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From dynamic to static large deviations in boundary driven exclusion particle systems

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Abstract

We consider the large deviations for the stationary measures associated to a boundary driven symmetric simple exclusion process. Starting from the large deviations for the hydrodynamics and following the Freidlin and Wentzell's strategy, we prove that the rate function is given by the quasi-potential of the Freidlin and Wentzell theory.

This result is motivated by the recent developments on the non-equilibrium stationary measures by Derrida et al. (J. Statist. Phys. 107 (2002) 599) and the more closely related dynamical approach by Bertini et al. (J. Statist. Phys. 107 (2002) 635).

MSC: 60K35

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1. Introduction

A rigorous understanding of the steady states associated to nonequilibrium systems is far from being complete. In particular, the transport phenomena which take place in some nonequilibrium systems induce, in general, long-range correlations in the stationary measures, see e.g. Spohn (1983). For the moment, there is no analog to the Gibbs equilibrium formalism and it is typically a very challenging problem to describe the stationary measures of systems which are defined only by dynamical prescriptions.

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A mathematical idealization of open systems is provided by stochastic models of interacting particles systems. Consider a system of particles performing a reversible hopping dynamics (*Kawasaki* dynamics) in a domain and some external mechanism of creation and annihilation of particles on the boundary of the domain which make the full process non-reversible. The hydrodynamic behavior, namely the law of large numbers followed by the stationary measures, has been derived for important general classes of models (we signal in particular Eyink et al. (1990) and Kipnis et al. (1995)). In the case of the symmetric simple exclusion process (SSEP) hydrodynamic behavior as well as further results on the fluctuations can be obtained by using the specific structure of the dynamics (see in particular Spohn (1983) and De Masi et al. (1982)).

More recently, breakthroughs were achieved by the derivation of a large deviation principle for the stationary measures of the one-dimensional boundary driven SSEP. Using exact computations, Derrida et al. (2002) obtained the explicit form of the rate function for the large deviation principle. Another approach, relying on the large deviations for the hydrodynamics, has been pursued by Bertini et al. (2002, 2003). By generalizing the Freidlin and Wentzell's theory in this context, they were able to formulate a dynamical fluctuation theory for the stationary non-equilibrium states. This approach relies on the hypotheses that the rate function associated to the steady states is given by a dynamical variational formula (the quasi-potential). As a consequence of these hypotheses, some general principles are deduced among which are an extension of the Onsager–Machlup theory and a nonlinear fluctuation dissipation relation.

The non-local structure of the rate function is extremely hard to interpret physically and therefore the result in Derrida et al. (2002) raises many questions for the generalization to a broader class of models. On the other hand, the dynamical approach seems to be very promising since the static rate function can be identified in a systematic way with the quasi-potential. Unfortunately, the quasi-potential provides a very indirect information and, at the moment, only partial results can be extracted from it. There is no general procedure to analyze the quasi-potential. Inspired by the exact formula in Derrida et al. (2002), Bertini et al. (2002, 2003) were able, in the case of SSEP, to integrate the dynamical information contained in the quasi-potential and to recover a tractable expression of the rate function by using a purely dynamical method. This important step may open the way towards further generalizations.

In this paper our modest goal is to address one of the hypotheses on which the dynamical theory in Bertini et al. (2002, 2003) rests. In fact, we implement the Freidlin–Wentzell theory in the context of the SSEP, by proving that the quasi-potential is the large deviation functional of the steady state. This complements the results in Bertini et al. (2002, 2003), providing thus an alternative proof of the result in Derrida et al. (2002). We stress that, contrary to the original heuristic in Bertini et al. (2002, 2003), the proof requires no hypotheses on the adjoint dynamics. We point out that this proof uses essentially nothing of the details of the SSEP dynamics: a *good large deviation principle* (Deuschel and Stroock, 1989) is the key ingredient, along with some properties of the macroscopic dynamics and therefore a good control of the hydrodynamic equation and, above all, of the large deviation functional would lead to the generalization of the result to a large class of interacting exclusion systems. We will address in detail this issue in the last section.

As a last remark, let us mention that an exact solution for the rate function of the totally asymmetric exclusion process has been also derived (Derrida et al., 2003) and it is an open problem to provide a dynamical counterpart similar to the results obtained for the SSEP.

2. The model and the results

2.1. Boundary driven SSEP

Let $\Lambda_N = \{-N, -N+1, \ldots, N\}$ and N be a positive integer. The configuration space is $\Omega_N = \{0,1\}^{\Lambda_N}$. The SSEP with reservoirs is defined as the Markov process $\{\eta_t\}_{t\geqslant 0}$, with $\eta_t \in \Omega_N$ for every $t \geqslant 0$, generated by

$$(L_N f)(\eta) = \frac{N^2}{2} \sum_{\substack{x,y\in\Lambda \\ |x-y|=1}} [f(\eta^{x,y}) - f(\eta)] + N^2 \sum_{x:|x|=N} c(x,\eta_x)[f(\eta^x) - f(\eta)], \quad (2.1)$$

where f is any function from Ω_N to $\mathbb{R}$ and η^x and $\eta^{x,y}$ are defined in the standard way, that is

$$\eta^x(z) = \begin{cases} 1-\eta(x) & \text{if } z = x, \\ \eta(z) & \text{otherwise,} \end{cases} \qquad \eta^{x,y}(z) = \begin{cases} \eta(x) & \text{if } z = y, \\ \eta(y) & \text{if } z = x, \\ \eta(z) & \text{otherwise.} \end{cases} \quad (2.2)$$

The rates $c(\pm N, \cdot) = c_\pm(\cdot)$ depend on the activities $\gamma^\pm \geqslant 0$ of the reservoirs

$$c_+(\eta_N) = \gamma^+ + (1-\gamma^+)\eta_N, \quad c_-(\eta_{-N}) = \gamma^- + (1-\gamma^-)\eta_{-N}. \quad (2.3)$$

Let us remark that if $\gamma^+ = \gamma^- (=\gamma)$ then the model is reversible. Therefore to every γ is naturally associated the value of the (uniform) density of the equilibrium measure in the infinite volume measure: we call ρ_+ (respectively ρ_-) the density associated to γ^+ (respectively γ^-)

$$\rho^+ = \frac{\gamma^+}{1+\gamma^+}, \quad \rho^- = \frac{\gamma^-}{1+\gamma^-}. \quad (2.4)$$

Call $\mathbb{P}_\eta \equiv \mathbb{P}_{N,\eta}$ the path measure of the process $\{\eta_t\}_{t\geqslant 0}$ with $\eta_0 = \eta$: it is of course a measure on $D([0,\infty);\Omega_N)$, the (Skorohod) space of CADLAG functions. If μ is a probability measure on (all subsets of) Ω_N, $\mathbb{P}_\mu(\cdot) = \int_{\Omega_N} \mathbb{P}_\eta(\cdot)\mu(\mathrm{d}\eta)$.

