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On differentiability of ruin functions under Markov-modulated models

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Abstract

This paper analyzes the continuity and differentiability of several classes of ruin functions under Markov-modulated insurance risk models with a barrier and threshold dividend strategy, respectively. Many ruin related functions in the literature, such as the expectation and the Laplace transform of the Gerber–Shiu discounted penalty function at ruin, of the total discounted dividends until ruin, and of the time-integrated discounted penalty and/or reward function of the risk process, etc, are special cases of the functions considered in this paper. Continuity and differentiability of these functions in the corresponding dual models are also studied.

Keywords: Ruin function; Markov-modulated model; Dual model; Strong Markov property; Differentiability; Gerber–Shiu function

1. Introduction

In risk theory, ruin probabilities and related quantities have been investigated extensively over the past century. There is a huge amount of literature in this area. The ruin related quantities include the time to ruin, the surplus immediately before ruin, the deficit at ruin, the maximal ruin severity, the aggregate severity of ruin until recovery, and the time-integrated penalty or rewards of the risk process, etc. When there are dividend payments in the model, the discounted aggregate dividends until ruin is another important quantity. To explore these random variables, we usually consider their expectations and/or Laplace transforms as functions of the initial surplus.

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One common way to deal with a risk function is to derive an integro-differential equation satisfied by the concerned ruin function first. Then by either solving or analyzing this integro-differential equation, we can obtain some interesting results. However, most of the papers in actuarial science literature do not provide conditions and proofs for the differentiability of the concerned ruin functions. In this paper, we are interested in the continuity and differentiability of several classes of general ruin functions. Justification of higher-order differentiability is also useful as in some particular cases where the claim size distributions fall into a certain class such that the integro-differential equations can be transformed into solvable high-order ordinary differential equations. Moreover, the existence of the derivatives of these ruin functions makes it possible to compute them as integrals of other functions. For some related discussions, see [1]. When dividend payments are taken into consideration, the problem becomes more complex and more important, since the continuity and differentiability of the functions concerned at the point of the level of the threshold or the barrier may be a question. Moreover, in these cases, the ruin functions usually satisfy piecewise integro-differential equations, which requires more boundary conditions in order to be solved. If we can prove continuity and /or differentiability of these functions at some of these points, these properties can serve as boundary conditions. Of course, the continuity and differentiability properties themselves are of great mathematical interest.

Cai [2] considered the continuity and differentiability of a penalty function under a classical insurance risk model with stochastic investment. For a similar model, Wang and Wu [3] proved the continuity and differentiability of the ruin probability and the distribution of surplus before and after ruin.

As an extension of the classical insurance risk model, the Markov-modulated (also called Markovian regime-switching) insurance risk model takes the impact of external environment into consideration. A time-homogeneous Markov chain with finite number of states is used to model the changes of the economic environment in this model. See [4,5] and the references therein for detailed discussions of this type of models. Another problem with the classical model is that it is very conservative. Under the positive loading condition the surplus tends to infinity with probability one as time goes to infinity. DeFinetti [6] proposed a model which pays dividends, and the insurance risk models with dividend payments have attracted considerable attention recently. Two most common dividend strategies are the barrier strategy and the threshold strategy. Under the barrier strategy, the excess of the surplus over a barrier is paid out immediately as dividends but no dividends are paid when the surplus is below the barrier. Under the threshold strategy, dividends are paid at a constant rate no greater than the premium rate when the surplus exceeds a barrier but no dividends are paid when the surplus is below the barrier.

In this paper, we consider the Markov-modulated risk model with a barrier or threshold strategy and the corresponding dual models. We analyze the continuity and differentiability of several classes of ruin functions based on these models. In order to make the ruin functions general enough to include most quantities in the ruin theory as special cases, we introduce two unified random variables and study the expectations and the Laplace transforms of them.

2. Markov-modulated model

In this section we deal with the Markov-modulated model. We consider two kinds of dividend strategies, namely the threshold strategy and the barrier strategy. Let R_t denote the surplus of an insurance company at time t, and J_t be the environment state at time t. Assume that $\{J_t\}$ is a time-homogeneous Markov process with the state space $\mathcal{E} = \{1, \ldots, m\}$, and that it is governed by the intensity matrix $Q = (q_{ij})_{i,j\in\mathcal{E}}$ with $q_i := -q_{ii}$. Then the Markov-modulated risk process can be defined by

$$R_t = R_0 + \int_0^t c_{J_s} \mathrm{d}s - \sum_{j=1}^{N(t)} U_j$$

where R_0 denotes the initial reserve, c_i $(i \in \mathcal{E})$ are premium rates, U_j is the size of the jth claim, and $N(t)$ counts the number of claims arriving before or on time t. Assume that given the environment state at time t $J_t = i$, the claim arrival intensity at this moment is λ_i, and the claim size distribution is $F_i(\cdot)$ if a claim occurs at this moment. For $j \in \mathbb{N}$, define S_j to be the arrival time of the jth claim. Then we have $N(t) = \#\{j : S_j \le t\}$ and given $J_{S_j} = i$, the conditional distribution function of U_j is $F_i(\cdot)$. Furthermore, under the model, it is assumed that given the environmental state J_{S_j}, U_j is independent of $\{U_i, i \ne j\}$ and $\{N(t)\}$.

When there are dividend payments in the model, we use R_t^b to denote the surplus at time t for the risk model modified by dividend payments with either a barrier or threshold strategy, where the superscript b denotes the level of the barrier or of the threshold. For the model under a threshold dividend strategy, let d_{J_t} stand for the dividend payout rate at time t, if any. Let $I\{\cdot\}$ be the indicator function. Then if the threshold dividend strategy is applied, the surplus process is given by

$$R_t^b = R_0^b + \int_0^t \left(c_{J_s} - d_{J_s} I\{R_{s-}^b \ge b\} \right) \mathrm{d}s - \sum_{j=1}^{N(t)} U_j. \tag{2.1}$$

And if the barrier dividend strategy is adopted, the surplus process can be represented as below

$$R_t^b = R_0^b \wedge b + \int_0^t c_{J_s} I\{R_{s-}^b < b\} \mathrm{d}s - \sum_{j=1}^{N(t)} U_j. \tag{2.2}$$

For convenience, consider $\{(R_t^b, J_t)\}_{t \in \mathbb{R}^+}$ as a canonical process on the path space $\Omega = (\mathbb{R} \times \mathcal{E})^{\mathbb{R}^+}$ equipped with the σ-field $\mathcal{A} = (\mathcal{B}(\mathbb{R}) \otimes 2^{\mathcal{E}})^{\mathbb{R}^+}$ generated by all evaluation maps $\pi_t : \Omega \to \mathbb{R} \times \mathcal{E}$ where $\mathcal{B}(\mathbb{R})$ and $2^{\mathcal{E}}$ denote the Borel σ-field of $\mathbb{R}$ and the collection of all subsets of $\mathcal{E}$, respectively. Consider the probability space $(\Omega, \mathcal{A}, \mathrm{P})$ and define $\mathcal{F}_t = \sigma\{(R_s^b, J_s) : 0 \le s \le t\}$. Write $\overline{\mathcal{A}}$ for the P-completion of $\mathcal{A}$ and $\overline{\mathcal{F}_t}$ for the P-completion of $\mathcal{F}_t$ in $\overline{\mathcal{A}}$. Throughout this paper, we base our study on the filtered probability space $(\Omega, \overline{\mathcal{A}}, \{\overline{\mathcal{F}_t}\}, \overline{\mathrm{P}})$ where $\overline{\mathrm{P}}$ is the extension of the probability measure P to the σ-field $\overline{\mathcal{A}}$. We introduce the shift operator $\theta_t : \Omega \to \Omega$, $t \ge 0$, given by $(\theta_t \omega)_s = \omega_{s+t}$, $s, t \ge 0$, $\omega = (\omega_s : s \ge 0) \in \Omega$.

We will show that for the Markov-modulated risk model without dividends, if the claim size distributions are nice, some frequently concerned risk functions are usually continuous and differentiable (up to high order) with respect to the initial reserve x except for the point $x = 0$. When there are dividends in the model, the problem is more complex. For instance, for a threshold or barrier dividend strategy with threshold or barrier level b, the point $x = b$ is sometimes not a differentiable point.

For convenience, we let (R^b, J) stand for the process $\{(R_t^b, J_t) : t \ge 0\}$. Define the time to ruin by $T_b = \inf\{t \ge 0 : R_t^b < 0\}$. It can be shown that T_b is an $\{\overline{\mathcal{F}}_t\}$-stopping time. For any fixed $t \ge 0$, write $(\mathbb{R} \times \mathcal{E})^{[0,t]}$ for the class of functions $f : [0, t] \to \mathbb{R} \times \mathcal{E}$. For any set $A \subseteq \mathbb{R}^+$, let $\pi_A : (\mathbb{R} \times \mathcal{E})^{\mathbb{R}^+} \to (\mathbb{R} \times \mathcal{E})^A$ be the projection map given by $\pi_A(\{X_s : s \ge 0\}) = \{X_s : s \in A\}$. Consider a measurable function $w : \mathbb{R} \times \mathbb{R}^+ \times \mathcal{E} \to \mathbb{R}$ such that for $i \in \mathcal{E}$, $w(x, y; i) \equiv 0$ for all $x < 0$ and $w(0, 0; i) = 0$. For any fixed t $(0 \le t \le \infty)$, let K_t^1 be a measurable map from the space $(\mathbb{R}\times\mathcal{E})^{[0,t]}$ to $\mathbb{R}$, and K_t^2 be a measurable map from the space $(\mathbb{R}\times\mathcal{E})^{[0,t]}\times\mathbb{R}^+$ to $\mathbb{R}$, such that both K_t^1 and K_t^2 have left limits for $t>0$, that $K_\infty^1(R^b,J)$ and $K_{T_b}^2(\pi_{[0,T_b]}(R^b,J),T_b)$ are bounded, and that for $0\le t\le s\le\infty$,

$$K_s^1(\pi_{[0,s]}(R^b,J)) = K_{t-}^1(\pi_{[0,t)}(R^b,J)) + g(t)K_{s-t}^1(\pi_{[0,s-t]}(\theta_t(R^b,J))), \tag{2.3}$$

$$\begin{aligned} K_{s\wedge T_b}^2(\pi_{[0,s\wedge T_b]}(R^b,J),T_b) &= K_{(t\wedge T_b)-}^2(\pi_{[0,t\wedge T_b)}(R^b,J),T_b) \\ &\quad + g(T_b)w(R_{T_b-}^b,|R_{T_b}^b|;J_{T_b})I(T_b\le t) \\ &\quad + g(t)K_{(s-t)\wedge(T_b\circ\theta_t)}^2\left(\pi_{[0,(s-t)\wedge(T_b\circ\theta_t)]}(\theta_t(R^b,J),T_b\circ\theta_t)\right)I(T_b>t), \end{aligned} \tag{2.4}$$

where $K_{0-}^l(\cdot)\equiv 0$ $(l=1,2)$, and $g(t)$ $(t\ge 0)$ is a continuous function with $g(0)=1$. Write for $0\le t<\infty$,

$$\overline{K}_t^1 = K_t^1(\pi_{[0,t]}(R^b,J)), \qquad \overline{K}_t^2 = K_{t\wedge T_b}^2(\pi_{[0,t\wedge T_b]}(R^b,J),T_b), \tag{2.5}$$

$$\overline{K}^1 = K_\infty^1(R^b,J), \qquad \overline{K}^2 = K_{T_b}^2(\pi_{[0,T_b]}(R^b,J),T_b), \tag{2.6}$$

$$\overline{K}_{0-}^1 = \overline{K}_{0-}^2 = 0.$$

Then $\{\overline{K}_t^1 : t\ge 0\}$ and $\{\overline{K}_t^2 : t\ge 0\}$ are two $\{\overline{\mathcal{F}}_t\}$-adapted stochastic processes, of which the left limits with respect to t exist for all $t>0$, and $\overline{K}^1$ and $\overline{K}^2$ are two random variables with respect to $\overline{\mathcal{F}}$.

