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Hydrostatics and dynamical large deviations of boundary driven gradient symmetric exclusion processes

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Abstract

We prove the hydrostatics of boundary driven gradient exclusion processes, Fick's law and we present a simple proof of the dynamical large deviations principle which holds in any dimension.

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1. Introduction

Statical and dynamical large deviations principles of boundary driven interacting particle systems have attracted attention recently as a first step in the understanding of nonequilibrium thermodynamics (cf. [5,7,8] and references therein).

This article has two purposes. First, inspired by the dynamical approach to stationary large deviations, introduced by Bertini et al. in the context of boundary driven interacting particle systems [3], we present a proof of the hydrostatics based on the hydrodynamic behavior of the system and on the fact that the stationary profile is a global attractor of the hydrodynamic equation.

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More precisely, if $\bar{\rho}$ represents the stationary density profile and π^N the empirical measure, to prove that π^N converges to $\bar{\rho}$ under the stationary state μ_{ss}^N, we first prove the hydrodynamic limit stated as follows. If we start from an initial configuration which has a density profile γ, on the diffusive scale the empirical measure converges to an absolutely continuous measure, $\pi(t, \mathrm{d}u) = \rho(t, u)\mathrm{d}u$, whose density ρ is the solution of the parabolic equation

$$\begin{cases} \partial_t \rho = \nabla \cdot D(\rho)\nabla\rho, \\ \rho(0, \cdot) = \gamma(\cdot), \\ \rho(t, \cdot) = b(\cdot) \quad \text{on } \Gamma, \end{cases} \tag{1.1}$$

where D is the diffusivity of the system, ∇ the gradient, b the boundary condition imposed by the stochastic dynamics and Γ the boundary of the macroscopic domain Ω in which the particles evolve. Since for all initial profile $0 \le \gamma \le 1$, the solution ρ_t is bounded above (resp. below) by the solution with initial condition equal to 1 (resp. 0), and since these solutions converge, as $t \uparrow \infty$, to the stationary profile $\bar{\rho}$, the hydrostatics follows from the hydrodynamics and the weak compactness of the space of measures.

The second contribution of this article is an important simplification of the proof of the dynamical large deviations principle of the empirical measure around the hydrodynamic limit. The original proof [15,9,13] relies on the convexity of the rate functional, a very special property only fulfilled by very few interacting particle systems as the symmetric simple exclusion process. The extension to general processes [19,20,6] is relatively technical. The main difficulty appears in the proof of the lower bound where one needs to show that any trajectory λ_t, $0 \le t \le T$, with finite rate function, $I_T(\lambda) < \infty$, can be approximated by a sequence of smooth trajectories $\{\lambda^n : n \ge 1\}$ such that

$$\lambda^n \longrightarrow \lambda \quad \text{and} \quad I_T(\lambda^n) \longrightarrow I_T(\lambda). \tag{1.2}$$

This property is proved by approximating in several steps a general trajectory λ by a sequence of profiles, smoother at each step, the main ingredient being the regularizing effect of the hydrodynamic equation. This part of the proof is quite elaborate and relies on properties of the Green kernel associated with the second-order differential operator.

We propose here a simpler proof. It is well known that a path λ with finite rate function may be obtained from the hydrodynamical path through an external field. More precisely, if $I_T(\lambda) < \infty$, there exists H such that

$$I_T(\lambda) = \frac{1}{4} \int_0^T \mathrm{d}t \int \sigma(\lambda_t)[\nabla H_t]^2 \mathrm{d}x, \tag{1.3}$$

where σ is the mobility of the system and H is related to λ by the equation

$$\begin{cases} \partial_t \lambda - \nabla \cdot D(\lambda)\nabla\lambda = -\nabla \cdot [\sigma(\lambda)\nabla H_t] \\ H(t, \cdot) = 0 \quad \text{at the boundary.} \end{cases} \tag{1.4}$$

This is an elliptic equation for the unknown function H for each $t \ge 0$. Note that the left hand side of the first equation is the hydrodynamical equation. Instead of approximating λ by a sequence of smooth trajectories, we show that on approximating H by a sequence of smooth functions, the corresponding smooth solutions of (1.4) converge in the sense (1.2) to λ.

This approach, closer to the original one in the convex case, simplifies considerably the proof of the large deviations of the empirical measure from the hydrodynamic limit. Indeed, the previous approach (cf. Lemma 5.6 and the proof of Theorem 5.1 in [6], as well as the proof of Theorem 6.4 in [20]) requires the selection of an appropriate space and time mollifier to smooth the trajectory through a convolution and relies on sharp estimate of the mollifiers. This is not easy in the case of systems in contact with reservoirs where the boundary prevents the use of Gaussian mollifiers, which have to be replaced by the resolvents of Brownian motions killed at the boundary. This step becomes much simpler here on approximating, as we said above, the function H appearing in (1.3) with a sequence of smooth functions.

Finally, the approach presented here to prove the hydrostatics of the empirical measure has an important advantage with respect to the original one [11,12,14]. In these articles, to derive the hydrostatics one proves first that all weak limits of the empirical measure are concentrated on weak solutions of a coupled differential equation on the product space $\Omega \times \Omega$, and then that there exists at most one weak solution of the coupled equation (cf. Theorem 1 of [11]). Since uniqueness of weak solutions of the coupled equation has been proven only in dimension 1, all results hold only in this dimension.

In this article we present a method which holds in all dimensions provided that the hydrodynamic equation has a unique fixed point and that the solutions are monotone in the sense that $\rho(t, \cdot) \le \lambda(t, \cdot)$ a.s. for all $t \ge s$ if this inequality holds for $t = s$ a.s. In particular, the approach proposed here extends the hydrostatics to higher dimension for several different kinds of dynamics.

2. Notation and results

Fix a positive integer $d \ge 2$. Denote by Ω the open set $(-1, 1) \times \mathbb{T}^{d-1}$, where $\mathbb{T}^k$ is the k-dimensional torus $[0, 1)^k$, and by Γ the boundary of Ω: $\Gamma = \{(u_1, \dots, u_d) \in [-1, 1] \times \mathbb{T}^{d-1} : u_1 = \pm 1\}$.

For an open subset Λ of $\mathbb{R} \times \mathbb{T}^{d-1}$, $\mathcal{C}^m(\Lambda)$, $1 \le m \le +\infty$, stands for the space of real functions that are m times continuously differentiable, defined on Λ. Let $\mathcal{C}^m_0(\Lambda)$ (resp. $\mathcal{C}^m_c(\Lambda)$), $1 \le m \le +\infty$, be the subset of functions in $\mathcal{C}^m(\Lambda)$ which vanish at the boundary of Λ (resp. with compact support in Λ).

Fix a positive function $b : \Gamma \to \mathbb{R}_+$. Assume that there exists a neighborhood V of Ω and a smooth function $\beta : V \to (0, 1)$ in $\mathcal{C}^2(V)$ such that β is bounded below by a strictly positive constant, bounded above by a constant smaller than 1 and such that the restriction of β to Γ is equal to b.

For an integer $N \ge 1$, denote by $\mathbb{T}^{d-1}_N = \{0, \dots, N-1\}^{d-1}$, the discrete $(d-1)$-dimensional torus of length N. Let $\Omega_N = \{-N+1, \dots, N-1\} \times \mathbb{T}^{d-1}_N$ be the cylinder in $\mathbb{Z}^d$ of length $2N-1$ and basis $\mathbb{T}^{d-1}_N$ and let $\Gamma_N = \{(x_1, \dots, x_d) \in \mathbb{Z} \times \mathbb{T}^{d-1}_N \mid x_1 = \pm(N-1)\}$ be the boundary of Ω_N. The elements of Ω_N are denoted by letters x, y and the elements of Ω by the letters u, v.

We consider boundary driven symmetric exclusion processes on Ω_N. A configuration is described as an element η in $X_N = \{0, 1\}^{\Omega_N}$, where $\eta(x) = 1$ (resp. $\eta(x) = 0$) if site x is occupied (resp. vacant) for the configuration η. At the boundary, particles are created and removed in order for the local density to agree with the given density profile b.

The infinitesimal generator of this Markov process can be decomposed into two pieces:

$$\mathcal{L}_N = \mathcal{L}_{N,0} + \mathcal{L}_{N,b}, \tag{2.1}$$

where $\mathcal{L}_{N,0}$ corresponds to the bulk dynamics and $\mathcal{L}_{N,b}$ to the boundary dynamics. The action of the generator $\mathcal{L}_{N,0}$ on functions $f : X_N \to \mathbb{R}$ is given by

$$\big(\mathcal{L}_{N,0} f\big)(\eta) = \sum_{i=1}^{d} \sum_{x} r_{x,x+e_i}(\eta) \left[f(\eta^{x,x+e_i}) - f(\eta) \right],$$

where $(e_1, \dots, e_d)$ stands for the canonical basis of $\mathbb{R}^d$ and where the second sum is performed over all $x \in \mathbb{Z}^d$ such that $x, x+e_i \in \Omega_N$. For $x, y \in \Omega_N$, $\eta^{x,y}$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(y)$:

$$\eta^{x,y}(z) = \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{if } z \neq x, y. \end{cases}$$

For $a > -1/2$, the rate functions $r_{x,x+e_i}(\eta)$ are given by

$$r_{x,x+e_i}(\eta) = 1 + a\big\{\eta(x-e_i) + \eta(x+2e_i)\big\}$$

if $x - e_i$ and $x + 2e_i$ belong to Ω_N. At the boundary, the rates are defined as follows. Let $\check{x} = (x_2, \dots, x_d) \in \mathbb{T}_N^{d-1}$. Then,

$$r_{(-N+1,\check{x}),(-N+2,\check{x})}(\eta) = 1 + a\big\{\eta(-N+3, \check{x}) + b(-1, \check{x}/N)\big\},$$
$$r_{(N-2,\check{x}),(N-1,\check{x})}(\eta) = 1 + a\big\{\eta(N-3, \check{x}) + b(1, \check{x}/N)\big\}.$$

The non-conservative boundary dynamics can be described as follows. For any function $f : X_N \to \mathbb{R}$,

$$\big(\mathcal{L}_{N,b} f\big)(\eta) = \sum_{x \in \Gamma_N} C^b(x, \eta)\big[f(\eta^x) - f(\eta)\big],$$

where η^x is the configuration obtained from η by flipping the occupation variable at site x:

$$\eta^x(z) = \begin{cases} \eta(z) & \text{if } z \neq x \\ 1 - \eta(x) & \text{if } z = x \end{cases}$$

and the rates $C^b(x, \cdot)$ are chosen in order for the Bernoulli measure with density $b(\cdot)$ to be reversible for the flipping dynamics restricted to this site:

$$\begin{aligned} C^b\big((-N+1, \check{x}), \eta\big) \;&=\; \eta(-N+1, \check{x})\big[1 - b(-1, \check{x}/N)\big] \\ &\quad + \big[1 - \eta(-N+1, \check{x})\big]b(-1, \check{x}/N), \\ C^b\big((N-1, \check{x}), \eta\big) &= \eta(N-1, \check{x})\big[1 - b(1, \check{x}/N)\big] + \big[1 - \eta(N-1, \check{x})\big]b(1, \check{x}/N), \end{aligned}$$

where $\check{x} = (x_2, \dots, x_d) \in \mathbb{T}_N^{d-1}$, as above.

Denote by $\{\eta_t : t \geq 0\}$ the Markov process associated with the generator $\mathcal{L}_N$ *speeded up* by N^2. For a smooth function $\rho : \Omega \to (0, 1)$, let $\nu^N_{\rho(\cdot)}$ be the Bernoulli product measure on X_N with marginals given by

$$\nu^N_{\rho(\cdot)}(\eta(x) = 1) = \rho(x/N).$$

It is easy to see that the Bernoulli product measure associated with any constant function is invariant for the process with generator $\mathcal{L}_{N,0}$. Moreover, if $b(\cdot) \equiv b$ for some constant b then the Bernoulli product measure associated with the constant density b is reversible for the full dynamics $\mathcal{L}_N$.

2.1. *Hydrostatics*

Denote by μ^N_{ss} the unique stationary state of the irreducible Markov process $\{\eta_t : t \geq 0\}$. We examine in Section 3 the asymptotic behavior of the empirical measure under the stationary state μ^N_{ss}.

Let $\mathcal{M} = \mathcal{M}(\Omega)$ be the space of positive measures on Ω with total mass bounded by 2 endowed with the weak topology. For each configuration η, denote by $\pi^N = \pi^N(\eta)$ the positive measure obtained by assigning mass N^{-d} to each particle of η:

$$\pi^N = N^{-d} \sum_{x \in \Omega_N} \eta(x) \delta_{x/N},$$

where δ_u is the Dirac measure concentrated on u.

To define rigorously the quasi-linear elliptic problem that the empirical measure is expected to solve, we need to introduce some Sobolev spaces. Let $L^2(\Omega)$ be the Hilbert space of functions $G : \Omega \to \mathbb{C}$ such that $\int_\Omega |G(u)|^2 \mathrm{d}u < \infty$ equipped with the inner product

$$\langle G, J \rangle_2 = \int_\Omega G(u) \bar{J}(u) \mathrm{d}u,$$

where, for $z \in \mathbb{C}$, $\bar{z}$ is the complex conjugate of z and $|z|^2 = z\bar{z}$. The norm of $L^2(\Omega)$ is denoted by $\| \cdot \|_2$.

Let $H^1(\Omega)$ be the Sobolev space of functions G with generalized derivatives $\partial_{u_1} G, \ldots, \partial_{u_d} G$ in $L^2(\Omega)$. $H^1(\Omega)$ endowed with the scalar product $\langle \cdot, \cdot \rangle_{1,2}$, defined by

$$\langle G, J \rangle_{1,2} = \langle G, J \rangle_2 + \sum_{j=1}^{d} \langle \partial_{u_j} G,\ \partial_{u_j} J \rangle_2,$$

is a Hilbert space. The corresponding norm is denoted by $\| \cdot \|_{1,2}$.

Let $\varphi : [0, 1] \to \mathbb{R}_+$ be given by $\varphi(r) = r(1 + ar)$, let $\nabla \rho$ represent the gradient of some function ρ in $H^1(\Omega)$: $\nabla \rho = (\partial_{u_1} \rho, \ldots, \partial_{u_d} \rho)$, and let $\| \cdot \|$ be the Euclidean norm: $\|(v_1, \ldots, v_d)\|^2 = \sum_{1 \le i \le d} v_i^2$. A function $\rho : \Omega \to [0, 1]$ is said to be a weak solution of the elliptic boundary value problem

$$\begin{cases} \Delta \varphi(\rho) = 0 & \text{on } \Omega, \\ \rho = b & \text{on } \Gamma, \end{cases} \tag{2.2}$$

if:

(**S1**) ρ belongs to $H^1(\Omega)$:

$$\int_\Omega \|\nabla \rho(u)\|^2 \mathrm{d}u < \infty.$$

(**S2**) For every function $G \in \mathcal{C}_0^2(\Omega)$,

$$\int_\Omega \big(\Delta G\big)(u) \varphi\big(\rho(u)\big) \mathrm{d}u = \int_\Gamma \varphi(b(u)) \mathbf{n}_1(u) (\partial_{u_1} G)(u) \mathrm{d}S,$$

where $\mathbf{n} = (\mathbf{n}_1, \ldots, \mathbf{n}_d)$ stands for the outward unit vector normal to the boundary surface Γ and $\mathrm{d}S$ for an element of surface on Γ.

We prove in Section 7 existence and uniqueness of weak solutions of (2.2). The first main result of this article establishes a law of large numbers for the empirical measure under μ_{ss}^N. Let $\overline{\Omega} = [-1, 1] \times \mathbb{T}^{d-1}$ and denote by E^μ the expectation with respect to a probability measure μ. Moreover, for a measure m in $\mathcal{M}$ and a continuous function $G : \Omega \to \mathbb{R}$, denote by $\langle m, G \rangle$ the integral of G with respect to m:

$$\langle m, G \rangle = \int_\Omega G(u) m(\mathrm{d}u).$$

Theorem 2.1. *For any continuous function $G : \overline{\Omega} \to \mathbb{R}$,*

$$\lim_{N\to\infty} E^{\mu_{ss}^N}\left[\left|\langle \pi^N, G\rangle - \int_{\overline{\Omega}} G(u)\bar{\rho}(u)\mathrm{d}u\right|\right] = 0,$$

where $\bar{\rho}(u)$ is the unique weak solution of (2.2).

