between the functions f_n and g_n^σ :

$$\partial_\alpha g_n^\sigma(x, \cdot) = (-1)^{d-|\sigma|} f_n(x, \cdot)$$

with $\alpha = (1, \dots, d)$.

From (2.14) and using a conditional version of Proposition 1, which can be proved easily using (2.12), yields

$$\begin{aligned} E(f(Y)^2 Z | \mathcal{F}_a) &= \sum_{\sigma \subset \{1, \dots, d\}} (-1)^{d-|\sigma|} \lim_n \int_{Q_\sigma(A)} f(x)^2 E(\partial_\alpha g_n^\sigma(x, Y) Z | \mathcal{F}_a) dx \\ &= \sum_{\sigma \subset \{1, \dots, d\}} (-1)^{d-|\sigma|} \lim_n \int_{Q_\sigma(A)} f(x)^2 E(g_n^\sigma(x, Y) H_\alpha^{a,b}(Y, Z) | \mathcal{F}_a) dx \\ &= \sum_{\sigma \subset \{1, \dots, d\}} (-1)^{d-|\sigma|} \int_{Q_\sigma(A)} f(x)^2 E(\mathbf{1}_{\{Y^i > x^i, i \in \sigma, Y^i < x^i, i \notin \sigma\}} H_\alpha^{a,b}(Y, Z) | \mathcal{F}_a) dx, \end{aligned}$$

which completes the proof. \square

Definition 1. For any function $f \in L^2([0, T]^n)$, any random variable $F \in \mathbb{D}^{k,p}$, and any process u such that $u_t \in \mathbb{D}^{k,p}$ for all $t \in [0, T]$, we define $\|\cdot\|_{a, H^{\otimes n}}$, $\|\cdot\|_{k,p}^{\mathcal{F}_a}$ and $\|\cdot\|_{k,p,H}^{\mathcal{F}_a}$ as

$$\|f\|_{a, H^{\otimes n}} = \left(\int_{[0, T]^n} f(s)^2 ds \right)^{1/2},$$

$$\|F\|_{k,p}^{\mathcal{F}_a} = \left\{ E(|F|^p | \mathcal{F}_a) + \sum_{j=1}^k E(\|D^j F\|_{a,H^{\otimes j}}^p | \mathcal{F}_a) \right\}^{1/p},$$

$$\|u\|_{k,p,H}^{\mathcal{F}_a} = \left\{ E(\|u\|_{a,H}^p | \mathcal{F}_a) + \sum_{j=1}^k E(\|D^j u\|_{a,H^{\otimes(j+1)}}^p | \mathcal{F}_a) \right\}^{1/p}.$$

Then the following conditional version of inequality (2.11) holds

$$\|\delta(u\mathbf{1}_{[a,T]})\|_{k,p}^{\mathcal{F}_a} \leq c_p \|u\|_{k+1,p,H}^{\mathcal{F}_a}. \quad (2.15)$$

3. Existence of the quadratic covariation and an extension of Itô's formula

Let $X = \{X_t, t \in [0, T]\}$ be a d -dimensional continuous and adapted stochastic process. Consider a sequence D_n of partitions of $[0, T]$. The points of a partition D_n will be denoted by $0 = t_0 < t_1 < \dots < t_{k(n)} < t_{k(n)+1} = T$. We will assume that this sequence satisfies the following conditions:

$$\lim_n \sup_{t_i \in D_n} (t_{i+1} - t_i) = 0, \quad L := \sup_n \sup_{t_i \in D_n} \frac{t_{i+1}}{t_i} < \infty. \quad (3.1)$$

Definition 2. Given two stochastic processes $Y = \{Y_t, t \in [0, T]\}$ and $Z = \{Z_t, t \in [0, T]\}$ we define their *quadratic covariation* as the stochastic process $[Y, Z]$ given by the following limit in probability, if it exists,

$$[Y, Z]_t = \lim_n \sum_{t_i \in D_n, t_i < t} (Y_{t_{i+1}} - Y_{t_i})(Z_{t_{i+1}} - Z_{t_i}).$$

Let $\mathcal{W}^{1,p}(\mathbb{R}^d)$ denote the Sobolev space of functions in $L^p(\mathbb{R}^d)$ such that the weak first derivatives belong to $L^p(\mathbb{R}^d)$. We denote by $\mathcal{W}_{\text{loc}}^{1,p}(\mathbb{R}^d)$ the space of functions that coincide on each compact set with a function in $\mathcal{W}^{1,p}(\mathbb{R}^d)$. For any $F \in \mathcal{W}_{\text{loc}}^{1,p}(\mathbb{R}^d)$ we denote by $f_k = \partial F / \partial x_k$ the k th weak partial derivative of F .

The next result provides sufficient conditions for the existence of the quadratic covariation $[f(X), X^k]$ for all $k = 1, \dots, d$, when $f: \mathbb{R}^d \rightarrow \mathbb{R}$ is in $L_{\text{loc}}^p(\mathbb{R}^d)$ and X is a d -dimensional martingale. Under these conditions we can write a change-of-variable formula for a process of the form $F(X_t)$ with F in $\mathcal{W}^{1,p}(\mathbb{R}^d)$, where the last term of the formula is the sum with respect to k of the quadratic covariations $\frac{1}{2}[f_k(X), X^k]$, f_k being the k th weak partial derivative of F .

Theorem 5. Let $X = \{(X_t^1, \dots, X_t^d), t \in [0, T]\}$ be a continuous and adapted stochastic process of the form $X_t^k = \sum_{i=1}^d \int_0^t u_s^{k,i} dW_s^i$, where for all $k, i = 1, \dots, d$, $u^{k,i}$ is adapted and $\int_0^T (u_s^{k,i})^2 ds < \infty$ a.s. Suppose that for all $\delta > 0$ there exist constants $c_j^\delta, j = 1, 2$, such that for any n , for any k and for any $t \in [0, T]$, we have,

$$P \left\{ \int_0^T f(X_s^k)^2 |u_s^k|^2 ds > \delta \right\} \leq c_1^\delta \|f\|_p^2, \quad (3.2)$$

$$P \left\{ \left| \sum_{\substack{t_i \in D_n, \\ t_i < t}} [f(X_{t_{i+1}}) - f(X_{t_i})](X_{t_{i+1}}^k - X_{t_i}^k) \right| > \delta \right\} \leq c_2^\delta \|f\|_p, \quad (3.3)$$

for any function f in $C_K^\infty(\mathbb{R}^d)$ (infinitely differentiable with compact support). Then the quadratic covariation $[f(X), X^k]$ exists for any function f in $L_{\text{loc}}^p(\mathbb{R}^d)$ and for any k . Moreover, for any function $F \in \mathcal{W}_{\text{loc}}^{1,p}(\mathbb{R}^d)$, the following Itô's formula holds

$$F(X_t) = F(0) + \sum_{k=1}^d \int_0^t f_k(X_s) dX_s^k + \frac{1}{2} \sum_{k=1}^d [f_k(X), X^k]_t \quad (3.4)$$

for all $t \in [0, T]$, where f_k denotes the k th weak partial derivative of F .

Proof. Notice that by an easy approximation argument inequalities (3.2) and (3.3) hold for any function f in $L^p(\mathbb{R}^d)$.

Fix $t \in [0, T]$, and set for all $k = 1, \dots, d$,

$$V_n^k(f) = \sum_{t_i \in D_n, t_i < t} [f(X_{t_{i+1}}) - f(X_{t_i})](X_{t_{i+1}}^k - X_{t_i}^k).$$

For each $n \geq 0$ set $K_n = \{x \in \mathbb{R}^d, |x| \leq n\}$ and consider the stopping time $T_n = \inf\{t: X_t \notin K_n\}$. Let $\varepsilon > 0$ and take n_0 in such a way that $P(T_{n_0} \leq t) \leq \varepsilon$. Let g be an infinitely differentiable function with support included in K_{n_0} such that

$$\int_{K_{n_0}} |g(x) - f(x)|^p dx \leq \varepsilon^p.$$

For all $k = 1, \dots, d$, and $n, m \geq n_0$ we have that

$$\begin{aligned} P(|V_n^k(f) - V_m^k(f)| > \delta) &\leq P(T_{n_0} \leq t) + P\left(T_{n_0} > t, |V_n^k(f - g)| > \frac{\delta}{3}\right) \\ &\quad + P\left(T_{n_0} > t, |V_m^k(f - g)| > \frac{\delta}{3}\right) \\ &\quad + P\left(T_{n_0} > t, |V_n^k(g) - V_m^k(g)| > \frac{\delta}{3}\right) \\ &\leq \varepsilon + 2c_2^\delta \varepsilon + P\left(|V_n^k(g) - V_m^k(g)| > \frac{\delta}{3}\right). \end{aligned}$$

We know that $\lim_{n,m} P(|V_n^k(g) - V_m^k(g)| > \delta/3) = 0$ for all $k = 1, \dots, d$. As a consequence, the quadratic covariation $[f(X), X^k]$ exists for any function f in $L_{\text{loc}}^p(\mathbb{R}^d)$, and

$$P(|[f(X), X^k]_t| > \delta) \leq c_2^\delta \|f\|_p \quad (3.5)$$

for all $k = 1, \dots, d$ and for any f in $L^p(\mathbb{R}^d)$.

We can approximate F by functions $F_n \in C^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ in such a way that the partial derivatives satisfy that $\|f_k^n - f_k\|_p$ converges to zero as n tends to infinity. In order to show Itô's formula we can assume, by a localization argument, that the

process X_t takes values in a compact set $K \subset \mathbb{R}^d$ and that F and f_k have support in this set. We know that for each n Itô's formula holds, that is,

$$F_n(X_t) = F(0) + \sum_{k=1}^d \int_0^t f_k^n(X_s) dX_s^k + \frac{1}{2} \sum_{k=1}^d [f_k^n(X), X^k]_t. \quad (3.6)$$

By (3.5) we have that $[f_k^n(X), X^k]_t$ converges in probability to $[f_k(X), X^k]_t$ for all $k = 1, \dots, d$, as n tends to infinity. On the other hand, we need to prove that $\int_0^t f_k^n(X_s) dX_s^k$ converges in probability to $\int_0^t f_k(X_s) dX_s^k$. This follows from the inequalities

$$\begin{aligned} P \left(\left| \int_0^t (f_k - f_k^n)(X_s) dX_s^k \right| > \delta \right) &\leq \frac{M}{\delta^2} + P \left(\int_0^t (f_k - f_k^n)^2(X_s) |u_s^k|^2 ds > M \right) \\ &\leq \frac{M}{\delta^2} + c_1^M \|f_k^n - f_k\|_p^2. \end{aligned}$$

Then taking the limit in (3.6) we obtain (3.4) and this completes the proof of the theorem. \square

4. Basic estimates for stochastic integrals and Malliavin matrix

Let $u = (u^{i,j})_{1 \leq i,j \leq d}$ be a matrix of adapted processes $u^{i,j} = \{u_t^{i,j}, t \in [0, T]\}$ such that $E \int_0^T |u_s|^2 ds < \infty$. Set $X_t^k = \sum_{i=1}^d \int_0^t u_s^{k,i} dW_s^i$. Let us introduce the following hypotheses on the process u :

(H1) $_{n,p}$ For each $t \in [0, T]$ we have $u_t \in \mathbb{D}^{n,2}(\mathbb{R}^{d^2})$, and for some $p \geq 2$ we have

$$E|u_r|^p + E|D_{t_1} u_r|^p + \dots + E|D_{t_1, t_2, \dots, t_n} u_r|^p \leq K_{n,p}$$

for any $r, t_1, \dots, t_n \in [0, T]$.

(H2) $\sum_{i=1}^d |\sum_{k=1}^d u_t^{k,i} v_k|^2 \geq \rho^2 > 0$ for some constant ρ , for all $t \in [0, T]$ and for all $v \in \mathbb{R}^d$ such that $|v| = 1$.

(H3) $|u_t| \leq M$ for some constant M , for all $t \in [0, T]$.

This section will be devoted to obtain some estimations of the $\|\cdot\|_{n,p}$ -norm of the inverse of the Malliavin matrix $\gamma_{X_p}^{a,b}$ (Lemma 10 for $n = 0$ and Lemma 11) and of $H_{\alpha}^{a,b}(X_p, Z)$ (Lemma 12), plus the conditional versions of these results (Lemmas 13 and 14). Lemmas 6–9, are previous estimates which are needed in order to prove the above-mentioned results.

