ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

stochastic processes and their applications

www.elsevier.com/locate/spa

Quasi-continuous random variables and processes under the G-expectation framework

Mingshang Hu[a], Falei Wang[b,*], Guoqiang Zheng[c]

[a] Zhongtai Institute of Finance, Shandong University, China
[b] Institute for Advanced Research and School of Mathematics, Shandong University, China
[c] School of Mathematics, Shandong University, China

Received 31 March 2015; received in revised form 9 November 2015; accepted 8 February 2016

Abstract

In this paper, we first use PDE techniques and probabilistic methods to identify a kind of quasi-continuous random variables. Then we give a characterization of the G-integrable processes and get a kind of quasi-continuous processes by Krylov's estimates. This result is useful for the development of G-stochastic analysis theory. Moreover, it also provides a tool for the study of the non-Markovian Itô processes.

MSC: 60H10; 60H30

Keywords: G-expectation; G-Brownian motion; Quasi-continuous; Krylov's estimates

1. Introduction

Motivated by model uncertainty in finance, Peng [11,12] firstly constructed a kind of dynamically consistent fully nonlinear expectations through PDE approach. An important case is the G-expectation $\hat{\mathbb{E}}[\cdot]$ and the corresponding canonical process $(B_t)_{t\geq 0}$ is called G-Brownian

* Corresponding author.
E-mail addresses: humingshang@sdu.edu.cn (M. Hu), flwang2011@gmail.com (F. Wang), zhengguoqiang.ori@gmail.com (G. Zheng).

http://dx.doi.org/10.1016/j.spa.2016.02.003

motion analogous to the classical Wiener process. Under the G-expectation framework, the corresponding stochastic calculus of Itô's type was also established in Peng [13,14].

The G-expectation can be also seen as a upper expectation. Indeed, Denis et al. [1] obtained a representation theorem of G-expectation $\hat{\mathbb{E}}[\cdot]$ by stochastic control method:

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X] \quad \text{for } X \in L_{ip}(\Omega),$$

where $\mathcal{P}$ is a family of weakly compact probability measures on $(\Omega, \mathcal{B}(\Omega))$. Moreover, they gave a characterization of the space $L_G^p(\Omega)$ and proved that every random variable in $L_G^p(\Omega)$ is quasi-continuous. The representation theorem was also obtained in [6] by a simple probabilistic method.

The present article is devoted to the study of integrable random variables and stochastic processes in the G-expectation framework. The classical Lusin's theorem indicates each random variable is "quasi-continuous" in a probability space. However, it is difficult to verify a random variable is quasi-continuous in the G-expectation framework, since the measures in $\mathcal{P}$ may be mutually singular. This problem has restricted the development of the G-stochastic analysis theory. For example, it is difficult to construct the approximation of an admissible control to get the dynamic programming principle for G-stochastic control problems and we cannot use the approximation theory of measurable function to prove the Markov property of the G-stochastic differential equations.

To overcome this difficult, we use PDE techniques and stochastic control methods to obtain some polar sets associated to X, which is a multi-dimensional G-Itô process. Based on these polar sets, we prove some "irregular" Borel measurable functions on $(\Omega, \mathcal{B}(\Omega))$ are quasi-continuous, which implies the space $L_G^p(\Omega)$ contains enough elements such as $I_{\{X_t \in [a,b]\}}$. Thus the approximation of quasi-continuous random variables through simple functions is possible. Indeed, Hu and Ji [2] studied the G-stochastic control problems with the help of this result. In 1-dimensional case, Martini [10] also got some polar sets by a pure probabilistic approach. By our arguments, we also obtain the convergence rate, which enables us to study the sample path properties of the non-Markovian Itô processes, such as the differentiability and the maxima.

The similar questions arise for the G-integrable processes, and the rest of this paper is devoted to studying the space $M_G^p(0, T)$. First, we give a characterization of $M_G^p(0, T)$, which non-trivially generalizes the result of [1]. Moreover, we establish a monotone convergence theorem for quasi-continuous processes. Next we apply Krylov's estimates to get a kind of quasi-continuous processes. In particular, these estimates induce a weak dominated convergence theorem for G-Itô processes, which is useful for the study of G-stochastic analysis. For example, this result can be used to deal with the well-posedness of G-backward stochastic differential equations under non-Lipschitz condition.

This paper is organized as follows. In Section 2, we recall some necessary notations and results of G-expectation theory. In Section 3, we study the polar sets and give some useful quasi-continuous random variables. In Section 4, we obtain the characterization of $M_G^p(0, T)$ and get some useful quasi-continuous progressively measurable processes by Krylov's estimates.

2. Preliminaries

The main purpose of this section is to recall some basic notions and results of G-expectation, which are needed in the sequel. The readers may refer to [13–16] for more details.

Please cite this article in press as: M. Hu, et al., Quasi-continuous random variables and processes under the G-expectation framework, Stochastic Processes and their Applications (2016), http://dx.doi.org/10.1016/j.spa.2016.02.003

Let $\Omega = C_0^d(\mathbb{R}^+)$ be the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t\geq 0}$, with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i}[(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2|) \wedge 1].$$

For each $t \in [0, \infty)$, we denote

- $B_t(\omega) := \omega_t$ for each $\omega \in \Omega$;
- $\mathcal{B}(\Omega)$: the Borel σ-algebra of Ω, $\Omega_t := \{\omega_{\cdot\wedge t} : \omega \in \Omega\}$, $\mathcal{F}_t := \mathcal{B}(\Omega_t)$;
- $L^0(\Omega)$: the space of all $\mathcal{B}(\Omega)$-measurable real functions;
- $L^0(\Omega_t)$: the space of all $\mathcal{B}(\Omega_t)$-measurable real functions;
- $B_b(\Omega)$: all bounded elements in $L^0(\Omega)$; $B_b(\Omega_t) := B_b(\Omega) \cap L^0(\Omega_t)$;
- $C_b(\Omega)$: all continuous elements in $B_b(\Omega)$; $C_b(\Omega_t) := C_b(\Omega) \cap L^0(\Omega_t)$;
- $L_{ip}(\Omega) := \{\varphi(B_{t_1}, \ldots, B_{t_k}) : k \in \mathbb{N}, t_1, \ldots, t_k \in [0, \infty), \varphi \in C_{b.Lip}(\mathbb{R}^{k\times d})\}$, where $C_{b.Lip}(\mathbb{R}^{k\times d})$ denotes the space of bounded and Lipschitz functions on $\mathbb{R}^{k\times d}$; $L_{ip}(\Omega_t) := L_{ip}(\Omega) \cap L^0(\Omega_t)$.

For each given monotonic and sublinear function $G : \mathbb{S}(d) \to \mathbb{R}$, let the canonical process $B_t = (B_t^i)_{i=1}^d$ be the d-dimensional G-Brownian motion in the G-expectation space $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}}[\cdot], (\hat{\mathbb{E}}_t[\cdot])_{t\geq 0})$, where $\mathbb{S}(d)$ denotes the space of all $d \times d$ symmetric matrices. For each $p \geq 1$, the completion of $L_{ip}(\Omega)$ under the norm $\|X\|_{L_G^p} := (\hat{\mathbb{E}}[|X|^p])^{1/p}$ is denoted by $L_G^p(\Omega)$. Similarly, we can define $L_G^p(\Omega_T)$ for each fixed $T \geq 0$. In this paper, we always assume that G is non-degenerate, i.e., there exist two constants $0 < \underline{\sigma}^2 \leq \bar{\sigma}^2 < \infty$ such that

$$\frac{1}{2}\underline{\sigma}^2 \mathrm{tr}[A - B] \leq G(A) - G(B) \leq \frac{1}{2}\bar{\sigma}^2 \mathrm{tr}[A - B], \quad \text{for } A \geq B.$$

Then we deduce that $|G(A)| \leq \frac{1}{2}\bar{\sigma}^2\sqrt{d}\sqrt{\mathrm{tr}[AA^T]}$ for any $A \in \mathbb{S}(d)$.

Denis et al. [1] proved that the completions of $C_b(\Omega)$ and $L_{ip}(\Omega)$ under $\|\cdot\|_{L_G^p}$ are the same.

Theorem 2.1 ([1,6]). *There exists a weakly compact set of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$, such that*

$$\hat{\mathbb{E}}[\xi] = \sup_{P\in\mathcal{P}} E_P[\xi], \quad \textit{for all } \xi \in L_G^1(\Omega).$$

$\mathcal{P}$ is called a set that represents $\hat{\mathbb{E}}$.

Remark 2.2. Denis et al. [1] constructed a concrete set $\mathcal{P}_M$ that represents $\hat{\mathbb{E}}$. For simplicity's sake, we consider the 1-dimensional case, thus $G(a) = \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$ for each $a \in \mathbb{R}$. Suppose B is a Brownian motion defined on $(\Omega, L^0(\Omega), P)$, then

$$\mathcal{P}_M := \left\{ P_\theta : P_\theta = P \circ X^{-1},\ X_t = \int_0^t \theta_s dB_s,\ \theta \in L^2_{\mathcal{F}}([0, T]; [\underline{\sigma}^2, \bar{\sigma}^2]) \right\}$$

represents $\hat{\mathbb{E}}$, where $L^2_{\mathcal{F}}([0, T]; [\underline{\sigma}^2, \bar{\sigma}^2])$ is the collection of all adapted measurable processes with $\underline{\sigma}^2 \leq |\theta_s|^2 \leq \bar{\sigma}^2$.

Let $\mathcal{P}$ be a weakly compact set that represents $\hat{\mathbb{E}}$. For this $\mathcal{P}$, we define capacity

$$c(A) := \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

Please cite this article in press as: M. Hu, et al., Quasi-continuous random variables and processes under the G-expectation framework, Stochastic Processes and their Applications (2016), http://dx.doi.org/10.1016/j.spa.2016.02.003

An important property of this capacity is that $c(F_n) \downarrow c(F)$ for any closed sets $F_n \downarrow F$.

A set $A \subset \mathcal{B}(\Omega)$ is polar if $c(A) = 0$. A property holds "*quasi-surely*" (q.s.) if it holds outside a polar set. In the following, we do not distinguish between random variables X and Y if $X = Y$ q.s.

Definition 2.3. A real function X on Ω is said to be quasi-continuous if for each $\varepsilon > 0$, there exists an open set O with $c(O) < \varepsilon$ such that $X|_{O^c}$ is continuous.

Definition 2.4. We say that $X : \Omega \mapsto \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega \mapsto \mathbb{R}$ such that $X = Y$, q.s.