Denote by Γ_-, Γ_+ the left and right boundaries of Ω:

$$\Gamma_{\pm} = \{(u_1, \ldots, u_d) \in \Omega | u_1 = \pm 1\}$$

and denote by $W_{x,x+e_i}$, $x, x+e_i \in \Omega_N$, the instantaneous current over the bond $(x, x+e_i)$. This is the rate at which a particle jumps from x to $x+e_i$ minus the rate at which a particle jumps from $x+e_i$ to x. A simple computation shows that

$$W_{x,x+e_i} = \{h_{i,x}(\eta) - h_{i,x+e_i}(\eta)\} + \{g_{i,x}(\eta) - g_{i,x+2e_i}(\eta)\} \tag{2.3}$$

provided $x - e_i$ and $x + 2e_i$ belongs to Ω_N. Here, $h_{i,x}(\eta) = \eta(x) - a\eta(x+e_i)\eta(x-e_i)$ and $g_{i,x}(\eta) = a\eta(x-e_i)\eta(x)$.

Theorem 2.2 (*Fick's Law*)**.** *Fix $-1 < u < 1$. Then,*

$$\lim_{N\to\infty} E^{\mu_{ss}^N}\left[\frac{2N}{N^{d-1}} \sum_{y\in\mathbb{T}_N^{d-1}} W_{([uN],y),([uN]+1,y)}\right]$$
$$= \int_{\Gamma_-} \varphi(b(v))\mathrm{S}(\mathrm{d}v) - \int_{\Gamma_+} \varphi(b(v))\mathrm{S}(\mathrm{d}v).$$

Remark 2.3. We could have considered different bulk dynamics. The important feature used here to avoid serious technical problems is that the process is gradient, which means that the currents can be written as a sum of differences of a local function and its translation, as in (2.3).

The gradient assumption is restrictive, with consequences on the hydrodynamic equations. The jump rates of the known gradient dynamics are of the form

$$\begin{aligned} r_{x,x+e_i}(\eta) &= 1 + a_1(i)[\eta(x-e_i) + \eta(x+2e_i)] + a_2(i)\Big\{\eta(x-2e_i)\eta(x-e_i) \\ &\quad + \eta(x-e_i)\eta(x+2e_i) + \eta(x+2e_i)\eta(x+3e_i)\Big\} + \cdots \\ &\quad + a_m(i)\Big\{\eta(x-me_i)\cdots\eta(x-e_i) + \cdots + \eta(x+2e_i)\cdots\eta(x+[m+1]e_i)\Big\}, \end{aligned}$$

where the constants $a_1(i), \ldots, a_m(i)$ are chosen such that $r_{x,x+e_i}(\eta) > 0$ for all configurations η. The associated hydrodynamic equation is given by

$$\partial_t \rho = \sum_{i=1}^{d} \partial_{u_i}[D_i(\rho)\partial_{u_i}\rho], \tag{2.4}$$

where $D_i(\rho) = 1 + \sum_{1\le k\le m} a_k(i)(k+1)\rho^k$. In particular, the gradient condition restricts the hydrodynamic equations to parabolic equations of type (1.1) with diagonal matrices D whose entries are strictly positive polynomials. All results stated in this article hold in this context with exactly the same proofs.

2.2. Dynamical large deviations

Fix $T > 0$. Let $\mathcal{M}^0$ be the subset of $\mathcal{M}$ of all absolutely continuous measures with respect to the Lebesgue measure with positive density bounded by 1:

$$\mathcal{M}^0 = \big\{\pi \in \mathcal{M} : \pi(\mathrm{d}u) = \rho(u)\mathrm{d}u \text{ and } 0 \le \rho(u) \le 1 \text{ a.e.}\big\},$$

and let $D([0,T],\mathcal{M})$ be the set of right continuous with left limit trajectories $\pi : [0,T] \to \mathcal{M}$, endowed with the Skorohod topology. $\mathcal{M}^0$ is a closed subset of $\mathcal{M}$ and $D([0,T],\mathcal{M}^0)$ is a closed subset of $D([0,T],\mathcal{M})$.

Let $\Omega_T = (0,T) \times \Omega$ and $\overline{\Omega_T} = [0,T] \times \overline{\Omega}$. For $1 \le m, n \le +\infty$, denote by $\mathcal{C}^{m,n}(\overline{\Omega_T})$ the space of functions $G = G_t(u) : \overline{\Omega_T} \to \mathbb{R}$ with m continuous derivatives in time and n continuous derivatives in space. We also denote by $\mathcal{C}_0^{m,n}(\overline{\Omega_T})$ (resp. $\mathcal{C}_c^\infty(\Omega_T)$) the set of functions in $\mathcal{C}^{m,n}(\overline{\Omega_T})$ (resp. $\mathcal{C}^{\infty,\infty}(\overline{\Omega_T})$) which vanish at $[0,T] \times \Gamma$ (resp. with compact support in Ω_T).

Let the energy $\mathcal{Q} : D([0,T],\mathcal{M}^0) \to [0,\infty]$ be given by

$$\mathcal{Q}(\pi) = \sum_{i=1}^{d} \sup_{G \in \mathcal{C}_c^\infty(\Omega_T)} \left\{ 2\int_0^T \mathrm{d}t \langle \rho_t, \partial_{u_i} G_t \rangle - \int_0^T \mathrm{d}t \int_\Omega G(t,u)^2 \mathrm{d}u \right\}.$$

For each $G \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})$ and each measurable function $\gamma : \overline{\Omega} \to [0,1]$, let $\hat{J}_G = \hat{J}_{G,\gamma,T} : D([0,T],\mathcal{M}^0) \to \mathbb{R}$ be the functional given by

$$\begin{aligned}\hat{J}_G(\pi) = {} & \langle \pi_T, G_T \rangle - \langle \gamma, G_0 \rangle - \int_0^T \langle \pi_t, \partial_t G_t \rangle \mathrm{d}t \\ & - \int_0^T \langle \varphi(\rho_t), \Delta G_t \rangle \mathrm{d}t + \int_0^T \mathrm{d}t \int_{\Gamma^+} \varphi(b) \partial_{u_1} G \mathrm{d}S \\ & - \int_0^T \mathrm{d}t \int_{\Gamma^-} \varphi(b) \partial_{u_1} G \mathrm{d}S - \int_0^T \langle \sigma(\rho_t), \|\nabla G_t\|^2 \rangle \mathrm{d}t,\end{aligned}$$

where $\sigma(r) = r(1-r)(1+2ar)$ is the mobility and $\pi_t(\mathrm{d}u) = \rho_t(u)\mathrm{d}u$. Define $J_G = J_{G,\gamma,T} : D([0,T],\mathcal{M}) \to \mathbb{R}$ by

$$J_G(\pi) = \begin{cases} \hat{J}_G(\pi) & \text{if } \pi \in D([0,T],\mathcal{M}^0), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.5}$$

We define the rate functional $I_T(\cdot|\gamma) : D([0,T],\mathcal{M}) \to [0,+\infty]$ as

$$I_T(\pi|\gamma) = \begin{cases} \sup\limits_{G \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})} \big\{ J_G(\pi) \big\} & \text{if } \mathcal{Q}(\pi) < \infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.6}$$

Theorem 2.4. *Fix $T > 0$ and a measurable function $\rho_0 : \Omega \to [0,1]$. Consider a sequence η^N of configurations in X_N associated with ρ_0 in the sense that*

$$\lim_{N\to\infty} \langle \pi^N(\eta^N), G \rangle = \int_\Omega G(u)\rho_0(u)\mathrm{d}u$$

for every continuous function $G : \overline{\Omega} \to \mathbb{R}$. Then, the measure $Q_{\eta^N} = \mathbb{P}_{\eta^N}(\pi^N)^{-1}$ on $D([0,T]$, $\mathcal{M})$ satisfies a large deviation principle with speed N^d and rate function $I_T(\cdot|\rho_0)$. Namely, for *each closed set $\mathcal{C} \subset D([0,T],\mathcal{M})$,*

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q_{\eta^N}(\mathcal{C}) \le - \inf_{\pi\in\mathcal{C}} I_T(\pi|\rho_0)$$

and for each open set $\mathcal{O} \subset D([0,T],\mathcal{M})$,

$$\underline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q_{\eta^N}(\mathcal{O}) \ge - \inf_{\pi\in\mathcal{O}} I_T(\pi|\rho_0).$$

Moreover, the rate function $I_T(\cdot|\rho_0)$ is lower semicontinuous and has compact level sets.

3. Hydrodynamics, hydrostatics and Fick's law

We prove in this section Theorem 2.1. The idea is to couple three copies of the process, the first one starting from the configuration with all sites empty, the second one starting from the stationary state and the third one from the configuration with all sites occupied. The hydrodynamic limit states that the empirical measures of the first and third copies converge to the solution of the initial boundary value problem (3.2) with initial condition equal to 0 and 1. Denote these solutions by ρ_t^0, ρ_t^1, respectively. In turn, the empirical measure of the second copy converges to the solution of the same boundary value problem, denoted by ρ_t, with an unknown initial condition. Since all solutions are bounded below by ρ^0 and bounded above by ρ^1, and since ρ^j converges to a profile $\bar\rho$ as $t \uparrow \infty$, ρ_t also converges to this profile. However, since the second copy starts from the stationary state, the distribution of its empirical measure is independent of time. Hence, as ρ_t converges to $\bar\rho$, $\rho_0 = \bar\rho$. As we shall see in the proof, this argument does not require attractiveness of the underlying interacting particle system. This approach has been followed in [18] to prove hydrostatics for interacting particle systems with Kac interaction and random external field.

We first describe the hydrodynamic behavior. Consider a *symmetric* diffusion matrix $D(\rho) = \{D_{i,j}(\rho) : 1 \le i, j \le d\}$ such that:

(a) $D_{i,j} : [0,1] \to \mathbb{R}$ is a Lipschitz continuous function, $1 \le i, j \le d$. There exists $M > 0$ such that $|D_{i,j}(\rho) - D_{i,j}(\lambda)| \le M|\rho - \lambda|$ for all $\rho, \lambda \in [0,1]$.
(b) The matrix D is strictly elliptic. There exists $a > 0$ such that $\lambda \cdot D(\rho)\lambda \ge a|\lambda|^2$ for all $\lambda \in \mathbb{R}^d$, $0 \le \rho \le 1$.

Observe that there exists $\hat a < \infty$ such that $\lambda \cdot D(\rho)\lambda \le \hat a|\lambda|^2$ for all $\lambda \in \mathbb{R}^d$, $0 \le \rho \le 1$, because the entries are continuous.

Denote by $\chi(\rho)$ the static compressibility and by $\sigma(\rho)$ the mobility. We shall assume throughout this article that the static compressibility $\chi(\rho)$ is the scalar function $\rho(1-\rho)$ of exclusion processes. By the Einstein relation and since the compressibility is a scalar, $\sigma(\rho) = \chi(\rho)D(\rho)$. Denote by $d_{i,j}$ the primitive of $D_{i,j}$ such that $d_{i,j}(0) = 0$, so $d'_{i,j} = D_{i,j}$.

For a Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ and $T > 0$ we denote by $L^2([0,T],\mathbb{B})$ the Banach space of measurable functions $U : [0,T] \to \mathbb{B}$ for which

$$\|U\|^2_{L^2([0,T],\mathbb{B})} = \int_0^T \|U_t\|^2_{\mathbb{B}} \mathrm{d}t < \infty$$

holds.

Fix $T > 0$ and a profile $\rho_0 : \overline{\Omega} \to [0, 1]$. A measurable function $\rho : [0, T] \times \overline{\Omega} \to [0, 1]$ is said to be a weak solution of the initial boundary value problem

$$\begin{cases} \partial_t \rho = \nabla \cdot D(\rho) \nabla \rho, \\ \rho(0, \cdot) = \rho_0(\cdot), \\ \rho(t, \cdot)|_\Gamma = b(\cdot) \quad \text{for } 0 \le t \le T, \end{cases} \tag{3.1}$$

in the layer $[0, T] \times \Omega$ if:

(**H1**) ρ belongs to $L^2\big([0, T], H^1(\Omega)\big)$:

$$\int_0^T \mathrm{d}s \Big(\int_\Omega \|\nabla \rho(s, u)\|^2 \mathrm{d}u \Big) < \infty.$$

(**H2**) For every function $G = G_t(u)$ in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$,

$$\begin{aligned} &\int_\Omega \mathrm{d}u \big\{ G_T(u)\rho(T, u) - G_0(u)\rho_0(u) \big\} - \int_0^T \mathrm{d}s \int_\Omega \mathrm{d}u (\partial_s G_s)(u)\rho(s, u) \\ &\quad = \sum_{i,j=1}^d \int_0^T \mathrm{d}s \int_\Omega \mathrm{d}u (\partial^2_{u_i, u_j} G_s)(u) d_{i,j}\big(\rho(s, u)\big) \\ &\qquad - \sum_{i=1}^d \int_0^T \mathrm{d}s \int_\Gamma d_{i,1}(b(u)) \mathbf{n}_1(u) (\partial_{u_i} G_s(u)) \mathrm{d}S. \end{aligned}$$

The hydrodynamic equation of the boundary driven gradient symmetric exclusion process on Ω is the parabolic equation (3.1) with $D(\rho) = \varphi'(\rho)\mathbf{I}$, where $\mathbf{I}$ is the identity:

$$\begin{cases} \partial_t \rho = \Delta \varphi\big(\rho\big), \\ \rho(0, \cdot) = \rho_0(\cdot), \\ \rho(t, \cdot)|_\Gamma = b(\cdot) \quad \text{for } 0 \le t \le T. \end{cases} \tag{3.2}$$

We prove in Section 7 the existence of weak solutions of (3.2) and the uniqueness of weak solutions of (3.1).

For a measure μ on X_N, denote by $\mathbb{P}_\mu = \mathbb{P}_\mu^N$ the probability measure on the path space $D(\mathbb{R}_+, X_N)$ corresponding to the Markov process $\{\eta_t : t \ge 0\}$ with generator $N^2 \mathcal{L}_N$ starting from μ, and by $\mathbb{E}_\mu$ the expectation with respect to $\mathbb{P}_\mu$. Recall the definition of the empirical measure π^N and let $\pi_t^N = \pi^N(\eta_t)$:

$$\pi_t^N = N^{-d} \sum_{x \in \Omega_N} \eta_t(x) \delta_{x/N}.$$

Theorem 3.1. *Fix a profile $\rho_0 : \Omega \to (0, 1)$. Let μ^N be a sequence of measures on X_N associated with ρ_0 in the sense that*

$$\lim_{N \to \infty} \mu^N \left\{ \left| \langle \pi^N, G \rangle - \int_\Omega G(u) \rho_0(u) \mathrm{d}u \right| > \delta \right\} = 0, \tag{3.3}$$

for every continuous function $G : \Omega \to \mathbb{R}$ and every $\delta > 0$. Then, for every $t > 0$,

$$\lim_{N \to \infty} \mathbb{P}_\mu^N \left\{ \left| \langle \pi_t^N, G \rangle - \int_\Omega G(u) \rho(t, u) \mathrm{d}u \right| > \delta \right\} = 0,$$

where $\rho(t, u)$ is the unique weak solution of (3.2).

The proof of this result can be found in [12]. Denote by $\mathbf{Q}^N_{\rm ss}$ the probability measure on the Skorohod space $D([0,T],\mathcal{M})$ induced by the stationary measure $\mu^N_{\rm ss}$ and the process $\{\pi^N(\eta_t) : 0 \le t \le T\}$. Note that, in contrast with the case for the usual set-up of hydrodynamics, we do not know that the empirical measure at time 0 converges. We cannot prove, in particular, that the sequence $\mathbf{Q}^N_{\rm ss}$ converges, but only that this sequence is tight and that all limit points are concentrated on a weak solution of the hydrodynamic equation for some unknown initial profile.

We first show that the sequence of probability measures $\{\mathbf{Q}^N_{\rm ss} : N \ge 1\}$ is weakly relatively compact:

Proposition 3.2. *The sequence $\{\mathbf{Q}^N_{ss},\ N \ge 1\}$ is tight and all its limit points $\mathbf{Q}^*_{ss}$ are concentrated on absolutely continuous paths $\pi(t,\mathrm{d}u) = \rho(t,u)\mathrm{d}u$ whose density ρ is positive and bounded above by* 1*:*

$$\mathbf{Q}^*_{ss}\Big\{\pi : \pi(t,\mathrm{d}u) = \rho(t,u)\mathrm{d}u, \textit{for } 0 \le t \le T\Big\} = 1,$$

$$\mathbf{Q}^*_{ss}\Big\{\pi : 0 \le \rho(t,u) \le 1, \textit{for } (t,u) \in \overline{\Omega_T}\Big\} = 1.$$

The proof of this statement is similar to that of Proposition 3.2 in [16] and is thus omitted. Actually, the proof is even simpler because the model considered here is gradient.

