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Averaging along irregular curves and regularisation of ODEs

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Abstract

We consider the ordinary differential equation (ODE) $\mathrm{d}x_t = b(t, x_t)\mathrm{d}t + \mathrm{d}w_t$ where w is a continuous driving function and b is a time-dependent vector field which possibly is only a distribution in the space variable. We quantify the regularising properties of an arbitrary continuous path w on the existence and uniqueness of solutions to this equation. In this context we introduce the notion of ρ-*irregularity* and show that it plays a key role in some instances of the regularisation by noise phenomenon. In the particular case of a function w sampled according to the law of the fractional Brownian motion of Hurst index $H \in (0, 1)$, we prove that almost surely the ODE admits a solution for all b in the Besov–Hölder space $B^{\alpha+1}_{\infty,\infty}$ with $\alpha > -1/2H$. If $\alpha > 1 - 1/2H$ then the solution is unique among a natural set of continuous solutions. If $H > 1/3$ and $\alpha > 3/2 - 1/2H$ or if $\alpha > 2 - 1/2H$ then the equation admits a unique Lipschitz flow. Note that when $\alpha < 0$ the vector field b is only a distribution, nonetheless there exists a natural notion of solution for which the above results apply.

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1. Introduction

In [7] A. M. Davie showed that the integral equation

$$x_t = x_0 + \int_0^t b(s, x_s)\mathrm{d}s + w_t, \quad t \in [0, 1], \tag{1}$$

with $x, w \in C([0, 1]; \mathbb{R}^d)$ and $b : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ bounded and measurable has a unique continuous solution for almost every path w sampled from the law of the d-dimensional Brownian motion. This result can be interpreted as a phenomenon of regularisation by noise, in the sense that it is well known that the same equation without w can show non-uniqueness.

Regularisation by noise in the case of stochastic differential equations (SDEs) driven by Brownian motion is nowadays a well understood subject: see for example Veretennikov, Krylov and Roeckner [15], Flandoli, Gubinelli and Priola [9], Zhang, Flandoli and Da Prato [6]. All these work are essentially based of the use of Itô calculus to highlight the regularising properties of Brownian paths. Meyer-Brandis and Proske [17] use Malliavin calculus to derive similar conclusions. Davie's contribution [7] is more subtle in the sense that it is a result for an ordinary differential equation (ODE) and not for the related SDE, i.e. the existence and uniqueness of solutions is studied in the space of continuous paths and not in the more common probabilistic framework of continuous adapted processes on a given filtered probability space. This has been clearly pointed out by Flandoli [8] which called these more general solutions *path-by-path*. In this respect Davie's contribution is purely analytical and one of the aim of the present work is to *analytically* characterize the regularisation effect for general continuous perturbation w (whether random or not) to the evolution dictated by an irregular vector field.

Regularisation by "fast" or "dispersive" motions is an interesting phenomenon which appears also in some deterministic PDE situations, for example for Korteweg–de-Vries equation [13,1] and for fast-rotating Euler and Navier–Stokes equations [2]. In particular the technique of Young integration we employ in the present work is essentially the same used in the paper [13] to study the periodic Korteweg–de-Vries equation and take inspiration in the theory of rough paths [16,14,11].

In a recent paper [4,5] Chouk and Gubinelli analyse the regularisation phenomenon in the context of non-linear dispersive PDEs modulated by an irregular signal. In particular they considered equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = A\varphi_t \frac{\mathrm{d}w_t}{\mathrm{d}t} + \mathscr{N}(\varphi_t), \quad t \geq 0 \tag{2}$$

where $w : \mathbb{R}_+ \to \mathbb{R}$ is an arbitrary continuous function, A is an unbounded linear operator (like the Schrödinger operator $i\partial^2$ or the Airy operator ∂^3 acting on periodic or non-periodic functions) and $\mathscr{N}$ some local polynomial non-linearity with possibly derivative terms. The unifying theme of this last study and the present one is the fact that the regularising properties of $w \in C([0, 1]; \mathbb{R}^d)$ are analysed in terms of the *averaging operator* T_t^w defined as

$$T_t^w f(x) = \int_0^t f(x + w_r)\mathrm{d}r, \quad x \in \mathbb{R}^d \tag{3}$$

for any measurable functions $f : \mathbb{R}^d \to \mathbb{R}$. Characterising the mapping properties of T^w for various kind of perturbations w seems very interesting and not straightforward. Mapping properties of T^w for deterministic smooth curves w are, for reasons not related to the regularisation by noise phenomenon, an interesting subject in analysis: we have in mind, for example the work of Tao and Wright [19] on L^p improving bounds for averages along curves (we thank F. Flandoli and V.M. Tortorelli for having pointed us the existence of these results).

The averaging operator can be seen as the convolution against the *occupation measure* L_t^w of the path w defined as

$$L_t^w(\mathrm{d}y) = \int_0^t \delta_{w_u}(\mathrm{d}y)\mathrm{d}u.$$

Indeed, for continuous b, the following computation holds

$$T_t^w b(x) = \int_0^t b(x + w_u)\mathrm{d}u = \int_0^t \mathrm{d}u \int_{\mathbb{R}^d} b(x - y)\delta_{w_u}(\mathrm{d}y) = (b * L_t^w)(x).$$

The basic observation contained in Davie's paper [7] is that if $b : \mathbb{R}^d \to \mathbb{R}^d$ is a given bounded function then for almost every d-dimensional Brownian path $w : [0, T] \to \mathbb{R}^d$ and for all $0 \leqslant t \leqslant T$ the function $x \mapsto T_t^w b(x)$ has almost Lipschitz regularity (its modulus of continuity is of the type $|x| \log^{1/2}(1/|x|)$). Morally this is a gain of almost 1 degree of the regularity and one of the key steps to prove uniqueness of the ODE (1) for a bounded measurable drift b.

In this paper we analyse the behaviour of the averaging operator T^w in the scale of Hölder–Besov spaces $\mathscr{C}^\alpha = \mathscr{C}^\alpha(\mathbb{R}^d, \mathbb{R}^n) = B^\alpha_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^n)$ for arbitrary regularity $\alpha \in \mathbb{R}$. We consider a class of perturbations w given by the sample paths of the d-dimensional fractional Brownian motion (fBm) of Hurst index $H \in (0, 1)$, that is the unique centred Gaussian process $(B_t^H)_{t\geqslant 0}$ with values on $\mathbb{R}^d$ and covariance function

$$\mathbb{E}[B_t^H B_s^H] = c_H(|t + s|^{2H} - |t|^{2H} - |s|^{2H})Id$$

for all $t, s \geqslant 0$.

As an application of the averaging properties we obtain various existence and uniqueness results for solutions of the ODE (1) and relative flow properties for distributional vector field b.

The choice of fBm has the advantage of being a simple process for which many other results about existence and uniqueness of associated SDE are available [18]. More interestingly, the approach based on Itô calculus, used in most of the papers on the regularisation effect for Brownian motion, does not easily extend to the fBm case, nor does the explicit computations of Davie [7]. The freedom in the choice of the Hurst parameter gives us the possibility to explore the effect of different degrees of irregularity of the perturbation on the regularisation phenomenon and the quasi-invariance of the law of the fBm will allow us to study the effect of perturbations on the averaging properties of the paths.

Returning to the averaging behaviour of fBm paths we obtain the following result.

Theorem 1.1. *Take $H \in (0, 1)$ and $\rho < 1/2H$. Then there exists $\gamma > 1/2$ such that for all $f \in C(\mathbb{R}^d; \mathbb{R})$ there exists a Borel set $\mathscr{N}_{f,\gamma} \subseteq C([0, 1], \mathbb{R}^d)$ (which depends on f, γ) of zero measure with respect to the law of the d-dimensional fractional Brownian motion (fBm) of Hurst index H such that for all $w \notin \mathscr{N}_{f,\gamma}$ we have for $\alpha > -\rho$,*

$$\|T_t^w f - T_s^w f\|_{\mathscr{C}^{\alpha+\rho,\psi}} \lesssim_w \|f\|_{\mathscr{C}^\alpha} |t - s|^\gamma$$

for all $0 \leqslant s < t \leqslant 1$.

In this statement the weighted space $\mathscr{C}^{\alpha,\psi}$ is a subspace of the space of local Hölder continuous functions with given grow at infinity described by the weight ψ, and its precise definition is given in Definition 1.16. The space $\mathscr{C}^\alpha$ is the usual Besov–Hölder defined below in (6).

Letting for a moment aside the time regularity, this result shows that the averaging against fBm paths gains almost $1/2H$ derivatives in the space variable. Unfortunately the result stated in Theorem 1.1 is not very satisfying since one would really like to have the almost sure boundedness of $T_t^w : \mathscr{C}^\alpha \to \mathscr{C}^{\alpha+\rho,\psi}$. The difficulty is, of course, the fact that the exceptional set $\mathscr{N}_f$ of Theorem 1.1 depends itself on the function f. Using the Littlewood–Paley decomposition of Besov–Hölder distributions and the scaling of the fractional Brownian motion, the problem of finding a version of T^w which is almost surely continuous can be related to the following conjecture:

Conjecture 1.2. *Let $(B_t^H)_{t\geqslant 0}$ be a d-dimensional fBm of Hurst index $H \in (0,1)$. Let $K : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that*

$$|K(x)| \lesssim (1+|x|)^{-N}, \qquad \int_{\mathbb{R}^d} K(x)\mathrm{d}x = 0,$$

where $N > d$ can be chosen arbitrarily large. Then

$$\mathbb{E}(\|T_{0,t}^{B^H} K\|_{L^1(\mathbb{R}^d)}^p) = \mathbb{E}\left[\left(\int_{\mathbb{R}^d}\left|\int_0^t K(x+B_s^H)\mathrm{d}s\right|\mathrm{d}x\right)^p\right] \lesssim t^{p/2}$$

as $t \to +\infty$.

If the function K has a bounded support the estimation is true as an easy consequence of our results, however currently we are unable to prove or disprove this conjecture.