Write $\mathrm{E}_{(x,i)}[\cdot] = \mathrm{E}[\cdot|(R_0^b,J_0)=(x,i)]$, and define functions for $l=1,2$

$$V_l(x,b;i) = \mathrm{E}_{(x,i)}[\overline{K}^l], \tag{2.7}$$

$$L_l(x,b,r;i) = \mathrm{E}_{(x,i)}[\mathrm{e}^{-r\overline{K}^l}] \quad \text{for } r\ge 0. \tag{2.8}$$

These functions $V_l(x,b;i)$ $(l=1,2, i\in\mathcal{E})$ and $L_l(x,b,r;i)$ $(l=1,2, i\in\mathcal{E})$ are very general in the sense that they include most of the risk functions considered in the literature of risk theory as special cases. A few examples are listed below.

Example 1. For $0\le t\le\infty$, let $K_t^2(\pi_{[0,t]}(R^b,J),T_b) = \int_0^t d_{J_s} I\{R_{s-}^b\ge b\}\mathrm{e}^{-\delta s}\mathrm{d}s$, $g(t)=\mathrm{e}^{-\delta t}$ and $w(x,y;i)\equiv 0$ $(i\in\mathcal{E})$. The random variable $\overline{K}^2 = \int_0^{T_b} d_{J_s} I\{R_{s-}^b\ge b\}\mathrm{e}^{-\delta s}\mathrm{d}s$ is the aggregate discounted dividends until ruin, and hence for each i, $V_2(x,b;i)$ and $L_2(x,b,r;i)$ are the conditional expectation and the conditional Laplace transform of the aggregate discounted dividends until ruin, respectively, given that the initial surplus is x and that the initial environment state is i.

Example 2. Define a bounded measurable function $\omega:\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}^+$ such that $\omega(x,y)\equiv 0$ for $x<0$ and $y\ge 0$. For $0\le t\le\infty$, let $K_t^2(\pi_{[0,t]}(R^b,J),T_b) = \mathrm{e}^{-\delta T_b}\omega(R_{T_b-}^b,|R_{T_b}^b|)I(T_b\le t)$, $g(t)=\mathrm{e}^{-\delta t}$ for $\delta\ge 0$, and $w(x,y;i)=\omega(x,y)$ $(i\in\mathcal{E})$. The random variable $\overline{K}^2 = \mathrm{e}^{-\delta T_b}\omega(R_{T_b-}^b,|R_{T_b}^b|)$ is the discounted penalty at ruin and its expectation is the so-called Gerber–Shiu discounted penalty function. This function covers a lot of important quantities in risk theory (see [7]).

Example 3. Let $\mu:\mathbb{R}^+\to\mathbb{R}^+$ and $\nu:\mathbb{R}^+\to\mathbb{R}^+$ be two bounded measurable functions. For $0\le t\le\infty$, let $K_t^1(\pi_{[0,t]}(R^b,J)) = \int_0^t \mathrm{e}^{-\delta s}(\mu(|R_s^b|)I\{R_s^b<0\} - \nu(|R_s^b|)I\{R_s^b\ge 0\})\mathrm{d}s$, and

$g(t) = \mathrm{e}^{-\delta t}$. Then the random variable $\overline{K}^1 = \int_0^\infty \mathrm{e}^{-\delta s}(\mu(|R_s^b|)I\{R_s^b < 0\} - \nu(|R_s^b|)I\{R_s^b \geq 0\})\mathrm{d}s$ is the time-integrated discounted penalty. For more information about this quantity, refer to [1].

Introduce $R_{0-}^b = -1$. For $l = 1, 2$, define functions $u_l^b : \mathbb{R} \times \mathbb{R}^+ \times \mathcal{E} \to \mathbb{R}$ and $v_l^b : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{E} \to \mathbb{R}^+$ such that for $t > 0$ and $l = 1, 2$,

$$u_l^b(x, t; i) = \begin{cases} \mathrm{E}\left[\overline{K}_{t-}^l | R_s^b = (x + c_i s) \wedge b + (c_i - d_i)\left(\left(s - \dfrac{b-x}{c_i}\right) \vee 0\right), \right. \\ \qquad \left. J_s = i \text{ for } 0 \leq s < t\right] & x < b \\ \mathrm{E}\left[\overline{K}_{t-}^l | R_s^b = x + (c_i - d_i)s,\, J_s = i \text{ for } 0 \leq s < t\right] & x \geq b, \end{cases} \tag{2.9}$$

$$v_l^b(x, t, r; i) = \begin{cases} \mathrm{E}\left[\mathrm{e}^{-r\overline{K}_{t-}^l} | R_s^b = (x + c_i s) \wedge b + (c_i - d_i)\left(\left(s - \dfrac{b-x}{c_i}\right) \vee 0\right), \right. \\ \qquad \left. J_s = i \text{ for } 0 \leq s < t\right] & x < b \\ \mathrm{E}\left[\mathrm{e}^{-r\overline{K}_{t-}^l} | R_s^b = x + (c_i - d_i)s,\, J_s = i \text{ for } 0 \leq s < t\right] & x \geq b. \end{cases} \tag{2.10}$$

Since $\overline{K}_{0-}^l = 0$ $(l = 1, 2)$, from the definitions (2.9) and (2.10) it follows that for $l = 1, 2$, $i \in \mathcal{E}$, and $x \in \mathbb{R}$,

$$u_l^b(x, 0; i) \equiv 0, \qquad v_l^b(x, 0, r; i) \equiv 1,$$

and by noticing that ruin occurs immediately if the initial value is less than 0, from (2.9) and (2.10) we have that for $x < 0$, $t \geq 0$ and $i \in \mathcal{E}$,

$$u_2^b(x, t; i) \equiv 0, \qquad v_2^b(x, t, r; i) \equiv 1.$$

Noting that in (2.9) and (2.10) the path of the process over time period $[0, t)$ has been specified, it can be shown that for $l = 1, 2$ and $i \in \mathcal{E}$,

$$v_l^b(x, t, r; i) = \mathrm{e}^{-r u_l^b(x,t;i)}.$$

Since $R_{0-}^b = -1$, and $w(x, t; i) \equiv 0$ ($x < 0$ and $i \in \mathcal{E}$), from (2.4) we have that given $T_b = 0$, $\overline{K}^2 = w(R_{0-}^b, |R_0^b|; J_0) \equiv 0$. Noting that ruin occurs immediately if the initial value is less than 0, it follows from (2.7) and (2.8) that for $x < 0$ and $i \in \mathcal{E}$,

$$V_2^b(x, b; i) \equiv 0, \qquad L_2^b(x, b, r; i) \equiv 1. \tag{2.11}$$

Define for $i \in \mathcal{E}$, $\rho(x; i) = \int_x^\infty w(x, y - x; i)\mathrm{d}F_i(y)$ and $\alpha(x, r; i) = \int_x^\infty \mathrm{e}^{-rw(x,y-x;i)}\mathrm{d}F_i(y)$. Let $\phi(x, t; i)$ and $\varphi(x; i)$ be functions on $\mathbb{R} \times \mathbb{R}^+ \times \mathcal{E}$ and $\mathbb{R} \times \mathcal{E}$, respectively. We first introduce a few assumptions that will be needed later. We use C to denote any constant and C^+ any strictly positive constant and add a superscript (k) to a function to represent the kth-order derivative of it for $k = 1, 2, \ldots$, where a function with a superscript (0) is itself.

Assumptions

I: (a) $\phi(x,t;i)$ $(i\in\mathcal{E})$ are continuous for $x\in\mathbb{R}-\{0,b\}$ and $t>0$; $\lim_{x\uparrow b}\phi(x,\frac{b-x}{c_i};i)=0$;

(b) $\int_0^\infty \mathrm{e}^{-(q_i+\lambda_i)t}|\phi(x,t;i)|\mathrm{d}t$ $(i\in\mathcal{E})$ are bounded for $x\in\mathbb{R}$;

(c) $g(t)$ is continuous for $t>0$ and right continuous at $t=0$ i.e., $\lim_{t\downarrow 0}g(t)=1$;

(d)$\int_0^\infty \mathrm{e}^{-\min_i(q_i+\lambda_i)t}|g(t)|\mathrm{d}t<\infty$.

I′: $\rho(x;i)$ $(i\in\mathcal{E})$ are bounded for $x\in\mathbb{R}$.

II: For any $i\in\mathcal{E}$,

(a) $F_i(x)$ is continuously differentiable up to $(n-1)$th order and all $F_i^{(k)}(x)$ $(k=1,\dots,n-1)$ are bounded for $x>0$;

(b) For $(x,t)\in\{(x,t):x<0,0<t<-\frac{x}{c_i}\}\cup\{(x,t):0<x<b,0<t<\frac{b-x}{c_i}\}\cup\{(x,t):x>b,t>0\}$, and $k=0,1,\dots,n$, $\frac{\partial^k\phi(x,t;i)}{\partial x^k}$ exists and is continuous, and $|\frac{\partial^k\phi(x,t;i)}{\partial x^k}|\le C^+\mathrm{e}^{(q_i+\lambda_i-\epsilon)t}$ for some $\epsilon>0$;

(c) For $x\in(0,b)$: $\lim_{x\uparrow b}\phi(x,\frac{b-x}{c_i};i)=0$; $k,k'=0,1,2,\dots,n-1$, and $k+k'\le n-1$, $\frac{\partial^{k'}}{\partial x^{k'}}(\frac{\partial^k\phi(x,t;i)}{\partial x^k}|_{t=\frac{b-x}{c_i}-})$ exists and is continuous, and $|\frac{\partial^{k'}}{\partial x^{k'}}(\frac{\partial^k\phi(x,t;i)}{\partial x^k}|_{t=\frac{b-x}{c_i}-})|\le C^+\mathrm{e}^{(q_i+\lambda_i-\epsilon)\frac{b-x}{c_i}}$ for some $\epsilon>0$; $\frac{\partial^n\phi(x,\frac{b-x}{c_i}-;i)}{\partial x^n}$ exists and is continuous, and $|\frac{\partial^n\phi(x,\frac{b-x}{c_i}-;i)}{\partial x^n}|\le C^+\mathrm{e}^{(q_i+\lambda_i-\epsilon)\frac{b-x}{c_i}}$ for some $\epsilon>0$;

(d) $g(t)$ is right continuous at $t=0$; $g(t)$ is continuously differentiable up to nth order for $t>0$, and for $k=0,\dots,n$, $\int_0^\infty \mathrm{e}^{-\min_i(q_i+\lambda_i)t}|g^{(k)}(t)|\mathrm{d}t<\infty$

II′: For $i\in\mathcal{E}$ and $x\in(-\infty,0)$:

for $k,k'=0,1,2,\dots,n-1$, and $k+k'\le n-1$, $\frac{\partial^{k'}}{\partial x^{k'}}(\frac{\partial^k\phi(x,t;i)}{\partial x^k}|_{t=(-\frac{x}{c_i})-})$ exists and is continuous, and $|\frac{\partial^{k'}}{\partial x^{k'}}(\frac{\partial^k\phi(x,t;i)}{\partial x^k}|_{t=(-\frac{x}{c_i})-})|\le C^+\mathrm{e}^{-(q_i+\lambda_i-\epsilon)\frac{x}{c_i}}$ for some $\epsilon>0$; $\frac{\partial^n\phi(x,(-\frac{x}{c_i})-;i)}{\partial x^n}$ exists and is continuous, and $|\frac{\partial^n\phi(x,(-\frac{x}{c_i})-;i)}{\partial x^n}|\le C^+\mathrm{e}^{-(q_i+\lambda_i-\epsilon)\frac{x}{c_i}}$ for some $\epsilon>0$;

II″: $\varphi(x;i)$ $(i\in\mathcal{E})$ are continuously differentiable up to $(n-1)$th order and for $k=0,1,\dots,n-1$, the kth-order derivatives are bounded.

Note: All the ϵ that appear in the above assumptions are not necessarily the same one. The statement that a function is continuously differentiable up to 0th order is interpreted as that the function is continuous.

Remark 2.1. The assumptions regarding the functions $\phi(x,t;i)$ $(i\in\mathcal{E})$ look formidable. However, it is not difficult to verify these conditions for $\phi(x,t;i)=u_l^b(x,t;i)$ or $\phi(x,t;i)=v_l^b(x,t,r;i)$ corresponding to those frequently concerned ruin functions. We illustrate this by reconsidering the three examples introduced before.