The next two propositions show that all limit points of the sequence $\{\mathbf{Q}^N_{\rm ss} : N \ge 1\}$ are concentrated on absolutely continuous measures $\pi(t,\mathrm{d}u) = \rho(t,u)\mathrm{d}u$ whose densities ρ are weak solutions of (3.2) in the layer $[0,T]\times\Omega$. Denote by $\mathcal{A}_T \subset D\big([0,T],\mathcal{M}^0\big)$ the set of trajectories $\{\rho(t,u)\mathrm{d}u : 0 \le t \le T\}$ whose density ρ satisfies condition (**H2**) for some initial profile ρ_0.

Proposition 3.3. *All limit points $\mathbf{Q}^*_{ss}$ of the sequence $\{\mathbf{Q}^N_{ss},\ N > 1\}$ are concentrated on paths $\pi(t,\mathrm{d}u) = \rho(t,u)\mathrm{d}u$ in $\mathcal{A}_T$:*

$$\mathbf{Q}^*_{ss}\{\mathcal{A}_T\} = 1.$$

The proof of this proposition is similar to that of Proposition 3.3 in [16]. The next result states that every limit point $\mathbf{Q}^*_{\rm ss}$ of the sequence $\{\mathbf{Q}^N_{\rm ss}, N > 1\}$ is concentrated on paths whose density ρ belongs to $L^2([0,T], H^1(\Omega))$:

Proposition 3.4. *Let $\mathbf{Q}^*_{ss}$ be a limit point of the sequence $\{\mathbf{Q}^N_{ss}, N > 1\}$. Then,*

$$E_{\mathbf{Q}^*_{ss}}\left[\int_0^T \mathrm{d}s \left(\int_\Omega \|\nabla\rho(s,u)\|^2 \mathrm{d}u\right)\right] < \infty.$$

The proof of this proposition is similar to that of Lemma A.1.1 in [14]. We are now ready to prove the first main result of this article.

Proof of Theorem 2.1. Fix a continuous function $G : \overline{\Omega} \to \mathbb{R}$. We claim that

$$\lim_{N\to\infty} E^{\mu^N_{\rm ss}}\Big[\Big|\langle\pi,G\rangle - \langle\bar\rho(u)\mathrm{d}u, G\rangle\Big|\Big] = 0.$$

Note that the expectations are bounded. Consider a subsequence N_k along which the left hand side converges. It is enough to prove that the limit vanishes. Fix $T > 0$. Since $\mu^N_{\rm ss}$ is stationary, by definition of $\mathbf{Q}^{N_k}_{\rm ss}$,

$$E^{\mu^{N_k}_{\rm ss}}\Big[\Big|\langle\pi,G\rangle - \langle\bar\rho(u)\mathrm{d}u, G\rangle\Big|\Big] = \mathbf{Q}^{N_k}_{\rm ss}\Big[\Big|\langle\pi_T,G\rangle - \langle\bar\rho(u)\mathrm{d}u, G\rangle\Big|\Big].$$

Let $\mathbf{Q}^*_{ss}$ stand for a limit point of $\{\mathbf{Q}^{N_k}_{ss} : k \ge 1\}$. Since the expression inside the expectation is bounded, by Proposition 3.3,

$$\lim_{k\to\infty} \mathbf{Q}^{N_k}_{ss}\Big[\Big|\langle \pi_T, G\rangle - \langle \bar\rho(u)\mathrm{d}u, G\rangle\Big|\Big] = \mathbf{Q}^*_{ss}\Big[\Big|\langle \pi_T, G\rangle - \langle \bar\rho(u)\mathrm{d}u, G\rangle\Big| \mathbf{1}\{\mathcal{A}_T\}\Big]$$
$$\le \|G\|_\infty \mathbf{Q}^*_{ss}\Big[\|\rho(T,\cdot) - \bar\rho(\cdot)\|_1 \mathbf{1}\{\mathcal{A}_T\}\Big],$$

where $\|\cdot\|_1$ stands for the $L^1(\Omega)$ norm. Denote by $\rho^0(\cdot,\cdot)$ (resp. $\rho^1(\cdot,\cdot)$) the weak solution of the boundary value problem (3.2) with initial condition $\rho(0,\cdot) \equiv 0$ (resp. $\rho(0,\cdot) \equiv 1$). By Lemma 7.4, each profile ρ in $\mathcal{A}_T$, including the stationary profile $\bar\rho$, is bounded below by ρ^0 and above by ρ^1. Therefore

$$\overline{\lim_{k\to\infty}}\, E^{\mu^{N_k}_{ss}}\Big[\Big|\langle \pi, G\rangle - \langle \bar\rho(u)\mathrm{d}u, G\rangle\Big|\Big] \le \|G\|_\infty \big\|\rho^0(T,\cdot) - \rho^1(T,\cdot)\big\|_1.$$

Note that the left hand side does not depend on T. To conclude the proof it remains to let $T \uparrow \infty$ and to apply Lemma 7.6. □

Fick's law, announced in Theorem 2.2, follows from the hydrostatics and elementary computations presented in the Proof of Theorem 2.2 in [14]. The arguments here are even simpler and more explicit since the process is gradient.

4. The rate function $I_T(\cdot|\gamma)$

We examine in this section the rate function $I_T(\cdot|\gamma)$. The main result, presented in Theorem 4.6 below, states that $I_T(\cdot|\gamma)$ has compact level sets. The proof relies on two ingredients. The first one, stated in Lemma 4.2, is an estimate of the energy and of the H_{-1} norm of the time derivative of a trajectory in terms of the rate function. The second one, stated in Lemma 4.5, establishes that sequences of trajectories with rate function uniformly bounded which converges weakly in L^2 in fact converge strongly.

For each $G \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})$ and each measurable function $\gamma : \overline{\Omega} \to [0,1]$, let $\hat J_G = \hat J_{G,\gamma,T} : D([0,T], \mathcal{M}^0) \to \mathbb{R}$ be the functional given by

$$\begin{aligned}\hat J_G(\pi) &= \langle \pi_T, G_T\rangle - \langle \gamma, G_0\rangle - \int_0^T \langle \pi_t, \partial_t G_t\rangle \mathrm{d}t \\ &\quad - \sum_{i,j=1}^d \int_0^T \langle d_{i,j}(\rho_t), \partial^2_{u_i,u_j} G_t\rangle \mathrm{d}t + \sum_{i=1}^d \int_0^T \mathrm{d}t \int_{\Gamma^+} d_{i,1}(b)\partial_{u_i} G \mathrm{d}S \\ &\quad - \int_0^T \mathrm{d}t \int_{\Gamma^-} d_{i,1}(b)\partial_{u_i} G\mathrm{d}S - \int_0^T \langle \nabla G_t \cdot \sigma(\rho_t)\nabla G_t\rangle \mathrm{d}t,\end{aligned}$$

where $\pi_t(\mathrm{d}u) = \rho_t(u)\mathrm{d}u$. Define the functionals $J_G = J_{G,\gamma,T} : D([0,T],\mathcal{M}) \to \mathbb{R}$ and $I_T(\cdot|\gamma) : D([0,T],\mathcal{M}) \to [0,+\infty]$ by Eqs. (2.5) and (2.6).

Some Sobolev spaces play an important role in this section. Recall that we denote by $\mathcal{C}_c^\infty(\Omega)$ the set of infinitely differentiable functions $G : \Omega \to \mathbb{R}$, with compact support in Ω. Recall from Section 2.1 the definition of the Sobolev space $H^1(\Omega)$ and of the norm $\|\cdot\|_{1,2}$. Denote by $H_0^1(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$. Since Ω is bounded, by Poincaré's inequality, there exists a finite constant C_1 such that for all $G \in H^1_0(\Omega)$,

$$\|G\|_2^2 \le C_1 \|\partial_{u_1} G\|_2^2 \le C_1 \sum_{j=1}^{d} \langle \partial_{u_j} G, \partial_{u_j} G\rangle_2 .$$

This implies that, in $H^1_0(\Omega)$,

$$\|G\|_{1,2,0} = \left\{ \sum_{j=1}^{d} \langle \partial_{u_j} G, \partial_{u_j} G\rangle_2 \right\}^{1/2}$$

is a norm equivalent to the norm $\|\cdot\|_{1,2}$. Moreover, $H^1_0(\Omega)$ is a Hilbert space with inner product given by

$$\langle G, J\rangle_{1,2,0} = \sum_{j=1}^{d} \langle \partial_{u_j} G, \partial_{u_j} J\rangle_2 .$$

To assign boundary values along the boundary Γ of Ω to any function G in $H^1(\Omega)$, recall, from the trace theorem [22, Theorem 21.A.(e)], that there exists a continuous linear operator $B : H^1(\Omega) \to L^2(\Gamma)$, called the trace, such that $BG = G|_\Gamma$ if $G \in H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Moreover, the space $H^1_0(\Omega)$ is the space of functions G in $H^1(\Omega)$ with zero trace [22, Appendix (48b)]:

$$H^1_0(\Omega) = \left\{ G \in H^1(\Omega) : \; BG = 0 \right\}.$$

Since $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$ [22, Corollary 21.15.(a)], for functions F, G in $H^1(\Omega)$, the product FG has generalized derivatives $\partial_{u_i}(FG) = F\partial_{u_i} G + G\partial_{u_i} F$ in $L^1(\Omega)$ and

$$\begin{aligned} &\int_\Omega F(u)\partial_{u_1} G(u)\mathrm{d}u + \int_\Omega G(u)\partial_{u_1} F(u)\mathrm{d}u \\ &\quad = \int_{\Gamma_+} BF(u)BG(u)\mathrm{d}u - \int_{\Gamma_-} BF(u)BG(u)\mathrm{d}u. \end{aligned} \tag{4.1}$$

Moreover, if $G \in H^1(\Omega)$, $f \in \mathcal{C}^1(\mathbb{R})$ is such that f' is bounded, then $f \circ G$ belongs to $H^1(\Omega)$ with generalized derivatives $\partial_{u_i}(f \circ G) = (f' \circ G)\partial_{u_i} G$ and trace $B(f \circ G) = f \circ (BG)$.

Denote by $H^{-1}(\Omega)$ the dual of $H^1_0(\Omega)$. $H^{-1}(\Omega)$ is a Banach space with norm $\|\cdot\|_{-1}$ given by

$$\|v\|_{-1}^2 = \sup_{G \in \mathcal{C}^\infty_c(\Omega)} \left\{ 2\langle v, G\rangle_{-1,1} - \int_\Omega \|\nabla G(u)\|^2 \mathrm{d}u \right\},$$

where $\langle v, G\rangle_{-1,1}$ stands for the values of the linear form v at G.

For each $G \in \mathcal{C}^\infty_c(\Omega_T)$ and each integer $1 \le i \le d$, let $\mathcal{Q}_i^G : D([0,T], \mathcal{M}^0) \to \mathbb{R}$ be the functional given by

$$\mathcal{Q}_i^G(\pi) = 2\int_0^T \mathrm{d}t \langle \rho_t, \partial_{u_i} G_t\rangle - \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}u G(t,u)^2 ,$$

where $\pi(t, \mathrm{d}u) = \rho(t,u)\mathrm{d}u$, and recall, from Section 2.2, that the energy $\mathcal{Q}(\pi)$ was defined as

$$\mathcal{Q}(\pi) = \sum_{i=1}^{d} \mathcal{Q}_i(\pi) \quad \text{with} \;\; \mathcal{Q}_i(\pi) = \sup_{G \in \mathcal{C}^\infty_c(\Omega_T)} \mathcal{Q}_i^G(\pi).$$

The functional $\mathcal{Q}_i^G$ is convex and continuous in the Skorohod topology. Therefore $\mathcal{Q}_i$ and $\mathcal{Q}$ are convex and lower semicontinuous. Furthermore, it is well known that a measure $\pi(t, \mathrm{d}u) = \rho(t, u)\mathrm{d}u$ in $D([0, T], \mathcal{M})$ has finite energy, $\mathcal{Q}(\pi) < \infty$, if and only if its density ρ belongs to $L^2([0, T], H^1(\Omega))$, in which case

$$\hat{\mathcal{Q}}(\pi) := \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}u \|\nabla \rho_t(u)\|^2 < \infty$$

and $\mathcal{Q}(\pi) = \hat{\mathcal{Q}}(\pi)$.

Let $D_\gamma = D_{\gamma,b}$ be the subset of $C([0, T], \mathcal{M}^0)$ consisting of all paths $\pi(t, \mathrm{d}u) = \rho(t, u)\mathrm{d}u$ with initial profile $\rho(0, \cdot) = \gamma(\cdot)$, finite energy $\mathcal{Q}(\pi)$ (in which case ρ_t belongs to $H^1(\Omega)$ for almost all $0 \le t \le T$ and so $B(\rho_t)$ is well defined for those t) and such that $B(\rho_t) = b$ for almost all t in $[0, T]$.

Lemma 4.1. *Let π be a trajectory in $D([0, T], \mathcal{M})$ such that $I_T(\pi|\gamma) < \infty$. Then π belongs to D_γ.*

Proof. Fix a path π in $D([0, T], \mathcal{M})$ with finite rate function, $I_T(\pi|\gamma) < \infty$. By definition of I_T, π belongs to $D([0, T], \mathcal{M}^0)$. Denote its density by ρ: $\pi(t, \mathrm{d}u) = \rho(t, u)\mathrm{d}u$.

The proof that $\rho(0, \cdot) = \gamma(\cdot)$ is similar to that of Lemma 3.5 in [4]. To prove that $B(\rho_t) = b$ for almost all $t \in [0, T]$, since the function $d_{1,1} : [0, 1] \to \mathbb{R}_+$ belongs to $\mathcal{C}^1([0, 1])$ and is strictly increasing, and since $B(d_{1,1} \circ \rho_t) = d_{1,1}(B\rho_t)$ (for those t such that ρ_t belongs to $H^1(\Omega)$), it is enough to show that $B(d_{1,1} \circ \rho_t) = d_{1,1}(b)$ for almost all $t \in [0, T]$. To this end, we just need to show that, for any function $H_\pm \in \mathcal{C}^{1,2}([0, T] \times \Gamma_\pm)$,

$$\int_0^T \mathrm{d}t \int_{\Gamma_\pm} \mathrm{d}u \big\{ B(d_{1,1}(\rho_t))(u) - d_{1,1}(b(u)) \big\} H_\pm(t, u) = 0. \tag{4.2}$$

Fix a function $H \in \mathcal{C}^{1,2}([0, T] \times \Gamma_-)$. For each $0 < \theta < 1$, let $h_\theta : [-1, 1] \to \mathbb{R}$ be the function given by

$$h_\theta(r) = \begin{cases} r + 1 & \text{if } -1 \le r \le -1 + \theta, \\ \dfrac{-\theta r}{1 - \theta} & \text{if } -1 + \theta \le r \le 0, \\ 0 & \text{if } 0 \le r \le 1, \end{cases}$$

and define the function $G_\theta : \overline{\Omega_T} \to \mathbb{R}$ as $G(t, (u_1, \check{u})) = h_\theta(u_1) H(t, (-1, \check{u}))$ for all $\check{u} \in \mathbb{T}^{d-1}$. Of course, G_θ can be approximated by functions in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$. From the integration by parts formula (4.1) and the definition of J_{G_θ}, since π has finite energy,

$$\lim_{\theta \to 0} J_{G_\theta}(\pi) = \int_0^T \mathrm{d}t \int_{\Gamma_-} \mathrm{d}u \big\{ B(d_{1,1}(\rho_t))(u) - d_{1,1}(b(u)) \big\} H(t, u),$$

which proves (4.2) because $I_T(\pi|\gamma) < \infty$.