On the positive side if we replace $\mathscr{C}^\alpha$ by the Fourier–Lebesgue spaces $\mathcal{F}L^{\alpha,p}$ defined as

$$\mathcal{F}L^{\alpha,p}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : N_{\alpha,p}(f)^p = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^p (1+|\xi|)^{\alpha p}\mathrm{d}\xi < \infty \right\},$$

with $\mathcal{F}L^\alpha = \mathcal{F}L^{\alpha,1}$, then it is easy to see that for $0 \leqslant \gamma \leqslant 1$ and $\rho \in \mathbb{R}$:

$$\|T_t^w - T_s^w\|_{\mathcal{L}(\mathcal{F}L^\alpha;\mathcal{F}L^{\alpha+\rho})} = \sup_{f\in\mathcal{F}L^\alpha} \frac{N_{\alpha+\rho}(T_t^w f - T_s^w f)}{N_\alpha(f)} \lesssim \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-s|^\gamma,$$

where $\Phi_t^w(a) = \int_0^t e^{i\langle a,w_r\rangle}\mathrm{d}r = e^{-i\langle a,x\rangle} T_t^w(e^{i\langle a,\cdot\rangle})(x)$ and where we introduced the norm

$$\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} = \sup_{a\in\mathbb{R}^d}\ \sup_{0\leqslant s<t\leqslant T} (1+|a|)^\rho \frac{|\Phi_t^w(a) - \Phi_s^w(a)|}{|s-t|^\gamma}.$$

This observation reduces the question of the boundedness of T^w to that of the decay of the Fourier transform $a \mapsto \Phi_t^w(a)$ of the *occupation measure* of w (for generalities about occupation measures and densities for deterministic and random functions see for example the review of Geman and Horowitz [12]). This suggests to introduce the following novel notion of "irregularity" of the perturbation w:

Definition 1.3. Let $\rho > 0$ and $\gamma > 0$. We say that a function $w \in C([0,T];\mathbb{R}^d)$ is (ρ,γ)-irregular if

$$\|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} < +\infty.$$

Moreover we say that w is ρ-irregular if there exists $\gamma > 1/2$ such that w is (ρ,γ)-irregular.

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The time regularity of this Fourier transform, measured by the Hölder exponent γ, will also be crucial in our analysis. The notion of ρ-irregularity is also relevant to the boundedness of T^w in other functional spaces, for example we easily see that for all $\alpha \in \mathbb{R}$:

$$\|T_t^w f - T_s^w f\|_{H^{\alpha+\rho}(\mathbb{R}^d)} \leqslant \|\Phi^w\|_{\mathcal{W}_T^{\rho,\gamma}} |t-s|^\gamma \|f\|_{H^\alpha(\mathbb{R}^d)},$$

where $H^\alpha(\mathbb{R}^d) = \mathscr{F}L^{\alpha,2}$ are the usual Sobolev spaces on $\mathbb{R}^d$ and in general similar inequalities holds in Fourier–Lebesgue spaces $\mathscr{F}L^{\alpha,p}$ of arbitrary integrability $p \in [1,+\infty]$. However the notion of ρ-irregularity does not seem enough to control the boundedness of the averaging operator in Besov spaces.

The limiting value $1/2$ for γ does not seem to have any special meaning, as far as the occupation measure is concerned, however if $\gamma > 1/2$ we are able to develop a quite simple integration theory for the averaging operator using Young integral techniques and quite surprisingly it turns out that this is sufficient for the purpose of this paper. Indeed a proof similar to that of Theorem 1.1 gives the existence of (plenty of) perturbations w which are ρ-irregular:

Theorem 1.4. *Let $(B_t^H)_{t\geqslant 0}$ be a fractional Brownian motion of Hurst index $H \in (0,1)$ then for any $\rho < 1/2H$ there exist $\gamma > 1/2$ so that with probability one the sample paths of B^H are (ρ,γ)-irregular.*

In particular there exist continuous paths which are ρ-irregular for arbitrarily large ρ and thus paths which deliver an arbitrary degree of regularisation. Using well known properties of support of the law of the fractional Brownian motion it is also possible to show that there exists ρ-irregular trajectories which are arbitrarily close in the supremum norm to any smooth path.

As a direct corollary of Theorem 1.4 we have the boundedness of T^w in the Fourier–Lebesgue spaces $\mathscr{F}L^\alpha$:

Corollary 1.5. *Let $H \in (0,1)$ and $\rho < 1/2H$. Then almost surely with respect to the law of the fBm of Hurst index H we have that for all $0 \leqslant s \leqslant t \leqslant T$ the averaging operator T^w is bounded from $\mathscr{F}L^\alpha$ to $\mathscr{F}L^{\alpha+\rho}$ and satisfy*

$$\|T_t^w - T_s^w\|_{\mathcal{L}(\mathscr{F}L^\alpha;\mathscr{F}L^{\alpha+\rho})} \leqslant C_{w,\gamma,\rho}|t-s|^\gamma$$

for some constant $C_{w,\gamma,\rho}$ which depends only on w, γ, ρ. This means that

$$T^w \in \mathscr{C}^\gamma\left([0,T]; \mathcal{L}\left(\mathscr{F}L^\alpha, \mathscr{F}L^{\alpha+\rho}\right)\right).$$

One of the contributions of our work is the observation that the regularity of the occupation measure of w seems to play a major role in the understanding of the regularising properties of w in a non-linear context and it would be desirable to understand more deeply the link of the notion of ρ-irregularity with the pathwise properties of w, for example linking them to the notion of true roughness appearing in the literature on densities for differential equations driven by rough paths [10].

It would also be interesting to study more deeply the notion of irregularity for "generic" continuous paths (for example in the class of Hölder continuous paths). Indeed, set aside the classic contribution of Geman and Horowitz [12] mentioned above, the authors are not aware of any systematic study of occupation measures of random processes from the point of view of their action on spaces of functions or distributions, topic which seems central to our analysis.

An open problem is, for example, understanding what happens if we replace w with a regularised version w^ε or with a perturbed version. In this respect we conjecture that if w is

(ρ, γ)-irregular then for any smooth function $\varphi \in C([0,1];\mathbb{R}^d)$ the perturbed path $w^\varphi = w + \varphi$ is still (ρ,γ)-irregular. In relation to this last problem we have obtained the following general result:

Theorem 1.6. *Let $\rho \in \mathbb{R}$ and $\varphi \in \mathscr{C}^\beta([0,T];\mathbb{R}^d)$ with $1/2 \leqslant \beta < 1$. Then if w is ρ-irregular the path $w^\varphi = w + \varphi$ is $(\rho - 1/2\beta)$-irregular. Moreover for $\gamma > 1/2$ we have*

$$\|T^{w+\varphi} f\|_{\mathscr{C}^\gamma([0,T];\mathscr{C}^{\alpha+\rho-1/2\beta})} \lesssim_{T,\beta,\gamma} \|T^w f\|_{\mathscr{C}^{\gamma,\psi}([0,T];\mathscr{C}^{\alpha+\rho})} \|\varphi\|_{\mathscr{C}^\beta}.$$

In particular if $T^w \in \mathscr{C}^\gamma\left([0,T];\mathcal{L}\left(\mathscr{C}^\alpha;\mathscr{C}^{\alpha+\rho,\psi}\right)\right)$ then

$$T^{w+\varphi} \in \mathscr{C}^\gamma\left([0,T];\mathcal{L}\left(\mathscr{C}^\alpha;\mathscr{C}^{\alpha+\rho-1/2\beta}\right)\right).$$

In particular the irregularity property is preserved at the price of a loss at least $1/2$ in regularity (which happens when β is close to 1).

If w is sampled according to the law of a fBm and if the perturbation φ is adapted to the natural filtration of w then it is possible to exploit the quasi-invariance of the fBm measure with respect to adapted shifts to prove the irregularity of the perturbed path without any loss on the irregularity exponent:

Theorem 1.7. *Let B^H be a fBm of Hurst index $H \in (0,1)$ and let $\Phi : [0,T] \to \mathbb{R}^d$ be an Hölder continuous process which is adapted to the natural filtration of B^H. Then, for all $\rho < 1/2H$ almost surely the process $B^H + \Phi$ is ρ-irregular and for any $f \in \mathscr{C}^\alpha$*

$$\|T^{B^H+\Phi} f\|_{\mathscr{C}^\gamma([0,T];\mathscr{C}^{\alpha+\rho,\psi})} < +\infty \quad \textit{almost surely}.$$

The disadvantage of this result is that the exceptional set where the irregularity property fails depends a priori on Φ and this poses problems in applications to pathwise results valid for a large class of perturbations (for example smooth and adapted Φ).

One of our aims is to apply these results on the averaging properties of paths w and of its perturbations to the study of existence and uniqueness of solutions to the ODE (1) for distributional b. Two main situations will be considered:

1. $b \in \mathscr{C}^\alpha$ (or $b \in \mathscr{F}L^\alpha$) for some $\alpha > 0$. In this case b will be a bounded continuous function and the ODE (1) has a natural meaning and allows for continuous solution, we will then consider the related uniqueness problem and the existence of a Lipschitz flow.
2. $b \in \mathscr{C}^\alpha$ (or $b \in \mathscr{F}L^\alpha$) for some $\alpha < 0$. In this case even the appropriate meaning to give to the ODE (1) is not clear and we will investigate this problem and the related well-posedness and continuity issues.

In the case $\alpha \geqslant 0$ we have the following results:

Theorem 1.8. *Let $b \in \mathscr{C}(\mathbb{R}^d)$ and assume that $\|T^w b\|_{\mathscr{C}^\gamma([0,T];\mathscr{C}^{3/2,\psi})} < +\infty$. Then for any $x_0 \in \mathbb{R}^d$ there exists a unique continuous solution $x \in C([0,1];\mathbb{R}^d)$ of the ODE (1) and the flow map $x_0 \mapsto x_t$ of the equation is locally Lipschitz continuous in space uniformly in $t \in [0,1]$.*

Theorem 1.9. *Let $b \in \mathscr{C}^\alpha$ and assume that $\alpha > 1 - 1/2H$. Then for any $x_0 \in \mathbb{R}^d$ there exists a measurable set of perturbations $\mathscr{N}_{b,x_0} \subseteq C([0,1];\mathbb{R}^d)$ which is of zero measure with respect to the law of the fBm with index $H \in (0,1)$ and such that, for all $w \notin \mathscr{N}_{b,x_0}$ there exists a unique continuous solution $x \in C([0,1];\mathbb{R}^d)$ of the ODE (1).*

As we already remarked, in the case where $b \in \mathscr{C}^\alpha$ for $\alpha < 0$, the ODE (1) is not well defined since in general the evaluation of the distribution b along a continuous curve is not possible. However if we take into account a particular class of continuous paths we can show that this coupling has a meaning. A suitable class of continuous functions is given by a space of paths which are perturbations of w:

Definition 1.10. Let $\gamma \in (0, 1)$. The space $\mathscr{Q}^w_\gamma$ of (w, γ)-controlled paths is the space

$$\mathscr{Q}^w_\gamma = \{x \in C([0, 1]; \mathbb{R}^d) : x - w \in \mathscr{C}^\gamma([0, 1]; \mathbb{R}^d)\}.$$

Then for *controlled paths* we can prove the following result.

Theorem 1.11. *Let $b \in \mathcal{S}'(\mathbb{R}^d; \mathbb{R}^d)$, $\gamma > \frac{1}{2}$ and assume that $\|T^w b\|_{\mathscr{C}^\gamma([0,T];\mathscr{C}^{0,\psi})} < +\infty$. Let $\rho \in \mathcal{S}(\mathbb{R}^d)$ be a positive function with $\rho(0) = 1$ and let $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon)$. Then, for all $x \in \mathscr{Q}^w_\gamma$,*

$$\lim_{\varepsilon \to 0} \int_0^t (\rho_\varepsilon * b)(x_s)\mathrm{d}s =: \int_0^t b(x_s)\mathrm{d}s \tag{4}$$

exists uniformly in $t \in [0, T]$, is independent of ρ and extends the usual definition of the right hand side for continuous b. Moreover the function $t \mapsto \int_0^t b(x_s)\mathrm{d}s$ is Hölder continuous of exponent γ.

