

Transform formulae for linear functionals of affine processes and their bridges on positive semidefinite matrices

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Abstract

In this paper, we derive transform formulae for linear functionals of affine processes and their bridges whose state space is the set of positive semidefinite $d \times d$ matrices. Particularly, we investigate the relationship between such transforms and certain integral equations. Our findings extend and unify the well known results of Cuchiero et al. (2011) [5] and Pitman and Yor (1982) [19], who analysed affine processes on positive semidefinite matrices and transforms of linear functionals of squared Bessel processes, respectively. We are, then, able to derive analytic expressions for Laplace transforms of some functionals of Wishart bridges.

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1. Introduction

Recently, affine Markov processes have drawn much attention from the field of stochastic processes and their financial applications. At the center of this growing interest lies the affine property of the processes which states that their Laplace transforms are exponentially affine in their initial values. The systematic studies on affine processes were initiated by Duffie and Kan [10], and then Duffie et al. [9] gave a complete characterization of time-homogeneous affine processes on $\mathbb{R}_+^m \times \mathbb{R}^n$.

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Cuchiero et al. [5] complement the results of Duffie et al. [9], namely they fully characterized time-homogeneous affine processes on the cone of symmetric positive semidefinite matrices. The affine property for these processes is defined via the Laplace transform of the marginals:

$$\mathbb{E}_x[e^{-\text{tr}(uX_t)}] = e^{-\phi(t,u) - \text{tr}(\psi(t,u)x)},$$

for symmetric positive semidefinite matrices u . The functions ϕ and ψ solve a certain system of nonlinear ordinary differential equations, which are called generalized Riccati differential equations.

The importance of these mathematically interesting processes has also been exhibited through financial applications. In the very complex world of financial derivatives and highly volatile financial markets, the need for models that can handle multiple facets of risks while providing computationally viable formulae has never been higher. In this regard, transform formulae of many existing processes have been hailed by academics as well as practitioners because one can obtain semi-closed form expressions for certain important financial options via integral transforms [11]. Some recent successful applications of affine processes on positive semidefinite matrices include multi-asset option pricing with stochastic covariance [15], fixed-income models with stochastically correlated risk factors [14], and single asset multifactor stochastic volatility modelling [6].

The simplest 1-dimensional affine diffusion process is a classical squared Bessel process. For $x \geq 0$ and $\delta \geq 0$, a δ -dimensional squared Bessel process is defined by a nonnegative weak solution of the stochastic differential equation

$$dX_t = \delta dt + 2\sqrt{X_t}dW_t, \quad \text{with } X_0 = x \geq 0,$$

where W is a 1-dimensional standard Brownian motion. For a positive integer δ , one can easily see $X_t = \|B_t\|^2$, with a δ -dimensional Brownian motion B starting at $B_0 \in \mathbb{R}^\delta$ such that $\|B_0\|^2 = x$, is a δ -dimensional squared Bessel process. In the early 1980s, Pitman and Yor [19] showed that Laplace transforms of certain squared Bessel functionals can be computed via solving Sturm–Liouville equations.

Theorem 1 (Pitman and Yor [19]). *Let X be a δ -dimensional squared Bessel process starting at $x \geq 0$ and κ be a positive Radon measure on $(0, \infty)$ such that, for all $n, \kappa(0, n) < \infty$. Then*

$$\mathbb{E}_x \left[\exp \left\{ - \int_0^\infty X_t \kappa(dt) \right\} \right] = \Phi(\infty)^{\delta/2} \exp \left(\frac{x}{2} \Phi^+(0) \right),$$

where $\Phi(\infty)$ and $\Phi^+(0)$ are respectively the limit at ∞ , and the right derivative at 0 of the unique solution (in the distribution sense) of:

$$\Phi'' = 2\kappa \Phi \quad \text{on } (0, \infty), \quad \Phi(0) = 1, \quad 0 \leq \Phi \leq 1.$$

The above result provides a general recipe for computing Laplace transforms of squared Bessel functionals, and this has been the source of many important formulae, such as Cameron–Martin formula for integrated squared Bessel processes and a transform formula for squared Bessel bridges [19]. The above result of Pitman and Yor has been extended to a subclass of Wishart processes, which are the most fundamental and simplest affine processes on positive semidefinite matrices, by Bru [3] and Donati-Martin [7]. In her paper, Bru [3] proved that an analogous formula holds for Wishart processes of the form

$$dX_t = \delta I_d dt + \sqrt{X_t} dW_t + dW_t^\top \sqrt{X_t}, \quad \text{with } X_0 = x,$$

where x is a $d \times d$ symmetric positive semidefinite matrix, and W is a $d \times d$ matrix of independent standard 1-dimensional Brownian motions, henceforth called a standard $d \times d$ matrix Brownian motion. On the other hand, Donati-Martin [7] found that the same Laplace transforms can be obtained by solving certain quadratic integral equations.

The main contribution of this paper is the extension and unification of the affine transform formula and some other results of Pitman and Yor [19] to a general class of conservative affine processes on positive semidefinite matrices. Specifically, we prove that for a conservative affine process X on positive semidefinite matrices and a positive semidefinite matrix valued measure κ on $(0, T]$, the following formula holds

$$\mathbb{E}_x \left[\exp \left\{ - \int_t^T \text{tr} (X_s \kappa(ds)) \right\} \middle| \mathcal{F}_t \right] = \exp \{ -\phi(t, \kappa) - \text{tr} (\psi(t, \kappa) X_t) \}, \quad (1)$$

where $\phi(\cdot, \kappa)$ and $\psi(\cdot, \kappa)$ are solutions of certain integral equations. Further, we show that those equations always have solutions with some desirable properties, and the solutions are unique under a mild technical condition.

Another contribution of this work is a study on Laplace transforms of form (1) for bridges of conservative affine processes. We find that the transforms for the bridges are closely related to marginal distributions under equivalent probability measures, as well as the functions ϕ and ψ . Then, we obtain some analytic formulae for Wishart bridges. We first derive a transform formula of an integral of a Wishart bridge, which generalizes a formula obtained by Pitman and Yor in the case of Bessel bridges (see (2.m) of Pitman and Yor [19]), and we compute the Laplace transform of the marginal distribution of a Wishart bridge.

Formulae and analysis developed in this paper have potential applications in computational finance. For instance, consider a time-inhomogeneous multifactor short rate model

$$r_t = \text{tr} (g(t) X_t)$$

where $g(t)$ is a deterministic positive semidefinite matrix valued function representing a predictable time effect, and an affine process X_t on positive semidefinite matrices captures volatile risk factors (cf. Buraschi et al. [4]). Assuming the probability measure is risk-neutral, (1) yields the price of a zero-coupon bond with expiry date T :

$$\begin{aligned} P(t, T) &= \mathbb{E}_x \left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right] = \mathbb{E}_x \left[e^{-\int_t^T \text{tr} (g(s) X_s) ds} \middle| \mathcal{F}_t \right] \\ &= \exp \{ -\phi(t, \kappa) - \text{tr} (\psi(t, \kappa) X_t) \}, \end{aligned}$$

where $\kappa(ds) = g(s)ds$. Other standard derivatives, e.g. call and put options on bonds, can also be priced via integral transforms. In order to price such options, it is sufficient to find the \mathcal{F}_t -conditional transforms of X_T under the T -forward measure \mathbb{Q}^T (e.g. see Section 10.3 of Filipović [12]). It can be written as

$$\mathbb{E}_x^{\mathbb{Q}^T} \left[e^{-\text{tr} (u X_T)} \middle| \mathcal{F}_t \right] = \frac{1}{P(t, T)} \mathbb{E}_x \left[e^{-\int_t^T \text{tr} (g(s) X_s) ds - \text{tr} (u X_T)} \middle| \mathcal{F}_t \right].$$

The above transform is nested in (1) with $\kappa(ds) = g(s)ds + u \varepsilon_T(ds)$, where ε_T is the Dirac measure at time T .

This paper is organized as follows: Section 2 provides a brief review of conservative affine processes on positive semidefinite matrices. Section 3 presents transform formulae for conservative affine processes and their bridges, which are the main result of the paper, and proves

the existence and uniqueness of the related integral equations. Section 4 is devoted to Laplace transforms of Wishart functionals and provides some explicit computational results.

1.1. Notation

Let $d \in \mathbb{N}$ denote a dimension. In this paper, we use the following notations:

- M_d : the vector space of $d \times d$ matrices
- S_d : the vector space of $d \times d$ symmetric matrices
- $S_d^+(S_d^{++})$: the cone of $d \times d$ symmetric positive semidefinite (definite) matrices
- \sqrt{x} : the square root of the symmetric positive semidefinite matrix x
- $\chi : S_d \rightarrow S_d$: a bounded truncation function on S_d such that $\chi(\xi) = \xi$ near 0
- $\text{tr}(x)$: the trace of the square matrix x
- $\|x\|$: the norm defined on S_d by $\|x\| = \sqrt{\text{tr}(x^2)}$
- The cones S_d^+ and S_d^{++} induce a partial and strict order relation on S_d , respectively: $x \preceq y$ if and only if $y - x \in S_d^+$, and $x \prec y$ if and only if $y - x \in S_d^{++}$.

Note that $(x, y) \mapsto \text{tr}(xy)$ is an inner-product on S_d . We use the trace notation $\text{tr}(xy)$ instead of inner-product notation $\langle x, y \rangle$ since it is more convenient in the calculations regarding Wishart processes.

2. A brief review on conservative affine processes

In this section, we recall the definition of the conservative affine processes on S_d^+ and some fundamental properties of affine processes. The definitions and theorems in this section are simplified versions of those in the paper of Cuchiero et al. [5].

We consider a time-homogeneous Markov process $(X, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in S_d^+})$ with state space S_d^+ . We denote by \mathbb{E}_x the expectation with respect to \mathbb{P}_x .

Definition 1 (Cuchiero et al. [5]). The Markov process X is called *conservative affine* if

- (i) it is stochastically continuous, and
- (ii) $\mathbb{P}_x(X_t \in S_d^+ \text{ and } X_{t-} \in S_d^+; \forall t \geq 0) = 1$ for all $x \in S_d^+$, and
- (iii) its Laplace transform has exponential-affine dependence on the initial state

$$\mathbb{E}_x[e^{-\text{tr}(uX_t)}] = e^{-\phi(t,u) - \text{tr}(\psi(t,u)x)}, \quad (2)$$

for all $t \in \mathbb{R}_+$ and $u, x \in S_d^+$, for some functions $\phi : \mathbb{R}_+ \times S_d^+ \rightarrow \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \times S_d^+ \rightarrow S_d^+$.

In their paper, Cuchiero et al. [5] provide an equivalent characterization of the affine property in terms of *admissible parameter sets*.

Definition 2 (Cuchiero et al. [5]). An *admissible parameter set* $(\alpha, b, \beta^{ij}, m, \mu)$ associated with the truncation function χ consists of:

- a linear diffusion coefficient : $\alpha \in S_d^+$,
- a constant drift term : $b \succeq (d-1)\alpha$,
- a constant jump term: a Borel measure m on $S_d^+ \setminus \{0\}$ satisfying

$$\int_{S_d^+ \setminus \{0\}} (\|\xi\| \wedge 1) m(d\xi) < \infty,$$

- a linear jump coefficient: a $d \times d$ -matrix $\mu = (\mu_{ij})$ of finite signed measures on $S_d^+ \setminus \{0\}$ such that $\mu(E) \in S_d^+$ for all $E \in \mathcal{B}(S_d^+ \setminus \{0\})$ and the kernel

$$M(x, d\xi) = \frac{\text{tr}(x\mu(d\xi))}{\|\xi\|^2 \wedge 1}$$

satisfies

$$\int_{S_d^+ \setminus \{0\}} \text{tr}(\chi(\xi)u)M(x, d\xi) < \infty \quad \text{for all } x, u \in S_d^+ \text{ with } \text{tr}(xu) = 0,$$

- a linear drift coefficient: a family $\beta^{ij} = \beta^{ji} \in S_d$ such that the linear map $B : S_d \rightarrow S_d$ of the form

$$B(x) = \sum_{ij} \beta^{ij} x_{ij}$$

satisfies

$$\text{tr}(B(x)u) - \int_{S_d^+ \setminus \{0\}} \text{tr}(\chi(\xi)u)M(x, d\xi) \geq 0$$

for all $x, u \in S_d^+$ with $\text{tr}(xu) = 0$.