For the proof of the following results we will need Burkholder's inequality for Hilbert space valued martingales (see Métivier, 1982, E.2, p. 212). That is, if $\{M_t, t \in [0, T]\}$ is a continuous local martingale with values in a Hilbert space H , then for any $p > 0$ we have

$$E\|M_t\|_H^p \leq b_p E([M]_t^{p/2}), \quad (4.1)$$

where

$$[M]_t = \sum_{i=1}^{\infty} [\langle M, e_i \rangle_H]_t$$

$\{e_i, i \geq 1\}$ being a complete orthonormal system in H .

Lemma 6. Assume that u satisfies condition $(H1)_{n,p}$ for some $p \geq 2$ and $n \geq 1$. Then, we have

$$\sup_{t_1, \dots, t_n \in [0, T]} E|D_{t_1, \dots, t_n} X_s|^p \leq c_{n,p}^1, \quad (4.2)$$

for some constant $c_{n,p}^1$ of the form $c_p K_{n,p}$, where c_p depends on p and T .

Proof. Using the properties of the derivative operator we can write for $t_1, \dots, t_n \leq t$

$$D_{t_1, \dots, t_n} X_s = \sum_{i=1}^n D_{t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n} u_{t_i} + \int_{t_1 \vee \dots \vee t_n}^s D_{t_1, \dots, t_n} u_r dW_r.$$

As a consequence, applying Burkholder's inequality (4.1) we obtain

$$\begin{aligned} E|D_{t_1, \dots, t_n} X_s|^p &\leq 2^{p-1} \left(E \left| \sum_{i=1}^n D_{t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n} u_{t_i} \right|^p + E \left| \int_{t_1 \vee \dots \vee t_n}^s D_{t_1, \dots, t_n} u_r dW_r \right|^p \right) \\ &\leq 2^{p-1} \left(n^p \sup_{t_1, \dots, t_n} E|D_{t_1, \dots, t_{n-1}} u_{t_n}|^p + E \left| \int_{t_1 \vee \dots \vee t_n}^s D_{t_1, \dots, t_n} u_r dW_r \right|^p \right) \\ &\leq 2^{p-1} n^p K_{n,p} + 2^{p-1} b_p E \left(\int_{t_1 \vee \dots \vee t_n}^s |D_{t_1, \dots, t_n} u_r|^2 dr \right)^{p/2} \\ &\leq 2^{p-1} K_{n,p} (n^p + b_p T^{p/2}), \end{aligned}$$

where b_p is the Burkholder constant. This proves (4.2). \square

Remark 1. The following inequality can be proved in an analogous way for any $s \geq a$:

$$\begin{aligned} E \left(\int_{[a, T]^n} |D_{t_1, \dots, t_n} X_s|^p dt_1 \dots dt_n | \mathcal{F}_a \right) &\leq \alpha_{n,p}^1 E \left(\int_{[a, T]^n} |D_{t_1, \dots, t_{n-1}} u_{t_n}|^p dt_1 \dots dt_n | \mathcal{F}_a \right) \\ &\quad + \alpha_{n,p}^2 E \left(\int_{[a, T]^{n+1}} |D_{t_1, \dots, t_n} u_r|^p dt_1 \dots dt_n dr | \mathcal{F}_a \right), \end{aligned}$$

where $\alpha_{n,p}^1 = 2^{p-1} n^p$ and $\alpha_{n,p}^2 = 2^{p-1} b_p T^{p/2-1}$.

Lemma 7. Fix $0 \leq a < b \leq T$. We define $\Delta_{a,b} X^k = X_b^k - X_a^k$. Then

(i) If u satisfies condition $(H1)_{1,p}$ for some $p \geq 2$, then

$$\sup_{t \in [0, a]} E|D_t(\Delta_{a,b} X)|^p \leq c_1 (b-a)^{p/2} \quad (4.3)$$

for some constant c_1 depending on $K_{1,p}$, p and T .

(ii) If u satisfies condition $(H1)_{2,p}$ for some $p \geq 2$, then

$$\sup_{t \in [0, a]} E \left(\int_0^T |D_s D_t(\Delta_{a,b} X)|^2 ds \right)^{p/2} \leq c_2 (b-a)^{p/2} \quad (4.4)$$

for some constant c_2 depending on $K_{2,p}$, p and T .

Proof. Let us show (4.3). We know that for $t \leq a$

$$D_t(\Delta_{a,b}X) = \int_a^b D_t u_s \, dW_s. \quad (4.5)$$

Hence, we can write using Burkholder's inequality (4.1)

$$\begin{aligned} E|D_t(\Delta_{a,b}X)|^p &\leq b_p E \left(\int_a^b |D_t u_s|^2 \, ds \right)^{p/2} \\ &\leq b_p (b-a)^{p/2} \sup_{t,s} E|D_t u_s|^p \end{aligned}$$

and (4.3) holds. In the same way, from (4.5) we have that if $t < a$

$$\begin{aligned} E \left(\int_0^T |D_s D_t(\Delta_{a,b}X)|^2 \, ds \right)^{p/2} &= E \left(\int_0^T \left| D_t u_s \mathbf{1}_{[a,b]}(s) + \int_a^b D_s D_t u_\theta \, dW_\theta \right|^2 \, ds \right)^{p/2} \\ &\leq 2^{p-1} \left\{ (b-a)^{p/2} \sup_{s \in [0,T]} E|D_s u_t|^p + T^{p/2-1} E \int_0^T \left| \int_a^b D_s D_t u_\theta \, dW_\theta \right|^2 \, ds \right\} \\ &\leq 2^{p-1} \left\{ K_{2,p} (b-a)^{p/2} + T^{p/2-1} b_p E \int_0^T \left(\int_a^b |D_s D_t u_\theta|^2 \, d\theta \right)^{p/2} \, ds \right\} \\ &\leq 2^{p-1} K_{2,p} (b-a)^{p/2} (1 + T^{p/2} b_p) \end{aligned}$$

and we obtain (4.4). \square

Lemma 8. Assume u satisfies condition (H1) $_{n,p}$ for some $n \geq 0$ and some $p \geq 2$. Then, we have

$$\|\Delta_{a,b}X^k\|_{n,p} \leq c_{n,p}^2 (b-a)^{1/2}, \quad (4.6)$$

for some constant $c_{n,p}^2$ depending on $K_{n,p}$, n , p and T .

Proof. By the definition of $\|\cdot\|_{n,p}$ -norm we have

$$\|\Delta_{a,b}X^k\|_{n,p}^p = E|\Delta_{a,b}X^k|^p + \sum_{j=1}^n \|D^j(\Delta_{a,b}X^k)\|_{H^{\otimes j}}^p. \quad (4.7)$$

For the first summand using Burkholder's inequality (4.1) yields

$$\begin{aligned} E|\Delta_{a,b}X^k|^p &= E \left| \int_a^b u_s^k \, dW_s \right|^p \leq b_p E \left(\int_a^b |u_s^k|^2 \, ds \right)^{p/2} \\ &\leq b_p (b-a)^{p/2} \sup_{0 \leq s \leq T} E|u_s^k|^p. \end{aligned} \quad (4.8)$$

For the other terms, using again Burkholder's inequality, we have that for all $1 \leq j \leq n$

$$\begin{aligned}
 & E \left\| D^j \left(\int_a^b u_s^k dW_s \right) \right\|_{H^{\otimes j}}^p \\
 &= E \left\| \sum_{i=1}^j D_{r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_j} u_{r_i}^k \mathbf{1}_{[a,b]}(r_i) + \int_a^b D_{r_1, \dots, r_j} u_\theta^k dW_\theta \right\|_{H^{\otimes j}}^p \\
 &\leq 2^{p-1} j^p E \left(\int_{[0,T]^{j-1}} \int_a^b |D_{r_1, \dots, r_{j-1}} u_{r_j}^k|^2 dr_1 \dots dr_j \right)^{p/2} \\
 &\quad + 2^{p-1} b_p E \left(\int_a^b \|D^j u_\theta^k\|_{H^{\otimes j} \otimes \mathbb{R}^d}^2 d\theta \right)^{p/2} \\
 &\leq 2^{p-1} (b-a)^{p/2} \left(j^p T^{(j-1)p/2} K_{n,p} + b_p \sup_\theta E \|D^j u_\theta^k\|_{H^{\otimes j} \otimes \mathbb{R}^d}^p \right) \\
 &\leq 2^{p-1} K_{n,p} T^{(j-1)p/2} (b-a)^{p/2} (j^p + b_p T^{p/2}). \tag{4.9}
 \end{aligned}$$

Finally from (4.8) and (4.9) we obtain (4.7). \square

Lemma 9. Let $x = \{x_s, s \in [0, T]\}$ and $y = \{y_s, s \in [0, T]\}$ be d -dimensional stochastic processes satisfying

$$\begin{aligned}
 K_{n,2p}^x &:= \sup_{s \in [a,b]} \sum_{i=0}^n E(\|D^i x_s\|_{H^{\otimes i} \otimes \mathbb{R}^d}^{2p}) < \infty, \\
 K_{n,2p}^y &:= \sup_{s \in [a,b]} \sum_{i=0}^n E(\|D^i y_s\|_{H^{\otimes i} \otimes \mathbb{R}^d}^{2p}) < \infty
 \end{aligned} \tag{4.10}$$

for some $n \geq 0$ and some $p \geq 1$. Then, for any $0 \leq a < b \leq T$ we have

$$\left\| \int_a^b \langle x_s, y_s \rangle ds \right\|_{n,p} \leq c_{n,p}^3 (K_{n,2p}^x K_{n,2p}^y)^{1/2p} (b-a) \tag{4.11}$$

for some constant $c_{n,p}^3$ depending on n and p .

Proof. In order to simplify the proof we will suppose $d = 1$. On one hand we have that

$$\begin{aligned}
 E \left| \int_a^b x_s y_s ds \right|^p &\leq \left(E \left| \int_a^b x_s^2 ds \right|^p \right)^{1/2} \left(E \left| \int_a^b y_s^2 ds \right|^p \right)^{1/2} \\
 &\leq (b-a)^p \sup_{s \in [a,b]} (E|x_s|^{2p})^{1/2} \sup_{s \in [a,b]} (E|y_s|^{2p})^{1/2} \\
 &\leq (K_{n,2p}^x K_{n,2p}^y)^{1/2} (b-a)^p,
 \end{aligned}$$

where $K_{n,2p}^x$ and $K_{n,2p}^y$ are the constants defined by (4.10). On the other hand, for each $1 \leq j \leq n$ we have

$$\begin{aligned}
 & E \left\| D^j \left(\int_a^b x_s y_s \, ds \right) \right\|_{H^{\otimes j}}^p \\
 &= E \left\| \sum_{i=0}^j \binom{j}{i} \int_a^b D^i x_s D^{j-i} y_s \, ds \right\|_{H^{\otimes j}}^p \\
 &\leq (j+1)^{p-1} \sum_{i=0}^j \binom{j}{i}^p E \left\| \int_a^b D^i x_s D^{j-i} y_s \, ds \right\|_{H^{\otimes j}}^p \\
 &\leq (j+1)^{p-1} (b-a)^p \\
 &\quad \times \sum_{i=0}^j \binom{j}{i}^p \left\{ \sup_{s \in [a,b]} E \|D^i x_s\|_{H^{\otimes i}}^{2p} \sup_{s \in [a,b]} E \|D^{j-i} y_s\|_{H^{\otimes (j-i)}}^{2p} \right\}^{1/2} \\
 &\leq (j+1)^{p-1} \sum_{i=0}^j \binom{j}{i}^p (K_{n,2p}^x K_{n,2p}^y)^{1/2} (b-a)^p
 \end{aligned}$$

and (4.11) holds. \square

Lemma 10. Let $(\gamma_{X_b}^{a,b})^{-1}$ the inverse of the matrix $\gamma_{X_b}^{a,b}$ defined by (2.5). Suppose that u satisfies hypotheses (H1) $_{1,p'}$ for some $p' > 12$, and (H2). Then, for any $1 \leq p < (p' - 4)/4d$ we have

$$E[(\gamma_{X_b}^{a,b})_{ij}^{-1}]^p \leq \left(k_1 + k_2 E \int_{[0,T]^2} |D_t u_r|^{p'} \, dt \, dr \right) (b-a)^{-p}, \quad (4.12)$$

for some constants k_1, k_2 depending on p, p', T, d and ρ .