Theorem 1.11 allows to give a natural meaning to $\int_0^t b(x_s)\mathrm{d}s$ for all $x \in \mathscr{Q}^w_\gamma$ and from this we can say that $x \in \mathscr{Q}^w_\gamma$ is a solutions of the ODE (1) if

$$x_t - w_t = \int_0^t b(x_s)\mathrm{d}s$$

for all $t \in [0, 1]$. That is the ODE has a meaning not in the space of all continuous functions, as it was when b is a function, but in the more restricted space of functions which can be seen as "not too irregular" additive modifications of w. In this context we have natural generalisations of Theorems 1.8 and 1.9 provided we restrict the space of allowed functions to $\mathscr{Q}^w_\gamma$:

Theorem 1.12. *Assume that $\|T^w b\|_{\mathscr{C}^\gamma([0,T];\mathscr{C}^{3/2,\psi})} < +\infty$. Then for any $x_0 \in \mathbb{R}^d$ there exists a unique continuous solution $x \in \mathscr{Q}^w_\gamma$ of the ODE (1) and the flow map $x_0 \mapsto x_t$ of the equation is Lipschitz continuous uniformly in $t \in [0, 1]$.*

Theorem 1.13. *Let $b \in \mathscr{C}^{\alpha+1}$ and assume that $\alpha > -1/2H$. Then for any $x_0 \in \mathbb{R}^d$ there exists a measurable set of perturbations $\mathscr{N}_{b,x_0} \subseteq C([0, 1]; \mathbb{R}^d)$ which is of zero measure with respect to the law of the fBm with index $H \in (0, 1)$ and such that, for all $w \notin \mathscr{N}_{b,x_0}$ there exists a unique continuous solution $x \in \mathscr{Q}^w_\gamma$ of the ODE (1).*

Note that Theorems 1.12 and 1.13 are applicable also when $\alpha \geqslant 0$. In this case existence of solutions is simply a result of a compactness argument in $C([0, 1]; \mathbb{R}^d)$ and given a continuous solution it belongs necessarily to $\mathscr{Q}^w_\gamma$ so, in this case, Theorems 1.12 and 1.13 are natural generalisations of Theorems 1.8 and 1.9.

When w is sampled according to the law of the fBm with Hurst parameter H Theorem 1.12 give the following corollary.

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Theorem 1.14. *Fix $H \in (0,1)$ and assume that $b \in \mathscr{C}^{\alpha+3/2}$ for some $\alpha > -1/2H$. Then there exists a measurable set of perturbations $\mathscr{N}_b \subseteq C([0,1];\mathbb{R}^d)$ which is of zero measure with respect to the law of the fBm with index $H \in (0,1)$ and such that, for all $w \notin \mathscr{N}_b$ and for all $x_0 \in \mathbb{R}^d$ there exists a unique continuous solution $x \in \mathscr{Q}^w_\gamma$ of the ODE (1) and the corresponding flow map $\Phi_t : x_0 \mapsto x_t$ is globally Lipschitz. Moreover the exceptional set $\mathscr{N}_b$ can be chosen to be the same for all $b \in \mathscr{F}L^\alpha$.*

An interesting consequence of Theorem 1.14 is the fact that if one consider the ODE (1) as a strong SDE (that is an equation for stochastic processes adapted to the filtration generated by the process w) and if w has the law of the fBm of index H then we can allow general random $b \in \mathscr{F}L^\alpha$ and still retain uniqueness under the regularity conditions of the theorem. This was one of our main motivation to introduce the scale of Fourier–Lebesgue regularities $(\mathscr{F}L^\alpha)_\alpha$. Similar results for the Besov scale $(\mathscr{C}^\alpha)_\alpha$ are not known since we are not able to prove the corresponding mapping properties for the averaging operator T^w. Note that even in the case of the Brownian motion this was an open problem [9] since the standard approach using stochastic calculus cannot be applied in this case. Allowing random b could open the way to the study of a general class of stochastic transport equations where the drift itself depends on the solution.

The key to obtain these results (the existence part when $\alpha < 0$ and the uniqueness part for $\alpha \geqslant 0$ or $\alpha < 0$) lies in the fact that in all cases the ODE (1) is equivalent to an equation of Young type (YE) of the form

$$\theta_t = \theta_0 + \int_0^t X_{\mathrm{d}s}(\theta_s), \tag{5}$$

where here $X : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ plays the role of a time-varying, integrated, vector field and $\theta_t = x_t - w_t$ is the perturbation which by the hypothesis $x \in \mathscr{Q}^w_\gamma$ belongs to $\mathscr{C}^\gamma([0,1];\mathbb{R}^d)$. The integral operation featuring in (5) has to be understood as a natural non-linear generalisation of the Young integral [20] defined as limit of Riemann sums:

$$\int_0^t X_{\mathrm{d}s}(\theta_s) = \lim_{|\Pi| \to 0} \sum X_{t_i,t_{i+1}}(\theta_{t_i}),$$

where $X_{s,t}(x) = X_t(x) - X_s(x)$. In the case of the ODE (1) the integrated vector field X corresponds to the average of the original vector field b given by $X_t(x) = T^w_{0,t} b(x)$ for all $t \in [0,1]$ and $x \in \mathbb{R}^d$. Young differential equations of the type (5) are used also in [4,5] to study the regularisation phenomenon for some non-linear dispersive equations. The theory of such equations is very similar to the theory for standard Young-type equation but for the sake of the reader we rederive here the main results in our slightly non standard setting.

This paper is then divided naturally into two parts: in the first we study the non-linear Young integral and the YE (5) and derive the results announced above about existence and uniqueness for the ODE (1). In the second we analyse the averaging properties of fBm sample paths and apply the results to the study of the regularisation phenomenon for Eq. (1) driven by fBm paths.

1.1. Notations

Several function spaces are involved in the rest of the article. In this section we define those spaces, and specify some notations. Let $\psi, \varphi \in \mathcal{D}$ be a nonnegative radial functions such that

1. The support of ψ is contained in a ball and the support of φ is contained in an annulus;
2. $\psi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^d$;

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3. $\operatorname{supp}(\psi) \cap \operatorname{supp}(\varphi(2^{-j}.)) = \emptyset$ for $i \geq 1$ and if $|i-j| > 1$, then $\operatorname{supp}(\varphi(2^{-i}.)) \cap \operatorname{supp}(\varphi(2^{-j}.)) = \emptyset$.

For the existence of ψ and φ see [3]. The Littlewood–Paley blocks are now defined as

$$\Delta_{-1} u = \mathscr{F}^{-1}(\psi \mathscr{F} u) \quad \text{and for } j \geq 0 \ \Delta_j u = \mathscr{F}(\varphi(2^{-j}.)\mathscr{F} u).$$

The $\Delta_j u$ are smooth function with Fourier transform with compact support. We define the Hölder–Besov space $\mathscr{C}^\alpha$ by

$$\begin{aligned} \mathscr{C}^\alpha(\mathbb{R}^d;\mathbb{R}^n) &= B^\alpha_{\infty,\infty}(\mathbb{R}^d;\mathbb{R}^n) \\ &= \left\{ u \in \mathcal{S}'(\mathbb{R}^d,\mathbb{R}^n) : \|u\|_\alpha = \|(2^{j\alpha}\|\Delta_j u\|_\infty)_j\|_\infty < \infty \right\}. \end{aligned} \tag{6}$$

While the norm $\|.\|_\alpha$ depends on the choice of ψ and φ, the space $\mathscr{C}^\alpha$ does not and each choice of ψ, φ correspond to an equivalent semi-norm on $\mathscr{C}^\alpha$. If $\alpha \in \mathbb{R}_+ - \mathbb{N}$, then the space $\mathscr{C}^\alpha$ is the space of $[\alpha]$ times differentiable functions, whose partial derivatives up to order $[\alpha]$ are bounded, and whose partial derivatives of order $[\alpha]$ are $(\alpha - [\alpha])$-Hölder continuous. Note that we have the following continuous embedding, for $\alpha' \leqslant \alpha$ then $\mathscr{C}^\alpha \subset \mathscr{C}^{\alpha'}$ and $\|u\|_{\alpha'} \lesssim \|u\|_\alpha$. When $f \in C([0,T],\mathscr{C}^\alpha)$, we denote abusively $\|f\|_\alpha = \sup_{t\in I}\|u(t,.)\|_\alpha$. When $\alpha > 0$, the space $\mathscr{C}^\alpha = B^\alpha_{\infty,\infty}$ is the space of bounded Hölder continuous functions, indeed, for $m \in \mathbb{N}\setminus\{0\}$ and $m-1 \leqslant \alpha < m$, when we define $[\![f]\!]_\nu = \sup_{x\neq y}|f(x)-f(y)|/|x-y|^\nu$ for $\nu \in (0,1]$ and

$$\mathscr{C}^\alpha = \{f : \mathbb{R}^d \to \mathbb{R}^d : \|f\|_{\infty,\mathbb{R}^d} + [\![D^{m-1} f]\!]_{m-\nu} < +\infty\}.$$

Furthermore $\|.\|_\alpha$ and $\|f\|_\infty + [\![f]\!]_{m-\alpha}$ are equivalent norms. We will equally use either one or the other. We will also need some localised Hölder spaces described as follows:

Definition 1.15. Let $\nu \in [0,1)$. A weight is a continuous non-decreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for $c > 0$, there exists a constant $C_{c,\psi} > 0$ such that

$$\psi(cx) \leqslant C_{c,\psi}\psi(x).$$

A ν-weight is a weight such that

$$x^{-(1-\nu)}\psi(x) \xrightarrow[x\to+\infty]{} 0.$$

Hence, in that setting we define some weighted Hölder spaces as

Definition 1.16. Let ψ be a weight, $\nu \in (0,1]$ and V and W be two Banach spaces. The ψ-weighted Hölder space of index ν is the space $\mathscr{C}^{\nu,\psi}(V;W)$ defined by

$$\mathscr{C}^{\nu,\psi}(V;W) = \left\{ f : \mathscr{C}^\nu_{\text{loc}} : [\![f]\!]_{\nu,\psi} = \sup_{x\neq y\in V} \frac{|f(x)-f(y)|_W}{|x-y|_V\,\psi(|x|_V+|y|_V)} < +\infty \right\}.$$

When $n \in \mathbb{N}$ we say that a continuously n-times (Fréchet) differentiable function $f \in C^n(V;W)$ is in the ψ-weighted Hölder space of order $n+\nu$ if $D^n f \in \mathscr{C}^{\nu,\psi}(V;\mathcal{L}^n(W;W))$, where $\mathcal{L}^n(W;W)$ denote the space of n-linear continuous applications from W into itself.