Theorem 2 (Cuchiero et al. [5]). Let X be a conservative affine process on S_d^+ , and let ϕ and ψ be as in (2). Then there exists an admissible parameter set $(\alpha, b, \beta^{ij}, m, \mu)$ such that $\phi(t, u)$ and $\psi(t, u)$ solve the generalized Riccati differential equations, for $u \in S_d^+$,

$$\frac{\partial \phi(t, u)}{\partial t} = F(\psi(t, u)), \quad \phi(0, u) = 0, \quad (3)$$

$$\frac{\partial \psi(t, u)}{\partial t} = R(\psi(t, u)), \quad \psi(0, u) = u, \quad (4)$$

with

$$F(u) = \text{tr}(bu) - \int_{S_d^+ \setminus \{0\}} (e^{-\text{tr}(u\xi)} - 1)m(d\xi), \quad (5)$$

$$R(u) = -2u\alpha u + B^\top(u) - \int_{S_d^+ \setminus \{0\}} \left(\frac{e^{-\text{tr}(u\xi)} - 1 + \text{tr}(u\chi(\xi))}{\|\xi\|^2 \wedge 1} \right) \mu(d\xi), \quad (6)$$

where $B_{ij}^\top(u) = \text{tr}(\beta^{ij}u)$, i.e., $u \mapsto B^\top(u)$ is the adjoint operator of $x \mapsto B(x)$. Moreover, $\psi(t, 0) \equiv 0$ is the only S_d^+ -valued local solution of (4).

Conversely, let $(\alpha, b, \beta^{ij}, m, \mu)$ be an admissible parameter set such that $\psi(t, 0) \equiv 0$ is the only S_d^+ -valued local solution of (4). Then there exists a unique conservative affine process X on S_d^+ such that the function $\phi(t, u)$ and $\psi(t, u)$ solve Eqs. (3) and (4).

Moreover, they showed that a conservative affine process X is a semimartingale.

Theorem 3 (Cuchiero et al. [5]). Let X be a conservative affine process on S_d^+ and let $(\alpha, b, \beta^{ij}, m, \mu)$ be the related admissible parameter set associated with the truncation function χ . Then X is a semimartingale and there exists, possibly on an enlargement of the probability

space, a standard $d \times d$ -matrix Brownian motion W such that X admits the following representation:

$$\begin{aligned} X_t = & x + B_t + \int_0^t \left(\sqrt{X_s} dW_s \Sigma + \Sigma^\top dW_s^\top \sqrt{X_s} \right) \\ & + \int_0^t \int_{S_d^+ \setminus \{0\}} \chi(\xi) \left(\mu^X(ds, d\xi) - \nu(ds, d\xi) \right) \\ & + \int_0^t \int_{S_d^+ \setminus \{0\}} (\xi - \chi(\xi)) \mu^X(ds, d\xi), \end{aligned} \quad (7)$$

where $\Sigma \in M_d$ satisfies $\Sigma^\top \Sigma = \alpha$, μ^X denotes the random measure associated with the jumps of X , and $\nu(dt, d\xi)$ is the compensator of $\mu^X(dt, d\xi)$ and it satisfies

$$\nu(dt, d\xi) = (m(d\xi) + M(X_t, d\xi)) dt,$$

and

$$B_t = \int_0^t \left(b + \int_{S_d^+ \setminus \{0\}} \chi(\xi) m(d\xi) + B(X_s) \right) ds.$$

Remark 1. Recently, Mayerhofer [17] proved that if $d \geq 2$ then affine processes on S_d^+ have jumps of only finite variation. In this case, the truncation function can be set to identically zero, and Definition 2 and the function R can be simplified. But, for $S_1^+ = \mathbb{R}_+$, affine processes with jumps of infinite total variation actually exist (e.g., see Section 1 of Mayerhofer [17]). In order to not exclude the case \mathbb{R}_+ , we follow the general setting of Cuchiero et al. [5].

3. Main results

Throughout the paper, we consider a conservative affine process X on S_d^+ with the admissible parameter set $(\alpha, b, \beta^{ij}, m, \mu)$ associated with χ . As shown in [5], affine processes with nonzero linear diffusion coefficients are not infinitely decomposable. However, an additivity property holds for affine processes.

Proposition 1. Let \mathbb{P}_x and $\mathbb{Q}_{\tilde{x}}$ be the laws of the conservative affine processes with the admissible parameter sets $(\alpha, b, \beta^{ij}, m, \mu)$ and $(\alpha, \tilde{b}, \beta^{ij}, \tilde{m}, \mu)$. Then their convolution $\mathbb{P}_x * \mathbb{Q}_{\tilde{x}}$ is the law of the affine process with the admissible parameter set $(\alpha, b + \tilde{b}, \beta^{ij}, m + \tilde{m}, \mu)$ and the initial condition $x + \tilde{x}$.

Proof. Recall that the Laplace transform of the convolution of two measures is the same as the product of the Laplace transforms of each measures. In particular, we have for all $u \in S_d^{++}$ and $t \geq 0$,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_x * \mathbb{Q}_{\tilde{x}}} [e^{-\text{tr}(uX_t)}] &= \mathbb{E}^{\mathbb{P}_x} [e^{-\text{tr}(uX_t)}] \mathbb{E}^{\mathbb{Q}_{\tilde{x}}} [e^{-\text{tr}(uX_t)}] \\ &= \exp \left\{ -(\phi(t, u) + \tilde{\phi}(t, u)) - \text{tr} \left(x\psi(t, u) + \tilde{x}\tilde{\psi}(t, u) \right) \right\}, \end{aligned}$$

where $\phi, \tilde{\phi}, \psi$, and $\tilde{\psi}$ satisfy (3) and (4) corresponding to their admissible parameter sets. Since the linear coefficients of the admissible parameter sets for \mathbb{P}_x and $\mathbb{Q}_{\tilde{x}}$ are the same, their R functions are identical. According to Proposition 5.3 of [5], (4) has a unique global solution for

every $u \in S_d^{++}$. Thus $\psi = \tilde{\psi}$ on $\mathbb{R}_+ \times S_d^{++}$. It follows that

$$\mathbb{E}^{\mathbb{P}_x * \mathbb{Q}_{\tilde{x}}} [e^{-\text{tr}(uX_t)}] = \exp \left\{ -(\phi(t, u) + \tilde{\phi}(t, u)) - \text{tr}((x + \tilde{x})\psi(t, u)) \right\}.$$

And $\phi + \tilde{\phi}$ solves (3) corresponding to the admissible parameter $(\alpha, b + \tilde{b}, \beta^{ij}, m + \tilde{m}, \mu)$. Hence the proposition is proved. \square

From Proposition 1, we expect that the Laplace transform of any linear functional of an affine process is exponentially affine in x , b , and m (cf. Theorem 3.1 of Shiga and Watanabe [21] and Theorem 2.1 of Pitman and Yor [19]). We go further in this direction and look for an explicit formula for the Laplace transforms of linear functionals of affine processes. More precisely, we are establishing a formula for

$$\mathbb{E}_x \left[\exp \left\{ - \int_t^T \text{tr}(X_s \kappa(ds)) \right\} \middle| \mathcal{F}_t \right] \quad (8)$$

where $\kappa = (\kappa_{ij})$ is a $d \times d$ matrix of finite signed measures on $(0, T]$ such that $\kappa(E) \in S_d^+$ for all $E \in \mathcal{B}((0, T])$, i.e., κ is an S_d^+ -valued measure on $(0, T]$. Note that if we set $\kappa(dt) = u\varepsilon_T(dt)$, where ε_T is the Dirac measure at T , then (8) reduces to the Laplace transform of X_T . Furthermore, we also consider the Laplace transforms of the bridges

$$\mathbb{E}_x \left[\exp \left\{ - \int_0^T \text{tr}(X_s \kappa(ds)) \right\} \middle| X_T = y \right]. \quad (9)$$

Laplace transforms (8) and (9) are closely related to the following integral equations:

$$\phi(t, \kappa) = \int_t^T F(\psi(s, \kappa)) ds, \quad (10)$$

$$\psi(t, \kappa) = \kappa(t, T] + \int_t^T R(\psi(s, \kappa)) ds, \quad (11)$$

for $0 \leq t \leq T$, where the functions F and R are as in (5) and (6). The remainder of this section is devoted to investigating the relationship among the Laplace transforms (8)–(11).

Remark 2. In case $\kappa(ds) = u\varepsilon_T(ds)$, where $u \in S_d^+$ and $\varepsilon_T(ds)$ is the Dirac measure at T , the solutions $\phi(\cdot, u\varepsilon_T)$ and $\psi(\cdot, u\varepsilon_T)$ of integral equations (10) and (11) are equivalent to the solutions $\phi(T - \cdot, u)$ and $\psi(T - \cdot, u)$ of differential equations (3) and (4). Our main transform formula (17) reduces to the affine transform formula (2). For this reason, we use the same notation for slightly different functions.

3.1. Existence and uniqueness of the solution of (10) and (11)

This subsection establishes the existence and uniqueness of the solutions of (10) and (11). Notice that $\phi(\cdot, \kappa)$ is uniquely determined by $\psi(\cdot, \kappa)$, and $\phi(\cdot, \kappa) \geq 0$ if $\psi(\cdot, \kappa) \in S_d^+$. Thus we consider only (11).

In order to utilize classical results on ordinary differential equations, we first consider the following differential equation

$$-\frac{d}{dt}\psi(t) = g(t) + R(\psi(t)), \quad \text{for } 0 \leq t \leq T, \quad (12)$$

with a terminal condition $\psi(T) = u \in S_d^{++}$. The key tool for proving the existence of S_d^{++} -valued solution to Eq. (12) is the Volkmann's comparison theorem [5]. In order to state the Volkmann's comparison theorem, we need to review the notion of quasi-monotonicity.

Definition 3. Let $U \subset S_d$ be an open set. A function $f : U \rightarrow S_d$ is called quasi-monotone increasing if for all $x, y \in U, u \in S_d^+$ which satisfy $x \leq y$ and $\text{tr}(xu) = \text{tr}(yu)$,

$$\text{tr}(f(x)u) \leq \text{tr}(f(y)u)$$

holds true.

Since (12) is a backward equation, we reformulate Volkmann's comparison theorem (Theorem 4.8 of Cuchiero et al. [5]) in a backward form.

Theorem 4. Let $U \subset S_d$ be an open set. Let $f : (0, T] \times U \rightarrow S_d$ be a continuous locally Lipschitz map such that $f(t, \cdot)$ is quasi-monotone increasing on U for all $t \in (0, T]$. Let $0 \leq t_0 < T$ and $x, y : (t_0, T] \rightarrow U$ be differentiable maps such that $x(T) \geq y(T)$ and

$$-\frac{d}{dt}x(t) - f(t, x(t)) \geq -\frac{d}{dt}y(t) - f(t, y(t)), \quad \text{for all } t \in (t_0, T].$$

Then we have $x(t) \geq y(t)$ for all $t \in (t_0, T]$.

We recall some properties of the function R , which is used in the proof of the existence and the uniqueness of the solution of (12). For the proofs of the properties, refer to Cuchiero et al. [5].

Proposition 2 (Cuchiero et al. [5]). The function R enjoys the following properties:

- (i) R is analytic on S_d^{++} ,
- (ii) R is quasi-monotone increasing on S_d^+ ,
- (iii) there exists a constant $K > 0$ such that

$$\text{tr}(uR(u)) \leq \frac{K}{2}(\|u\|^2 + 1), \quad u \in S_d^+. \quad (13)$$

To show the existence and uniqueness of the solution of (11), we need two lemmas that show the existence of unique solutions for equations relevant to (11) and (12).

