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Thick points for a Gaussian Free Field in 4 dimensions

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Abstract

This article is concerned with the study of fractal properties of thick points for a 4-dimensional Gaussian Free Field. We adopt the definition of Gaussian Free Field on $\mathbb{R}^4$ introduced by Chen and Jakobson (2012) viewed as an abstract Wiener space with underlying Hilbert space $H^2(\mathbb{R}^4)$. We can prove that for $0 \le a \le 4$, the Hausdorff dimension of the set of a-high points is $4 - a$. We also show that the thick points give full mass to the Liouville Quantum Gravity measure on $\mathbb{R}^4$.

MSC: 60G60; 60G15

Keywords: KPZ; Liouville quantum gravity; Thick points; Hausdorff dimension; Abstract Wiener space; Bilaplacian

1. Introduction

Random measures defined by means of log-correlated Gaussian fields X and that can be formally written as "$m(\mathrm{d}\omega) = e^{aX(\omega)}\mathrm{d}\omega$" arise in conformal field theory and in the theory of Gaussian multiplicative Chaos (GMC). When X is an instance of the Gaussian Free Field (GFF) these measures are referred to as Liouville quantum gravity (LQG) measures. The interest around such objects comes from physics and in particular from the understanding and proving the KPZ relation, formulated by Knizhnik, Polyakov and Zamolodchikov [17], which gives the relation between volume exponents derived using the quantum metric induced by $m(\mathrm{d}\omega)$ and the

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Euclidean metric. Several interesting papers have been written to show this relation, and the first result is given by Duplantier and Sheffield [10] proving the formula for the planar case. For a clear explanation of this and other aspects of the KPZ we refer to Garban [11]. To construct such measures one has to rely on an approximation (cut-off) of the field and there are various methods to construct this approximation. While on the one hand a more geometric approach (which explicitly relies on the structure of the field) is present in the work Duplantier and Sheffield [10], the perspective of Robert and Vargas [26,27,24] dates back to the definition of Mandelbrot [21], Kahane [16] of multiplicative chaos, which deals with properties of the covariance kernel. These works extended the concept of multiplicative chaos of Kahane to a more general class of covariance kernels.

In this paper we focus our attention on the *multifractal formalism* of the underpinned Gaussian field, or with an equivalent terminology on its so-called *thick points*. To our knowledge the first rigorous study in this direction was made by Mandelbrot [22] in the context of one-dimensional log-correlated Gaussian fields. Hu et al. [13] showed that the Hausdorff dimension of the set of a-thick points is $2 - a$ for $0 \leq a \leq 2$ for the planar GFF (case of sphere average process). In Kahane [16], Rhodes and Vargas [24] such a result is shown for certain covariance kernels leading to multiplicative chaos. In this article we extend the results of Hu et al. [13] to 4 dimensions using the sphere average process introduced by Chen and Jakobson [3]. The set of thick points is relevant in understanding the support of the LQG. In fact it was shown in Duplantier and Sheffield [10] that the LQG measure is almost surely supported on the thick points, in analogy to Kahane's similar results [16] on 1D Gaussian multiplicative chaos and to Rhodes and Vargas [24, Theorem 4.1] in higher dimensions.

To give an analogy in $\mathbb{Z}^d$, one might look at the discrete Gaussian free field. It undergoes a phase transition at $d = 2$ in terms of the existence of an infinite-volume limit measure. Similarly the discrete membrane model (whose covariance is the inverse of the discrete Bilaplacian) shows the same change of phase in $d = 4$ (further results about it can be found for instance in Kurt [18,20]). In the critical dimension both fields possess logarithmically growing variances, and moreover the results contained in Daviaud [5] and Cipriani [4] show a similar fractal behavior of the thick points. In the continuum case, a natural analogue of the membrane model would be the Gaussian field arising from the inverse (in the sense of distributions) of the Bilaplacian operator. However, it is still an open problem to derive for it an appropriate sphere average in the sense of Duplantier and Sheffield [10]. In this direction, Chen and Jakobson [3] first constructed the sphere average process for the massive Bilaplacian Gaussian free field.

The construction of the set of thick points relies on the choice of cut-offs. One of them is the sphere average process $X_\epsilon(x)$, which can be taken as the average of the field over a ball of radius ϵ around x (in the rest of the paper we will assume the parameters denoted by ϵ, ϵ_1 etc. to be small). The main advantage of such cut-offs is that they enjoy the spatial Markov property, that is, informally, the processes $(X_{t+s}(x) - X_s(x))_{t\geq 0}$ and $(X_{t+s}(y) - X_s(y))_{t\geq 0}$ are independent whenever $\|x - y\|$ is large enough. Cut-offs can also be created by truncating appropriately the covariance function [24], or using the orthonormal basis representation for generalized Gaussian fields [14]. We prefer to stick to the more geometrical construction of the Gaussian free field, as in Chen and Jakobson [3] rather than handling it as an instance of multiplicative chaos, in the framework of Rhodes and Vargas [24, Theorem 4.2] although both approaches prove to be fruitful to investigate thick points. Differences between the two approaches are discussed in Section 2.1.

Main results and structure of the article: In Section 2 we recall the model introduced by Chen and Jakobson [3] and state our main result more precisely. We show in Theorem 2.1 that the set of thick points gives full mass to the LQG measure. Moreover, we show in Theorem 2.2 that the set of a-thick points has Hausdorff dimension $4 - a$ when $0 \le a \le 4$. When $a > 4$, the set of thick points is almost surely empty. In Section 3 we list some basic properties of the sphere average process and also provide a proof of Theorem 2.1 using a so-called *rooted* or *Peyrière measure*. The proof of Theorem 2.2 is given in Sections 4 and 5 and relies on proving two different bounds. For the upper bound we use the version of the Kolmogorov–Chentsov theorem derived by Hu et al. [13]. For the lower bound we use a standard finite-energy method and the spatial Markov property of the field. We discuss some comparison and open issues in Section 2.1.

2. GFF model and statement of the main results

To keep the paper self contained we review in this section some definitions of the GFF on $\mathbb{R}^4$ from Chen and Jakobson [3] and state some properties of the sphere average process which will be useful in deriving our main result. In order to do so we begin with the definition of abstract Wiener space.

Definition 2.1 (*Abstract Wiener Space, Stroock [30]*)**.** An *abstract Wiener space* is a triple $(\Theta, H, \mathcal{W})$, where

- Θ is a separable Banach space,
- H is a Hilbert space which is continuously embedded as a dense subspace of Θ, equipped with the scalar product $(\cdot, \cdot)_H$,
- $\mathcal{W}$ is a Gaussian probability measure on Θ defined as follows.

Let Θ^* be the dual space of Θ. Given any $x^* \in \Theta^*$ there exists a unique $h_{x^*} \in H$ such that for all $h \in H$, $\langle h, x^* \rangle = (h, h_{x^*})_H$ where $\langle \cdot, x^* \rangle$ denotes the action of x^* on Θ. The sigma algebra $\mathcal{B}(\Theta)$ on Θ is such that all the maps $\theta \mapsto \langle \theta, x^* \rangle$ are measurable. $\mathcal{W}$ is a probability measure such that for all $x^* \in \Theta^*$,

$$\mathsf{E}_{\mathcal{W}}\left[\exp\left(i\left\langle \cdot, x^*\right\rangle\right)\right] = \exp\left(-\frac{\|h_{x^*}\|_H^2}{2}\right). \tag{2.1}$$

Although the introduction of the set Θ is evidently important for the definition of the GFF, its choice is not unique; moreover $\mathcal{W}(H) = 0$ as explained in Stroock [30, Corollary 8.3.2 and Page 311]. In our setting, we consider the underlying Hilbert space to be $H := H^2\left(\mathbb{R}^4\right)$ which is the completion of the Schwartz space $\mathcal{S}\left(\mathbb{R}^4\right)$ equipped with the inner product

$$(f_1, f_2)_H = \int_{\mathbb{R}^4} (I - \Delta)^2 f_1(x) f_2(x)\, \mathrm{d}x \quad \text{for all } f_1, f_2 \in \mathcal{S}\left(\mathbb{R}^4\right).$$

$H^{-2}\left(\mathbb{R}^4\right)$ is the Hilbert space consisting of tempered distributions μ such that

$$\|\mu\|_{H^{-2}}^2 = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \left(1 + |\xi|^2\right)^{-2} \left|\hat{\mu}(\xi)\right|^2 \mathrm{d}\xi < \infty$$

where $\hat{\mu}$ is the Fourier transform. It is possible to identify H with H^{-2} through the linear isometry $(I - \Delta)^{-2} : H^{-2} \to H$. By abuse of notation we will call h_ν the image of $\nu \in H^{-2}$ under $(I - \Delta)^{-2}$, that is, h_ν is the unique element in H such that $\langle h, \nu \rangle = (h, h_\nu)_H$ for all $h \in H$. At this point we have to introduce another fundamental object for our work, the *Paley–Wiener integral* $\mathcal{I}(h_\nu)$. $\mathcal{I}$ is viewed as a mapping

$$\begin{aligned} &\mathcal{I} : x^* \in \Theta^* \mapsto \mathcal{I}(h_{x^*}) \in L^2(\mathcal{W}) \\ &\theta \in \Theta \mapsto [I(h_{x^*})](\theta) := \langle \theta, x^* \rangle . \end{aligned}$$

By (2.1), we have that $\left\{\mathcal{I}(h_\nu) : \nu \in H^{-2}\right\}$ is also a Gaussian family whose covariance is given by

$$\mathsf{E}_{\mathcal{W}}\left[\mathcal{I}\left(h_{\nu_1}\right)\mathcal{I}\left(h_{\nu_2}\right)\right] = \left\langle h_{\nu_1}, h_{\nu_2}\right\rangle_H = \langle \nu_1, \nu_2\rangle_{H^{-2}} .$$

Therefore $\mathcal{I}$ is an isometry from $\{h_{x^*} : x^* \in \Theta^*\} \to L^2(\mathcal{W})$, and since its domain is dense in H, it admits a unique extension to the whole of H.