To simplify the notation, we introduce also the following spaces related to time dependent nonlinear mappings between Banach spaces V and W.

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Definition 1.17. Let $0 < \gamma, \nu \leqslant 1$ and ψ be a weight. Let $I = [0, T]$ and V and W be two Banach spaces. For all $n \in \mathbb{N}$ and any $G : I \times V \to W$ we define

$$[\![G]\!]_{\gamma,\nu,\psi} = \sup_{s\neq t} \sup_{x\neq y} \frac{|(G_t(x) - G_s(x)) - (G_t(y) - G_s(y))|}{|t-s|^\gamma |x-y|^\nu \psi(|x|+|y|)},$$

$$\|G\|_{\gamma,n+\nu,\psi} = [\![D^n G]\!]_{\gamma,\nu,\psi} + \sum_{k=0}^{n} \sup_{s\neq t} \frac{|D^k G_t(0) - D^k G_s(0)|}{|t-s|^\gamma}$$

and

$$\mathscr{C}^{\gamma,n+\nu,\psi}(I, V, W) = \left\{ G \in L^\infty\left(I; \mathscr{C}^{\nu,\psi}(V; W)\right) : \|G\|_{\gamma,n+\nu,\psi} < \infty \right\}.$$

When $V = W$ we write $\mathscr{C}^{\gamma,\nu,\psi}(I, V, V) = \mathscr{C}^{\gamma,\nu,\psi}(I, V)$. Furthermore, when it is not ambiguous we only use $\mathscr{C}^{\gamma,\nu,\psi}$. When $\psi = 1$ and there is no ambiguity, we only write $\mathscr{C}^{\gamma,\nu}$.

As stated in the introduction, in order to have estimates for the averaging operator T^w which will not depend on the functions f, we introduce the following Fourier–Lebesgue spaces.

Definition 1.18. Let $\alpha \in \mathbb{R}$,

$$N_{\alpha,p}(f) = \left(\int_{\mathbb{R}^d} |\hat{f}(\xi)|^p (1+|\xi|)^{p\alpha} \mathrm{d}\xi \right)^{1/p}$$

and $\mathscr{F}L^{\alpha,p}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : N_\alpha(f) < \infty\}$. Then $N_{\alpha,p}$ is a norm on $\mathscr{F}L^{\alpha,p}(\mathbb{R}^d)$. When $p = 1$ we only write $\mathscr{F}L^\alpha$ and N_α.

When $\alpha \geqslant 0$ and $f \in \mathscr{F}L^\alpha$ implies that $\hat{f}$ is in L_1 and f is bounded continuous function. Furthermore if $\alpha \geq 1$, $f \in \mathscr{F}L^\alpha$ is globally Lipschitz continuous in the second variable. Furthermore for $\alpha \in (0, 1)$, $f \in \mathscr{F}L^\alpha$ is globally Hölder continuous in the second variable. Note that when $\alpha < 0$ the vector fields are only distributions.

Remark 1.19. An easy computation gives $\mathscr{F}L^\alpha \subset \mathscr{C}^\alpha$ for all $\alpha \in \mathbb{R}$, and for $\alpha > 0$ and ψ a weight, $\mathscr{C}^\alpha \subset \mathscr{C}^{\alpha,\psi}$.

It is natural to make some approximations in $\mathscr{C}^\alpha$ and in $\mathscr{F}L^\alpha$. Although the quantity $\sum_i \Delta_i f$ does not converge in $\mathscr{C}^\alpha$, it converges in all $\mathscr{C}^{\alpha'}$ with $\alpha' < \alpha$, which gives the following lemma:

Lemma 1.20. *Let $\alpha \in \mathbb{R}$ and $u \in \mathscr{C}^\alpha(\mathbb{R}^d)$. The sequence $(\pi_{\leqslant N} u)_{N\geq -1} = (\sum_{j\leqslant N} \Delta_j u)_N$ converges to u in $\mathscr{C}^{\alpha'}$ for all $\alpha' < \alpha$. Furthermore, for all α, $\pi_{\leqslant N} f \xrightarrow{\mathscr{F}L^\alpha} f$ for $f \in \mathscr{F}L^\alpha$.*

Finally if $G : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ we write $G_{s,t}(x) = G_t(x) - G_s(x)$.

2. The non-linear young integral and young-type equations

As already said, we intend to study the ODE (1) where $w : \mathbb{R}_+ \to \mathbb{R}^d$ is a continuous function (with $w_0 = 0$) and $b : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a (time-dependent, distributional) vector field. We think w as a very rough function whose oscillations dominate in small time scales the effects of the integrated vector field b. In this situation the function x behaves at small scales very much like w and the effects of b are seen only via a average over these fast oscillations. All this will cook up some regularisation effect which will allow to prove existence and uniqueness even when the vector field b does not enjoy sufficient space regularity.

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To highlight the effect of the translations induced by w on the flow of b let us introduce the change of variables $\theta_t = x_t - w_t$ so that the above equation now reads:

$$\theta_t = \theta_0 + \int_0^t b(w_s + \theta_s)\mathrm{d}s.$$

If we believe that w oscillate faster than θ then it seems reasonable to approximate the integral in the right hand side by a sum over a partition $t_0 = 0, \ldots, t_n = t$ of $[0, t]$ where we have fixed the θ parameter at the initial time of each segment:

$$\int_0^t b(s, w_s + \theta_s)\mathrm{d}s \simeq \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} b(s, w_s + \theta_{t_i})\mathrm{d}s = \sum_{i=0}^{n-1} (T^w_{t_i,t_{i+1}} b)(\theta_{t_i}) \tag{7}$$

where $T^w_{s,t} = T^w_t - T^w_s$.

Under appropriate conditions the expression on the right hand side of Eq. (7) will have a well defined limit as the size of the partition goes to zero and it defines a kind of integral which we naturally denote by

$$\int_0^t (T^w_{\mathrm{d}s} b)(\theta_s) = \lim \sum_{i=0}^{n-1} (T^w_{t_i,t_{i+1}} b)(\theta_{t_i})$$

and will enable us to set up an alternative formulation of the above ODE as an integral equation involving the time-dependent integrated vector field $G_t = T^w_t b$ which is an averaged version of b. The integral appearing in this equation is a kind of non-linear Young integral [20]. Existence and uniqueness of solutions for equations involving Young integrals are by now standard [16,14,11] and easily extended to this context as shown below. In particular the equation

$$\theta_t = \theta_0 + \int_0^t G_{\mathrm{d}s}(\theta_s)$$

will have a solution $\theta \in \mathscr{C}^\gamma([0, 1], \mathbb{R}^d)$ (the space of γ-Hölder continuous functions from $[0, T]$ to $\mathbb{R}^d$) provided $(x, t) \mapsto G_t(x)$ is a γ-Hölder function of time, locally Lipschitz in space with $\gamma > 1/2$, that is

$$|G_{s,t}(x) - G_{s,t}(y)| \lesssim |x - y||t - s|^\gamma \psi(|x| + |y|)$$

for all $x, y \in \mathbb{R}^d$ and $0 \leqslant s \leqslant t \leqslant 1$. Note that some space regularity is already needed to have existence (to be compared with the classical setup where bounded vector fields are sufficient for existence).

A strategy to prove uniqueness is to consider the difference between a solutions θ and a solution θ' of a similar equation

$$\theta'_t = x_0 + \int_0^t G'_{\mathrm{d}u}(\theta'_u).$$

It is necessary to estimate the difference

$$(\theta_t - \theta'_t) - (\theta_s - \theta'_s) = \int_s^t G_{\mathrm{d}u}(\theta_u) - G'_{\mathrm{d}u}(\theta'_u).$$

To deal with such an estimates, we will need an averaged translation operator $\tau_f G_{s,t}(z) = \int_s^t G_{\mathrm{d}u}(f_u + z)$ in order to have an equation on $\theta - \theta'$.

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In order for these estimates to be useful we need a way to link the regularity of the original vector field b with its averaged version $T^w_{s,t}b$ along an arbitrary continuous path w.

Theorem 2.1. *Assume that for $\alpha \in \mathbb{R}$, $f \to T^w f$ is defined on the whole space $\mathscr{F}L^{\alpha+\nu}$ for all $\nu \geqslant 0$. Assume also that there exists $\gamma > 1/2$ such that for all $\nu > 0$, there exists a ν-weight ψ such that T^w maps $\mathscr{C}^{\alpha+\nu}$ into $\mathscr{C}^{\gamma,\nu,\psi}$. Then there exists a solution $\theta(x_0) \in \mathscr{C}^\gamma([0,1],\mathbb{R}^d)$ to the Young-type equation*

$$\theta_t(x_0) = x_0 + \int_0^t (T^w_{\mathrm{d}s} b)(\theta_s(x_0))$$

for any $b \in \mathscr{C}^{\alpha+\nu}$ for ν such that $\gamma(1+\nu) > 1$. If $b \in \mathscr{C}^{\alpha+2}$ (or $\alpha + 3/2 > 0$ and $b \in \mathscr{C}^{\alpha+3/2}$) this is the unique γ-Hölder solution to this equation, and for all $t \in [0,1]$, the flow map $x_0 \to \theta_t(x_0)$ is well defined and locally Lipschitz continuous, uniformly in time.

Remark 2.2. To prove such a theorem, we need the two hypothesis about T^w. The first one is that this map is well defined. This will follow either from the definition of the map (when $\alpha \geq 0$) or from Section 3. The second one is to prove that T^w maps $\mathscr{C}^\alpha$ into $\mathscr{C}^{\gamma,\nu,\psi}$. We also need a theory of integration for vector fields in $\mathscr{C}^{\gamma,\nu,\psi}$. In the next section we will build such a theory.

This theorem is obtained when we apply Theorem 2.9, Remark 2.10, Theorem 2.17 and Corollary 2.18 to the operator T^w with the wanted hypothesis.

When $\alpha \geqslant -1$ the vector field $b \in \mathscr{C}^{\alpha+1}$ is continuous and the solutions are simply solutions to the classical ODE

$$\theta_t = \theta_0 + \int_0^t b(u, w_u + \theta_u)\mathrm{d}u.$$

In the case that $\alpha < -1$ the vector field b is a distribution and the previous ODE does not make sense. In that situation the natural meaning of these solution is the following. Let $b_n = \pi_{\leqslant n} b$ for $b \in \mathscr{C}^\alpha$ then

$$\int_0^t b_n(u, w_u + \theta_u)\mathrm{d}u = \int_0^t (T^w_{\mathrm{d}s} b_n)(\theta_s) \to \int_0^t (T^w_{\mathrm{d}s} b)(\theta_s)$$

by continuity of the Young integral and of the averaging with respect to the norm of $\mathscr{F}L^{\alpha+1}$. Then θ solves the equation

$$\theta_t = \theta_0 + \lim_n \int_0^t b_n(u, w_u + \theta_u)\mathrm{d}u,$$

where the right hand side is well defined for any $\theta \in \mathscr{C}^\gamma([0,1],\mathbb{R}^d)$. At this point we can identify

$$\int_0^t b(u, w_u + \theta_u)\mathrm{d}u = \lim_n \int_0^t b_n(u, w_u + \theta_u)\mathrm{d}u$$

and give meaning to the ODE with a distributional drift b.