Theorem 2.5 ([1,6]). *We have*

$$L_G^p(\Omega) = \{X \in L^0(\Omega) : \lim_{N\to\infty} \hat{\mathbb{E}}[|X|^p I_{|X|\geq N}] = 0 \text{ and}$$
$$X \text{ has a quasi-continuous version}\}.$$

Theorem 2.6 ([1,6]). *Let $(X_k)_{k\geq 1} \subset L_G^1(\Omega)$, be such that $X_k \downarrow X$ q.s. Then $\hat{\mathbb{E}}[X_k] \downarrow \hat{\mathbb{E}}[X]$. In particular, if $X \in L_G^1(\Omega)$, then $\hat{\mathbb{E}}[|X_k - X|] \downarrow 0$.*

Definition 2.7 ([5]). Assume $X_\theta \in L_G^1(\Omega_t)$ for each $\theta \in \Theta$. Then the essential supremum of $\{X_\theta \mid \theta \in \Theta\}$, denoted by $\mathrm{esssup}_{\theta\in\Theta} X_\theta$, is a random variable $\zeta \in L_G^1(\Omega_t)$ satisfying:

(i) $\forall\theta \in \Theta,\ \zeta \geq X_\theta$ q.s.;
(ii) if ξ is a random variable satisfying $\xi \geq X_\theta$ q.s. for any $\theta \in \Theta$, then $\zeta \leq \xi$ q.s.

Definition 2.8. Let $M_G^0(0, T)$ be the collection of processes of the following form: for a given partition $\{t_0, \ldots, t_N\} = \pi_T$ of $[0, T]$,

$$\eta_t(\omega) = \sum_{i=0}^{N-1} \xi_i(\omega) I_{[t_i,t_{i+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0, 1, 2, \ldots, N-1$. For each $p \geq 1$, denote by $M_G^p(0, T)$ the completion of $M_G^0(0, T)$ under the norm $\|\eta\|_{M_G^p} := (\hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt])^{1/p}$.

For each $\eta \in M_G^2(0, T)$, the G-Itô integral $\{\int_0^t \eta_s dB_s^i\}_{t\in[0,T]}$ is well defined, see Peng [16] and Li–Peng [9].

3. Quasi-continuous random variables

In this section, we shall prove some "irregular" Borel measurable functions on Ω are quasi-continuous by virtue of a PDE approach. We consider the following G-Itô processes (in this paper we always use Einstein's summation convention): for each given $x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$ and $1 \leq i \leq n$,

$$X_t^{x_i;i} = x_i + \int_0^t \alpha_i(s)ds + \int_0^t \beta_i^{jk}(s)d\langle B^j, B^k\rangle_s + \int_0^t \sigma_i(s)dB_s,$$

where $\beta^{jk}(t) = \beta^{kj}(t)$ and σ_i is the ith row of σ. Denote by $X_t^x = (X_t^{x_1;1}, \ldots, X_t^{x_n;n})^\top$, $\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))^\top$ and $\beta^{jk}(t) = (\beta_1^{jk}(t), \ldots, \beta_n^{jk}(t))^\top$. Then the above G-Itô processes can

Please cite this article in press as: M. Hu, et al., Quasi-continuous random variables and processes under the G-expectation framework, Stochastic Processes and their Applications (2016), http://dx.doi.org/10.1016/j.spa.2016.02.003

be written as

$$X_t^x = x + \int_0^t \alpha_s ds + \int_0^t \beta_s^{jk} d\langle B^j, B^k \rangle_s + \int_0^t \sigma_s dB_s. \tag{1}$$

In this paper, we shall use the following assumptions:

(H1) For each $s > 0$, $(\alpha_t)_{0 \le t \le s}$ and $(\beta_t^{jk})_{0 \le t \le s}$ are in $M_G^2(0, s; \mathbb{R}^n)$, $(\sigma_t)_{0 \le t \le s}$ are in $M_G^2(0, s; \mathbb{R}^{n \times d})$.

(H2) There exists a constant $L > 0$ such that for each $t \in [0, \infty)$,

$$|\alpha_i(t)| \le L, \qquad |\beta_i^{jk}(t)| \le L, \qquad |\sigma_i(t)| \le L, \quad \text{for } j, k \le d \text{ and } i \le n.$$

(H3) There exist two constants $0 < \lambda < \Lambda < \infty$ such that for each $t \in [0, \infty)$,

$$\lambda I_{n \times n} \le \sigma_t (\sigma_t)^\top \le \Lambda I_{n \times n}, \quad \text{if } n \le d,$$
$$\lambda I_{d \times d} \le (\sigma_t)^\top \sigma_t \le \Lambda I_{d \times d}, \quad \text{if } n > d.$$

(H4) There exist two constants $0 < \gamma < \Gamma < \infty$ such that for each $(t, x) \in [0, \infty) \times \mathbb{R}^n$,

$$\gamma \le |\sigma_i(t)|^2 = \sigma_i(t)(\sigma_i(t))^\top \le \Gamma, \quad \text{for } i \le n.$$

Remark 3.1. If $n \le d$, then (H3) is stronger than (H4).

In order to state the main results of this section, we shall use the stochastic representation for the HJB equation. For this purpose, we denote the following sets:

$$\mathcal{V} = \{v = (\alpha, \beta, \sigma) | \, \alpha, \beta \text{ and } \sigma \text{ satisfy assumptions (H1), (H2) and (H3)}\}$$

and

$$\mathcal{V}_0 = \{v \in \mathcal{V} | \, v \text{ is a constant process, i.e., } v(t) = v(0) \text{ for each } t > 0 \}.$$

Now for each fixed $t \ge 0$, $v \in \mathcal{V}$ and for each given $\xi \in L_G^2(\Omega_t; \mathbb{R}^n)$, consider the following G-Itô process:

$$X_s^{t,\xi,v} = \xi + \int_t^s \alpha_r dr + \int_t^s \beta_r^{jk} d\langle B^j, B^k \rangle_r + \int_t^s \sigma_r dB_r. \tag{2}$$

Then for each fixed $T > 0$ and $\Phi \in C_{b.Lip}(\mathbb{R}^n)$, we define

$$Y_t^{t,\xi} = \operatorname*{esssup}_{v \in \mathcal{V}} \hat{\mathbb{E}}_t[\Phi(X_T^{t,\xi,v})], \quad t \in [0, T].$$

Next, for each $x \in \mathbb{R}^n$, we set

$$u(t, x) := Y_t^{t,x}.$$

It is important to note that $u(0, x) = \sup_{v \in \mathcal{V}} \hat{\mathbb{E}}[\Phi(X_T^{0,x,v})]$.

Theorem 3.2 ([5]). *For each fixed $T > 0$, we have*

(1) *$u(t, x)$ is a deterministic continuous function of (t, x);*
(2) *For each $\xi \in L_G^2(\Omega_t; \mathbb{R}^n)$, $Y_t^{t,\xi} = u(t, \xi)$;*

(3) *u is the unique viscosity solution of the following PDE:*

$$\begin{cases} \partial_t u + \sup_{v\in\mathcal{V}_0}\{G(\sigma^\top D_x^2 u\sigma + 2\langle\beta^{jk}, D_x u\rangle) + \langle\alpha, D_x u\rangle\} = 0, \\ \quad (t,x)\in[0,T)\times\mathbb{R}^n, \\ u(T,x) = \Phi(x). \end{cases} \tag{3}$$

Remark 3.3. In the definition of $\mathcal{V}$, we can also assume α, β and σ satisfy assumptions (H1), (H2) and (H4). In this case, the set $\mathcal{V}_0$, value function u and Eq. (3) have to be redefined accordingly.

3.1. Polar sets associated to G-Itô processes

In the following, we first prove that $c(\{X_t^x = a\}) = 0$ for any $t > 0$ and $a \in \mathbb{R}^n$, i.e. the G-Itô process X_t^x does not weight single point. The proof is based on an estimate of the solution to PDE (3).

Lemma 3.4. *Let $T > 0$, $\rho = (n\wedge d)\lambda\underline{\sigma}^2(8d\bar{\sigma}^2\Lambda)^{-1}$, $\theta = (2d\bar{\sigma}^2\Lambda)^{-1}$, $\varepsilon = (8\kappa)^{-1}\wedge T$, $m \geq 8\kappa$ and u_m be the solution of PDE (3) with the terminal condition $u_m(T,x) = \exp(-\frac{m\theta|x-a|^2}{2})$, where $a = (a_1,\dots,a_n)^\top \in \mathbb{R}^n$, $n\wedge d = \min\{n,d\}$,*

$$\kappa = L^2(\bar{\sigma}^2 d\sqrt{d}+1)^2((n\wedge d)\lambda\underline{\sigma}^2)^{-1}.$$

Then for any $(t,x)\in[T-\varepsilon,T)\times\mathbb{R}^n$, we have

$$0 \leq u_m(t,x) \leq (1+m(T-t))^{-\rho}. \tag{4}$$

Proof. It is easy to check that $\bar{u}_m(t,x) = 0$ is a viscosity subsolution of PDE (3). Thus by comparison theorem we get $u_m(t,x)\geq 0$ for each $(t,x)\in[0,T]\times\mathbb{R}^n$. Set

$$\tilde{u}_m(t,x) = (1+m(T-t))^{-\rho}\exp\left(-\frac{m\theta|x-a|^2}{2(1+m(T-t))}\right). \tag{5}$$

It is obvious that $\tilde{u}_m(T,x) = \exp(-\frac{m\theta|x-a|^2}{2})$. In the following, we shall show that $\tilde{u}_m$ is a viscosity supersolution of PDE (3) if $t \geq T-\varepsilon$. It is easy to verify that

$$\begin{aligned} \partial_t\tilde{u}_m &= \frac{\rho m}{1+m(T-t)}\tilde{u}_m - \frac{m^2\theta|x-a|^2}{2(1+m(T-t))^2}\tilde{u}_m, \\ \partial_{x_i}\tilde{u}_m &= -\frac{m\theta(x_i-a_i)}{1+m(T-t)}\tilde{u}_m, \\ \partial^2_{x_ix_i}\tilde{u}_m &= -\frac{m\theta}{1+m(T-t)}\tilde{u}_m + \frac{m^2\theta^2|x_i-a_i|^2}{(1+m(T-t))^2}\tilde{u}_m, \\ \partial^2_{x_ix_j}\tilde{u}_m &= \frac{m^2\theta^2(x_i-a_i)(x_j-a_j)}{(1+m(T-t))^2}\tilde{u}_m, \quad i\neq j. \end{aligned}$$