Example 1. We have for $x<0$, $u_2^b(x,t;i)\equiv 0$ and $v_2^b(x,t,r;i)\equiv 1$. For $0<x<b$ and $0<t\le\frac{b-x}{c_i}$, we also have $u_2^b(x,t;i)\equiv 0$ and $v_2^b(x,t,r;i)\equiv 1$. When $x>b$, we have $u_2^b(x,t;i)=d_i\int_0^t \mathrm{e}^{-\delta s}\mathrm{d}s$ and $v_2^b(x,t,r;i)=\mathrm{e}^{-d_i\int_0^t \mathrm{e}^{-\delta s}\mathrm{d}s}$. It is easy to see that all the assumptions hold for $\phi(x,t;i)=u_l^b(x,t;i)$ or $\phi(x,t;i)=v_l^b(x,t,r;i)$.

Example 2. For $\phi(x,t;i)=u_l^b(x,t;i)$ or $\phi(x,t;i)=v_l^b(x,t,r;i)$, all the assumptions on $\phi(x,t;i)$ are satisfied by noting that $u_2^b(x,t;i)\equiv 0$ and $v_2^b(x,t,r;i)\equiv 1$ for all $x\in\mathbb{R}$ and $t\ge 0$.

Example 3. Suppose that for all $x>0$, $\mu(x)$ and $\nu(x)$ are both continuously differentiable up to $(n-1)$th order, and for any $k=0,1,\dots,n-1$, the kth-order derivatives are bounded. By the definition (2.9), we have that for $x < 0$ and $0 < t < -\frac{x}{c_i}$,

$$u_1^b(x,t;i) = \int_0^t \mathrm{e}^{-\delta s}\mu(-x - c_i s)\mathrm{d}s = \frac{1}{c_i}\int_x^{x+c_i t} \mathrm{e}^{-\delta\frac{s-x}{c_i}}\mu(-s)\mathrm{d}s,$$
$$u_1^b\left(x, -\frac{x}{c_i}; i\right) = \frac{1}{c_i}\int_x^0 \mathrm{e}^{-\delta\frac{s-x}{c_i}}\mu(-s)\mathrm{d}s.$$

So for any fixed $i \in \mathcal{E}$, it is easy to see that for $x < 0$, $0 < t < -\frac{x}{c_i}$, and $k = 1, \ldots, n-1$, $\frac{\partial^k}{\partial x^k}u_1^b(x,t;i)$ and $\frac{\partial^k}{\partial x^k}u_1^b(x, -\frac{x}{c_i}; i)$ exist, and

$$\begin{aligned}
\frac{\partial^k}{\partial x^k}u_1^b(x,t;i) \;&=\; (-1)^{k-1}\frac{1}{c_i}\left(\mu^{(k-1)}(-x-c_i t)\mathrm{e}^{-\delta t} - \mu^{(k-1)}(-x)\right)\\
&\quad + \frac{\delta}{c_i}\frac{\partial^{k-1}}{\partial x^{k-1}}u_1^b(x,t;i),\\
\frac{\partial^k}{\partial x^k}u_1^b(x,t;i)|_{t=(-\frac{x}{c_i})-} \;&=\; (-1)^{k-1}\frac{1}{c_i}\left(\mu^{(k-1)}(0-)\mathrm{e}^{\delta\frac{x}{c_i}} - \mu^{(k-1)}(-x)\right)\\
&\quad + \frac{\delta}{c_i}\frac{\partial^{k-1}}{\partial x^{k-1}}u_1^b(x,t;i)|_{t=(-\frac{x}{c_i})-},\\
\frac{\partial^k}{\partial x^k}u_1^b\left(x,-\frac{x}{c_i};i\right) &= (-1)^k\frac{1}{c_i}\mu^{(k-1)}(-x) + \frac{\delta}{c_i}\frac{\partial^{k-1}}{\partial x^{k-1}}u_1^b\left(x,-\frac{x}{c_i};i\right).
\end{aligned} \tag{2.12}$$

It follows from (2.12) that for any $k + k' \le n-1$, if $\frac{\partial^{k'}}{\partial x^{k'}}(\frac{\partial^{k-1}}{\partial x^{k-1}}u_1^b(x,t;i)|_{t=(-\frac{x}{c_i})-})$ exists, so does $\frac{\partial^{k'}}{\partial x^{k'}}(\frac{\partial^{k}}{\partial x^{k}}u_1^b(x,t;i)|_{t=(-\frac{x}{c_i})-})$. Hence, by noticing that $u_1^b(x,(-\frac{x}{c_i})-;i) = u_1^b(x,-\frac{x}{c_i};i)$ is differentiable up to $(n-1)$th order, it can be shown that for any k and k' with $k + k' \le n-1$, $\frac{\partial^{k'}}{\partial x^{k'}}(\frac{\partial^{k}}{\partial x^{k}}u_1^b(x,t;i)|_{t=(-\frac{x}{c_i})-})$ exists and

$$\begin{aligned}
&\frac{\partial^{k'}}{\partial x^{k'}}\left(\frac{\partial^k}{\partial x^k}u_1^b(x,t;i)|_{t=(-\frac{x}{c_i})-}\right)\\
&\quad= (-1)^{k-1}\frac{1}{c_i}\left(\left(\frac{\delta}{c_i}\right)^{k'}\mu^{(k-1)}(0-)\mathrm{e}^{\delta\frac{x}{c_i}} - (-1)^{k'}\mu^{(k+k'-1)}(-x)\right)\\
&\qquad+ \frac{\delta}{c_i}\frac{\partial^{k'}}{\partial x^{k'}}\left(\frac{\partial^{k-1}}{\partial x^{k-1}}u_1^b(x,t;i)|_{t=-(\frac{x}{c_i})-}\right).
\end{aligned}$$

Since for all $i \in \mathcal{E}$, $u_1^b(x,-\frac{x}{c_i};i)$ $(x < 0)$, $u_1^b(x,t;i)$ $(x < 0)$ and $\mu^{(k)}(x)$ $(x > 0, k = 0,\ldots,n-1)$ are bounded, proceeding recursively it can be shown that all the derivatives considered above are bounded. So for $\phi(x,t;i) = u_l^b(x,t;i)$, II′ is satisfied here.

For $0 \le x < b$, $0 < t < \frac{b-x}{c_i}$ and $i \in \mathcal{E}$,

$$u_1^b(x,t;i) = -\int_0^t \mathrm{e}^{-\delta s}\nu(x + c_i s)\mathrm{d}s = -\frac{1}{c_i}\int_x^{x+c_i t}\mathrm{e}^{-\delta\frac{s-x}{c_i}}\nu(s)\mathrm{d}s$$

and $u_1^b(x, \frac{b-x}{c_i}; i) = -\frac{1}{c_i}\int_x^b \mathrm{e}^{-\delta\frac{s-x}{c_i}}\nu(s)\mathrm{d}s$.

For $x \geq b, t > 0$ and $i \in \mathcal{E}$,

$$u_1^b(x,t;i) = -\int_0^t \mathrm{e}^{-\delta s}\nu(x+(c_i-d_i)s)\mathrm{d}s = -\frac{1}{c_i-d_i}\int_x^{x+(c_i-d_i)t} \mathrm{e}^{-\delta\frac{s-x}{c_i-d_i}}\nu(s)\mathrm{d}s.$$

Noticing the similarity between the case $x < 0$ and the two cases $0 < x < b$ and $x > b$, we can therefore apply a similar argument to show that the assumptions II(b), (c) are satisfied for $\phi(x,t;i) = u_l^b(x,t;i)$.

Noting that $v_1^b(x,t,r;i) = \mathrm{e}^{-ru_1^b(x,t;i)}$, it is easy to prove that all the assumptions II(b), (c) as well as II$'$ are satisfied for $\phi(x,t;i) = v_l^b(x,t,r;i)$.

In what follows, we use A1 to represent the assumptions listed in I, and A2 to represent the assumptions included in I and I$'$. Let B1 denote the assumptions II and II$'$, and B2 denote the assumptions II and II$''$.

2.1. Threshold dividend strategy

Assume that the company adopts a threshold dividend strategy with the threshold level b and that it pays dividends at the rate d_{J_t} $(0 \leq d_{J_t} \leq c_{J_t})$ at time t given the environment state J_t if the surplus at this moment exceeds b. This model can be described by (2.1). We study the risk functions $V_l(x,b;i)$ $(l = 1,2, i \in \mathcal{E})$ and $L_l(x,b,r;i)$ $(l = 1,2, i \in \mathcal{E})$ based on this model. In the next theorem, we present the conditions sufficient for the functions to be continuous or differentiable (up to high order) with respect to the initial reserve x. Define σ_1 to be the first transition time of the Markov chain $\{J_t\}$. Let $\frac{\partial^-}{\partial x}$, $\frac{\partial^+}{\partial x}$ and $\frac{\partial}{\partial x}$ stand for the left-partial derivative, the right-partial derivative and the partial derivative, respectively, with respect to x.

Theorem 2.1. *For fixed $l = 1$ or 2, if* Al *holds for $\phi(x,t;i) = u_l^b(x,t;i)$, then the functions $V_l(x,b;i)$ $(i \in \mathcal{E})$ and $L_l(x,b,r;i)$ $(i \in \mathcal{E})$ are continuous in x for $x \in \mathbb{R} - \{0\}$.*

Proof. For any fixed i and any a.s. finite stopping time η, by conditioning on $\overline{\mathcal{F}}_\eta$ first and then using the strong Markov property of $\{(R_t^b, J_t)\}$, from (2.3)–(2.7) we have

$$V_1(x,b;i) = \mathrm{E}_{(x,i)}[\overline{K}^1_{\eta-} + g(\eta)\mathrm{E}_{(R_\eta^b,J_\eta)}[\overline{K}^1]] = \mathrm{E}_{(x,i)}[\overline{K}^1_{\eta-} + g(\eta)V_1(R_\eta^b,b;J_\eta)], \tag{2.13}$$

$$\begin{aligned} &V_2(x,b;i) \\ &= \mathrm{E}_{(x,i)}\left[\overline{K}^2_{\eta-} + g(\eta)\left(\mathrm{E}_{(R_\eta^b,J_\eta)}[\overline{K}^2]I(T_b > \eta) + w(R^b_{T_b-}, |R^b_{T_b}|; J_{T_b})I(T_b = \eta)\right)\right] \\ &= \mathrm{E}_{(x,i)}\left[\overline{K}^2_{\eta-} + g(\eta)\left(V_2(R_\eta^b,b;J_\eta)I(T_b > \eta) + w(R^b_{T_b-}, |R^b_{T_b}|; J_{T_b})I(T_b = \eta)\right)\right]. \end{aligned} \tag{2.14}$$

Note that conditioning on $J_0 = i$ and $\sigma_1 = t$, the conditional probability of the event $S_1 > t$ is $\mathrm{e}^{-\lambda_i t}$ and the conditional density of S_1 is $\lambda_i \mathrm{e}^{-\lambda_i s}$ for $s < t$. Given $(R_0^b, J_0) = (x,i)$ and $\sigma_1 = t$, we have for $0 \leq s < S_1$ and $s \leq t$,

$$R_s^b = x + c_i\left(s \wedge \left(\frac{b-x}{c_i} \vee 0\right)\right) + (c_i - d_i)\left(\left(s - \frac{b-x}{c_i} \vee 0\right) \vee 0\right).$$

By (2.9) it follows that for any $0 \leq \eta \leq S_1 \wedge \sigma_1$, $\overline{K}^l_{\eta-} = u_l^b(R_0^b, \eta; J_0)$ $(l = 1,2)$. Then, for $x < 0$, setting $\eta = \sigma_1 \wedge S_1 \wedge (-\frac{x}{c_i})$ in (2.13) and distinguishing three cases $\eta = \sigma_1$, $\eta = S_1$ and