Remark 2.3. When the vector field b is in $\mathscr{F}L^\alpha$, the limiting procedure does not depend on the choice of the sequence. That is the principal reason of the introduction of that spaces.

One of the aims of this paper is to show that the above program can be carried out successfully in the case of w given by a sample path of a fractional Brownian motion B^H of Hurst parameter $H \in (0,1)$.

2.1. Definition of the Young integral

We define now the Young integral [20,16,14] for nonlinear operators.

Theorem 2.4. *Let $\gamma, \rho, \nu > 0$ with $\gamma + \nu\rho > 1$, a ν-weight ψ, and V and W two Banach spaces and I a finite interval on $\mathbb{R}$. Let $G \in \mathscr{C}^{\gamma,\nu,\psi}(I, V, W)$ and $f \in \mathscr{C}^{\rho}(I; V)$. Let $s, t \in I$ with $s \leqslant t$. Then the following limit exists and is independent of the partition*

$$\int_s^t G_{\mathrm{d}u}(f_u) := \lim_{\substack{\Pi \text{ partition of } [s,t] \\ |\Pi|\to 0}} \sum_i G_{t_i,t_{i+1}}(f_{t_i}).$$

Furthermore

1. *For all $s \leqslant u \leqslant t$ with $s, u, t \in I$ we have*
$$\int_s^t G_{\mathrm{d}r}(f_r) = \int_s^u G_{\mathrm{d}r}(f_r) + \int_u^t G_{\mathrm{d}r}(f_r).$$
2. $$\left|\int_s^t G_{\mathrm{d}r}(f_r) - G_{s,t}(f_s)\right|_W \leqslant C_{\gamma,\rho,\nu} [\![G]\!]_{\gamma,\nu,\psi} [\![f]\!]_{\rho,I}^{\nu} |t-s|^{\gamma+\nu\rho} \psi(\|f\|_{\infty,I}).$$
3. *For all $s \leqslant t \in I$ and $R > 0$, the map $(f, G) \mapsto \int_s^t G_{\mathrm{d}r}(f_r)$ is continuous as a function of $\left(\left\{g \in \mathscr{C}^{\rho}(I, V), \|g\|_{\gamma,[s,t]} \leqslant R\right\}, \|.\|_{\infty,[s,t]}\right) \times (\mathscr{C}^{\gamma,\nu,\psi}(I, V, W), \|.\|_{\gamma,\nu,\psi})$ to W.*

Proof. Let $s, t \in I$ with $s \leqslant t$ be fixed until the end of the proof. Suppose first that G is differentiable (in time) and $G' \in \mathscr{C}^{\gamma,\nu,\psi}(I, V, W)$ and $G \in \mathscr{C}^{\gamma,\nu,\psi}$. For simplicity, in all the proof we write $\|G\|$ and $[\![G]\!]$ instead of $\|G\|_{\gamma,\nu,\psi}$ and $[\![G]\!]_{\gamma,\nu,\psi}$. Then we define for $s \leqslant t$

$$\int_s^t G_{\mathrm{d}u}(f_u) := \int_s^t G'_u(f_u)\mathrm{d}u := I_{s,t}(f, G)$$

and also define $J_{s,t}(f, G) := I_{s,t}(f, G) - G_{s,t}(f_s)$. For $u \in [s, t]$ we have

$$J_{s,t}(f, G) = J_{s,u}(f, G) + J_{u,t}(f, G) + G_{u,t}(f_u) - \delta G_{u,t}(f_s),$$

hence, for $n \geq 1$, $i \in \{0, \ldots, 2^n\}$ and $t_i^n = s + (t-s)i2^{-n}$,

$$J_{s,t}(f, G) = \sum_{i=0}^{2^n-1} J_{t_i^n,t_{i+1}^n}(f, G) + \sum_{k=1}^{n} \sum_{i=1}^{2^k-1} (G_{t_{2i-1}^k,t_{2i}^k}(f_{t_{2i-1}^k}) - G_{t_{2i-1}^k,t_{2i}^k}(f_{t_{2(i-1)}^k})).$$

But, as G is differentiable, the following computation holds

$$\begin{aligned}
|J_{t_i^n,t_{i+1}^n}(f, G)|_W &\leqslant \int_{t_i^n}^{t_{i+1}^n} |G'_u(f_u) - G'_u(f_{t_i^n})|_W \mathrm{d}u \\
&\leqslant \int_{t_i^n}^{t_{i+1}^n} [\![G'_u]\!]_{\nu,\psi} |f_u - f_{t_i^n}|_V^{\nu} \psi(|f_u| + |f_{t_i^n}|)\mathrm{d}u \\
&\leqslant 2\|[\![G']\!]_{\nu,\psi}\|_\infty \int_{t_i^n}^{t_{i+1}^n} |f|_\rho^{\nu} |u - t_i^n|^{\nu\rho} \psi(\|f\|_{\infty,[s,t]})\mathrm{d}u \\
&\lesssim 2^{-(1+\nu\rho)n}.
\end{aligned}$$

Hence

$$\left|\sum_{i=0}^{2^n-1} J_{t_i^n,t_{i+1}^n}(f,G)\right| \lesssim 2^{-n\nu\rho} \underset{n\to\infty}{\to} 0,$$

and then

$$|J_{s,t}(f,G)| \leqslant \sum_{k=1}^{\infty}\sum_{i=1}^{2^k-1} |G_{t^k_{2i-1},t^k_{2i}}(f_{t^k_{2i-1}}) - G_{t^k_{2i-1},t^k_{2i}}(f_{t^k_{2(i-1)}})|_W.$$

For $k \geq 1$ and $i \in \{1,\dots,2^k-1\}$, we have

$$\begin{aligned}
&|G_{t^k_{2i-1},t^k_{2i}}(f_{t^k_{2i-1}}) - G_{t^k_{2i-1},t^k_{2i}}(f_{t^k_{2(i-1)}})| \\
&\quad \leqslant [\![G]\!] |t^k_{2i-1} - t^k_{2i}|^{\gamma} |f_{t^k_{2i-1}} - f_{t^k_{2i-2}}|^{\nu} \psi(|f_{t^k_{2i-1}}| + |f_{t^k_{2i-2}}|) \\
&\quad \lesssim [\![G]\!][\![f]\!]^{\nu}_{\rho,[s,t]} \psi(\|f\|_{\infty,[s,t]}) |t-s|^{\gamma+\nu\rho} 2^{-(\gamma+\nu\rho)k}.
\end{aligned}$$

Hence, the following bound holds

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{i=1}^{2^k-1} |G_{t^k_{2i-1},t^k_{2i}}(f_{t^k_{2i-1}}) - G_{t^k_{2i-1},t^k_{2i}}(f_{t^k_{2(i-1)}})|_W \\
&\quad \lesssim [\![G]\!][\![f]\!]^{\nu}_{\rho,[s,t]} \psi(\|f\|_{\infty,[s,t]}) |t-s|^{\nu\rho+\gamma} \sum_{k=1}^{\infty}\sum_{i=1}^{2^k-1} 2^{-(\gamma+\nu\rho)k} \\
&\quad \lesssim \frac{2^{-(\nu\rho+\gamma-1)}}{1-2^{-(\nu\rho+\gamma-1)}} [\![G]\!][\![f]\!]^{\nu}_{\rho,[s,t]} \psi(\|f\|_{\infty,[s,t]}) |t-s|^{\nu\rho+\gamma}.
\end{aligned} \tag{8}$$

The result is proved for differentiable G. Let us now take $G \in \mathscr{C}^{\gamma,\nu,\psi}(I,V,W)$ and f as wanted. Let G^n be differentiable as above such that $G^n_{s,t}(f_s) \to G_{s,t}(f_s)$ as $n \to \infty$; for all $\gamma' < \gamma$ $\lim_{n\to\infty} \|G - G^n\|_{\gamma',\nu,\psi} = 0$ and for all $n \geq 0$, $\|G^n\|_{\gamma,\nu,\psi} \leqslant \|G\|_{\gamma,\nu,\psi}$. As $I_{s,t}$ is linear in the second variable, we have, for $\gamma' < \gamma$

$$\begin{aligned}
|J_{s,t}(f,G^n) - J_{s,t}(f,G^{n+m})|_W &= |J_{s,t}(f,G^n - G^{n+m})|_W \\
&\lesssim [\![G^n - G^{n+m}]\!]_{\gamma',\nu} \\
&\underset{n\to\infty}{\to} 0.
\end{aligned}$$

The sequence $(J_{s,t}(f,G^n))_n$ is Cauchy in W which is a Banach space. Let us say it converges to a number $J_{s,t}(f,G)$. Furthermore, the sequence $G^n_{s,t}(f_s)$ converges obviously to $G_{s,t}(f_s)$. Then as $I_{s,t}(f,G^n) = J_{s,t}(f,G^n) + G^n_{s,t}(f_s)$ the sequence $(I_{s,t}(f,G^n))_n$ converges to a limit called $I_{s,t}(f,G)$. Furthermore,

$$\begin{aligned}
|J_{s,t}(f,G^n)|_W &\lesssim_{\gamma,\rho} [\![G^n]\!][\![f]\!]_{\rho,[s,t]} \psi(\|f\|_{\infty}) |t-s|^{\gamma+\nu\rho} \\
&\lesssim [\![G]\!][\![f]\!]_{\rho} \psi(\|f\|_{\infty}) |t-s|^{\gamma+\nu\rho}
\end{aligned}$$

Q4 and so does $|I_{s,t}(f,G) - G_{s,t}(f_s)|_W$. The Chasles property and the triangular inequality are obvious with the definition of I. Moreover since $I(f,G)$ is linear in G it is easy to see that the definition does not depend on the particular sequence G^n.