Proof. We have that

$$|(\gamma_{X_b}^{a,b})_{ij}^{-1}| = |A_{ij}(\det \gamma_{X_b}^{a,b})^{-1}|,$$

where A_{ij} is the adjoint of $(\gamma_{X_b}^{a,b})_{ij}$. Hence,

$$E|(\gamma_{X_b}^{a,b})_{ij}^{-1}|^p \leq c_{d,p} E[(\det \gamma_{X_b}^{a,b})^{-2p}]^{1/2} E[\|DX_b \mathbf{1}_{[a,b]}\|_H^{4p(d-1)}]^{1/2}. \quad (4.13)$$

For the second factor, using Lemma 6 with $n = 1$ and $p = 4p(d-1)$ yields

$$\begin{aligned}
 E(\|DX_b \mathbf{1}_{[a,b]}\|_H^{4p(d-1)}) &= E \left(\int_a^b |D_s X_b|^2 \, ds \right)^{2p(d-1)} \\
 &\leq (b-a)^{2p(d-1)} \sup_s E |D_s X_b|^{4p(d-1)} \\
 &\leq c_{1,4p(d-1)}^1 (b-a)^{2p(d-1)}.
 \end{aligned} \quad (4.14)$$

In order to estimate the first factor we write

$$\det \gamma_{X_b}^{a,b} \geq \inf_{|v|=1} (v^T \gamma_{X_b}^{a,b} v)^d = \inf_{|v|=1} \left(\int_a^b \sum_{k=1}^d \left| \sum_{j=1}^d D_s^{(k)} X_b^j v_j \right|^2 ds \right)^d.$$

Then we have for any $h \in [0, 1]$ and using (H2)

$$\begin{aligned} & \int_a^b \sum_{k=1}^d \left| \sum_{j=1}^d D_s^{(k)} X_b^j v_j \right|^2 ds \\ &= \int_a^b \sum_{k=1}^d \left| \sum_{j=1}^d v_j (u_s^{j,k} + \sum_{i=1}^d \int_s^b D_s^{(k)} u_r^{j,i} dW_r^i) \right|^2 ds \\ &\geq \frac{1}{2} \int_{a+(b-a)(1-h)}^b \sum_{k=1}^d \left| \sum_{j=1}^d v_j u_s^{j,k} \right|^2 ds - \int_{a+(b-a)(1-h)}^b \sum_{k=1}^d \left| \sum_{i,j=1}^d v_j \int_s^b D_s^{(k)} u_r^{j,i} dW_r^i \right|^2 ds \\ &\geq \frac{\rho^2(b-a)h}{2} - \int_{a+(b-a)(1-h)}^b \left| \int_s^b D_s u_r dW_r \right|^2 ds \\ &= \frac{\rho^2(b-a)h}{2} - I_h, \end{aligned}$$

where

$$I_h = \int_{a+(b-a)(1-h)}^b \left| \int_s^b D_s u_r dW_r \right|^2 ds.$$

We choose $h = 4/(b-a)\rho^2 y^{1/d}$, where $y \geq c := 4^d/(b-a)^d \rho^{2d}$. Then, we can write for any $q \geq 2$

$$\begin{aligned} E|I_h|^q &\leq b_q(b-a)^{q-1} h^{q-1} E \int_{a+(b-a)(1-h)}^b \left(\int_s^b |D_s u_r|^2 dr \right)^q ds \\ &\leq b_q(b-a)^{2q-2} h^{2q-2} E \int_{[0,T]^2} |D_s u_r|^{2q} dr ds. \end{aligned}$$

As a consequence,

$$\begin{aligned} E[(\det \gamma_{X_b}^{a,b})^{-2p}] &= \int_0^\infty 2p y^{2p-1} P\{(\det \gamma_{X_b}^{a,b})^{-1} > y\} dy \\ &\leq c^{2p} + 2p \int_c^\infty y^{2p-1} P\left\{\det \gamma_{X_b}^{a,b} < \frac{1}{y}\right\} dy \\ &\leq \left(\frac{4^d}{(b-a)^d \rho^{2d}}\right)^{2p} + 2p \int_c^\infty E|I_h|^q y^{2p-1+q/d} dy \\ &\leq \left(\frac{4^d}{\rho^{2d}}\right)^{2p} (b-a)^{-2dp} + 2pb_q \left(\frac{4}{\rho^2}\right)^{2q-2} \end{aligned}$$

$$\begin{aligned}
& \times E \int_{[0,T]^2} |D_s u_r|^{2q} \, dr \, ds \int_c^\infty y^{2p-1-(q/d)+2/d} \, dy \\
& \leq (b-a)^{-2dp} 4^{2dp} \rho^{-4dp} \left\{ 1 + 2pb_q 4^q \rho^{-2q} (b-a)^{q-2} \right. \\
& \quad \times \left. \frac{d}{q-2dp-2} \left(E \int_{[0,T]^2} |D_s u_r|^{2q} \, ds \, dr \right) \right\} \\
& \leq \left(c_1 + c_2 E \int_{[0,T]^2} |D_s u_r|^{2q} \, ds \, dr \right) (b-a)^{-2dp}, \tag{4.15}
\end{aligned}$$

where c_1 and c_2 are constants depending on q, ρ, T, d and p , and provided $q > 2dp+2$. We will take $p' = 2q > 4(dp+1) \geq 12$.

Finally, from (4.13)–(4.15) we get (4.12). \square

Remark 2. With the additional hypothesis (H3) the following conditional version of the previous result holds:

$$\begin{aligned}
E[(\gamma_{X_b}^{a,b})_{ij}^{-1}]^p | \mathcal{F}_a] & \leq (b-a)^{-p} \left\{ a_1 + a_2 E \left(\int_{[0,T]^2} |D_t u_r|^{p'} \, dt \, dr | \mathcal{F}_a \right) \right\} \\
& \times \left\{ b_1 + b_2 E \left(\int_{[0,T]^2} |D_s u_r|^{4p(d-1)} \, ds \, dr | \mathcal{F}_a \right) \right\}
\end{aligned}$$

for some constants a_1, a_2, b_1, b_2 depending on M, ρ, d, T, p and p' .

Proof. The proof is similar to that of Lemma 10. We have that

$$E[(\gamma_{X_b}^{a,b})_{ij}^{-1}]^p | \mathcal{F}_a] \leq c_{d,p} E[(\det \gamma_{X_b}^{a,b})^{-2p} | \mathcal{F}_a]^{1/2} E[\|DX_b \mathbf{1}_{[a,b]}\|_H^{4p(d-1)} | \mathcal{F}_a]^{1/2}.$$

The following conditional version of inequality (4.15) holds:

$$E[(\det \gamma_{X_b}^{a,b})^{-2p} | \mathcal{F}_a] \leq (b-a)^{-2dp} \left\{ k_1 + k_2 E \left(\int_{[0,T]^2} |D_s u_r|^{p'} \, ds \, dr | \mathcal{F}_a \right) \right\}. \tag{4.16}$$

On the other hand, we have for some $q \geq 4$,

$$\begin{aligned}
& E[\|DX_b \mathbf{1}_{[a,b]}\|_H^q | \mathcal{F}_a] \\
& = E \left(\left(\int_a^b |D_s X_b|^2 \, ds \right)^{q/2} \, ds | \mathcal{F}_a \right) \\
& \leq E \left(\left(\int_a^b \left| u_s + \int_a^b D_s u_\theta \, dW_\theta \right|^2 \, ds \right)^{q/2} \, ds | \mathcal{F}_a \right) \\
& \leq 2^{q-1} \left\{ M^q (b-a)^{q/2} + (b-a)^{q/2-1} E \left(\int_a^b \left| \int_a^b D_s u_\theta \, dW_\theta \right|^2 \, ds | \mathcal{F}_a \right)^q \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq 2^{q-1} \left\{ M^q (b-a)^{q/2} + b_q (b-a)^{q/2-1} E \left(\int_a^b \left(\int_a^b |D_s u_\theta|^2 d\theta \right)^{q/2} ds | \mathcal{F}_a \right) \right\} \\
&\leq 2^{q-1} (b-a)^{q/2} \left\{ M^q + b_q T^{q/2-2} E \left(\int_a^b \int_a^b |D_s u_\theta|^q d\theta ds | \mathcal{F}_a \right) \right\}. \quad (4.17)
\end{aligned}$$

Hence, from (4.16) and using (4.17) with $q = 4p(d-1)$ we obtain

$$\begin{aligned}
E[(\gamma_{X_b}^{a,b})_{ij}^{-1}]^p | \mathcal{F}_a] &\leq (b-a)^{-p} \left\{ k_1 + k_2 E \left(\int_{[0,T]^2} |D_s u_r|^{p'} ds dr | \mathcal{F}_a \right) \right\}^{1/2} \\
&\quad \times \left\{ k'_1 + k'_2 E \left(\int_{[0,T]^2} |D_s u_r|^{4p(d-1)} ds dr | \mathcal{F}_a \right) \right\}^{1/2} \\
&\leq (b-a)^{-p} \left\{ a_1 + a_2 E \left(\int_{[0,T]^2} |D_s u_r|^{p'} ds dr | \mathcal{F}_a \right) \right\} \\
&\quad \times \left\{ b_1 + b_2 E \left(\int_{[0,T]^2} |D_s u_r|^{4p(d-1)} ds dr | \mathcal{F}_a \right) \right\},
\end{aligned}$$

where for the last inequality we have used the fact that $\sqrt{x} \leq 1 + x$. \square

Lemma 11. Assume u satisfies $(H1)_{n+1,p}$ for all $p \geq 2$ and some fixed $n \geq 0$, and $(H2)$. Then, for all $p \geq 2$

$$\|(\gamma_{X_b}^{a,b})^{-1}\|_{n,p} \leq c_{n,p}^4 (b-a)^{-1}$$

for some constant $c_{n,p}^4$ depending on n, p, ρ, T and $K_{n+1,p'}$, where $p' > 4dp(n+1)^2 + 4$.

Proof. For any $0 \leq k \leq n$ we can write

$$\begin{aligned}
&E \|D^k((\gamma_{X_b}^{a,b})^{-1})\|_{H^{\otimes k}}^p \\
&\leq c \sum_{i_1 + \dots + i_r = k} E \|(\gamma_{X_b}^{a,b})^{-1} D^{i_1} \gamma_{X_b}^{a,b} \dots (\gamma_{X_b}^{a,b})^{-1} D^{i_r} \gamma_{X_b}^{a,b} (\gamma_{X_b}^{a,b})^{-1}\|_{H^{\otimes k}}^p \\
&\leq c \sum_{i_1 + \dots + i_r = k} E (\|D^{i_1} \gamma_{X_b}^{a,b}\|_{H^{\otimes i_1}}^p \dots \|D^{i_r} \gamma_{X_b}^{a,b}\|_{H^{\otimes i_r}}^p | (\gamma_{X_b}^{a,b})^{-1}|^{p(1+r)}) \\
&\leq c \sum_{i_1 + \dots + i_r = k} E (\|D^{i_1} \gamma_{X_b}^{a,b}\|_{H^{\otimes i_1}}^{p(r+1)})^{1/(r+1)} \dots E (\|D^{i_r} \gamma_{X_b}^{a,b}\|_{H^{\otimes i_r}}^{p(1+r)})^{1/(r+1)} \\
&\quad \times E(|(\gamma_{X_b}^{a,b})^{-1}|^{p(r+1)^2})^{1/(r+1)}. \quad (4.18)
\end{aligned}$$

In order to estimate the first factors we put

$$E \|D^k(\gamma_{X_b}^{a,b})_{ij}\|_{H^{\otimes k}}^p \leq \|(\gamma_{X_b}^{a,b})_{ij}\|_{k,p}^p$$

$$= \left\| \int_a^b \langle D_s X_b^i, D_s X_b^j \rangle ds \right\|_{k,p}^p \leq c'(b-a)^p, \quad (4.19)$$

where the last inequality has been obtained using Lemma 9 with $x = DX_b^i$ and $y = DX_b^j$ (x and y satisfy the required hypotheses due to the Lemma 6). Finally, from Lemma 10, (4.18) and (4.19) we obtain the desired result. \square

Lemma 12. Fix $n, m \geq 1$, $p \geq 2$ and $0 \leq a < b \leq T$. Suppose that u satisfies hypotheses (H1) $_{n+m+1, p'}$ for all $p' \geq 2$ and (H2). Let $Z \in \mathbb{D}^{n+m, 2^m p}$. Then, for any multi-index $\alpha \in \{1, \dots, d\}^m$ we have

$$\|H_{\alpha}^{a,b}(X_b, Z)\|_{n,p} \leq c_{n,p}^5 (b-a)^{-m/2} \|Z\|_{n+m, 2^m p}, \quad (4.20)$$

where $c_{n,p}^5$ is a constant depending on p, T, d and ρ .