$\eta = -\frac{x}{c_i}$ yield

$$\begin{aligned} V_1(x,b;i) &= \int_0^{-\frac{x}{c_i}} q_i \mathrm{e}^{-(q_i+\lambda_i)t} \left(u_1^b(x,t;i) + g(t) \sum_{j\neq i} \frac{q_{ij}}{q_i} V_1(x+c_i t, b; j) \right) \mathrm{d}t \\ &\quad + \int_0^{-\frac{x}{c_i}} \lambda_i \mathrm{e}^{-(q_i+\lambda_i)s} \mathrm{d}s \left(u_1^b(x,s;i) + g(s) \int_0^{\infty} V_1(x+c_i s - y, b; i) \mathrm{d}F_i(y) \right) \\ &\quad + \mathrm{e}^{(q_i+\lambda_i)\frac{x}{c_i}} \left(u_1^b \left(x, -\frac{x}{c_i}; i \right) + g\left(-\frac{x}{c_i} \right) V_1(0,b;i) \right) \quad \text{for } x < 0, \end{aligned} \tag{2.15}$$

where the first term on the right-hand side is obtained from the case $\eta = \sigma_1$ by conditioning on $\sigma_1 = t$ $(0 \le t \le -\frac{x}{c_i})$ and making use of the fact that conditioning on J_0 and σ_1, $I\{\sigma_1 < S_1\}$ is independent of $(R^b_{\sigma_1}, J_{\sigma_1})$, the second term comes from the case $\eta = S_1$ by first conditioning on $\sigma_1 = t$ $(0 \le t \le \infty)$ and then on $S_1 = s$ $(0 \le s \le t \wedge (-\frac{x}{c_i}))$, and the last term is regarding the case $\eta = -\frac{x}{c_i}$. Similarly, for $0 \le x < b$, by letting $\eta = \sigma_1 \wedge S_1 \wedge \frac{b-x}{c_i}$ in (2.13) and (2.14), and then distinguishing the cases $\eta = \sigma_1$, $\eta = S_1$ and $\eta = \frac{b-x}{c_i}$, we have

$$\begin{aligned} V_l(x,b;i) &= \int_0^{\frac{b-x}{c_i}} q_i \mathrm{e}^{-(q_i+\lambda_i)t} \left(u_l^b(x,t;i) + g(t) \sum_{j\neq i} \frac{q_{ij}}{q_i} V_l(x+c_i t, b; j) \right) \mathrm{d}t \\ &\quad + \int_0^{\frac{b-x}{c_i}} \lambda_i \mathrm{e}^{-(q_i+\lambda_i)s} \mathrm{d}s \left(u_l^b(x,s;i) + g(s) \right. \\ &\quad \times \left. \left[\int_0^{x+c_i s} V_l(x+c_i s - y, b; i) \mathrm{d}F_i(y) + \rho(x+c_i s; i) I\{l=2\} \right] \right) \\ &\quad + \mathrm{e}^{-(q_i+\lambda_i)\frac{b-x}{c_i}} \left(u_l^b \left(x, \frac{b-x}{c_i}; i \right) + g\left(\frac{b-x}{c_i} \right) V_l(b,b;i) \right) \quad \text{for } 0 \le x < b. \end{aligned} \tag{2.16}$$

When $x \ge b$, letting $\eta = \sigma_1 \wedge S_1$ in (2.13) and (2.14), and then distinguishing $\eta = \sigma_1$ and $\eta = S_1$ gives

$$\begin{aligned} V_l(x,b;i) &= \int_0^{\infty} q_i \mathrm{e}^{-(q_i+\lambda_i)t} \left(u_l^b(x,t;i) + g(t) \sum_{j\neq i} \frac{q_{ij}}{q_i} V_l(x+(c_i-d_i)t, b; j) \right) \mathrm{d}t \\ &\quad + \int_0^{\infty} \lambda_i \mathrm{e}^{-(q_i+\lambda_i)s} \mathrm{d}s \left(u_l^b(x,s;i) \right. \\ &\quad + g(s) \left[\int_0^{x+(c_i-d_i)s} V_l(x+(c_i-d_i)s - y, b; i) \mathrm{d}F_i(y) \right. \\ &\quad \left. \left. + \rho(x+(c_i-d_i)s; i) I\{l=2\} \right] \right). \end{aligned} \tag{2.17}$$

Recall from (2.11) that

$$V_2(x,b;i) \equiv 0 \quad \text{for } x < 0.$$

Rearranging the terms in (2.15)–(2.17), we obtain that for $l = 1, 2$ and $i, j \in \mathcal{E}$,

$$V_l(x, b; i) = C_l(x; i) + \sum_{j \neq i} q_{ij} A_l(x; i, j) + B_l(x; i) \tag{2.18}$$

where

$$A_1(x; i, j) = \begin{cases} \int_0^{-\frac{x}{c_i}} e^{-(q_i+\lambda_i)t} g(t) V_1(x + c_i t, b; j) dt & x < 0 \\ \int_0^{\frac{b-x}{c_i}} e^{-(q_i+\lambda_i)t} g(t) V_1(x + c_i t, b; j) dt & 0 \leq x < b \\ \int_0^{\infty} e^{-(q_i+\lambda_i)t} g(t) V_1(x + (c_i - d_i)t, b; j) dt & x \geq b, \end{cases} \tag{2.19}$$

$$A_2(x; i, j) = \begin{cases} 0 & x < 0 \\ \int_0^{\frac{b-x}{c_i}} e^{-(q_i+\lambda_i)t} g(t) V_2(x + c_i t, b; j) dt & 0 \leq x < b \\ \int_0^{\infty} e^{-(q_i+\lambda_i)t} g(t) V_2(x + (c_i - d_i)t, b; j) dt & x \geq b, \end{cases} \tag{2.20}$$

$$B_1(x; i) = \begin{cases} \int_0^{-\frac{x}{c_i}} \lambda_i e^{-(q_i+\lambda_i)s} g(s) ds \int_0^{\infty} V_1(x + c_i s - y, b; i) dF_i(y), & x < 0 \\ \int_0^{\frac{b-x}{c_i}} \lambda_i e^{-(q_i+\lambda_i)s} g(s) ds \int_0^{\infty} V_1(x + c_i s - y, b; i) dF_i(y), & 0 \leq x < b \\ \int_0^{\infty} \lambda_i e^{-(q_i+\lambda_i)s} g(s) ds \int_0^{\infty} V_1(x + (c_i - d_i)s - y, b; i) dF_i(y), & x \geq b, \end{cases} \tag{2.21}$$

$$B_2(x; i) = \begin{cases} 0, \quad x < 0 \\ \int_0^{\frac{b-x}{c_i}} \lambda_i e^{-(q_i+\lambda_i)s} g(s) ds \\ \quad \times \left(\int_0^{x+c_i s} V_2(x + c_i s - y, b; i) dF_i(y) + \rho(x + c_i s; i) \right), & 0 \leq x < b \\ \int_0^{\infty} \lambda_i e^{-(q_i+\lambda_i)s} g(s) ds \left(\int_0^{x+(c_i-d_i)s} V_2(x + (c_i - d_i)s - y, b; i) dF_i(y) \right. \\ \quad \left. + \rho(x + (c_i - d_i)s) \right), & x \geq b, \end{cases} \tag{2.22}$$

$$\begin{aligned} C_1(x; i) \; = \; & (q_i + \lambda_i) \int_0^{-\frac{x}{c_i}} e^{-(q_i+\lambda_i)t} u_1^b(x, t; i) dt + e^{(q_i+\lambda_i)\frac{x}{c_i}} u_1^b\left(x, -\frac{x}{c_i}; i\right) \\ & + e^{(q_i+\lambda_i)\frac{x}{c_i}} g\left(-\frac{x}{c_i}\right) V_1(0, b; i) \quad \text{for } x < 0, \end{aligned} \tag{2.23}$$

$$C_2(x; i) \equiv 0 \quad \text{for } x < 0, \tag{2.24}$$

$$C_l(x;i) = (q_i+\lambda_i)\int_0^{\frac{b-x}{c_i}} \mathrm{e}^{-(q_i+\lambda_i)t} u_l^b(x,t;i)\mathrm{d}t + \mathrm{e}^{-(q_i+\lambda_i)\frac{b-x}{c_i}} u_l^b\left(x, \frac{b-x}{c_i}; i\right) + \mathrm{e}^{-(q_i+\lambda_i)\frac{b-x}{c_i}} g\left(\frac{b-x}{c_i}\right) V_l(b,b;i) \quad \text{for } 0 \le x < b, \tag{2.25}$$

and

$$C_l(x;i) = (q_i+\lambda_i)\int_0^{\infty} \mathrm{e}^{-(q_i+\lambda_i)t} u_l^b(x,t;i)\mathrm{d}t \quad \text{for } x \ge b. \tag{2.26}$$

For $l = 1$ or 2, by I(a), (b), (c) for $\phi(x,t;i) = u_l^b(x,t;i)$ and the dominated convergence theorem, it follows from (2.23)–(2.26) that $C_l(x;i)$ $(i \in \mathcal{E})$ are continuous for $x \in \mathbb{R} - \{0, b\}$ and bounded for all $x \in \mathbb{R}$.

Note from (2.19) and (2.20) that for $i, j \in \mathcal{E}$,

$$A_1(x;i,j) = \begin{cases} \dfrac{1}{c_i}\displaystyle\int_x^0 \mathrm{e}^{-(q_i+\lambda_i)\frac{t-x}{c_i}} g\left(\frac{t-x}{c_i}\right) V_1(t,b;j)\mathrm{d}t & x < 0 \\ \dfrac{1}{c_i}\displaystyle\int_x^b \mathrm{e}^{-(q_i+\lambda_i)\frac{t-x}{c_i}} g\left(\frac{t-x}{c_i}\right) V_1(t,b;j)\mathrm{d}t & 0 \le x < b \\ \dfrac{1}{c_i-d_i}\displaystyle\int_x^\infty \mathrm{e}^{-(q_i+\lambda_i)\frac{t-x}{c_i-d_i}} g\left(\frac{t-x}{c_i-d_i}\right) V_1(t,b;j)\mathrm{d}t & x \ge b. \end{cases} \tag{2.27}$$

$$A_2(x;i,j) = \begin{cases} 0 & x < 0 \\ \dfrac{1}{c_i}\displaystyle\int_x^b \mathrm{e}^{-(q_i+\lambda_i)\frac{t-x}{c_i}} g\left(\frac{t-x}{c_i}\right) V_2(t,b;j)\mathrm{d}t & 0 \le x < b \\ \dfrac{1}{c_i-d_i}\displaystyle\int_x^\infty \mathrm{e}^{-(q_i+\lambda_i)\frac{t-x}{c_i-d_i}} g\left(\frac{t-x}{c_i-d_i}\right) V_2(t,b;j)\mathrm{d}t & x \ge b. \end{cases} \tag{2.28}$$

Since $\overline{K}^l$ is bounded, $V_l(x,b;i)$ $(i \in \mathcal{E})$ are also bounded. The boundedness of $V_l(x,b;i)$ $(i \in \mathcal{E})$ and the assumptions I(c), (d) for $\phi(x,t;i) = u_l^b(x,t;i)$, imply that $A_l(x;i,j)$ $(i,j \in \mathcal{E})$ are continuous for $x \in \mathbb{R} - \{0,b\}$ and bounded for $x \in \mathbb{R}$.