We deal now with the continuity of π. We claim that there exists a positive constant C_0 such that, for any $g \in \mathcal{C}_c^\infty(\Omega)$, and any $0 \le s < r < T$,

$$|\langle \pi_r, g\rangle - \langle \pi_s, g\rangle| \le C_0 (r - s)^{1/2} \Big\{ I_T(\pi|\gamma) + \|g\|_{1,2,0}^2 + (r - s)^{1/2} \sum_{i,j=1}^d \|\partial^2_{u_i,u_j} g\|_1 \Big\}. \tag{4.3}$$

Indeed, for each $\delta > 0$, let $\psi^\delta : [0, T] \to \mathbb{R}$ be the function given by

$$(r-s)^{1/2}\psi^\delta(t) = \begin{cases} 0 & \text{if } 0 \le t \le s \text{ or } r+\delta \le t \le T, \\ \dfrac{t-s}{\delta} & \text{if } s \le t \le s+\delta, \\ 1 & \text{if } s+\delta \le t \le r, \\ 1 - \dfrac{t-r}{\delta} & \text{if } r \le t \le r+\delta, \end{cases}$$

and let $G^\delta(t, u) = \psi^\delta(t)g(u)$. Of course, G^δ can be approximated by functions in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$ and then

$$(r-s)^{1/2} \lim_{\delta\to 0} J_{G^\delta}(\pi) = \langle \pi_r, g\rangle - \langle \pi_s, g\rangle - \sum_{i,j=1}^{d} \int_s^r \mathrm{d}t \langle d_{i,j}(\rho_t), \partial^2_{u_i,u_j} g\rangle$$
$$- \frac{1}{(r-s)^{1/2}} \int_s^r \mathrm{d}t \langle \nabla g \cdot \sigma(\rho_t)\nabla g\rangle.$$

To conclude the proof, it remains to observe that the left hand side is absolutely bounded by $(r-s)^{1/2} I_T(\pi|\gamma)$, and to note that $d_{i,j}$, $\sigma_{i,j}$ are absolutely bounded on [0, 1]. □

Denote by $L^2([0, T], H_0^1(\Omega))^*$ the dual of $L^2([0, T], H_0^1(\Omega))$. By Proposition 23.7 in [22], $L^2([0, T], H_0^1(\Omega))^*$ corresponds to $L^2([0, T], H^{-1}(\Omega))$ and for v in $L^2([0, T], H_0^1(\Omega))^*$, G in $L^2([0, T], H_0^1(\Omega))$,

$$\langle\langle v, G\rangle\rangle_{-1,1} = \int_0^T \langle v_t, G_t\rangle_{-1,1} \mathrm{d}t, \tag{4.4}$$

where the left hand side stands for the value of the linear functional v at G. Moreover, if we denote by $|||v|||_{-1}$ the norm of v,

$$|||v|||_{-1}^2 = \int_0^T \|v_t\|_{-1}^2 \mathrm{d}t.$$

Fix a path $\pi(t, \mathrm{d}u) = \rho(t, u)\mathrm{d}u$ in D_γ and suppose that

$$\sup_{H\in\mathcal{C}_c^\infty(\Omega_T)} \left\{ 2\int_0^T \mathrm{d}t \langle \rho_t, \partial_t H_t\rangle - \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}u \|\nabla H_t\|^2 \right\} < \infty. \tag{4.5}$$

In this case $\partial_t\rho : C_c^\infty(\Omega_T) \to \mathbb{R}$ defined by

$$\partial_t\rho(H) = -\int_0^T \langle \rho_t, \partial_t H_t\rangle \mathrm{d}t$$

can be extended to a bounded linear operator $\partial_t\rho : L^2([0, T], H_0^1(\Omega)) \to \mathbb{R}$. It belongs therefore to $L^2([0, T], H_0^1(\Omega))^* = L^2([0, T], H^{-1}(\Omega))$. In particular, there exists $v = \{v_t : 0 \le t \le T\}$ in $L^2([0, T], H^{-1}(\Omega))$, which we denote by $v_t = \partial_t\rho_t$, such that for any H in $L^2([0, T], H_0^1(\Omega))$,

$$\langle\langle \partial_t\rho, H\rangle\rangle_{-1,1} = \int_0^T \langle \partial_t\rho_t, H_t\rangle_{-1,1} \mathrm{d}t.$$

Moreover,

$$\begin{aligned}|||\partial_t \rho|||_{-1}^2 &= \int_0^T \|\partial_t \rho_t\|_{-1}^2 \mathrm{d}t \\ &= \sup_{H \in \mathcal{C}_c^\infty(\Omega_T)} \left\{ 2 \int_0^T \mathrm{d}t \langle \rho_t, \partial_t H_t \rangle - \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}u \|\nabla H_t\|^2 \right\}.\end{aligned}$$

Let W be the set of paths $\pi(t, \mathrm{d}u) = \rho(t, u)\mathrm{d}u$ in D_γ such that (4.5) holds, i.e., such that $\partial_t \rho$ belongs to $L^2\big([0, T], H^{-1}(\Omega)\big)$. For G in $L^2\big([0, T], H_0^1(\Omega)\big)$, let $\mathbb{J}_G : W \to \mathbb{R}$ be the functional given by

$$\begin{aligned}\mathbb{J}_G(\pi) = \langle\!\langle \partial_t \rho, G \rangle\!\rangle_{-1,1} &+ \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}u \nabla G_t(u) \cdot D(\rho_t(u)) \nabla \rho_t(u) \\ &- \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}u \nabla G_t(u) \cdot \sigma(\rho_t(u)) \nabla G_t(u).\end{aligned}$$

Note that $\mathbb{J}_G(\pi) = J_G(\pi)$ for every G in $\mathcal{C}_c^\infty(\Omega_T)$. Moreover, since $\mathbb{J}_\cdot(\pi)$ is continuous in $L^2\big([0, T], H_0^1(\Omega)\big)$ and since $\mathcal{C}_c^\infty(\Omega_T)$ is dense in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$ and in $L^2([0, T], H_0^1(\Omega))$, for every π in W,

$$I_T(\pi|\gamma) = \sup_{G \in \mathcal{C}_c^\infty(\Omega_T)} \mathbb{J}_G(\pi) = \sup_{G \in L^2([0,T], H_0^1)} \mathbb{J}_G(\pi). \tag{4.6}$$

Lemma 4.2. *There exists a constant $C_0 > 0$ such that if the density ρ of some path $\pi(t, \mathrm{d}u) = \rho(t, u)\mathrm{d}u$ in $D([0, T], \mathcal{M}^0)$ has a generalized gradient, $\nabla \rho$, then*

$$\int_0^T \mathrm{d}t \, \|\partial_t \rho_t\|_{-1}^2 \le C_0 \{I_T(\pi|\gamma) + \mathcal{Q}(\pi)\}, \tag{4.7}$$

$$\int_0^T \mathrm{d}t \int_\Omega \mathrm{d}u \frac{\|\nabla \rho_t(u)\|^2}{\chi(\rho_t(u))} \le C_0 \{I_T(\pi|\gamma) + 1\}. \tag{4.8}$$

Proof. Fix a path $\pi(t, \mathrm{d}u) = \rho(t, u)\mathrm{d}u$ in $D([0, T], \mathcal{M}^0)$. By Lemma 4.1, we may assume that $\pi(t, \mathrm{d}u)$ belongs to D_γ. In view of the discussion presented before the lemma, we need to show that the left hand side of (4.5) is bounded by the right hand side of (4.7). Such an estimate follows from the definition of the rate function $I_T(\cdot|\gamma)$ and from the elementary inequality $2ab \le Aa^2 + A^{-1}b^2$.

We turn now to the proof of (4.8). We may of course assume that $I_T(\pi|\gamma) < \infty$, in which case $\mathcal{Q}(\pi) < \infty$. Fix a function β as at the beginning of Section 2. For each $\delta > 0$, let $h^\delta : [0, 1]^2 \to \mathbb{R}$ be the function given by

$$h^\delta(x, y) = (x + \delta) \log\left(\frac{x + \delta}{y + \delta}\right) + (1 - x + \delta) \log\left(\frac{1 - x + \delta}{1 - y + \delta}\right).$$

By (4.7), $\partial_t \rho$ belongs to $L^2([0, T], H^{-1}(\Omega))$. We claim that

$$\int_0^T \mathrm{d}t \langle \partial_t \rho_t, \partial_x h^\delta(\rho_t, \beta) \rangle_{-1,1} = \int_\Omega h^\delta(\rho_T(u), \beta(u))\mathrm{d}u - \int_\Omega h^\delta(\rho_0(u), \beta(u))\mathrm{d}u, \tag{4.9}$$

where $\partial_x h^\delta$ stands for the derivative of h^δ with respect to the first coordinate.

Indeed, by Lemma 4.1 and (4.7), $\rho - \beta$ belongs to $L^2\left([0,T], H_0^1(\Omega)\right)$ and $\partial_t(\rho-\beta) = \partial_t \rho$ belongs to $L^2([0,T], H^{-1}(\Omega))$. Then, there exists a sequence $\{\widetilde{G}^n : n \geq 1\}$ of smooth functions $\widetilde{G}^n : \overline{\Omega_T} \to \mathbb{R}$ such that $\widetilde{G}^n_t$ belongs to $\mathcal{C}_c^\infty(\Omega)$ for every t in $[0,T]$, $\widetilde{G}^n$ converges to $\rho-\beta$ in $L^2([0,T], H_0^1(\Omega))$ and $\partial_t \widetilde{G}^n$ converges to $\partial_t(\rho-\beta)$ in $L^2([0,T], H^{-1}(\Omega))$ (cf. [22], Proposition 23.23(ii)). For each positive integer n, let $G^n = \widetilde{G}^n + \beta$ and for each $\delta > 0$, fix a smooth function $\tilde{h}^\delta : \mathbb{R}^2 \to \mathbb{R}$ with compact support and such that its restriction to $[0,1]^2$ is h^δ. It is clear that

$$\int_0^T \mathrm{d}t \langle \partial_t G^n_t, \partial_x \tilde{h}^\delta(G^n_t, \beta)\rangle = \int_\Omega \tilde{h}^\delta(G^n_T(u), \beta(u))\mathrm{d}u - \int_\Omega \tilde{h}^\delta(G^n_0(u), \beta(u))\mathrm{d}u. \quad (4.10)$$

On the one hand, $\partial_x h^\delta : [0,1]^2 \to \mathbb{R}$ is given by

$$\partial_x h^\delta(x,y) = \log\left(\frac{x+\delta}{1-x+\delta}\right) - \log\left(\frac{y+\delta}{1-y+\delta}\right).$$

Hence, $\partial_x h^\delta(\rho,\beta)$ and $\partial_x \tilde{h}^\delta(G^n,\beta)$ belongs to $L^2\left([0,T], H_0^1(\Omega)\right)$. Moreover, since $\partial_x \tilde{h}^\delta$ is smooth with compact support and G^n converges to ρ in $L^2([0,T], H^1(\Omega))$, $\partial_x \tilde{h}^\delta(G^n,\beta)$ converges to $\partial_x h^\delta(\rho,\beta)$ in $L^2([0,T], H_0^1(\Omega))$. From this fact and since $\partial_t G^n$ converges to $\partial_t \rho$ in $L^2([0,T], H^{-1}(\Omega))$, if we let $n \to \infty$, the left hand side in (4.10) converges to

$$\int_0^T \mathrm{d}t \langle \partial_t \rho_t, \partial_x h^\delta(\rho_t, \beta)\rangle_{-1,1}.$$

On the other hand, by Proposition 23.23(ii) in [22], G^n_0 (resp. G^n_T) converges to ρ_0 (resp. ρ_T) in $L^2(\Omega)$. Then, if we let $n \to \infty$, the right hand side in (4.10) goes to

$$\int_\Omega h^\delta(\rho_T(u), \beta(u))\mathrm{d}u - \int_\Omega h^\delta(\rho_0(u), \beta(u))\mathrm{d}u,$$

which proves claim (4.9).

Notice that, since β is bounded away from 0 and 1, there exists a positive constant $C = C(\beta)$ such that for δ small enough,

$$h^\delta(\rho(t,u), \beta(u)) \leq C \quad \text{for all } (t,u) \text{ in } \overline{\Omega_T}. \quad (4.11)$$

For each $\delta > 0$, let $H^\delta : \overline{\Omega_T} \to \mathbb{R}$ be the function given by

$$H^\delta(t,u) = \frac{\partial_x h^\delta(\rho(t,u), \beta(u))}{2(1+2\delta)}.$$

A simple computation shows that

$$\begin{aligned}\mathbb{J}_{H^\delta}(\pi) \geq{}& \int_0^T \mathrm{d}t \left\langle \partial_t \rho_t, H^\delta_t\right\rangle_{-1,1} + \frac{1}{8}\int_0^T \mathrm{d}t \int_\Omega \mathrm{d}u \frac{1}{\chi_\delta(\rho_t(u))} \nabla\rho_t(u) \cdot D(\rho_t(u))\nabla\rho_t(u) \\ & - 3\int_0^T \mathrm{d}t \int_\Omega \mathrm{d}u \frac{\chi_\delta(\rho_t(u))}{\chi_\delta(\beta(u))^2} \nabla\beta(u) \cdot D(\rho_t(u))\nabla\beta(u),\end{aligned}$$

where $\chi_\delta(r) = (r+\delta)(1-r+\delta)$. In view of the strict ellipticity of the diffusion matrix D, this last inequality together with (4.6), (4.9) and (4.11) shows that there exists a positive constant $C_0 = C_0(\beta)$ such that for δ small enough

$$C_0 \{I_T(\pi|\gamma) + 1\} \geq \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}u \frac{\|\nabla \rho(t,u)\|^2}{\chi_\delta(\rho(t,u))}.$$

We conclude the proof by letting $\delta \downarrow 0$ and by using Fatou's lemma. □

Corollary 4.3. *The density ρ of a path $\pi(t, \mathrm{d}u) = \rho(t,u)\mathrm{d}u$ in $D([0,T], \mathcal{M}^0)$ is the weak solution of Eq. (3.1) with initial profile γ if and only if the rate function $I_T(\pi|\gamma)$ vanishes. Moreover, in that case*

$$\int_0^T \mathrm{d}t \int_\Omega \mathrm{d}u \frac{\|\nabla \rho_t(u)\|^2}{\chi(\rho_t(u))} < \infty.$$

Proof. On the one hand, if the density ρ of a path $\pi(t, \mathrm{d}u) = \rho(t,u)\mathrm{d}u$ in $D([0,T], \mathcal{M}^0)$ is the weak solution of Eq. (3.1), by assumption (**H1**), the energy $Q(\pi)$ is finite. Moreover, since the initial condition is γ, in the formula of $\hat{J}_G(\pi)$, the linear part in G vanishes which proves that the rate functional $I_T(\pi|\gamma)$ vanishes. On the other hand, if the rate functional vanishes, the path ρ belongs to $L^2([0,T], H^1(\Omega))$ and the linear part in G of $J_G(\pi)$ has to vanish for all functions G. In particular, ρ is a weak solution of (3.1). Moreover, in that case, by the previous lemma, the bound claimed holds. □

For each $q > 0$, let E_q be the level set of $I_T(\pi|\gamma)$ defined by

$$E_q = \{\pi \in D([0,T], \mathcal{M}) : I_T(\pi|\gamma) \leq q\}.$$

By Lemma 4.1, E_q is a subset of $C([0,T], \mathcal{M}^0)$. Thus, from the previous lemma, it is easy to deduce the next result.

Corollary 4.4. *For every $q \geq 0$, there exists a finite constant $C(q)$ such that*

$$\sup_{\pi \in E_q} \left\{ \int_0^T \mathrm{d}t \, \|\partial_t \rho_t\|_{-1}^2 + \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}u \frac{\|\nabla \rho(t,u)\|^2}{\chi(\rho(t,u))} \right\} \leq C(q).$$

The next result together with the previous estimates provides the compactness needed in the proof of the lower semicontinuity of the rate function.

Lemma 4.5. *Let $\{\rho^n : n \geq 1\}$ be a sequence of functions in $L^2(\Omega_T)$ such that uniformly on n,*

$$\int_0^T \mathrm{d}t \, \left\|\rho_t^n\right\|_{1,2}^2 + \int_0^T \mathrm{d}t \, \left\|\partial_t \rho_t^n\right\|_{-1}^2 < C$$

for some positive constant C. Suppose that $\rho \in L^2(\Omega_T)$ and that $\rho^n \to \rho$ weakly in $L^2(\Omega_T)$. Then $\rho^n \to \rho$ strongly in $L^2(\Omega_T)$.

Proof. Since $H^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ with compact embedding $H^1(\Omega) \to L^2(\Omega)$, from Corollary 8.4, [21], the sequence $\{\rho^n\}$ is relatively compact in $L^2\big([0,T], L^2(\Omega)\big)$. Therefore the weak convergence implies the strong convergence in $L^2\big([0,T], L^2(\Omega)\big)$. □

Theorem 4.6. *The functional $I_T(\cdot|\gamma)$ is lower semicontinuous and has compact level sets.*

Proof. We have to show that, for all $q \geq 0$, E_q is compact in $D([0,T],\mathcal{M})$. Since $E_q \subset C([0,T],\mathcal{M}^0)$ and $C([0,T],\mathcal{M}^0)$ is a closed subset of $D([0,T],\mathcal{M})$, we just need to show that E_q is compact in $C([0,T],\mathcal{M}^0)$.