For every $x \in \mathbb{R}^4$ and $\epsilon > 0$ denote as $\sigma_\epsilon^x \in H^{-2}$ the tempered distribution given by

$$\left\langle f, \sigma_\epsilon^x \right\rangle = \frac{1}{2\pi^2\epsilon^3}\int_{D(x,\epsilon)} f(y)\,\mathrm{d}\sigma(y), \quad \text{for all } f \in \mathcal{S}\left(\mathbb{R}^4\right),$$

where $\mathrm{d}\sigma$ is the surface area measure on $D(x, \epsilon)$, the sphere of radius ϵ around x. Interestingly, Chen and Jakobson [3] noted that $\left\{\mathcal{I}\left(h_{\sigma_\epsilon^x}\right) : \epsilon > 0\right\}$ fails to possess the Markov property and considered instead another Gaussian family:

$$\left\{\mathcal{I}\left(h_{\sigma_\epsilon^x}\right), \mathcal{I}\left(h_{\mathrm{d}\sigma_\epsilon^x}\right) : x \in \mathbb{R}^4, \epsilon > 0\right\},$$

with $\mathrm{d}\sigma_\epsilon^x$ the tempered distribution given by $\left\langle f, \mathrm{d}\sigma_\epsilon^x\right\rangle := \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left\langle f, \sigma_\epsilon^x\right\rangle$ for all $f \in \mathcal{S}\left(\mathbb{R}^4\right)$. It is important to point out at this juncture that such a collection is reminiscent of the double boundary conditions needed for the membrane model in the discrete case [19]. Let $\zeta := (1, 1)^T$ and

$$\mathbf{B}(r) := \begin{pmatrix} I_1(r)/r & I_1'(r) \\ I_2(r)/r & I_1''(r) \end{pmatrix},$$

where I_k are the modified Bessel functions of the first kind of order $k \in \mathbb{N}$ (for definitions of the Bessel functions that appear throughout the article a good reference is for example Abramowitz and Stegun [1, Chapter 9]). Define

$$\mu_\epsilon^x := \zeta^\top \mathbf{B}^{-1}(\epsilon)\begin{pmatrix} \sigma_\epsilon^x \\ \mathrm{d}\sigma_\epsilon^x \end{pmatrix}. \tag{2.2}$$

It was shown in Chen and Jakobson [3] that $\mu_\epsilon^x \in H^{-2}(\mathbb{R}^4)$ and $\left\{\mathcal{I}\left(h_{\mu_\epsilon^x}\right) : x \in \mathbb{R}^4, \epsilon > 0\right\}$ forms a Gaussian family with the correct Markovian properties and is the suitable candidate for the sphere average process in four dimensions.

Definition 2.2 (*Thick Points of the Sphere Average*)**.** For the sphere average process the set of *a-thick points* is defined as

$$T(a) := \left\{x \in \mathbb{R}^4 : \lim_{\epsilon\to 0}\frac{\mathcal{I}\left(h_{\mu_\epsilon^x}\right)}{\sqrt{2\pi^2 G(\epsilon)}} = \sqrt{2a}\right\}. \tag{2.3}$$

Here $G(\epsilon) = \mathsf{Var}_{\mathcal{W}}\left(\mathcal{I}\left(h_{\mu_\epsilon^x}\right)\right)$ and an explicit expression using Bessel functions is given in (3.1).

We would also need the definition of a set quite similar to the above:

$$T_{\geq}(a) = \left\{ x \in \mathbb{R}^4 : \limsup_{\epsilon \to 0} \frac{\mathcal{I}\left(h_{\mu_\epsilon^x}\right)}{\sqrt{2\pi^2} G(\epsilon)} \geq \sqrt{2a} \right\}. \tag{2.4}$$

It is easy to see that

$$T(a) \subset T_{\geq}(a).$$

One of the main results of Chen and Jakobson [3, Theorem 5] was to show the existence of the Liouville quantum gravity measure and the validity of the KPZ relation in $\mathbb{R}^4$. Define to this purpose a random measure on $\mathbb{R}^4$ by

$$m_\epsilon^\theta(\mathrm{d}x) := E_\epsilon^\theta(x)\mathrm{d}x,$$

where

$$E_\epsilon^\theta = \exp\left(\gamma \mathcal{I}\left(h_{\mu_\epsilon^x}\right) - \frac{\gamma^2}{2} G(\epsilon)\right), \quad \gamma > 0.$$

If $\epsilon_n = \epsilon_0^n$ with $\epsilon_0 \in (0, 1)$ and $0 < \gamma^2 < 2\pi^2$, then there exists a non-negative measure m^θ on $\mathbb{R}^4$ such that the following convergence holds for every $f \in C_c(\mathbb{R}^4)$:

$$\lim_{n \to +\infty} \int_{\mathbb{R}^4} f(x) m_{\epsilon_n}^\theta(\mathrm{d}x) = \int_{\mathbb{R}^4} f(x) m^\theta(\mathrm{d}x) \tag{2.5}$$

$\mathcal{W}$-almost surely and also in $L^2(\mathcal{W})$. It is also known that this measure is almost surely positive.

In the following theorem we show that the set of thick points gives full measure to the LQG measure in $\mathbb{R}^4$.

Theorem 2.1. *Let $0 < \gamma^2 < 2\pi^2$, then for $a = \gamma^2/4\pi^2$ we have*

$$m^\theta(T(a)^c) = 0 \quad \mathcal{W} - a.s.$$

That is, the set $T(a)$ gives full mass to the measure $m^\theta(\cdot)$.

For the proof of Theorem 2.1 we construct an auxiliary tool, the *rooted* or *Peyrière measure*. For the use of rooted measures see [10,24].

Before we state our main result on fractal properties of thick points, we recall the definition of Hausdorff dimension and Hausdorff measure.

Definition 2.3 (*Hausdorff Dimension*)**.** Let X be a metric space and $S \subseteq X$. For every $d \geq 0$ and $\delta > 0$ define the Hausdorff-d-measure in the following way:

$$C_\delta^d(S) := \inf\left\{ \sum_i \mathrm{diam}(E_i)^d : E_1, E_2, E_3, \ldots, \text{ cover } S \text{ and } \mathrm{diam}(E_i) \leq \delta \right\},$$

i.e. we are considering coverings of S by sets of diameter no more than δ. Then

$$C_{\mathcal{H}}^d(S) = \sup_{\delta > 0} C_\delta^d(S) = \lim_{\delta \downarrow 0} C_\delta^d(S)$$

is the Hausdorff-d-measure of the set S. The *Hausdorff dimension* of S is defined by

$$\dim_{\mathcal{H}}(S) := \inf\{d \geq 0 : C_{\mathcal{H}}^d(S) = 0\}.$$

Theorem 2.2. *For $0 \le a \le 4$, the Hausdorff dimension of $T(a)$ is $4-a$. For $a > 4$, we have that $T(a)$ is empty.*

Remark 2.1. The above result shows similarity with the membrane model. In Cipriani [4] it was shown that discrete fractal dimension of the a-high points is $4-4a^2$.

To prove Theorem 2.2 we apply some of the techniques implemented in Dembo et al. [7,6] to show similar results for occupation measures of planar or spatial Brownian motion.

2.1. Some comments and remarks

In this subsection we point out some open questions and comments on the existing literature on different cut-offs for Gaussian free fields.

1. Different cut-offs can be considered for the whole plane massive GFF. For example, note that the Green's function $G(x, y)$ for the Bessel operator $(\mathbb{I}-\Delta)^{d/2}$ on $\mathbb{R}^d$ is given by $K_0(\|x-y\|)$ where $K_0(r)$ denotes the modified Bessel function of second kind of order zero and is given by
$$G(x, y) = K_0(\|x-y\|) = \int_0^\infty e^{-u/2} e^{-\frac{\|x-y\|^2}{2u}} \frac{\mathrm{d}u}{u},$$
see Stein [28, Page 133]. This falls in the category of σ-positive kernels [16] and star-scale invariant kernels [2]. Now as in Garban et al. [12] one can consider an increasing sequence $(c_n)_{n\ge 1}$ with $c_1 = 1$ and an independent (in n) Gaussian sequence $(Y_n(x))$ with covariance given by
$$\mathbb{E}[Y_n(x)Y_n(y)] = \int_{c_{n-1}}^{c_n} k(u\|x-y\|)\frac{\mathrm{d}u}{u}, \tag{2.6}$$
where $k(r) = \frac{1}{2}\int_0^\infty e^{-\frac{r^2}{2v}} e^{-v/2}\mathrm{d}v$. Formally, the field can be represented as $X(x) = \sum_{n\ge 1} Y_n(x)$ and hence one can consider integral cut-offs of the form $X_n(x) = \sum_{k=1}^n Y_k(x)$. The main advantage of working with such cut-offs is that the exponential of the field becomes a positive martingale and logarithmic bounds due to the covariance structure give uniform integrability of the martingale, with the goal of proving the existence/uniqueness of the limiting quantum gravity measure. One can exploit the methods developed in Kahane [16] to conclude about the lower bound of thick points in such cases. However not all known cut-offs can be written in such fashion (as also pointed out in some examples in Robert and Vargas [26]). For instance, we could not show that the sphere average process can be related to any integral cut-off.
2. We also note that for $(X_t(x))$ a Gaussian process with covariance kernel given by
$$\mathbb{E}[X_t(x)X_t(y)] = \int_1^\infty k(u\|x-y\|)\frac{\mathrm{d}y}{y}$$
there is no long range independence, but still exponential decay for correlations. It would be interesting to see whether a modification of the proof for the lower bound presented in Section 5 could be adapted to these fields. Indeed, the present proof crucially depends on Lemma 3.1 and thus the independence of the Brownian motions becomes the most important aspect of it. The use of the exponential decay of the cut-off was recently exploited in the works of Duplantier et al. [8,9] and Rhodes and Vargas [25].