Transformation of variables in (2.21) and (2.22) gives

$$B_1(x;i) = \begin{cases} \displaystyle\int_x^0 \frac{\lambda_i}{c_i}\mathrm{e}^{-(q_i+\lambda_i)\frac{z-x}{c_i}} g\left(\frac{z-x}{c_i}\right)\mathrm{d}z \int_0^\infty V_1(z-y,b;i)\mathrm{d}F_i(y), & x < 0, \\ \displaystyle\int_x^b \frac{\lambda_i}{c_i}\mathrm{e}^{-(q_i+\lambda_i)\frac{z-x}{c_i}} g\left(\frac{z-x}{c_i}\right)\mathrm{d}z \int_0^\infty V_1(z-y,b;i)\mathrm{d}F_i(y), & 0 \le x < b, \\ \displaystyle\int_x^\infty \frac{\lambda_i}{c_i-d_i}\mathrm{e}^{-(q_i+\lambda_i)\frac{z-x}{c_i-d_i}} g\left(\frac{z-x}{c_i-d_i}\right)\mathrm{d}z & \\ \quad \times \displaystyle\int_0^\infty V_1(z-y,b;i)\mathrm{d}F_i(y), & x \ge b, \end{cases} \tag{2.29}$$

$$B_2(x;i) = \begin{cases} 0, & x < 0, \\ \displaystyle\int_x^b \frac{\lambda_i}{c_i}\mathrm{e}^{-(q_i+\lambda_i)\frac{z-x}{c_i}} g\left(\frac{z-x}{c_i}\right)\xi(z;i)\mathrm{d}z, & 0 \le x < b, \\ \displaystyle\int_x^\infty \frac{\lambda_i}{c_i-d_i}\mathrm{e}^{-(q_i+\lambda_i)\frac{z-x}{c_i-d_i}} g\left(\frac{z-x}{c_i-d_i}\right)\xi(z;i)\mathrm{d}z, & x \ge b, \end{cases} \tag{2.30}$$

where $\xi(x; i) = \int_0^x V_2(x - y, b; i)\mathrm{d}F_i(y) + \rho(x; i)$. By the dominated convergence theorem, it follows from the boundedness of V_1 and the assumptions I(c), (d) for $\phi(x, t; i) = u_1^b(x, t; i)$, that $B_1(x; i)$ $(i \in \mathcal{E})$ are continuous and bounded for $x \in \mathbb{R} - \{0, b\}$. Since if I′ holds, $\xi(x; i)$ $(i \in \mathcal{E})$ are continuous and bounded, then $B_2(x; i)$ $(i \in \mathcal{E})$ are also continuous for $x \in \mathbb{R} - \{0, b\}$ and bounded for all $x \in \mathbb{R}$. Therefore, $V_l(x, t; i)$ $(i \in \mathcal{E})$ are continuous for $x \in \mathbb{R} - \{0, b\}$ and bounded for $x \in \mathbb{R}$.

Now we look at the continuity at $x = b$. By using a similar argument in proving the continuity of V_l in x for $x > b$, from (2.17) we can show that $V_l(x, b; i)$ $(i \in \mathcal{E})$ are right continuous with respect to x at $x = b$. Letting $x \uparrow b$ in (2.16), from the fact that $\lim_{x\uparrow b} u_l^b(x, \frac{b-x}{c_i}; i) = 0$ and $\lim_{t\downarrow 0} g(t) = 1$, it follows that $\lim_{x\uparrow b} V_l(x, b; i) = V_l(b, b; i)$ $(i \in \mathcal{E})$. Therefore, $V_l(x, b; i)$ $(i \in \mathcal{E})$ are continuous with respect to x at $x = b$.

The desired results concerning $L(x, r, b; i)$ can be proved similarly. $\square$

Next, we consider the differentiability of the ruin functions $V_l(x, b; i)$ and $L_l(x, r, b; i)$ with respect to the initial reserve x.

Theorem 2.2. *For fixed $l = 1$ or 2, if for some n $(n \geq 1)$, Bl holds for $\phi(x, t; i) = u_l^b(x, t; i)$ and $\varphi(x; i) = \rho(x; i)$, then all the functions $V_l(x, b; i)$ $(i \in \mathcal{E})$ are continuously differentiable up to nth order with respect to x for $x \in \mathbb{R} - \{0, b\}$, and the right and left derivatives with respect to x of the functions $V_l(x, b; i)$ $(i \in \mathcal{E})$ at the point $x = b$ exist and satisfy*

$$\frac{\partial^-}{\partial x} V_l(x, b; i)|_{x=b} = \lim_{h\downarrow 0} \frac{u_l^b(b, \frac{h}{c_i}; i) - u_l^b(b - h, \frac{h}{c_i}; i)}{h} + \frac{c_i - d_i}{c_i} \frac{\partial^+}{\partial x} V_l(x, b; i)|_{x=b}$$

for $i \in \mathcal{E}$.

Proof. Let $\phi(x, t; i) = u_l^b(x, t; i)$ and let the functions $A_l(x; i, j)$, $B_l(x; i)$ and $C_l(x; i)$ be defined in the proof of Theorem 2.1. Before we proceed to study the differentiability, we first consider the functions $C_l(x; i)$ $(i \in \mathcal{E}, l = 1, 2)$. If B$l$ holds for $n = 1$, from (2.23)–(2.26) it is not difficult to see that for $x \in \mathbb{R} - \{0, b\}$, $\frac{\mathrm{d}}{\mathrm{d}x} C_l(x; i)$ $(i \in \mathcal{E})$ exist, and that they are continuous and bounded. Moreover, for $i \in \mathcal{E}$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} C_1(x; i) &= C \int_0^{-\frac{x}{c_i}} \mathrm{e}^{-(q_i+\lambda_i)t} \frac{\partial}{\partial x} u_1^b(x, t; i)\mathrm{d}t + C\mathrm{e}^{(q_i+\lambda_i)\frac{x}{c_i}} u_1^b\left(x, -\frac{x}{c_i}; i\right) \\
&\quad + C\mathrm{e}^{(q_i+\lambda_i)\frac{x}{c_i}} \frac{\partial}{\partial x} u_1^b\left(x, -\frac{x}{c_i}; i\right) + C\mathrm{e}^{(q_i+\lambda_i)\frac{x}{c_i}} g\left(-\frac{x}{c_i}\right) + C\mathrm{e}^{(q_i+\lambda_i)\frac{x}{c_i}} g'\left(-\frac{x}{c_i}\right) \\
&\quad \text{for } x < 0, \\
\frac{\mathrm{d}}{\mathrm{d}x} C_2(x; i) &= 0 \quad \text{for } x < 0, \\
\frac{\mathrm{d}}{\mathrm{d}x} C_l(x; i) &= C \int_0^{\frac{b-x}{c_i}} \mathrm{e}^{-(q_i+\lambda_i)t} \frac{\partial}{\partial x} u_l^b(x, t; i)\mathrm{d}t + C\mathrm{e}^{-(q_i+\lambda_i)\frac{b-x}{c_i}} u_l^b\left(x, \frac{b-x}{c_i}; i\right) \\
&\quad + C\mathrm{e}^{-(q_i+\lambda_i)\frac{b-x}{c_i}} \frac{\partial}{\partial x} u_l^b\left(x, \frac{b-x}{c_i}; i\right) + C\mathrm{e}^{-(q_i+\lambda_i)\frac{b-x}{c_i}} g\left(\frac{b-x}{c_i}\right) \\
&\quad + C\mathrm{e}^{-(q_i+\lambda_i)\frac{b-x}{c_i}} g'\left(\frac{b-x}{c_i}\right) \quad \text{for } 0 < x < b,
\end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}C_l(x;i) = C\int_0^\infty \mathrm{e}^{-(q_i+\lambda_i)t}\frac{\partial}{\partial x}u_l^b(x,t;i)\mathrm{d}t \quad \text{for } x > b.$$

Proceeding repeatedly, we can conclude that if Bl ($l = 1$ or 2) holds for some n, $\frac{\mathrm{d}^k}{\mathrm{d}x^k}C_l(x;i)$ ($i \in \mathcal{E}$) exist for $k \leq n$, and they are continuous and bounded for $x \in \mathbb{R} - \{0, b\}$, and for all $k \leq n$, $l = 1, 2$ and $i \in \mathcal{E}$,

$$\begin{aligned}
\frac{\mathrm{d}^k}{\mathrm{d}x^k}C_1(x;i) &= C\int_0^{-\frac{x}{c_i}} \mathrm{e}^{-(q_i+\lambda_i)t}\frac{\partial^k}{\partial x^k}u_1^b(x,t;i)\mathrm{d}t \\
&\quad + \sum_{k'=0}^{k-1}\sum_{s=0}^{k-k'-1} C\mathrm{e}^{(q_i+\lambda_i)\frac{x}{c_i}}\frac{\partial^s}{\partial x^s}\left(\frac{\partial^{k'}}{\partial x^{k'}}u_1^b(x,t;i)|_{t=\left(-\frac{x}{c_i}\right)-}\right) \\
&\quad + \sum_{k'=0}^{k} C\mathrm{e}^{(q_i+\lambda_i)\frac{x}{c_i}}\frac{\partial^{k'}}{\partial x^{k'}}u_1^b\left(x, -\frac{x}{c_i}; i\right) + \sum_{k'=0}^{k} C\mathrm{e}^{-(q_i+\lambda_i)\frac{b-x}{c_i}} g^{(k')}\left(-\frac{x}{c_i}\right) \quad \text{for } x < 0, \\
\frac{\mathrm{d}^k}{\mathrm{d}x^k}C_2(x;i) &\equiv 0 \quad \text{for } x < 0, \\
\frac{\mathrm{d}^k}{\mathrm{d}x^k}C_l(x;i) &= C\int_0^{\frac{b-x}{c_i}} \mathrm{e}^{-(q_i+\lambda_i)t}\frac{\partial^k}{\partial x^k}u_l^b(x,t;i)\mathrm{d}t \\
&\quad + \sum_{k'=0}^{k-1}\sum_{s=0}^{k-k'-1} C\mathrm{e}^{-(q_i+\lambda_i)\frac{b-x}{c_i}}\frac{\partial^s}{\partial x^s}\left(\frac{\partial^{k'}}{\partial x^{k'}}u_l^b(x,t;i)|_{t=\frac{b-x}{c_i}-}\right) \\
&\quad + \sum_{k'=0}^{k} C\mathrm{e}^{-(q_i+\lambda_i)\frac{b-x}{c_i}}\frac{\partial^{k'}}{\partial x^{k'}}u_l^b\left(x, \frac{b-x}{c_i}; i\right) \\
&\quad + \sum_{k'=0}^{k} C\mathrm{e}^{-(q_i+\lambda_i)\frac{b-x}{c_i}} g^{(k')}\left(\frac{b-x}{c_i}\right) \quad \text{for } 0 < x < b,
\end{aligned}$$

and

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}C_l(x;i) = (q_i+\lambda_i)\int_0^\infty \mathrm{e}^{-(q_i+\lambda_i)t}\frac{\partial^k}{\partial x^k}u_l^b(x,t;i)\mathrm{d}t \quad \text{for } x > b.$$

Assume that Bl holds for $n = 1$. For any fixed i and j, from the boundedness and the continuity of $V(x, b; i)$ for $x \in \mathbb{R} - \{0\}$, and the assumption II (d), we can conclude that the derivatives of the integrands in (2.27) and (2.28) with respect to x are both continuous on the associated area and they are bounded for $x \in \mathbb{R} - \{0, b\}$. So from (2.27) and (2.28), we can see that for $x \in \mathbb{R} - \{0, b\}$ and $i, j \in \mathcal{E}$, $\frac{\mathrm{d}}{\mathrm{d}x}A_l(x;i,j)$ exists and it is continuous with respect to x.