We will show first that E_q is closed in $C([0,T],\mathcal{M}^0)$. Fix $q \in \mathbb{R}$ and let $\{\pi^n : n \geq 1\}$ be a sequence in E_q converging to some π in $C([0,T],\mathcal{M}^0)$. Then, for all $G \in \mathcal{C}(\overline{\Omega_T})$,

$$\lim_{n\to\infty} \int_0^T \mathrm{d}t \langle \pi_t^n, G_t\rangle = \int_0^T \mathrm{d}t \langle \pi_t, G_t\rangle.$$

Notice that this means that $\pi^n \to \pi$ weakly in $L^2(\Omega_T)$, which together with Corollary 4.4 and Lemma 4.5 implies that $\pi^n \to \pi$ strongly in $L^2(\Omega_T)$. From this fact and the definition of J_G it is easy to see that, for all G in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$,

$$\lim_{n\to\infty} J_G(\pi_n) = J_G(\pi).$$

This limit, Corollary 4.4 and the lower semicontinuity of $\mathcal{Q}$ permit us to conclude that $\mathcal{Q}(\pi) \leq C(q)$ and that $I_T(\pi|\gamma) \leq q$.

We prove now that E_q is relatively compact. To this end, it is enough to prove that for every continuous function $G : \overline{\Omega} \to \mathbb{R}$,

$$\lim_{\delta\to 0} \sup_{\pi\in E_q} \sup_{\substack{0\leq s,r\leq T\\ |r-s|<\delta}} |\langle \pi_r, G\rangle - \langle \pi_s, G\rangle| = 0. \tag{4.12}$$

Since $E_q \subset C([0,T],\mathcal{M}^0)$, we may assume by approximations of G in $L^1(\Omega)$ that $G \in \mathcal{C}_c^\infty(\Omega)$. In which case, (4.12) follows from (4.3). $\square$

We conclude this section with an explicit formula for the rate function $I_T(\cdot|\gamma)$. For each $\pi(t,\mathrm{d}u) = \rho(t,u)\mathrm{d}u$ in $D([0,T],\mathcal{M}^0)$, denote by $H_0^1(\sigma(\rho))$ the Hilbert space induced by $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$ endowed with the inner product $\langle\cdot,\cdot\rangle_{\sigma(\rho)}$ defined by

$$\langle H, G\rangle_{\sigma(\rho)} = \int_0^T \mathrm{d}t \langle \nabla H_t \cdot \sigma(\rho_t)\nabla G_t\rangle.$$

Induced means that we first declare two functions F, G in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$ to be equivalent if $\langle F-G, F-G\rangle_{\sigma(\rho)} = 0$ and then we complete the quotient space with respect to the inner product $\langle\cdot,\cdot\rangle_{\sigma(\rho)}$. The norm of $H_0^1(\sigma(\rho))$ is denoted by $\|\cdot\|_{\sigma(\rho)}$.

Fix a path ρ in $D([0,T],\mathcal{M}^0)$ and a function H in $H_0^1(\sigma(\rho))$. A measurable function $\lambda : [0,T]\times\Omega \to [0,1]$ is said to be a weak solution of the nonlinear boundary value parabolic equation

$$\begin{cases} \partial_t\lambda = \nabla\cdot D(\lambda)\nabla\lambda - \nabla\cdot\sigma(\lambda)\nabla H, \\ \lambda(0,\cdot) = \gamma, \\ \lambda(t,\cdot)|_\Gamma = b \quad \text{for } 0 \leq t \leq T, \end{cases} \tag{4.13}$$

if it satisfies the following two conditions:

(**H1′**) λ belongs to $L^2\left([0,T], H^1(\Omega)\right)$:

$$\int_0^T \mathrm{d}s \left(\int_\Omega \|\nabla\lambda(s,u)\|^2 \mathrm{d}u\right) < \infty.$$

(**H2′**) For every function $G(t,u) = G_t(u)$ in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$,

$$\begin{aligned}
&\int_\Omega \mathrm{d}u \big\{G_T(u)\lambda(T,u) - G_0(u)\gamma(u)\big\} - \int_0^T \mathrm{d}s \int_\Omega \mathrm{d}u (\partial_s G_s)(u)\lambda(s,u) \\
&\quad = \sum_{i,j=1}^d \int_0^T \mathrm{d}s \int_\Omega \mathrm{d}u (\partial^2_{u_i,u_j} G_s)(u) d_{i,j}\big(\lambda(s,u)\big) \\
&\qquad - \sum_{i=1}^d \int_0^T \mathrm{d}s \int_\Gamma d_{i,1}(b(u))\mathbf{n}_1(u)(\partial_{u_i} G_s(u))\mathrm{d}S \\
&\qquad + \int_0^T \mathrm{d}s \int_\Omega \mathrm{d}u \nabla G_s(u)\cdot\sigma(\lambda(s,u))\nabla H_s(u).
\end{aligned}$$

Note that in this definition we assumed that the solutions take values in the bounded set $[0, 1]$.

In Section 7 we prove the uniqueness of weak solutions of Eq. (4.13) when H belongs to $L^2\big([0,T], H^1(\Omega)\big)$, i.e., provided

$$\int_0^T \mathrm{d}t \int_\Omega \mathrm{d}u \|\nabla H_t(u)\|^2 < \infty.$$

Lemma 4.7. *Assume that $\pi(t,\mathrm{d}u) = \rho(t,u)\mathrm{d}u$ in $D([0,T],\mathcal{M}^0)$ has finite rate function: $I_T(\pi|\gamma) < \infty$. Then, there exists a function H in $H_0^1(\sigma(\rho))$ such that ρ is a weak solution to (4.13). Moreover,*

$$I_T(\pi|\gamma) = \frac{1}{4}\|H\|^2_{\sigma(\rho)}. \tag{4.14}$$

The proof of this lemma is similar to that of Lemma 5.3 in [13] and is therefore omitted.

5. $I_T(\cdot|\gamma)$-density

The main result of this section, stated in Theorem 5.3, asserts that any trajectory λ_t, $0 \le t \le T$, with finite rate function, $I_T(\lambda|\gamma) < \infty$, can be approximated by a sequence of smooth trajectories $\{\lambda^n : n \ge 1\}$ such that

$$\lambda^n \longrightarrow \lambda \quad \text{and} \quad I_T(\lambda^n|\gamma) \longrightarrow I_T(\lambda|\gamma).$$

This is one of the main steps in the proof of the lower bound of the large deviations principle for the empirical measure. The proof rests mainly on the regularizing effects of the hydrodynamic equation and is one of the main contributions of this article, since it considerably simplifies the existing methods.

A subset A of $D([0,T],\mathcal{M})$ is said to be $I_T(\cdot|\gamma)$-dense if for every π in $D([0,T],\mathcal{M})$ such that $I_T(\pi|\gamma) < \infty$, there exists a sequence $\{\pi^n : n \ge 1\}$ in A such that π^n converges to π and $I_T(\pi^n|\gamma)$ converges to $I_T(\pi|\gamma)$.

Let Π_1 be the subset of $D([0,T],\mathcal{M}^0)$ consisting of paths $\pi(t,\mathrm{d}u) = \rho(t,u)\mathrm{d}u$ whose density ρ is a weak solution of the hydrodynamic equation (3.1) in the time interval $[0,\delta]$ for some $\delta > 0$.

Lemma 5.1. *The set Π_1 is $I_T(\cdot|\gamma)$-dense.*

Proof. Fix $\pi(t, \mathrm{d}u) = \rho(t, u)\mathrm{d}u$ in $D([0, T], \mathcal{M})$ such that $I_T(\pi|\gamma) < \infty$. By Lemma 4.1, π belongs to $C([0, T], \mathcal{M}^0)$. For each $\delta > 0$, let ρ^δ be the path defined as

$$\rho^\delta(t, u) = \begin{cases} \lambda(t, u) & \text{if } 0 \le t \le \delta, \\ \lambda(2\delta - t, u) & \text{if } \delta \le t \le 2\delta, \\ \rho(t - 2\delta, u) & \text{if } 2\delta \le t \le T, \end{cases}$$

where λ is the weak solution of the hydrodynamic equation (3.1) with initial condition γ. It is clear that $\pi^\delta(t, \mathrm{d}u) = \rho^\delta(t, u)\mathrm{d}u$ belongs to D_γ, because so do π and λ, and that $\mathcal{Q}(\pi^\delta) \le \mathcal{Q}(\pi) + 2\mathcal{Q}(\lambda) < \infty$. Moreover, π^δ converges to π as $\delta \downarrow 0$ because π belongs to $\mathcal{C}([0, T], \mathcal{M})$. By the lower semicontinuity of $I_T(\cdot|\gamma)$, $I_T(\pi|\gamma) \le \underline{\lim}_{\delta\to 0} I_T(\pi^\delta|\gamma)$. Then, in order to prove the lemma, it is enough to prove that $I_T(\pi|\gamma) \ge \overline{\lim}_{\delta\to 0} I_T(\pi^\delta|\gamma)$. To this end, decompose the rate function $I_T(\pi^\delta|\gamma)$ as the sum of the contributions on each time interval $[0, \delta]$, $[\delta, 2\delta]$ and $[2\delta, T]$. The first contribution vanishes because π^δ solves the hydrodynamic equation in this interval. On the time interval $[\delta, 2\delta]$, $\partial_t \rho_t^\delta = -\partial_t \lambda_{2\delta - t} = -\nabla \cdot D(\lambda_{2\delta-t})\nabla\lambda_{2\delta-t} = -\nabla \cdot D(\rho_t^\delta)\nabla\rho_t^\delta$. In particular, the second contribution is equal to

$$\sup_{G\in\mathcal{C}_0^{1,2}(\overline{\Omega_T})} \left\{ 2\int_0^\delta \mathrm{d}s \int_\Omega \mathrm{d}u \nabla G \cdot D(\lambda)\nabla\lambda - \int_0^\delta \mathrm{d}s \int_\Omega \mathrm{d}u \nabla G \cdot \sigma(\lambda)\nabla G \right\}$$

which, by the Schwarz inequality, is bounded above by

$$\int_0^\delta \mathrm{d}s \int_\Omega \mathrm{d}u \frac{1}{\chi(\lambda)} \nabla\lambda \cdot D(\lambda)\nabla\lambda \,.$$

By Corollary 4.3, this last expression converges to zero as $\delta \downarrow 0$. Finally, the third contribution is bounded by $I_T(\pi|\gamma)$ because π^δ in this interval is just a time translation of the path π. □

Let Π_2 be the set of all paths π in Π_1 with the property that for every $\delta > 0$ there exists $\epsilon > 0$ such that $\epsilon \le \pi_t(\cdot) \le 1 - \epsilon$ for all $t \in [\delta, T]$.

Lemma 5.2. *The set Π_2 is $I_T(\cdot|\gamma)$-dense.*

Proof. By the previous lemma, it is enough to show that each path $\pi(t, \mathrm{d}u) = \rho(t, u)\mathrm{d}u$ in Π_1 can be approximated by paths in Π_2. Fix π in Π_1 and let λ be as in the proof of the previous lemma. For each $0 < \varepsilon < 1$, let $\rho^\varepsilon = (1 - \varepsilon)\rho + \varepsilon\lambda$, $\pi^\varepsilon(t, \mathrm{d}u) = \rho^\varepsilon(t, u)\mathrm{d}u$. Note that $\mathcal{Q}(\pi^\varepsilon) < \infty$ because $\mathcal{Q}$ is convex and both $\mathcal{Q}(\pi)$ and $\mathcal{Q}(\lambda)$ are finite. Hence, π^ε belongs to D_γ since both ρ and λ satisfy the boundary conditions. Moreover, it is clear that π^ε converges to π as $\varepsilon \downarrow 0$. By the lower semicontinuity of $I_T(\cdot|\gamma)$, in order to conclude the proof, it is enough to show that

$$\overline{\lim_{\varepsilon\to 0}}\, I_T(\pi^\varepsilon|\gamma) \le I_T(\pi|\gamma). \tag{5.1}$$

By Lemma 4.7, there exists $H \in H_0^1(\sigma(\rho))$ such that ρ solves the Eq. (4.13). Let $\mathbf{P} = \sigma(\rho)\nabla H - D(\rho)\nabla\rho$ and $\mathbf{P}^\lambda = -D(\lambda)\nabla\lambda$. For each $0 < \varepsilon < 1$, let $\mathbf{P}^\varepsilon = (1 - \varepsilon)\mathbf{P} + \varepsilon\mathbf{P}^\lambda$. Since ρ solves Eq. (4.13), for every $G \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})$,

$$\int_0^T \mathrm{d}t \langle \mathbf{P}_t^\varepsilon \cdot \nabla G_t \rangle = \langle \pi_T^\varepsilon, G_T \rangle - \langle \pi_0^\varepsilon, G_0 \rangle - \int_0^T \mathrm{d}t \langle \pi_t^\varepsilon, \partial_t G_t \rangle.$$

Hence, by (4.6), $I_T(\pi^\varepsilon|\gamma)$ is equal to

$$\sup_{G\in\mathcal{C}_0^{1,2}(\overline{\Omega_T})}\left\{\int_0^T \mathrm{d}t\int_\Omega \left\{\mathbf{P}^\varepsilon + D(\rho^\varepsilon)\nabla\rho^\varepsilon\right\}\cdot\nabla G\mathrm{d}u - \int_0^T \mathrm{d}t\int_\Omega \nabla G\cdot\sigma(\rho^\varepsilon)\nabla G\mathrm{d}u\right\}.$$

This expression can be rewritten as

$$\frac{1}{4}\int_0^T \mathrm{d}t\int_\Omega \mathrm{d}u\big[\mathbf{P}^\varepsilon + D(\rho^\varepsilon)\nabla\rho^\varepsilon\big]\cdot\sigma(\rho^\varepsilon)^{-1}\big[\mathbf{P}^\varepsilon + D(\rho^\varepsilon)\nabla\rho^\varepsilon\big] \\ -\frac{1}{4}\inf_G\left\{\left\|\sigma(\rho^\varepsilon)^{-1}\big[\mathbf{P}^\varepsilon + D(\rho^\varepsilon)\nabla\rho^\varepsilon\big]-\nabla G\right\|^2_{\sigma(\rho^\varepsilon)}\right\}.$$

Hence,

$$I_T(\pi^\varepsilon|\gamma)\le\frac{1}{4}\int_0^T \mathrm{d}t\int_\Omega \mathrm{d}u\big[\mathbf{P}^\varepsilon + D(\rho^\varepsilon)\nabla\rho^\varepsilon\big]\cdot\sigma(\rho^\varepsilon)^{-1}\big[\mathbf{P}^\varepsilon + D(\rho^\varepsilon)\nabla\rho^\varepsilon\big].$$

In view of this inequality and (4.14), in order to prove (5.1), it is enough to show that

$$\lim_{\varepsilon\to 0}\int_0^T \mathrm{d}t\int_\Omega \mathrm{d}u\big[\mathbf{P}^\varepsilon + D(\rho^\varepsilon)\nabla\rho^\varepsilon\big]\cdot\sigma(\rho^\varepsilon)^{-1}\big[\mathbf{P}^\varepsilon + D(\rho^\varepsilon)\nabla\rho^\varepsilon\big] \\ =\int_0^T \mathrm{d}t\int_\Omega \mathrm{d}u\big[\mathbf{P} + D(\rho)\nabla\rho\big]\cdot\sigma(\rho)^{-1}\big[\mathbf{P} + D(\rho)\nabla\rho\big].$$

By the continuity of $D(\cdot)$ and $\chi(\cdot)$, the strict ellipticity of D, and from the definition of $\mathbf{P}^\varepsilon$,

$$\lim_{\varepsilon\to 0}\big[\mathbf{P}^\varepsilon + D(\rho^\varepsilon)\nabla\rho^\varepsilon\big]\cdot\sigma(\rho^\varepsilon)^{-1}\big[\mathbf{P}^\varepsilon + D(\rho^\varepsilon)\nabla\rho^\varepsilon\big] \\ =\big[\mathbf{P} + D(\rho)\nabla\rho\big]\cdot\sigma(\rho)^{-1}\big[\mathbf{P} + D(\rho)\nabla\rho\big]$$

almost everywhere. Therefore, to prove (5.1), it remains to show the uniform integrability of

$$\left\{\frac{\|\mathbf{P}^\varepsilon\|^2}{\chi(\rho^\varepsilon)}:\ \varepsilon>0\right\}\quad\text{and}\quad\left\{\frac{\|\nabla\rho^\varepsilon\|^2}{\chi(\rho^\varepsilon)}:\ \varepsilon>0\right\}.$$

Since $I_T(\pi|\gamma)<\infty$, by (4.8), (4.14) and Corollary 4.3, the functions $\frac{\|\mathbf{P}\|^2}{\chi(\rho)}$, $\frac{\|\mathbf{P}_\lambda\|^2}{\chi(\lambda)}$, $\frac{\|\nabla\rho\|^2}{\chi(\rho)}$ and $\frac{\|\nabla\lambda\|^2}{\chi(\lambda)}$ belong to $L^1(\Omega_T)$. In particular, the function

$$g=\max\left\{\frac{\|\mathbf{P}\|^2}{\chi(\rho)},\frac{\|\mathbf{P}_\lambda\|^2}{\chi(\lambda)},\frac{\|\nabla\rho\|^2}{\chi(\rho)},\frac{\|\nabla\lambda\|^2}{\chi(\lambda)}\right\},$$

also belongs to $L^1(\Omega_T)$. By the convexity of $\|\cdot\|^2$ and the concavity of $\chi(\cdot)$,

$$\frac{\|\mathbf{P}^\varepsilon\|^2}{\chi(\rho^\varepsilon)}\le\frac{(1-\varepsilon)\|\mathbf{P}\|^2+\varepsilon\|\mathbf{P}_\lambda\|^2}{(1-\varepsilon)\chi(\rho)+\varepsilon\chi(\lambda)}\le g,$$

which proves the uniform integrability of the family $\frac{\|\mathbf{P}^\varepsilon\|^2}{\chi(\rho^\varepsilon)}$. The uniform integrability of the family $\frac{\|\nabla\rho_\varepsilon\|^2}{\chi(\rho_\varepsilon)}$ follows from the same estimate with $\nabla\rho_\varepsilon$, $\nabla\rho$ and $\nabla\lambda$ in place of $\mathbf{P}_\varepsilon$, $\mathbf{P}$ and $\mathbf{P}_\lambda$, respectively. □

Let Π be the subset of Π_2 consisting of all those paths π which are solutions of the Eq. (4.13) for some $H\in\mathcal{C}_0^{1,2}(\overline{\Omega_T})$.