For each $v\in\mathcal{V}_0$, by the assumptions (H1)–(H3), we obtain that

$$G(-\sigma^\top\sigma) \leq -\frac{\underline{\sigma}^2}{2}\mathrm{tr}[\sigma^\top\sigma] \leq -\frac{1}{2}(n\wedge d)\lambda\underline{\sigma}^2,$$

Please cite this article in press as: M. Hu, et al., Quasi-continuous random variables and processes under the G-expectation framework, Stochastic Processes and their Applications (2016), http://dx.doi.org/10.1016/j.spa.2016.02.003

$$G(\sigma^\top(x-a)(x-a)^T\sigma) \le \frac{\bar{\sigma}^2}{2}|x-a|^2\mathrm{tr}[\sigma^\top\sigma] \le \frac{1}{2}d\Lambda\bar{\sigma}^2|x-a|^2,$$
$$G((-\langle\beta^{jk}, x-a\rangle)_{j,k=1}^d) \le \frac{1}{2}L\bar{\sigma}^2 d\sqrt{d}|x-a|, \qquad -\langle\alpha, x-a\rangle \le L|x-a|.$$

Note that $L(\bar{\sigma}^2 d\sqrt{d}+1)|x-a| \le L^2(\bar{\sigma}^2 d\sqrt{d}+1)^2|x-a|^2((n\wedge d)\lambda\underline{\sigma}^2)^{-1} + \frac{1}{4}(n\wedge d)\lambda\underline{\sigma}^2$. Then for $(t,x)\in[T-\varepsilon,T)\times\mathbb{R}^n$, we have

$$\begin{aligned}
&\partial_t\tilde{u}_m + \sup_{v\in\mathcal{V}_0}\{G(\sigma^\top D_x^2\tilde{u}_m\sigma + (2\langle\beta^{jk}(t,x), D_x\tilde{u}_m\rangle)_{j,k=1}^d) + \langle\alpha, D_x\tilde{u}_m\rangle\} \\
&\le \partial_t\tilde{u}_m + \frac{m\theta\tilde{u}_m}{1+m(T-t)}\sup_{v\in\mathcal{V}_0} G(-\sigma^\top\sigma) \\
&\quad + \frac{m^2\theta^2\tilde{u}_m}{(1+m(T-t))^2}\sup_{v\in\mathcal{V}_0} G(\sigma^\top(x-a)(x-a)^\top\sigma) \\
&\quad + \frac{2m\theta\tilde{u}_m}{1+m(T-t)}\sup_{v\in\mathcal{V}_0} G((-\langle\beta^{jk}, x-a\rangle)_{j,k=1}^d) + \frac{m\theta\tilde{u}_m}{1+m(T-t)}\sup_{v\in\mathcal{V}_0}\{-\langle\alpha, x-a\rangle\} \\
&\le -\frac{m\theta\tilde{u}_m}{1+m(T-t)}|x-a|^2\left(\frac{m}{4(1+m(T-t))}-\kappa\right) \\
&\le -\frac{m\theta\tilde{u}_m}{1+m(T-t)}|x-a|^2\left(\frac{m}{4(1+m\varepsilon)}-\kappa\right) \\
&= -\frac{m\theta\tilde{u}_m}{1+m(T-t)}|x-a|^2\times\frac{m-8\kappa}{8(1+m\varepsilon)} \\
&\le 0,
\end{aligned}$$

which implies that $\tilde{u}_m$ is a viscosity supersolution of PDE (3) if $t\ge T-\varepsilon$. Thus by comparison theorem we obtain for $(t,x)\in[T-\varepsilon,T]\times\mathbb{R}^n$,

$$u_m(t,x) \le \tilde{u}_m(t,x) \le (1+m(T-t))^{-\rho}.$$

The proof is complete. ■

Remark 3.5. If $\alpha=\beta^{jk}=0$. From the above proof, we can take $\rho=(n\wedge d)\lambda\underline{\sigma}^2(2d\bar{\sigma}^2\Lambda)^{-1}$, $\theta=(d\bar{\sigma}^2\Lambda)^{-1}$, $\varepsilon=T$ $(\kappa=0)$, $m\ge 0$ and the results also hold true.

Remark 3.6. We remark that there is a potential to extend our results to a much more general nonlinear expectation setting. In particular, by slightly more involved estimates, our results still hold for the following PDE (see [3–5]):

$$\begin{cases}\partial_t u + \sup_{v\in\mathcal{V}_0}\{G(\sigma^\top D_x^2 u\sigma + 2\langle\beta^{jk}, D_x u\rangle + f_1(t, D_x u, v)) \\ \quad + \langle\alpha, D_x u\rangle + f_2(t, D_x u, v)\} = 0, \\ u(T,x) = \Phi(x),\end{cases}$$

where f_i $(i=1,2)$ is a Lipschitz continuous function satisfying $f_i(t,0,v)=0$. The proof is the same without any difficulty.

Please cite this article in press as: M. Hu, et al., Quasi-continuous random variables and processes under the G-expectation framework, Stochastic Processes and their Applications (2016), http://dx.doi.org/10.1016/j.spa.2016.02.003

Theorem 3.7. *Assume* (H1)–(H3) *hold. Then we have for each* $T > 0$

$$\hat{\mathbb{E}}\left[\exp\left(-\frac{m\theta|X_T^x - a|^2}{2}\right)\right] \le (1 + m(T \wedge \varepsilon))^{-\rho}, \tag{6}$$

where X_t^x *is the* G*-Itô process* (1) *and* $\theta, \rho, \varepsilon$ *are given in Lemma* 3.4. *In particular, we have*

$$c(\{X_T^x = a\}) = 0. \tag{7}$$

Proof. If $T \le \varepsilon$, it follows from Lemma 3.4 and $\hat{\mathbb{E}}[\exp(-\frac{m\theta|X_T^x-a|^2}{2})] \le u_m(0, x)$ that $\hat{\mathbb{E}}[\exp(-\frac{m\theta|X_T^x-a|^2}{2})] \le (1 + mT)^{-\rho}$. If $T > \varepsilon$, by Theorem 3.2(2) and Lemma 3.4, we get that

$$\begin{aligned}\hat{\mathbb{E}}[\exp(-\frac{m\theta|X_T^x - a|^2}{2})] &= \hat{\mathbb{E}}\left[\hat{\mathbb{E}}_{T-\varepsilon}\left[\exp\left(-\frac{m\theta|X_T^{T-\varepsilon, X_{T-\varepsilon}^x} - a|^2}{2}\right)\right]\right] \\ &\le \hat{\mathbb{E}}[u_m(T - \varepsilon, X_{T-\varepsilon}^x)] \\ &\le \hat{\mathbb{E}}[(1 + m\varepsilon)^{-\rho}] \\ &= (1 + m\varepsilon)^{-\rho}.\end{aligned}$$

Thus we obtain Eq. (6). Note that $\exp(-\frac{m\theta|X_T^x-a|^2}{2}) \ge I_{\{X_T^x=a\}}$, then

$$c(\{X_T^x = a\}) \le \hat{\mathbb{E}}\left[\exp\left(-\frac{m\theta|X_T^x - a|^2}{2}\right)\right] \le (1 + m(T \wedge \varepsilon))^{-\rho}.$$

Thus we can get $c(\{X_T^x = a\}) = 0$ by letting $m \to \infty$. ■

We remark that Martini [10] proved a similar result in the one dimensional case. By a probabilistic method, he obtained that the Itô process does not weight single point under strict ellipticity condition. In Theorem 3.7, we also obtain the convergence rate (6), which can be used to estimate the quality of the G-Itô processes staying in a ball.

Corollary 3.8. *Assume* (H1)–(H3) *hold and* $\alpha = \beta^{jk} = 0$. *Then for each* $t > 0$, $y \in \mathbb{R}^n$ *and* $\epsilon > 0$, *we have*

$$c(\{|X_t^x - y| \le \epsilon\}) \le \exp\left(\frac{\theta}{2}\right)\frac{\epsilon^{2\rho}}{t^\rho},$$

where $\rho = (n \wedge d)\lambda\underline{\sigma}^2(2d\bar{\sigma}^2\Lambda)^{-1}$, $\theta = (d\bar{\sigma}^2\Lambda)^{-1}$. *In particular,*

$$\lim_{\epsilon\downarrow 0}\sup_{y\in\mathbb{R}^d} c(\{|X_t^x - y| \le \epsilon\}) = 0.$$

Proof. By Remark 3.5 and Theorem 3.7, we obtain for each $y \in \mathbb{R}^n$ and $m \ge 0$,

$$\hat{\mathbb{E}}\left[\exp\left(-\frac{m\theta|X_t^x - y|^2}{2}\right)\right] \le \frac{1}{(1 + mt)^\rho}.$$

Please cite this article in press as: M. Hu, et al., Quasi-continuous random variables and processes under the G-expectation framework, Stochastic Processes and their Applications (2016), http://dx.doi.org/10.1016/j.spa.2016.02.003

Thus we get for each m and $\epsilon > 0$,

$$\hat{\mathbb{E}}[I_{\{|X_t^x - y| \le \epsilon\}}] \le \exp\left(\frac{m\theta\epsilon^2}{2}\right)\hat{\mathbb{E}}\left[\exp\left(-\frac{m\theta|X_t^x - y|^2}{2}\right)\right] \le \exp\left(\frac{m\theta\epsilon^2}{2}\right)\frac{1}{(1+mt)^\rho}.$$

In particular, taking $m = \frac{1}{\epsilon^2}$, we get for each $y \in \mathbb{R}^n$,

$$c(\{|X_t^x - y| \le \epsilon\}) \le \exp\left(\frac{\theta}{2}\right)\frac{\epsilon^{2\rho}}{t^\rho},$$

which completes the proof. ■

Example 3.9. From Corollary 3.8, we can obtain that for each $t > 0$, $y \in \mathbb{R}^d$ and $\epsilon > 0$,

$$c(\{|B_t - y| \le \epsilon\}) \le \exp\left(\frac{\theta}{2}\right)\frac{\epsilon^{2\rho}}{t^\rho},$$

where $\rho = \frac{\underline{\sigma}^2}{2\bar{\sigma}^2}$, $\theta = (d\bar{\sigma}^2)^{-1}$. This inequality provides a way to study the sample path properties of non-Markovian Itô process in the Wiener space. Indeed by Remark 2.2, we have

$$P(\{|X_t - y| \le \epsilon\}) \le \exp\left(\frac{\theta}{2}\right)\frac{\epsilon^{2\rho}}{t^\rho}$$

and $X_t = \int_0^t \theta_s dB_s$ is non-differentiable almost everywhere (see [19]).

By Remark 3.3, we conclude also the value function u is the viscosity solution of PDE (3) under the assumptions (H1), (H2) and (H4). Then we have the following result.