3. On $\mathbb{R}^4$, the Bilaplacian model on domains is still an open area of research and especially the construction of the associated sphere average process and its thick points. It would be interesting to derive any geometrical property which is analogous to the conformal invariance of thick points derived by Hu et al. [13]. We believe the methods used in Chen and Jakobson [3] and here can act as a starting point of such a study. The sphere average process in fact appears as a combination of two measures and clearly indicates the role of boundary conditions in a Bilaplacian boundary value problem.
4. Although not directly related to the present article, the problem of the choice of cut-offs is not irrelevant. In general the almost sure uniqueness of the limiting quantum gravity measure is an open problem. Such universality results date back to Kahane [16] who showed uniqueness in law of GMC under certain conditions. The conditions were relaxed in Robert and Vargas [26]. The almost sure question in the case of the planar GFF was partially dealt with in Duplantier and Sheffield [10] where it was shown that the LQG measures arising out of sphere averages and orthonormal basis truncations are almost surely equal. It remains to be investigated whether there are sufficient conditions on cut-offs which lead to the same LQG measure or GMC almost surely. Also we note that the proof of Duplantier and Sheffield [10] uses conformal properties and hence it is not clear how one could extend such results in higher dimensions. Similarly, one may inquire if these uniqueness results are also true for thick points.

We are looking at some of the issues pointed out here and we intend to address them in a future paper.

3. GFF model and some estimates

This section is devoted to providing some details about the behavior of the sphere average process, such as the covariance structure. We then use them to derive a proof of Theorem 2.1.

3.1. Some more properties of the sphere average process: covariance structure

Let us denote as $D(0, R)$ the sphere centered at 0 with radius $R > 0$. Let I_r, K_r be the modified Bessel functions of order $r \in \mathbb{N} \cup \{0\}$. Define the positive function $G : (0, \infty) \mapsto (0, \infty)$ by

$$G(r) := \left(-\frac{1}{4\pi^2}\right) \frac{2I_1(r) K_1(r) + 2I_2(r) K_0(r) - 1}{I_1^2(r) - I_0(r) I_2(r)}. \tag{3.1}$$

It can be shown that G is strictly decreasing and smooth, with $\lim_{r\to 0} G(r) = +\infty$ and $\lim_{r\to+\infty} G(r) = 0$. It also follows from the properties of the Bessel functions that as r decreases to 0, $G(r)$ asymptotically behaves like $-\frac{1}{2\pi^2}\log r$. Then, we have that

1. given $x \in \mathbb{R}^4$ and $\epsilon_1 \geq \epsilon_2 > 0$,

$$\mathsf{E}_{\mathcal{W}}\left[\mathcal{I}\left(h_{\mu^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{\mu^x_{\epsilon_2}}\right)\right] = \mathsf{E}_{\mathcal{W}}\left[\mathcal{I}^2\left(h_{\mu^x_{\epsilon_1}}\right)\right] = G(\epsilon_1). \tag{3.2}$$

2. Given $x, y \in \mathbb{R}^4$, $x \neq y$, and $\epsilon_1, \epsilon_2 > 0$ with $\overline{D(x, \epsilon_1)} \cap \overline{D(y, \epsilon_2)} = \emptyset$,

$$\mathsf{E}_{\mathcal{W}}\left[\mathcal{I}\left(h_{\mu^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{\mu^y_{\epsilon_2}}\right)\right] = \frac{1}{2\pi^2} K_0(\|x - y\|). \tag{3.3}$$

3. Given $x, y \in \mathbb{R}^4$, $x \neq y$, and $\epsilon_1, \epsilon_2 > 0$ with $D(y, \epsilon_2) \subseteq D(x, \epsilon_1)$,

$$\mathsf{E}_{\mathcal{W}}\left[\mathcal{I}\left(h_{\mu^x_{\epsilon_1}}\right)\mathcal{I}\left(h_{\mu^y_{\epsilon_2}}\right)\right] = I_0\left(\|x-y\|\right)G\left(\epsilon_1\right) - \frac{1}{4\pi^2}\frac{I_2\left(\|x-y\|\right)}{I_1^2\left(\epsilon_1\right) - I_0\left(\epsilon_1\right)I_2\left(\epsilon_1\right)}. \tag{3.4}$$

We would like to point out that $K_0(x) = \int_0^\infty e^{\frac{x^2}{4v}} e^{-v}\frac{\mathrm{d}v}{v}$. Now using the saddle point method it is easy to show that this integral is bounded and in fact decays exponentially to 0 as $\|x-y\| \to +\infty$. On the other hand for $\|x - y\|$ bounded it can be shown, using Abramowitz and Stegun [1, Equation 9. 6. 54], that

$$K_0(x) = -\left(\log\frac{x}{2} + \gamma\right)I_0(x) + 2\sum_{k=1}^{+\infty}\frac{I_{2k}(x)}{k} = \log\frac{1}{\|x\|} + H(x), \tag{3.5}$$

with H uniformly bounded and γ the Euler–Mascheroni constant, using the expansion (see Abramowitz and Stegun [1, Equation 9. 6. 10])

$$I_\nu(z) = \left(\frac{1}{2}z^2\right)\sum_{k=0}^{+\infty}\frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)}. \tag{3.6}$$

Hence (3.3) shows that sphere average processes indexed by disjoint spheres have logarithmic decay of correlations. The independence of ϵ_1 and ϵ_2 on the right-hand side of (3.3) is also crucial to prove Lemma 3.1.

The next lemma states one of the most useful and important properties of the spherical average process and is analogous to the properties of the two dimensional circle average studied in Duplantier and Sheffield [10], Hu et al. [13]. It shows that for fixed $x \in \mathbb{R}^4$, the spherical average after a time change is a Brownian motion and in disjoint annuli two such motions are independent. We briefly sketch the proof of the following lemma as it is an easy consequence after one compares the covariance structure.

Lemma 3.1. (a) *Let $G(\cdot)$ be as in (3.1) and for $x \in \mathbb{R}^4$, let $B(x,t) = \mathcal{I}\left(h_{\mu^x_{G^{-1}(t)}}\right)$. Then $B(x,t) - B(x,t_1)$ has the same distribution as a standard Brownian motion for $t \geq t_1$.*

(b) *Given $x, y \in \mathbb{R}^4$ and $t_1 \leq t \leq t_2$ and $s_1 \leq s \leq s_2$ be such that $D(x, G^{-1}(s_1)) \setminus D(x, G^{-1}(s_2))$ and $D(y, G^{-1}(t_1)) \setminus D(y, G^{-1}(t_2))$ are disjoint, then $\{B(x,s) - B(x,s_1)\}_{s_1 \leq s \leq s_2}$ is independent of $\{B(y,t) - B(y,t_1)\}_{t_1 \leq t \leq t_2}$.*

Proof. (a) It follows from (3.2) that for $t_1 \leq s \leq t$ one has

$$\begin{aligned}&\mathrm{Cov}_{\mathcal{W}}(B(x,t) - B(x,t_1), B(x,s) - B(x,t_1))\\ &\quad = G(G^{-1}(s)) - G(G^{-1}(t_1)) - G(G^{-1}(t_1)) + G(G^{-1}(t_1)) = s - t_1.\end{aligned}$$

Here we have used the fact that $G(\cdot)$ and $G^{-1}(\cdot)$ are decreasing functions and hence, as $t_1 \leq s \leq t$ we have $G^{-1}(t_1) \geq G^{-1}(s) \geq G^{-1}(t)$.

(b) As the annuli are disjoint it follows that $\|x-y\| > G^{-1}(t_1) + G^{-1}(s_1) \geq G^{-1}(t) + G^{-1}(s) \geq G^{-1}(t_1) + G^{-1}(s_1)$ and hence using (3.3) we obtain

$$\mathrm{Cov}_{\mathcal{W}}\left(B(y,t) - B(y,t_1), B(x,s) - B(x,s_1)\right) = 0. \quad \square$$

3.2. Proof of Theorem 2.1

Let Γ be a compact subset of $\mathbb{R}^4$. Let $\mathcal{B}(\Gamma)$ be the Borel sigma algebra of Γ. We define a rooted measure on $\mathcal{B}(\Theta) \otimes \mathcal{B}(\Gamma)$ as

$$\mathcal{M}(\mathrm{d}x\mathrm{d}\theta) := \frac{m^\theta(\mathrm{d}x)\mathcal{W}(\mathrm{d}\theta)}{|\Gamma|}.$$

Here $|\Gamma|$ denotes the volume of the set Γ with respect to the Lebesgue measure. Note that $\mathcal{M}(\Theta \times \Gamma) = \mathsf{E}_{\mathcal{W}}\left[m^\theta(\Gamma)\right]|\Gamma|^{-1} = 1$ and as such $\mathcal{M}$ is a probability measure on the space $\Gamma \times \Theta$.

Let $r(t) := G^{-1}(t + G(R))$, $R > 0$ fixed and define

$$\widetilde{B}(x,t)(\theta) := \mathcal{I}\left(h_{\mu^x_{r(t)}}\right)(\theta) - \mathcal{I}\left(h_{\mu^x_R}\right)(\theta).$$

The following lemma allows us to view the random measure m^θ in a different way. We show that the joint distribution of $(x, \widetilde{B}(x,t))$ under $\mathcal{M}(\mathrm{d}x\mathrm{d}\theta)$ is nothing but the distribution of $(x, \widetilde{B}(x,t) + \gamma t)$ under $\mathcal{W}(\mathrm{d}\theta)\mathrm{d}x$ (where $\gamma \in (0, 2\pi^2)$ as in the statement of the theorem) and in the latter case the marginal on Θ does not depend on x.