Proof. Using the continuity of the operator δ we have

$$\begin{aligned} & \|H_{\alpha}^{a,b}(X_b, Z)\|_{n,p} \\ &= \|H_{(\alpha_m)}^{a,b}(X_b, H_{(1, \dots, \alpha_{m-1})}^{a,b}(X_b, Z))\|_{n,p} \\ &= \left\| \sum_{j=1}^d \delta(H_{(1, \dots, \alpha_{m-1})}^{a,b}(X_b, Z)) (\gamma_{X_b}^{a,b})_{ij}^{-1} DX_b^j \mathbf{1}_{[a,b]} \right\|_{n,p} \\ &\leq d^{p-1} \|H_{(1, \dots, \alpha_{m-1})}^{a,b}(X_b, Z)\|_{n+1, 2p} \sum_{j=1}^d \|(\gamma_{X_b}^{a,b})_{ij}^{-1}\|_{n+1, 4p} \|DX_b^j \mathbf{1}_{[a,b]}\|_{n+1, 4p} \\ &\leq d^p \|H_{(1, \dots, \alpha_{m-1})}^{a,b}(X_b, Z)\|_{n+1, 2p} \|(\gamma_{X_b}^{a,b})^{-1}\|_{n+1, 4p} \|DX_b \mathbf{1}_{[a,b]}\|_{n+1, 4p}. \end{aligned} \quad (4.21)$$

Using Lemma 6 it is easy to see that

$$\|DX_b \mathbf{1}_{[a,b]}\|_{n+1, 4p} \leq c(b-a)^{1/2} \quad (4.22)$$

and then, by Lemma 11, (4.21) and (4.22) we get

$$\|H_{(\alpha_m)}^{a,b}(X_b, Z)\|_{n,p} \leq c'(b-a)^{-1/2} \|H_{(1, \dots, \alpha_{m-1})}^{a,b}(X_b, Z)\|_{n+1, 2p}.$$

By an iteration procedure we obtain (4.20). \square

Now we will deduce the conditional versions of the last two results.

Lemma 13. Fix $n \geq 0$, $p \geq 2$. Assume u satisfies (H1) $_{n+1, p'}$ for all $p' \geq 2$, (H2) and (H3). Let $0 \leq a < b \leq T$. Then there exists a random variable $Z_{n,p}^a$ such that

$$\|(\gamma_{X_b}^{a,b})^{-1}\|_{n,p}^{\mathcal{F}_a} \leq (b-a)^{-1} Z_{n,p}^a, \quad (4.23)$$

where $Z_{n,p}^a$ has the form

$$Z_{n,p}^a = c \sum_{r=1}^n \left\{ 1 + \lambda_r^1 E \left(\int_{[0,T]^2} |D_t u_s|^{p'_r} dt ds | \mathcal{F}_a \right) \right\}$$

$$\begin{aligned}
& \times \left\{ 1 + \lambda_r^2 E \left(\int_{[0,T]^2} |D_t u_s|^{4p(r+1)^2(d-1)} dt ds | \mathcal{F}_a \right) \right\} \\
& \times \left\{ 1 + \sum_{m=0}^{n+1} \mu_{r,m} E \left(\int_{[a,T]^{m+1}} |D_{s_1, \dots, s_m} u_s|^{p(r+1)} ds_1 \dots ds_m ds | \mathcal{F}_a \right) \right\} \quad (4.24)
\end{aligned}$$

for some constants p'_r such that $p(r+1)^2 < (p'_r - 4)/4d$ and $\lambda_r^1, \lambda_r^2, \mu_{r,m}$ depending on ρ, M, p, d, r, p'_r and T .

Proof. As in proof of (4.18) in Lemma 11 we can write for $1 \leq k \leq n$

$$\begin{aligned}
& E(\|D^k((\gamma_{X_b}^{a,b})^{-1})\|_{a,H^{\otimes k}}^p | \mathcal{F}_a) \\
& \leq c \sum_{i_1 + \dots + i_r = k} \{E(\|D^{i_1} \gamma_{X_b}^{a,b}\|_{a,H^{\otimes i_1}}^{(r+1)p} | \mathcal{F}_a) \dots E(\|D^{i_r} \gamma_{X_b}^{a,b}\|_{a,H^{\otimes i_r}}^{(r+1)p} | \mathcal{F}_a) \\
& \quad \times E(\|(\gamma_{X_b}^{a,b})^{-1}\|_{a,H^{\otimes k}}^{p(r+1)^2} | \mathcal{F}_a)\}^{1/(r+1)}. \quad (4.25)
\end{aligned}$$

We can obtain the following conditional version of inequality (4.19):

$$\begin{aligned}
& E(\|D^k(\gamma_{X_b}^{a,b})_{ij}\|_{a,H^{\otimes k}}^q | \mathcal{F}_a) \\
& = E \left(\left\| D^k \left(\int_a^b \langle D_s X_b^i, D_s X_b^j \rangle ds \right) \right\|_{a,H^{\otimes k}}^q | \mathcal{F}_a \right) \\
& = E \left(\left\| \sum_{m=0}^k \binom{k}{m} \int_a^b \langle D^m D_s X_b^i, D^{k-m} D_s X_b^j \rangle ds \right\|_{a,H^{\otimes k}}^q | \mathcal{F}_a \right) \\
& \leq (k+1)^{q-1} \sum_{m=0}^k \binom{k}{m}^q E \left(\left\| \int_a^b \langle D^m D_s X_b^i, D^{k-m} D_s X_b^j \rangle ds \right\|_{a,H^{\otimes k}}^q | \mathcal{F}_a \right) \\
& \leq c(b-a)^{q-1} (b-a)^{k(\frac{q}{2}-1)} \\
& \quad \times \sum_{m=0}^k E \left(\int_{[a,T]^{m+1}} |D_{s_1, \dots, s_{m+1}} X_b^i|^{2q} ds_1 \dots ds_{m+1} | \mathcal{F}_a \right)^{1/2} \\
& \quad \times E \left(\int_{[a,T]^{k-m+1}} |D_{s_1, \dots, s_{k-m+1}} X_b^j|^{2q} ds_1 \dots ds_{k-m+1} | \mathcal{F}_a \right)^{1/2} \\
& \leq c(b-a)^q T^{(kq/2)-k-1} \sum_{m=0}^{k+1} E \left(\int_{[a,T]^m} |D_{s_1, \dots, s_m} X_b|^{2q} ds_1 \dots ds_m | \mathcal{F}_a \right). \quad (4.26)
\end{aligned}$$

Notice that we need $q \geq 4$. From (4.26) taking $q = (r+1)p$ and using Remark 1 we obtain

$$E(\|D^k \gamma_{X_b}^{a,b}\|_{a,H^{\otimes k}}^q | \mathcal{F}_a) \leq (b-a)^q \times \sum_{m=0}^{k+1} \alpha_{m,q} E \left(\int_{[a,T]^{m+1}} |D_{s_1, \dots, s_m} u_s|^q ds_1 \dots ds_m ds | \mathcal{F}_a \right), \quad (4.27)$$

where the constants $\alpha_{m,q}$ have the form $\alpha_{m,q} = C(\alpha_{m-1,q}^2 + \alpha_{m,q}^1)$, with C depending on k, q and T and $\alpha_{m,q}^1, \alpha_{m,q}^2$ the constants of the Remark 1.

Using Remark 2 with exponent $p(r+1)^2$ we have that there exist constants $p'_r, a_{1,r}, a_{2,r}, b_{1,r}, b_{2,r}$, such that $p(r+1)^2 < (p'_r - 4)/4d$ and

$$\begin{aligned} E(|(\gamma_{X_b}^{a,b})^{-1}|^{p(r+1)^2} | \mathcal{F}_a) \\ \leq (b-a)^{-p(r+1)^2} \left\{ a_{1,r} + a_{2,r} E \left(\int_{[0,T]^2} |D_t u_s|^{p'_r} dt ds | \mathcal{F}_a \right) \right\} \\ \times \left\{ b_{1,r} + b_{2,r} E \left(\int_{[0,T]^2} |D_t u_s|^{q_r} dt ds | \mathcal{F}_a \right) \right\}, \end{aligned} \quad (4.28)$$

where $q_r = 4p(r+1)^2(d-1)$. Hence, from (4.25), (4.27) and (4.28) we obtain for any $1 \leq j \leq n$

$$\begin{aligned} E(\|D^j((\gamma_{X_b}^{a,b})^{-1})\|_{a,H^{\otimes j}}^p | \mathcal{F}_a) \\ \leq c(b-a)^p \sum_{r=1}^j \left\{ a_{1,r} + a_{2,r} E \left(\int_{[0,T]^2} |D_t u_s|^{p'_r} dt ds | \mathcal{F}_a \right) \right\}^{1/(r+1)} \\ \times \left\{ b_{1,r} + b_{2,r} E \left(\int_{[0,T]^2} |D_t u_s|^{4p(r+1)^2(d-1)} dt ds | \mathcal{F}_a \right) \right\}^{1/(r+1)} \\ \times \left\{ \sum_{m=0}^{j+1} \mu_{r,m} E \left(\int_{[a,T]^{m+1}} |D_{s_1, \dots, s_m} u_s|^{p(r+1)} ds_1 \dots ds_m ds | \mathcal{F}_a \right) \right\}^{r/(r+1)} \\ \leq c(b-a)^p \sum_{r=1}^n \left\{ 1 + a_{1,r} + a_{2,r} E \left(\int_{[0,T]^2} |D_t u_s|^{p'_r} dt ds | \mathcal{F}_a \right) \right\} \\ \times \left\{ 1 + \sum_{m=0}^{n+1} \mu_{r,m} E \left(\int_{[a,T]^{m+1}} |D_{s_1, \dots, s_m} u_s|^{p(r+1)} ds_1 \dots ds_m ds | \mathcal{F}_a \right) \right\} \\ \times \left\{ 1 + b_{1,r} + b_{2,r} E \left(\int_{[0,T]^2} |D_t u_s|^{4p(r+1)^2(d-1)} dt ds | \mathcal{F}_a \right) \right\}, \end{aligned} \quad (4.29)$$

where $\mu_{r,m} = \alpha_{m,p(r+1)}$ and for the last inequality we have used that $x^\varepsilon \leq 1 + x$ for all $\varepsilon < 1$ and $x \geq 0$. From (4.29) and Remark 2 we obtain (4.23). \square

Lemma 14. Fix $n \geq 1$, $p \geq 2$. Suppose that u satisfies hypotheses (H2) and (H1) $_{n+m+1, p'}$ for all $p' \geq 2$. Then, for any multi-index $\alpha \in \{1, \dots, d\}^m$ we have

$$\|H_{\alpha}^{a,b}(X_b, 1)\|_{n,p}^{\mathcal{F}_a} \leq c(b-a)^{-m/2} Y_{n,p}^a, \quad (4.30)$$

where

$$Y_{n,p}^a = \prod_{i=1}^m Z_{n,p}^i V_{n,p}^i \quad (4.31)$$

with $Z_{n,p}^i = Z_{n+i, 2^{i+1}p}^a$ which is defined by (4.24) and $V_{n,p}^i$ defined as

$$V_{n,p}^i = \beta_0^i + \sum_{j=1}^{n+i+1} \beta_j^i E \left(1 + \int_{[a,b]^{j+1}} |D_{t_1 \dots t_j} u_r|^{2^{i+1}p} dt_1 \dots dt_j dr \middle| \mathcal{F}_a \right)$$

for some constants β_j^i depending on M, d, T, n and p .