Lemma 1. Let $g : [0, T] \rightarrow S_d^+$ be a Lipschitz continuous function. Then (12) with the terminal condition $\psi(T) = u \in S_d^{++}$ has a unique solution on $[0, T]$. Moreover there exists a constant $\tilde{K} > 0$ which depends only on $u, \int_0^T \|g(t)\| dt, T$ and K in (13) such that

$$\sup_{t \in [0, T]} \|\psi(t)\| \leq \tilde{K},$$

and there exists a constant $\epsilon > 0$ which depends only on u, T , and the function R (not on g) such that

$$\psi(t) \geq \epsilon I_d, \quad \text{for all } 0 \leq t \leq T.$$

Proof. Since $u \mapsto g(t) + R(u)$ is locally Lipschitz continuous on S_d^{++} and the Lipschitz constant can be chosen uniformly in $t \in [0, T]$, standard ODE results (e.g. II.6.VII Existence and Uniqueness Theorem in [22]) yield that (12) has a unique maximal S_d^{++} -valued solution ψ

defined on $(t_0, T]$ with the maximal lifetime $t_0 \geq 0$ in backward such that

$$t_0 = 0, \quad \lim_{t \downarrow t_0} \|\psi(t)\| = \infty, \quad \text{or} \quad \lim_{t \downarrow t_0} \psi(t) \in \partial S_d^+.$$

By Proposition 5.3 of [5],

$$-\frac{d}{dt} \tilde{\psi}(t) = R(\tilde{\psi}(t)), \quad \tilde{\psi}(T) = u$$

has a unique S_d^{++} -valued solution on $[0, T]$. Since $g(\cdot)$ is S_d^+ -valued,

$$-\frac{d}{dt} \psi(t) - R(\psi(t)) = g(t) \geq 0 = -\frac{d}{dt} \tilde{\psi}(t) - R(\tilde{\psi}(t))$$

holds for all $t \in (t_0, T]$. It follows from Theorem 4 and Proposition 2 that $\psi(t) \geq \tilde{\psi}(t)$ on $(t_0, T]$. Since $t \mapsto \tilde{\psi}(t)$ is continuous, $\{\tilde{\psi}(t) : t \in [0, T]\}$ is compact. Notice that $(\epsilon I_d + S_d^{++})_{\epsilon > 0}$ is an open covering of $\{\tilde{\psi}(t) : t \in [0, T]\}$. Therefore, there exists $\epsilon > 0$ which depends only on u, T , and the function R such that

$$\psi(t) \geq \tilde{\psi}(t) \succ \epsilon I_d, \quad \text{for all } 0 \leq t \leq T.$$

Now, we proceed to prove that $\psi(\cdot)$ is bounded on $(t_0, T]$, which leads to $t_0 = 0$. By applying integration by parts, we obtain

$$\begin{aligned} \|\psi(t)\|^2 &= \text{tr}(\psi(t)\psi(t)) \\ &= \|u\|^2 + 2 \int_t^T \text{tr}(\psi(s)g(s)) ds + 2 \int_t^T \text{tr}(\psi(s)R(\psi(s))) ds \\ &\leq \|u\|^2 + \int_t^T (2\|g(s)\| + K) (\|\psi(s)\|^2 + 1) ds \\ &\leq \hat{K} + \int_t^T \eta(s) \|\psi(s)\|^2 ds, \end{aligned}$$

where $\eta(s) = 2\|g(s)\| + K$ and $\hat{K} = \|u\|^2 + \int_0^T \eta(s) ds$. By Gronwall's inequality, we have

$$\|\psi(t)\|^2 \leq \hat{K} + \hat{K} \int_t^T \eta(s) e^{\int_t^s \eta(r) dr} ds.$$

Hence $\psi(\cdot)$ is bounded and $\tilde{K} > 0$ can be chosen by

$$\tilde{K}^2 = \hat{K} \left(1 + e^{\int_0^T \eta(s) ds} \int_0^T \eta(s) ds \right). \quad \square$$

The following lemma proves the existence and uniqueness of the solution of (11) under the assumption $\kappa\{T\} \in S_d^{++}$. The idea of proof is to regularize (11) and then to apply Lemma 1.

Lemma 2. Let κ be an S_d^+ -valued measure on $(0, T]$ with $\kappa\{T\} \in S_d^{++}$. Then there exists a unique bounded S_d^{++} -valued solution $\psi(\cdot, \kappa)$ of (11). Moreover, if $\hat{\kappa}$ is another S_d^+ -valued measure satisfying $\kappa(A) = \hat{\kappa}(A)$ for $A \in \mathcal{B}(0, T)$ and $\kappa\{T\} \geq \hat{\kappa}\{T\} \in S_d^{++}$, then $\psi(t, \kappa) \geq \psi(t, \hat{\kappa})$ for all $0 \leq t \leq T$ where $\psi(\cdot, \hat{\kappa})$ is the solution of (11) corresponding to $\hat{\kappa}$.

Proof. Let f be a nonnegative real valued C^1 function whose support is contained in $[-1, 0]$ and $\int_{-1}^0 f(s)ds = 1$. We set $f_n(s) = nf(ns)$,

$$g_n(t) = \int_{(0,T)} f_n(t-s)\kappa(ds), \quad \kappa_n(dt) = g_n(t)dt,$$

for $n = 1, 2, \dots$. Then g_1, g_2, \dots are C^1 functions and

$$\lim_{n \rightarrow \infty} \kappa_n(t, T) = \kappa(t, T) \quad \text{for all } t \in [0, T]. \quad (14)$$

And for all $n = 1, 2, \dots$,

$$\begin{aligned} \int_0^T \|g_n(t)\|dt &\leq \int_0^T \int_{(0,T)} f_n(t-s)|\kappa|(ds)dt \\ &\leq \int_{(0,T)} \int_0^T f_n(t-s)dt|\kappa|(ds) \leq |\kappa|(0, T), \end{aligned} \quad (15)$$

where $|\kappa|(ds) = \sum_{ij} |\kappa_{ij}|(ds)$ and $|\kappa_{ij}|(ds)$ is the total variation of the signed measure $\kappa_{ij}(ds)$. By Lemma 1, there exist functions $\psi_1, \psi_2, \dots : [0, T] \rightarrow S_d^{++}$ such that

$$-\frac{d}{dt}\psi_n(t) = g_n(t) + R(\psi_n(t)) \quad \text{on } [0, T], \quad \psi_n(T) = \kappa\{T\}, \quad (16)$$

or, equivalently,

$$\psi_n(t) = \kappa\{T\} + \kappa_n(t, T) + \int_t^T R(\psi_n(s))ds, \quad t \in [0, T].$$

The sequence ψ_1, ψ_2, \dots is uniformly bounded away from both infinity and the boundary ∂S_d^+ on $[0, T]$ by Lemma 1 and (15). In other words, $\{\psi_n(t) : n \geq 1, 0 \leq t \leq T\}$ is contained in a compact subset of S_d^{++} . Since R is locally Lipschitz continuous on S_d^{++} , there exists $C > 0$ such that for all m, n and $t \in [0, T]$,

$$\|R(\psi_m(t)) - R(\psi_n(t))\| \leq C\|\psi_m(t) - \psi_n(t)\|.$$

It follows that for all $m \geq n$ and $t \in [0, T]$,

$$\|\psi_m(t) - \psi_n(t)\| \leq \|\kappa_m(t, T) - \kappa_n(t, T)\| + C \int_t^T \|\psi_m(s) - \psi_n(s)\|ds.$$

From Gronwall's inequality, we deduce that for all $m \geq n$ and $t \in [0, T]$,

$$\|\psi_m(t) - \psi_n(t)\| \leq \|\kappa_m(t, T) - \kappa_n(t, T)\| + C \int_t^T \|\kappa_m(s, T) - \kappa_n(s, T)\|e^{C(s-t)}ds.$$

Hence $(\psi_n(t))_{n \geq 1}$ is a Cauchy sequence and we set $\psi(t, \kappa) = \lim_{n \rightarrow \infty} \psi_n(t)$. The dominated convergence theorem and (14) show that $\psi(\cdot, \kappa)$ is a solution of (11). The uniqueness is an immediate consequence of the local Lipschitzness of $u \mapsto R(u)$ on S_d^{++} .

For the second assertion, we repeat the construction of a sequence $\hat{\psi}_1, \hat{\psi}_2, \dots$ for the measure $\hat{\kappa}$. Since $\kappa(A) = \hat{\kappa}(A)$ for all $A \in \mathcal{B}(0, T)$, the resulting differential equations are identical to (16) except terminal condition $\hat{\psi}_n(T) = \hat{\kappa}\{T\}$. Therefore, $\psi_n(T) \succeq \hat{\psi}_n(T)$ and

$$-\frac{d}{dt}\psi_n(t) - R(\psi_n(t)) = g_n(t) = -\frac{d}{dt}\hat{\psi}_n(t) - R(\hat{\psi}_n(t))$$

for all $n \geq 1$ and $0 \leq t \leq T$. Then we apply [Theorem 4](#) to have $\psi_n(t) \geq \hat{\psi}_n(t)$ for all $n \geq 1$ and $0 \leq t \leq T$. Hence, the same order relation holds for their limits, i.e., $\psi(t, \kappa) \geq \psi(t, \hat{\kappa})$ for $0 \leq t \leq T$. \square

In the case of $\kappa\{T\} \in \partial S_d^+$, the uniqueness of the solution of [\(11\)](#) does not hold in general. Nevertheless, it has a S_d^+ -valued solution.

Theorem 5. For every S_d^+ -valued measure κ on $(0, T]$, there exists a S_d^+ -valued solution of [\(11\)](#).

Proof. We set $\kappa_n(dt) = \frac{1}{n}I_d\varepsilon_T(dt) + \kappa(dt)$ for $n = 1, 2, \dots$. Then

$$\kappa_n\{T\} \geq \kappa_m\{T\} \in S_d^{++}, \quad \kappa_n(A) = \kappa_m(A),$$

for $1 \leq n \leq m$ and $A \in \mathcal{B}(0, T)$. Therefore, by [Lemma 2](#), there exist $\psi(\cdot, \kappa_1), \psi(\cdot, \kappa_2), \dots$ such that $\psi(t, \kappa_n) \geq \psi(t, \kappa_m) \geq 0$ and

$$\psi(t, \kappa_n) = \kappa_n(t, T] + \int_t^T R(\psi(s, \kappa_n))ds,$$

for all $n \leq m$ and $0 \leq t \leq T$. For each $t \in [0, T]$, $\{\psi(t, \kappa_n)\}_{n \geq 1}$ converges to a limit in S_d^+ because $\{\psi(t, \kappa_n)\}_{n \geq 1}$ is a nonincreasing sequence in S_d^+ . Put $\psi(t, \kappa) = \lim_{n \rightarrow \infty} \psi(t, \kappa_n)$ for $0 \leq t \leq T$. Note that R is continuous on S_d^+ and the sequence $\{\psi(\cdot, \kappa_n)\}_{n \geq 1}$ is uniformly bounded. Hence, by the dominated convergence theorem, $\psi(\cdot, \kappa)$ is a solution of [\(11\)](#). \square

Remark 3. If the function R is locally Lipschitz continuous on S_d^+ , the S_d^+ -valued solution of [\(11\)](#) uniquely exists. A sufficient condition for Lipschitz continuity of R is

$$\int_{S_d^+ \cap \{\|\xi\| \geq 1\}} \|\xi\| |\mu_{ij}|(d\xi) < \infty \quad \text{for all } 1 \leq i \leq j \leq d.$$

Lipschitz continuity of R on S_d^+ can be shown similarly as in Section 9 of Duffie et al. [\[9\]](#).

3.2. Main Laplace transform formulae

This subsection gives the formulae for Laplace transforms [\(8\)](#) and [\(9\)](#) in terms of the solutions of [\(10\)](#) and [\(11\)](#). We start with the formula for [\(8\)](#) which extends the affine transform formula [\(2\)](#) in a natural way.

Theorem 6. Let X be a conservative affine process on S_d^+ and let $(\alpha, b, \beta^{ij}, m, \mu)$ be the related admissible parameter set associated with the truncation function χ . Then for all S_d^+ -valued measures κ on $(0, T]$, $x \in S_d^+$, and $0 \leq t \leq T$, we have

$$\mathbb{E}_x \left[\exp \left\{ - \int_t^T \text{tr}(X_s \kappa(ds)) \right\} \middle| \mathcal{F}_t \right] = \exp \{ -\phi(t, \kappa) - \text{tr}(\psi(t, \kappa)X_t) \}, \quad (17)$$

where $(\phi(\cdot, \kappa), \psi(\cdot, \kappa))$ is an $\mathbb{R}_+ \times S_d^+$ -valued solution on $[0, T]$ to [\(10\)](#) and [\(11\)](#).