2.2. Hydrodynamics, invariant measure and hydrostatics

In Eyink et al. (1991) the hydrodynamic limit scaling for this system has been proven. More precisely, we introduce the empirical measure for $r \in [-1,1]$

$$\pi_\eta^N(r) = \sum_{x=-N}^{N-1} \eta_x \mathbf{1}_{[x,x+1)}(rN), \quad (2.5)$$

so $\pi_\eta^N \in \mathscr{M} \equiv \{\rho \in \mathbb{L}^\infty([-1,1])\colon 0 \leqslant \rho \leqslant 1 \text{ a.e.}\}$,[1] and we assume that we are given a sequence $\{v_N\}_N$ and a function $\rho_0 \in C^0([-1,1];[0,1])$ such that the law of $v_N \circ (\pi_\cdot^N)^{-1}$, measure on $\mathscr{M}$, tends weakly, as $N \to \infty$, to the measure concentrated on ρ_0. Then the law of the process $\{\phi_{\pi_{tN^2}}\}_{t\in[0,T]}$, under $\mathbb{P}_{v_N}$, converges weakly to the measure on $C^0([0,T];[0,1])$ concentrated on ρ, satisfying the energy condition $\int_{[0,T]\times[-1,1]} |\nabla \rho(t,r)|^2 \, \mathrm{d}r \, \mathrm{d}t < \infty$, unique weak solution of

$$\begin{cases} \partial_t \rho(t,r) = \Delta \rho(t,r) & \text{for every } (t,r) \in \mathbb{R}^+ \times (-1,1), \\ \rho(\pm 1, t) = \rho_\pm & \text{for every } t \in \mathbb{R}^+, \\ \rho(0) = \rho_0. \end{cases} \tag{2.6}$$

Moreover for every fixed N, the unique invariant measure (*steady state*) is denoted by μ_N. It has in fact been proven that the law of π_η^N, under $\mu_N(\mathrm{d}\eta)$ converges weakly as N tends to infinity to the measure on $\mathscr{M}$ concentrated on the stationary solution of (2.6) which is

$$\bar{\rho}(r) = \frac{(\rho_+ - \rho_-)}{2} r + \frac{(\rho_+ + \rho_-)}{2}. \tag{2.7}$$

For a proof see Spohn (1983) or Eyink et al. (1990).

2.3. From dynamic to static large deviations

Call $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{L}^2([-1,1])$. For $H \in C_0^{1,2}([0,T] \times [-1,1])$ (that is, $H(\cdot, \pm 1) \equiv 0$) let

$$\begin{aligned} J_H(\pi) = {} & \langle \pi(T), H(T) \rangle - \langle \rho_0, H(0) \rangle - \int_0^T \langle \pi(t), \partial_t H(t) + \Delta H(t) \rangle \, \mathrm{d}t \\ & + \rho^+ \int_0^T \nabla H(t,1) \, \mathrm{d}t - \rho^- \int_0^T \nabla H(t,-1) \, \mathrm{d}t \\ & - \frac{1}{2} \int_0^T \langle \sigma(\pi(t)), (\nabla H(t))^2 \rangle \, \mathrm{d}t, \end{aligned} \tag{2.8}$$

where $\sigma(x) = x(1-x)$ is the mobility. Set also

$$I_T(\pi) = \sup_{H \in C_0^{1,2}([0,T]\times[-1,1])} J_H(\pi). \tag{2.9}$$

For $\rho_0 \in \mathscr{M}$ we define the *LD rate function* as

$$I_T(\pi \mid \rho_0) = \begin{cases} I_T(\pi) & \text{if } \pi(0) = \rho_0 \\ +\infty & \text{otherwise.} \end{cases} \tag{2.10}$$

[1] The topology of $\mathscr{M}$ is induced by the weak convergence: $\rho_n \overset{n\to\infty}{\longrightarrow} \rho$ if $\int_{-1}^1 \rho_n(r) f(r) \, \mathrm{d}r$ tends to $\int_{-1}^1 \rho(r) f(r) \, \mathrm{d}r$ for every $f \in C_b^0([-1,1];\mathbb{R})$. See later for more on this (metrizable) topology. The σ-algebra of the measurable sets of $\mathscr{M}$ is chosen to be the Borel one.

By exploiting the concavity of σ one can show that I_T is convex. Moreover, one can show also that it is lower semicontinuous (l.s.c.) and that the level sets $\{\pi \in D([0,T];\mathscr{M}) : I_{[0,T]}(\pi) \leqslant a\}$ are compact for every $a \geqslant 0$. These properties extend to $I_T(\cdot \mid \rho_0)$. In particular $I_T(\cdot \mid \rho_0)$ is a *good* rate function (i.e., it is l.s.c. and it has compact level sets, Deuschel and Stroock (1989). A proof of these properties can be found in Kipnis and Landim (1999, Chapter 10) and in Bertini et al. (2002, 2003).

The large deviation principle for boundary driven SSEP is derived in Bertini et al. (2002, 2003).