Theorem 5.3. *The set Π is $I_T(\cdot|\gamma)$-dense.*

Proof. By the previous lemma, it is enough to show that each path π in Π_2 can be approximated by paths in Π. Fix $\pi(t, du) = \rho(t, u)du$ in Π_2. By Lemma 4.7, there exists $H \in H_0^1(\sigma(\rho))$ such that ρ solves the Eq. (4.13). Since π belongs to $\Pi_2 \subset \Pi_1$, ρ is the weak solution of (3.1) in some time interval $[0, 2\delta]$ for some $\delta > 0$. In particular, $\nabla H = 0$ a.e. in $[0, 2\delta] \times \Omega$. On the other hand, since π belongs to Π_1, there exists $\epsilon > 0$ such that $\epsilon \le \pi_t(\cdot) \le 1 - \epsilon$ for $\delta \le t \le T$. Therefore,

$$\int_0^T dt \int_\Omega \|\nabla H_t(u)\|^2 du < \infty. \tag{5.2}$$

Since H belongs to $H_0^1(\sigma(\rho))$, there exists a sequence of functions $\{H^n : n \ge 1\}$ in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$ converging to H in $H_0^1(\sigma(\rho))$. We may assume of course that $\nabla H_t^n \equiv 0$ in the time interval $[0, \delta]$. In particular,

$$\lim_{n\to\infty} \int_0^T dt \int_\Omega du \|\nabla H_t^n(u) - \nabla H_t(u)\|^2 = 0. \tag{5.3}$$

For each integer $n > 0$, let ρ^n be the weak solution of (4.13) with H^n in place of H and set $\pi^n(t, du) = \rho^n(t, u)du$. By (4.14) and since σ is bounded above in $[0, 1]$ by a finite constant,

$$I_T(\pi^n|\gamma) = \frac{1}{4}\int_0^T dt \langle \nabla H_t^n \cdot \sigma(\rho_t^n)\nabla H_t^n\rangle \le C_0 \int_0^T dt \int_\Omega du \|\nabla H_t^n(u)\|^2.$$

In particular, by (5.2) and (5.3), $I_T(\pi^n|\gamma)$ is uniformly bounded. Thus, by Theorem 4.6, the sequence π^n is relatively compact in $D([0, T], \mathcal{M})$.

Let $\{\pi^{n_k} : k \ge 1\}$ be a subsequence of π^n converging to some π^0 in $D([0, T], \mathcal{M}^0)$. For every G in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$,

$$\begin{aligned}\langle \pi_T^{n_k}, G_T\rangle - \langle \gamma, G_0\rangle - \int_0^T dt \langle \pi_t^{n_k}, \partial_t G_t\rangle &= \sum_{i,j=1}^d \int_0^T dt \langle d_{i,j}(\rho_t^{n_k}), \partial^2_{u_i,u_j} G_t\rangle \\ &\quad - \sum_{i=1}^d \int_0^T dt \int_\Gamma d_{i,1}(b)\mathbf{n}_1(\partial_{u_i} G)dS + \int_0^T dt \langle \sigma(\rho_t^n)\nabla H_t^{n_k} \cdot \nabla G_t\rangle.\end{aligned}$$

Letting $k \to \infty$ in this equation, we obtain the same equation with π^0 and H in place of π^{n_k} and H^{n_k}, respectively, if

$$\begin{aligned}&\lim_{k\to\infty} \int_0^T dt \langle d_{i,j}(\rho_t^{n_k}), \partial^2_{u_i,u_j} G_t\rangle = \int_0^T dt \langle d_{i,j}(\rho_t^0), \partial^2_{u_i,u_j} G_t\rangle, \\ &\lim_{k\to\infty} \int_0^T dt \langle \sigma(\rho_t^{n_k})\nabla H_t^{n_k} \cdot \nabla G_t\rangle = \int_0^T dt \langle \sigma(\rho_t^0)\nabla H_t \cdot \nabla G_t\rangle.\end{aligned} \tag{5.4}$$

We prove the second claim, the first one being simpler. Note first that we can replace H^{n_k} by H in the previous limit, because σ is bounded in $[0, 1]$ by some positive constant and (5.3) holds. Now, ρ^{n_k} converges to ρ^0 weakly in $L^2(\Omega_T)$ because π^{n_k} converges to π^0 in $D([0, T], \mathcal{M}^0)$. Since $I_T(\pi^n|\gamma)$ is uniformly bounded, by Corollary 4.4 and Lemma 4.5, ρ^{n_k} converges to ρ^0 strongly in $L^2(\Omega_T)$ which implies (5.4). In particular, since (5.2) holds, by the uniqueness of weak solutions of Eq. (4.13), $\pi^0 = \pi$ and we are done. □

6. Large deviations

We prove in this section the dynamical large deviations principle for the empirical measure of boundary driven symmetric exclusion processes in dimension $d \geq 1$. The proof relies on the results presented in the previous section and is quite similar to the original one presented in [15,9]. There are just three additional difficulties. On the one hand, the lack of explicitly known stationary states hinders the derivation of the usual estimates of the entropy and the Dirichlet form, so important in the proof of the hydrodynamic behavior. On the other hand, due to the definition of the rate function, we have to show that trajectories with infinite energy can be neglected in the large deviations regime. Finally, since we are working with the empirical measure, instead of the empirical density, we need to show that trajectories which are not absolutely continuous with respect to the Lebesgue measure and whose density is not bounded by one can also be neglected. The first two problems have already been faced and solved, the first one in [17,4] and the second in [19,6]. The approach here is quite similar; we thus only sketch the main steps for the sake of completeness.

6.1. Superexponential estimates

It is well known that one of the main steps in the derivation of the upper bound is a superexponential estimate which allows the replacement of local functions by functionals of the empirical density in the large deviations regime. Essentially, the problem consists in bounding expressions such as $\langle V, f^2\rangle_{\mu_{ss}^N}$ in terms of the Dirichlet form $\langle -N^2\mathcal{L}_N f, f\rangle_{\mu_{ss}^N}$. Here V is a local function and $\langle\cdot,\cdot\rangle_{\mu_{ss}^N}$ indicates the inner product with respect to the invariant state μ_{ss}^N. In our context, the fact that the invariant state is not known explicitly introduces a technical difficulty.

Let β be as at the beginning of Section 2. Following [17,4], we use $\nu_{\beta(\cdot)}^N$ as the reference measure and estimate everything with respect to $\nu_{\beta(\cdot)}^N$. However, since $\nu_{\beta(\cdot)}^N$ is not the invariant state, there are no reasons for $\langle -N^2\mathcal{L}_N f, f\rangle_{\nu_{\beta(\cdot)}^N}$ to be positive. The next statement shows that this expression is almost positive.

For each function $f : X_N \to \mathbb{R}$, let

$$D_{N,0}(f) = \sum_{i=1}^{d}\sum_{x}\int r_{x,x+e_i}(\eta)\left[f(\eta^{x,x+e_i}) - f(\eta)\right]^2 \mathrm{d}\nu_{\beta(\cdot)}^N(\eta),$$

where the second sum is carried out over all x such that $x, x+e_i \in \Omega_N$.

Lemma 6.1. *There exists a finite constant C depending only on β such that*

$$\langle N^2\mathcal{L}_{N,0}f, f\rangle_{\nu_{\beta(\cdot)}^N} \leq -\frac{N^2}{4}D_{N,0}(f) + CN^d\langle f, f\rangle_{\nu_{\beta(\cdot)}^N},$$

for every function $f : X_N \to \mathbb{R}$.

The proof of this lemma is elementary and is thus omitted. Further, we may choose β for which there exists a constant $\theta > 0$ such that

$$\begin{aligned} &\beta(u_1,\check{u}) = b(-1,\check{u}) \quad \text{if } -1 \leq u_1 \leq -1+\theta,\\ &\beta(u_1,\check{u}) = b(1,\check{u}) \qquad \text{if } 1-\theta \leq u_1 \leq 1, \end{aligned}$$

for all $\check{u} \in \mathbb{T}^{d-1}$. In that case, for every N large enough, $\nu^N_{\beta(\cdot)}$ is reversible for the process with generator $\mathcal{L}_{N,b}$ and then $\langle -N^2\mathcal{L}_{N,b} f, f\rangle_{\nu^N_{\beta(\cdot)}}$ is positive.

This lemma together with the computation presented in [2], p. 78, for nonreversible processes, permits us to prove the superexponential estimate. For a cylinder function Ψ, denote the expectation of Ψ with respect to the Bernoulli product measure ν^N_α by $\widetilde{\Psi}(\alpha)$:

$$\widetilde{\Psi}(\alpha) = E^{\nu^N_\alpha}[\Psi].$$

For a positive integer l and $x \in \Omega_N$, denote the empirical mean density on a box of size $2l+1$ centered at x by $\eta^l(x)$:

$$\eta^l(x) = \frac{1}{|\Lambda_l(x)|} \sum_{y \in \Lambda_l(x)} \eta(y),$$

where

$$\Lambda_l(x) = \Lambda_{N,l}(x) = \{y \in \Omega_N : |y-x| \le l\}.$$

For each $G \in \mathcal{C}(\overline{\Omega_T})$, each cylinder function Ψ and each $\varepsilon > 0$, let

$$V^{G,\Psi}_{N,\varepsilon}(s,\eta) = \frac{1}{N^d} \sum_x G(s, x/N) \left[\tau_x \Psi(\eta) - \widetilde{\Psi}(\eta^{\varepsilon N}(x))\right],$$

where the sum is carried out over all x such that the support of $\tau_x \Psi$ belongs to Ω_N.

For a continuous function $H : [0,T] \times \Gamma \to \mathbb{R}$, let

$$V^{\pm}_{N,H} = \int_0^T \mathrm{d}s \frac{1}{N^{d-1}} \sum_{x \in \Gamma^{\pm}_N} V^{\pm}(x,\eta_s) H\left(s, \frac{x \pm e_1}{N}\right),$$

where Γ^-_N (resp. Γ^+_N) stands for the left (resp. right) boundary of Ω_N:

$$\Gamma^{\pm}_N = \{(x_1,\dots,x_d) \in \Gamma_N : x_1 = \pm(N-1)\}$$

and where

$$V^{\pm}(x,\eta) = \left[\eta(x) + b\left(\frac{x \pm e_1}{N}\right)\right]\left[\eta(x \mp e_1) - b\left(\frac{x \pm e_1}{N}\right)\right].$$

Proposition 6.2. *Fix $G \in \mathcal{C}(\overline{\Omega_T})$, H in $\mathcal{C}([0,T] \times \Gamma)$, a cylinder function Ψ and a sequence $\{\eta^N : N \ge 1\}$ of configurations with η^N in X_N. For every $\delta > 0$,*

$$\overline{\lim_{\varepsilon \to 0}}\ \overline{\lim_{N \to \infty}} \frac{1}{N^d} \log \mathbb{P}_{\eta^N}\left[\left|\int_0^T V^{G,\Psi}_{N,\varepsilon}(s,\eta_s)\mathrm{d}s\right| > \delta\right] = -\infty,$$
$$\overline{\lim_{N \to \infty}} \frac{1}{N^d} \mathbb{P}_{\eta^N}\left[|V^{\pm}_{N,H}| > \delta\right] = -\infty.$$

For each $\varepsilon > 0$ and π in $\mathcal{M}$, denote by $\Xi_\varepsilon(\pi) = \pi^\varepsilon$ the absolutely continuous measure obtained by smoothing the measure π:

$$\Xi_\varepsilon(\pi)(dx) = \pi^\varepsilon(dx) = \frac{1}{U_\varepsilon} \frac{\pi(\boldsymbol{\Lambda}_\varepsilon(x))}{|\boldsymbol{\Lambda}_\varepsilon(x)|} \mathrm{d}x,$$

where $\boldsymbol{\Lambda}_\varepsilon(x) = \{y \in \Omega : |y - x| \le \varepsilon\}$, $|A|$ stands for the Lebesgue measure of the set A, and $\{U_\varepsilon : \varepsilon > 0\}$ is a strictly decreasing sequence converging to 1: $U_\varepsilon > 1$, $U_\varepsilon > U_{\varepsilon'}$ for $\varepsilon > \varepsilon'$, and $\lim_{\varepsilon \downarrow 0} U_\varepsilon = 1$. Let

$$\pi^{N,\varepsilon} = \Xi_\varepsilon(\pi^N).$$

A simple computation shows that $\pi^{N,\varepsilon}$ belongs to $\mathcal{M}^0$ for N sufficiently large because $U_\varepsilon > 1$, and that for each continuous function $H : \Omega \to \mathbb{R}$,

$$\langle \pi^{N,\varepsilon}, H\rangle = \frac{1}{N^d} \sum_{x \in \Omega_N} H(x/N)\eta^{\varepsilon N}(x) + O(N, \varepsilon),$$

where $O(N, \varepsilon)$ is absolutely bounded by $C_0\{N^{-1} + \varepsilon\}$ for some finite constant C_0 depending only on H.

For each H in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$ consider the exponential martingale M_t^H defined by

$$\begin{aligned} M_t^H = \exp\Big\{ N^d \Big[&\langle \pi_t^N, H_t\rangle - \langle \pi_0^N, H_0\rangle \\ &- \frac{1}{N^d} \int_0^t e^{-N^d\langle \pi_s^N, H_s\rangle} \big(\partial_s + N^2 \mathcal{L}_N\big) e^{N^d\langle \pi_s^N, H_s\rangle} \mathrm{d}s \Big]\Big\}. \end{aligned}$$

Recall from Section 2.2 the definition of the functional $\hat{J}_H$. An elementary computation shows that

$$M_T^H = \exp\left\{ N^d \left[\hat{J}_H(\pi^{N,\varepsilon}) + \mathbb{V}_{N,\varepsilon}^H + c_H^1(\varepsilon) + c_H^2(N^{-1}) \right]\right\}. \tag{6.1}$$

In this formula,

$$\begin{aligned} \mathbb{V}_{N,\varepsilon}^H = &- \sum_{i=1}^d \int_0^T V_{N,\varepsilon}^{\partial_{u_i}^2 H, h_i}(s, \eta_s)\mathrm{d}s - \frac{1}{2} \sum_{i=1}^d \int_0^T V_{N,\varepsilon}^{(\partial_{u_i} H)^2, g_i}(s, \eta_s)\mathrm{d}s \\ &+ a V_{N,\partial_{u_1} H}^{+} - a V_{N,\partial_{u_1} H}^{-} + \langle \pi_0^N, H_0\rangle - \langle \gamma, H_0\rangle; \end{aligned}$$

the cylinder functions h_i, g_i are given by

$$\begin{aligned} h_i(\eta) &= \eta(0) + a\Big\{\eta(0)[\eta(-e_i) + \eta(e_i)] - \eta(-e_i)\eta(e_i)\Big\}, \\ g_i(\eta) &= r_{0,e_i}(\eta)[\eta(e_i) - \eta(0)]^2; \end{aligned}$$

and $c_H^j : \mathbb{R}_+ \to \mathbb{R}$, $j = 1, 2$, are functions depending only on H such that $c_H^j(\delta)$ converges to 0 as $\delta \downarrow 0$. In particular, the martingale M_T^H is bounded by $\exp\big\{C(H, T)N^d\big\}$ for some finite constant $C(H, T)$ depending only on H and T. Therefore, Proposition 6.2 holds for $\mathbb{P}_{\eta^N}^H = \mathbb{P}_{\eta^N} M_T^H$ in place of $\mathbb{P}_{\eta^N}$.