Proof. In the same way as in Lemma 12 and using also Lemma 13, we can obtain the following inequality:

$$\begin{aligned} \|H_{\alpha}^{a,b}(X_b, 1)\|_{n,p}^{\mathcal{F}_a} &\leq d^p \|H_{(1, \dots, \alpha_{m-1})}^{a,b}(X_b, 1)\|_{n+1, 2p}^{\mathcal{F}_a} \|(\gamma_{X_b}^{a,b})^{-1}\|_{n+1, 4p}^{\mathcal{F}_a} \\ &\quad \times \|DX_b \mathbf{1}_{[a,b]}\|_{n+1, 4p, H}^{\mathcal{F}_a} \\ &\leq d^p (b-a)^{-1} Z_{n+1, 4p}^a \|H_{(1, \dots, \alpha_{m-1})}^{a,b}(X_b, 1)\|_{n+1, 2p}^{\mathcal{F}_a} \\ &\quad \times \|DX_b \mathbf{1}_{[a,b]}\|_{n+1, 4p, H}^{\mathcal{F}_a}, \end{aligned} \quad (4.32)$$

where $Z_{n+1, 4p}^a$ is the random variable defined in Lemma 13. On the other hand, we have

$$\|DX_b \mathbf{1}_{[a,b]}\|_{n+1, 4p, H}^{\mathcal{F}_a} = \left\{ E \left(\left(\int_a^b |D_s X_b|^2 ds \right)^{2p} \middle| \mathcal{F}_a \right) + \sum_{j=2}^{n+2} E(\|D^j X_b\|_{H^{\otimes j}}^{4p} | \mathcal{F}_a) \right\}^{1/4p}.$$

For the first term, using inequality (4.17) with the exponent $4p$ yields

$$\begin{aligned} &E \left(\left(\int_a^b |D_s X_b|^2 ds \right)^{2p} \middle| \mathcal{F}_a \right) \\ &\leq 2^{4p-1} (b-a)^{2p} \left\{ M^{4p} + b_{4p} T^{2p-2} E \left(\int_a^b \int_a^b |D_s u_{\theta}|^{4p} d\theta ds \middle| \mathcal{F}_a \right) \right\}. \end{aligned} \quad (4.33)$$

For the other terms we make use of Remark 1. Then, for all $2 \leq j \leq n+2$ we get

$$\begin{aligned} &E(\|D^j X_b\|_{a, H^{\otimes j}}^{4p} | \mathcal{F}_a) \\ &\leq E \left(\left(\int_{[a,b]^j} |D_{s_1, \dots, s_j} X_b|^2 ds_1 \dots ds_j \right)^{2p} \middle| \mathcal{F}_a \right) \end{aligned}$$

$$\begin{aligned}
&\leq (b-a)^{j(2p-1)} E \left(\int_{[a,b]^j} |D_{s_1, \dots, s_j} X_b|^{4p} ds_1 \dots ds_j | \mathcal{F}_a \right) \\
&\leq (b-a)^{2p} T^{j(2p-1)-2p} \left\{ \alpha_{j,4p}^1 E \left(\int_{[a,T]^j} |D_{t_1, \dots, t_{j-1}} u_{t_j}|^{4p} dt_1 \dots dt_j | \mathcal{F}_a \right) \right. \\
&\quad \left. + \alpha_{j,4p}^2 E \left(\int_{[a,T]^{j+1}} |D_{t_1, \dots, t_j} u_r|^{4p} dt_1 \dots dt_j dr | \mathcal{F}_a \right) \right\}. \tag{4.34}
\end{aligned}$$

Hence, from (4.33) and (4.34) we obtain

$$\|DX_b \mathbf{1}_{[a,b]}\|_{n+1,4p,H}^{\mathcal{F}_a} \leq (b-a)^{2p} V_{n,p}^1,$$

where

$$V_{n,p}^1 = \beta_0^1 + \sum_{j=1}^{n+2} \beta_j^1 E \left(1 + \int_{[a,b]^{j+1}} |D_{t_1 \dots t_j} u_r|^{4p} dt_1 \dots dt_j dr | \mathcal{F}_a \right) \tag{4.35}$$

or some constants β_j^1 depending on T, j, M and p . Then, from (4.32) we have

$$\|H_{\alpha}^{a,b}(X_b, 1)\|_{n,p}^{\mathcal{F}_a} \leq d^p (b-a)^{-1/2} Z_{n+1,4p}^a V_{n,p}^1 \|H_{(1, \dots, \alpha_{m-1})}^{a,b}(X_b, 1)\|_{n+1,2p}^{\mathcal{F}_a}.$$

Applying recurrently the last inequality we obtain (4.30). \square

Remark 3. Notice that if u satisfies $(H1)_{n+m+d+1, p'}$ for all $p' \geq 1$, then by the properties of the derivative operator we have that $Y_{k,p}^a \in \mathbb{D}^{d, p'}$ for all $p' \geq 2$.

5. Existence of quadratic covariation and Itô's formula for Brownian martingales

Let $u = (u^{i,j})_{1 \leq i, j \leq d}$ be a matrix of adapted processes $u^{i,j} = \{u_t^{i,j}, t \in [0, T]\}$ such that $E \int_0^T |u_s|^2 ds < \infty$. Set $X_t^k = \sum_{i=1}^d \int_0^t u_s^{k,i} dW_s^i$.

We will assume henceforth that u satisfies hypothesis $(H1)_{n,p}$ for all $p \geq 2$ and all $0 \leq n \leq 2d+1$. We will call that hypothesis $(H1)$. We will also suppose henceforth that u satisfies $(H2)$ and $(H3)$.

Consider a partition $\pi = \{0 = t_0 < t_1 < \dots < t_{n+1} = t\}$ of the interval $[0, t]$ for some fixed $t \in [0, T]$ satisfying

$$L := \sup_{0 \leq i \leq n} \frac{t_{i+1}}{t_i} < \infty. \tag{5.1}$$

Set $\Delta_i X^k = X_{t_{i+1}}^k - X_{t_i}^k$, for $0 \leq i \leq n$. The main result of this section are the following estimates, which, as we have seen before, imply the existence of the quadratic covariation and the Itô's formula for the process X .

We will denote c and c_p general constants which may change along of all this section.

Lemma 15. *There exists a constant c such that for any function $f \in L^p(\mathbb{R}^d)$ for some $p > 2$ and $G \in \mathbb{D}^{1, 2^{d+1}}$ we have*

$$E(f(X_t)^2 G) \leq c t^{-d/p} \|f\|_p^2 \|G\|_{d, 2^{d+1}}. \tag{5.2}$$

Proof. By Corollary 2 with $a = 0$ and $b = t$, we have

$$\begin{aligned} E(f(X_t)^2 G) &= \int_{\mathbb{R}^d} f(x)^2 E(H_{(1,\dots,d)}^{0,t}(X_t, G) \mathbf{1}_{\{X_t > x\}}) dx \\ &\leq \int_{\mathbb{R}^d} f(x)^2 (E(H_{(1,\dots,d)}^{0,t}(X_t, G))^2)^{1/2} (E \mathbf{1}_{\{X_t > x\}})^{1/2} dx. \end{aligned}$$

Applying Lemma 12 with $a = 0$, $b = t$, $n = 0$ and $p = 2$ yields

$$E(H_{(1,\dots,d)}^{0,t}(X_t, G)^2)^{1/2} \leq ct^{-d/2} \|G\|_{d,2^{d+1}}. \quad (5.3)$$

On the other hand, using the exponential inequality for martingales and Hölder's inequality we have that

$$E(\mathbf{1}_{\{X_t > x\}}) \leq \prod_{k=1}^d P(X_t^k > x^k)^{1/d} \leq \prod_{k=1}^d e^{-(x^k)^2/2tM^2d} = e^{-\|x\|^2/2tM^2d}, \quad (5.4)$$

where M is the constant of hypothesis (H3). Then, from (5.3) and (5.4) we obtain

$$\begin{aligned} E(f(X_t)^2 G) &\leq ct^{-d/2} \|G\|_{d,2^{d+1}} \int_{\mathbb{R}^d} f(x)^2 e^{-\|x\|^2/2tM^2d} dx \\ &\leq ct^{-d/2} \|G\|_{d,2^{d+1}} \|f\|_p^2 \left(\int_{\mathbb{R}^d} e^{-\|x\|^2 q/2tM^2d} dx \right)^{1/q} \\ &\leq c \left(\frac{2\pi M^2 d}{q} \right)^{d/2q} t^{-d/2+d/2q} \|G\|_{d,2^{d+1}} \|f\|_p^2, \end{aligned} \quad (5.5)$$

where q is such that $(2/p) + (1/q) = 1$, and as a consequence (5.2) holds. \square

Corollary 16. *There exists a constant c such that for any function $f \in L^p(\mathbb{R}^d)$ with $p > d$ we have*

$$E \int_0^T f(X_s)^2 |u_s^k|^2 ds \leq c \|f\|_p^2. \quad (5.6)$$

Proof. Applying Lemma 15 with $G = |u_s^k|^2$ yields

$$\begin{aligned} E \int_0^T f(X_s)^2 |u_s^k|^2 ds &\leq c \|f\|_p^2 \int_0^t s^{-d/p} \| |u_s^k|^2 \|_{d,2^{d+1}} ds \\ &\leq c \|f\|_p^2 \sup_s \| |u_s^k|^2 \|_{d,2^{d+1}} \int_0^T s^{-d/p} ds. \end{aligned}$$

It only remains to prove that $\sup_s \| |u_s^k|^2 \|_{d,2^{d+1}} < \infty$. For any $1 \leq j \leq d$ we have

$$\begin{aligned} E \|D^j(|u_s^k|^2)\|_{H^{\otimes j}}^p &= E \left\| 2 \sum_{i=1}^d \sum_{m=0}^{j-1} \binom{j-1}{m} D^m u_s^{k,i} D^{j-m} u_s^{k,i} \right\|_{H^{\otimes j}}^p \\ &\leq c \sum_{i=1}^d \sum_{m=0}^{j-1} E (\|D^m u_s^{k,i}\|_{H^{\otimes m+1}}^{2p})^{1/2} E (\|D^{j-m} u_s^{k,i}\|_{H^{\otimes j-m+1}}^{2p})^{1/2} \\ &\leq c' K_{d,2p}, \end{aligned}$$

and that completes the proof. \square

Proposition 17. *There exists a constant c such that for any function $f \in L^p(\mathbb{R}^d)$ with $p > d$ we have*

$$E \left| \sum_{i=0}^n f(X_{t_i}) \Delta_i X^k \right|^2 \leq c \|f\|_p^2. \quad (5.7)$$

Proof. By the isometry of Itô's stochastic integral we have

$$E \left| \sum_{i=0}^n f(X_{t_i}) \Delta_i X^k \right|^2 = E \sum_{i=0}^n f(X_{t_i})^2 \left(\int_{t_i}^{t_{i+1}} |u_s^k|^2 ds \right).$$

Then using Lemma 15 with $G = \int_{t_i}^{t_{i+1}} |u_s^k|^2 ds$ yields

$$E \sum_{i=0}^n f(X_{t_i})^2 \left(\int_{t_i}^{t_{i+1}} |u_s^k|^2 ds \right) \leq c \|f\|_p^2 \sum_{i=0}^n t_i^{-d/p} \left\| \int_{t_i}^{t_{i+1}} |u_s^k|^2 ds \right\|_{d, 2^{d+1}}. \quad (5.8)$$

In order to estimate the last factor, we make use of the Lemma 9 with $x = y = u^k$, $a = t_i$ and $b = t_{i+1}$, and we get

$$\left\| \int_{t_i}^{t_{i+1}} |u_s^k|^2 ds \right\|_{d, 2^{d+1}} \leq c'(t_{i+1} - t_i). \quad (5.9)$$

Finally, from (5.8) and (5.9) we obtain

$$\begin{aligned} E \left| \sum_{i=0}^n f(X_{t_i}) \Delta_i X^k \right|^2 &\leq cc' \|f\|_p^2 \sum_{i=0}^n t_i^{-d/p} (t_{i+1} - t_i) \\ &\leq cc' L^{d/p} \|f\|_p^2 \int_0^t s^{-d/p} ds \\ &\leq c'' \|f\|_p^2, \end{aligned}$$

where L is the constant appearing in condition (5.1). \square

Proposition 18. *There exists a constant c such that for any function $f \in C_K^\infty(\mathbb{R}^d)$ we have*

$$E \left| \sum_{i=0}^n f(X_{t_{i+1}}) \Delta_i X^k \right|^2 \leq c \|f\|_p^2. \quad (5.10)$$

Proof. In order to prove the proposition we will establish the following inequalities:

$$S_1^k := E \sum_{i=0}^n f(X_{t_{i+1}})^2 (\Delta_i X^k)^2 \leq c_1 \|f\|_p^2, \quad (5.11)$$

$$S_2^k := E \sum_{i < j} f(X_{t_{i+1}}) f(X_{t_{j+1}}) \Delta_i X^k \Delta_j X^k \leq c_2 \|f\|_p^2. \quad (5.12)$$

Proof of (5.11). Using Lemma 15 with $G = (\Delta_i X^k)^2$ and $t = t_{i+1}$ we have

$$S_1^k \leq c \|f\|_p^2 \sum_{i=0}^n t_{i+1}^{-d/p} \|(\Delta_i X^k)^2\|_{d, 2^{d+1}}. \quad (5.13)$$