Let us show that $I_{s,t}(f,G)$ is the limit of Riemann sum. Let $\Pi = \{s = t_0 < t_1 < \cdots < t_n = t\}$ a partition of $[s,t]$. Let

$$S_\Pi = \sum_{k=0}^{n-1} G_{t_i,t_{i+1}}(f_{t_{i+1}})$$

be the Riemann sum corresponding to this partition. As $G_{t_i,t_{i+1}}(f_{t_{i+1}}) = I_{t_i,t_{i+1}}(f,G) - J_{t_i,t_{i+1}}(f,G)$ the following equality holds

$$S_\Pi - I_{s,t}(f,G) = -\sum_{i=0}^{n-1} J_{t_i,t_{i+1}}(f,G).$$

Hence

$$\begin{aligned}|S_\Pi - I_{s,t}(f,G)|_W &\leqslant \sum_{i=0}^{n-1} |J_{t_i,t_{i+1}}(f,G)|_W \\ &\lesssim \sum_{i=0}^{n-1} |t_{i+1}-t_i|^{\gamma+\nu\rho} \lesssim |\Pi|^{\gamma+\nu\rho-1} \to_{|\Pi|\to 0} 0.\end{aligned}$$

It remains to show the continuity of the map $(f,G) \mapsto I(f,G)$. Take f, f', G, G' and assume for simplicity that $G(0) = G'(0) = 0$ then

$$I_{s,t}(f,G) - I_{s,t}(f',G') = [I_{s,t}(f,G) - I_{s,t}(f',G)] + I_{s,t}(f', G-G')$$

and

$$\begin{aligned}&|I_{s,t}(f', G-G')|_W \\ &\quad \leqslant |(G-G')_{s,t}(f_s) - (G-G')_{s,t}(0)|_W + |(G-G')_{s,t}(0)| + |J_{s,t}(f, G-G')|_W \\ &\quad \lesssim \|G-G'\| |t-s|^\gamma (|f_s|^\gamma \psi(|f_s|) + 1 + |t-s|^{\nu\rho} [\![f]\!]_\rho^\nu \psi(\|f\|_\infty)) \\ &\quad \lesssim \|G-G'\| |t-s|^\gamma (\|f\|_\infty^\nu \psi(\|f\|_\infty) + 1 + |t-s|^{\nu\rho} [\![f]\!]_\rho^\nu \psi(\|f\|_\infty)) \\ &\quad \lesssim \|G-G'\| |t-s|^\gamma (1 + \|f\|_\rho^\nu \psi(\|f\|_\rho)).\end{aligned}$$

Furthermore

$$I_{s,t}(f,G) - I_{s,t}(f',G) = G_{s,t}(f_s) - G_{s,t}(f'_s) + J_{s,t}(f,G) - J_{s,t}(f',G).$$

We have also

$$\begin{aligned}|I_{s,t}(f,G) - I_{s,t}(f',G)|_W &\lesssim \|G\| \|f-f'\|_\infty^\nu \psi(\|f\|_\infty + \|f'\|_\infty) |t-s|^\gamma \\ &\qquad + ([\![f]\!]_\rho^\nu \psi(\|f\|_\infty) + [\![f']\!]_\rho^\nu \psi(\|f'\|_\infty)) \|G\| |t-s|^{\nu\rho+\gamma} \\ &\lesssim \|G\| \|f-f'\|_\infty^\nu \psi(\|f\|_\rho + \|f'\|_\rho) |t-s|^\gamma \\ &\qquad + (\|f\|_\rho^\nu \psi(\|f\|_\rho) + \|f'\|_\rho^\nu \psi(\|f'\|_\rho)) \|G\| |t-s|^{\nu\rho+\gamma}.\end{aligned}$$

By partitioning the interval $[s,t]$ in subintervals $[t_i, t_{i+1}]$ of size 2^{-n} and summing up the contributions according to these bounds we obtain an improved estimate

$$\begin{aligned}&|I_{s,t}(f,G) - I_{s,t}(f',G)|_W \\ &\quad \lesssim \|G\| \|f-f'\|_\infty^\nu \psi(\|f\|_\rho + \|f'\|_\rho) 2^{(1-\gamma)n} |t-s|^\gamma \\ &\qquad + \|G\| (\|f\|_\rho^\nu \psi(\|f\|_\rho) + \|f'\|_\rho^\nu \psi(\|f'\|_\rho)) (2^{-n}|t-s|)^{\nu\rho+\gamma} 2^n.\end{aligned}$$

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Taking n large enough so that

$$2^{-\nu\rho n} \leqslant \frac{\|f-f'\|_\infty^\nu \psi(\|f\|_\rho + \|f'\|_\rho)}{(\|f\|_\rho^\nu \psi(\|f\|_\rho) + \|f'\|_\rho^\nu \psi(\|f'\|_\rho))|t-s|^{\nu\rho}} \leqslant 2^{-\nu\rho(n-1)},$$

we have

$$|I_{s,t}(f,G) - I_{s,t}(f',G)|_W \lesssim \|G\| \|f-f'\|_\infty^\nu \psi(\|f\|_\rho + \|f'\|_\rho) 2^{(1-\gamma)n} |t-s|^\gamma,$$

which means that it is possible to choose n such that

$$\begin{aligned} |I_{s,t}(f,G) - I_{s,t}(f',G)|_W &\lesssim \|G\| \|f-f'\|_\infty^\nu \psi(\|f\|_\rho + \|f'\|_\rho) |t-s|^\gamma \\ &\quad \times \left(\frac{(\|f\|_\rho^\nu \psi(\|f\|_\rho) + \|f'\|_\rho^\nu \psi(\|f'\|_\rho))|t-s|^{\nu\rho}}{\|f-f'\|_\infty^\nu \psi(\|f\|_\rho + \|f'\|_\rho)} \right)^{\frac{1-\gamma}{\nu\rho}} \\ &\lesssim \|G\| \{\|f-f'\|_\infty^\nu \psi(\|f\|_\rho + \|f'\|_\rho)\}^{(\gamma+\nu\rho-1)/\nu\rho} |t-s| \\ &\quad \times (\|f\|_\rho^\nu \psi(\|f\|_\rho) + \|f'\|_\rho^\nu \psi(\|f'\|_\rho))^{(1-\gamma)/(\nu\rho)} \end{aligned}$$

and this allows us to infer the continuity of $I(f,G)$. □

Remark 2.5. It is easy to construct a suitable sequence $(G^n)_{n\geq 1}$. Let $h : \mathbb{R} \to \mathbb{R}$ be a compactly supported, smooth positive function with integral 1. Define $h_n(t) = nh(nt)$ and define for all $v \in V$ and all $t \in \mathbb{R}$

$$G_t^n(v) = \int_{\mathbb{R}} h_n(t-s) G_s(v) \mathrm{d}s = G_t^n(v) = \int_{\mathbb{R}} h_n(s) G_{t-s}(v) \mathrm{d}s.$$

Then G^n is as wanted. Indeed,

$$\begin{aligned} |G_{s,t}^n(v) - G_{s,t}^n(w)|_W &\leqslant \int_{\mathbb{R}} h_n(r) |(G_{t-r} - G_{s-r})(v) - (G_{t-r} - G_{s-r})(w)|_W \mathrm{d}r \\ &\leqslant \int_{\mathbb{R}} h_n(r) |t-s|^\gamma |v-w|_V^\nu \psi(|v|+|w|) \mathrm{d}r \\ &\leqslant \|G\|_{\gamma,\nu,\psi} |t-s|^\gamma |v-w|_V^\nu \psi(|v|+|w|) \end{aligned} \tag{9}$$

Q5 which proves that $G^n \in \mathscr{C}^{\gamma,\nu,\psi}(\mathbb{R}, V, W)$ and that $\|G^n\|_{\gamma,\nu,\psi} \leqslant \|G\|_{\gamma,\nu,\psi}$. Furthermore G^n is differentiable and $(G^n)' \in \mathscr{C}^{\gamma,\nu,\psi}(\mathbb{R}, V, W)$. As we can choose h_n to be a good kernel, all the properties required on G^n are satisfied.

Definition 2.6. The limit functional I defined in the last theorem is obviously an integral and then we will refer to it as $\int_s^t G_{du}(f_u)$.

Remark 2.7. Let $g \in \mathscr{C}^\gamma(I, V')$ and $f \in \mathscr{C}^\rho(I, V)$ with $\gamma + \rho > 1$, where V and V' are (finite-dimensional) Banach spaces. Let $W = V \otimes V'$ and for all $x \in V$, $G_t(x) = x \otimes g_t$. Then $G \in \mathscr{C}^{\gamma,1}(I, V, W)$ and the above integral is the standard Young integral.

Remark 2.8. The bound in Theorem 2.4 is

$$\left| \int_s^t G_{\mathrm{du}}(f_u) - G_{s,t}(f_s) \right| \lesssim [\![G]\!]_{\gamma,\nu,\psi} |t-s|^{\gamma+\rho\nu} [\![f]\!]_\gamma^\nu \psi(\|f\|_\infty).$$

But as $[\![f]\!]_\gamma \leqslant \|f\|_\gamma$ and $\|f\|_\infty \leqslant (1+|I|)\|f\|_\gamma$ and ψ is a weight, we also have this other useful bound

$$\left|\int_s^t G_{\mathrm{d}u}(f_u) - G_{s,t}(f_s)\right| \lesssim [\![G]\!]_{\gamma,\nu,\psi}\, |t-s|^{\gamma+\rho\nu} \|f\|_\gamma^\nu \psi(\|f\|_\gamma),$$

where the new constant depends on the length of the interval $|I|$ and ψ. In the following, we will exploit these three bounds equally and without further notice.

We intend to solve differential equations driven by such G. Thanks to the definition of the integral and the bound in Theorem 2.4, we are able to define the equation, prove the existence of solutions and give an a priori bound on the norm of the solutions. Here we will use the notion of ν-weight, in order to control the growth of the norm.