6.2. Energy estimates

To exclude paths with infinite energy in the large deviations regime, we need an energy estimate. We state first the following technical result.

Lemma 6.3. *There exists a finite constant C_0, depending on T, such that for every G in $C_c^\infty(\Omega_T)$, every integer $1 \le i \le d$ and every sequence $\{\eta^N : N \ge 1\}$ of configurations with η^N in X_N,*

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log \mathbb{E}_{\eta^N} \left[\exp \left\{ N^d \int_0^T \mathrm{d}t \langle \pi_t^N, \partial_{u_i} G \rangle \right\} \right] \le C_0 \left\{ 1 + \int_0^T \|G_t\|_2^2 \mathrm{d}t \right\}.$$

The proof of this proposition is similar to that of Lemma A.1.1 in [14].

Fix throughout the rest of the subsection a constant C_0 satisfying the statement of Lemma 6.3. For each G in $\mathcal{C}_c^\infty(\Omega_T)$ and each integer $1 \le i \le d$, let $\tilde{\mathcal{Q}}_i^G : D([0,T], \mathcal{M}) \to \mathbb{R}$ be the function given by

$$\tilde{\mathcal{Q}}_i^G(\pi) = \int_0^T \mathrm{d}t \langle \pi_t, \partial_{u_i} G_t \rangle - C_0 \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}u G(t,u)^2.$$

Notice that

$$\sup_{G \in \mathcal{C}_c^\infty(\Omega_T)} \left\{ \tilde{\mathcal{Q}}_i^G(\pi) \right\} = \frac{\mathcal{Q}_i(\pi)}{4C_0}. \tag{6.2}$$

Fix a sequence $\{G_k : k \ge 1\}$ of smooth functions dense in $L^2([0,T], H^1(\Omega))$. For any positive integers r, l, let

$$B_{r,l} = \left\{ \pi \in D([0,T], \mathcal{M}) : \max_{\substack{1\le k\le r\\ 1\le i\le d}} \tilde{\mathcal{Q}}_i^{G_k}(\pi) \le l \right\}.$$

Since, for fixed G in $\mathcal{C}_c^\infty(\Omega_T)$ and $1 \le i \le d$ integer, the function $\tilde{\mathcal{Q}}_i^G$ is continuous, $B_{r,l}$ is a closed subset of $D([0,T], \mathcal{M})$.

Lemma 6.4. *There exists a finite constant C_0, depending on T, such that for any positive integers r, l and any sequence $\{\eta^N : N \ge 1\}$ of configurations with η^N in X_N,*

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q_{\eta^N} \left[(B_{r,l})^c \right] \le -l + C_0.$$

Proof. For integers $1 \le k \le r$ and $1 \le i \le d$, by the Chebyshev inequality and by Lemma 6.3,

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log \mathbb{P}_{\eta^N} \left[\tilde{\mathcal{Q}}_i^{G_k} > l \right] \le -l + C_0.$$

Hence, from

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log(a_N + b_N) \le \max \left\{ \overline{\lim_{N\to\infty}} \frac{1}{N^d} \log a_N, \overline{\lim_{N\to\infty}} \frac{1}{N^d} \log b_N \right\}, \tag{6.3}$$

we obtain the desired inequality. □

6.3. The upper bound

Fix a sequence $\{F_k : k \ge 1\}$ of smooth nonnegative functions dense in $\mathcal{C}(\overline{\Omega})$ for the uniform topology. For $k \ge 1$ and $\delta > 0$, let

$$D_{k,\delta} = \left\{ \pi \in D([0,T], \mathcal{M}) : 0 \le \langle \pi_t, F_k \rangle \le \int_\Omega F_k(x)\mathrm{d}x + C_k\delta, 0 \le t \le T \right\},$$

where $C_k = \|\nabla F_k\|_\infty$ and ∇F is the gradient of F. Clearly, the set $D_{k,\delta}$, $k \geq 1$, $\delta > 0$, is a closed subset of $D([0, T], \mathcal{M})$. Moreover, if

$$E_{m,\delta} = \bigcap_{k=1}^{m} D_{k,\delta},$$

we have that $D([0, T], \mathcal{M}^0) = \cap_{n\geq 1} \cap_{m\geq 1} E_{m,1/n}$. Note, finally, that for all $m \geq 1$, $\delta > 0$,

$$\pi^{N,\varepsilon} \text{ belongs to } E_{m,\delta} \text{ for } N \text{ sufficiently large.} \tag{6.4}$$

Fix a sequence of configurations $\{\eta^N : N \geq 1\}$ with η^N in X_N and such that $\pi^N(\eta^N)$ converges to $\gamma(u)\mathrm{d}u$ in $\mathcal{M}$. Let A be a subset of $D([0, T], \mathcal{M})$:

$$\frac{1}{N^d} \log \mathbb{P}_{\eta^N} \left[\pi^N \in A\right] = \frac{1}{N^d} \log \mathbb{E}_{\eta^N} \left[M_T^H (M_T^H)^{-1} \mathbf{1}\{\pi^N \in A\}\right].$$

Maximizing over π^N in A, we get from (6.1) that the last term is bounded above by

$$-\inf_{\pi\in A} \hat{J}_H(\pi^\varepsilon) + \frac{1}{N^d} \log \mathbb{E}_{\eta^N}\left[M_T^H e^{-N^d \mathbb{V}_{N,\varepsilon}^H}\right] - c_H^1(\varepsilon) - c_H^2(N^{-1}).$$

Since $\pi^N(\eta^N)$ converges to $\gamma(u)\mathrm{d}u$ in $\mathcal{M}$ and since Proposition 6.2 holds for $\mathbb{P}_{\eta^N}^H = \mathbb{P}_{\eta^N} M_T^H$ in place of $\mathbb{P}_{\eta^N}$, the second term of the previous expression is bounded above by some $C_H(\varepsilon, N)$ such that

$$\varlimsup_{\varepsilon\to 0} \varlimsup_{N\to\infty} C_H(\varepsilon, N) = 0.$$

Hence, for every $\varepsilon > 0$, and every H in $\mathcal{C}_0^{1,2}(\overline{\Omega_T})$,

$$\varlimsup_{N\to\infty} \frac{1}{N^d} \log \mathbb{P}_{\eta^N}[A] \leq -\inf_{\pi\in A} \hat{J}_H(\pi^\varepsilon) + C'_H(\varepsilon), \tag{6.5}$$

where $\lim_{\varepsilon\to 0} C'_H(\varepsilon) = 0$.

For each $H \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})$, each $\varepsilon > 0$ and any $r, l, m, n \in \mathbb{Z}_+$, let $J_{H,\varepsilon}^{r,l,m,n} : D([0, T], \mathcal{M}) \to \mathbb{R} \cup \{\infty\}$ be the functional given by

$$J_{H,\varepsilon}^{r,l,m,n}(\pi) = \begin{cases} \hat{J}_H(\pi^\varepsilon) & \text{if } \pi \in B_{r,l} \cap E_{m,1/n}, \\ +\infty & \text{otherwise.} \end{cases}$$

This functional is lower semicontinuous because $\hat{J}_H \circ \Xi_\varepsilon$ is too and because $B_{r,l}$, $E_{m,1/n}$ are closed subsets of $D([0, T], \mathcal{M})$.

Let $\mathcal{O}$ be an open subset of $D([0, T], \mathcal{M})$. By Lemma 6.4 and (6.3)–(6.5),

$$\begin{aligned}
\varlimsup_{N\to\infty} \frac{1}{N^d} \log Q_{\eta^N}[\mathcal{O}] &\leq \max\Big\{ \varlimsup_{N\to\infty} \frac{1}{N^d} \log Q_{\eta^N}[\mathcal{O} \cap B_{r,l} \cap E_{m,1/n}], \\
&\qquad \varlimsup_{N\to\infty} \frac{1}{N^d} \log Q_{\eta^N}[(B_{r,l})^c] \Big\} \\
&\leq \max\Big\{ -\inf_{\pi\in\mathcal{O}\cap B_{r,l}\cap E_{m,1/n}} \hat{J}_H(\pi^\varepsilon) + C'_H(\varepsilon), -l + C_0 \Big\} \\
&= -\inf_{\pi\in\mathcal{O}} L_{H,\varepsilon}^{r,l,m,n}(\pi),
\end{aligned}$$

where

$$L_{H,\varepsilon}^{r,l,m,n}(\pi) = \min\left\{J_{H,\varepsilon}^{r,l,m,n}(\pi) - C'_H(\varepsilon), l - C_0\right\}.$$

In particular,

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q_{\eta^N}[\mathcal{O}] \le - \sup_{H,\varepsilon,r,l,m,n} \inf_{\pi\in\mathcal{O}} L_{H,\varepsilon}^{r,l,m,n}(\pi).$$

Note that, for each $H \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})$, each $\varepsilon > 0$ and $r, l, m, n \in \mathbb{Z}_+$, the functional $L_{H,\varepsilon}^{r,l,m,n}$ is lower semicontinuous. Then, by Lemma A2.3.3 in [13], for each compact subset $\mathcal{K}$ of $D([0,T],\mathcal{M})$,

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q_{\eta^N}[\mathcal{K}] \le - \inf_{\pi\in\mathcal{K}} \sup_{H,\varepsilon,r,l,m,n} L_{H,\varepsilon}^{r,l,m,n}(\pi).$$

By (6.2) and since $D([0,T],\mathcal{M}^0) = \cap_{n\ge1}\cap_{m\ge1} E_{m,1/n}$,

$$\begin{aligned}&\overline{\lim_{\varepsilon\to0}}\,\overline{\lim_{l\to\infty}}\,\overline{\lim_{r\to\infty}}\,\overline{\lim_{m\to\infty}}\,\overline{\lim_{n\to\infty}}\, L_{H,\varepsilon}^{r,l,m,n}(\pi)\\ &\quad= \begin{cases}\hat{J}_H(\pi) & \text{if } \mathcal{Q}(\pi) < \infty \text{ and } \pi \in D([0,T],\mathcal{M}^0),\\ +\infty & \text{otherwise.}\end{cases}\end{aligned}$$

This result and the last inequality imply the upper bound for compact sets because $\hat{J}_H$ and J_H coincide on $D([0,T],\mathcal{M}^0)$. To pass from compact sets to closed sets, we have to obtain exponential tightness for the sequence $\{Q_{\eta^N}\}$. This means that there exists a sequence of compact sets $\{\mathcal{K}_n : n \ge 1\}$ in $D([0,T],\mathcal{M})$ such that

$$\overline{\lim_{N\to\infty}} \frac{1}{N^d} \log Q_{\eta^N}({\mathcal{K}_n}^c) \le -n.$$

The proof presented in [1] for the non-interacting zero-range process is easily adapted to our context.

6.4. The lower bound

The proof of the lower bound is similar to that in the convex periodic case. We just sketch it and refer the reader to [13], Section 10.5. Fix a path π in Π and let $H \in \mathcal{C}_0^{1,2}(\overline{\Omega_T})$ be such that π is the weak solution of Eq. (4.13). Recall from the previous section the definition of the martingale M_t^H and denote by $\mathbb{P}_{\eta^N}^H$ the probability measure on $D([0,T],X_N)$ given by $\mathbb{P}_{\eta^N}^H[A] = \mathbb{E}_{\eta^N}[M_T^H \mathbf{1}\{A\}]$. Under $\mathbb{P}_{\eta^N}^H$ and for each $0 \le t \le T$, the empirical measure π_t^N converges in probability to π_t. Further,

$$\lim_{N\to\infty} \frac{1}{N^d} H\left(\mathbb{P}_{\eta^N}^H | \mathbb{P}_{\eta^N}\right) = I_T(\pi|\gamma),$$

where $H(\mu|\nu)$ stands for the relative entropy of μ with respect to ν. From these two results we can obtain that for every open set $\mathcal{O} \subset D([0,T],\mathcal{M})$ which contains π,

$$\underline{\lim_{N\to\infty}} \frac{1}{N^d} \log \mathbb{P}_{\eta^N}\left[\mathcal{O}\right] \ge -I_T(\pi|\gamma).$$

The lower bound follows from this and the $I_T(\cdot|\gamma)$-density of Π established in Theorem 5.3.

7. Existence and uniqueness of weak solutions

We examine in this section the existence and uniqueness of weak solutions of the boundary value problems (2.2), (3.2) and (4.13), as well as some properties of the solutions.

Proposition 7.1. *Let $\rho_0 : \overline{\Omega} \to [0, 1]$ be a measurable function. There exists a unique weak solution of* (3.2).

Proof. Existence of weak solutions of (3.2) is guaranteed by the tightness of the sequence $\mathbf{Q}^N_{\text{ss}}$ proved in Section 3. Indeed, fix a profile $\rho_0 : \Omega \to [0, 1]$ and consider a sequence $\{\mu^N : N \geq 1\}$ of probability measures in $\mathcal{M}$ associated with ρ_0 in the sense (3.3). Fix $T > 0$ and denote by Q^N the probability measure on $D([0, T], \mathcal{M})$ induced by the measure μ^N and the process π^N_t. Repeating the arguments of Section 3, one can prove that the sequence $\{Q^N : N \geq 1\}$ is tight and that any limit point of $\{Q^N : N \geq 1\}$ is concentrated on weak solutions of (3.2). This proves the existence. The uniqueness follows from Lemma 7.2 below. $\square$

Denote by $\|\cdot\|_1$ the $L^1(\Omega)$ norm. The next lemma states that the $L^1(\Omega)$-norm of the difference of two weak solutions of the boundary value problem (3.1) decreases in time:

Lemma 7.2. *Fix two profiles $\rho^1_0, \rho^2_0 : \Omega \to [0, 1]$. Let ρ^j, $j = 1, 2$, be weak solutions of* (3.1) *with initial condition ρ^j_0. Then, $\|\rho^1_t - \rho^2_t\|_1$ decreases in time. In particular, there is at most one weak solution of* (3.1).