Lemma 3.10. *Let $T > 0$, $\rho = \gamma\underline{\sigma}^2(8\bar{\sigma}^2\Gamma)^{-1}$, $\theta = (2\bar{\sigma}^2\Gamma)^{-1}$, $\varepsilon = (8\kappa)^{-1} \wedge T$, $m \ge 8\kappa$ and u_m be the solution of PDE (3) with terminal condition $u_m(T, x) = \exp(-\frac{m\theta|x_i - a_i|^2}{2})$, where $a_i \in \mathbb{R}$, $\kappa = L^2(\bar{\sigma}^2 d\sqrt{d} + 1)^2(\gamma\underline{\sigma}^2)^{-1}$. Then for any $(t, x) \in [T - \varepsilon, T) \times \mathbb{R}^n$, we have*

$$0 \le u_m(t, x) \le (1 + m(T - t))^{-\rho}. \tag{8}$$

Proof. The proof of $u_m(t, x) \ge 0$ is the same as in Lemma 3.4. Set

$$\tilde{u}_m(t, x) = (1 + m(T - t))^{-\rho}\exp\left(-\frac{m\theta|x_i - a_i|^2}{2(1 + m(T - t))}\right). \tag{9}$$

It is obvious that $\tilde{u}_m(T, x) = \exp(-\frac{m\theta|x_i - a_i|^2}{2})$. In the following, we show that $\tilde{u}_m$ is a viscosity supersolution of PDE (3) if $t \ge T - \varepsilon$. It is easy to verify that, for each $v \in \mathcal{V}_0$

$$\begin{aligned}
\partial_t \tilde{u}_m &= \frac{\rho m}{1 + m(T - t)}\tilde{u}_m - \frac{m^2\theta|x_i - a_i|^2}{2(1 + m(T - t))^2}\tilde{u}_m,\\
\partial_{x_i}\tilde{u}_m &= -\frac{m\theta(x_i - a_i)}{1 + m(T - t)}\tilde{u}_m,\\
\partial^2_{x_i x_i}\tilde{u}_m &= -\frac{m\theta}{1 + m(T - t)}\tilde{u}_m + \frac{m^2\theta^2|x_i - a_i|^2}{(1 + m(T - t))^2}\tilde{u}_m,\\
\partial_{x_j}\tilde{u}_m &= 0, \qquad \partial^2_{x_i x_j}\tilde{u}_m = 0, \qquad j \ne i,\\
\sigma^\top D_x^2\tilde{u}_m\sigma &= (\partial^2_{x_i x_i}\tilde{u}_m)\sigma_i^\top\sigma_i,
\end{aligned}$$

$$G(-\sigma_i^\top\sigma_i) \le -\frac{\gamma\underline{\sigma}^2}{2}; \qquad G(\sigma_i^\top\sigma_i) \le \frac{\bar{\sigma}^2\Gamma}{2},$$
$$(\langle\beta^{jk}, D_x\tilde{u}_m\rangle)_{j,k=1}^d = (\partial_{x_i}\tilde{u}_m)(\beta_i^{jk})_{j,k=1}^d.$$

Then for each $(t,x) \in [T-\varepsilon, T) \times \mathbb{R}^n$, we have

$$\begin{aligned}
&\partial_t\tilde{u}_m + \sup_{v\in\mathcal{V}_0}\{G(\sigma^\top D_x^2\tilde{u}_m\sigma + (2\langle\beta^{jk}, D_x\tilde{u}_m\rangle)_{j,k=1}^d) + \langle\alpha, D_x\tilde{u}_m\rangle\} \\
&\quad\le \partial_t\tilde{u}_m + \frac{m\theta\tilde{u}_m}{1+m(T-t)}\sup_{v\in\mathcal{V}_0} G(-\sigma_i^\top\sigma_i) + \frac{m^2\theta^2\tilde{u}_m|x_i-a_i|^2}{(1+m(T-t))^2}\sup_{v\in\mathcal{V}_0} G(\sigma_i^\top\sigma_{i^*}) \\
&\qquad + \frac{2m\theta\tilde{u}_m}{1+m(T-t)}\sup_{v\in\mathcal{V}_0} G((-(x_i-a_i)\beta_i^{jk}(t,x))_{j,k=1}^d) \\
&\qquad + \frac{m\theta\tilde{u}_m}{1+m(T-t)}\sup_{v\in\mathcal{V}_0}(a_i-x_i)\alpha_i \\
&\quad\le -\frac{m\theta\tilde{u}_m}{1+m(T-t)}|x_i-a_i|^2\left(\frac{m}{4(1+m\varepsilon)}-\kappa\right) \\
&\quad\le 0,
\end{aligned}$$

which implies that $\tilde{u}_m$ is a viscosity supersolution of PDE (3) if $t \ge T-\varepsilon$. Thus by comparison theorem we obtain for $(t,x) \in [T-\varepsilon, T) \times \mathbb{R}^n$,

$$u_m(t,x) \le \tilde{u}_m(t,x) \le (1+m(T-t))^{-\rho}.$$

The proof is complete. ■

Note that the above result still holds if assumption (H4) is valid only for some i. By a similar analysis as in Theorem 3.7, we can show that $c(\{X_t^{x_i;i} = a_i\}) = 0$ for any $t > 0$ and $a_i \in \mathbb{R}$. We remark that one can also obtained this result by Martini's approach and Girsanov's theorem. However, we can also get the convergence rate. Indeed,

Theorem 3.11. *Under the assumptions* (H1), (H2) *and* (H4), *we obtain that for each* $T > 0$

$$\hat{\mathbb{E}}\left[\exp\left(-\frac{m\theta|X_T^{x_i;i}-a_i|^2}{2}\right)\right] \le (1+m(T\wedge\varepsilon))^{-\rho}, \tag{10}$$

where θ, ρ *and* ε *are given in Lemma* 3.10.

By the above result, we can show that the maximal process does not weight a single point.

Corollary 3.12. *Assume* $d = 1$. *Then we have* $c(\{B_t^* = a\}) = 0$ *for each* $a \in \mathbb{R}$, *where* $B_t^* = \sup_{0\le s\le t} B_s$.

Proof. Without loss of generality, assume $t = 1$. For each $m \ge 1$, set $\varphi_m(x) = \exp(-\frac{m^{\frac{2(1+\rho)}{\rho}}\theta|x-a|^2}{2})$, where θ, ρ are given in Lemma 3.10. Then applying Fatou's lemma yields that

$$c(\{B_t^* = a\}) \le \liminf_{m\to\infty}\hat{\mathbb{E}}[\varphi_m(\sup\{B_{t_1^m}, B_{t_2^m}, \ldots, B_1\})],$$

where $t_i^m = \frac{i}{m}$ for each $i \le m$.

By Remark 3.5 and Theorem 3.11, we conclude that

$$\begin{aligned}\hat{\mathbb{E}}[\varphi_m(\sup\{B_{t_1^m}, B_{t_2^m}\})] &\le \hat{\mathbb{E}}[\varphi_m(B_{t_1^m} + \sup\{0, B_{t_2^m} - B_{t_1^m}\})] \\ &\le \hat{\mathbb{E}}[\varphi_m(B_{t_1^m})] + \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi_m(y + B_{t_2^m} - B_{t_1^m})]_{y=B_{t_1^m}}] \\ &\le 2\left(1 + m^{\frac{2(1+\rho)}{\rho}} m^{-1}\right)^{-\rho} \le \frac{2}{m^{2+\rho}}.\end{aligned}$$

Iterating the procedure for m times implies that

$$\hat{\mathbb{E}}[\varphi_m(\sup\{B_{t_1^m}, B_{t_2^m}, \dots, B_1\})] \le \frac{1}{m^{1+\rho}}$$

and this completes the proof. ■

Example 3.13. By Remark 2.2, we have $P(\{X_t^* = y\}) = 0$, where X_t^* is the maximal process of $X_t = \int_0^t \theta_s dB_s$ and this provides a way to study the maxima of non-Markovian Itô process. Moreover, one can get that X_t has a unique maxima in the interval $[0, t]$.

Finally, we shall study the capacity of the G-Itô process staying in a curve.

Theorem 3.14. *Assume* (H1), (H2) *and* (H4) *hold. Suppose f satisfies $\partial_{x_i} f, \partial^2_{x_i x_j} f \in C_{b,Lip}(\mathbb{R}^n)$ and there exist two constants $0 < \delta \le \Delta < \infty$ such that*

$$\delta \le \left|\sum_{i=1}^n \partial_{x_i} f \sigma_i\right|^2 \le \Delta.$$

Then for each $T > 0$ we have

$$c(\{f(X_T^x) = 0\}) = 0.$$

Proof. Applying the G-Itô formula yields that

$$\begin{aligned}f(X_t^x) =& f(x) + \int_0^t \partial_{x_i} f \alpha_i(s) ds + \int_0^t \left[\partial_{x_i} f \beta_i^{jk} + \frac{1}{2}\partial^2_{x_i x_l} f \sigma_{ij}\sigma_{lk}\right](s) d\langle B^j, B^k\rangle_s \\ &+ \int_0^t \partial_{x_i} f \sigma_i(s) dB_s.\end{aligned}$$

Thus $\tilde{X}_t^x = ((X_t^x)^\top, f(X_t^x))^\top$ can be seen as the G-Itô process (1) corresponding to

$$\tilde{\alpha}(t) = \begin{pmatrix} \alpha(t) \\ \partial_{x_i} f \alpha_i(t) \end{pmatrix}, \qquad \tilde{\sigma}(t) = \begin{pmatrix} \sigma(t) \\ \partial_{x_i} f \sigma_i(t) \end{pmatrix}$$

and

$$\tilde{\beta}^{jk}(t) = \begin{pmatrix} \beta^{jk}(t) \\ \left[\partial_{x_i} f \beta_i^{jk} + \frac{1}{2}\partial^2_{x_i x_l} f \sigma_{ij}\sigma_{lk}\right](t) \end{pmatrix}.$$

Thus we have $c(\{f(X_T^x) = 0\}) = 0$ and this completes the proof. ■

Example 3.15. The property required upon the gradient of the curve f is necessary. Indeed, we take $n = 2$, $d = 1$, $x = 0$, $b = 0$, $h^{jk} = 0$, $\sigma = (1, -1)^T$ and $f(x, y) = x - y$. Then $f(B_T, B_T) = 0$, $q.s.$ However $\partial_x f \sigma_1 + \partial_y f \sigma_2 = 0$.

3.2. Some applications

In this subsection, we shall identify some non-trivial quasi-continuous Borel measurable functions on Ω and we always assume (H1), (H2) and (H4) hold.