Moreover, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}A_1(x;i,j)$$
$$=\begin{cases} CV_1(x,b;j)+C\int_x^0 \mathrm{e}^{-(q_i+\lambda_i)\frac{t-x}{c_i}} \\ \times\left(\frac{q_i+\lambda_i}{c_i}g\left(\frac{t-x}{c_i}\right)-\frac{1}{c_i}g'\left(\frac{t-x}{c_i}\right)\right)V_1(t,b;j)\mathrm{d}t & x<0, \\ CV_1(x,b;j)+C\int_x^b \mathrm{e}^{-(q_i+\lambda_i)\frac{t-x}{c_i}} \\ \times\left(\frac{q_i+\lambda_i}{c_i}g\left(\frac{t-x}{c_i}\right)-\frac{1}{c_i}g'\left(\frac{t-x}{c_i}\right)\right)V_1(t,b;j)\mathrm{d}t & 0<x<b, \\ CV_1(x,b;j)+C\int_x^\infty \mathrm{e}^{-(q_i+\lambda_i)\frac{t-x}{c_i-d_i}} \\ \times\left(\frac{q_i+\lambda_i}{c_i-d_i}g\left(\frac{t-x}{c_i-d_i}\right)-\frac{1}{c_i-d_i}g'\left(\frac{t-x}{c_i-d_i}\right)\right)V_1(t,b;j)\mathrm{d}t & x>b, \end{cases} \tag{2.31}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}A_2(x;i,j)$$
$$=\begin{cases} 0 & x<0 \\ CV_2(x,b;j)+C\int_x^b \mathrm{e}^{-(q_i+\lambda_i)\frac{t-x}{c_i}} \\ \times\left(\frac{q_i+\lambda_i}{c_i}g\left(\frac{t-x}{c_i}\right)-\frac{1}{c_i}g'\left(\frac{t-x}{c_i}\right)\right)V_2(t,b;j)\mathrm{d}t & 0<x<b, \\ CV_2(x,b;j)+C\int_x^\infty \mathrm{e}^{-(q_i+\lambda_i)\frac{t-x}{c_i-d_i}} \\ \times\left(\frac{q_i+\lambda_i}{c_i-d_i}g\left(\frac{t-x}{c_i-d_i}\right)-\frac{1}{c_i-d_i}g'\left(\frac{t-x}{c_i-d_i}\right)\right)V_2(t,b;j)\mathrm{d}t & x>b. \end{cases} \tag{2.32}$$

For $x\in\mathbb{R}-\{0,b\}$, by the continuity and boundedness of $\int_0^\infty V_1(x-y,b;i)\mathrm{d}F_i(y)$ ($i\in\mathcal{E}$), and the assumption II(d), it follows from (2.29) that $B_1(x;i)$ ($i\in\mathcal{E}$) are continuously differentiable and the derivatives are given as below: for $i\in\mathcal{E}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}B_1(x;i)&=C\int_0^\infty V_1(x-y,b;i)\mathrm{d}F_i(y)\\ &+C\int_x^0 \mathrm{e}^{-(q_i+\lambda_i)\frac{z-x}{c_i}}g\left(\frac{z-x}{c_i}\right)\mathrm{d}z\int_0^\infty V_1(z-y,b;i)\mathrm{d}F_i(y)\\ &+C\int_x^0 \mathrm{e}^{-(q_i+\lambda_i)\frac{z-x}{c_i}}g'\left(\frac{z-x}{c_i}\right)\mathrm{d}z\int_0^\infty V_1(z-y,b;i)\mathrm{d}F_i(y)\quad\text{for } x<0,\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}B_1(x;i)&=C\int_0^\infty V_1(x-y,b;i)\mathrm{d}F_i(y)\\ &+C\int_x^b \mathrm{e}^{-(q_i+\lambda_i)\frac{z-x}{c_i}}g'\left(\frac{z-x}{c_i}\right)\mathrm{d}z\int_0^\infty V_1(z-y,b;i)\mathrm{d}F_i(y)\\ &+C\int_x^b \mathrm{e}^{-(q_i+\lambda_i)\frac{z-x}{c_i}}g\left(\frac{z-x}{c_i}\right)\mathrm{d}z\int_0^\infty V_1(z-y,b;i)\mathrm{d}F_i(y)\quad\text{for } 0<x<b,\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}B(x;i) = C\int_0^\infty V_1(x-y,b;i)\mathrm{d}F_i(y)$$
$$+ C\int_x^\infty \mathrm{e}^{-(q_i+\lambda_i)\frac{z-x}{c_i-d_i}} g\left(\frac{z-x}{c_i-d_i}\right)\mathrm{d}z\int_0^\infty V_1(z-y,b;i)\mathrm{d}F_i(y)$$
$$+ C\int_x^\infty \mathrm{e}^{-(q_i+\lambda_i)\frac{z-x}{c_i-d_i}} g'\left(\frac{z-x}{c_i-d_i}\right)\mathrm{d}z\int_0^\infty V_1(z-y,b;i)\mathrm{d}F_i(y) \quad \text{for } x > b.$$

For $i \in \mathcal{E}$ and for $x \in \mathbb{R} - \{0, b\}$, from the continuity and boundedness of $\int_0^x V_2(x-y,b;i)\mathrm{d}F_i(y)$ and the assumption II″, it follows that $\xi(x;i)$ is continuous and bounded. Hence, by II d), from (2.30) we can conclude that $B_2(x;i)$ $(i \in \mathcal{E})$ are continuously differentiable and that for $i \in \mathcal{E}$,

$$\frac{\mathrm{d}}{\mathrm{d}x}B_2(x;i) \equiv 0 \quad \text{for } x < 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}x}B_2(x;i) = C\xi(x;i) + C\int_x^b \mathrm{e}^{-(q_i+\lambda_i)\frac{z-x}{c_i}} g'\left(\frac{z-x}{c_i}\right)\xi(z;i)\mathrm{d}z$$
$$+ C\int_x^b \mathrm{e}^{-(q_i+\lambda_i)\frac{z-x}{c_i}} g\left(\frac{z-x}{c_i}\right)\xi(z;i)\mathrm{d} \quad \text{for } 0 < x < b,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}B_2(x;i) = C\xi(x;i) + C\int_x^\infty \mathrm{e}^{-(q_i+\lambda_i)\frac{z-x}{c_i-d_i}} g\left(\frac{z-x}{c_i-d_i}\right)\xi(z;i)\mathrm{d}z$$
$$+ C\int_x^\infty \mathrm{e}^{-(q_i+\lambda_i)\frac{z-x}{c_i-d_i}} g'\left(\frac{z-x}{c_i-d_i}\right)\xi(z;i)\mathrm{d}z \quad \text{for } x > b.$$

Further, by II(d) it can be seen that for $x \in \mathbb{R} - \{0, b\}$, $l = 1, 2$, and $i \in \mathcal{E}$,

$$\frac{\mathrm{d}}{\mathrm{d}x}B_l(x;i) \le C^+ + C^+\int_0^\infty \mathrm{e}^{-(q_i+\lambda_i)t} g(t)\mathrm{d}t + C^+\int_0^\infty \mathrm{e}^{-(q_i+\lambda_i)t} g'(t)\mathrm{d}t < \infty.$$

As a result, all $B_l(x;i)$ $(i \in \mathcal{E})$ are bounded. Thus, from (2.18) it follows that for $i \in \mathcal{E}$ and $x \in \mathbb{R} - \{0, b\}$, $V_l(x, b; i)$ is continuously differentiable with respect to x and $\frac{\partial}{\partial x}V_l(x, b; i)$ is bounded in x.

Now we look at the differentiability at $x = b$. The approach used to prove the differentiability with respect to x for $x \in \mathbb{R} - \{0, b\}$ can be applied to show that the right and left derivatives with respect to x of the functions $V_l(x, b; i)$ $(i \in \mathcal{E})$ exist for $x \in \mathbb{R} - \{0\}$, and the right and left derivatives are continuous in x for $x \in \mathbb{R} - \{0\}$. Let $\eta = \sigma_1 \wedge S_1 \wedge \frac{h}{c_i}$ with $h > 0$, which is also a stopping time. Invoking (2.13) and (2.14), and distinguishing three cases $\eta = \sigma_1$, $\eta = S_1$ and $\eta = \frac{h}{c_i}$, from the same argument as in deriving (2.16) and (2.17), it follows that for $i \in \mathcal{E}$,

$$V_l(b, b; i) - V_l(b-h, b; i)$$
$$= \int_0^{\frac{h}{c_i}} \mathrm{e}^{-(q_i+\lambda_i)t}\mathrm{d}t\left[(q_i+\lambda_i)\left(u_l^b(b,t;i) - u_l^b(b-h,t;i)\right)\right.$$
$$+ g(t)\left(\sum_{j\neq i} q_{ij}\left(V_l(b+(c_i-d_i)t, b; j) - V_l(b-h+c_it, b; j)\right)\right.$$
$$+ \lambda_i I\{l=1\}\int_0^\infty \left(V_1(b+(c_i-d_i)t-z, b; j) - V_1(b-h+c_it, b; j)\right)\mathrm{d}F_i(z)$$

$$+ \lambda_i I\{l=2\} (\xi(b+(c_i t - d_i t); i) - \xi(b-h+c_i t; i)) \Bigg) \Bigg] + e^{-(q_i+\lambda_i)\frac{h}{c_i}} \left(u_l^b \left(b, \frac{h}{c_i}; i \right) - u_l^b \left(b-h, \frac{h}{c_i}; i \right) + g\left(\frac{h}{c_i}\right) \times \left(V_l(b+(c_i-d_i)\frac{h}{c_i}, b; i) - V_l(b,b;i) \right) \right).$$

Dividing both sides of the equation (2.32) by h and then letting $h \downarrow 0$ yield that for $i \in \mathcal{E}$,

$$\frac{\partial^-}{\partial x} V_l(x,b;i)|_{x=b} = \lim_{h\downarrow 0} \frac{u_l^b(b, \frac{h}{c_i}; i) - u_l^b(b-h, \frac{h}{c_i}; i)}{h} + \frac{c_i - d_i}{c_i} \frac{\partial^+}{\partial x} V_l(x,b;i)|_{x=b}.$$

Therefore, for any fixed i, $V_l(x,b;i)$ is not necessarily differentiable at $x = b$ and the differentiability holds if and only if $\frac{\partial^-}{\partial x} V_l(x,b;i)|_{x=b} = \frac{c_i}{d_i} \lim_{h\downarrow 0} \frac{u_l^b(b,\frac{h}{c_i};i)-u_l^b(b-h,\frac{h}{c_i};i)}{h}$ or if and only if $\frac{\partial^+}{\partial x} V_l(x,b;i)|_{x=b} = \frac{c_i}{d_i} \lim_{h\downarrow 0} \frac{u_l^b(b,\frac{h}{c_i};i)-u_l^b(b-h,\frac{h}{c_i};i)}{h}$.

Next, let us consider the case that Bl holds for $n = 2$. From the previous part, it is obvious that for $x \in \mathbb{R} - \{0, b\}$, the first terms of all the expressions on the right-hand side of (2.31) and (2.32) are continuously differentiable with respect to x, and they have bounded derivatives. By the assumption II(d), a similar argument shows that for $x \in \mathbb{R} - \{0, b\}$, the second terms (2.31) and (2.32) are also continuously differentiable with respect to x, and the corresponding derivatives are bounded in x. As a consequence, for $x \in \mathbb{R} - \{0, b\}$, $A_l(x; i, j)$ $(i, j \in \mathcal{E})$ are twice continuously differentiable with respect to x and the corresponding second-order derivatives are bounded.

Since for $x \in \mathbb{R} - \{0, b\}$, $\frac{\partial}{\partial x} V(x,b;i)$ $(i \in \mathcal{E})$ exist, and are continuous and bounded in x, by II′ and by applying a similar argument as in proving the first-order differentiability, it follows that $\xi(x;i)$ $(i \in \mathcal{E})$ and $\xi'(x;i)$ $(i \in \mathcal{E})$ are continuous and bounded on $x \in \mathbb{R} - \{0, b\}$. Hence, the assumption Bl (for $n = 2$) guarantees that $B_l(x;i)$ $(i \in \mathcal{E})$ are twice continuously differentiable for $x \in \mathbb{R} - \{0, b\}$, and these second-order derivatives are bounded.

Therefore, by (2.18), we can conclude that for $x \in \mathbb{R} - \{0, b\}$, $V_l(x,b;i)$ $(i \in \mathcal{E})$ are twice continuously differentiable with respect to x and the second-order derivatives are bounded in x.

We can proceed analogously to prove that for $l \in \{1, 2\}$ and $x \in \mathbb{R} - \{0, b\}$, if B$l$ hold for some $n = 1, 2, \ldots,$ $V_l(x,b;i)$ $(i \in \mathcal{E})$ are continuously differentiable up to nth order with respect to x, and for $k \le n$, all the kth-order derivatives are bounded. □

The same technique can be applied to prove the following theorem concerning the differentiability of the functions $L_l(x,b,r;i)$ with respect to x.

Theorem 2.3. *For fixed $l = 1$ or 2, if for some n $(n \ge 1)$,* Bl *holds for $\phi(x,t;i) = v_l^b(x,t,r;i)$ and $\varphi(x;i) = \alpha(x,r;i)$, then the functions $L_l(x,b,r;i)$ $(i \in \mathcal{E})$ are continuously differentiable up to nth order with respect to x for $x \in \mathbb{R} - \{0, b\}$, and the right and left derivatives of the functions $L_l(x,b,r;i)$ $(i \in \mathcal{E})$ with respect to x at $x = b$ exist and satisfy*

$$\frac{\partial^-}{\partial x} L_l(x,b,r;i)|_{x=b} = \lim_{h\downarrow 0} \frac{v_l^b(b, \frac{h}{c_i}, r; i) - v_l^b(b-h, \frac{h}{c_i}, r; i)}{h} L_l(b,b,r;i) + \frac{c_i - d_i}{c_i} \frac{\partial^+}{\partial x} L_l(x,b,r;i)|_{x=b} \quad \text{for } i \in \mathcal{E}.$$

The next corollary states the conditions for guaranteeing the continuity and differentiability of those frequently concerned ruin functions considered in the previous examples.