Proof. According to [Theorem 3](#), X is a semimartingale and it has the representation [\(7\)](#). Since $\psi(\cdot, \kappa)$ is a deterministic finite variation function (so, it is predictable), Proposition I.4.49(b) of Jacod and Shiryaev [\[16\]](#) yields

$$X_t \psi(t, \kappa) = x \psi(0, \kappa) + \int_0^t X_s d\psi(s, \kappa) + \int_0^t dX_s \psi(s, \kappa).$$

Taking the trace of both sides,

$$\mathrm{tr}(X_t \psi(t, \kappa)) = \mathrm{tr}(x \psi(0, \kappa)) + \int_0^t \mathrm{tr}(X_s d\psi(s, \kappa)) + \int_0^t \mathrm{tr}(dX_s \psi(s, \kappa)).$$

By Theorem 3 and (11),

$$\begin{aligned} \mathrm{tr}(X_t \psi(t, \kappa)) &= \mathrm{tr}(x \psi(0, \kappa)) - \int_0^t \mathrm{tr}(X_{s-\kappa}(ds)) - \int_0^t \mathrm{tr}(X_s R(\psi(s, \kappa)))ds \\ &\quad + \int_0^t \mathrm{tr}(dB_s \psi(s, \kappa)) + \int_0^t \mathrm{tr}(dX_s^c \psi(s, \kappa)) \\ &\quad + \int_0^t \int_{S_d^+ \setminus \{0\}} \mathrm{tr}(\chi(\xi) \psi(s, \kappa))(\mu^X - \nu)(ds, d\xi) \\ &\quad + \int_0^t \int_{S_d^+ \setminus \{0\}} \mathrm{tr}((\xi - \chi(\xi)) \psi(s, \kappa))\mu^X(ds, d\xi), \end{aligned} \quad (18)$$

where $X^c = \int_0^\cdot (\sqrt{X_s} dW_s \Sigma + \Sigma^\top dW_s^\top \sqrt{X_s})$ is the continuous martingale part of X . Note that $\mathrm{tr}(uB^\top(v)) = \mathrm{tr}(B(u)v)$ for all $u, v \in S_d$. Therefore, by the definition of R, F, B_t , and $\nu(dt, d\xi)$, we have

$$\begin{aligned} &\int_0^t \mathrm{tr}(X_s R(\psi(s, \kappa)))ds - \int_0^t \mathrm{tr}(dB_s \psi(s, \kappa)) \\ &= -2 \int_0^t \mathrm{tr}(X_s \psi(s, \kappa) \alpha \psi(s, \kappa))ds + \int_0^t \mathrm{tr}(X_s B^\top(\psi(s, \kappa)))ds \\ &\quad - \int_0^t \int_{S_d^+ \setminus \{0\}} \left(e^{-\mathrm{tr}(\xi \psi(s, \kappa))} - 1 + \mathrm{tr}(\chi(\xi) \psi(s, \kappa)) \right) M(X_s, d\xi)ds \\ &\quad - \int_0^t \mathrm{tr}(b \psi(s, \kappa))ds - \int_0^t \int_{S_d^+ \setminus \{0\}} \mathrm{tr}(\chi(\xi) \psi(s, \kappa))m(d\xi)ds \\ &\quad - \int_0^t \mathrm{tr}(B(X_s) \psi(s, \kappa))ds \\ &= -2 \int_0^t \mathrm{tr}(X_s \psi(s, \kappa) \alpha \psi(s, \kappa))ds \\ &\quad - \int_0^t \int_{S_d^+ \setminus \{0\}} \left(e^{-\mathrm{tr}(\xi \psi(s, \kappa))} - 1 + \mathrm{tr}(\chi(\xi) \psi(s, \kappa)) \right) \nu(ds, d\xi) \\ &\quad - \int_0^t \mathrm{tr}(b \psi(s, \kappa))ds + \int_0^t \int_{S_d^+ \setminus \{0\}} \left(e^{-\mathrm{tr}(\xi \psi(s, \kappa))} - 1 \right) m(d\xi)ds \\ &= -2 \int_0^t \mathrm{tr}(X_s \psi(s, \kappa) \alpha \psi(s, \kappa))ds + \phi(t, \kappa) - \phi(0, \kappa) \\ &\quad - \int_0^t \int_{S_d^+ \setminus \{0\}} \left(e^{-\mathrm{tr}(\xi \psi(s, \kappa))} - 1 + \mathrm{tr}(\chi(\xi) \psi(s, \kappa)) \right) \nu(ds, d\xi). \end{aligned}$$

Substituting the above quantity into (18) and rearranging the terms, we have

$$Y_t^\kappa := -\mathrm{tr}(X_t \psi(t, \kappa)) + \mathrm{tr}(x \psi(0, \kappa)) + \phi(0, \kappa) - \phi(t, \kappa) - \int_0^t \mathrm{tr}(X_{s-\kappa}(ds))$$

$$\begin{aligned}
&= - \int_0^t \text{tr} (dX_s^c \psi(s, \kappa)) - 2 \int_0^t \text{tr} (X_s \psi(s, \kappa) \alpha \psi(s, \kappa)) ds \\
&\quad - \int_0^t \int_{S_d^+ \setminus \{0\}} \text{tr} (\chi(\xi) \psi(s, \kappa)) (\mu^X - \nu)(ds, d\xi) \\
&\quad - \int_0^t \int_{S_d^+ \setminus \{0\}} \text{tr} ((\xi - \chi(\xi)) \psi(s, \kappa)) \mu^X(ds, d\xi) \\
&\quad - \int_0^t \int_{S_d^+ \setminus \{0\}} \left(e^{-\text{tr}(\xi \psi(s, \kappa))} - 1 + \text{tr}(\chi(\xi) \psi(s, \kappa)) \right) \nu(ds, d\xi).
\end{aligned}$$

We take, for $0 \leq t \leq T$,

$$\begin{aligned}
Z_t^\kappa = e^{Y_t^\kappa} = \exp \Big\{ & - \text{tr} (X_t \psi(t, \kappa)) + \text{tr} (x \psi(0, \kappa)) + \phi(0, \kappa) - \phi(t, \kappa) \\
& - \int_0^t \text{tr} (X_{s-} \kappa(ds)) \Big\}.
\end{aligned}$$

By Itô's formula, we have

$$Z_t^\kappa = 1 + \int_0^t Z_{s-}^\kappa dY_s^\kappa + \frac{1}{2} \int_0^t Z_{s-}^\kappa d[Y^\kappa, Y^\kappa]_s^c + \sum_{0 < s \leq t} \{Z_s^\kappa - Z_{s-}^\kappa - Z_{s-}^\kappa \Delta Y_s^\kappa\}. \quad (19)$$

Here, $[Y^\kappa, Y^\kappa]_t^c = 4 \int_0^t \text{tr} (X_s \psi(s, \kappa) \alpha \psi(s, \kappa)) ds$, and

$$\begin{aligned}
\sum_{0 < s \leq t} \{Z_s^\kappa - Z_{s-}^\kappa - Z_{s-}^\kappa \Delta Y_s^\kappa\} &= \sum_{0 < s \leq t} Z_{s-}^\kappa (e^{\Delta Y_s^\kappa} - 1 - \Delta Y_s^\kappa) \\
&= \sum_{0 < s \leq t} Z_{s-}^\kappa \left(e^{-\text{tr}(\Delta X_s \psi(s, \kappa))} - 1 + \text{tr}(\Delta X_s \psi(s, \kappa)) \right) \\
&= \int_0^t \int_{S_d^+ \setminus \{0\}} Z_{s-}^\kappa \left(e^{-\text{tr}(\xi \psi(s, \kappa))} - 1 + \text{tr}(\xi \psi(s, \kappa)) \right) \mu^X(ds, d\xi) \\
&= \int_0^t \int_{S_d^+ \setminus \{0\}} Z_{s-}^\kappa \left(e^{-\text{tr}(\xi \psi(s, \kappa))} - 1 + \text{tr}(\chi(\xi) \psi(s, \kappa)) \right) \mu^X(ds, d\xi) \\
&\quad + \int_0^t \int_{S_d^+ \setminus \{0\}} Z_{s-}^\kappa \text{tr}((\xi - \chi(\xi)) \psi(s, \kappa)) \mu^X(ds, d\xi).
\end{aligned}$$

By substituting the above quantities into (19), we have

$$\begin{aligned}
Z_t^\kappa &= 1 - \int_0^t Z_s^\kappa \text{tr} (dX_s^c \psi(s, \kappa)) - \int_0^t \int_{S_d^+ \setminus \{0\}} Z_{s-}^\kappa \text{tr} (\chi(\xi) \psi(s, \kappa)) (\mu^X - \nu)(ds, d\xi) \\
&\quad + \int_0^t \int_{S_d^+ \setminus \{0\}} Z_{s-}^\kappa \left(e^{-\text{tr}(\xi \psi(s, \kappa))} - 1 + \text{tr}(\chi(\xi) \psi(s, \kappa)) \right) (\mu^X - \nu)(ds, d\xi) \\
&= 1 - \int_0^t Z_s^\kappa \text{tr} (dX_s^c \psi(s, \kappa)) \\
&\quad + \int_0^t \int_{S_d^+ \setminus \{0\}} Z_{s-}^\kappa \left(e^{-\text{tr}(\xi \psi(s, \kappa))} - 1 \right) (\mu^X - \nu)(ds, d\xi).
\end{aligned}$$

Therefore, Z_t^κ is a local martingale. Indeed, it is a bounded martingale because $(\phi(\cdot, \kappa), \psi(\cdot, \kappa))$ is $\mathbb{R}_+ \times S_d^+$ -valued. It follows from $\psi(T, \kappa) = 0$ and $\mathbb{E}_x[Z_T^\kappa | \mathcal{F}_t] = Z_t^\kappa$ that

$$\mathbb{E}_x \left[\exp \left\{ - \int_t^T \text{tr} (X_{s-\kappa}(ds)) \right\} | \mathcal{F}_t \right] = \exp \{ -\phi(t, \kappa) - \text{tr} (\psi(t, \kappa) X_t) \}.$$

Since the affine process X is stochastically continuous, $\Delta X_s = 0$ \mathbb{P}_x -a.s. for all $s \geq 0$. Notice that $\kappa\{s\} = 0$ for all but countably many s . Therefore,

$$\exp \left\{ - \int_t^T \text{tr} (X_s \kappa(ds)) \right\} = \exp \left\{ - \int_t^T \text{tr} (X_{s-\kappa}(ds)) \right\}, \quad \mathbb{P}_x\text{-a.s.}$$

Hence the theorem is proved. \square

Remark 4. For conservative affine processes with special admissible parameter set $(I_d, \delta I_d, 0, 0, 0)$, a transform formula, which is a special case of (17), was already known in Donati-Martin [7]. In this case, the existence and uniqueness of the solution to (11) can be easily shown by diagonalizing the solution ψ .

For the remainder of this subsection, we consider Laplace transform (9) of the bridges. In order to find a formula for (9), we apply the change of measure techniques (e.g., see section XI.3 of Revuz and Yor [20]). We consider a fixed S_d^+ -valued measure κ on $(0, T]$, and let (ϕ, ψ) be a fixed solution of (10) and (11) corresponding to κ . In the proof of Theorem 6, we verified that

$$Z_t^\kappa = \exp \left\{ - \text{tr} (X_t \psi(t, \kappa)) + \text{tr} (x \psi(0, \kappa)) + \phi(0, \kappa) - \phi(t, \kappa) - \int_0^t \text{tr} (X_{s-\kappa}(ds)) \right\},$$

is a martingale. We define a probability measure \mathbb{P}_x^κ on \mathcal{F}_T by

$$\frac{d\mathbb{P}_x^\kappa}{d\mathbb{P}_x} = Z_T^\kappa. \quad (20)$$

We write expectation with respect to \mathbb{P}_x^κ by \mathbb{E}_x^κ . The following proposition shows that process X still has the affine property under \mathbb{P}_x^κ .