Theorem 3.16. *Let $\xi \in L_G^1(\Omega;\mathbb{R}^k)$ and $A \in \mathcal{B}(\mathbb{R}^k)$ with $c(\{\xi \in \partial A\}) = 0$. Then $I_{\{\xi\in A\}} \in L_G^1(\Omega)$.*

Proof. For each $\epsilon > 0$, since $\xi \in L_G^1(\Omega;\mathbb{R}^k)$, we can find an open set $O \subset \Omega$ with $c(O) \leq \frac{\epsilon}{2}$ such that $\xi|_{O^c}$ is continuous. Set $D_i = \{x \in \mathbb{R}^k : d(x,\partial A) \leq \frac{1}{i}\}$ and $A_i = \{x \in \mathbb{R}^k : d(x,\partial A) < \frac{1}{i}\}$, it is easy to check that $\{\xi \in D_i\} \cap O^c$ is closed, $\{\xi \in A_i\} \subset \{\xi \in D_i\}$ and $\{\xi \in D_i\} \cap O^c \downarrow \{\xi \in \partial A\} \cap O^c$. Then we have

$$c(\{\xi \in D_i\} \cap O^c) \downarrow c(\{\xi \in \partial A\} \cap O^c) = 0.$$

Thus we can find an i_0 such that $c(\{\xi \in A_{i_0}\} \cap O^c) \leq \frac{\epsilon}{2}$. Set $O_1 = \{\xi \in A_{i_0}\} \cup O$, it is easy to verify that $c(O_1) \leq \epsilon$, $O_1^c = \{\xi \in A_{i_0}^c\} \cap O^c$ is closed and $I_{\{\xi\in A\}}$ is continuous on O_1^c. Thus $I_{\{\xi\in A\}}$ is quasi-continuous, which implies $I_{\{\xi\in A\}} \in L_G^1(\Omega)$. ■

Now we consider the capacity of $X_s^{t,\xi}$ hitting the boundary of cubes, where $X^{t,\xi}$ is the G-Itô process (1) starting at t and from the random variable ξ. Then, by the above theorem, we can get a kind of quasi-continuous random variables associated to G-Itô processes.

Lemma 3.17. *Let $A = [a,b]$, where $a, b \in \mathbb{R}^n$ with $a \leq b$. Then for each given $t \geq 0$, $\xi \in L_G^2(\Omega_t;\mathbb{R}^n)$, $s > t$, we have $c(\{X_s^{t,\xi} \in \partial A\}) = 0$.*

Proof. It suffices to prove that $c(\{X_s^{t,\xi_i;i} = a_i\}) = c(\{X_s^{t,\xi_i;i} = b_i\}) = 0$. We shall only show that $c(\{X_s^{t,\xi_1;1} = a_1\}) = 0$ and the other cases can be proved in a similar way. For each m, set $\varphi_m(x) = \exp(-\frac{m\theta|x_1-a_1|^2}{2})$. Applying Theorems 3.2 and 3.11, we conclude that

$$\hat{\mathbb{E}}[\varphi_m(X_s^{t,\xi})] \leq (1 + m((s-t)\wedge\varepsilon))^{-\rho}.$$

Letting $m \to \infty$ yields the desired result and this completes the proof. ■

Theorem 3.18. *Let $A_i = [a^i, b^i]$ with $a^i, b^i \in \mathbb{R}^n$, $a^i \leq b^i$ for $i \geq 1$ and $D \in \mathcal{B}(\mathbb{R}^n)$ with $\partial D \subset \cup_{i=1}^{\infty} \partial A_i$. Then for each given $t \geq 0$, $\xi \in L_G^2(\Omega_t;\mathbb{R}^n)$, $s > t$, we have $I_{\{X_s^{t,\xi}\in D\}} \in L_G^1(\Omega_s)$. In particular, $I_{\{X_s^x\in D\}} \in L_G^1(\Omega_s)$.*

Proof. This is a direct consequence of Lemma 3.17 and Theorem 3.16. ■

In the following, we only consider the capacity of B_t on the sphere. But the method can be applied to deal with $X_s^{t,\xi}$.

Lemma 3.19. *Let D be a d-dimensional sphere. Then we have for each $t > 0$,*

$$c(\{B_t \in \partial D\}) = 0.$$

Proof. Without loss of generality, we assume D is the unit sphere. Set $\bar{x} = (x_1,\ldots,x_{d-1})$ and denote functions

$$f(\bar{x}) := \sqrt{1-|\bar{x}|^2} I_{\{|\bar{x}|^2\leq 1\}}.$$

Please cite this article in press as: M. Hu, et al., Quasi-continuous random variables and processes under the G-expectation framework, Stochastic Processes and their Applications (2016), http://dx.doi.org/10.1016/j.spa.2016.02.003

For each $\epsilon > 0$, there exists a nonnegative function $J^\epsilon(\bar{x}) \in C_0^\infty(\mathbb{R}^{d-1})$ such that

$$J^\epsilon(\bar{x}) = \begin{cases} 1, & \text{if } |\bar{x}| \le 1 - 2\epsilon; \\ 0, & \text{if } |\bar{x}| \ge 1 - \epsilon. \end{cases}$$

Then define function $f^\epsilon(x) := x_d - J^\epsilon(\bar{x}) f(\bar{x})$. It is easy to check that $J^\epsilon(\bar{x}) f(\bar{x}) \in C_0^\infty(\mathbb{R}^{d-1})$. Moreover, $|\sum_{i=1}^d \partial_{x_i} f^\epsilon(x) e_i|^2 = \sum_{i=1}^{d-1} |\partial_{x_i} f^\epsilon(x)|^2 + 1$. Then applying Theorem 3.14, we obtain for each given $t \ge 0$,

$$c(\{B_t^d - J^\epsilon(\tilde{B}_t) f(\tilde{B}_t) = 0\}) = 0,$$

where $\tilde{B}_t = (B_t^1, \dots, B_t^{d-1})$. Consequently,

$$c(\{B_t^d - f(\tilde{B}_t) = 0\} \cap \{|\tilde{B}_t|^2 \le 1 - 2\epsilon\}) = 0.$$

Note that $\{B_t^d - f(\tilde{B}_t) = 0\} \cap \{|\tilde{B}_t|^2 \le 1 - 2\epsilon\} \uparrow \{B_t^d - f(\tilde{B}_t) = 0\} \cap \{|\tilde{B}_t|^2 < 1\}$, then by taking $\epsilon \downarrow 0$ we get that

$$c(\{B_t^d - f(\tilde{B}_t) = 0\} \cap \{|\tilde{B}_t|^2 < 1\}) = 0.$$

From Theorem 3.11, we get $c(\{B_t^d = 0\}) = 0$. Therefore, we deduce that

$$c(\{B_t^d - f(\tilde{B}_t) = 0\}) \le c(\{B_t^d - f(\tilde{B}_t) = 0\} \cap \{|\tilde{B}_t|^2 < 1\}) + c(\{B_t^d = 0\}) = 0.$$

By a similar analysis, we also get $c(\{B_t^d + f(\tilde{B}_t) = 0\}) = 0$. Thus

$$c(\{B_t \in \partial D\}) \le c(\{B_t^d - f(\tilde{B}_t) = 0\}) + c(\{B_t^d + f(\tilde{B}_t) = 0\}) = 0,$$

which is the desired result. ■

The following result is a direct consequence of Theorem 3.16, Lemmas 3.17 and 3.19.

Theorem 3.20. *Suppose A_i is a d-dimensional sphere or $[a^i, b^i]$ with $a^i, b^i \in \mathbb{R}^d$, $a^i \le b^i$ for $i \ge 1$. If D is in $\mathcal{B}(\mathbb{R}^d)$ with $\partial D \subset \cup_{i=1}^\infty \partial A_i$, then $I_{\{B_t \in D\}} \in L_G^1(\Omega_t)$ for any $t > 0$.*

Example 3.21. Assume $d = 1$. Given a function $u \in C_{b,Lip}(\mathbb{R})$. Then for each given $n \in \mathbb{N}$, we take

$$h_i^n(x) = \mathbf{1}_{[-n+\frac{i}{n}, -n+\frac{i+1}{n})}(x), \quad i = 0, \dots, 2n^2 - 1, \qquad h_{2n^2}^n = 1 - \sum_{i=0}^{2n^2-1} h_i^n.$$

We denote $u^n(B_t) := \sum_{i=0}^{2n^2} u(-n + \frac{i}{n}) h_i^n(B_t)$. Then by Theorem 3.20 and a direct calculation, we conclude $u^n(B_t) \in L_G^1(\Omega_t)$ and

$$\lim_{n \mapsto \infty} \hat{\mathbb{E}}[|u^n(B_t) - u(B_t)|] = 0,$$

which can be seen as a counterpart of the approximation of function in the nonlinear expectation theory. In particular, it provides a method to construct the approximation of an admissible control under the G-expectation framework, more details can be founded in [2].

Please cite this article in press as: M. Hu, et al., Quasi-continuous random variables and processes under the G-expectation framework, Stochastic Processes and their Applications (2016), http://dx.doi.org/10.1016/j.spa.2016.02.003

4. Quasi-continuous processes

In this section, we shall study the integrable processes under the G-expectation framework. First, we consider the characterization of $M_G^p(0,T)$. Then we apply Krylov's estimates to get some quasi-continuous processes.

4.1. Characterization of $M_G^p(0,T)$

We shall give a characterization of the space $M_G^p(0,T)$ for each $T>0$ and $p\geq 1$, which generalizes the results in [1].

Set $\mathcal{F}_t=\mathcal{B}(\Omega_t)$ for $t\in[0,T]$ and the distance

$$\rho((t,\omega),(t',\omega'))=|t-t'|+\max_{s\in[0,T]}|\omega_s-\omega'_s|,\quad \text{for } (t,\omega),(t',\omega')\in[0,T]\times\Omega_T.$$

Define, for each $p\geq 1$,

$$\mathbb{M}^p(0,T)=\left\{\eta:\text{progressively measurable on }[0,T]\times\Omega_T\text{ and }\hat{\mathbb{E}}\left[\int_0^T|\eta_t|^p dt\right]<\infty\right\}$$

and the corresponding capacity

$$\hat{c}(A)=\frac{1}{T}\hat{\mathbb{E}}\left[\int_0^T I_A(t,\omega)dt\right],\ \text{for each progressively measurable set } A\subset[0,T]\times\Omega_T.$$

Proposition 4.1. *Let A be a progressively measurable set in $[0,T]\times\Omega_T$. Then $I_A=0$ $\hat{c}$-q.s. if and only if $\int_0^T I_A(t,\cdot)dt=0$ c-q.s.*

Proof. It is obvious $\int_0^T I_A(t,\cdot)dt\geq 0$. Thus we can easily get $\hat{\mathbb{E}}[\int_0^T I_A(t,\omega)dt]=0$ if and only if $c(\{\int_0^T I_A(t,\cdot)dt>0\})=0$, which completes the proof. ■

In the following, we do not distinguish the progressively measurable process η from η' if $\hat{c}(\{\eta\neq\eta'\})=0$.