Lemma 3.2. *Let $0 < \gamma^2 < 2\pi^2$. For any compact set Γ and any $F \in C_c(\mathbb{R}^4 \times \mathbb{R})$ we have*

$$\int_\Theta \int_\Gamma F(x, \widetilde{B}(x,t)(\theta))\mathcal{M}(\mathrm{d}x\mathrm{d}\theta) = \frac{1}{|\Gamma|}\int_\Gamma \int_\Theta F(x, \widetilde{B}(x,t)(\theta) + \gamma t)\mathcal{W}(\mathrm{d}\theta)\mathrm{d}x. \tag{3.7}$$

Proof. Note that for almost every θ, the map $\Gamma \ni x \mapsto F(x, \widetilde{B}(x,t)(\theta))$ is continuous by Corollary 3 of [3]. So from the vague convergence in (2.5) we have $\mathcal{W}$-a.s.

$$\lim_{n\to\infty} \int_\Gamma F(x, \widetilde{B}(x,t))m^\theta_{\epsilon_n}(\mathrm{d}x) = \int_\Gamma F(x, \widetilde{B}(x,t))m^\theta(\mathrm{d}x).$$

Since the function in the integral is bounded we have for some constant C and for all n

$$\int_\Theta \int_\Gamma F(x, \widetilde{B}(x,t))m^\theta_{\epsilon_n}(\mathrm{d}x)\mathcal{W}(\mathrm{d}\theta) \le C|\Gamma|.$$

So by dominated convergence

$$\lim_{n\to\infty} \frac{1}{|\Gamma|}\int_\Theta \int_\Gamma F(x, \widetilde{B}(x,t))m^\theta_{\epsilon_n}(\mathrm{d}x)\mathcal{W}(\mathrm{d}\theta) = \int_\Theta \int_\Gamma F(x, \widetilde{B}(x,t))\mathcal{M}(\mathrm{d}x\mathrm{d}\theta). \tag{3.8}$$

Note that for small enough $\epsilon > 0$

$$\mathrm{Cov}(\widetilde{B}(x,t), h_{\mu^x_\epsilon}) = G(r(t)) - G(R) = t$$

holds, so

$$\begin{aligned}\int_\Theta \int_\Gamma F(x, \widetilde{B}(x,t))\mathcal{M}(\mathrm{d}x\mathrm{d}\theta) &\overset{(3.8)}{=} \lim_{n\to+\infty} \frac{1}{|\Gamma|}\int_\Theta \int_\Gamma F(x, \widetilde{B}(x,t))m^\theta_{\epsilon_n}(\mathrm{d}x)\mathcal{W}(\mathrm{d}\theta) \\ &= \lim_{n\to+\infty} \frac{1}{|\Gamma|}\int_\Theta \int_\Gamma F(x, \widetilde{B}(x,t))E^\theta_{\epsilon_n}(x)\mathrm{d}x\mathcal{W}(\mathrm{d}\theta) \\ &= \frac{1}{|\Gamma|}\int_\Gamma \int_\Theta F(x, \widetilde{B}(x,t)(\theta) + \gamma t)\mathcal{W}(\mathrm{d}\theta)\mathrm{d}x\end{aligned}$$

where in the third equality we have employed the Cameron–Martin theorem. $\square$

Proof of Theorem 2.1. Using the fact that $\mathrm{E}_{\mathcal{W}}\left[m^{\theta}(A)\right] = |A|$ for any bounded set A it follows that the marginal of $\mathcal{M}$ on Γ is nothing but the normalized Lebesgue measure on Γ. Hence by Theorem 9.2.2. of [30] there exists a Borel measurable map

$$\Gamma \ni x \to \mathcal{L}_x(\cdot) \in M_1(\Theta),$$

where $M_1(\Theta)$ is the set of probability measures on Θ for which the following holds:

$$\mathcal{M}(\mathrm{d}x\mathrm{d}\theta) = \mathcal{L}_x(\mathrm{d}\theta)\frac{\mathrm{d}x}{|\Gamma|}.$$

Note that $\mathcal{L}_x(\mathrm{d}\theta)$ is nothing but a regular conditional probability. Now using the above decomposition we have that

$$\int_{\Theta}\int_{\Gamma} F(x, \widetilde{B}(x,t))\mathcal{M}(\mathrm{d}x\mathrm{d}\theta) = \frac{1}{|\Gamma|}\int_{\Gamma}\int_{\Theta} F(x, \widetilde{B}(x,t))\mathcal{L}_x(\mathrm{d}\theta)\mathrm{d}x.$$

So from Lemma 3.2 we have for any compact set Γ and $F \in C_c(\mathbb{R}^4 \times \mathbb{R})$

$$\frac{1}{|\Gamma|}\int_{\Gamma}\int_{\Theta} F(x, \widetilde{B}(x,t))\mathcal{L}_x(\mathrm{d}\theta)\mathrm{d}x = \frac{1}{|\Gamma|}\int_{\Gamma}\int_{\Theta} F(x, \widetilde{B}(x,t) + \gamma t)\mathcal{W}(\mathrm{d}\theta)\mathrm{d}x. \tag{3.9}$$

If we denote by μ_x the law of $\widetilde{B}(x,t)$ under $\mathcal{L}_x(\mathrm{d}\theta)$ and by ν the law of $\widetilde{B}(x,t) + \gamma t$ under $\mathcal{W}(\mathrm{d}\theta)$ on $\mathbb{R}$, it is easy to show that for almost every $x \in \mathbb{R}^4$, $\mu_x = \nu$, since (3.9) holds for any compact set Γ. It is also possible to see that ν is the law of a standard Brownian motion with a drift. If we take $a = \gamma^2/4\pi^2$ and use the fact that the sphere average process is a time inversion of a Brownian motion (due to Lemma 3.1), then the set of thick points can also be written as

$$T(a) = \left\{x \in \mathbb{R}^4 : \lim_{t\to\infty}\frac{\widetilde{B}(x,t)}{t} = \gamma\right\}.$$

Now from the discussion above we have that

$$\mathcal{M}(T(a)^c) = \frac{1}{|\Gamma|}\int_{\Gamma}\mathcal{L}_x(T(a)^c)\mathrm{d}x.$$

Since the law of $\widetilde{B}(x,t)$ under $\mathcal{L}_x$ is the same as the law of Brownian motion with a drift, the condition for the thick points gets satisfied with probability 1. So we have $\mathcal{M}(T(a)^c) = 0$, which, together with the fact that $m^{\theta}(\cdot)$ is a. s. a positive measure, proves the result. □

4. Upper bound of Theorem 2.2

In this section we prove the upper bound. By the countable stability property, viz.

$$\dim_{\mathcal{H}}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_{1\le i\le\infty}\dim_{\mathcal{H}}(E_i)$$

it is enough to show that for $R \ge 1$

$$\dim_{\mathcal{H}} T_{\ge}(a,R) = \dim_{\mathcal{H}}\left\{x \in D(0,R) : \limsup_{\epsilon\to 0}\frac{\mathcal{I}\left(h_{\mu_\epsilon^x}\right)}{\sqrt{2\pi^2 G(\epsilon)}} \ge \sqrt{2a}\right\} \le 4 - a \tag{4.1}$$

almost surely. The next proposition gives the local Hölder continuity of the process and through it we can determine a modification of the process which has some uniform estimates on the increments. It is similar to Proposition 2.1 of Hu et al. [13] and uses Lemma C.1 of Hu et al. [13]. The proof also uses some finer estimates on the covariance functions and some bounds on Bessel functions which are provided in the Appendix.

Proposition 4.1. *There exists a modification* $\widetilde{X}$ *of the process* $\{\mathcal{I}\left(h_{\mu_t^z}\right) : z \in D(0,R),\ t \in (0,1)\}$ *such that for every* $0 < \gamma < \frac{1}{2}$ *and* $\epsilon, \zeta > 0$ *there exists* $M > 0$ *such that the following holds:*

$$|\widetilde{X}(z,r) - \widetilde{X}(w,s)| \le M\left(\log\frac{1}{r}\right)^{\zeta}\frac{|(z,r)-(w,s)|^{\gamma}}{r^{(1+\epsilon)\gamma}}, \tag{4.2}$$

for all $z, w \in D(0,R)$ *and* $r, s \in (0,1]$ *with* $1/2 \le r/s \le 2$.

Proof. Consider $x,\ y \in D(0,R)$, $\epsilon_1,\ \epsilon_2 \in (0,1)$ and we abbreviate

$$H_{\epsilon_1,\epsilon_2}(x,y) := \mathsf{Cov}_{\mathcal{W}}\left(\mathcal{I}\left(h_{\mu^x_{\epsilon_1}}\right), \mathcal{I}\left(h_{\mu^y_{\epsilon_2}}\right)\right).$$

We distinguish between three cases:

Case 1. Let $x = y$. By Lemma A.1, we have

$$\begin{aligned}
&|H_{\epsilon_1,\epsilon_1}(x,x) - H_{\epsilon_2,\epsilon_1}(x,x)| \\
&\quad\le |H_{\epsilon_1,\epsilon_1}(x,x) - H_{\epsilon_1,\epsilon_2}(x,x)| + |H_{\epsilon_2,\epsilon_1}(x,x) - H_{\epsilon_1,\epsilon_2}(x,x)| \\
&\quad\overset{(3.2)}{\le} |G(\epsilon_1) - G(\epsilon_1\vee\epsilon_2)| + |G(\epsilon_2) - G(\epsilon_1\vee\epsilon_2)| \\
&\quad\le C\frac{|\epsilon_1-\epsilon_2|}{\epsilon_1\wedge\epsilon_2}.
\end{aligned}$$

Here we have used that $|\log(z/w)| \le \frac{\|z-w\|}{z\wedge w}$ for all $z,\ w \in (0,+\infty)$.