Proposition 3. *The process X is a time-inhomogeneous affine process under the probability measure \mathbb{P}_x^κ in the sense that the Laplace transform under \mathbb{P}_x^κ is given by, for $u \in S_d^+$,*

$$\mathbb{E}_x^\kappa [e^{-\text{tr}(u X_t)} | \mathcal{F}_s] = e^{-\phi_\kappa(s, t, u) - \text{tr}(\psi_\kappa(s, t, u) X_s)}, \quad 0 \leq s \leq t \leq T,$$

where $\phi_\kappa(\cdot, \cdot, \cdot)$ and $\psi_\kappa(\cdot, \cdot, \cdot)$ solves the following differential equations

$$-\frac{\partial}{\partial s} \phi_\kappa(s, t, u) = F_\kappa(s, \psi_\kappa(s, t, u)), \quad \phi_\kappa(t, t, u) = 0, \quad (21)$$

$$-\frac{\partial}{\partial s} \psi_\kappa(s, t, u) = R_\kappa(s, \psi_\kappa(s, t, u)), \quad \psi_\kappa(t, t, u) = u, \quad (22)$$

for $0 \leq s \leq t \leq T$. Here, $F_\kappa : \mathbb{R}_+ \times S_d \rightarrow \mathbb{R}$ and $R_\kappa : \mathbb{R}_+ \times S_d \rightarrow S_d$ are defined by

$$F_\kappa(s, u) = \text{tr}(bu) - \int_{S_d^+ \setminus \{0\}} (e^{-\text{tr}(u\xi)} - 1) m_\kappa(s, d\xi),$$

$$R_{\kappa}(s, u) = -2u\alpha u + B_{\kappa}^{\top}(s, u) - \int_{S_d^+ \setminus \{0\}} \left(\frac{e^{-\text{tr}(u\xi)} - 1 + \text{tr}(u\chi_{\kappa}(s, \xi))}{\|\xi\|^2 \wedge 1} \right) \mu_{\kappa}(s, d\xi),$$

where $\text{tr}(B_{\kappa}^{\top}(s, u)x) = \text{tr}(uB_{\kappa}(s, x))$, with the modified time-dependent parameters and truncation function

$$\begin{aligned} m_{\kappa}(s, d\xi) &= e^{-\text{tr}(\xi\psi(s, \kappa))} m(d\xi), & \mu_{\kappa}(s, d\xi) &= e^{-\text{tr}(\xi\psi(s, \kappa))} \mu(d\xi), \\ \chi_{\kappa}(s, \xi) &= e^{\text{tr}(\xi\psi(s, \kappa))} \chi(\xi), & B_{\kappa}(s, x) &= B(x) - 2\alpha\psi(s, \kappa)x - 2x\psi(s, \kappa)\alpha. \end{aligned}$$

Proof. Let $t \in [0, T]$ and $u \in S_d^+$ be fixed. Observe that

$$\begin{aligned} \mathbb{E}_x^{\kappa}[e^{-\text{tr}(uX_t)} | \mathcal{F}_s] &= \frac{1}{Z_s^{\kappa}} \mathbb{E}_x[Z_t^{\kappa} e^{-\text{tr}(uX_t)} | \mathcal{F}_s] \\ &= \exp\{\text{tr}(\psi(s, \kappa)X_s) + \phi(s, \kappa) - \phi(t, \kappa)\} \\ &\quad \times \mathbb{E}_x\left[\exp\left\{-\int_s^t \text{tr}(X_r \tilde{\kappa}(dr))\right\} \middle| \mathcal{F}_s\right], \end{aligned}$$

where $\tilde{\kappa}(dr) = (u + \psi(t, \kappa))\varepsilon_t(dr) + \mathbb{1}_{(0, t]}(r)\kappa(dr)$. As indicated in the proof of [Theorem 6](#),

$$\exp\left\{-\int_s^t \text{tr}(X_r \tilde{\kappa}(dr))\right\} = \exp\left\{-\int_s^t \text{tr}(X_r \tilde{\kappa}(dr))\right\}, \quad \mathbb{P}_x\text{-a.s.}$$

By [Theorem 6](#), we have

$$\mathbb{E}_x\left[\exp\left\{-\int_s^t \text{tr}(X_r \tilde{\kappa}(dr))\right\} \middle| \mathcal{F}_s\right] = \exp\{-\phi(s, \tilde{\kappa}) - \text{tr}(\psi(s, \tilde{\kappa})X_s)\},$$

for $0 \leq s \leq t$. Therefore,

$$\begin{aligned} \mathbb{E}_x^{\kappa}[e^{-\text{tr}(uX_t)} | \mathcal{F}_s] &= \exp\{-\phi(s, \tilde{\kappa}) + \phi(t, \kappa) - \phi(s, \kappa) \\ &\quad - \text{tr}((\psi(s, \tilde{\kappa}) - \psi(s, \kappa))X_s)\}. \end{aligned}$$

We set $\phi_{\kappa}(s, t, u) = \phi(s, \tilde{\kappa}) + \phi(t, \kappa) - \phi(s, \kappa)$ and $\psi_{\kappa}(s, t, u) = \psi(s, \tilde{\kappa}) - \psi(s, \kappa)$. It remains to show that $\phi_{\kappa}(s, t, u)$ and $\psi_{\kappa}(s, t, u)$ satisfy [\(21\)](#) and [\(22\)](#). Note that

$$\begin{aligned} \psi_{\kappa}(s, t, u) &= u + \psi(t, \kappa) + \kappa(s, t] + \int_s^t R(\psi(r, \tilde{\kappa}))dr - \psi(s, \kappa) \\ &= u + \int_s^t (R(\psi(r, \tilde{\kappa})) - R(\psi(r, \kappa)))dr \\ &= u + \int_s^t R_{\kappa}(r, \psi(r, \tilde{\kappa}) - \psi(r, \kappa))dr \\ &= u + \int_s^t R_{\kappa}(r, \psi_{\kappa}(r, t, u))dr, \end{aligned}$$

and

$$\begin{aligned} \phi_{\kappa}(s, t, u) &= \int_s^t (F(\psi(r, \tilde{\kappa})) - F(\psi(r, \kappa)))dr \\ &= \int_s^t F_{\kappa}(r, \psi(r, \tilde{\kappa}) - \psi(r, \kappa))dr \\ &= \int_s^t F_{\kappa}(r, \psi_{\kappa}(r, t, u))dr. \quad \square \end{aligned}$$

The next and final proposition of this section gives a formula for Laplace transform (9) in terms of ψ , ϕ , and transition kernels under \mathbb{P}_x and \mathbb{P}_x^κ . The transition kernels of general affine processes are not known in explicit form, but they can be derived for some special affine processes including Wishart processes, as we shall see in the next section.

Proposition 4. *Let X be a conservative affine process on S_d^+ and let $(\alpha, b, \beta^{ij}, m, \mu)$ be the related admissible parameter set associated with the truncation function χ . Then for all S_d^+ -valued measures κ on $(0, T]$ and for all $x \in S_d^+$ we have*

$$\begin{aligned} & \mathbb{E}_x \left[\exp \left\{ - \int_0^T \text{tr} (X_s \kappa(ds)) \right\} \middle| X_T = y \right] \\ &= \exp \{ -\phi(0, \kappa) - \text{tr} (\psi(0, \kappa)x) \} \frac{p_{0,T}^\kappa(x, dy)}{p_{0,T}(x, dy)}, \end{aligned} \quad (23)$$

$p_{0,T}(x, dy)$ -a.s., where (ϕ, ψ) is an $\mathbb{R}_+ \times S_d^+$ -valued solution on $[0, T]$ to (10) and (11), and $\frac{p_{0,T}^\kappa(x, dy)}{p_{0,T}(x, dy)}$ is the Radon–Nikodym derivative of the transition kernel $p_{0,T}^\kappa(x, dy)$ of X under \mathbb{P}_x^κ with respect to the transition kernel $p_{0,T}(x, dy)$ of X under \mathbb{P}_x . The probability law \mathbb{P}_x^κ is defined by (20).

Proof. For any nonnegative measurable function f on S_d^+ , we have

$$\begin{aligned} & \int_{S_d^+} \mathbb{E}_x \left[\exp \left\{ - \int_0^T \text{tr} (X_t \kappa(dt)) \right\} \middle| X_T = y \right] f(y) p_{0,T}(x, dy) \\ &= \mathbb{E}_x \left[\exp \left\{ - \int_0^T \text{tr} (X_t \kappa(dt)) \right\} f(X_T) \right] \\ &= \exp \{ -\phi(0, \kappa) - \text{tr} (\psi(0, \kappa)x) \} \mathbb{E}_x [Z_T f(X_T)] \\ &= \exp \{ -\phi(0, \kappa) - \text{tr} (\psi(0, \kappa)x) \} \mathbb{E}_x^\kappa [f(X_T)] \\ &= \exp \{ -\phi(0, \kappa) - \text{tr} (\psi(0, \kappa)x) \} \int_{S_d^+} f(y) p_{0,T}^\kappa(x, dy). \end{aligned}$$

Hence, the Laplace transforms of bridges are in the desired form. \square

4. Laplace transforms of Wishart functionals

In this section, we restrict ourselves to special conservative affine processes, which are called Wishart processes. We consider a conservative affine process with the admissible parameter set $(\Sigma^\top \Sigma, \delta \Sigma^\top \Sigma, \beta^{ij}, 0, 0)$ satisfying that $\Sigma \in M_d$, $\delta \geq d - 1$ and the linear drift coefficient β^{ij} is of the form:

$$B(x) = \sum_{ij} \beta^{ij} x_{ij} = Hx + xH^\top, \quad \text{for all } x \in S_d,$$

for some $H \in M_d$. Existence of such process can be verified by Theorem 2. Note that the parameter set satisfies the admissibility condition described in Definition 2. The corresponding F and R functions are as follows:

$$\begin{aligned} F(u) &= \delta \text{tr} (\Sigma^\top \Sigma u), \\ R(u) &= -2u \Sigma^\top \Sigma u + uH + H^\top u. \end{aligned}$$

It is obvious that R is Lipschitz continuous on S_d , and the equation

$$\frac{\partial}{\partial t} \psi(t, 0) = -2\psi(t, 0) \Sigma^\top \Sigma \psi(t, 0) + \psi(t, 0)H + H^\top \psi(t, 0), \quad \psi(0, 0) = 0,$$

has a unique solution $\psi(t, 0) \equiv 0$. Therefore, by [Theorem 2](#), there exists a unique conservative affine process with the admissible parameter set $(\Sigma^\top \Sigma, \delta \Sigma^\top \Sigma, \beta^{ij}, 0, 0)$. And, by [Theorem 3](#), it has the following representation

$$dX_t = (\delta \Sigma^\top \Sigma + H X_t + X_t H^\top) dt + \sqrt{X_t} dW_t \Sigma + \Sigma^\top dW_t^\top \sqrt{X_t}, \\ X_0 = x \in S_d^+.$$

The affine diffusion process X is called the Wishart process with the parameter set (Σ, δ, H) and is denoted by $\text{WIS}(\Sigma, \delta, H)$. For Wishart processes, the Eqs. (10) and (11) reduce to

$$\phi(t, \kappa) = \delta \int_t^T \text{tr} \left(\Sigma^\top \Sigma \psi(s, \kappa) \right) ds, \quad (24)$$

$$\psi(t, \kappa) = \kappa(t, T] + \int_t^T \left(-2\psi(s, \kappa) \Sigma^\top \Sigma \psi(s, \kappa) + \psi(s, \kappa)H + H^\top \psi(s, \kappa) \right) ds, \quad (25)$$

for $0 \leq t \leq T$. In the following subsections, we consider refinements of [Proposition 4](#) and simplification of (17) for Wishart processes.

4.1. Explicit formulae for Wishart bridges

This subsection concerns refinements of [Proposition 4](#) for Wishart bridges. In order to write the formula (23) in explicit form, we should find the transition kernels under probability laws \mathbb{P}_x and \mathbb{P}_x^κ . Affine characterization under \mathbb{P}_x^κ , which is described in [Proposition 3](#), plays a key role in finding them.

Corollary 1. *Let X be a Wishart process $\text{WIS}(\Sigma, \delta, H)$, κ be an S_d^+ -valued measure on $(0, T]$, ψ be the solution of (25), and \mathbb{P}_x^κ be the probability law defined by (20). Then, for all $x, u \in S_d^+$ and $0 \leq s \leq t \leq T$,*

$$\mathbb{P}_x^\kappa[e^{-\text{tr}(uX_t)} | \mathcal{F}_s] = \det(I_d + 2uV_\kappa(s, t))^{-\delta/2} \\ \times \exp \left\{ -\text{tr} \left(\Psi_\kappa(s, t)u(I_d + 2V_\kappa(s, t)u)^{-1} \Psi_\kappa(s, t)^\top X_s \right) \right\}, \quad (26)$$

where $\Psi_\kappa(\cdot, \cdot)$ is the solution of a linear equation

$$-\frac{\partial}{\partial s} \Psi_\kappa(s, t) = (H^\top - 2\psi(s, \kappa) \Sigma^\top \Sigma) \Psi_\kappa(s, t), \quad \Psi_\kappa(t, t) = I_d, \quad (27)$$

and

$$V_\kappa(s, t) = \int_s^t \Psi_\kappa(r, t)^\top \Sigma^\top \Sigma \Psi_\kappa(r, t) dr. \quad (28)$$