Theorem 2.1. *For every choice of* $\{\eta^N\}_N$, $\eta^N \in \Omega_N$ *such that* $\pi^N_{\eta^N} \in \mathscr{M}$ *converges to* ρ_0, *the sequence of random functions* $\{\pi^N_{\eta_{N^2\cdot}}\}_{N=1,2,\ldots}$ *in* $D([0,T],\mathscr{M})$, $\eta_0 = \eta^N$, *obeys a full Large Deviations principle with speed* N *and rate function* $I_T(\cdot \mid \rho_0)$. *That is, for every* $A \subset D([0,T],\mathscr{M})$, *we have that*

$$-\inf_{\pi\in A^\circ} I_T(\pi \mid \rho_0) \leqslant \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}(\pi^N_\cdot \in A^\circ)$$
$$\leqslant \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}(\pi^N_\cdot \in \bar{A}) \leqslant -\inf_{\pi\in\bar{A}} I_T(\pi \mid \rho_0), \tag{2.11}$$

where A° *denotes the interior of* A *and* $\bar{A}$ *its closure.*

Let us introduce the *quasipotential* (Freidlin and Wentzell, 1998; Bertini et al., 2002, 2003): for every $\rho \in \mathscr{M}$

$$V(\rho) = \inf\{I_T(\pi \mid \bar{\rho}) : \pi(T) = \rho \text{ and } T > 0\}. \tag{2.12}$$

Of course, the infimum can be restricted to $\pi \in D([0,T];\mathscr{M})$ such that $\pi(0) = \bar{\rho}$. Moreover, it is not too difficult to show (cf. Bertini et al. (2002, 2003) and Deuschel and Stroock (1989)) that $I_T(\pi) = +\infty$ unless $\pi \in C^0([0,T];\mathscr{M})$. Therefore, we may restrict further this extremum to trajectories which are continuous in time. Starting with the next statement, we will abuse of notation calling μ_N also the measure $\mu_N \circ (\pi^N_\cdot)^{-1}$ on $\mathscr{M}$.

We can now state the main result of this paper.

Theorem 2.2. *The stationary measure* μ_N *obeys a full Large Deviations principle with rate function* V *and speed* N.

3. The proof

The scheme of the proof follows closely the one introduced by Freidlin and Wentzell (1998, Section 4). This requires further notations on the topology of the functional spaces.

Recall that the space

$$\mathscr{M} \equiv \{\rho \in \mathbb{L}^\infty([-1,1]):\ 0 \leqslant \rho \leqslant 1\}$$

was introduced in Section 2.2. The space $\mathscr{M}$ is metrizable: if we set $f_{2n+1}(r)=\sin(\pi nr)$ and $f_{2n}(r)=\cos(\pi nr)$, $n=0,1,\ldots$, we may define the distance as

$$\mathrm{dist}(\rho_1,\rho_2)=\sum_{k=1}^{\infty}\frac{1}{2^k}\,|\langle\rho_1,f_k\rangle-\langle\rho_2,f_k\rangle|, \tag{3.1}$$

for $\rho_1,\ \rho_2\in\mathscr{M}$. Of course $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{L}^2$. Moreover for $\varepsilon>0$ and $\rho\in\mathscr{M}$, then the closed ε-ball around ρ in the weak topology is denoted by

$$\mathbb{B}_\varepsilon(\rho)=\{\varphi\in\mathscr{M}\mid \mathrm{dist}(\rho,\varphi)\leqslant\varepsilon\}. \tag{3.2}$$

On the dynamical level we will work with several spaces, but the basic one is the Skorohod space $D([0,T],\mathscr{M})$: observe that $\pi^N_\cdot\in D([0,T],\mathscr{M})$. Let us be more precise about this space and let us recall that it is a metric space: if we let Λ be the set of increasing continuous functions λ of $[0,T]$ into itself, then a distance associated to the Skorohod topology is given by

$$\mathrm{d}(\pi,\pi')=\inf_{\lambda\in\Lambda}\sup_{t\in[0,T]}\{\mathrm{dist}(\pi_t,\pi'_{\lambda(t)})+|\lambda(t)-t|\},\qquad \pi,\pi'\in D([0,T],\mathscr{M}). \tag{3.3}$$

For any π in $D([0,T],\mathscr{M})$, the ε-neighborhood of π in the Skorohod topology is denoted by $\mathscr{V}^\varepsilon_{[0,T]}(\pi)$. In the same way if $A\subset D([0,T],\mathscr{M})$, $\mathscr{V}^\varepsilon_{[0,T]}(A)=\bigcup_{\pi\in A}\mathscr{V}^\varepsilon_{[0,T]}(\pi)$.

3.1. Lower bound

It is sufficient to check that for any $\varepsilon>0$ and any ρ in $\mathscr{M}$

$$\liminf_{N\to\infty}\frac{1}{N}\log\mu_N(\mathbb{B}_\varepsilon(\rho))\geqslant -V(\rho). \tag{3.4}$$

By definition of V (recall that the infimum may be restricted to continuous functions), for every $\delta>0$, there exists T and $\pi\in C^0([0,T];\mathscr{M})$ such that

$$I_T(\pi\mid\bar\rho)\leqslant V(\rho)+\delta\quad\text{ and }\quad \pi(T)=\rho. \tag{3.5}$$

By using definition (3.3), for any trajectory v in $\mathscr{V}^\varepsilon_{[0,T]}(\pi)$ we see that $v_T\in\mathbb{B}_\varepsilon(\rho)$ (because $\lambda(T)=T$). Since μ_N is the stationary measure of the dynamics

$$\mu_N(\mathbb{B}_\varepsilon(\rho))\geqslant\mathbb{P}_{\mu_N}(\pi^N_{\eta_T}\in\mathbb{B}_\varepsilon(\rho))\geqslant\mathbb{E}_{\mu_N}[\mathbb{P}_{\eta_0}(\pi^N_{\eta_\cdot}\in\mathscr{V}^\varepsilon_{[0,T]}(\pi));\ \pi^N_{\eta_0}\in\mathbb{B}_{\varepsilon_N}(\bar\rho)], \tag{3.6}$$

for every $\varepsilon_N>0$, the hydrostatics results recalled in subsection 2.2 can be rephrased as

$$\text{for every }\delta>0,\qquad \lim_{N\to\infty}\mu_N(\mathbb{B}_\delta(\bar\rho))=1. \tag{3.7}$$