Case 2. Let $\overline{D(x,\ \epsilon_1)} \cap \overline{D(y,\ \epsilon_2)} = \emptyset$. In this case $\|x-y\| > \epsilon_1+\epsilon_2 > \epsilon_1$. Then

$$\begin{aligned}
|H_{\epsilon_1,\epsilon_1}(x,x) - H_{\epsilon_1,\epsilon_2}(x,y)| &= \left|G(\epsilon_1) - \frac{1}{2\pi^2}K_0(\|x-y\|)\right| \\
&\overset{(3.5)}{\le} -C(\log\epsilon_1 + \log(\|x-y\|)) \le \frac{\|x-y\|}{\epsilon_1}.
\end{aligned}$$

Note that in applying (3.5) a bounded constant does not affect the statement of the theorem. Similarly one can show that $|H_{\epsilon_2,\epsilon_2}(y,y) - H_{\epsilon_1,\epsilon_2}(x,y)| \le \frac{\|x-y\|}{\epsilon_1}$.

Case 3. For $\overline{D(y,\ \epsilon_2)} \subseteq D(x,\ \epsilon_1)$ one obtains

$$\begin{aligned}
&|H_{\epsilon_1,\epsilon_1}(x,x) - H_{\epsilon_1,\epsilon_2}(x,y)| \\
&\quad\le |G(\epsilon_1)(1 - I_0(\|x-y\|))| + C\frac{I_2(\|x-y\|)}{I_1^2(\epsilon_1) - I_0(\epsilon_1)I_2(\epsilon_1)} \\
&\quad\le -C\log\epsilon_1\|x-y\|^2 + \frac{\|x-y\|^2}{\epsilon_1^2} \le C\frac{\|x-y\|}{\epsilon_1}.
\end{aligned}$$

Here we have used the series expansion (3.6).

Note that in Cases 1 and 3 we can choose ϵ_1 and ϵ_2 small so that the asymptotics of the Bessel functions are justified. Combining these three cases yields

$$\mathsf{Var}_{\mathcal{W}}\left(\mathcal{I}\left(h_{\mu^x_{\epsilon_1}}\right) - \mathcal{I}\left(h_{\mu^y_{\epsilon_2}}\right)\right) \le C\frac{\|x-y\| + |\epsilon_1-\epsilon_2|}{\epsilon_1\wedge\epsilon_2}. \tag{4.3}$$

Since $\mathcal{I}\left(h_{\mu^x_{\epsilon_1}}\right) - \mathcal{I}\left(h_{\mu^y_{\epsilon_2}}\right)$ is Gaussian,

$$\mathsf{E}_{\mathcal{W}}\left[\left|\mathcal{I}\left(h_{\mu^x_{\epsilon_1}}\right) - \mathcal{I}\left(h_{\mu^y_{\epsilon_2}}\right)\right|^{\alpha}\right] \le C\left(\frac{\|x-y\| + |\epsilon_1 - \epsilon_2|}{\epsilon_1 \wedge \epsilon_2}\right)^{\alpha/2}.$$

We can find α and β large enough such that $|\frac{\beta}{\alpha} - \frac{1}{2}| < \delta$, and consequently by Hu et al. [13, Lemma C.1] there exists a modification $\widetilde{X}(x,\epsilon) = \mathcal{I}\left(h_{\mu^x_\epsilon}\right)$ a.s. on $L^2(\mathcal{W})$ satisfying (4.2). $\square$

In this section for the proof of the upper bound we work with this modification which we also denote by $\mathcal{I}\left(h_{\mu^x_t}\right)$. Recall that $B(x,t) = \mathcal{I}\left(h_{\mu^x_{G^{-1}(t)}}\right)$.

Proof of the upper bound. Let $\varepsilon > 0$ and $\gamma \in (0, 1/2)$, $\zeta \in (0,1)$ and denote $\tilde{\gamma} := (1+\varepsilon)\gamma$. Also let $K := \varepsilon^{-1}$, $r_n := n^{-K}$.

Define the set

$$U_R := \left\{x \in D(0,R) : \limsup_{n\to+\infty} \frac{\mathcal{I}\left(h_{\mu^x_{r_n}}\right)}{\sqrt{2\pi^2}G(r_n)} \ge \sqrt{2a}\right\}.$$

We first show that

$$T_{\ge}(a,R) \subset U_R. \tag{4.4}$$

For $x \in T_{\ge}(a,R)$ and for $t \in (G(r_n),\ G(r_{n+1}))$ we write $B(x, G(r_n)) = B(x, G(r_n)) - B(x,t) + B(x,t)$. By Proposition 4.1 we have

$$\begin{aligned}|B(x,t) - B(x,G(r_n))| &\le M\left(\log\left(\frac{1}{G^{-1}(t)}\right)\right)^{\zeta}\frac{\left(G^{-1}(t) - r_n\right)^{\gamma}}{G^{-1}(t)^{\tilde{\gamma}}}\\ &\le M(\log(n+1))^{\zeta}\frac{(r_{n+1} - r_n)^{\gamma}}{r_{n+1}^{\tilde{\gamma}}} = \mathrm{O}\left((\log n)^{\zeta}\right).\end{aligned} \tag{4.5}$$

Hence using the fact that $G(r_n) \sim C\log n$ for $n \to +\infty$ and $\zeta < 1$ we have

$$\left|\frac{B(x,G(r_n)) - B(x,t)}{\sqrt{2\pi^2}G(r_n)}\right| = \mathrm{O}\left(\frac{(\log n)^{\zeta}}{G(r_n)}\right) = \mathrm{o}(1).$$

Now (4.4) follows as we have

$$\limsup_{n\to+\infty}\frac{B(x,G(r_n))}{\sqrt{2\pi^2}G(r_n)} \ge \limsup_{t\to+\infty}\frac{B(x,t)}{\sqrt{2\pi^2}t} \ge \sqrt{2a}.$$

The next step is to determine a cover for the set U_R. In view of that, let $\{x_{nj} : j = 1, \dots, \bar{k}_n\}$, be a maximal collection of points in $D(0,R)$ such that $\inf_{l\neq j}\|x_{nj} - x_{nl}\| \ge r_n^{1+\varepsilon}$. Note that there exists a constant c' such that $\bar{k}_n \le c' {r_n}^{-4(1+\epsilon)}$. Denote

$$\mathcal{A}_n := \left\{1 \le j \le \bar{k}_n : \frac{|B(x_{nj}, G(r_n))|}{\sqrt{2\pi^2}G(r_n)} \ge \sqrt{2a} - \delta(n)\right\}$$

with $\delta(n) = C(\log n)^{\zeta-1}$ (the constant C will be tuned later according to (4.6)).

We now show that for any $N \ge 1$, $\bigcup_{n\ge N}\bigcup_{j\in\mathcal{A}_N} D\left(x_{nj}, r_n^{1+\varepsilon}\right)$ covers U_R with sets having maximal diameter $2r_N^{1+\varepsilon}$. First note that for any $x \in D(0,R)$, there exists $j \in \left\{1, \dots, \bar{k}_n\right\}$

such that $x \in D\left(x_{nj}, r_n^{1+\varepsilon}\right)$. Otherwise, this would contradict the maximality of the set $\{x_{nj} : j = 1, \dots, \bar{k}_n\}$. For any $x \in U_R$ we have that for $N \geq 1$, there exists $n \geq N$ such that by Proposition 4.1 the follows holds:

$$\frac{|B(x_{nj}, G(r_n)) - B(x, G(r_n))|}{\sqrt{2\pi^2}G(r_n)} \leq C(\log n)^{\zeta} \frac{\|x - x_{nj}\|^{\gamma}}{G(r_n)^{\tilde{\gamma}+1}}$$
$$= \delta(n)\frac{\log n}{G(r_n)} \leq C\delta(n) \tag{4.6}$$

which implies, renaming possibly $\delta(n)$,

$$\frac{B(x_{nj}, G(r_n))}{2\pi^2 G(r_n)} \geq \sqrt{2a} - \delta(n).$$

Hence we have $x \in D(x_{nj}, r_n^{1+\epsilon})$ with $j \in \mathcal{A}_n$. Let us assume for the moment that, for any $a \in (0, 4]$, there exists a constant C' (the constant may vary later) such that

$$\mathsf{E}_{\mathcal{W}}[|\mathcal{A}_n|] \leq C'(\log n)^{-1/2} r_n^{a-4(1+\varepsilon)+v_n}, \tag{4.7}$$

where $v_n \to 0$ as $n \to \infty$. If we choose $\alpha := 4 - a + \varepsilon\frac{4+a}{1+\varepsilon}$ we have, setting N large,

$$\mathsf{E}_{\mathcal{W}}\left[\sum_{n\geq N}\sum_{j\in\mathcal{A}_n} \operatorname{diam}(D(x_{nj}, r_n^{1+\varepsilon}))^{\alpha}\right] \leq \sum_{n\geq N} \mathsf{E}_{\mathcal{W}}[|\mathcal{A}_n|]\,(2r_n^{1+\epsilon})^{\alpha}$$
$$\leq \sum_{n\geq N} (\log n)^{-1/2} r_n^{(1+\varepsilon)\alpha+a-4(1+\varepsilon)+v_n}$$
$$\leq \sum_{n\geq N} (\log n)^{-1/2} r_n^{4\varepsilon+v_n}. \tag{4.8}$$