Proposition 4.2. *For each $p\geq 1$, $\mathbb{M}^p(0,T)$ is a Banach space under the norm $\|\eta\|_{\mathbb{M}^p}:=(\hat{\mathbb{E}}[\int_0^T|\eta_t|^p dt])^{1/p}$.*

Proof. The proof is the same as the classical case and we omit it. ■

It is clear that $M_G^0(0,T)\subset\mathbb{M}^p(0,T)$ for any $p\geq 1$. Thus $M_G^p(0,T)$ is a closed subspace of $\mathbb{M}^p(0,T)$. Also we set

$$M_c(0,T)=\{\text{all adapted processes } \eta \text{ in } C_b([0,T]\times\Omega_T)\}.$$

Proposition 4.3. *For each $p\geq 1$, the completion of $M_c(0,T)$ under the norm $\|\cdot\|_{\mathbb{M}^p}$ is $M_G^p(0,T)$.*

Proof. We first prove that the completion of $M_c(0,T)$ under the norm $\|\cdot\|_{\mathbb{M}^p}$ is included in $M_G^p(0,T)$. For each fixed $\eta\in M_c(0,T)$, we set

$$\eta_t^k(\cdot)=\sum_{i=0}^{k-1}\eta_{(iT)/k}(\cdot)I_{[\frac{iT}{k},\frac{(i+1)T}{k})}(t).$$

Please cite this article in press as: M. Hu, et al., Quasi-continuous random variables and processes under the G-expectation framework, Stochastic Processes and their Applications (2016), http://dx.doi.org/10.1016/j.spa.2016.02.003

By Theorem 2.5, it is easy to verify that $\eta^k \in M_G^p(0,T)$. For each $\varepsilon > 0$, since $\mathcal{P}$ is weakly compact, there exists a compact set $K \subset \Omega_T$ such that $\hat{\mathbb{E}}[I_{K^c}] \le \varepsilon$. Thus

$$\hat{\mathbb{E}}\left[\int_0^T |\eta_t - \eta_t^k|^p dt\right] \le \hat{\mathbb{E}}\left[I_K \int_0^T |\eta_t - \eta_t^k|^p dt\right] + \hat{\mathbb{E}}\left[I_{K^c} \int_0^T |\eta_t - \eta_t^k|^p dt\right]$$
$$\le \sup_{(t,\omega)\in[0,T]\times K} T|\eta_t(\omega) - \eta_t^k(\omega)|^p + (2l)^p T\varepsilon,$$

where l is the upper bound of η. Noting that $[0,T]\times K$ is compact and $\eta \in C_b([0,T]\times\Omega_T)$, thus

$$\limsup_{k\to\infty} \hat{\mathbb{E}}\left[\int_0^T |\eta_t - \eta_t^k|^p dt\right] \le (2l)^p T\varepsilon.$$

Since ε is arbitrary, we get $\|\eta^k - \eta\|_{\mathbb{M}^p} \to 0$ as $k \to \infty$. Thus $\eta \in M_G^p(0,T)$, which implies the desired result.

Now we prove the converse part. For each given $\bar{\eta}_t = \sum_{i=0}^{N-1} \xi_i I_{[t_i,t_{i+1})}(t) \in M_G^0(0,T)$, we can find $\{\phi_k^i : k \ge 1\} \subset C([0,\infty))$, $i < N$, $k \ge 1$ so that $\mathrm{supp}(\phi_k^i) \subset (t_i, t_{i+1})$ and $\int_0^T |\phi_k^i(t) - I_{[t_i,t_{i+1})}(t)|^p dt \to 0$ as $k \to \infty$. Set $\bar{\eta}_t^k = \sum_{i=0}^{N-1} \xi_i \phi_k^i(t)$, it is easy to check that $\bar{\eta}^k \in M_c(0,T)$ and $\|\bar{\eta}^k - \bar{\eta}\|_{\mathbb{M}^p} \to 0$ as $k \to \infty$. Thus each element of $M_G^p(0,T)$ belongs to the completion of $M_c(0,T)$ under the norm $\|\cdot\|_{\mathbb{M}^p}$, which completes the proof. ■

Definition 4.4. A progressively measurable process $\eta : [0,T]\times\Omega_T \to \mathbb{R}$ is called quasi-continuous (q.c.), if for each $\varepsilon > 0$, there exists a progressively measurable open set G in $[0,T]\times\Omega_T$ such that $\hat{c}(G) < \varepsilon$ and $\eta|_{G^c}$ is continuous.

Remark 4.5. Our definition of quasi-continuous process is different from the ones in [17,18].

Definition 4.6. We say that a progressively measurable process $\eta : [0,T]\times\Omega_T \to \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous process η' such that $\hat{c}(\{\eta \ne \eta'\}) = 0$.

Theorem 4.7. *For each $p \ge 1$,*

$$M_G^p(0,T) = \left\{\eta \in \mathbb{M}^p(0,T) : \lim_{N\to\infty} \hat{\mathbb{E}}\left[\int_0^T |\eta_t|^p I_{\{|\eta_t|\ge N\}} dt\right] = 0 \textit{ and } \eta \textit{ has a quasi-continuous version}\right\}.$$

Proof. We denote

$$J_p = \left\{\eta \in \mathbb{M}^p(0,T) : \lim_{N\to\infty} \hat{\mathbb{E}}\left[\int_0^T |\eta_t|^p I_{\{|\eta_t|\ge N\}} dt\right] = 0 \text{ and } \eta \text{ has a quasi-continuous version}\right\}.$$

Noting that the completion of $M_c(0,T)$ under the norm $\|\cdot\|_{\mathbb{M}^p}$ is $M_G^p(0,T)$, then, by the same analysis as in Propositions 18 and 24 in [1], we can get $M_G^p(0,T) \subset J_p$.

On the other hand, for each $\eta \in J_p$, we need to prove that $\eta \in M_G^p(0,T)$. Without loss of generality, we assume that η is quasi-continuous. For each $N > 0$, set $\eta^N = (\eta \wedge N) \vee (-N)$, since

$\hat{\mathbb{E}}[\int_0^T |\eta_t - \eta_t^N|^p dt] \leq \hat{\mathbb{E}}[\int_0^T |\eta_t|^p I_{\{|\eta_t|\geq N\}} dt] \to 0$ as $N \to \infty$, it suffices to show that $\eta^N \in M_G^p(0,T)$ for each fixed $N > 0$. For each $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \Omega_T$ such that $\hat{\mathbb{E}}[I_{K_\varepsilon^c}] \leq \varepsilon$ and a progressively measurable open set $G_\varepsilon \subset [0,T] \times \Omega_T$ such that $\hat{c}(G_\varepsilon) < \varepsilon$ and $\eta^N|_{G_\varepsilon^c}$ is continuous. By Tietze's extension theorem, there exists a function $\tilde{\eta}^{N,\varepsilon} \in C_b([0,T] \times \Omega_T)$ such that $|\tilde{\eta}^{N,\varepsilon}| \leq N$ and $\tilde{\eta}^{N,\varepsilon}|_{G_\varepsilon^c} = \eta^N|_{G_\varepsilon^c}$. For each $k \geq 1$, we set $F^{i,k} = G_\varepsilon^c \cap ([t_i^k, t_{i+1}^k] \times \Omega_T)$ for $i \leq k-1$, where $t_i^k = \frac{iT}{k}$ for $i = 0, \ldots, k$. Since G_ε^c is progressively measurable, we can get $F^{i,k} \in \mathcal{B}([0, t_{i+1}^k]) \times \mathcal{B}(\Omega_{t_{i+1}^k})$. Since $F^{i,k}$ is closed, again by Tietze's extension theorem, there exists a function $\zeta^{N,i,k} \in C_b([0, t_{i+1}^k] \times \Omega_T)$ such that $\zeta^{N,i,k} \in \mathcal{B}([0, t_{i+1}^k]) \times \mathcal{B}(\Omega_{t_{i+1}^k})$, $|\zeta^{N,i,k}| \leq N$ and $\zeta^{N,i,k}|_{F^{i,k}} = \eta^N|_{F^{i,k}}$. We denote $\tilde{\eta}_t^{N,k}(\omega) = \sum_{i=0}^{k-1} \zeta^{N,i,k}(t,\omega) I_{[t_i^k, t_{i+1}^k)}(t)$ and

$$\bar{\eta}_t^{N,k}(\omega) = \tilde{\eta}^{N,k}\left(t - \frac{T}{k}, \omega\right) I_{[t_1^k, T)}(t), \qquad \bar{\eta}_t^{N,\varepsilon,k}(\omega) = \tilde{\eta}^{N,\varepsilon}\left(t - \frac{T}{k}, \omega\right) I_{[t_1^k, T)}(t).$$

A similar analysis as in Proposition 4.3 implies that $\bar{\eta}^{N,k} \in M_G^p(0,T)$. Moreover, we obtain that