Proof. Fix two profiles $\rho^1_0, \rho^2_0 : \Omega \to [0, 1]$. Let ρ^j, $j = 1, 2$, be weak solutions of (3.1) with initial condition ρ^j_0. Fix $0 \leq s < t$. For $\delta > 0$ small, denote by R_δ the function defined by

$$R_\delta(u) = \frac{u^2}{2\delta}\mathbf{1}\{|u| \leq \delta\} + \big(|u| - \delta/2\big)\mathbf{1}\{|u| > \delta\}.$$

Let $\psi : \mathbb{R}^d \to \mathbb{R}_+$ be a smooth approximation of the identity:

$$\psi(u) \geq 0, \quad \operatorname{supp} \psi \subset [-1, 1]^d, \qquad \int \psi(u)\mathrm{d}u = 1.$$

For each positive ϵ, define ψ_ϵ as

$$\psi_\epsilon(u) = \epsilon^{-d}\psi(u\epsilon^{-1}).$$

Taking the time derivative of the convolution of ρ^j_t with ψ_ϵ, after some elementary computations based on properties (**H1**$'$), (**H2**$'$) of weak solutions of (3.2), one can show that

$$\int_\Omega \mathrm{d}u R_\delta\big(\rho^1(t, u) - \rho^2(t, u)\big) - \int_\Omega \mathrm{d}u R_\delta\big(\rho^1(s, u) - \rho^2(s, u)\big)$$
$$= -\delta^{-1}\int_s^t \mathrm{d}\tau \int_{A_\delta} \mathrm{d}u \nabla(\rho^1 - \rho^2) \cdot \big\{D(\rho^1)\nabla\rho^1 - D(\rho^2)\nabla\rho^2\big\},$$

where A_δ stands for the subset of $[0, T] \times \Omega$ where $|\rho^1(t, u) - \rho^2(t, u)| \leq \delta$. We may rewrite the previous expression as

$$-\delta^{-1}\int_s^t \mathrm{d}\tau \int_{A_\delta} \mathrm{d}u \nabla(\rho^1 - \rho^2) \cdot D(\rho^1)\nabla(\rho^1 - \rho^2)$$
$$-\delta^{-1}\int_s^t \mathrm{d}\tau \int_{A_\delta} \mathrm{d}u \nabla(\rho^1 - \rho^2) \cdot \big\{D(\rho^1) - D(\rho^2)\big\}\nabla\rho^2.$$

By the strict ellipticity of the diffusion coefficient D, the first line is bounded above by

$$-a\delta^{-1}\int_s^t \mathrm{d}\tau \int_{A_\delta} \mathrm{d}u \|\nabla(\rho^1-\rho^2)\|^2.$$

On the other hand, since $D_{i,j}$ is Lipschitz continuous, on the set A_δ, $|D_{i,j}(\rho^1)-D_{i,j}(\rho^2)| \le M|\rho^1-\rho^2| \le M\delta$. In particular, by the Schwarz inequality, the second line of the previous formula is bounded by

$$\frac{MdA}{2\delta}\int_s^t \mathrm{d}\tau \int_{A_\delta} \mathrm{d}u \|\nabla(\rho^1-\rho^2)\|^2 + \frac{\delta M d}{2A}\int_s^t \mathrm{d}\tau \int_{A_\delta} \mathrm{d}u \|\nabla\rho^2\|^2$$

for every $A > 0$. Choose $A = (2a/Md)$ to obtain that

$$\int_\Omega \mathrm{d}u R_\delta\big(\rho^1(t,u)-\rho^2(t,u)\big) - \int_\Omega \mathrm{d}u R_\delta\big(\rho^1(s,u)-\rho^2(s,u)\big) \le \frac{\delta(dM)^2}{4a}\int_0^t \mathrm{d}\tau \int \mathrm{d}u \|\nabla\rho^2\|^2.$$

Letting $\delta \downarrow 0$, we conclude the proof of the lemma because $R_\delta(\cdot)$ converges to the absolute value function as $\delta \downarrow 0$. □

Lemma 7.3. *Fix two profiles $\rho_0^1, \rho_0^2 : \Omega \to [0,1]$. Let ρ^j, $j = 1, 2$, be weak solutions of (4.13) for the same H satisfying (5.2) and with initial condition ρ_0^j. Then, $\|\rho_t^1 - \rho_t^2\|_1$ decreases in time. In particular, there is at most one weak solution of (4.13) when H satisfies (5.2).*

Proof. Following the same procedure of the proof of the previous lemma, we get first

$$\begin{aligned}\int_\Omega \mathrm{d}u R_\delta\big(\rho^1(t,u)-\rho^2(t,u)\big) &- \int_\Omega \mathrm{d}u R_\delta\big(\rho^1(s,u)-\rho^2(s,u)\big)\\ &= -\delta^{-1}\int_s^t \mathrm{d}\tau \int_{A_\delta} \mathrm{d}u \nabla(\rho^1-\rho^2)\cdot\big\{D(\rho^1)\nabla\rho^1 - D(\rho^2)\nabla\rho^2\big\}\\ &\quad + \delta^{-1}\int_s^t \mathrm{d}\tau \int_{A_\delta} \mathrm{d}u \nabla(\rho^1-\rho^2)\big\{\sigma(\rho^1)-\sigma(\rho^2)\big\}\cdot\nabla H,\end{aligned}$$

and then

$$\begin{aligned}\int_\Omega \mathrm{d}u R_\delta\big(\rho^1(t,u)-\rho^2(t,u)\big) &- \int_\Omega \mathrm{d}u R_\delta\big(\rho^1(s,u)-\rho^2(s,u)\big)\\ &\le \delta C_1\int_0^t \mathrm{d}\tau \int \mathrm{d}u \|\nabla\rho^2\|^2 + \delta C_2\int_0^t \mathrm{d}\tau \int \mathrm{d}u \|\nabla H\|^2,\end{aligned}$$

for some positive constants C_1 and C_2. Hence, letting $\delta \downarrow 0$ we conclude the proof of the lemma. □

The same ideas permit to show the monotonicity of weak solutions of (3.2). This is the content of the next result which plays a fundamental role in proving the existence and uniqueness of weak solutions of (2.2).

Lemma 7.4. *Fix two profiles $\rho_0^1, \rho_0^2 : \Omega \to [0,1]$. Let ρ^j, $j = 1, 2$, be the weak solutions of (4.13) for some H satisfying (5.2) and with initial condition ρ_0^j. Assume that there exists $s \ge 0$ such that*

$$\lambda\Big\{u \in \Omega : \rho^1(s,u) \le \rho^2(s,u)\Big\} = 1,$$

where λ is the Lebesgue measure on Ω. Then, for all $t \ge s$

$$\lambda\Big\{u \in \Omega : \rho^1(t,u) \le \rho^2(t,u)\Big\} = 1.$$

Proof. We just need to repeat the same proof as for Lemma 7.2 but considering the function $R_\delta^+(u) = R_\delta(u)\mathbf{1}\{u \ge 0\}$ instead of R_δ. □

Corollary 7.5. *Denote by ρ^0 (resp. ρ^1) the weak solution of (3.1) with initial condition equal to 0 (resp. 1). Then, for $0 \le s \le t$, $\rho_t^1(\cdot) \le \rho_s^1(\cdot)$ and $\rho_s^0(\cdot) \le \rho_t^0(\cdot)$ a.e.*

Proof. Fix $s \ge 0$. Note that $\hat\rho(r,u)$ defined by $\hat\rho(r,u) = \rho^1(s+r,u)$ is a weak solution of (3.1) with initial condition $\rho^1(s,u)$. Since $\rho^1(s,u) \le 1 = \rho^1(0,u)$, by the previous lemma, for all $r \ge 0$, $\rho^1(r+s,u) \le \rho^1(r,u)$ for almost all u. □

We now turn to the existence and uniqueness of the boundary value problem (2.2). Recall the notation introduced at the beginning of Section 4. Consider the following classical boundary eigenvalue problem for the Laplacian:

$$\begin{cases} -\Delta U = \alpha U, \\ U \in H_0^1(\Omega). \end{cases} \tag{7.1}$$

By the Sturm–Liouville theorem (cf. [10], Section 9.12.3), problem (7.1) has a countable system $\{U_n, \alpha_n : n \ge 1\}$ of eigensolutions which contains all possible eigenvalues. The set $\{U_n : n \ge 1\}$ of eigenfunctions forms a complete orthonormal system in the Hilbert space $L^2(\Omega)$, each U_n belongs to $H_0^1(\Omega)$, all the eigenvalues, α_n, have finite multiplicity and

$$0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \cdots \to \infty.$$

The set $\{U_n/\alpha_n^{1/2} : n \ge 1\}$ is a complete orthonormal system in the Hilbert space $H_0^1(\Omega)$. Hence, a function V belongs to $L^2(\Omega)$ if and only if

$$V = \lim_{n\to\infty} \sum_{k=1}^n \langle V, U_k\rangle_2 U_k$$

in $L^2(\Omega)$. In this case,

$$\langle V, W\rangle_2 = \sum_{k=1}^\infty \langle V, U_k\rangle_2 \overline{\langle W, U_k\rangle_2}$$

for all W in $L^2(\Omega)$. Moreover, a function V belongs to $H_0^1(\Omega)$ if and only if

$$V = \lim_{n\to\infty} \sum_{k=1}^n \langle V, U_k\rangle_2 U_k$$

in $H_0^1(\Omega)$. In this case,

$$\langle V, W\rangle_{1,2,0} = \sum_{k=1}^\infty \alpha_k \langle V, U_k\rangle_2 \overline{\langle W, U_k\rangle_2} \tag{7.2}$$

for all W in $H_0^1(\Omega)$.

Lemma 7.6. *Fix two profiles ρ_0^1, $\rho_0^2 : \Omega \to [0,1]$. Let ρ^j, $j = 1, 2$, be the weak solutions of (3.2) with initial condition ρ_0^j. Then,*

$$\int_0^\infty \|\rho_t^1 - \rho_t^2\|_1^2 \mathrm{d}t < \infty.$$

In particular,

$$\lim_{t\to\infty} \|\rho_t^1 - \rho_t^2\|_1 = 0.$$

Proof. Fix two profiles $\rho_0^1, \rho_0^2 : \Omega \to [0, 1]$ and let ρ^j, $j = 1, 2$, be the weak solutions of (3.2) with initial condition ρ_0^j. Let $\rho_t^j(\cdot) = \rho^j(t, \cdot)$. For $n \geq 1$ let $F_n : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by

$$F_n(t) = \sum_{k=1}^n \frac{1}{\alpha_k} \left|\langle \rho_t^1 - \rho_t^2, U_k\rangle_2\right|^2.$$

Since ρ^1, ρ^2 are weak solutions, F_n is time differentiable. Since $\Delta U_k = -\alpha_k U_k$ and since $\alpha_k > 0$, for $t > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} F_n(t) = -\sum_{k=1}^n \Big\{ \langle \rho_t^1 - \rho_t^2, U_k\rangle_2 \overline{\langle \varphi(\rho_t^1) - \varphi(\rho_t^2), U_k\rangle_2} + \langle \varphi(\rho_t^1) - \varphi(\rho_t^2), U_k\rangle_2 \overline{\langle \rho_t^1 - \rho_t^2, U_k\rangle_2} \Big\}. \tag{7.3}$$

Fix $t_0 > 0$. Integrating (7.3) over time, applying identity (7.2), and letting $n \uparrow \infty$, we get

$$\begin{aligned}\int_{t_0}^T \mathrm{d}t \int_\Omega \big[\varphi(\rho_t^1(u)) - \varphi(\rho_t^2(u))\big]\big[\rho_t^1(u) - \rho_t^2(u)\big]\mathrm{d}u &= \lim_{n\to\infty} \frac{1}{2}\Big\{F_n(t_0) - F_n(T)\Big\} \\ &\leq \frac{1}{2\alpha_1} \|\rho_{t_0}^1 - \rho_{t_0}^2\|_2^2\end{aligned}$$

for all $T > t_0$. Since $\rho_{t_0}^1 - \rho_{t_0}^2$ belongs to $L^2(\Omega)$,

$$\int_{t_0}^\infty \mathrm{d}t \int_\Omega \big[\varphi(\rho_t^1(u)) - \varphi(\rho_t^2(u))\big]\big[\rho_t^1(u) - \rho_t^2(u)\big]\mathrm{d}u < \infty.$$

There exists a positive constant C_2 such that, for all $a, b \in [0, 1]$,

$$C_2(b - a)^2 \leq \big(\varphi(b) - \varphi(a)\big)(b - a).$$

On the other hand, by the Schwarz inequality, for all $t \geq t_0$,

$$\|\rho_t^1 - \rho_t^2\|_1^2 \leq 2\|\rho_t^1 - \rho_t^2\|_2^2.$$

Therefore

$$\int_{t_0}^\infty \|\rho_t^1 - \rho_t^2\|_1^2 \mathrm{d}t < \infty$$

and the first statement of the lemma is proved because the integral in $[0, t_0]$ is bounded by $4t_0$. The second statement of the lemma follows from the first one and from Lemma 7.2. □

Proposition 7.7. *There exists a unique weak solution of the boundary value problem (2.2).*

Proof. We start with existence. Let $\rho^1(t,u)$ (resp. $\rho^0(t,u)$) be the weak solution of the boundary value problem (3.2) with initial profile constant equal to 1 (resp. 0). By Lemma 7.4, the sequence of profiles $\{\rho^1(n,\cdot) : n \geq 1\}$ (resp. $\{\rho^0(n,\cdot) : n \geq 1\}$) decreases (resp. increases) to a limit denoted by $\rho^+(\cdot)$ (resp. $\rho^-(\cdot)$). In view of Lemma 7.6, $\rho^+ = \rho^-$ almost surely. Denote this profile by $\bar\rho$ and by $\bar\rho(t,\cdot)$ the solution of (3.2) with initial condition $\bar\rho$. Since $\rho^0(t,\cdot) \leq \bar\rho(\cdot) \leq \rho^1(t,\cdot)$ for all $t \geq 0$, by Lemma 7.4, $\rho^0(t+s,\cdot) \leq \bar\rho(s,\cdot) \leq \rho^1(t+s,\cdot)$ a.e. for all $s, t \geq 0$. Letting $t \uparrow \infty$, we obtain that $\bar\rho(s,\cdot) = \bar\rho(\cdot)$ a.e. for all s. In particular, $\bar\rho$ is a solution of (2.2).

Uniqueness is simpler. Assume that $\rho^1, \rho^2 : \Omega \to [0,1]$ are two weak solutions of (2.2). Then, $\rho^j(t,u) = \rho^j(u)$, $j = 1, 2$, are two stationary weak solutions of (3.2). By Lemma 7.6, $\rho^1 = \rho^2$ almost surely. □

Lemma 7.6 holds for weak solutions of Eq. (2.4) and Proposition 7.7 for the associated elliptic boundary value problem.

References

[1] O. Benois, Large deviations for the occupation times of independent particle systems, Ann. Appl. Probab. 6 (1996) 269–296.

[2] O. Benois, C. Kipnis, C. Landim, Large deviations from the hydrodynamical limit of mean zero asymmetric zero range processes, Stochastic Process. Appl. 55 (1995) 65–89.

[3] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, Macroscopic fluctuation theory for stationary non equilibrium state, J. Stat. Phys. 110 (2002) 635–675.

[4] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, Large deviations for the boundary driven symmetric simple exclusion process, Math. Phys. Anal. Geom. 6 (2003) 231–267.

[5] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, Large deviation approach to non equilibrium processes in stochastic lattice gases, Bull. Braz. Math. Soc., (N.S.) 37 (2006) 611–643.

[6] L. Bertini, C. Landim, M. Mourragui, Dynamical large deviations for the boundary driven weakly asymmetric exclusion process, Ann. Probab 37 (6) (2009) 2357–2403.

[7] T. Bodineau, B. Derrida, Current large deviations for asymmetric exclusion processes with open boundaries, J. Stat. Phys. 123 (2006) 277–300.

[8] B. Derrida, Non equilibrium steady states: fluctuations and large deviations of the density and of the current, J. Stat. Mech. Theory Exp. P07023 (2007).

[9] M.D. Donsker, S.R.S. Varadhan, Large deviations from a hydrodynamic scaling limit, Comm. Pure Appl. Math. XLII (1989) 243–270.

[10] R.E. Edwards, Functional Analysis. Theory and Applications, Holt, Rinehart and Winston, New York, 1965.

[11] G. Eyink, J.L. Lebowitz, H. Spohn, Hydrodynamics of stationary nonequilibrium states for some lattice gas models, Comm. Math. Phys. 132 (1990) 252–283.

[12] G. Eyink, J.L. Lebowitz, H. Spohn, Lattice gas models in contact with stochastic reservoirs: local equilibrium and relaxation to the steady state, Commun. Math. Phys. 140 (1991) 119–131.

[13] C. Kipnis, C. Landim, Hydrodynamic Limit of Interacting Particle Systems, Springer-Verlag, 1999.

[14] C. Kipnis, C. Landim, S. Olla, Macroscopic properties of a stationary nonequilibrium distribution for a non-gradient interacting particle system, Ann. Inst. H. Poincaré 31 (1995) 191–221.

[15] C. Kipnis, S. Olla, S.R.S. Varadhan, Hydrodynamics and large deviations for simple exclusion processes, Commun. Pure Appl. Math. 42 (1989) 115–137.

[16] C. Landim, M. Mourragui, S. Sellami, Hydrodynamical limit for a nongradient interacting particle system with stochastic reservoirs, Teor. Veroyatnost. i Primenen. 45 (2000) 694–717; Translation in Theory Probab. Appl. 45, 604–623 (2002).

[17] C. Landim, S. Olla, S. Volchan, Driven tracer particle in one dimensional symmetric simple exclusion process, Comm. Math. Phys. 192 (1998) 287–307.

[18] M. Mourragui, E. Orlandi, Lattice gas model in random medium and open boundaries: hydrodynamic and relaxation to the steady state, J. Stat. Phys. 136 (2009) 685–714.

[19] J. Quastel, Large deviations from a hydrodynamic scaling limit for a nongradient system, Ann. Probab. 23 (1995) 724–742.

[20] J. Quastel, F. Rezakhanlou, S.R.S Varadhan, Large deviations for the symmetric simple exclusion process in dimension $d \geq 3$, Probab. Theory Related Fields 113 (1999) 1–84.
[21] J. Simon, Compact sets in the space $L^p(0, T; B)$, Ann. Mat. Pura Appl. (4) CXLVI (1987) 65–96.
[22] E. Zeidler, Nonlinear Functional Analysis and its Applications. II/A, in: Linear Monotone Operators and II/B Nonlinear Monotone Operators, Springer-Verlag, Berlin, 1980.