Now as $v_n \to 0$ one can choose n large enough, so that $v_n \leq \varepsilon/2$. Using $r_n = n^{-1/\varepsilon}$ it follows that the RHS of (4.8) is finite. Hence we have $\sum_{n\geq N}\sum_{j\in\mathcal{A}_n} \operatorname{diam}(D(x_{nj}, r_n^{1+\varepsilon}))^{\alpha} < +\infty$ almost surely. Therefore, using notation from Definition 2.3 we get,

$$C^{\alpha}_{2r_N^{1+\varepsilon}}(U_R) \leq \sum_{n\geq N}\sum_{j\in\mathcal{A}_n} \operatorname{diam}(D(x_{nj}, r_n^{1+\varepsilon}))^{\alpha}.$$

Now taking $N \to \infty$, since the right hand side is finite, we have $C^{\alpha}_{\mathcal{H}}(U_R) = 0$ almost surely. So we have that,

$$\dim_{\mathcal{H}}(U_R) \leq 4 - a + \varepsilon\frac{4+a}{1+\varepsilon}.$$

Now letting $\varepsilon \downarrow 0$ implies $\dim_{\mathcal{H}}(T_{\geq}(a, r)) \leq 4 - a$ a.s. This completes the proof of the upper bound provided we show (4.7). We first estimate $\mathcal{W}(j \in \mathcal{A}_n)$ as follows:

$$\mathcal{W}(j \in \mathcal{A}_n) = \mathcal{W}\left(\frac{|B(x_{nj}, G(r_n))|}{\sqrt{G(r_n)}} \geq (\sqrt{2a} - \delta(n))\sqrt{2\pi^2}\sqrt{G(r_n)}\right)$$
$$\leq C'(a + v_n)^{-1/2} G(r_n)^{-1/2} \exp\left(-a\,(1 + v_n)\,2\pi^2 G(r_n)\right)$$
$$\leq C'(\log n)^{-1/2} r_n^{a+v_n},$$

since $G(r_n) \sim -\frac{\log r_n}{2\pi^2}$ as $n \to +\infty$ and $v_n = c_1(\log n)^{2(\zeta-1)} - c_2(\log n)^{\zeta-1}$ for some constants c_1 and c_2. Since $\zeta \in (0,1)$, we have $v_n \to 0$. Furthermore

$$\mathsf{E}_{\mathcal{W}}[|\mathcal{A}_n|] \le C'(\log n)^{-1/2}\overline{k}_n r_n^{(a+v_n)} \le C'(\log n)^{-1/2} r_n^{a+v_n-4(1+\varepsilon)}.$$

This proves (4.7) and hence the upper bound.

Now we show that for every $R > 1$, $T_{\geq}(a, R)$ is empty for $a > 4$ using the above estimates. Note that

$$\sum_{n\geq 1} \mathcal{W}(|\mathcal{A}_n| > 1) \le \sum_{n\geq 1} \mathsf{E}_{\mathcal{W}}[|\mathcal{A}_n|] \le \sum_{n\geq 1} r_n^{a-4(1+\varepsilon)} \le C' \sum_{n\geq 1} n^{-4} < +\infty$$

and hence by the Borel–Cantelli lemma we can conclude that, if ε becomes arbitrarily small, $|\mathcal{A}_n| = 0$ eventually and so $T_{\geq}(a, R)$ is empty for $a > 4$ with probability one. □

5. Lower bound of Theorem 2.2

To derive the lower bound we use the energy method, for whose details we refer to Section 4.3 of Mörters et al. [23]. The α-th energy of a measure μ is given by

$$\widetilde{I}_\alpha(\mu) = \iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{\|x-y\|^\alpha}.$$

Given a set A, the method allows to say that if we can find a measure ρ such that $\widetilde{I}_\alpha(\rho) < \infty$, then $\dim_H(A) > \alpha$. For this, partition the hypercube $J := [0,1]^4$ into s_n^{-4} smaller hypercubes of radius $s_n = \frac{1}{n!}$. Let x_{ni} be the centers of these hypercubes and C_n be the set of these centers. Define $t_m := G(s_m)$ for all $m \le n$. Note that since G is decreasing we have that t_m is increasing and also using the asymptotic expansion of G we have $t_m = -\frac{\log s_m}{2\pi^2}(1+\mathrm{o}(1))$. Let $A_m(x)$, $B_m(x)$ be the events

$$A_m(x) := \left\{ \sup_{t_m < t \le t_{m+1}} |B(x,t) - B(x,t_m) - \sqrt{4a\pi^2}(t - t_m)| \le \sqrt{t_{m+1} - t_m} \right\},$$

$$B_m(x) := \left\{ \sup_{t \geq t_m} |B(x,t) - B(x,t_m)| - t \le 1 - t_m \right\}.$$

We say that x is an *n-perfect a-thick point* if $E^n(x) := \bigcap_{m\le n} A_m(x) \cap B_{n+1}(x)$ occurs. Note that $B_{n+1}(x)$ is independent of the other events. We introduce a random variable Y_{ni} for $i = 1, \ldots, |C_n|$ such that

$$Y_{ni} = \begin{cases} 1 & \text{if } x_{ni} \text{ is an } n\text{-perfect } a\text{-thick point}, \\ 0 & \text{otherwise}. \end{cases}$$

Fix $t_m < t \le t_{m+1}$ and on the event $E^n(x)$ we have, as $m \to \infty$,

$$|B(x,t) - B(x,t_1) - \sqrt{4a\pi^2}(t - t_1)| = \mathrm{o}(m \log m) = \mathrm{o}(t). \tag{5.1}$$

Define now the set of perfect a-thick points as

$$P(a) := \bigcap_{k\geq 1} \overline{\bigcup_{n\geq k} \bigcup_{z\in C_n(a)} S(z, s_n)},$$

where $C_n(a)$ is the set of centers of which x_{ni} is a n-perfect thick point and $S(z, r)$ is a hypercube of radius r centered around z. Let

$$T(a, J) := \left\{ x \in J : \lim_{t\to\infty} \frac{\mathcal{I}\left(h_{\mu^x_{G^{-1}(t)}}\right)}{\sqrt{2\pi^2 t}} = a \right\} \subset T(a).$$

Lemma 5.1.

$$P(a) \subseteq T(a, J). \tag{5.2}$$

Proof. If $z \in P(a)$ there exists a sequence $(z_{n_k})_{k\in\mathbb{N}}$ of points such that $z_{n_k} \in C_n(a)$ for all k and $\|z - z_{n_k}\| \le s_n$. For m such that $t_m < t \le t_{m+1}$

$$\left| B(z_{n_k}, t) - B(z_{n_k}, t_1) - \sqrt{4a\pi^2}(t - t_1) \right| = \mathrm{o}(t)$$

follows as in (5.1). Since the Brownian motion is a.s. continuous taking the limit for $k \to +\infty$

$$\left| B(z, t) - B(z, t_1) - \sqrt{4a\pi^2}(t - t_1) \right| = \mathrm{o}(t)$$

and dividing by $\sqrt{2\pi^2 t}$

$$\left| \frac{\mathcal{I}\left(h_{\mu^z_{G^{-1}(t)}}\right)}{\sqrt{2\pi^2 t}} - \sqrt{2a} \right| = \mathrm{o}(1)$$

which is an equivalent formulation of the set of thick points. $\square$

Next we make preparations to define a measure μ supported on $P(a)$ with positive probability. For this purpose define a sequence of measures μ_n on J supported on n-perfect thick points as such:

$$\mu_n(\cdot) = \sum_{i=1}^{|C_n|} \frac{1}{\mathcal{W}\left(E^n(x_{ni})\right)} \mathbb{1}_{\{Y_{ni}=1\}} \lambda\left(\cdot \cap S(x_{ni}, s_n)\right), \tag{5.3}$$

where $\lambda(\cdot)$ is the Lebesgue measure.

In the following lemma we list down some important properties of this measure.

Lemma 5.2. *Let $\mu_n(\cdot)$ be as in (5.3). Then the following hold:*

(a) $\mathsf{E}_{\mathcal{W}}\left[\mu_n(J)\right] = 1$;
(b) $\sup_n \mathsf{E}_{\mathcal{W}}\left[\mu_n(J)^2\right] < \infty$;
(c) $\sup_n \mathsf{E}_{\mathcal{W}}\left[\widetilde{I}_\alpha(\mu_n)\right] < \infty$;
(d) *there exist $a, b \in (0, \infty)$ such that for all n we have*

$$\mathcal{W}\left(b \le \mu_n(J) < b^{-1}, \widetilde{I}_\alpha(\mu_n) < a\right) > 0$$

for any $\alpha \le 4 - a$.

The proof of Lemma 5.2 requires a correlation inequality and a lower bound depends on the following lemma.

Lemma 5.3. *Let $A_m(x)$, $B_m(x)$ be as above with $s_m = \frac{1}{m!}$. Let*

$$E^n(x) = \bigcap_{m \le n} A_m(x) \cap B_{n+1}(x).$$

Then for every $y \in S(x, s_\ell) \setminus S(x, s_{\ell+1})$, $\ell > 2$, we have

$$\mathcal{W}\left(E^n(x) \cap E^n(y)\right) \le \mathscr{C}_\ell\, \mathcal{W}\left(E^n(x)\right) \mathcal{W}\left(E^n(y)\right), \tag{5.4}$$

where $\mathscr{C}_\ell$ is defined by

$$\mathscr{C}_\ell := C \prod_{j \le \ell+1} \frac{1}{c_j},$$

and $c_j = \exp\left(\frac{1}{2}\sqrt{4a\pi^2}\sqrt{t_{j+1} - t_j} - 4a\pi^2(t_{j+1} - t_j)\right)$.