This is equivalent to the existence of a sequence $\{\varepsilon_N\}_{N=1,2,\ldots}$, $\varepsilon_N\searrow 0$ as $N\to\infty$, such that $\mu_N(\mathbb{B}_{\varepsilon_N}(\bar\rho))$ converges to 1. Therefore, for N sufficiently large

$$\mu_N(\mathbb{B}_\varepsilon(\rho))\geqslant\tfrac{1}{2}\inf\{\mathbb{P}_{\eta^N}(\pi^N_{\eta_\cdot}\in\mathscr{V}^\varepsilon_{[0,T]}(\pi)):\ \eta^N\ \text{s.t.}\ \pi^N_{\eta^N}\in\mathbb{B}_{\varepsilon_N}(\bar\rho)\}, \tag{3.8}$$

where the factor $\frac{1}{2}$ refers to the bound (3.7). Since $\varepsilon_N \searrow 0$ as $N \nearrow \infty$, we may apply the lower bound in Theorem 2.1 to obtain that

$$\liminf_{N\to\infty} \frac{1}{N} \log \mu_N(\mathbb{B}_\varepsilon(\rho)) \geqslant - \inf_{\pi' \in \mathscr{V}^{\varepsilon/2}_{[0,T]}(\pi)} I_T(\pi' \mid \bar\rho)$$

$$\geqslant -I_T(\pi \mid \bar\rho) \geqslant - V(\rho) - \delta. \tag{3.9}$$

Since δ can be chosen arbitrarily small, (3.4) is proven and the lower bound in Theorem 2.2 is established. □

3.2. Upper bound

We are now going to check that for any closed subset $\mathscr{C}$ of $\mathscr{M}$

$$\limsup_{N\to\infty} \frac{1}{N} \log \mu_N(\mathscr{C}) \leqslant - V(\mathscr{C}), \tag{3.10}$$

where $V(\mathscr{C}) = \inf_{\pi\in\mathscr{C}} V(\pi)$. Since, $\bar\rho \in \mathscr{C}$ the result is trivial, let us assume $\bar\rho \notin \mathscr{C}$. Therefore, there exists $\delta > 0$ such that $\mathbb{B}_{4\delta}(\bar\rho) \cap \mathscr{C} = \emptyset$. We fix δ throughout the proof and set

$$\vartheta = \mathbb{B}_\delta(\bar\rho) \quad \text{and} \quad \Gamma = \{\rho \in \mathscr{M} \mid 3\delta \leqslant \text{dist}(\bar\rho, \rho) \leqslant 4\delta\}.$$

For any subset A of $\mathscr{M}$, let τ_A be the first return time in A of the process $\{\pi^N_{\eta_t}\}_{t\geqslant 0}$. We also introduce τ_1 defined as follows:

$$\tau_1 = \inf\{t > 0\colon \text{ there exists } s \in [0,t) \text{ such that } \pi^N_{\eta_s} \in \Gamma \text{ and } \pi^N_{\eta_t} \in \vartheta\}. \tag{3.11}$$

In order to state a classical representation of the invariant measure μ_N for the Markov chain $\{\eta_t\}_{t\geqslant 0}$, we need to introduce some more notation. The first step is to define a notion of *discrete external boundary* for ϑ. Let $\partial\vartheta^N$ be the set of configurations η^N such that there exists $k \in \mathbb{N}$ and a sequence of configurations $\eta^{N,0}, \ldots, \eta^{N,k} = \eta^N$ which satisfy the following constraints:

(1) for every i, the configuration $\eta^{N,i+1}$ can be deduced from the configuration $\eta^{N,i}$ by spin exchange or spin creation according to the rule prescribed by the dynamics.
(2) $\eta^{N,0} \in \Gamma$ and for every $i < k$, we have that $\eta^{N,i} \notin \mathbb{B}_{2\delta}(\bar\rho)$ and $\eta^{N,k} \in \mathbb{B}_{2\delta}(\bar\rho)$.

For any N, we consider a variant of the stopping time τ_1

$$\tau_1^N = \inf\{t > 0\colon \text{ there exists } s \in (0,t) \text{ such that } \pi^N_{\eta_s} \in \Gamma \text{ and } \eta_t \in \partial\vartheta^N\}. \tag{3.12}$$

The sequence of stopping times obtained by iterating this procedure is denoted by $\{\tau_k^N\}$. In this way an irreducible Markov chain $\{X_k\}_{k=1,2,\ldots}$ is defined on $\partial\vartheta^N$ by setting $X_k = X^\eta_{\tau_k^N}$ (see Remark 3.3 at the end of the proof).

Since the irreducible chain $\{X_k\}_{k=1,2,\ldots}$ evolves on a finite state space, it has a unique stationary measure v_N on $\partial\vartheta^N$. Following Freidlin and Wentzell (1998), we represent the stationary measure of the process $\{\eta_t\}_{t\geqslant 0}$ as

$$\mu_N(A) = \frac{1}{C_N} \int_{\partial\vartheta^N} \mathbb{E}_\eta \left(\int_0^{\tau_1^N} \mathbf{1}_A(\eta_s)\,\mathrm{d}s \right) \mathrm{d}v_N(\eta), \tag{3.13}$$

for every $A \subset \mathscr{M}$, with

$$C_N = \int_{\partial\vartheta^N} \mathbb{E}_\eta(\tau_1^N)\, \mathrm{d}\nu_N(\eta). \tag{3.14}$$

In order to estimate the probability of the set $\mathscr{C}$, we observe that the strong Markov property implies

$$\mu_N(\mathscr{C}) \leqslant \frac{1}{C_N} \sup_{\eta\in\partial\vartheta^N} \mathbb{P}_\eta(\tau_{\mathscr{C}} < \tau_1^N) \sup_{\eta\in\mathscr{C}} \mathbb{E}_\eta(\tau_1^N). \tag{3.15}$$

Moreover, we notice that there is $c > 0$ such that $C_N \geqslant 1/cN^2$: this comes from the fact that the process, which jumps with jump rates of the order of N^2, has to leave $\partial\vartheta^N$ before returning to it. By construction $\tau_1^N \leqslant \tau_1$, thus

$$\mu_N(\mathscr{C}) \leqslant cN^2 \sup_{\eta\in\partial\vartheta^N} \mathbb{P}_N^\eta(\tau_{\mathscr{C}} < \tau_1) \sup_{\eta\in\mathscr{C}} \mathbb{E}_N^\eta(\tau_1). \tag{3.16}$$

The upper bound of the large deviations (3.10) will therefore follow from the following lemma. Recall that most of the definitions we gave depend on a positive (and sufficiently small) parameter δ.