Corollary 2.4. (i) *All the conditional expectations and Laplace transforms of the aggregate discounted dividends until ruin defined in Example 1, are continuous with respect to x on* $\mathbb{R}-\{0\}$. *Furthermore, if all* $F_i(x)$ $(i \in \mathcal{E})$ *are continuously differentiable up to* $(n-1)$*th order, all these conditional expectations and Laplace transforms are continuously differentiable up to nth order with respect to x for* $x \in \mathbb{R}-\{0, b\}$.

(ii) *If* $\int_0^\infty \omega(x, y)\mathrm{d}F_i(y)$ $(i \in \mathcal{E})$ *are bounded, all the conditional expectations and Laplace transforms of the discounted penalty function at ruin defined in Example 2, are continuous with respect to x on* $\mathbb{R}-\{0\}$. *Furthermore, if for all* $i \in \mathcal{E}$, $F_i(x)$ *and* $\int_0^\infty \omega(x, y)\mathrm{d}F_i(y)$ *are continuously differentiable up to* $(n-1)$*th order and nth order respectively, and the derivatives* $F_i^{(k)}(x)$ $(k = 1, \dots, n-1)$ *and* $\frac{\partial k}{\partial x^k}\int_0^\infty \omega(x, y)\mathrm{d}F_i(y)$ $(k = 1, \dots, n)$ *are bounded for* $x > 0$, *then all the conditional expectations and Laplace transforms are continuously differentiable up to nth order with respect to x for* $x \in \mathbb{R}-\{0, b\}$.

(iii) *All the conditional expectations and Laplace transforms of the time-integrated discounted penalty studied in Example 3, are continuous with respect to x on* $\mathbb{R}-\{0\}$. *If for* $x > 0$, *all* $F_i(x)$ $(i \in \mathcal{E})$ *are continuously differentiable up to* $(n-1)$*th order,* $\mu(x)$ *and* $\nu(x)$ *are continuously differentiable up to nth order, and the derivatives* $F_i^{(k)}(x)$ $(i \in \mathcal{E}, k = 0, 1, \dots, n-1)$, $\mu^{(k)}(x)$ *and* $\nu^{(k)}(x)$ $(k = 0, 1, \dots, n)$ *are bounded for* $x > 0$, *then all the conditional expectations and Laplace transforms are continuously differentiable up to nth order with respect to x for* $x \in \mathbb{R}-\{0, b\}$.

Proof. The assertions follow immediately from Remark 2.1 and Theorems 2.1–2.3. □

2.2. Barrier strategy

In this subsection, we assume that dividends are paid according to a barrier strategy with barrier level b. Then the surplus process can be described by (2.2). When the initial reserve x is less than or equal to the barrier b, this process coincides with the risk process under the threshold dividend strategy with the dividend payment rates equaling the premium income rates at the same time ($d_{J_t} = c_{J_t}$). But this is not the case if the initial reserve is greater than the barrier i.e. $x > b$. Because, when the barrier strategy is applied, all the excess amount over b is immediately paid out as dividends to make the surplus always stay at level b before the arrival of the first claim, while the surplus will be greater than x ($x > b$) until the arrival of the first claim if the threshold strategy is used. Therefore, when $x \le b$, the functions $V_l(x, b; i)$ $(i \in \mathcal{E})$ and $L_l(x, b, r; i)$ $(i \in \mathcal{E})$ based on the model (2.2) have same expressions as in the threshold strategy case with $d_i = c_i$ for all $i \in \mathcal{E}$. For $x > b$, the evolution of the risk process under the barrier strategy with the initial value x is same as that of the process under the threshold strategy with initial value b, except that we have an amount of dividends $x - b$ at time 0 in the former case. Similarly, we define functions for $l = 1, 2$, $u_l^b : \mathbb{R} \times \mathbb{R}^+ \times \mathcal{E} \to \mathbb{R}$ and $v_l^b : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{E} \to \mathbb{R}^+$ such that for $t > 0$ and $l = 1, 2$,

$$u_l^b(x, t; i) = \begin{cases} \mathrm{E}\left[\overline{K}_{t-}^l | R_s^b = (x + c_i s) \wedge b,\, J_s = i \text{ for } 0 \le s < t\right] & x < b \\ \mathrm{E}\left[\overline{K}_{t-}^l | R_s^b = b,\ J_s = i \text{ for } 0 \le s < t\right] & x \ge b, \end{cases}$$

$$v_l^b(x, t, r; i) = \begin{cases} \mathrm{E}\left[\mathrm{e}^{-r\overline{K}_{t-}^l} | R_s^b = (x + c_i s) \wedge b,\, J_s = i \text{ for } 0 \le s < t\right] & x < b \\ \mathrm{E}\left[\mathrm{e}^{-r\overline{K}_{t-}^l} | R_s^b = b,\, J_s = i \text{ for } 0 \le s < t\right] & x \ge b. \end{cases}$$

It is clear that when $x \le b$, the above functions coincide with the corresponding ones based on the risk model under the threshold strategy with $c_i = d_i$ $(i \in \mathcal{E})$. These lead to the following theorem.

Theorem 2.5. *Fix $l = 1$ or 2.*

(i) *If* Al *holds for $\phi(x,t;i) = u_l^b(x,t;i)$, then for all $i \in \mathcal{E}$, the functions $V_l(x,b;i)$ and $L_l(x,b,r;i)$ are continuous with respect to x for $x \in \mathbb{R} - \{0\}$.*

(ii) *If for some $n \ge 1$,* Bl *holds for $\phi(x,t;i) = u_l^b(x,t;i)$ $(\phi(x,t;i) = v_l^b(x,t,r;i))$ and $\varphi(x;i) = \rho(x;i)$ $(\varphi(x;i) = \alpha(x,r;i))$, then for all $i \in \mathcal{E}$, $V_l(x,b;i)$ $(L_l(x,b,r;i))$ are continuously differentiable up to nth order with respect to x for $x \in \mathbb{R} - \{0,b\}$, and the right and left derivatives of the functions $V_l(x,b;i)$ $(L_l(x,b,r;i))$ with respect to x exist and satisfy*

$$\frac{\partial^-}{\partial x} V_l(x,b;i)|_{x=b} = \lim_{h\downarrow 0} \frac{u_l^b(b,\frac{h}{c_i};i) - u_l^b(b-h,\frac{h}{c_i};i)}{h}$$

$$\left(\frac{\partial^-}{\partial x} L_l(x,b,r;i)|_{x=b} = \lim_{h\downarrow 0} \frac{v_l^b\left(b,\frac{h}{c_i},r;i\right) - v_l^b(b-h,\frac{h}{c_i},r;i)}{h} L_l(b,b,r;i)\right).$$

Proof. The continuity and differentiability are obviously true when $x > b$. For $x \le b$, proceeding similarly as in the model under the threshold strategy gives us the desired assertions. $\square$

Remark 2.2. The continuity and differentiability of the functions $V_l(x,b;i)$ $(i \in \mathcal{E})$ and $L_l(x,b,r;i)$ $(i \in \mathcal{E})$ with respect to x for $x \in \mathbb{R} - \{0,b\}$ under the barrier dividend strategy are same as those under the threshold strategy. As a direct result, Corollary 2.4 still holds for the model under barrier strategy. We will show in the next corollary, that under the barrier strategy, the ruin functions may have better properties at the point $x = b$.

Corollary 2.6. (i) *Under the barrier strategy, if all the claim size distribution functions $F_i(x)$ $(i \in \mathcal{E})$ are absolutely continuous, all the conditional expectations and Laplace transforms of the aggregate discounted dividends until ruin defined in Example 1 are continuously differentiable with respect to x at $x = b$.*

(ii) *If all the functions $F_i(x)$ $(i \in \mathcal{E})$ and $\int_0^\infty \omega(x,y)\mathrm{d}F_i(y)$ $(i \in \mathcal{E})$ are continuous and bounded, all the conditional expectations and Laplace transforms of the discounted penalty at ruin defined in Example 2 are continuously differentiable with respect to x at $x = b$.*

(iii) *If all the functions $F_i(x)$ $(i \in \mathcal{E})$, $\mu(x)$ and $\nu(x)$ are continuous and bounded, all the conditional expectations and Laplace transforms of the time-integrated discounted penalty studied in Example 3 are continuously differentiable with respect to x at $x = b$.*

Proof. By Theorem 2.5, it is easy to see that if these functions are differentiable with respect to x at $x = b$, then the derivatives are continuous at $x = b$.

(i) For the aggregate discounted dividends until ruin defined in Example 1, we have

$$w(x,y;i) \equiv 0 \quad \text{for all } x \in \mathbb{R},$$

$$u_2^b(x,t;i) = \int_{\frac{b-x}{c_i}\wedge t}^t c_i \mathrm{e}^{-\delta s}\mathrm{d}s, \qquad v_2^b(x,t,r;i) = \mathrm{e}^{-r\int_{\frac{b-x}{c_i}\wedge t}^t c_i \mathrm{e}^{-\delta s}\mathrm{d}s} \quad \text{for } 0 \le x \le b,$$

$$V_2(x,b;i) = x - b + V_2(b,b;i), \qquad L_2(x,b,r;i) = \mathrm{e}^{-r(x-b)} L_2(b,b,r;i) \quad \text{for } x > b.$$

It follows by Theorem 2.5 that

$$\frac{\partial^-}{\partial x}V_2(x,b;i)|_{x=b} = \frac{c_i\int_0^{\frac{h}{c_i}} \mathrm{e}^{-\delta s}\mathrm{d}s}{h} = 1,$$

$$\frac{\partial^-}{\partial x}L_2(x,b,r;i)|_{x=b} = \lim_{h\downarrow 0}\frac{\mathrm{e}^{-r\int_0^{\frac{h}{c_i}} c_i\mathrm{e}^{-\delta t}\mathrm{d}t}-1}{h}L_2(b,b,r;i) = -rL_2(b,b,r;i).$$

Since from (2.32) we have $\frac{\partial^+}{\partial x}V_2(x,b;i)|_{x=b}=1$, and $\frac{\partial^+}{\partial x}L_2(x,b,r;i)|_{x=b}=-rL_2(b,b,r;i)$, the assertion follows immediately.

(ii) For the discounted penalty at ruin considered in Example 2, we have

$$u_2^b(x,t;i)\equiv 0, \qquad v_2^b(x,t,r;i)\equiv 1, \quad \text{for all } x\geq 0,$$
$$V_2(x,b;i)=V_2(b,b;i), \qquad L_2(x,b,r;i)=L_2(b,b,r;i), \quad \text{for } x\geq b.$$

Therefore, we have

$$\frac{\partial^+}{\partial x}V_2(x,b;i)|_{x=b}=0, \qquad \frac{\partial^+}{\partial x}L_2(x,b,r;i)|_{x=b}=0.$$

By Theorem 2.5 it is easy to see that

$$\frac{\partial^-}{\partial x}V_2(x,b;i)|_{x=b}=0, \qquad \frac{\partial^-}{\partial x}L_2(x,b,r;i)|_{x=b}=0.$$

This proves the desired differentiability at $x=b$.