Proof. We put

$$\psi_\kappa(s, t, u) = \Psi_\kappa(s, t)u(I_d + 2V_\kappa(s, t)u)^{-1} \Psi_\kappa(s, t)^\top$$

for $0 \leq s \leq t \leq T$. By the terminal conditions $\Psi_\kappa(t, t) = I_d$ and $V_\kappa(t, t) = 0$,

$$\psi_\kappa(t, t, u) = I_d u(I_d + 0)^{-1} I_d = u.$$

And, by differentiating with respect to s , we have

$$\begin{aligned} -\frac{\partial}{\partial s} \psi_\kappa(s, t, u) &= -\left(\frac{\partial}{\partial s} \Psi_\kappa(s, t)\right) u(I_d + 2V_\kappa(s, t)u)^{-1} \Psi_\kappa(s, t)^\top \\ &\quad + \Psi_\kappa(s, t) u(I_d + 2V_\kappa(s, t)u)^{-1} 2\left(\frac{\partial}{\partial s} V_\kappa(s, t)\right) u \\ &\quad \times (I_d + 2V_\kappa(s, t)u)^{-1} \Psi_\kappa(s, t)^\top \\ &\quad - \Psi_\kappa(s, t) u(I_d + 2V_\kappa(s, t)u)^{-1} \left(\frac{\partial}{\partial s} \Psi_\kappa(s, t)^\top\right) \\ &= -2\psi_\kappa(s, t, u) \Sigma^\top \Sigma \psi_\kappa(s, t, u) \\ &\quad + (H^\top - 2\psi(s, \kappa) \Sigma^\top \Sigma) \psi_\kappa(s, t, u) + \psi_\kappa(s, t, u) \\ &\quad \times (H - 2\Sigma^\top \Sigma \psi(s, \kappa)). \end{aligned}$$

Therefore, $\psi_\kappa(s, t, u)$ is the solution of (22). In a similar manner, one can easily check that

$$\phi_\kappa(s, t, u) = \frac{\delta}{2} \ln \det(I_d + 2uV_\kappa(s, t)), \quad 0 \leq s \leq t \leq T,$$

is the solution of (21). Hence, the Laplace transform of X_t under \mathbb{P}_x^κ satisfies (26). \square

Remark 5. If $\kappa(ds) = 0$, then the solution of (25) is $\psi(t, \kappa) \equiv 0$, and the probability law \mathbb{P}_x^κ reduces to the original law \mathbb{P}_x . Thus, Corollary 1 tells that Laplace transform of X_t under \mathbb{P}_x is

$$\begin{aligned} \mathbb{E}_x[e^{-\text{tr}(uX_t)} | \mathcal{F}_s] &= \det(I_d + 2uV(s, t))^{-\delta/2} \\ &\quad \times \exp\left\{-\text{tr}\left(e^{(t-s)H^\top} u(I_d + 2V(s, t)u)^{-1} e^{(t-s)H} X_s\right)\right\}, \end{aligned}$$

where

$$V(s, t) = \int_s^t e^{(t-r)H} \Sigma^\top \Sigma e^{(t-r)H^\top} dr. \quad (29)$$

For this case, the above formula, with some technical conditions on parameters H and Σ , was already known in the original paper of Bru [3], and Ahdida and Alfonsi [1] seem to be the first to prove it for general parameters.

Formula (26) tells us that \mathcal{F}_s -conditional distribution of X_t , under \mathbb{P}_x^κ , is noncentral Wishart with degree of freedom δ , covariance matrix $V_\kappa(s, t)$, and matrix of noncentrality parameters $V_\kappa(s, t)^{-1} \Psi_\kappa(s, t)^\top X_s \Psi_\kappa(s, t)$. In particular, the transition density of X can be written through well-known probability densities of noncentral Wishart distributions. These densities involve some multivariate special functions, namely multivariate gamma functions Γ_d and hypergeometric functions ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; \cdot)$ with matrix arguments:

$$\begin{aligned} \Gamma_d(c) &= \int_{S_d^{++}} e^{-\text{tr}(y)} (\det y)^{c-(d+1)/2} (dy), \quad \text{for } \Re c > \frac{1}{2}(d-1), \\ {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; y) &= \sum_{k=0}^{\infty} \sum_{\iota} \frac{(a_1)_\iota \cdots (a_p)_\iota}{(b_1)_\iota \cdots (b_q)_\iota} \frac{C_\iota(y)}{k!}, \end{aligned}$$

where \sum_ι denotes summation over all partitions $\iota = (k_1, \dots, k_d)$, $k_1 \geq \dots \geq k_d$, of $k = k_1 + \dots + k_d$, $C_\iota(y)$ is the zonal polynomial of y corresponding to ι and the generalized

hypergeometric coefficient $(a)_t$ is given by

$$(a)_t = \prod_{i=1}^d \left(a - \frac{1}{2}(i-1) \right)_{k_i},$$

where $(a)_k = a(a+1)\cdots(a+k-1)$, $(a)_0 = 1$. Here y is a complex symmetric $d \times d$ matrix and the parameters a_i, b_j are complex numbers. For detailed discussion on Wishart distributions, gamma functions, hypergeometric functions of matrix argument, and zonal polynomial, refer to Muirhead [18]. We write the transition densities in the following corollary.

Corollary 2 (Muirhead [18]). *Let X be a Wishart process $\text{WIS}(\Sigma, \delta, H)$ with $\delta > d-1$ and $\Sigma^\top \Sigma \in S_d^{++}$. The transition density $p_{s,t}^\kappa(x_s, x_t)$ under \mathbb{P}_x^κ is given by*

$$\begin{aligned} p_{s,t}^\kappa(x_s, x_t) &= \frac{(\det x_t)^{(\delta-d-1)/2}}{2^{d\delta/2} \Gamma_d\left(\frac{1}{2}\delta\right) (\det V_\kappa(s, t))^{\delta/2}} \\ &\times \exp\left\{-\frac{1}{2}\text{tr}\left(V_\kappa(s, t)^{-1}(x_t + \Psi_\kappa(s, t)^\top x_s \Psi_\kappa(s, t))\right)\right\} \\ &\times {}_0F_1\left(\frac{1}{2}\delta; \frac{1}{4}V_\kappa(s, t)^{-1}\Psi_\kappa(s, t)^\top x_s \Psi_\kappa(s, t)V_\kappa(s, t)^{-1}x_t\right) \end{aligned}$$

for $0 \leq s \leq t \leq T$ and $x_s, x_t \in S_d^{++}$, where $\Psi_\kappa(\cdot, \cdot)$ and $V_\kappa(\cdot, \cdot)$ are given by (27) and (28), respectively.

Then we combine Proposition 4 and Corollary 2 to have an explicit transform formula for Laplace transforms of Wishart bridges.

Corollary 3. *Let X be a Wishart process $\text{WIS}(\Sigma, \delta, H)$ with $\delta > d-1$ and $\Sigma^\top \Sigma \in S_d^{++}$, and (ϕ, ψ) be the solution of (24) and (25) corresponding to an S_d^+ -valued measure κ on $(0, T]$. Then, for all $x \in S_d^+$, and $y \in S_d^{++}$, we have*

$$\begin{aligned} \mathbb{E}_x \left[\exp \left\{ - \int_0^T \text{tr}(X_t \kappa(dt)) \right\} \middle| X_T = y \right] \\ = \left(\frac{\det V(0, T)}{\det V_\kappa(0, T)} \right)^{\delta/2} \exp \{ -\phi(0, \kappa) - \text{tr}(\psi(0, \kappa)x) \} \\ \times \exp \left\{ \frac{1}{2} \text{tr} \left(\left(e^{TH^\top} V(0, T)^{-1} e^{TH} - \Psi_\kappa(0, T) V_\kappa(0, T)^{-1} \Psi_\kappa(0, T)^\top \right) x \right) \right\} \\ \times \exp \left\{ \frac{1}{2} \text{tr} \left(\left(V(0, T)^{-1} - V_\kappa(0, T)^{-1} \right) y \right) \right\} \\ \times \frac{{}_0F_1\left(\frac{1}{2}\delta; \frac{1}{4}V_\kappa(0, T)^{-1}\Psi_\kappa(0, T)^\top x \Psi_\kappa(0, T)V_\kappa(0, T)^{-1}y\right)}{{}_0F_1\left(\frac{1}{2}\delta; \frac{1}{4}V(0, T)^{-1}e^{TH}xe^{TH^\top}V(0, T)^{-1}y\right)}, \end{aligned} \quad (30)$$

where $\Psi_\kappa(\cdot, \cdot)$, $V_\kappa(\cdot, \cdot)$, and $V(\cdot, \cdot)$ are given by (27)–(29), respectively.

In some cases, the Eqs. (24), (25) and (27)–(29) admit closed-form solutions, which in turn gives a closed-form expression for Laplace transform (30). Example 1 considers an absolutely continuous measure with a constant derivative, i.e., $\kappa(dt) = \frac{1}{2}\lambda^2 \mathbb{1}_{(0, T]}(t)dt$.

Example 1. Let X be a Wishart process $\text{WIS}(\Sigma, \delta, H)$ such that $\delta > d - 1$, $\Sigma^\top \Sigma \in S_d^{++}$, $H \Sigma^\top \Sigma = \Sigma^\top \Sigma H^\top$, and $\kappa(dt) = \frac{1}{2} \lambda^2 \mathbb{1}_{(0,T]}(t) dt$ where $\lambda \in S_d^+$. In this example, we are considering the problem of computing (30) in a closed-form. For notational convenience, we assume that both λ and H are invertible. The other case, i.e. either λ or H is singular, can be dealt with by appropriate limiting arguments. Computational details of this example are provided in Appendix.

(Step 1) Consider the case $\Sigma = I_d$ and H is symmetric. In this case, (25) reduces to

$$-\frac{\partial}{\partial t} \psi(t, \kappa) = -2\psi(t, \kappa)^2 + H\psi(t, \kappa) + \psi(t, \kappa)H + \frac{1}{2}\lambda^2, \quad 0 \leq t \leq T, \quad (31)$$

with the terminal value $\psi(T, \kappa) = 0$. The solutions $\phi(\cdot, \kappa)$ and $\psi(\cdot, \kappa)$ of (24) and (25) are given by

$$\begin{cases} \phi(t, \kappa) = \frac{\delta}{2} \ln \det \left(e^{(T-t)H} \psi_1(t, \kappa) \right), \\ \psi(t, \kappa) = \frac{1}{2} \left(\psi_1(t, \kappa)^{-1} \psi_2(t, \kappa) + H \right), \end{cases} \quad (32)$$

where

$$\begin{aligned} \psi_1(t, \kappa) &= \cosh((T-t)\xi) - H\xi^{-1} \sinh((T-t)\xi), \\ \psi_2(t, \kappa) &= \xi \sinh((T-t)\xi) - H \cosh((T-t)\xi), \end{aligned}$$

and $\xi = \sqrt{\lambda^2 + H^2}$. Since H and λ are symmetric and invertible, $\lambda^2 + H^2$ is symmetric positive definite. So, its square root ξ and the inverse ξ^{-1} are well-defined. For a nonsymmetric H , we do not expect that (24) and (25) have solutions of the form (32). The hyperbolic functions in this example are matrix hyperbolic functions in the usual sense, i.e.,

$$\begin{aligned} \cosh(A) &= \frac{1}{2}(e^A + e^{-A}), & \sinh(A) &= \frac{1}{2}(e^A - e^{-A}), \\ \text{sech}(A) &= \cosh(A)^{-1}, & \text{csch}(A) &= \sinh(A)^{-1}, \\ \tanh(A) &= \cosh(A)^{-1} \sinh(A), & \coth(A) &= \tanh(A)^{-1}, \end{aligned}$$

where e^A and e^{-A} are matrix exponentials of A and $-A$, respectively. The solutions of (27) and (28) are

$$\Psi_\kappa(t, T) = \psi_1(t, \kappa)^{-1}, \quad V_\kappa(t, T) = (\psi_1(t, \kappa)^{-1})^\top \xi^{-1} \sinh((T-t)\xi). \quad (33)$$

A direct integration shows

$$V(0, T) = \int_0^T e^{2(T-r)H} dr = \frac{1}{2} H^{-1} (e^{2TH} - I_d) = e^{TH} H^{-1} \sinh(TH).$$

We substitute quantities $\phi(0, \kappa)$, $\psi(0, \kappa)$, $\Psi_\kappa(0, T)$, $V_\kappa(0, T)$ and $V(0, T)$ into (30) to have

$$\begin{aligned} \mathbb{E}_x \left[\exp \left\{ -\frac{1}{2} \int_0^T \text{tr}(\lambda^2 X_t) dt \right\} \middle| X_T = y \right] &= \left(\frac{\det(\xi \text{csch}(T\xi))}{\det(\xi \text{csch}(T\xi))} \right)^{\delta/2} \\ &\times \exp \left\{ \frac{1}{2} \text{tr}((x+y)(\xi \coth(T\xi) - \xi \coth(T\xi))) \right\} \end{aligned}$$

$$\times \frac{{}_0F_1\left(\frac{1}{2}\delta; \frac{1}{4}\xi \operatorname{csch}(T\xi) x \operatorname{csch}(T\xi)\xi y\right)}{{}_0F_1\left(\frac{1}{2}\delta; \frac{1}{4}\zeta \operatorname{csch}(T\zeta) x \operatorname{csch}(T\zeta)\zeta y\right)}, \quad (34)$$

where $\xi = \sqrt{\lambda^2 + H^2}$ and $\zeta = \sqrt{H^2}$.