Theorem 2.9. *Let $\gamma > \frac{1}{2}$, $\nu \in [0,1)$ such that $\gamma(1+\nu) > 1$ and ψ be a ν-weight as in Definition 1.15. Let $G \in \mathscr{C}^{\gamma,\nu,\psi}([0,T],\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. There exists a solution $\theta \in \mathscr{C}^\gamma([0,T];\mathbb{R}^d)$ to the non-linear Young differential equation*

$$\theta_t = \theta_0 + \int_0^t G_{\mathrm{d}u}(\theta_u).$$

Furthermore, there exists two universal constants K_1 and K_2 depending on γ, ν, ψ and T such that

$$\|\theta\|_{\infty,[0,T]} \leqslant K_1 (1+\|G\|_{\gamma,\nu,\psi})^{K_2 \|G\|_{\gamma,\nu,\psi}^{1/\nu\gamma}} (|\theta_0|+1)$$

and

$$\|\theta\|_{\gamma,[0,T]} \leqslant \tilde{K}_1 \|G\|^{1/\nu\gamma} (1+\|G\|_{\gamma,\nu,\psi})^{\tilde{K}_2 \|G\|_{\gamma,\nu,\psi}^{1/\nu\gamma}} (1+|\theta_0|).$$

Proof. Let us first deal with the existence of the solutions. Let $t_0 \in I = [0,T]$, $K > 0$ and $0 < S \leqslant T$ to be specify later. Let $J = [t_0, (t_0+S)\wedge T]$ and let us define for all $x \in V$,

$$\mathcal{C}_{t_0,x} = \{\theta \in \mathscr{C}^\gamma(J) : \theta_{t_0} = x, \|\theta\|_{\gamma,J} \leqslant K\}$$

and

$$\Phi_{t_0,x} : \begin{array}{ccc} \mathscr{C}^\gamma(J,V) & \to & \mathscr{C}^\gamma(J,V) \\ \theta & \to & x + \int_{t_0}^{\cdot} G_{\mathrm{d}u}(\theta_u). \end{array}$$

By Theorem 2.4 the map $\Phi_{t_0,x}$ is well defined. Furthermore we always have

$$|\theta_s| \leqslant |\theta_{t_0}| + T^\nu [\![\theta]\!]_{\gamma,J} \lesssim_T \|\theta\|_{\gamma,J}.$$

Hence for $s < t \in J$ we have

$$\begin{aligned} |\delta(\Phi_{t_0,x}(\theta))_{s,t}| &\leqslant \left|\int_s^t G_{\mathrm{d}u}(\theta_u) - G_{s,t}(\theta_s)\right| + |G_{s,t}(\theta_s) - G_{s,t}(0)| + |G_{s,t}(0)| \\ &\lesssim \|G\|_{\gamma,\nu,\psi} |t-s|^\gamma (S^{\nu\gamma} \|\theta\|_{\gamma,J}^\nu \psi(\|\theta\|_{\gamma,J}) + |\theta_s|^\nu \psi(|\theta_s|) + 1) \\ &\lesssim \|G\|_{\gamma,\nu,\psi} |t-s|^\gamma (S^{\nu\gamma} \|\theta\|_{\gamma,J}^\nu \psi(\|\theta\|_{\gamma,J}) + |\theta_{t_0}|^\nu \psi(|\theta_0|) + 1). \qquad (10) \end{aligned}$$

Now take $\theta \in \mathcal{C}_{t_0,x}$,

$$[\![\Phi_{t_0,x}(\theta)]\!]_{\gamma,J} \lesssim \|G\|_\gamma (S^{\nu\gamma} \|\theta\|_{\gamma,J}^\nu \psi(\|\theta\|_{\gamma,J}) + |x|^\nu \psi(|x|) + 1).$$

But since $\nu < 1$ and ψ is a ν-weight, there exists a constant $C_{\nu,\psi}$ such that $\|\theta\|_{\gamma,J}^{\nu}\psi(\|\theta\|_{\gamma,J}) \leqslant C_{\nu,\psi}(1+|x|)$. Hence, there is a universal constant $C > 0$ such that

$$\|\Phi_{t_0,x}(\theta)\|_{\gamma,J} = [\![\Phi_{t_0,x}(\theta)]\!]_{\gamma,J} + |x| \leqslant |x| + C\|G\|_{\gamma}(S^{\nu\gamma}\|\theta\|_{\gamma,J} + |x|^{\nu}\psi(|x|) + 1).$$

For S such that $C\|G\|_{\gamma}S^{\nu\gamma} < 1/2$, and for $K \geqslant 2\{|x| + C(|x|^{\nu}\psi(|x|)+1)\}$, we have

$$\|\Phi_{t_0,x}(\theta)\|_{\gamma,J} \leqslant K.$$

Then $\Phi_{t_0,x}(\theta) \in \mathcal{C}_{t_0,x}$, moreover by the property of the Young integral the map $\Phi_{t_0,x}$ is continuous on $\mathcal{C}_{t_0,x}$ for the norm $\|.\|_{\infty,[t_0,(t_0+T)\wedge 1]}$. By its definition $\mathcal{C}_{t_0,x}$ is immediately a closed convex set of $C(J)$. Let us show that $\Phi_{t_0,x}(\mathcal{C}_{t_0,x})$ is relatively compact in $\mathcal{C}^0$. It is obviously equicontinuous as $\|\Phi_{t_0,x}(\theta)\|_{\gamma} \leqslant K$ and relatively bounded as $|\Phi_{t_0,x}(\theta)_t| \leqslant |x| + K^{\nu}\psi(K)(t-t_0)^{\gamma}$. Hence by Ascoli theorem $\Phi_{t_0,x}(\mathcal{C}_{t_0,x})$ is relatively compact. Thanks to Leray–Schauder–Tychonoff fixed point theorem, there exists $\theta^{t_0,x}$ such that $\theta^{t_0,x} = \Phi_{t_0,x}(\theta^{t_0,x}) = x + \int_{t_0}^{t} G_{\mathrm{d}u}(\theta_u^{t_0,x})$. We then construct by induction a solution on the whole interval. For n such that $nS \leqslant T$ let $\theta^0 = \theta^{0,x_0}$ and $\theta^n = \theta^{nS,\theta_S^{n-1}}$. Let us define $\theta_t = \theta_t^n$ if $t \in [nS,(n+1)S]$. By an immediate induction, θ is solution of the equation $\theta_t = x_0 + \int_0^t G_{\mathrm{d}u}(\theta_u)$ and then is obviously in $\mathscr{C}^{\gamma}$.

We have all the tools to bound the norm of a solution of the equation. Again take t_0 and S to be specified later, and θ a solution of the non-linear Young differential equation. And take $J = [t_0,(t_0+S)\wedge T]$. We have

$$\begin{aligned}|\delta\theta_{s,t}| &\leqslant \left|\int_s^t G_{\mathrm{d}u}(\theta_u) - G_{s,t}(\theta_s)\right| + |G_{s,t}(\theta_s) - G_{s,t}(\theta_{t_0})| \\ &\quad + |G_{s,t}(\theta_{t_0}) - G_{s,t}(0)| + |G_{s,t}(0)| \\ &\lesssim |t-s|^{\gamma}\|G\|_{\gamma,\nu,\psi}(S^{\gamma\nu}[\![\theta]\!]_{\gamma,J}^{\nu}\,\psi(\|\theta\|_{\gamma,J}) + |\theta_{t_0}|^{\nu}\psi(|\theta_{t_0}|)+1),\end{aligned}$$

hence

$$[\![\theta]\!]_{\gamma,[t_0,t_0+S]} \lesssim \|G\|_{\gamma,\nu,\psi}(S^{\gamma\nu}\|\theta\|_{\gamma,J}^{\nu}\psi(\|\theta\|_{\gamma,J}) + |\theta_{t_0}|^{\nu}\psi(|\theta_{t_0}|)+1).$$

Let S be such that $C\|G\|_{\gamma,\nu,\psi}S^{\gamma,\nu} \leqslant 1$, and we have

$$\|\theta\|_{\gamma,[t_0,t_0+S]} \lesssim \|\theta\|_{\gamma,J}^{\nu}\psi(\|\theta\|_{\gamma,J}) + |\theta_{t_0}| + C\|G\|_{\gamma,\nu,\psi}|\theta_{t_0}|^{\nu}\psi(|\theta_{t_0}|) + C\|G\|_{\gamma,\nu}.$$

As $x \to x^{\nu}\psi(x)$ is sublinear (as before), there exists a constant depending on ν and ψ such that

$$\|\theta\|_{\gamma,[t_0,t_0+S]} \lesssim_{\nu,\psi} 1 + |\theta_{t_0}| + \|G\|_{\gamma,\nu,\psi}|\theta_{t_0}|^{\nu}\psi(|\theta_{t_0}|) + \|G\|_{\gamma,\nu,\psi}.$$

There also exists a constant C such that

$$|\theta_{t_0}|^{\nu}\psi(|\theta_{t_0}|) \leqslant C + |\theta_{t_0}|$$

and

$$\|\theta\|_{\gamma,[t_0,t_0+S]} \leqslant (1+\|G\|_{\gamma,\nu,\psi})|\theta_{t_0}| + C(\|G\|_{\gamma,\nu,\psi}+1). \tag{11}$$

From this we deduce

$$|\theta_t| \leqslant |\theta_t - \theta_{t_0}| + |\theta_{t_0}| \lesssim_T \|\theta\|_{\gamma,J} + |\theta_{t_0}| \lesssim (1+\|G\|_{\gamma,\nu,\psi})|\theta_{t_0}| + \|G\|_{\gamma,\nu,\psi} + 1$$

and then

$$\|\theta\|_{\infty,J} \lesssim (1+\|G\|_{\gamma,\nu,\psi})|\theta_{t_0}| + \|G\|_{\gamma,\nu,\psi} + 1.$$

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Now let n be such that $S = T/n$ and $1/2 \leqslant C\|G\|T^{\nu\gamma}n^{-\nu\gamma} \leqslant 1$ hence $n \geqslant T(C\|G\|)^{1/\nu\gamma}$ and we have for $J_i = [iT/n, (i+1)T/n]$

$$\|\theta\|_{\infty,J_i} \lesssim (1 + \|G\|_{\gamma,\nu,\psi})\|\theta\|_{\infty,J_{i-1}} + \|G\|_{\gamma,\nu,\psi} + 1$$

and

$$\|\theta\|_{\infty,J_0} \lesssim (1 + \|G\|_{\gamma,\nu,\psi})|\theta_0| + \|G\|_{\gamma,\nu,\psi} + 1.$$

Hence

$$\|\theta\|_{\infty,J_i} \lesssim C^i(1 + \|G\|_{\gamma,\nu,\psi})^i((1 + \|G\|_{\gamma,\nu,\psi})|\theta_0| + \|G\|_{\gamma,\nu,\psi} + 1)$$

and finally

$$\|\theta\|_{\infty,[0,T]} \leqslant K_1(1 + \|G\|_{\gamma,\nu,\psi})^{K\|G\|^{1/\nu\gamma}_{\gamma,\nu,\psi}}(|\theta_{t_0}| + 1),$$

where K_1 and K_2 are two universal constants depending on ν, γ, ψ and T. From the Eq. (11), we can deduce, with the same induction argument, that

$$\|\theta\|_{\gamma,[0,T]} \leqslant |\theta_0| + C\|G\|^{1/\nu\gamma}T((1 + \|G\|_{\gamma,\nu,\psi})\|\theta\|_{\infty,[0,T]} + \|G\|_{\gamma,\nu,\psi} + 1)$$

and the result follows. □

Remark 2.10. The bounds on the solutions of the differential equation allows us to get rid of the ν-weight ψ. Indeed, we have, for a solution θ of the non-linear Young equation we have

$$\|\theta\|_{\infty,[0,T]} \leqslant K_1(1 + \|G\|_{\gamma,\nu,\psi})^{K_2\|G\|^{1/\nu\gamma}_{\gamma,\nu,\psi}}(|\theta_{t_0}| + 1).$$

Then for $R > 0$, and $\theta_0 \in B(0, R)$ $\|\theta\|_{\infty,[0,T]} \leqslant K_1(1 + \|G\|_{\gamma,\nu,\psi})^{K\|G\|^{1/\nu\gamma}_{\gamma,\nu,\psi}}(R + 1)$. Hence, it is enough to consider the localised norm of G

$$\|G\|^R_{\gamma,\nu} = \sup_{t\neq s\in I}\ \sup_{x\neq y\in B(0,R^{G,R})} \frac{|G_{s,t}(x) - G_{s,t}(y)|}{|x-y|^\nu|t-s|^\gamma} + \sup_{s\neq t}\frac{|G_{s,t}(0)|}{|t-s|^\gamma},$$

where

$$R^{G,R} = K_1(1 + \|G\|_{\gamma,\nu,\psi})^{K\|G\|^{1/\nu\gamma}_{\gamma,\nu,\psi}}(R + 1).$$

From now we will consider only bounded G, namely $G \in \mathscr{C}^{\gamma,\nu}$, and we will extend the results to $\mathscr{C}^{\gamma,\nu,\psi}$ thanks to the previous remark.