Lemma 3.1. *We have that* (1) *for every* δ

$$\limsup_{N\to\infty} \frac{1}{N} \log \sup_{\eta\in\mathscr{C}} \mathbb{E}_\eta(\tau_1) \leqslant 0, \tag{3.17}$$

for every $\varepsilon > 0$ *there is* δ_0 *such that for* $\delta \in (0, \delta_0)$

$$\limsup_{N\to\infty} \frac{1}{N} \log \sup_{\eta\in\partial\vartheta^N} \mathbb{P}_\eta(\tau_{\mathscr{C}} < \tau_1) \leqslant -V(\mathscr{C}) + \varepsilon. \tag{3.18}$$

In the proof of Lemma 3.1, we will make use of the following technical result:

Lemma 3.2. *There exists* $T_0 > 0,\ c > 0$ *and* $N_0 > 0$ *such that*

$$\sup_{\eta\notin\vartheta} \mathbb{P}_\eta(\tau_\vartheta > T) \leqslant \exp(-c(T - T_0)N), \tag{3.19}$$

for every $T \geqslant T_0$ *and* $N \geqslant N_0$.

Proof. The first step is to check that there is $T_0 > 0$ and $a > 0$ such that if $\pi \in D([0, T_0], \mathscr{M})$ is such that $\pi(t) \in \mathscr{M}\backslash\vartheta$ for every t, then

$$I_{[0,T_0]}(\pi) > a. \tag{3.20}$$

To establish this start by considering the following Cauchy problem: for given $\rho_0 \in \mathscr{M}$, we look for $\rho(\cdot) \in C^0([0, T], \mathscr{M})$ such that

$$\langle J(T), \rho(T)\rangle - \langle J(0), \rho_0\rangle - \int_0^T \langle(\partial_t + \Delta)J(t), \rho(t)\rangle\, \mathrm{d}t$$

$$+ \rho_+ \int_0^T \nabla J(t,1)\mathrm{d}t - \rho_- \int_0^T \nabla J(t,-1)\mathrm{d}t = 0, \quad \text{for every } J \in C_0^{1,2}. \tag{3.21}$$

This Cauchy problem is well posed and the solution is classical for positive times. We can see this by first observing, for example via Fourier analysis, that there exists a solution $\rho \in C^{1,2}((0,T] \times [-1,1])$ satisfying $\partial_t \rho(t) = \Delta \rho(t)$ for $t \in (0,T]$, $\rho(t, \pm 1) = \rho^{\pm}$ and $\lim_{t \searrow 0} \rho(t) = \rho_0$ (in $\mathscr{M}$, but also in $\mathbb{L}^1$). Uniqueness follows from the following argument: for any function $f_0 \in C_0^2([-1,1])$, we consider the classical solution $f \in C_0^{1,2}([0,T] \times [-1,1])$ of the heat equation with $f(0) = f_0$. We are now going to insert into equation (3.21) the test function $J(r,t) = f(r, T-t)$. Since

$$\partial_t J(t) + \Delta J(t) = 0 \quad \text{for every } t \in [0,T],$$

the differential term in (3.21) disappears. Let us then assume that ρ, $\bar{\rho} \in C^0([0,T']; \mathscr{M})$, $T \leqslant T'$, are two solutions of (3.21) with the same initial data. Let us set $\tilde{\rho} = \rho - \bar{\rho}$. By using the test function J and by linearity, we obtain

$$\int_{-1}^{+1} f_0(r) \tilde{\rho}(T,r)\, \mathrm{d}r = 0. \tag{3.22}$$

Finally, by approximation we can extend the validity of (3.22) to every $f_0 \in C_b^0([-1,1])$. This implies that $\tilde{\rho}(T) = 0$ and, since T is arbitrary, $\tilde{\rho} \equiv 0$.

We claim now that the solution to (3.21) relaxes in $\mathbb{L}^2([-1,1])$ exponentially fast to the equilibrium profile. In fact, since by uniqueness $\rho(\cdot)$ is smooth, for $t > 0$, we have

$$\frac{1}{2}\, \partial_t \| \rho(t) - \bar{\rho} \|_2^2 = - \int_{-1}^{+1} [\nabla(\rho(t,r) - \bar{\rho}(r))]^2\, \mathrm{d}r. \tag{3.23}$$

But the spectral gap of the Laplacian with Dirichlet boundary conditions is strictly positive. So we get that for some $c_1 > 0$

$$\partial_t \| \rho(t) - \bar{\rho} \|_2^2 \leqslant -c_1 \| \rho(t) - \bar{\rho} \|_2^2 \tag{3.24}$$

for every $t > 0$, and therefore the exponentially fast convergence to equilibrium. Since $0 \leqslant \rho_0 \leqslant 1$, (3.24) implies that for every $\varepsilon > 0$ there exists $T > 0$ such that

$$\forall t \geqslant T, \qquad \sup_{\rho_0 \in \mathscr{M}} \| \rho(t) - \bar{\rho} \|_2 \leqslant \varepsilon. \tag{3.25}$$

This ensures the existence of $T > 0$ such that $\rho(t) \in \vartheta$ for every $t > T$.

We set $T_0 = 2T$ and we want to show that (3.20) holds with this choice. Let us assume that this is not the case: then there exists a sequence π_k of trajectories in $D([0,T_0], \mathscr{M} \backslash \vartheta)$ such that $I_{[0,T_0]}(\pi_k) \leqslant 1/k$. This, together with the fact that I_{T_0} is l.s.c. and has compact level sets, implies the existence of π in $D([0,T_0], \mathscr{M})$ taking values in the closure of $\mathscr{M} \backslash \vartheta$ and such that $I_{T_0}(\pi) = 0$. Then π solves (3.21), and, as we saw in (3.25), $\pi(t) \in \vartheta$ for $t \geqslant T$, which contradicts the assumption and we are done with proving (3.20).