(Step 2) We consider a Wishart process X with parameter set (Σ, δ, H) satisfying $\delta > d - 1$, $\Sigma^\top \Sigma \in S_d^{++}$, and $H \Sigma^\top \Sigma = \Sigma^\top \Sigma H^\top$. By Proposition 4.13 of Cuchiero et al. [5], $Y = (\Sigma^{-1})^\top X \Sigma^{-1}$ is a Wishart process with parameter set $(I_d, \delta, (\Sigma^{-1})^\top H \Sigma^\top)$. Notice that by the assumption $H \Sigma^\top \Sigma = \Sigma^\top \Sigma H^\top$,

$$\left((\Sigma^{-1})^\top H \Sigma^\top\right)^\top = (\Sigma^{-1})^\top H \Sigma^\top.$$

Initial and terminal values correspond in the following way

$$X_0 = x, \quad X_T = y \iff Y_0 = (\Sigma^{-1})^\top x \Sigma^{-1}, \quad Y_T = (\Sigma^{-1})^\top y \Sigma^{-1}.$$

Therefore, by step 1, we have

$$\begin{aligned} & \mathbb{E}_x \left[\exp \left\{ -\frac{1}{2} \int_0^T \operatorname{tr}(\lambda^2 X_t) dt \right\} \middle| X_T = y \right] \\ &= \mathbb{E}_{(\Sigma^{-1})^\top x \Sigma^{-1}} \left[\exp \left\{ -\frac{1}{2} \int_0^T \operatorname{tr}(\Sigma \lambda^2 \Sigma^\top Y_t) dt \right\} \middle| Y_T = (\Sigma^{-1})^\top y \Sigma^{-1} \right] \\ &= \left(\frac{\det(\xi \operatorname{csch}(T\xi))}{\det(\zeta \operatorname{csch}(T\zeta))} \right)^{\delta/2} \\ &\quad \times \exp \left\{ \frac{1}{2} \operatorname{tr} \left((\Sigma^{-1})^\top (x + y) \Sigma^{-1} (\zeta \coth(T\zeta) - \xi \coth(T\xi)) \right) \right\} \\ &\quad \times \frac{{}_0F_1\left(\frac{1}{2}\delta; \frac{1}{4}\xi \operatorname{csch}(T\xi) (\Sigma^{-1})^\top x \Sigma^{-1} \operatorname{csch}(T\xi) \xi (\Sigma^{-1})^\top y \Sigma^{-1}\right)}{{}_0F_1\left(\frac{1}{2}\delta; \frac{1}{4}\zeta \operatorname{csch}(T\zeta) (\Sigma^{-1})^\top x \Sigma^{-1} \operatorname{csch}(T\zeta) \zeta (\Sigma^{-1})^\top y \Sigma^{-1}\right)}, \end{aligned} \quad (35)$$

where

$$\begin{aligned} \zeta &= \sqrt{((\Sigma^{-1})^\top H \Sigma^\top)^2} = \sqrt{\Sigma H^\top (\Sigma^\top \Sigma)^{-1} H \Sigma^\top}, \\ \xi &= \sqrt{\Sigma \lambda^2 \Sigma^\top + ((\Sigma^{-1})^\top H \Sigma^\top)^2} = \sqrt{\Sigma (\lambda^2 + H^\top (\Sigma^\top \Sigma)^{-1} H) \Sigma^\top}. \end{aligned}$$

Remark 6. For squared Bessel processes (i.e. 1-dimensional Wishart processes with vanishing linear drift), the formula (35) was known in Pitman and Yor [19]. And the formula was extended to 1-dimensional Wishart processes with non-vanishing linear drift in Broadie and Kaya [2] study of exact simulation of Heston's stochastic volatility model.

Remark 7. Donati-Martin et al. [8] derived a formula similar to (35); if X is a Wishart process $\operatorname{WIS}(I_d, d + 1, 0)$ and if λ and ν are nonnegative real numbers, then the following holds

$$\begin{aligned} & \mathbb{E}_x \left[\exp \left\{ -\frac{\lambda^2}{2} \int_0^T \operatorname{tr}(X_t) dt - \frac{\nu^2}{2} \int_0^T \operatorname{tr}(X_t^{-1}) dt \right\} \middle| X_T = y \right] \\ &= \left(\frac{\lambda T}{\sinh(\lambda T)} \right)^{d(d+1)/2} \exp \left\{ -\frac{\lambda T \coth(\lambda T) - 1}{2T} \operatorname{tr}(x + y) \right\} \frac{\tilde{\mathbf{I}}_\nu \left(\frac{\lambda^2 xy}{4 \sinh^2(\lambda T)} \right)}{\tilde{\mathbf{I}}_0 \left(\frac{xy}{4T^2} \right)}, \end{aligned}$$

where

$$\tilde{\mathbf{I}}_v(z) = \frac{(\det z)^{v/2}}{\Gamma_d((d+1)/2 + v)} {}_0F_1((d+1)/2 + v; z).$$

4.2. Simplification of formula (17) for $\text{WIS}(\Sigma, \delta, 0)$

The aim of this subsection is to simplify the formula (17) for Wishart processes without linear drift $\text{WIS}(\Sigma, \delta, 0)$, and to show how our results are related to previous studies.

We consider the following linear differential equation

$$\frac{\partial}{\partial t} \Phi(t, \kappa) = -2\Sigma^\top \Sigma \psi(t, \kappa) \Phi(t, \kappa) \quad \text{with } \Phi(0, \kappa) = I_d. \quad (36)$$

Notice that $\frac{\partial}{\partial t} \Phi(t, \kappa)|_{t=T} = -2\Sigma^\top \Sigma \psi(T, \kappa) \Phi(T, \kappa) = 0$. By (25) and integration by parts, we have

$$\begin{aligned} \frac{\partial}{\partial t} \Phi(t, \kappa) &= \int_t^T 2\Sigma^\top \Sigma d\psi(s, \kappa) \Phi(s, \kappa) + \int_t^T 2\Sigma^\top \Sigma \psi(s, \kappa) \frac{\partial}{\partial t} \Phi(s, \kappa) ds \\ &= - \int_t^T 2\Sigma^\top \Sigma \kappa(ds) \Phi(s, \kappa). \end{aligned}$$

Therefore, the function Φ is a solution (in the distribution sense) to the following initial and terminal value problem

$$\frac{\partial^2}{\partial t^2} \Phi = 2\Sigma^\top \Sigma \kappa \Phi \quad \text{on } [0, T], \text{ with } \Phi(0, \kappa) = I_d, \quad \frac{\partial}{\partial t} \Phi(t, \kappa)|_{t=T} = 0. \quad (37)$$

Sometimes the above linear equation is simpler to solve than the quadratic equation (25). Moreover, ϕ in (24) also can be represented via Φ . Observe that $\ln \det(\Phi(0, \kappa)) = 0$ and

$$\frac{\partial}{\partial t} \left(\frac{\delta}{2} \ln \det(\Phi(t, \kappa)) \right) = \frac{\delta}{2} \text{tr} \left(\Phi(t, \kappa)^{-1} \frac{\partial}{\partial t} \Phi(t, \kappa) \right) = -\delta \text{tr} \left(\Sigma^\top \Sigma \psi(t, \kappa) \right).$$

Therefore,

$$\frac{\delta}{2} \ln \det(\Phi(T, \kappa)) = - \int_0^T \delta \text{tr} \left(\Sigma^\top \Sigma \psi(t, \kappa) \right) dt = -\phi(0, \kappa).$$

We summarize these observations in the following corollary.

Corollary 4. Let X be a Wishart process $\text{WIS}(\Sigma, \delta, 0)$, κ be an S_d^+ measure on $(0, T]$, and ψ, Φ be the solutions to (25), (37), respectively. Then, for all $x \in S_d^+$, we have

$$\mathbb{E}_x \left[\exp \left\{ - \int_0^T \text{tr}(X_s \kappa(ds)) \right\} \right] = \det(\Phi(T, \kappa))^{\delta/2} \exp \{ -\text{tr}(\psi(0, \kappa)x) \}.$$

Moreover, if Σ is invertible then

$$\begin{aligned} \mathbb{E}_x \left[\exp \left\{ - \int_0^T \text{tr}(X_s \kappa(ds)) \right\} \right] &= \det(\Phi(T, \kappa))^{\delta/2} \\ &\quad \times \exp \left\{ \frac{1}{2} \text{tr} \left((\Sigma^\top \Sigma)^{-1} \frac{\partial}{\partial t} \Phi(t, \kappa)|_{t=0} x \right) \right\}. \end{aligned} \quad (38)$$

Studies of the relationship between the solution of (37) and the Laplace transforms of Wishart functionals was initiated by Pitman and Yor [19]. They characterized the Laplace transforms of squared Bessel functionals in terms of the 1-dimensional version of (37), and they derived the univariate version of (38). Indeed, their formula is stronger than (38) for squared Bessel processes because their formula deals with positive Radon measures not on $(0, T]$, but on $(0, \infty)$. Bru considered Laplace transforms of Wishart functionals in her paper Bru [3] and she essentially found the same formula as (38).

We illustrate the usefulness of (38) in the following two examples. **Example 2** computes the joint Laplace transform of a marginal distribution and an integral of the Wishart process. **Example 3** computes the Laplace transform of the marginal distribution of a Wishart bridge.

Example 2. Let $\kappa(dt) = \frac{1}{2}\lambda^2 \mathbb{1}_{(0,T]}(t)dt + \frac{1}{2}u\varepsilon_T(dt)$ where ε_T is the Dirac measure at T , $\lambda \in S_d^{++}$, and $u \in S_d^+$. Consider (37) with $\Sigma = I_d$. Then it is equivalent to

$$\frac{\partial^2}{\partial t^2} \Phi(t, \kappa) = \lambda^2 \Phi(t, \kappa) \quad \text{on } (0, T),$$

with the boundary conditions $\Phi(0, \kappa) = I_d$ and $\frac{\partial}{\partial t} \Phi(t, \kappa)|_{t=T-} = -u\Phi(T, \kappa)$. The unique solution to the above equation is given by

$$\begin{aligned} \Phi(t, \kappa) &= \left(\cosh((T-t)\lambda) + \sinh((T-t)\lambda) \lambda^{-1} u \right) \\ &\quad \times \left(\cosh(T\lambda) + \sinh(T\lambda) \lambda^{-1} u \right)^{-1}, \end{aligned}$$

for $0 \leq t \leq T$. This gives the matrix Cameron–Martin formula in Bru [3]. For a Wishart process X with $\text{WIS}(I_d, \delta, 0; x)$, $\lambda \in S_d^{++}$, and $u \in S_d^+$, the following holds

$$\begin{aligned} \mathbb{E}_x \left[\exp \left\{ -\frac{1}{2} \int_0^T \text{tr}(\lambda^2 X_t) dt - \frac{1}{2} \text{tr}(u X_T) \right\} \right] \\ = \left(\det \left(\cosh(T\lambda) + \sinh(T\lambda) \lambda^{-1} u \right) \right)^{-\delta/2} \\ \times \exp \left\{ -\frac{1}{2} \text{tr} \left(x \left(\sinh(T\lambda) \lambda + \cosh(T\lambda) u \right) \left(\cosh(T\lambda) + \sinh(T\lambda) \lambda^{-1} u \right)^{-1} \right) \right\}. \end{aligned}$$

Recently, the above formula is extended by Gnoatto and Grasselli [13]. They derived a closed-form formula for the joint Laplace transform of the Wishart process $\text{WIS}(\Sigma, \delta, H; x)$ and its time integral under the conditions $H\Sigma^\top \Sigma = \Sigma^\top \Sigma H^\top$ and $\delta \geq d+1$.