Using Hölder's inequality for the $\|\cdot\|_{k,p}$ -norms and Lemma 8 with $p=2^{d+2}$ and $n=d$ we obtain

$$\begin{aligned} \|(\Delta_i X^k)^2\|_{d,2^{d+1}} &\leq \|\Delta_i X^k\|_{d,2^{d+2}}^2 \\ &\leq c(t_{i+1} - t_i) \end{aligned}$$

and hence, from (5.13) we get

$$\begin{aligned} S_1^k &\leq c\|f\|_p^2 \sum_{i=0}^n (t_{i+1} - t_i) t_{i+1}^{-d/p} \\ &\leq c\|f\|_p^2 \int_0^t s^{-d/p} ds \\ &\leq c'\|f\|_p^2. \end{aligned}$$

Proof of (5.12). Our objective is to transform the martingale increments $\Delta_i X^k$ and $\Delta_j X^k$ into terms which involve only Lebesgue integrals. More precisely, if $S_{ij}^k := E(f(X_{t_{i+1}})f(X_{t_{j+1}})\Delta_i X^k \Delta_j X^k)$, we derive an equality of the form

$$S_{ij}^k = E(f(X_{t_{i+1}})f(X_{t_{j+1}})C_{ij}^k),$$

where

$$\|C_{ij}^k\|_2 \leq c \frac{(t_{i+1} - t_i)(t_{j+1} - t_j)}{\sqrt{t_{i+1}(t_{j+1} - t_{i+1})}}. \quad (5.14)$$

Using the duality relationship between the derivative operator D and the Itô stochastic integral we can write for $i < j$

$$\begin{aligned} S_{ij}^k &= E(f(X_{t_{i+1}})f(X_{t_{j+1}})\Delta_i X^k \Delta_j X^k) \\ &= E\left(f(X_{t_{i+1}})f(X_{t_{j+1}})\Delta_i X^k \int_{t_j}^{t_{j+1}} u_t^k dW_t\right) \\ &= E\left(\int_{t_j}^{t_{j+1}} \sum_{l=1}^d u_t^{k,l} D_t^{(l)}(f(X_{t_{i+1}})f(X_{t_{j+1}})\Delta_i X^k) dt\right) \\ &= \sum_{m=1}^d E\left(f(X_{t_{i+1}})\Delta_i X^k (\partial_m f)(X_{t_{j+1}}) \int_{t_j}^{t_{j+1}} \sum_{l=1}^d u_t^{k,l} D_t^{(l)} X_{t_{j+1}}^m dt\right) \\ &= \sum_{m=1}^d E(f(X_{t_{i+1}})\Delta_i X^k (\partial_m f)(X_{t_{j+1}})(\nabla_j^{u^k} X_{t_{j+1}}^m)), \end{aligned}$$

where for any random variable F we write

$$\nabla_j^{u^k} F = \sum_{l=1}^d \int_{t_j}^{t_{j+1}} u_t^{k,l} D_t^{(l)} F dt.$$

We now apply Proposition 1 to $Y = X_{t_{j+1}}$ and to $Z = f(X_{t_{i+1}})\Delta_i X^k \nabla_j^{u^k} X_{t_{j+1}}^m$ and to the interval $[a, b] = [t_{i+1}, t_{j+1}]$ in order to get rid off the partial derivatives

of f . Of course, new derivatives will appear from the Skorohod integral $H_{(m)}^{t_{i+1}, t_{j+1}}(X_{t_{j+1}}, f(X_{t_{i+1}}) \Delta_i X^k \nabla_j^{u^k} X_{t_{j+1}}^m)$ and a further analysis will be necessary. Then, Proposition 1 yields

$$\begin{aligned} & E(f(X_{t_{i+1}}) \Delta_i X^k (\partial_m f)(X_{t_{j+1}}) (\nabla_j^{u^k} X_{t_{j+1}}^m)) \\ &= E(f(X_{t_{j+1}}) H_{(m)}^{t_{i+1}, t_{j+1}}(X_{t_{j+1}}, f(X_{t_{i+1}}) \Delta_i X^k (\nabla_j^{u^k} X_{t_{j+1}}^m))) \\ &= E(f(X_{t_{i+1}}) f(X_{t_{j+1}}) \Delta_i X^k B_{ij}^{k,m}), \end{aligned}$$

where

$$B_{ij}^{k,m} = H_{(m)}^{t_{i+1}, t_{j+1}}(X_{t_{j+1}}, \nabla_j^{u^k} X_{t_{j+1}}^m).$$

Applying again the duality relationship to the increment $\Delta_i X^k$ we obtain

$$\begin{aligned} & E(f(X_{t_{i+1}}) f(X_{t_{j+1}}) \Delta_i X^k B_{ij}^{k,m}) \\ &= E \left(\int_{t_i}^{t_{i+1}} \sum_{l=1}^d u_t^{k,l} D_t^{(l)}(f(X_{t_{i+1}}) f(X_{t_{j+1}}) B_{ij}^{k,m}) dt \right) \\ &= \sum_{n=1}^d E((\partial_n f)(X_{t_{i+1}}) f(X_{t_{j+1}}) B_{ij}^{k,m} (\nabla_i^{u^k} X_{t_{i+1}}^n)) \\ &\quad + \sum_{n=1}^d E(f(X_{t_{i+1}}) (\partial_n f)(X_{t_{j+1}}) B_{ij}^{k,m} (\nabla_i^{u^k} X_{t_{j+1}}^n)) \\ &\quad + E(f(X_{t_{i+1}}) f(X_{t_{j+1}}) (\nabla_i^{u^k} B_{ij}^{k,m})). \end{aligned} \quad (5.15)$$

Notice that we still have twice the derivative of the function f that must be eliminated. In order to do this, we write

$$\begin{aligned} D_t(f(X_{t_{i+1}}) f(X_{t_{j+1}})) &= f(X_{t_{j+1}}) \sum_{n=1}^d (\partial_n f)(X_{t_{i+1}}) D_t X_{t_{i+1}}^n \\ &\quad + f(X_{t_{i+1}}) \sum_{n=1}^d (\partial_n f)(X_{t_{j+1}}) D_t X_{t_{j+1}}^n. \end{aligned} \quad (5.16)$$

Multiplying both members of (5.16) by $D_t X_{t_{j+1}}^m$ and integrating in the interval $[t_{i+1}, t_{j+1}]$ yields

$$\int_{t_{i+1}}^{t_{j+1}} \langle D_t X_{t_{j+1}}^m, D_t(f(X_{t_{i+1}}) f(X_{t_{j+1}})) \rangle dt = f(X_{t_{i+1}}) \sum_{n=1}^d (\partial_n f)(X_{t_{j+1}}) (\gamma_{X_{j+1}}^{t_{i+1}, t_{j+1}})_{mn}$$

and as a consequence

$$f(X_{t_{i+1}}) (\nabla f)(X_{t_{j+1}}) = (\gamma_{X_{j+1}}^{t_{i+1}, t_{j+1}})^{-1} \left(\int_{t_{i+1}}^{t_{j+1}} \langle D_t X_{t_{j+1}}, D_t(f(X_{t_{i+1}}) f(X_{t_{j+1}})) \rangle dt \right) \quad (5.17)$$

where $\nabla f = (\partial_1 f, \dots, \partial_d f)'$. Multiplying both members of (5.16) by $D_t X_{t_{i+1}}^m$

and integrating in the interval $[0, t_{i+1}]$ yields

$$\begin{aligned} & \int_0^{t_{i+1}} \langle D_t X_{t_{i+1}}^m, D_t (f(X_{t_{i+1}}) f(X_{t_{j+1}})) \rangle dt \\ &= f(X_{t_{j+1}}) \sum_{n=1}^d (\partial_n f)(X_{t_{i+1}}) (\gamma_{X_{t_{i+1}}})_{mn} \\ & \quad + f(X_{t_{i+1}}) \sum_{n=1}^d (\partial_n f)(X_{t_{j+1}}) \int_0^{t_{i+1}} \langle D_t X_{t_{i+1}}^m, D_t X_{t_{j+1}}^n \rangle dt \end{aligned}$$

and as a consequence

$$\begin{aligned} & (\nabla f)(X_{t_{i+1}}) f(X_{t_{j+1}}) \\ &= \gamma_{X_{t_{i+1}}}^{-1} \left(\int_0^{t_{i+1}} \langle D_t X_{t_{i+1}}, D_t (f(X_{t_{i+1}}) f(X_{t_{j+1}})) \rangle dt - f(X_{t_{i+1}}) \Gamma_{ij} (\nabla f)(X_{t_{j+1}}) \right), \end{aligned} \quad (5.18)$$

where the matrix Γ_{ij} is defined by

$$(\Gamma_{ij})_{mn} = \int_0^{t_{i+1}} \langle D_t X_{t_{i+1}}^m, D_t X_{t_{j+1}}^n \rangle dt.$$

Substituting (5.17) into (5.18) we get

$$\begin{aligned} & (\nabla f)(X_{t_{i+1}}) f(X_{t_{j+1}}) \\ &= \gamma_{X_{t_{i+1}}}^{-1} \left(\int_0^{t_{i+1}} \langle D_t X_{t_{i+1}}, D_t (f(X_{t_{i+1}}) f(X_{t_{j+1}})) \rangle dt \right. \\ & \quad \left. - \Gamma_{ij} (\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})^{-1} \int_{t_{i+1}}^{t_{j+1}} \langle D_t X_{t_{j+1}}, D_t (f(X_{t_{i+1}}) f(X_{t_{j+1}})) \rangle dt \right). \end{aligned} \quad (5.19)$$

From (5.15) we have

$$\begin{aligned} S_{ij}^k &= \sum_{m=1}^d E((\nabla f)'(X_{t_{i+1}}) (\nabla_i^k X_{t_{i+1}}) f(X_{t_{j+1}}) B_{ij}^{k,m}) \\ & \quad + \sum_{m=1}^d E((\nabla f)'(X_{t_{j+1}}) (\nabla_i^k X_{t_{j+1}}) f(X_{t_{i+1}}) B_{ij}^{k,m}) \\ & \quad + E(f(X_{t_{i+1}}) f(X_{t_{j+1}}) (\nabla_i^k B_{ij}^{k,m})), \end{aligned} \quad (5.20)$$

where $\nabla_i^k X_{t_{j+1}} = (\nabla_i^k X_{t_{j+1}}^1, \dots, \nabla_i^k X_{t_{j+1}}^d)$ and $\nabla_i^k X_{t_{i+1}} = (\nabla_i^k X_{t_{i+1}}^1, \dots, \nabla_i^k X_{t_{i+1}}^d)$. Now substituting (5.19) and (5.17) into (5.20) we obtain

$$S_{ij}^k = \sum_{m=1}^d E \left\{ \left(\int_{t_{i+1}}^{t_{j+1}} \langle D_t X_{t_{j+1}}, D_t (f(X_{t_{i+1}}) f(X_{t_{j+1}})) \rangle dt \right)' (\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})^{-1} G_{ij}^{k,m} \right\}$$

$$\begin{aligned}
& + \sum_{m=1}^d E \left\{ \left(\int_0^{t_{i+1}} \langle D_t X_{t_{i+1}}, D_t (f(X_{t_{i+1}}) f(X_{t_{j+1}})) \rangle dt \right)' (\gamma_{X_{t_{i+1}}}^{-1}) (\nabla_i^{u^k} X_{t_{i+1}}) B_{ij}^{k,m} \right\} \\
& + E(f(X_{t_{i+1}}) f(X_{t_{j+1}}) (\nabla_i^{u^k} B_{ij}^{k,m})),
\end{aligned}$$

where

$$\begin{aligned}
G_{ij}^k &= \nabla_i^{u^k} X_{t_{j+1}} - \Gamma'_{ij}(\gamma_{X_{t_{i+1}}}^{-1}) \nabla_i^{u^k} X_{t_{i+1}} \\
&= \nabla_i^{u^k} (X_{t_{j+1}} - X_{t_{i+1}}) - \Psi_{ij}(\gamma_{X_{t_{i+1}}}^{-1}) \nabla_i^{u^k} X_{t_{i+1}}
\end{aligned} \tag{5.21}$$

and Ψ_{ij} is the matrix defined as

$$(\Psi_{ij})_{mn} = \int_0^{t_{i+1}} \langle D_t (X_{t_{j+1}}^m - X_{t_{i+1}}^m), D_t X_{t_{i+1}}^n \rangle dt. \tag{5.22}$$

Applying the duality relationship we obtain

$$S_{ij}^k = E(f(X_{t_{i+1}}) f(X_{t_{j+1}}) C_{ij}^k),$$

where

$$C_{ij}^k = \sum_{m,n=1}^d \left\{ H_{(n)}^{t_{i+1}, t_{j+1}}(X_{t_{j+1}}, G_{ij}^{k,n} B_{ij}^{k,m}) + H_{(n)}^{0, t_{i+1}}(X_{t_{i+1}}, \nabla_i^{u^k} X_{t_{i+1}}^n B_{ij}^{k,m}) + \nabla_i^{u^k} B_{ij}^{k,m} \right\}. \tag{5.23}$$