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\int_0^T |\eta_t^N - \bar{\eta}_t^{N,k}|^p dt\right] \\
&\leq 3^{p-1}\left(\hat{\mathbb{E}}\left[\int_0^T |\eta_t^N - \tilde{\eta}_t^{N,\varepsilon}|^p dt\right] + \hat{\mathbb{E}}\left[\int_0^T |\tilde{\eta}_t^{N,\varepsilon} - \bar{\eta}_t^{N,\varepsilon,k}|^p dt\right]\right. \\
&\quad \left. + \hat{\mathbb{E}}\left[\int_0^T |\bar{\eta}_t^{N,\varepsilon,k} - \bar{\eta}_t^{N,k}|^p dt\right]\right) \\
&\leq 3^{p-1}\left(\hat{\mathbb{E}}\left[\int_0^T |\eta_t^N - \tilde{\eta}_t^{N,\varepsilon}|^p dt\right] + \hat{\mathbb{E}}\left[\int_0^T |\tilde{\eta}_t^{N,\varepsilon} - \bar{\eta}_t^{N,\varepsilon,k}|^p dt\right]\right. \\
&\quad \left. + \hat{\mathbb{E}}\left[\int_0^T |\tilde{\eta}_t^{N,\varepsilon} - \tilde{\eta}_t^{N,k}|^p dt\right]\right) \\
&\leq 3^{p-1}\left(2(2N)^p T\varepsilon + \hat{\mathbb{E}}\left[\int_0^T |\tilde{\eta}_t^{N,\varepsilon} - \bar{\eta}_t^{N,\varepsilon,k}|^p dt\right]\right) \\
&\leq 3^{p-1}\left(2(2N)^p T\varepsilon + (2N)^p \frac{T}{k} + \hat{\mathbb{E}}\left[\int_{t_1^k}^T |\tilde{\eta}_t^{N,\varepsilon} - \bar{\eta}_t^{N,\varepsilon,k}|^p dt\right]\right) \\
&\leq 3^{p-1}\left(2(2N)^p T\varepsilon + (2N)^p \frac{T}{k} + \hat{\mathbb{E}}\left[I_{K_\varepsilon^c} \int_{t_1^k}^T |\tilde{\eta}_t^{N,\varepsilon} - \bar{\eta}_t^{N,\varepsilon,k}|^p dt\right]\right. \\
&\quad \left. + \hat{\mathbb{E}}\left[I_{K_\varepsilon} \int_{t_1^k}^T |\tilde{\eta}_t^{N,\varepsilon} - \bar{\eta}_t^{N,\varepsilon,k}|^p dt\right]\right) \\
&\leq 3^{p-1}\left(3(2N)^p T\varepsilon + (2N)^p \frac{T}{k} + \sup_{(t,\omega)\in[t_1^k, T]\times K_\varepsilon}\right. \\
&\quad \left. \times\ T\left|\tilde{\eta}^{N,\varepsilon}(t,\omega) - \tilde{\eta}^{N,\varepsilon}\left(t - \frac{T}{k}, \omega\right)\right|^p\right).
\end{aligned}$$

Noting that $[0, T] \times K_\varepsilon$ is compact and $\tilde{\eta}^{N,\varepsilon} \in C_b([0, T] \times \Omega_T)$, thus

$$\limsup_{k\to\infty} \hat{\mathbb{E}}\left[\int_0^T |\eta_t^N - \bar{\eta}_t^{N,k}|^p dt\right] \leq (6N)^p T\varepsilon,$$

which implies $\eta^N \in M_G^p(0, T)$. The proof is complete. ■

Remark 4.8. Note that the Tietze's extension theorem cannot ensure the extension of a progressively measurable process is also progressively measurable. Then we provide an alternative way to prove the characterization of $M_G^p(0, T)$, which is different from that of [1].

By Theorem 4.7, we immediately have the following result.

Corollary 4.9. *Let $\eta \in M_G^1(0, T)$ and $f \in C_b([0, T] \times \mathbb{R})$. Then $(f(t, \eta_t))_{t\leq T} \in M_G^p(0, T)$ for any $p \geq 1$.*

Theorem 4.10. *Let η^k be in $M_G^1(0, T)$, $k \geq 1$, such that $\eta^k \downarrow \eta$ $\hat{c}$-q.s. Then $\hat{\mathbb{E}}[\int_0^T \eta_t^k dt] \downarrow \hat{\mathbb{E}}[\int_0^T \eta_t dt]$. Moreover, if $\eta \in M_G^1(0, T)$, then $\hat{\mathbb{E}}[\int_0^T |\eta_t^k - \eta_t|dt] \downarrow 0$.*

Proof. Since $\eta^k \in M_G^1(0, T)$, we can choose $\eta^{k,N} \in M_G^0(0, T)$ such that $\hat{\mathbb{E}}[\int_0^T |\eta_t^k - \eta_t^{k,N}|dt] \to 0$ as $N \to \infty$. It is easy to check that $\int_0^T \eta_t^{k,N} dt \in L_G^1(\Omega_T)$ and $\hat{\mathbb{E}}[|\int_0^T \eta_t^{k,N} dt - \int_0^T \eta_t^k dt|] \leq \hat{\mathbb{E}}[\int_0^T |\eta_t^k - \eta_t^{k,N}|dt]$. Then we get $\int_0^T \eta_t^k dt \in L_G^1(\Omega_T)$ for $k \geq 1$. By Proposition 4.1 and Theorem 4.7, it is easy to verify that $\int_0^T \eta_t^k dt \downarrow \int_0^T \eta_t dt$ c-q.s. Thus, applying Theorem 2.6 yields that $\hat{\mathbb{E}}[\int_0^T \eta_t^k dt] \downarrow \hat{\mathbb{E}}[\int_0^T \eta_t dt]$. If $\eta \in M_G^1(0, T)$, then $|\eta^k - \eta| \in M_G^1(0, T)$ and $|\eta^k - \eta| \downarrow 0$ $\hat{c}$-q.s., which implies that $\hat{\mathbb{E}}[\int_0^T |\eta_t^k - \eta_t|dt] \downarrow 0$. ■

The following example shows that $M_G^p(0, T)$ is strictly contained in $\mathbb{M}^p(0, T)$.

Example 4.11. Suppose $0 < \underline{\sigma}^2 < \bar{\sigma}^2 < \infty$, $T > 0$. We consider 1-dimensional G-Brownian motion $(B_t)_{t\geq 0}$. $(\langle B\rangle_t)_{t\geq 0}$ is the quadratic process of $(B_t)_{t\geq 0}$. Let

$$\eta_t = I_{\{\langle B\rangle_t = \frac{(\underline{\sigma}^2+\bar{\sigma}^2)t}{2}\}} \quad \text{for } t \leq T.$$

Then we claim that $\eta \notin M_G^1(0, T)$. Indeed we can choose $f^k(t, x) \in C_b([0, T] \times \mathbb{R})$, $k \geq 1$, such that

$$f^k(t, x) = 1 \quad \text{for } \left|x - \frac{(\underline{\sigma}^2 + \bar{\sigma}^2)t}{2}\right| \leq \frac{T}{k};$$

$$f^k(t, x) = 0 \quad \text{for } \left|x - \frac{(\underline{\sigma}^2 + \bar{\sigma}^2)t}{2}\right| \geq \frac{2T}{k}.$$

Set $g^k = \wedge_{i=1}^k f^i$, it is easy to check that $g^k \in C_b([0, T] \times \mathbb{R})$, $g^k(t, x) = 1$ for $|x - \frac{(\underline{\sigma}^2+\bar{\sigma}^2)t}{2}| \leq \frac{T}{k}$ and $g^k \downarrow I_{\{x=\frac{(\underline{\sigma}^2+\bar{\sigma}^2)t}{2}\}}$. Since $g^k(t, \langle B\rangle_t) \downarrow \eta_t$, we have $g^k(t, \langle B\rangle_t) \in M_G^1(0, T)$ by Corollary 4.9. If $\eta \in M_G^1(0, T)$, then it follows from Theorem 4.10 that $\hat{\mathbb{E}}[\int_0^T |g^k(t, \langle B\rangle_t) - \eta_t|dt] \downarrow 0$. On the other hand, by the representation of $\hat{\mathbb{E}}[\cdot]$ in [1], there exists a probability measure $P \in \mathcal{P}$ such that $\langle B\rangle_t = ((\frac{(\underline{\sigma}^2+\bar{\sigma}^2)}{2} - \frac{1}{k}) \vee \underline{\sigma}^2)t$ P-a.s. Therefore we have $\hat{\mathbb{E}}[\int_0^T |g^k(t, \langle B\rangle_t) - \eta_t|dt] \geq E_P[\int_0^T |g^k(t, \langle B\rangle_t) - \eta_t|dt] = T$ and this contradiction implies that $\eta \notin M_G^1(0, T)$.

4.2. G-integrable processes

In the above subsection, we give the characterization of $M_G^p(0,T)$. However, it is also difficult to check that a progressively measurable process is quasi-continuous. Then the present section is devoted to finding some Borel measurable functions on $[0,T]\times\Omega_T$ are quasi-continuous processes.

In this section, we always assume $n \le d$ and (H1)–(H3) hold. For some fixed $x_0\in\mathbb{R}^n$, consider the G-Itô process X^{x_0} given by (1). For convenience, we set $X = X^{x_0}$.

Theorem 4.12 (*Krylov's Estimates*)**.** *For each $\delta > 0$ and $p \ge n$, there exists a constant N depending on $p, \lambda, \Lambda, L, G$ and δ such that for each Borel measurable function $f(t,x)$ and $g(x)$,*

$$\hat{\mathbb{E}}\left[\int_0^\infty \exp(-\delta t)|f(t,X_t)|dt\right] \le N\|f\|_{L^{p+1}([0,\infty)\times\mathbb{R}^n)},$$
$$\hat{\mathbb{E}}\left[\int_0^\infty \exp(-\delta t)|g(X_t)|dt\right] \le N\|g\|_{L^p(\mathbb{R}^n)}.$$

Proof. Let $\mathcal{P}$ be the weakly compact set that represents $\hat{\mathbb{E}}$. By Corollary 5.7 in Chapter 3 of [16], we obtain that $d\langle B^j, B^k\rangle_t = \hat{\gamma}_t^{jk}dt$ q.s. and $\underline{\sigma}^2 t I_{d\times d} \le \hat{\gamma}_t = (\hat{\gamma}_t^{jk})_{j,k=1}^d \le \bar{\sigma}^2 t I_{d\times d}$. Note that B is a martingale on the probability space $(\Omega, (\mathcal{F}_t)_{t\ge 0}, P)$ for each $P\in\mathcal{P}$. Then it is easy to check that

$$W_t^P := \int_0^t \hat{\gamma}_s^{-\frac{1}{2}} dB_s, \qquad P\text{-a.s.}$$

is a Brownian motion on $(\Omega, (\mathcal{F}_t)_{t\ge0}, P)$. Thus we have

$$X_t = x_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s^{jk}\hat{\gamma}_s^{jk} ds + \int_0^t \sigma_s \hat{\gamma}_s^{\frac{1}{2}} dW_s^P, \qquad P\text{-a.s.}$$

Applying Theorem 3.4 in Chapter 2 of Krylov [7] (see also [8]), we can find a constant N depending on $p, \lambda, \Lambda, L, G$ and δ such that for each Borel measurable function $f(t,x)$,

$$E_P\left[\int_0^\infty \exp(-\delta t)|f(t,X_t)|dt\right] \le \bar{N}\|f\|_{L^{p+1}([0,T]\times\mathbb{R}^n)}.$$

Therefore, we have

$$\hat{\mathbb{E}}\left[\int_0^\infty \exp(-\delta t)|f(t,X_t)|dt\right] = \sup_{P\in\mathcal{P}} E_P\left[\int_0^\infty \exp(-\delta t)|f(t,X_t)|dt\right]$$
$$\le N\|f\|_{L^{p+1}([0,T]\times\mathbb{R}^n)}$$

and the second inequality can be proved in a similar way. ■

The following estimates are from Theorem 4.12.