From (3.20), we know that for some $a > 0$ there exists T_a such that any trajectory in

$$\Phi(a) = \{\pi \in C^0([0,T_a); \mathscr{M}) : I_{T_a}(\pi) \leqslant a\}, \tag{3.26}$$

enters in the neighborhood $\mathbb{B}_{\delta/2}(\bar{\rho})$. Notice that the interior of the set $\Phi(a)$ is empty (recall that we are working with the Skorohod topology) and we therefore, choose to work with an open neighborhood of $\Phi(a)$:

$$\Phi'(a) = \mathscr{V}^{\delta/2}_{[0,T_a]}(\Phi(a)), \tag{3.27}$$

and if $\pi \in \Phi'(a)$, then $\pi(t) \in \vartheta$ for some $t \in [0, T_a]$. This implies that

$$\{\pi \in D([0,\infty); \mathscr{M}) : \tau_\vartheta > T_a\} \subset (\Phi'(a))^{\complement}.$$

Furthermore, by construction, for any π in $(\Phi'(a))^{\complement}$ we have $I_{T_a}(\pi) > a$.

We are now in the position of applying the dynamical large deviation principle: observe that we can select a sequence $\{\tilde{\eta}^N\}_{N=1,2,\dots}$ such that

$$\max_{\eta^N : \pi^N_{\eta^N} \in \mathscr{M}\setminus\vartheta} \mathbb{P}_{\eta^N}(\tau_\vartheta > T_a) = \mathbb{P}_{\tilde{\eta}^N}(\tau_\vartheta > T_a) \tag{3.28}$$

and by compactness of $\mathscr{M}$ we can apply the large deviations upper bound (2.11) to every subsequence of $\{\tilde{\eta}^N\}_{N=1,2,\dots}$ such that $\pi^N_{\tilde{\eta}^N}$ converges in $\mathscr{M}$ to obtain that there exists N_0 such that for $N > N_0$

$$\sup_{\eta^N:\ \pi^N_{\eta^N} \in \mathscr{M}\setminus\vartheta} \mathbb{P}_{\eta^N}(\tau_\vartheta > T_a) \leqslant \sup_{\eta^N:\ \pi^N_{\eta^N} \in \mathscr{M}\setminus\vartheta} \mathbb{P}_{\eta^N}((\Phi'(a))^{\complement}) \leqslant \exp\left(-\frac{a}{2}N\right). \tag{3.29}$$

By using the Markov property we can iterate this procedure to get that for $N > N_0$

$$\begin{aligned}\sup_{\eta^N:\ \pi^N_{\eta^N} \in \mathscr{M}\setminus\vartheta} \mathbb{P}_{\eta^N}(\tau_\vartheta > kT_a) &\leqslant \sup_{\eta^N:\ \pi^N_{\eta^N} \in \mathscr{M}\setminus\vartheta} \mathbb{E}_{\eta^N}(1_{\{\tau_\vartheta > (k-1)T_a\}} \mathbb{P}_{\eta_{(k-1)T_a}}(\tau_\vartheta > T_a)) \\ &\leqslant \exp\left(-\frac{ak}{2}N\right),\end{aligned} \tag{3.30}$$

where k is an arbitrary positive integer number. The proof is therefore complete. □

Proof of Lemma 3.1. By construction $\mathscr{C} \cap \mathbb{B}_{4\delta} = \emptyset$. Therefore, for N large enough, any trajectory $\pi^N_{\eta_\cdot}$ starting from $\mathscr{C}$ will cross Γ before touching ϑ (the jumps of $\mathrm{dist}(\pi^N_{\eta_\cdot}, \bar{\rho})$ are in fact of order $1/N$). This implies that τ_1 can be replaced by τ_ϑ in (3.17). By applying Lemma 3.2, we see that uniformly in η^N such that $\pi^N_{\eta^N} \in \mathscr{C}$ for $N > N_0$

$$\mathbb{E}_{\eta^N}(\tau_\vartheta) \leqslant T_a \left(1 + \sum_{k=1}^{\infty} \mathbb{P}_{\eta^N}(\tau_\vartheta \geqslant kT_a)\right) \leqslant T_a \sum_{k=0}^{\infty} \exp\left(-\frac{ak}{2}N\right). \tag{3.31}$$

Therefore, (3.17) holds and the first part of Lemma 3.1 is established.

In order to prove (3.18), it is enough to check that for every $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \log \sup_{\eta \in \tilde{\vartheta}} \mathbb{P}_\eta(\tau_{\mathscr{C}} < \tau_1) \leqslant -V(\mathscr{C}) + \varepsilon, \tag{3.32}$$

where $\tilde{\vartheta} = \mathbb{B}_{2\delta}(\bar{\rho})$. Lemma 3.2 ensures that there is $T > 0$ and $N_0 > 0$ large enough such that for $N > N_0$

$$\forall N \geqslant N_0, \quad \sup_{\eta \in \tilde{\vartheta}} \mathbb{P}_\eta(\tau_1 > T) \leqslant \exp(-N(V(\mathscr{C}) + 1)). \tag{3.33}$$

Thus it remains to check that for N large

$$\sup_{\eta\in\tilde{\vartheta}} \mathbb{P}_\eta(\tau_{\mathscr{C}} \leqslant \tau_1 \leqslant T) \leqslant \exp(-(V(\mathscr{C})-\varepsilon)N). \tag{3.34}$$