Proof of Lemma 5.3. Fix $\ell > 2$ and $y \in S(x, s_\ell) \setminus S(x, s_{\ell+1})$. First note that the collections $\{A_i(x) : 1 \le i \le \ell + 1\}$ and $\{A_i(x) : \ell + 2 \le i \le n\}$ are independent as they depend on disjoint annuli. Similarly, as $S(x, s_{\ell+2}) \cap S(x, s_j) \setminus S(x, s_{j+1}) = \emptyset$ the collection $\{A_j(y) : j \ne \ell - 1, \ell, \ell + 1\}$ and $\{A_i(x) : \ell + 2 \le i \le n\}$ are independent. Now note that by the assumption,

$$\begin{aligned}
&\mathcal{W}\left(\bigcap_{1 \le i \le \ell+1} A_i(x)\right) \mathcal{W}\left(\bigcap_{\ell-1 \le j \le \ell+1} A_j(y)\right) \\
&\quad = \prod_{1 \le i \le \ell+1} \mathcal{W}\left(A_i(x)\right) \prod_{\ell-1 \le j \le \ell+1} \mathcal{W}\left(A_j(y)\right) \ge \prod_{i=1}^{\ell+1} \mathscr{C}_\ell.
\end{aligned} \tag{5.5}$$

So we have,

$$\begin{aligned}
\mathcal{W}\left(E^n(x) \cap E^n(y)\right) &= \mathcal{W}\left(\bigcap_{i \le n} A_i(x) \cap B_{n+1}(x) \cap \bigcap_{j \le n} A_j(y) \cap B_{n+1}(y)\right) \\
&\le \mathcal{W}\left(\bigcap_{i \le n} A_i(x) \cap \bigcap_{j \le n} A_j(y)\right) \\
&\le \mathcal{W}\left(\bigcap_{\ell+2 \le i \le n} A_i(x) \cap \bigcap_{j \le n, j \ne \ell-1, \ell, \ell+1} A_j(y)\right) \\
&\le \mathcal{W}\left(\bigcap_{\ell+2 \le i \le n} A_i(x)\right) \mathcal{W}\left(\bigcap_{j \le n, j \ne \ell-1, \ell, \ell+1} A_j(y)\right).
\end{aligned}$$

If we now multiply and divide the last probability by

$$\mathcal{W}\left(\bigcap_{1 \le i \le \ell+1} A_i(x)\right) \mathcal{W}\left(\bigcap_{\ell-1 \le j \le \ell+1} A_j(y)\right)$$

and use independence we get

$$\mathcal{W}\left(E^n(x)\cap E^n(y)\right) \le \frac{\mathcal{W}\left(\bigcap_{i\le n} A_i(x)\right)\mathcal{W}\left(\bigcap_{i\le n} A_i(y)\right)}{\mathcal{W}\left(\bigcap_{1\le i\le \ell+1} A_i(x)\right)\mathcal{W}\left(\bigcap_{l-1\le j\le \ell+1} A_j(y)\right)}.$$

Now using the bound in (5.5) and the fact that $B_{n+1}(x)$ is independent from $\{A_i(x) : i \le n\}$ we get

$$\mathcal{W}\left(E^n(x)\cap E^n(y)\right) \le \mathscr{C}_\ell\, \mathcal{W}\left(E^n(x)\right)\mathcal{W}\left(E^n(y)\right).$$

We can adjust appropriately the constant $\mathscr{C}_\ell$ when $\ell \le 2$ to complete the proof. □

Using the above Lemma the proof of Lemma 5.2 is almost immediate.

Proof of Lemma 5.2. Note the series $\sum_{\ell=1}^{\infty} s_\ell^4 \mathscr{C}_\ell$ converges (absolutely) by the ratio test. By means of the same criterion one shows also that $\sum_{\ell=1}^{\infty} s_\ell^4 \mathscr{C}_\ell s_{\ell+1}^{-\alpha} < +\infty$ under the assumption $\alpha \le 4$. Keeping these facts in mind we proceed to the proof.

(a) As $S(x_{ni}, s_n)$ forms a cover of J it is easy to show that $\mathsf{E}_{\mathcal{W}}[\mu_n(J)] = 1$. In particular,

$$\begin{aligned}\mathsf{E}_{\mathcal{W}}[\mu_n(J)] &= \sum_{i=1}^{|C_n|} \frac{1}{\mathcal{W}\left(E^n(x_{ni})\right)} \mathcal{W}\left(Y_{ni}=1\right)\lambda(J\cap S(x_{ni}, s_n))\\ &= \sum_{i=1}^{|C_n|} \lambda(J\cap S(x_{ni}, s_n)) = 1.\end{aligned}$$

(b) Using Lemma 5.3 we have

$$\begin{aligned}\mathsf{E}_{\mathcal{W}}\left[\mu_n(J)^2\right] &= \sum_{i,j=1}^{|C_n|} \frac{\mathcal{W}\left(Y_{ni}=1, Y_{nj}=1\right)}{\mathcal{W}\left(E^n(x_{ni})\right)\mathcal{W}\left(E^n(x_{nj})\right)}\lambda(S(x_{ni}, s_n))\lambda(S(x_{nj}, s_n))\\ &\le s_n^8 \sum_{i=1}^{|C_n|}\sum_{\ell=1}^{n} \sum_{j=1, s_\ell \ge \|x_{nj}-x_{ni}\| > s_{\ell+1}}^{|C_n|} \frac{\mathcal{W}\left(E^n(x_{ni})\cap E^n(x_{nj})\right)}{\mathcal{W}\left(E^n(x_{ni})\right)\mathcal{W}\left(E^n(x_{nj})\right)}\\ &\le s_n^8 \sum_{i=1}^{|C_n|}\sum_{\ell=1}^{n} \frac{s_\ell^4}{s_n^4}\mathscr{C}_\ell \le \sum_{\ell=1}^{\infty} s_\ell^4 \mathscr{C}_\ell < \infty.\end{aligned}$$

Above we have used the fact that the number of hypercubes with center at x_{ni} and radius s_ℓ is proportional to s_ℓ^4/s_n^4.

(c) For the expected energy we follow the same procedure as above. Note that $\|x_{ni}-x_{nj}\| > s_{\ell+1}$ then if we take $x \in S(x_{ni}, s_n)$ and $y \in S(x_{nj}, s_n)$ then $\|x-y\| > s_{\ell+1}$.

$$\begin{aligned}\mathsf{E}_{\mathcal{W}}\left[\widetilde{I}_\alpha(\mu_n)\right] &= \sum_{i,j=1}^{|C_n|} \frac{\mathcal{W}\left(E^n(x_{ni})\cap E^n(x_{nj})\right)}{\mathcal{W}\left(E^n(x_{ni})\right)\mathcal{W}\left(E^n(x_{nj})\right)} \int_{S(x_{ni}, s_n)}\int_{S(x_{nj}, s_n)} \frac{\mathrm{d}x\mathrm{d}y}{\|x-y\|^\alpha}\\ &\le s_n^8 \sum_{i=1}^{|C_n|}\sum_{\ell=1}^{n} \frac{s_\ell^4}{s_n^4}\mathscr{C}_\ell s_{\ell+1}^{-\alpha} \le \sum_{\ell\ge 1} \mathscr{C}_\ell s_\ell^4 s_{\ell+1}^{-\alpha} < +\infty.\end{aligned}$$

(d) By Paley–Zygmund inequality and the fact that $\sup_{n\geq 2} \mathsf{E}_{\mathcal{W}}\left[\mu_n(J)^2\right] < \infty$, there exists $v > 0$

$$\mathcal{W}\left(\mu_n(J) \geq b\right) \geq (1-b)^2 \frac{1}{\mathsf{E}_{\mathcal{W}}\left[\mu_n(J)\right]} \geq \frac{(1-b)^2}{\sup\limits_{n\geq 2} \mathsf{E}_{\mathcal{W}}\left[\mu_n(J)^2\right]} \geq v > 0.$$

Also using Markov's inequality we have that

$$\mathcal{W}\left(\mu_n(J) \geq b^{-1}\right) \leq b\mathsf{E}_{\mathcal{W}}\left[\mu_n(J)\right] = b.$$

Hence choosing $b > 0$ and $v > 0$ appropriately we have

$$\mathcal{W}\left(b \leq \mu_n(J) \leq b^{-1}\right) = \mathcal{W}\left(\mu_n(J) \geq b\right) - \mathcal{W}\left(\mu_n(J) \geq b^{-1}\right) \geq 2v > 0.$$

Also note that since $\mathsf{E}_{\mathcal{W}}\left[\widetilde{I}_\alpha(\mu_n)\right]$ is uniformly bounded in n, using Markov's inequality we have

$$\mathcal{W}\left(\widetilde{I}_\alpha(\mu_n) > a\right) \leq v.$$

Hence (d) follows from the above observations and the fact that,

$$\begin{aligned}\mathcal{W}\left(b \leq \mu_n(J) \leq b^{-1}, \widetilde{I}_\alpha(\mu_n) \leq a\right) &\geq \mathcal{W}\left(b \leq \mu_n(J) \leq b^{-1}\right) - \mathcal{W}\left(\widetilde{I}_\alpha(\mu_n) > a\right)\\ &\geq 2v - v = v > 0. \quad \square\end{aligned}$$

Proof of the lower bound. Now using Lemma 5.2 we continue with the proof of lower bound. If we define

$$\mathcal{R} := \limsup_{n\to+\infty} \left\{b \leq \mu_n(J) < b^{-1}, \widetilde{I}_\alpha(\mu_n) < a\right\},$$

then by Lemma 5.2(d), $\mathcal{W}(\mathcal{R})$ is bounded away from zero. $\widetilde{I}_\alpha$ being a lower semicontinuous function, the set of measures μ for which $b \leq \mu(J) < b^{-1}$ and $\widetilde{I}_\alpha(\mu) < a$ is compact in the topology of weak convergence. Therefore the sequence $(\mu_n)_{n\in\mathbb{N}}$ admits surely along a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ a weak limit μ, which is a finite measure supported on $P(a)$ and whose α-energy is finite. Hence, we have

$$\mathcal{W}\left(C_{\mathcal{H}}^{4-a}(P(a)) > 0\right) > 0. \tag{5.6}$$

Now by the monotonicity of the Hausdorff-α-measure, if we can show that

$$\mathcal{W}\left(C_{\mathcal{H}}^{4-a}(T(a,J)) > 0\right) \in \{0,1\}$$

then by (5.6), the set $\left\{C_{\mathcal{H}}^{4-a}(T(a,J)) > 0\right\}$ will have probability one and hence the proof will be complete.