Corollary 4.13. *For each $T > 0$ and $p \ge n$, there exists a constant N_T depending on $p, \lambda, \Lambda, L, G$ and T such that for each Borel measurable function $f(t,x)$ and $g(x)$,*

$$\hat{\mathbb{E}}\left[\int_0^T |f(t,X_t)|dt\right] \le N_T\|f\|_{L^{p+1}([0,T]\times\mathbb{R}^n)},$$
$$\hat{\mathbb{E}}\left[\int_0^T |g(X_t)|dt\right] \le N_T\|g\|_{L^p(\mathbb{R}^n)}.$$

From now on, we shall use Krylov's estimates to generate some quasi-continuous processes.

Lemma 4.14. (i) *If ψ is in $L^p([0,T]\times\mathbb{R}^n)$ with $p \geq n+1$, then for each $T > 0$, we have $(\psi(t,X_t))_{t\leq T} \in M_G^1(0,T)$. Moreover, for each $\psi' = \psi$, a.e., we have $\psi'(\cdot, X_\cdot) = \psi(\cdot, X_\cdot)$.*
(ii) *If φ is in $L^p(\mathbb{R}^n)$ with $p \geq n$, then for each $T > 0$, we have $(\varphi(X_t))_{t\leq T} \in M_G^1(0,T)$. Moreover, for each $\varphi' = \varphi$, a.e., we have $\varphi'(X_\cdot) = \varphi(X_\cdot)$.*

Proof. We only prove (ii), since (i) can be proved in a similar way. Note that there exists a sequence of bounded continuous functions $(\varphi^k)_{k\geq 1}$, which converges to φ in $L^p(\mathbb{R}^n)$. Then by Corollary 4.13, we can find a constant C' so that

$$\lim_{k\to\infty} \hat{\mathbb{E}}\left[\int_0^T |\varphi^k - \varphi|(X_t)dt\right] \leq C' \lim_{k\to\infty} \|\varphi^k - \varphi\|_{L^p(\mathbb{R}^n)} = 0.$$

By Theorem 4.7, we can get $(\varphi^k(X_t))_{t\leq T} \in M_G^1(0,T)$ for each $k \geq 1$. Thus we obtain $(\varphi(X_t))_{t\leq T} \in M_G^1(0,T)$.

Assume $\varphi = \varphi'$, $a.e.$ Applying Corollary 4.13 again, we conclude that

$$\hat{\mathbb{E}}\left[\int_0^T |\varphi' - \varphi|(X_t)dt\right] \leq C'\|\varphi' - \varphi\|_{L^p(\mathbb{R}^n)} = 0,$$

which completes the proof. ■

Theorem 4.15. *Let $(\varphi^k)_{k\geq 1}$ be a sequence of $\mathbb{R}^n$-valued Borel measurable functions and $|\varphi^k(x)| \leq \bar{C}(1+|x|^l)$, $k \geq 1$ for some constants $\bar{C}$ and l. If $\varphi^k \to \varphi$, a.e., then for each $T > 0$ and $p \geq 1$,*

$$\lim_{k\to\infty} \hat{\mathbb{E}}\left[\int_0^T |\varphi^k(X_t) - \varphi(X_t)|^p dt\right] = 0.$$

Proof. By Lemma 4.14, we may assume that $|\varphi(x)| \leq \bar{C}(1+|x|^l)$. For each fixed $N > 0$, we have

$$\hat{\mathbb{E}}\left[\int_0^T |\varphi^k(X_t) - \varphi(X_t)|^p dt\right] \leq \hat{\mathbb{E}}\left[\int_0^T |\varphi^k(X_t) - \varphi(X_t)|^p I_{\{|X_t|\leq N\}} dt\right] + \hat{\mathbb{E}}\left[\int_0^T |\varphi^k(X_t) - \varphi(X_t)|^p I_{\{|X_t|\geq N\}} dt\right].$$

By Corollary 4.13, there exists a constant C' independent of k such that

$$\hat{\mathbb{E}}\left[\int_0^T |\varphi^k(X_t) - \varphi(X_t)|^p I_{\{|X_t|\leq N\}} dt\right] \leq C' \left|\int_{\{|x|\leq N\}} |\varphi^k(x) - \varphi(x)|^{np} dx\right|^{\frac{1}{n}}.$$

Then applying Lebesgue's dominated convergence theorem yields that

$$\hat{\mathbb{E}}\left[\int_0^T |\varphi^k(X_t) - \varphi(X_t)|^p I_{\{|X_t|\leq N\}} dt\right] \to 0 \quad \text{as } k \to \infty.$$

Noting that $|\varphi^k(X_t) - \varphi(X_t)|^p I_{\{|X_t|\geq N\}} \leq \frac{(2\bar{C})^p}{N}(1+|X_t|^l)^p |X_t|$, then we get

$$\limsup_{k\to\infty} \hat{\mathbb{E}}\left[\int_0^T |\varphi^k(X_t) - \varphi(X_t)|^p dt\right] \leq \frac{(2\bar{C})^p}{N} \int_0^T \hat{\mathbb{E}}[(1+|X_t|^l)^p |X_t|] dt.$$

Since N can be arbitrarily large, we obtain

$$\lim_{k\to\infty} \hat{\mathbb{E}}\left[\int_0^T |\varphi^k(X_t) - \varphi(X_t)|^p dt\right] = 0,$$

which is the desired result. ■

Theorem 4.15 can be seen as a weak dominated convergence theorem for the G-Itô processes. By this result, we obtain

Theorem 4.16. *If φ is a $\mathbb{R}^n$-valued Borel measurable function of polynomial growth, then we have $(\varphi(X_t))_{t\le T} \in M_G^2(0,T)$ for each $T > 0$.*

Proof. We can find a sequence of continuous functions $(\varphi^k)_{k\ge 1}$ with compact support, such that φ^k converges to φ a.e. and $|\varphi^k(x)| \le \bar{C}(1+|x|^l)$, where $\bar{C}$, l are constants independent of k. Then by Theorem 4.15, for each $T > 0$, we conclude that

$$\lim_{k\to\infty} \hat{\mathbb{E}}\left[\int_0^T |\varphi^k - \varphi|^2(X_t) dt\right] = 0.$$

Since $(\varphi^k(X_t))_{t\le T} \in M_G^2(0,T)$ for each k by Theorem 4.7, we derive that $(\varphi(X_t))_{t\le T} \in M_G^2(0,T)$ and this completes the proof. ■

Acknowledgments

The authors would like to thank Prof. Peng, Shige for his helpful discussions and suggestions. We also thank the anonymous reviewer for carefully reading the manuscript and giving many valuable suggestions.

Hu's research was supported by the National Natural Science Foundation of China (No. 11201262 and 11301068) and Shandong Province (No. BS2013SF020 and ZR2014AP005). Wang's research was supported by the China Postdoctoral Science Foundation (No. 2015M582068) and Fundamental Research Funds of Shandong University (No. 2015GN023). Hu, Wang and Zheng's research was partially supported by the National Natural Science Foundation of China (No. 10921101) and Programme of Introducing Talents of Discipline to Universities of China (No. B12023).

References

[1] L. Denis, M. Hu, S. Peng, Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion paths, Potential Anal. 34 (2011) 139–161.
[2] M. Hu, S. Ji, Dynamic Programming Principle for Stochastic Recursive Optimal Control Problem under G-framework, 2014, in arxiv:1410.3538.
[3] M. Hu, S. Ji, S. Peng, Y. Song, Backward stochastic differential equations driven by G-Brownian motion, Stochastic Process. Appl. 124 (2014) 759–784.
[4] M. Hu, S. Ji, S. Peng, Y. Song, Comparison theorem, Feynman–Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion, Stochastic Process. Appl. 124 (2014) 1170–1195.
[5] M. Hu, S. Ji, S. Yang, A stochastic recursive optimal control problem under the G-expectation framework, Appl. Math. Optim. 70 (2014) 253–278.
[6] M. Hu, S. Peng, On representation theorem of G-expectations and paths of G-Brownian motion, Acta Math. Appl. Sin. Engl. Ser. 25 (3) (2009) 539–546.
[7] N.V. Krylov, Controlled Diffusion Processes, Springer Verlag, 1980.
[8] N.V. Krylov, On estimates of the minimum of a solution of a parabolic equation and estimates of the distribution of a semimartingale, Math. GSSR Sb. 58 (1) (1987) 207–221.

[9] X. Li, S. Peng, Stopping times and related Itô's calculus with G-Brownian motion, Stochastic Process. Appl. 121 (2011) 1492–1508.

[10] C. Martini, On the marginal laws of one-dimensional stochastic integrals with uniformly elliptic integrand, Ann. Inst. H. Poincare (B) Probab. Statist. 36 (1) (2000) 35–43.

[11] S. Peng, Filtration consistent nonlinear expectations and evaluations of contingent claims, Acta Math. Appl. Sin. 20 (2004) 1–24.

[12] S. Peng, Nonlinear expectations and nonlinear Markov chains, Chinese Ann. Math. 26B (2) (2005) 159–184.

[13] S. Peng, G-expectation, G-Brownian Motion and Related Stochastic Calculus of Itô type, in: Stochastic Analysis and Applications, in: Abel Symp., vol. 2, Springer, Berlin, 2007, pp. 541–567.

[14] S. Peng, Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation, Stochastic Process. Appl. 118 (12) (2008) 2223–2253.

[15] S. Peng, Backward stochastic differential equation, nonlinear expectation and their applications, in: Proceedings of the International Congress of Mathematicians Hyderabad, India, 2010, pp. 281–307.

[16] S. Peng, Nonlinear expectations and stochastic calculus under uncertainty, 2010. arXiv:1002.4546v1.

[17] Y. Song, Some properties on G-evaluation and its applications to G-martingale decomposition, Sci. China Math. 54 (2011) 287–300.

[18] Y. Song, Uniqueness of the representation for G-martingales with finite variation, Electron. J. Probab. 17 (2012) 1–15.

[19] F. Wang, G. Zheng, Some sample path properties of G-Brownian motion, 2014, in arxiv:1407.0211.

Please cite this article in press as: M. Hu, et al., Quasi-continuous random variables and processes under the G-expectation framework, Stochastic Processes and their Applications (2016), http://dx.doi.org/10.1016/j.spa.2016.02.003