Since $\mathscr{C}$ and ϑ are closed sets, the set of trajectories such that $\{\tau_{\mathscr{C}} \leqslant \tau_1 \leqslant T\}$ is also a closed subset (of $D([0,T],\mathscr{M})$). Therefore, it is enough to check that for any π such that $\pi(0)\in\tilde{\vartheta}$ and $\pi(t)\in\mathscr{C}$ for some $t\in[0,T]$

$$I_T(\pi) \geqslant V(\mathscr{C}) - \varepsilon \tag{3.35}$$

for δ sufficiently small. If this is not true then one can choose $\delta = 1/k$ and a sequence π_k in $D([0,T],\mathscr{M})$ such that for some $\alpha > 0$

$$\limsup_{k\to\infty} I_T(\pi_k) < V(\mathscr{C}) - \alpha. \tag{3.36}$$

But $\{\pi_k(0)\}_{k=1,2,\dots}$ converges to $\bar{\rho}$ and, since I_T has compact level sets, one can extract a subsequence of $\{\pi_k\}_{k=1,2,\dots}$ which converges in $D([0,T];\mathscr{M})$ to π such that

$$\pi(0)=\bar{\rho},\ \tau_{\mathscr{C}} \leqslant T, \quad \text{and} \quad I_T(\pi) \leqslant V(\mathscr{C}) - \alpha, \tag{3.37}$$

by lower semicontinuity of the functional I_T. By the definition (2.12) of V, this is a contradiction and this completes the proof of Lemma 3.1 and, with it, the proof of the upper bound of Theorem 2.2. $\square$

Remark 3.3. While the irreducibility of the Markov process $\{\eta_t\}_{t\geqslant 0}$ is clear, we would like to comment on the irreducibility of the chain $\{X_k\}_k$ introduced right after (3.12).

Let $\eta^{(1)},\eta^{(2)}$ be in $\partial\vartheta^N$. By definition of $\eta^{(2)}$, there is a sequence of (particle) configurations $\{\zeta_1,\dots,\zeta_k=\eta^{(2)}\}$ leading from Γ to $\eta^{(2)}$, keeping out of $\partial\vartheta^N$ except for the last point (that is $\eta^{(2)}$). Therefore, it is enough to check that one can find a sequence of configurations $\{\sigma_1,\dots,\sigma_{k'}\}$ which does not touch Γ and which leads from $\eta^{(1)}$ to $\eta^{(2)}$: notice that we are allowed to go from one configuration to another only via the elementary steps of the dynamics. In fact, if we can find it by considering the sequence of configurations $\{\sigma_1=\eta^{(1)},\dots,\sigma_{k'},\zeta_{k-1},\dots,\zeta_2,\zeta_1,\zeta_2,\dots,\zeta_k=\eta^{(2)}\}$ that starts from η_1 and intersects $\partial\vartheta^N$ for the first time (after having touched Γ) at the point $\eta^{(2)}$, we are done.

As ϑ is convex, the functions $\{u_k \equiv (k/K)\pi^N_{\eta^{(1)}} + (1-(k/K))\pi^N_{\eta^{(2)}}\}_{0\leqslant k\leqslant K}$ belong to $\mathbb{B}_{2\delta}(\bar{\rho})$ for any $K\in\mathbb{N}$. Choose K much bigger than $1/\delta$ and consider only integer k's: it should be clear that we are done if we show how to go, for N sufficiently large, from η to σ, $\mathrm{dist}(\pi^N_\eta,\pi^N_\sigma)\leqslant 2/K$ passing through configurations ζ such that $\mathrm{dist}(\pi^N_\zeta,\pi^N_\sigma)\leqslant 4/K$. This is achieved by taking into account that:

(1) By choosing N sufficiently large we may assume that birth or death is allowed at any point of the system: for example for a birth at a site x, choose the first particle on the right and displace it by elementary hops till x and restart, till there is no particle on the right and just have one be born and displace it till the right position. Analogous reasoning for the death of a particle.
(2) Partition $[-1,1]$ in (say) at least K^2 (but no more than $2K^2$) intervals of equal length. Two functions in $\mathscr{M}$ which differ only on one of these subintervals are closer than $1/K^2$.

(3) Finally, by taking N sufficiently large we may assume that we can approximate two functions u and v in $\mathscr{M}$ which differ only on one of the subintervals via two particle configurations σ and η such that $\mathrm{dist}(\pi_\sigma^N, u) \leqslant 1/K^3$ and $\mathrm{dist}(\pi_\eta^N, v) \leqslant 1/K^3$.

By using the three steps above, one performs the requested path.

4. About more general exclusion processes

The aim of this short section is to stress that the proof of Theorem 2.2 is *very little model dependent*, once a result like Theorem 2.1 is known. Therefore, we expect it to be susceptible of generalization to a broad class of model. This however passes through clarifying a number of issues, that are of analytical rather than probabilistic nature. We will not attempt to solve these points here: we merely list them and connect them with the argument presented in this note.

4.1. Boundary driven exclusion processes: hydrodynamics and hydrostatics

A natural generalization of the boundary driven SSEP are boundary driven *Kawasaki* dynamics. By this we mean processes generated by operators of the form

$$L_N f(\eta) = \frac{N^2}{2} \sum_{x,y \in \Lambda_N} c(x,y,\eta)[f(\eta^{x,y}) - f(\eta)] + \sum_{x \in \Lambda_N:\ |x|=N} c(x,\eta)[f(\eta^x) - f(\eta)], \tag{4.1}$$

which clearly generalizes (2.1). The arising process is clearly the superposition of a dynamics with a conservation law (*Kawasaki dynamics*: the rates are $c(x,y,\eta)$), acting on the whole of Λ_N, and a dynamics without conservation laws (*Glauber dynamics* or *birth and death dynamics*: the rates are $c(x,\eta)$), acting only at the boundary. Some hypotheses on the rates should be imposed and we present them in a rather informal way, we refer to Eyink et al. (1990) for precise definitions: consider first the class of finite range non-degenerate models of particles hopping on Λ_N, with birth and death at the boundary, which are reversible (cf. Spohn, 1991, pp. 161–164) with respect to a finite volume Gibbs measure associated to a translation invariant family of specifications. Of course, the chemical potential of the Gibbs measure will be related to the (equal at $\pm N$!) activity of the birth and death process at the boundary. Moreover, the value of the mean density (or expected value of the occupation number, under the Gibbs measure), which will be independent of the space coordinate, is determined by the chemical potential. Under these prescriptions, the Kawasaki rates are (unlike the Glauber rates) independent of the chemical potential. The general class of dynamics of interest corresponds to choosing the Kawasaki rates exactly like in the previous example, but this time we allow the possibility of choosing Glauber rates $c(\pm N, \eta)$ with different activities at $\pm N$. Thus, while the dynamics is locally reversible, in general it is not globally reversible and one has no expression for the invariant measure.