Let us prove that the terms C_{ij}^k satisfy condition (5.14). Applying Lemma 12 with $n = 0$, $p = 2$ and $m = 1$, yields

$$\begin{aligned}
E|C_{ij}^k|^2 &\leq c_1 \sum_{m,n=1}^d \left\{ (t_{j+1} - t_{i+1})^{-1} \|B_{ij}^{k,m}\|_{1,8}^2 \|G_{ij}^{k,n}\|_{1,8}^2 \right. \\
&\quad \left. + t_{i+1}^{-1} \|B_{ij}^{k,m}\|_{1,8}^2 \|\nabla_i^{u^k} X_{t_{i+1}}^n\|_{1,8}^2 + E|\nabla_i^{u^k} B_{ij}^{k,m}|^2 \right\}.
\end{aligned} \tag{5.24}$$

Then, we will make use of the following estimates:

$$\|B_{ij}^{k,m}\|_{1,8} \leq \alpha_1 (t_{j+1} - t_j) (t_{j+1} - t_{i+1})^{-1/2}, \tag{5.25}$$

$$\|G_{ij}^k\|_{1,8} \leq \alpha_2 (t_{i+1} - t_i) (t_{j+1} - t_{i+1})^{1/2}, \tag{5.26}$$

$$\sup_{t \in [0, T]} E|D_t B_{ij}^{k,m}|^4 \leq \alpha_3 (t_{j+1} - t_j)^4 (t_{j+1} - t_{i+1})^{-2}, \tag{5.27}$$

$$E|\nabla_i^{u^k} B_{ij}^{k,m}|^2 \leq \alpha_4 (t_{i+1} - t_i)^2 (t_{j+1} - t_j)^2 (t_{j+1} - t_{i+1})^{-1}. \tag{5.28}$$

Proof of (5.25). Applying Lemma 12 to the random variable $Z = \nabla_j^{u^k} X_{t_{j+1}}$ and with $a = t_{i+1}$, $b = t_{j+1}$ and $\alpha = (m)$ we have

$$\|B_{ij}^{k,m}\|_{1,8} \leq c_{1,8}^5 (t_{j+1} - t_{i+1})^{-1/2} \|\nabla_j^{u^k} X_{t_{j+1}}^m\|_{2,16}. \tag{5.29}$$

Using Lemma 9 with $x = u^k$, $y = DX_{t_{j+1}}^m$, $a = t_j$, $b = t_{j+1}$, $n = 2$ and $p = 16$ we obtain

$$\|\nabla_j^k X_{t_{j+1}}^m\|_{2,16} \leq c(t_{j+1} - t_j). \quad (5.30)$$

Notice that hypothesis (H1) and Lemma 6 imply that x and y satisfy the hypotheses required to apply Lemma 9. That is $K_{2,32}^x < \infty$ and $K_{2,32}^y < \infty$. Substituting (5.30) into (5.29) yields (5.25).

Proof of (5.26). From the definition of G_{ij} and using Hölder's inequality for $\|\cdot\|_{k,p}$ -norms we have that

$$\|G_{ij}^k\|_{1,8} \leq \|\nabla_i^k (X_{t_{j+1}} - X_{t_{i+1}})\|_{1,8} + c\|\Psi_{ij}\|_{1,16} \|\gamma_{X_{t_{i+1}}}^{-1}\|_{1,16} \|\nabla_i^k X_{t_{i+1}}\|_{1,32}.$$

For the first term, we apply Lemma 9 with $x = u^k$, $y = D(X_{t_{j+1}} - X_{t_{i+1}})$, $a = t_i$ and $b = t_{i+1}$, $n = 1$ and $p = 8$. Notice that inequalities (4.3) and (4.4) imply that for all $p \geq 2$

$$\begin{aligned} K_{1,p}^y &= \sup_{t \in [0, t_{i+1}]} E|D_t(X_{t_{j+1}} - X_{t_{i+1}})|^p + \sup_{t \in [0, t_{i+1}]} E \left(\int_0^T |D_s D_t(X_{t_{j+1}} - X_{t_{i+1}})|^2 ds \right)^{p/2} \\ &\leq (c_1 + c_2)(t_{j+1} - t_{i+1})^{p/2} \end{aligned}$$

and that allows us to apply Lemma 9. Hence,

$$\begin{aligned} \|\nabla_i^k (X_{t_{j+1}} - X_{t_{i+1}})\|_{1,8} &\leq c(K_{1,16}^x K_{1,16}^y)^{1/16} (t_{i+1} - t_i) \\ &\leq c'(t_{j+1} - t_{i+1})^{1/2} (t_{i+1} - t_i). \end{aligned} \quad (5.31)$$

In order to estimate the term $\|\Psi_{ij}\|_{1,16}$ we apply again Lemma 9 with $x = D(X_{t_{j+1}} - X_{t_{i+1}})$, $y = DX_{t_{i+1}}$, $a = 0$ and $b = t_i$, $n = 1$ and $p = 16$. As we have seen before $K_{1,p}^x \leq c(t_{j+1} - t_{i+1})^{p/2}$. By Lemma 6 we have also that $K_{1,p}^y \leq c'$. Thus,

$$\begin{aligned} \|(\Psi_{ij})\|_{1,16} &\leq c(K_{n,32}^x K_{n,32}^y)^{1/32} t_{i+1} \\ &\leq c'' t_{i+1} (t_{j+1} - t_{i+1})^{1/2}. \end{aligned} \quad (5.32)$$

The term $\|\nabla_i^k X_{t_{i+1}}\|_{1,32}$ can be estimated in a similar way obtaining

$$\|\nabla_i^k X_{t_{i+1}}\|_{1,32} \leq c(t_{i+1} - t_i). \quad (5.33)$$

Finally, Lemma 11 implies $\|\gamma_{X_{t_{i+1}}}^{-1}\|_{1,16} \leq c t_{i+1}^{-1}$, and using inequalities (5.31), (5.32) and (5.33) we get (5.26).

Proof of (5.27). Using the definition of $B_{ij}^{k,m}$ we have that for all $s \in [t_i, t_{i+1}]$

$$\begin{aligned} D_s B_{ij}^{k,m} &= D_s (H_{(m)}^{t_{i+1}, t_{j+1}} (X_{t_{j+1}}, \nabla_j^k X_{t_{j+1}}^m)) \\ &= \sum_{l=1}^d \int_{t_{i+1}}^{t_{j+1}} D_s (D_t X_{t_{j+1}}^l (\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})_{ml}^{-1} \nabla_i^k X_{t_{i+1}}^m) dW_t. \end{aligned}$$

With s fixed we have, by the continuity of δ from $\mathbb{D}^{1,p}(H)$ into $L^p(\Omega)$

$$\begin{aligned}
 E|D_s B_{ij}^{k,m}|^4 &\leq c \sum_{l=1}^d \left\{ \|D_s D_{X_{t_{j+1}}}^l (\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})^{-1} \nabla_i^{u^k} X_{t_{i+1}}^m \mathbf{1}_{[t_{i+1}, t_{j+1}]}(\cdot)\|_{1,4}^4 \right. \\
 &\quad + \|D_s X_{t_{j+1}}^l D_s ((\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})^{-1}) \nabla_i^{u^k} X_{t_{i+1}}^m \mathbf{1}_{[t_{i+1}, t_{j+1}]}(\cdot)\|_{1,4}^4 \\
 &\quad \left. + \|D_s X_{t_{j+1}}^l (\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})^{-1} D_s (\nabla_i^{u^k} X_{t_{i+1}}^m) \mathbf{1}_{[t_{i+1}, t_{j+1}]}(\cdot)\|_{1,4}^4 \right\} \\
 &\leq c \|D_s D_{X_{t_{j+1}}} \mathbf{1}_{[t_{i+1}, t_{j+1}]}(\cdot)\|_{1,8}^4 \|(\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})^{-1}\|_{1,16}^4 \\
 &\quad \times \|\nabla_i^{u^k} X_{t_{i+1}}^m\|_{1,16}^4 + \|D_s X_{t_{j+1}} \mathbf{1}_{[t_{i+1}, t_{j+1}]}(\cdot)\|_{1,8}^4 \\
 &\quad \times \|D_s ((\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})^{-1})\|_{1,16}^4 \|\nabla_i^{u^k} X_{t_{i+1}}^m\|_{1,16}^4 \\
 &\quad + \|D_s X_{t_{j+1}} \mathbf{1}_{[t_{i+1}, t_{j+1}]}(\cdot)\|_{1,8}^4 \|(\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})^{-1}\|_{1,16}^4 \\
 &\quad \times \|D_s (\nabla_i^{u^k} X_{t_{i+1}}^m)\|_{1,16}^4 \\
 &:= A_1 + A_2 + A_3.
 \end{aligned}$$

Applying twice Lemma 6 with $p = 8$ and $n = 2, 3$ we have that

$$\begin{aligned}
 &\|D_s D_{X_{t_{j+1}}} \mathbf{1}_{[t_{i+1}, t_{j+1}]}(\cdot)\|_{1,8}^8 \\
 &= E \left(\int_{t_{i+1}}^{t_{j+1}} |D_s D_r X_{t_{j+1}}|^2 dr \right)^4 + E \left(\int_0^T \int_{t_{i+1}}^{t_{j+1}} |D_t D_s D_r X_{t_{j+1}}|^2 dr dt \right)^4 \\
 &\leq (t_{j+1} - t_{i+1})^4 (c_{2,8}^1 + T^4 c_{3,8}^1).
 \end{aligned} \tag{5.34}$$

On the other hand, using Lemma 9 with $x = u^k$, $y = DX_{t_{i+1}}^m$, $a = t_i$, $b = t_{i+1}$, $n = 1$ and $p = 16$ we have

$$\|\nabla_i^{u^k} X_{t_{i+1}}^m\|_{1,16}^4 \leq c(t_{i+1} - t_i)^4. \tag{5.35}$$

Finally, from (5.34), (5.35) and using also Lemma 11 with $a = t_{i+1}$ and $b = t_{j+1}$, $n = 1$ and $p = 16$ we get

$$A_1 \leq c(t_{j+1} - t_{i+1})^{-2}(t_{i+1} - t_i)^4.$$

The first factor of A_2 can be estimated in the same way as (5.34). That is

$$\|D_s X_{t_{j+1}} \mathbf{1}_{[t_{i+1}, t_{j+1}]}(\cdot)\|_{1,8}^4 \leq c(t_{j+1} - t_{i+1})^2. \tag{5.36}$$

For the second factor, applying again Lemma 11 with $a = t_{i+1}$ and $b = t_{j+1}$ yields

$$\|D_s ((\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})^{-1})\|_{1,16}^4 \leq \|(\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})^{-1} D_s (\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}}) (\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})^{-1}\|_{1,16}^4$$

$$\begin{aligned} &\leq \|(\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})^{-1}\|_{1,64}^8 \|D_s(\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})\|_{1,32}^4 \\ &\leq c(t_{j+1} - t_{i+1})^{-8} \|D_s(\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})\|_{1,32}^4. \end{aligned} \quad (5.37)$$

In order to estimate the term $\|D_s(\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})\|_{1,32}^4$, we have that

$$D_s((\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})_{mn}) = \int_{t_{i+1}}^{t_{j+1}} D_s D_r X_{t_{i+1}}^m D_r X_{t_{j+1}}^n dr + \int_{t_{i+1}}^{t_{j+1}} D_r X_{t_{i+1}}^m D_s D_r X_{t_{j+1}}^n dr$$

and applying Lemma 9 to the first summand with $x = D_s D X_{t_{i+1}}^m$, $y = D X_{t_{j+1}}^n$, $a = t_{i+1}$, $b = t_{j+1}$, $n = 1$ and $p = 32$, and to the second summand with $x = D X_{t_{i+1}}^m$, $y = D_s D X_{t_{j+1}}^n$, $a = t_{i+1}$, $b = t_{j+1}$, $n = 1$ and $p = 32$, we obtain

$$\|D_s(\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})\|_{1,32}^4 \leq c'(t_{j+1} - t_{i+1})^4 \quad (5.38)$$

and hence, substituting (5.38) into (5.37) yields

$$\|D_s((\gamma_{X_{t_{j+1}}}^{t_{i+1}, t_{j+1}})^{-1})\|_{1,16}^4 \leq c c'(t_{j+1} - t_{i+1})^{-4}. \quad (5.39)$$

We notice that x and y satisfy the hypotheses of Lemma 9 due to Lemma 6. Finally, from (5.36), (5.39) and (5.35) we get

$$A_2 \leq c(t_{j+1} - t_{i+1})^{-2}(t_{i+1} - t_i)^4.$$

For the third factor of A_3 , using twice Lemma 9, one with $x = D_s u^k$, $y = D X_{t_{i+1}}^m$, $a = t_i$, $b = t_{i+1}$, $n = 1$ and $p = 16$ and another with $x = u^k$, $y = D_s D X_{t_{i+1}}^m$, $a = t_i$, $b = t_{i+1}$, $n = 1$ and $p = 16$, we have

$$\begin{aligned} &\|D_s(\nabla_i^{u^k} X_{t_{i+1}}^m)\|_{1,16}^4 \\ &\leq 2^3 \left\| \int_{t_i}^{t_{i+1}} D_s u_r^k D_r X_{t_{i+1}}^m dr \right\|_{1,16}^4 + 2^3 \left\| \int_{t_i}^{t_{i+1}} u_r^k D_s D_r X_{t_{i+1}}^m dr \right\|_{1,16}^4 \\ &\leq c(t_{i+1} - t_i)^4, \end{aligned} \quad (5.40)$$

and from (5.36), (5.40) and using also Lemma 11 with $a = t_{i+1}$ and $b = t_{j+1}$ yields

$$A_3 \leq c(t_{j+1} - t_{i+1})^{-2}(t_{i+1} - t_i)^4.$$

Then, inequality (5.27) holds.