Example 3. This example computes the Laplace transform of the marginal distribution of Wishart bridges. More precisely, we are considering the problem of computing

$$\mathbb{E}_x \left[e^{-\text{tr}(u X_{T_0})} | X_T = y \right],$$

for a Wishart process X with $\text{WIS}(I_d, \delta, 0)$, $u, x \in S_d^+$, and $0 < T_0 < T$. This problem can be solved via applying Corollary 3 to $\kappa(dt) = u\varepsilon_{T_0}(dt)$. One can easily check that

$$\Phi(t, \kappa) = \begin{cases} (I_d + 2(T_0 - t)u)(I_d + 2T_0 u)^{-1} & \text{if } 0 \leq t < T_0 \\ (I_d + 2T_0 u)^{-1} & \text{if } T_0 \leq t \leq T \end{cases}$$

solves (37). By (36),

$$\psi(t, \kappa) = \begin{cases} u(I_d + 2(T_0 - t)u)^{-1} & \text{if } 0 \leq t < T_0 \\ 0 & \text{if } T_0 \leq t \leq T. \end{cases}$$

And the solution to the equation

$$\frac{\partial}{\partial t} \Psi_\kappa(t, T) = 2\psi(t, \kappa) \Psi_\kappa(t, T)^\top, \quad \text{with } \Psi_\kappa(T, T) = I_d$$

is given by

$$\Psi_\kappa(t, T) = \begin{cases} (I_d + 2(T_0 - t)u)^{-1} & \text{if } 0 < t < T_0 \\ I_d & \text{if } T_0 \leq t \leq T. \end{cases}$$

Notice that $I_d + 2T_0u$, $(I_d + 2(T_0 - t)u)^{-1}$ are symmetric and they commute. Using this observation, we can compute the covariance matrix $V_\kappa(0, T)$ as follows

$$\begin{aligned} V_\kappa(0, T) &= \int_0^T \Psi_\kappa(t, T)^\top \Psi_\kappa(t, T) dt = \int_0^{T_0} \left\{ (I_d + 2(T_0 - t)u)^{-1} \right\}^2 dt + \int_{T_0}^T I_d dt \\ &= \int_0^{T_0} \left\{ (I_d + 2tu)^{-1} \right\}^2 dt + (T - T_0)I_d = T_0(I_d + 2T_0u)^{-1} + (T - T_0)I_d \\ &= (TI_d + 2(T - T_0)T_0u)(I_d + 2T_0u)^{-1}. \end{aligned}$$

And it is obvious that $V(0, T) = TI_d$. Therefore,

$$\left(\frac{\det V(0, T)}{\det V_\kappa(0, T)} \right)^{\delta/2} \exp \{-\phi(0, \kappa)\} = \left(T^d \det (TI_d + 2(T - T_0)T_0u)^{-1} \right)^{\delta/2},$$

$$\begin{aligned} e^{TH^\top V(0, T)^{-1} e^{TH}} - \Psi_\kappa(0, T) V_\kappa(0, T)^{-1} \Psi_\kappa(0, T)^\top - 2\psi(0, \kappa) \\ = -\frac{2}{T} (T - T_0)^2 (TI_d + 2(T - T_0)T_0u)^{-1} u, \end{aligned}$$

$$V(0, T)^{-1} - V_\kappa(0, T)^{-1} = -\frac{2}{T} T_0^2 (TI_d + 2(T - T_0)T_0u)^{-1} u,$$

$$V_\kappa(0, T)^{-1} \Psi_\kappa(0, T)^\top = (TI_d + 2(T - T_0)T_0u)^{-1},$$

and $V(0, T)^{-1} e^{TH} = \frac{1}{T} I_d$. Then we substitute the above quantities into (30) to have

$$\begin{aligned} \mathbb{E}_X \left[e^{-\text{tr}(uX_{T_0})} | X_T = y \right] &= \left(T^d \det (U(T_0)) \right)^{\delta/2} \\ &\times \exp \left\{ -\frac{1}{T} \text{tr} \left(U(T_0)u \left((T - T_0)^2 x + T_0^2 y \right) \right) \right\} \frac{{}_0F_1 \left(\frac{1}{2}\delta; \frac{1}{4}U(T_0)xU(T_0)y \right)}{{}_0F_1 \left(\frac{1}{2}\delta; \frac{1}{4T^2}xy \right)}, \end{aligned}$$

where $U(T_0) = (TI_d + 2(T - T_0)T_0u)^{-1}$.

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Appendix

In this appendix, we provide the computational details of Step 1 in [Example 1](#). Note that

$$\frac{\partial}{\partial t} \psi_1(t, \kappa) = -\psi_2(t, \kappa), \quad \text{and} \quad \frac{\partial}{\partial t} \psi_2(t, \kappa) = -\psi_1(t, \kappa) \xi^2.$$

Using these identities, we can check that $\psi(\cdot, \kappa)$ and $\phi(\cdot, \kappa)$ given in (32) solve (31) and (24):

$$\begin{aligned} -\frac{\partial}{\partial t} \psi(t, \kappa) &= \frac{1}{2} \psi_1(t, \kappa)^{-1} \left(\frac{\partial}{\partial t} \psi_1(t, \kappa) \right) \psi_1(t, \kappa)^{-1} \psi_2(t, \kappa) \\ &\quad - \frac{1}{2} \psi_1(t, \kappa)^{-1} \left(\frac{\partial}{\partial t} \psi_2(t, \kappa) \right) \\ &= -\frac{1}{2} \psi_1(t, \kappa)^{-1} \psi_2(t, \kappa) \psi_1(t, \kappa)^{-1} \psi_2(t, \kappa) + \frac{1}{2} \psi_1(t, \kappa)^{-1} \psi_1(t, \kappa) \xi^2 \\ &= -\frac{1}{2} (2\psi(t, \kappa) - H)^2 + \frac{1}{2} (\lambda^2 + H^2) \\ &= -2\psi(t, \kappa)^2 + \psi(t, \kappa)H + H\psi(t, \kappa) + \frac{1}{2} \lambda^2, \\ -\frac{\partial}{\partial t} \phi(t, \kappa) &= -\frac{\delta}{2} \text{tr} \left(\left(e^{(T-t)H} \psi_1(t, \kappa) \right)^{-1} \frac{\partial}{\partial t} \left(e^{(T-t)H} \psi_1(t, \kappa) \right) \right) \\ &= -\frac{\delta}{2} \text{tr} \left(\psi_1(t, \kappa)^{-1} e^{-(T-t)H} \right. \\ &\quad \times \left. \left(-He^{(T-t)H} \psi_1(t, \kappa) - e^{(T-t)H} \psi_2(t, \kappa) \right) \right) \\ &= \frac{\delta}{2} \text{tr} \left(H + \psi_1(t, \kappa)^{-1} \psi_2(t, \kappa) \right) = \delta \text{tr} (\psi(t, \kappa)), \end{aligned}$$

and it is obvious that $\psi(T, \kappa) = 0$ and $\phi(T, \kappa) = 0$. The linear equation (27) reduces to

$$\frac{\partial}{\partial t} \Psi_\kappa(t, T) = \psi_1(t, \kappa)^{-1} \psi_2(t, \kappa) \Psi_\kappa(t, T), \quad \Psi_\kappa(T, T) = I_d.$$

A direct differentiation shows that $\Psi_\kappa(t, T) = \psi_1(t, \kappa)^{-1}$. We can check that $V_\kappa(t, T)$ given in (33) satisfies (28). Differentiation with respect to t yields

$$\begin{aligned} \frac{\partial}{\partial t} V_\kappa(t, T) &= (\psi_1(t, \kappa)^{-1})^\top \psi_1(t, \kappa)^{-1} \psi_2(t, \kappa) \xi^{-1} \sinh((T-t)\xi) \\ &\quad - (\psi_1(t, \kappa)^{-1})^\top \cosh((T-t)\xi) \\ &= -(\psi_1(t, \kappa)^{-1})^\top \psi_1(t, \kappa)^{-1} \\ &\quad \times \left(\psi_1(t, \kappa) \cosh((T-t)\xi) - \psi_2(t, \kappa) \xi^{-1} \sinh((T-t)\xi) \right) \\ &= -(\psi_1(t, \kappa)^{-1})^\top \psi_1(t, \kappa)^{-1} \left(\cosh^2((T-t)\xi) - \sinh^2((T-t)\xi) \right) \\ &= -(\psi_1(t, \kappa)^{-1})^\top \psi_1(t, \kappa)^{-1} \\ &= -\Psi_\kappa(t, T)^\top \Psi_\kappa(t, T), \end{aligned}$$

and $V_\kappa(T, T) = 0$. The inverses of $V_\kappa(0, T)$ and $V(0, T)$ are

$$\begin{aligned} V_\kappa(0, T)^{-1} &= \text{csch}(T\xi) \xi \psi_1(0, \kappa)^\top = \text{csch}(T\xi) \xi \left(\cosh(T\xi) - \xi^{-1} \sinh(T\xi) H \right) \\ &= \xi \coth(T\xi) - H, \end{aligned}$$

$$\begin{aligned} V(0, T)^{-1} &= \operatorname{csch}(TH) H e^{-TH} = \operatorname{csch}(TH) H (\cosh(TH) - \sinh(TH)) \\ &= H \coth(TH) - H. \end{aligned}$$

So,

$$V(0, T)^{-1} - V_{\kappa}(0, T)^{-1} = H \coth(TH) - \xi \coth(T\xi).$$

Observe that

$$\begin{aligned} &\Psi_{\kappa}(0, T) V_{\kappa}(0, T)^{-1} \Psi_{\kappa}(0, T)^{\top} + \psi_1(0, \kappa)^{-1} \psi_2(0, \kappa) \\ &= \psi_1(0, \kappa)^{-1} \operatorname{csch}(T\xi) \xi \psi_1(0, \kappa)^{\top} (\psi_1(0, \kappa)^{\top})^{-1} + \psi_1(0, \kappa)^{-1} \psi_2(0, \kappa) \\ &= \psi_1(0, \kappa)^{-1} \left(I_d + \psi_2(0, \kappa) \sinh(T\xi) \xi^{-1} \right) \operatorname{csch}(T\xi) \xi \\ &= \psi_1(0, \kappa)^{-1} \left(I_d + \sinh^2(T\xi) - H \cosh(T\xi) \sinh(T\xi) \xi^{-1} \right) \operatorname{csch}(T\xi) \xi \\ &= \psi_1(0, \kappa)^{-1} \left(\cosh^2(T\xi) - H \cosh(T\xi) \sinh(T\xi) \xi^{-1} \right) \operatorname{csch}(T\xi) \xi \\ &= \psi_1(0, \kappa)^{-1} \left(\cosh(T\xi) - H \sinh(T\xi) \xi^{-1} \right) \cosh(T\xi) \operatorname{csch}(T\xi) \xi \\ &= \xi \coth(T\xi), \end{aligned}$$

and

$$\begin{aligned} e^{TH^{\top}} V(0, T)^{-1} e^{TH} - H &= e^{TH} \operatorname{csch}(TH) H e^{-TH} e^{TH} - H \\ &= e^{TH} \operatorname{csch}(TH) H - H \\ &= \left(e^{TH} - \sinh(TH) \right) \operatorname{csch}(TH) H \\ &= \cosh(TH) \operatorname{csch}(TH) H = H \coth(TH). \end{aligned}$$

Therefore,

$$\begin{aligned} &e^{TH^{\top}} V(0, T)^{-1} e^{TH} - \Psi_{\kappa}(0, T) V_{\kappa}(0, T)^{-1} \Psi_{\kappa}(0, T)^{\top} - 2\psi(0, \kappa) \\ &= \left(e^{TH^{\top}} V(0, T)^{-1} e^{TH} - H \right) - \left(\Psi_{\kappa}(0, T) V_{\kappa}(0, T)^{-1} \Psi_{\kappa}(0, T)^{\top} \right. \\ &\quad \left. + \psi_1(0, \kappa)^{-1} \psi_2(0, \kappa) \right) \\ &= H \coth(TH) - \xi \coth(T\xi). \end{aligned}$$

Using the multiplicative property of determinants, we have

$$\begin{aligned} \left(\frac{\det V(0, T)}{\det V_{\kappa}(0, T)} \right)^{\delta/2} e^{-\phi(0, \kappa)} &= \left(\frac{1}{\det(e^{TH} \psi_1(0, \kappa))} \frac{\det(e^{TH} H^{-1} \sinh(TH))}{\det(\psi_1(0, \kappa)^{-1} \xi^{-1} \sinh(T\xi))} \right)^{\delta/2} \\ &= \left(\frac{\det(\xi \operatorname{csch}(T\xi))}{\det(H \operatorname{csch}(TH))} \right)^{\delta/2}. \end{aligned}$$

Also

$$V_{\kappa}(0, T)^{-1} \Psi_{\kappa}(0, T)^{\top} = \xi \operatorname{csch}(T\xi), \quad V(0, T)^{-1} e^{TH} = H \operatorname{csch}(TH).$$

Using orthogonal diagonalization, one can check that

$$H \coth(TH) = \sqrt{H^2} \coth\left(T\sqrt{H^2}\right), \quad H \operatorname{csch}(TH) = \sqrt{H^2} \operatorname{csch}\left(T\sqrt{H^2}\right).$$

Then we substitute the above quantities into (30) to establish (34).

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