4.2. Hydrodynamics, invariant measure and hydrostatics

In Eyink et al. (1991) it has been proven that the hydrodynamic limit of such systems are described by parabolic non-degenerate equations

$$\partial_t \rho(t,r) = \nabla[D(\rho(t,r))\nabla\rho(t,r)] \text{ for every } (t,r) \in \mathbb{R}^+ \times (-1,1), \tag{4.2}$$

with $\rho(t,\pm 1) = \rho_\pm$ for every $t \in \mathbb{R}^+$. We remark here that in Eyink et al. (1991) such a result is proven only for gradient models (Kipnis and Landim, 1999): in this case it is easy to see that $D(\cdot)$ is a smooth function, see Eyink et al. (1990) formula (3.5). The result may be extended to non-gradient models (Kipnis and Landim, 1999): then $D(\cdot)$ can be expressed in terms of a *Green–Kubo* formula, see e.g. Spohn (1991, p. 180), and it is not as easy to obtain its regularity properties. We would like to stress that, at least in one-dimensional cases, the hydrodynamic limit problem (the law of large numbers) with boundaries is rather well understood as long as the corresponding problem without boundaries (say, on a torus) is understood. Moreover, these results rely on the absence of phase transitions, which of course is ensured for local models in $\mathrm{d} = 1$.

Once again for every fixed N the assumptions we make on the rates are (largely) sufficient to ensure the existence of a unique invariant measure (steady state) that we will call μ_N. In Eyink et al. (1990), for the gradient case, and in Kipnis et al. (1995) and Landim et al. (2001) for some non-gradient ones, a law of large numbers for $\{\mu_N\}_N$ has been established. It has in fact been proven that the law of the empirical field on the steady state converges as N tends to infinity to the measure on $\mathscr{M}$ concentrated on the unique solution $\bar\rho$ of the non-degenerate elliptic equation

$$\nabla[D(\bar\rho(r))\nabla\bar\rho(r)] = 0, \quad \text{for every } r \in (-1,1), \tag{4.3}$$

and $\bar\rho(\pm 1) = \rho_\pm$.

4.3. From dynamic to static large deviations

It is not difficult to guess what the dynamical large deviation function should be in this general case: going back to Section 2.3, it suffices to replace formula (2.8) with

$$\begin{aligned} J_H(\pi) = {} & \langle \pi(T), H(T)\rangle - \langle \rho_0, H(0)\rangle - \int_0^T [\langle \pi(t), \partial_t H(t)\rangle + \langle \theta(\pi(t)), \Delta H(t)\rangle]\,\mathrm{d}t \\ & + \theta(\rho^+)\int_0^T \nabla H(t,1)\,\mathrm{d}t - \theta(\rho^-)\int_0^T \nabla H(t,-1)\,\mathrm{d}t \\ & - \frac{1}{2}\int_0^T \langle \sigma(\pi(t)), (\nabla H(t))^2\rangle\,\mathrm{d}t, \end{aligned} \tag{4.4}$$

where $\theta(0) = 0$, $\theta' = D$ and σ is a function from $[0,1]$ to $[0,\infty)$. The function σ (*mobility*, *conductivity*) is related to the diffusion coefficient D via the so called Einstein relation (Spohn, 1991). D and σ coincide up to a multiplicative density-dependent factor (*compressibility*), which is a thermodynamical coefficient which depends only on the equilibrium measure, and therefore it is regular. Of course, the expected large deviations functional for $\{\mu_N\}_N$ is still given by the quasi-potential (2.12).

The argument of this note goes through word by word if

(1) One has the generalization of Theorem 2.1. It should be noted that the *hydrodynamic limit* technology (Kipnis and Landim, 1999; Varadhan and Yau, 1997) naturally provides the *super-exponential* probabilistic estimates that allow to analyze large deviation events and leads to the proof of a full upper bound and a lower bound for neighbors of smooth trajectories. The full lower bound is recovered if one can show that $I_{[0,T]}(\rho_n) \to I_{[0,T]}(\rho)$ for a sequence of smooth functions ρ_n which tends to ρ in $C^0([0,T];\mathscr{M})$ (in the SSEP case this is shown by using some convexity properties that are absent in the general context). Moreover we require $I_{[0,T]}$ to be a good rate functional: while the compactness of the level sets follows by the standard arguments, one has to provide a proof of lower semicontinuity.

(2) One has uniqueness to the weak formulation of the limit PDE (4.2), that is there exists a unique $\rho \in C^0([0,T];\mathscr{M})$ such that $I_{[0,T]}(\rho|\rho_0) = 0$, $\rho_0 \in \mathscr{M}$. This result is already known, see Kipnis and Landim (1999, th. 4.1 page 365), with periodic boundary conditions. We remark that we used also the regularity of ρ for positive times, and therefore ρ is a classical solution to (4.2) for $t > 0$: however this requirement may be weakened and the argument goes through, once uniqueness is established, if there exists a standard weak solution (in the $\mathbb{H}_1$ sense) to (4.2) for positive times. Standard parabolic regularity results may be applied if D is differentiable and in this case there exists a classical solution to (4.2).

In higher dimensions $d \geqslant 2$, we can consider also a stochastic evolution with reservoirs in the domain $\Lambda_N^d = \{-N, \ldots, N\}^d$ by defining creation/annihilation rates equal to γ^+ on the face $\{x_1 = N\}$ and γ^- on the face $\{x_1 = -N\}$. In this case, if an hydrodynamic large deviation principle holds for which the previous assumptions on the large deviation functional are satisfied then our proof would also imply that the static large deviation functional is equal to the quasi-potential. Unfortunately besides the one-dimensional SEP, there is no other instance for which the quasi-potential can be explicitely computed (cf. Bertini et al. (2002, 2003); Derrida et al. (2002)) and higher-dimensional models remain an important source of open problems.

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