Proof of (5.28). Applying Lemma 9 with $x = u^k$, $y = D B_{ij}^{k,m}$, $a = t_i$, $b = t_{i+1}$, $n = 0$ and $p = 2$ we obtain (5.28). Notice that it can be applied since by (5.27) y satisfies the required hypotheses.

Finally, inequality (5.14) is a consequence of (5.25)–(5.28).

To finish the proof of Proposition 18, we have by (5.14)

$$\begin{aligned} S_2^k &= \sum_{i < j} E(f(X_{i+1}) f(X_{t_{j+1}}) C_{ij}^k) \\ &\leq \sum_{i < j} E(f(X_{i+1})^2 f(X_{t_{j+1}})^2)^{1/2} (E(C_{ij}^k)^2)^{1/2} \\ &\leq c \sum_{i < j} \frac{(t_{i+1} - t_i)(t_{j+1} - t_j)}{\sqrt{t_{i+1}} \sqrt{t_{j+1} - t_{i+1}}} E(f(X_{i+1})^2 f(X_{t_{j+1}})^2)^{1/2}. \end{aligned} \quad (5.41)$$

Using (2.13) and Lemma 14 we get

$$\begin{aligned}
 E(f(X_{t_{i+1}})^2 f(X_{t_{j+1}})^2) &= E(f(X_{t_{i+1}})^2 E(f(X_{t_{j+1}})^2 | \mathcal{F}_{t_{i+1}})) \\
 &= \sum_{\sigma \subset \{1, \dots, d\}} (-1)^{d-|\sigma|} E \left(f(X_{t_{i+1}})^2 \int_{Q_\sigma(X_{t_{i+1}})} f(x)^2 \right. \\
 &\quad \times E(\mathbf{1}_{\{X_{t_{j+1}}^k > x^k, k \in \sigma, X_{t_{j+1}}^k < x^k, k \notin \sigma\}} H_{(1, \dots, d)}^{t_{i+1}, t_{j+1}}(X_{t_{j+1}}, 1) | \mathcal{F}_{t_{i+1}}) dx \Big) \\
 &\leq c(t_{j+1} - t_{i+1})^{-d/2} \sum_{\sigma \subset \{1, \dots, d\}} E \left(f(X_{t_{i+1}})^2 Y_i \int_{Q_\sigma(X_{t_{i+1}})} f(x)^2 \right. \\
 &\quad \times E(\mathbf{1}_{\{X_{t_{j+1}}^k > x^k, k \in \sigma, X_{t_{j+1}}^k < x^k, k \notin \sigma\}} | \mathcal{F}_{t_{i+1}})^{1/2} dx \Big), \quad (5.42)
 \end{aligned}$$

where $Y_i = Y_{0,2}^{t_{i+1}}$ is defined by (4.31). On the other hand, using the properties of the conditional expectation, and also Txeibixev's inequality we have that for all $x \in Q_\sigma(X_{t_{i+1}})$ and for all $\lambda_k > 0$

$$\begin{aligned}
 &E(\mathbf{1}_{\{X_{t_{j+1}}^k > x^k, k \in \sigma, X_{t_{j+1}}^k < x^k, k \notin \sigma\}} | \mathcal{F}_{t_{i+1}}) \\
 &\leq \left\{ \prod_{k \in \sigma}^d P(X_{t_{j+1}}^k > x^k | \mathcal{F}_{t_{i+1}}) \prod_{k \notin \sigma}^d P(X_{t_{j+1}}^k < x^k | \mathcal{F}_{t_{i+1}}) \right\}^{1/d} \\
 &\leq \prod_{k \in \sigma}^d P \left\{ \exp \left(\lambda_k (X_{t_{j+1}}^k - X_{t_{i+1}}^k) - \frac{\lambda_k^2}{2} \langle X^k - X_{t_{i+1}}^k \rangle_{t_{j+1}} \right) \right. \\
 &\quad \left. > \exp \left(\lambda_k (x^k - X_{t_{i+1}}^k) - \frac{\lambda_k^2}{2} M^2(t_{j+1} - t_{i+1}) \right) \middle| \mathcal{F}_{t_{i+1}} \right\}^{1/d} \\
 &\quad \times \prod_{k \notin \sigma}^d P \left\{ \exp \left(\lambda_k (X_{t_{i+1}}^k - X_{t_{j+1}}^k) - \frac{\lambda_k^2}{2} \langle X_{t_{i+1}}^k - X^k \rangle_{t_{j+1}} \right) \right. \\
 &\quad \left. > \exp \left(\lambda_k (X_{t_{i+1}}^k - x^k) - \frac{\lambda_k^2}{2} M^2(t_{j+1} - t_{i+1}) \right) \middle| \mathcal{F}_{t_{i+1}} \right\}^{1/d} \\
 &\leq \prod_{k \in \sigma}^d \exp \left(-\frac{\lambda_k}{d} (x^k - X_{t_{i+1}}^k) + \frac{\lambda_k^2}{2d} M^2(t_{j+1} - t_{i+1}) \right)
 \end{aligned}$$

$$\begin{aligned}
& \times \left[E \left(\exp \left(\lambda_k (X_{t_{j+1}}^k - X_{t_{i+1}}^k) - \frac{\lambda_k^2}{2} \langle X^k - X_{t_{i+1}}^k \rangle_{t_{j+1}} \right) \middle| \mathcal{F}_{t_{i+1}} \right) \right]^{1/d} \\
& \times \prod_{k \notin \sigma}^d \exp \left(-\frac{\lambda_k}{d} (X_{t_{i+1}}^k - x^k) + \frac{\lambda_k^2}{2d} M^2(t_{j+1} - t_{i+1}) \right) \\
& \times \left[E \left(\exp \left(\lambda_k (X_{t_{i+1}}^k - X_{t_{j+1}}^k) - \frac{\lambda_k^2}{2} \langle X_{t_{i+1}}^k - X^k \rangle_{t_{j+1}} \right) \middle| \mathcal{F}_{t_{i+1}} \right) \right]^{1/d} \\
& = \exp \left(-\frac{1}{d} \sum_{k=1}^d \lambda_k S_k (x^k - X_{t_{i+1}}^k) + \frac{M^2(t_{j+1} - t_{i+1})}{2d} \sum_{k=1}^d \lambda_k^2 \right),
\end{aligned}$$

where $S_k = \text{sign}(x^k - X_{t_{i+1}}^k)$ and the last equality comes from the fact that $\{\exp(\pm \lambda_k (X_t^k - X_{t_{i+1}}^k) - \lambda_k^2/2 \langle X^k - X_{t_{i+1}}^k \rangle_t), t \geq t_{i+1}\}$ is a martingale for all k and for all λ_k . Then, choosing $\lambda_k = S_k(x^k - X_{t_{i+1}}^k)/M^2(t_{j+1} - t_{i+1})$ which is positive, we obtain

$$E(\mathbf{1}_{\{X_{t_{j+1}}^k > x^k, k \in \sigma, X_{t_{j+1}}^k < x^k, k \notin \sigma\}} \middle| \mathcal{F}_{t_{i+1}}) \leq \exp \left(-\frac{|x - X_{t_{i+1}}|^2}{2dM^2(t_{j+1} - t_{i+1})} \right). \quad (5.43)$$

From (5.42) and (5.43) and using Hölder's inequality with some p, q such that $1/p + 2/q = 1$, we get

$$\begin{aligned}
& E(f(X_{t_{i+1}})^2 f(X_{t_{j+1}})^2) \\
& \leq c(t_{j+1} - t_{i+1})^{-d/2} \\
& \times E \left(f(X_{t_{i+1}})^2 Y_i \int_{\mathbb{R}^d} f(x)^2 \exp \left(-\frac{|x - X_{t_{i+1}}|^2}{4dM^2(t_{j+1} - t_{i+1})} \right) dx \right) \\
& \leq c'(t_{j+1} - t_{i+1})^{-d/p} \|f\|_p^2 E(f(X_{t_{i+1}})^2 Y_i).
\end{aligned}$$

Applying Lemma 15 with $G = Y_i$ and $t = t_{i+1}$ we have

$$E(f(X_{t_{i+1}})^2 f(X_{t_{j+1}})^2) \leq c'(t_{j+1} - t_{i+1})^{-d/p} t_{i+1}^{-d/p} \|f\|_p^4 \|Y_i\|_{d, 2^{d+1}}. \quad (5.44)$$

From Remark 3 we have that $\|Y_i\|_{d, 2^{d+1}} < \infty$. Then, using (5.41) and (5.44) yields

$$\begin{aligned}
S_2^k & \leq C \|f\|_p^2 \sum_{i < j} \frac{(t_{i+1} - t_i)(t_{j+1} - t_j)}{(t_{i+1}(t_{j+1} - t_{i+1}))^{d/2p+1/2}} \\
& \leq C \|f\|_p^2 \sum_{i < j} (t_{i+1} - t_i) t_{i+1}^{-d/2p-1/2} \int_{t_{i+1}}^t \frac{1}{(s - t_{i+1})^{d/2p+1/2}} ds \\
& \leq CT^{-d/2p+1/2} \left(-\frac{d}{2p} + \frac{1}{2} \right)^{-1} \|f\|_p^2 \int_0^T t^{-d/2p-1/2} dt \\
& \leq C' \|f\|_p^2.
\end{aligned}$$

and this completes the proof. \square

From Theorem 5 it follows that Corollary 16, Propositions 17 and 18, imply the existence of the quadratic covariation and the Itô's formula for $F(X_t)$ where $X_t = (\sum_{i=1}^d \int_0^t u_s^{1,i} dW_s^i, \dots, \sum_{k=1}^d \int_0^t u_s^{d,i} dW_s^i)$. These results hold if the process u satisfies hypotheses (H1)–(H3). We can provide a local version of this result. Let us introduce the following class of processes.

We denote by \mathcal{U} the class of matrices $d \times d$ of adapted processes verifying the following conditions:

- (i) $\{\Omega_m\}$ are mesurable sets increasing to Ω , and u equals u_m on Ω_m .
- (ii) For each $m \geq 1$ the matrix of process u_m satisfies hypotheses (H1)–(H3) with constants $K_{n,p}^m$, ρ_m and M_m .

Now we can state the main result of this section.

Theorem 19. *Let u be a process in the class \mathcal{U} . Set $X_t = (X_t^1, \dots, X_t^d)$ where $X_t^k = \sum_{i=1}^d \int_0^t u_s^{k,i} dW_s^i$. Consider a sequence D_n of partitions of $[0, T]$ verifying conditions (3.1). Then, for any function f in $L^p(\mathbb{R}^d)$ with $p > d$ the quadratic covariation $[f(X), X^k]$ exists, and for any function F in $\mathcal{W}^{1,p}(\mathbb{R}^d)$ the following Itô's formula holds:*

$$F(X_t) = F(0) + \sum_{k=0}^d \int_0^t f_k(X_s) dX_s + \frac{1}{2} \sum_{k=0}^d [f_k(X), X^k]_t,$$

for all $t \in [0, T]$, where f_k are the weak partial derivatives of F .

Proof. The proof is straightforward consequence of Theorem 5, Corollary 16, Propositions 17 and 18. \square

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