Let $(h_m)_{m\geq 1}$ be an orthonormal basis of H. Let us denote the sigma field generated by the random variable $\mathcal{I}(h_m)$ by $\mathcal{F}_m$. Let $\mathcal{T}_m = \sigma(\bigcup_{j\geq m} \mathcal{F}_j)$. Note that the sigma fields $\mathcal{F}_m$ are independent since $(\mathcal{I}(h_m))_{m\geq 1}$ are i.i.d. Denote the tail sigma field by $\mathcal{T} = \bigcap_{m>0} \mathcal{T}_m$. We now claim that

$$\left\{\limsup_{\epsilon\to 0} \frac{\mathcal{I}\left(h_{\mu_\epsilon^x}\right)}{\sqrt{2\pi}G(\epsilon)} = \sqrt{2a}\right\} \in \mathcal{T}.$$

Now from the construction of μ_ϵ^x, it holds from Equation 7.9 of Chen and Jakobson [3] that $\mathcal{I}\left(h_{\mu_\epsilon^x}\right) = f_1(\epsilon)\mathcal{I}(h_{\sigma_\epsilon^x}) + f_2(\epsilon)\mathcal{I}(h_{\mathrm{d}\sigma_\epsilon^x})$, where

$$f_1(\epsilon) = \frac{\epsilon I_1(\epsilon) - 2I_2(\epsilon)}{I_1^2(\epsilon) - I_0(\epsilon)I_2(\epsilon)}, \quad f_2(\epsilon) = \frac{-\epsilon I_2(\epsilon)}{I_1^2(\epsilon) - I_0(\epsilon)I_2(\epsilon)}.$$

Since $\lim_{\epsilon\to 0} f_1(\epsilon) = 2$ and $\lim_{\epsilon\to 0} f_2(\epsilon) = 0$, $\mu_\epsilon^x \to 2\delta_x$ as $\epsilon \to 0$ in the sense of distributions. In fact, since $\widehat{\mathrm{d}\sigma_\epsilon^x}(\xi) = -\frac{2}{\epsilon}J_2(\epsilon|\xi|)\exp\left(i(\xi, x)_{\mathbb{R}^4}\right) \to 0$ for all ξ, $\mathrm{d}\sigma_\epsilon^x \to 0$ in the sense of distributions. Thus

$$\limsup_{\epsilon\to 0} \frac{\mathcal{I}\left(h_{\mu_\epsilon^x}\right)}{\sqrt{2\pi}G(\epsilon)} = \limsup_{\epsilon\to 0} \frac{f_1(\epsilon)\mathcal{I}(h_{\sigma_\epsilon^x})}{\sqrt{2\pi}G(\epsilon)}.$$

We now show $\left\{\limsup_{\epsilon\to 0} \frac{f_1(\epsilon)\mathcal{I}(h_{\sigma_\epsilon^x})}{\sqrt{2\pi}G(\epsilon)} = \sqrt{2a}\right\} \in \mathcal{T}$. By Stroock [29, Section 2], when $\{h_m\}_{m\in\mathbb{N}}$ is an orthonormal basis of H, any $\theta \in \Theta$ admits the series representation

$$\theta \overset{\mathcal{W}-\text{a.s.}}{=} \sum_{m\geq 1}[\mathcal{I}(h_m)(\theta)]h_m.$$

Hence

$$[\mathcal{I}(h_{\sigma_\epsilon^x})](\theta) = \left\langle\theta,\, \sigma_\epsilon^x\right\rangle \overset{\mathcal{W}-\text{a.s.}}{=} \left\langle \sum_{m\geq 1}[\mathcal{I}(h_m)(\theta)]h_m,\, \sigma_\epsilon^x \right\rangle.$$

Indeed, for all m, $\left\langle h_m,\, \sigma_\epsilon^x\right\rangle \to h_m(x)$ and $G(\epsilon) \to +\infty$ as $\epsilon \to 0$. Given any arbitrary $m_0 > 0$ large, one sees that

$$\lim_{\epsilon\to 0} \frac{1}{G(\epsilon)} \left\langle \sum_{m\leq m_0}[\mathcal{I}(h_m)(\theta)]h_m,\, \sigma_\epsilon^x \right\rangle = 0,$$

$\mathcal{W}$-a. s. Hence

$$\begin{aligned} &\left\{\limsup_{\epsilon\to 0} \frac{f_1(\epsilon)\mathcal{I}(h_{\sigma_\epsilon^x})}{\sqrt{2\pi}G(\epsilon)} = \sqrt{2a}\right\} \\ &= \left\{\limsup_{\epsilon\to 0} \frac{f_1(\epsilon)\left\langle \sum_{m\geq m_0}[\mathcal{I}(h_m)(\theta)]h_m,\, \sigma_\epsilon^x \right\rangle}{\sqrt{2\pi}G(\epsilon)} = \sqrt{2a}\right\} \in \mathcal{T}_{m_0}. \end{aligned}$$

Since m_0 is arbitrary, this allows one to apply Kolmogorov's 0–1 law to conclude. □

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Appendix

Here we will collect some of the bounds on the Bessel functions. These bounds are easy to derive but for completeness we provide a short proof for them. In the paper we have extensively used the Bessel function J_ν and the modified Bessel functions I_ν and K_ν of the first and second kind. A closed alternative representation can be found in [1] and it is

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^\pi \cos(z\cos\theta)(\sin\theta)^{2\nu}\mathrm{d}\theta,$$

$$I_\nu(z) := \frac{1}{\pi}\int_0^{+\infty} \mathrm{e}^{z\cos(\theta)}\cos(\nu\theta)\mathrm{d}\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^{+\infty} \mathrm{e}^{-z\cosh(t)-\nu t}\mathrm{d}t, \quad \nu \in \mathbb{R}$$

and

$$K_\nu(z) := \frac{\Gamma(\nu+\frac{1}{2})(2z)^\nu}{\sqrt{\pi}} \int_0^{+\infty} \frac{\cos(t)\mathrm{d}t}{(t^2+z^2)^{(\nu+1/2)}}, \quad \nu \in \mathbb{R}.$$

We will prove here an auxiliary bound.

Lemma A.1. (a) *For some constant* $C > 0$ *and* $x > 0$

$$|I_1^2(x) - I_0(x)I_2(x)| \geq Cx^2.$$

(b) *Let* $G(\cdot)$ *be as in* (3.1), *then* $G(x) \leq -C\log x$ *for all* $x \in [0,1]$, *with* $C > 0$ *uniform in* x.

Proof. (a) Following [15] we have,

$$I_1^2(x) - I_0(x)I_2(x) = \frac{I_1^2(x)}{x}\left(x\frac{I_1'(x)}{I_1(x)}\right)' = \frac{I_1^2(x)}{x}\sum_{n\geq 1}\frac{4xj_{1,n}}{(x^2+j_{1,n}^2)^2}$$

where we used the equality $\left(x\frac{I_1'(x)}{I_1(x)}\right)' = \sum_{n\geq 1}\frac{4xj_{1,n}}{(x^2+j_{1,n}^2)^2}$, $j_{i,n}$ being the n-th zero of $J_1(x)/x$ [31]. Now using the identity $I_1(x) = (x/C)\prod_{n\geq 1}\left(1+\frac{x^2}{j_{1,n}^2}\right)$ [31, Page 498] we derive

$$\begin{aligned} I_1^2(x) - I_0(x)I_2(x) &= \frac{I_1^2(x)}{x}\left(x\frac{I_1'(x)}{I_1(x)}\right)' \\ &= \frac{I_1^2(x)}{x}\frac{4xj_{1,1}}{(x^2+j_{1,1}^2)^2} + \frac{I_1^2(x)}{x}\sum_{n\geq 2}\frac{4xj_{1,n}}{(x^2+j_{1,n}^2)^2} \\ &> \frac{4I_1^2(x)j_{1,1}}{(x^2+j_{1,1}^2)^2} > C'x^2. \end{aligned}$$

(b) By part (a) and the series expansion of Bessel functions [1] one can find a bound for $G(\cdot)$ as follows (γ is the Euler–Mascheroni constant):

$$\begin{aligned} G(x) &\leq \frac{C}{x^2}(2I_1(x)K_1(x) + 2I_2(x)K_0(x) - 1) \\ &= \frac{C}{x^2}\left(2\left(\frac{x}{2}+\frac{x^3}{16}+\mathrm{O}\left(x^4\right)\right)\right. \end{aligned}$$

$$
\begin{aligned}
&\times \left(\frac{1}{x} + \frac{x}{4}(-1 + 2\gamma - 2\log 2 + 2\log x) + \mathrm{O}\left(x^3 \log x\right)\right) \\
&+ 2\left(\frac{x^2}{8} + \mathrm{O}\left(x^4\right)\right)\left((-\gamma + \log 2 - \log x) + \mathrm{O}\left(x^2 \log x\right)\right) - 1\Big) \\
= \frac{C}{x^2}&\left(1 + \frac{x^2}{8} + \mathrm{O}\left(x^3\right) + \frac{-1 + 2\gamma - 2\log 2}{4}x^2 + \frac{-1 + 2\gamma - 2\log 2}{32}x^4\right. \\
&\left. + \mathrm{O}\left(x^2 \log x\right) + \frac{x^2}{4}C + \mathrm{O}\left(x^4\right) - \frac{x^2 \log x}{4} - 1\right) = -C\log x + C'.
\end{aligned}
$$

Here C, C' denote positive constants that may vary from line to line. □

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