(iii) For the time-integrated discounted penalty concerned in Example 3, we have

$$u_1^b\left(b-h,\frac{h}{c_i};i\right) = -\int_0^{\frac{h}{c_i}}\mathrm{e}^{-\delta s}\nu(b-h+c_is)\mathrm{d}s = -\frac{1}{c_i}\mathrm{e}^{-\frac{\delta}{c_i}(h-b)}\int_{b-h}^{b}\mathrm{e}^{-\frac{\delta}{c_i}s}\nu(s)\mathrm{d}s,$$
$$u_1^b\left(b,\frac{h}{c_i};i\right) = -\int_0^{\frac{h}{c_i}}\mathrm{e}^{-\delta s}\nu(b+(c_i-d_i)s)\mathrm{d}s.$$

Hence, by Theorem 2.5 and noticing that $v_1^b(x,t,r;i)=\mathrm{e}^{-ru_1^b(x,t;i)}$, it follows that for $i\in\mathcal{E}$,

$$\frac{\partial^-}{\partial x}V_1(x,b;i)|_{x=b}=0, \qquad \frac{\partial^-}{\partial x}L_1(x,b,r;i)|_{x=b}=0.$$

Since

$$V_1(x,b;i)=V_1(b,b;i), \qquad L_1(x,b,r;i)=L_1(b,b,r;i), \quad \text{for } x\geq b,$$

we have

$$\frac{\partial^+}{\partial x}V_1(x,b;i)|_{x=b}=0, \qquad \frac{\partial^+}{\partial x}L_1(x,b,r;i)|_{x=b}=0.$$

Therefore, $V_1(x,b;i)$ $(i\in\mathcal{E})$ and $L_1(x,b,r;i)$ are differentiable with respect to x at the point $x=b$. □

3. Markov-modulated dual model

In this section, we consider a Markov-modulated dual model. The dual model has been studied by some authors before. Avanzi et al. [8] has considered the optimal dividends in the classical dual model. We continue to use R_t to denote the surplus of the dual model at time t. Let all the other notations be defined same as before unless stated otherwise. Define R_t by

$$R_t = x + \sum_{j=1}^{N(t)} U_j - \int_0^t c_{J_s} \mathrm{d}s.$$

Also let R_t^b represent the surplus of the corresponding modified process with a barrier or threshold dividend strategy with the level of the barrier or of the threshold being b. We consider the dual model with the threshold dividend strategy, the surplus process of which is described by

$$R_t^b = x + \sum_{j=1}^{N(t)} U_j - \int_0^t (c_{J_s} + d_{J_s} I\{R_{s-}^b \geq b\}) \mathrm{d}s.$$

Define for $l = 1, 2$ and $i \in \mathcal{E}$,

$$u_l^b(x, t; i) = \begin{cases} E\left[\overline{K}_{t-}^l | R_s^b = x - c_i s,\ J_s = i \text{ for } 0 \leq s < t\right] & \text{for } x \leq b, \\ E\left[\overline{K}_{t-}^l | R_s^b = x - (c_i + d_i) s I\left\{s < \dfrac{x - b}{c_i + d_i}\right\}\right. & \\ \left. \quad - c_i \left(\left(s - \dfrac{x-b}{c_i + d_i}\right) \vee 0\right),\ J_s = i \text{ for } 0 \leq s < t, \right] & \text{for } x > b, \end{cases}$$

$$v_l^b(x, t, r; i) = \begin{cases} E\left[\mathrm{e}^{-r\overline{K}_{t-}^l} | R_s^b = x - c_i s,\ J_s = i \text{ for } 0 \leq s < t\right] & \text{for } x \leq b, \\ E\left[\mathrm{e}^{-r\overline{K}_{t-}^l} | R_s^b = x - (c_i + d_i) s I\left\{s < \dfrac{x - b}{c_i + d_i}\right\}\right. & \\ \left. \quad - c_i \left(\left(s - \dfrac{x-b}{c_i + d_i}\right) \vee 0\right),\ J_s = i \text{ for } 0 \leq s < t, \right] & \text{for } x > b. \end{cases}$$

Since $\{(R_t^b, J_t)\}$ is still a Markov process, the approach used to deal with the modified Markov-modulated model applies to the dual model, too. Proceeding analogously, we can derive the following theorem.

Theorem 3.1. (i) *For $l = 1$ or 2, under assumption* I *for $\phi(x, t; i) = u_l^b(x, t; i)$ ($\phi(x, t; i) = v_l^b(x, t, r; i)$), for all $i \in \mathcal{E}$ the functions $V_l(x, b; i)$ ($L_l(x, b, r; i)$) are continuous with respect to x for $x \in \mathbb{R} - \{0\}$.*

(ii) *Under B1 for $\phi(x, t; i) = u_1^b(x, t; i)$ ($\phi(x, t; i) = v_1^b(x, t, r; i)$) and $\varphi(x; i) = \rho(x; i)$ ($\varphi(x; i) = \alpha(x, r; i)$), for all $i \in \mathcal{E}$, $V_1(x, b; i)$ ($L_1(x, b, r; i)$) are continuously differentiable up to nth order with respect to x for $x \in \mathbb{R} - \{0, b\}$. Under* II *for $\phi(x, t; i) = u_2^b(x, t; i)$ ($\phi(x, t; i) = v_2^b(x, t, r; i)$), all functions $V_2(x, b; i)$ ($L_2(x, b, r; i)$) are continuously differentiable up to nth order with respect to x for $x \in \mathbb{R} - \{0, b\}$. If for $l = 1$, B1 holds for some $n \geq 1$, and for $l = 2$,* II *holds for some $n \geq 1$ and $\phi(x, t; i) = u_2^b(x, t; i)$ ($\varphi(x; i) = \alpha(x, r; i)$), then for all $i \in \mathcal{E}$, the right and left derivatives of $V_l(x, b; i)$ ($L_l(x, b, r; i)$) with respect to x exist for $x \in \mathbb{R} - \{0\}$ and satisfy*

$$\frac{\partial^+}{\partial x} V_l(x, b; i)|_{x=b} = \lim_{h \downarrow 0} \frac{u_l^b(b + h, \frac{h}{c_i + d_i}; i) - u_l^b(b, \frac{h}{c_i + d_i}; i)}{h} + \frac{c_i}{c_i + d_i} \frac{\partial^-}{\partial x} V_l(x, b; i)|_{x=b}. \tag{3.34}$$

$$\left(\frac{\partial^+}{\partial x} L_l(x,b,r;i)|_{x=b} = \lim_{h\downarrow 0} \frac{v_l^b(b,\frac{h}{c_i},r;i) - v_l^b(b-h,\frac{h}{c_i},r;i)}{h} L_l(b,b,r;i) \right.$$
$$\left. + \frac{c_i}{c_i+d_i}\frac{\partial^-}{\partial x} L_l(x,b,r;i)|_{x=b}.\right)$$

Proof. We only give the proof for the results concerning $V_l(x,b;i)$. All the asserted results concerning $L_l(x,b,r;i)$ can be proved similarly.

Since for the dual model, (2.13) and (2.14) also hold. For $l = 1$, when $x \le 0$, setting $\eta = \sigma_1 \wedge S_1$ in (2.13) and distinguishing cases $\eta = \sigma_1$ and $\eta = S_1$ yield for $i \in \mathcal{E}$,

$$\begin{aligned} &V_1(x,b;i) \\ &= \int_0^\infty q_i e^{-(q_i+\lambda_i)t}\left(u_1^b(x,t;i) + g(t)\sum_{j\neq i}\frac{q_{ij}}{q_i} V_1(x-c_it,b;j)\right)dt \\ &\quad + \int_0^\infty \lambda_i e^{-(q_i+\lambda_i)s} ds \left(u_1^b(x,s;i) + g(s)\int_0^\infty V_1(x-c_is+y,b;i)dF_i(y)\right) \\ &\quad \text{for } x \le 0. \end{aligned}$$

Since $P_{(x,i)}(T_b = 0) \equiv 1$ for $x \le 0$ and that we have defined $R^b_{0-} \equiv -1$, $w(x,y;i) \equiv 0$ for $x < 0$ and $w(0,0;i) = 0$, we have for $i \in \mathcal{E}$,

$$V_2(x,b;i) \equiv 0 \quad \text{for } x \le 0.$$

For $0 < x \le b$, by setting $\eta = \sigma_1 \wedge S_1 \wedge \frac{x}{c_i}$ in (2.13) and (2.14), and noticing that $R^b_{T_b-} = R^b_{T_b} = 0$ a.s., we have for $i \in \mathcal{E}$,

$$\begin{aligned} &V_l(x,b;i) \\ &= \int_0^{\frac{x}{c_i}} q_i e^{-(q_i+\lambda_i)t}\left(u_l^b(x,t;i) + g(t)\sum_{j\neq i}\frac{q_{ij}}{q_i} V_l(x-c_it,b;j)\right)dt \\ &\quad + \int_0^{\frac{x}{c_i}} \lambda_i e^{-(q_i+\lambda_i)s} ds \left(u_l^b(x,s;i) + g(s)\int_0^\infty V_l(x-c_is+y,b;i)dF_i(y)\right) \\ &\quad + e^{-(q_i+\lambda_i)\frac{x}{c_i}}\left(u_l^b\left(x,\frac{x}{c_i};i\right) + g\left(\frac{x}{c_i}\right) V_1(0,b;i)I\{l=1\}\right. \\ &\quad \left. + g\left(\frac{x}{c_i}\right) w(0,0;i)I\{l=2\}\right) \quad \text{for } 0 < x \le b. \end{aligned}$$

When $x > b$, by setting $\eta = \sigma_1 \wedge S_1 \wedge \frac{x-b}{c_i+d_i}$ and noticing that $P_{(x,i)}(T^b \le \eta) \equiv 0$, we have for $l = 1, 2$, $i \in \mathcal{E}$,

$$\begin{aligned} &V_l(x,b;i) \\ &= \int_0^\infty q_i e^{-(q_i+\lambda_i)t}\left(u_l^b(x,t;i) + g(t)\sum_{j\neq i}\frac{q_{ij}}{q_i} V_l(x-(c_i+d_i)t,b;j)\right)dt \\ &\quad + \int_0^\infty \lambda_i e^{-(q_i+\lambda_i)s} ds \left(u_l^b(x,s;i) + g(s)\int_0^\infty V_l(x-(c_i+d_i)s+y,b;i)dF_i(y)\right) \\ &\quad + e^{-(q_i+\lambda_i)\frac{b-x}{c_i+d_i}}\left(u_l^b\left(x,\frac{b-x}{c_i+d_i};i\right) + g\left(\frac{b-x}{c_i+d_i}\right) V_l(b,b;i)\right) \quad \text{for } x > b. \end{aligned}$$

Then the desired assertions on the continuity with respect to x for $x \in \mathbb{R} - \{0\}$ and continuous differentiability with respect to x for $x \in \mathbb{R} - \{0, b\}$ follow from a similar argument as in the previous section. One difference is that in the dual model, we do not require restrictions on the functions $w(x, y; i)$ $(i \in \mathcal{E})$, because only $w(0, 0; i)$ $(i \in \mathcal{E})$ are involved in the expressions.

Set $\eta = \sigma_1 \wedge S_1 \wedge \frac{h}{c_i+d_i}$ in (2.13) and (2.14) for $x = b + h$ and $x = b$ respectively, and distinguish three cases $\eta = \sigma_1$, $\eta = S_1$ and $\eta = \frac{h}{c_i+d_i}$. Substituting the newly obtained expressions in quantities $\frac{V_l(b+h,b;i)-V_l(b,b;i)}{h}$ and then letting $h \downarrow 0$ yield the desired equations (3.34). □

Remark 3.1. For the Markov-modulated risk model under the barrier strategy, the same results in Theorem 3.1 concerning continuity with respect to x, and differentiability with respect to x for $x \in \mathbb{R} - \{0, b\}$ hold.

Remark 3.2. For the Markov-modulated model, the right continuity at $x = 0$, and the continuity and the equations involving the right and/or left derivatives at $x = b$ can serve as boundary conditions. For the Markov-modulated model with the barrier strategy, we need to solve a system of equations with derivatives at $x = b$ taking some specified value(s). For the model with a threshold strategy, the functions satisfy different systems of equations depending on whether $0 < x < b$ or $x > b$. So we need to solve two systems of equations. As a result, the number of boundary conditions required doubles. The extra boundary conditions are usually obtained by noticing that ruin of the dual model occurs immediately if the initial reserve is 0.

4. Conclusion

In this paper, we have introduced several classes of very general ruin functions, and have shown that many frequently considered ruin functions in the actuarial science literature are special cases of the functions studied in this paper. We have investigated the continuity and differentiability of the ruin functions.

For the purpose of illustrating the idea and avoiding tedious mathematics, we consider a Markov-modulated compound Poisson insurance risk process with dividend payment. The idea and methods in this paper can be used to obtain the same results for other models, such as the models with interest rate, the diffusion perturbed models, and the renewal type models.

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