2.2. Uniqueness of solutions

2.2.1. Comparison principle

From now, thanks to Remark 2.10 we can restrict the study of the properties of the solutions, their uniqueness and their regularity with respect to the parameters for bounded G. Hence, we define the space $\mathscr{C}^{\gamma,n+\nu}_b = \{G \in \mathscr{C}^{\gamma,\nu} : \|G\|^b_{\gamma,n+\nu} < +\infty\}$ with

$$\|G\|^b_{\gamma,n+\nu} := [\![D^n G]\!]_{\gamma,\nu} + \sum_{k=0}^{n} \sup_{s\neq t\in[0,T]}\ \sup_{x\in\mathbb{R}^d} |D^k G_{s,t}(x)|/|t-s|^\gamma.$$

As there will be no ambiguity in the following, we will usually avoid to mention explicitly the b in the norm on that space. Those spaces are nicer than the whole space $\mathscr{C}^{\gamma,\nu}$ as there are natural embeddings:

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Lemma 2.11. *Let* $0 < \gamma \leqslant 1$, $0 \leqslant \nu' \leqslant \nu$ *and* $G \in \mathscr{C}_b^{\gamma,\nu}$ *then* $\|G\|^b_{\gamma,\nu'} \lesssim \|G\|^b_{\gamma,\nu}$.

Proof. Let $x, y \in \mathbb{R}^d$ $s, t \in [0, T]$. For $0 \leqslant \nu \leqslant \nu' \leqslant 1$, we have

$$\begin{aligned}\frac{|G_{s,t}(x) - G_{s,t}(y)|}{|x-y|^{\nu'}} &\leqslant \left(\frac{|G_{s,t}(x) - G_{s,t}(y)|}{|x-y|^{\nu}}\right)^{\nu'/\nu} |G_{s,t}(x) - G_{s,t}(y)|^{1-\nu'/\nu} \\ &\leqslant 2^{1-\nu'}|t-s|^{\gamma} [\![G]\!]^{\nu'/\nu}_{\gamma,\nu} (\|G\|^b_{\gamma,\nu})^{1-\nu'/\nu} \\ &\lesssim 2^{1-\nu'/\nu}|t-s|^{\gamma}\|G\|^b_{\gamma,\nu}\end{aligned}$$

and the following bound holds

$$\|G\|^b_{\gamma,\nu'} = (1 + 2^{1-\nu'/\nu})\|G\|^b_{\gamma,\nu}.$$

Furthermore, we also have

$$\begin{aligned}|G_{s,t}(x) - G_{s,t}(y)| &\leqslant \int_0^1 \mathrm{d}r |DG_{s,t}(r(x-y)+y)||x-y| \\ &\leqslant \|DG\|^b_{\gamma,1+\nu}|x-y||t-s|^{\gamma}\end{aligned}$$

and

$$\|G\|^b_{\gamma,1} \leqslant 2\|DG\|^b_{\gamma,1+\nu}.$$

The general result follows by an easy induction. □

Remark 2.12. These embeddings allows us to state a result for the existence of the solutions when $G \in \mathscr{C}_b^{\gamma,1}$ with $\gamma > \frac{1}{2}$. Indeed, as for all $\nu < 1$, $G \in \mathscr{C}_b^{\gamma,\nu}$ and $\|G\|^b_{\gamma,\nu} \lesssim \|G\|^b_{\gamma,1}$, there exist a solution θ and the non-linear Young differential equation. Furthermore, for all $\nu < 1$, there exists a constant K_2 such that

$$\|\theta\|_\infty \lesssim (1 + \|G\|^b_{\gamma,\nu})^{K_2(\|G\|^b_{\gamma,\nu})^{1/\nu\gamma}}(|\theta_0| + 1) \lesssim (1 + \|G\|^b_{\gamma,1})^{K_2(\|G\|^b_{\gamma,1})^{1/\nu\gamma}}(|\theta_0| + 1).$$

In fact, a deeper look at the proof of Theorem 2.9, allows us to get rid of the ν, and state that there exists a constant K depending on T and γ such that

$$\|\theta\|_\infty \lesssim (1 + \|G\|^b_{\gamma,1})^{K(\|G\|^b_{\gamma,1})^{1/\gamma}}(|\theta_0| + 1)$$

and a similar bound holds for $\|\theta\|_\gamma$.

In order to study the properties of the solutions of the non-linear Young differential equation, we intend to compare two solutions θ^1 and θ^2. In the classical case (when G is differentiable in time), we would have

$$\begin{aligned}\theta^1_t - \theta^2_t &= (\theta^1_0 - \theta^2_0) + \int_0^t \left(G'_u(\theta^1_u) - G'_u(\theta^2_u)\right)\mathrm{d}u \\ &= (\theta^1_0 - \theta^2_0) + \int_0^t \left(G'_u(\theta^1_u - \theta^2_u + \theta^2_u) - G'_u(\theta^2_u)\right)\mathrm{d}u \\ &= (\theta^1_0 - \theta^2_0) + \int_0^t (\tau_{\theta^2}G)'_u(\theta^1_u - \theta^2_u)\mathrm{d}u - \int_0^t G'_u(\theta^2_u)\mathrm{d}u,\end{aligned}$$

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where $(\tau_{\theta^2} G)_t(x) = \int_0^t G'_u(\theta^2_u + x)\mathrm{d}u$. Hence $\theta^1 - \theta^2$ solve a differential equation, but with a translated and averaged function $(\tau_{\theta^2} G)$ and a second member. In order to prove some properties on the solutions, we then have to study this differential equation.

In the case of the Young differential equation, this strategy will be very profitable, we have to define the averaged translation and to study some of its properties. Hence, we define the natural action of the additive group of $\mathscr{C}^\rho$ paths on the integrated vector fields $C \in \mathscr{C}^{\gamma,\nu}_b$.

Definition 2.13. Let $\gamma, \nu, \rho \in [0, 1]$ such that $\gamma + \rho\nu > 1$, $G \in \mathscr{C}^{\gamma,\nu}_b$ and $f \in \mathscr{C}^\rho$. We define the average translation of G by f, and we write $\tau_f G$ the following quantity

$$\tau_f G : (t, x) \to \int_0^t G_{\mathrm{d}u}(f_u + x).$$

Due to the requirements of Young integration, the estimations for the translated integrated vector field $\tau_f G$ show a loss of regularity quantified by the next lemma.

Lemma 2.14. *For $\gamma + \nu\rho > 1$ and $\gamma + \eta\rho > 1$, $f \in \mathscr{C}^\rho$ and $G \in \mathscr{C}^{\gamma,\nu+\eta}_b$ we have $\tau_f G \in \mathscr{C}^{\gamma,\nu}$ and*

$$\|\tau_f G\|_{\gamma,\nu} \lesssim \|G\|_{\gamma,\nu+\eta}(1 + \|f\|^\eta_\rho).$$

Proof. Suppose first that $\eta + \nu \leqslant 1$. Let $x, y \in V$ and define $\tilde{G}(z) = G(x + z) - G(y + z)$. There is two bounds for the increments of $\tilde{G}$:

$$|\tilde{G}_{s,t}(z_1) - \tilde{G}_{s,t}(z_2)| \lesssim [\![G]\!]_{\gamma,\nu+\eta} |t - s|^\gamma |x - y|^{\nu+\eta}$$

and

$$|\tilde{G}_{s,t}(z_1) - \tilde{G}_{s,t}(z_2)| \lesssim [\![G]\!]_{\gamma,\nu+\eta} |t - s|^\gamma |z_1 - z_2|^{\nu+\eta}.$$

Hence, by interpolating these two inequalities

$$|\tilde{G}_{s,t}(z_1) - \tilde{G}_{s,t}(z_2)| \lesssim [\![G]\!]_{\gamma,\nu+\eta} |t - s|^\gamma |x - y|^\nu |z_1 - z_2|^\eta.$$

When $2 \geqslant \eta + \nu > 1$, we have

$$\begin{aligned}|\tilde{G}_{s,t}(z_1) - \tilde{G}_{s,t}(z_2)| &= \left| \int_0^1 \mathrm{d}r\{DG_{s,t}(x(r) + z_1) - DG_{s,t}(x(r) + z_2)\}.(x - y) \right| \\ &\lesssim [\![G]\!]_{\gamma,\nu+\eta} |t - s|^\gamma |x - y||z_1 - z_2|^{\nu+\eta-1},\end{aligned}$$

where $x(r) = y + r(x - y)$,

$$|\tilde{G}_{s,t}(z_1) - \tilde{G}_{s,t}(z_2)| \lesssim [\![G]\!]_{\gamma,\nu+\eta} |t - s|^\gamma |x - y|^{\nu+\eta-1} |z_1 - z_2|$$

and again

$$|\tilde{G}_{s,t}(z_1) - \tilde{G}_{s,t}(z_2)| \lesssim [\![G]\!]_{\gamma,\nu+\eta} |t - s|^\gamma |x - y|^\nu |z_1 - z_2|^\eta.$$

In these two cases, we have

$$[\![\tilde{G}]\!]_{\gamma,\eta} \lesssim [\![G]\!]_{\gamma,\nu+\eta} |t - s|^\gamma |x - y|^\nu.$$

Hence

$$\tau_f G_{s,t}(x) - \tau_f G_{s,t}(y) = \int_s^t G_{\mathrm{d}u}(f_u + x) - G_{\mathrm{d}u}(f_u + y)$$
$$\leqslant \int_s^t \tilde{G}_{\mathrm{d}u}(f_u) - \tilde{G}_{s,t}(f_s) + \tilde{G}_{s,t}(f_s).$$

Hence,

$$|\tau_f G_{s,t}(x) - \tau_f G_{s,t}(y)| \lesssim [\![\tilde{G}]\!]_{\gamma,\eta}(\|f\|_\gamma^\eta + 1) + |\tilde{G}_{s,t}(f_s)|$$

and as $|\tilde{G}_{s,t}(f_s)| \leqslant 2|t-s|^\gamma \|G\|_{\gamma,\nu+\eta}$,

$$[\![\tau_f G]\!]_{s,t} \leqslant \|G\|_{\gamma,\nu+\eta}(\|f\|_\gamma^\eta + 1).$$

Furthermore

$$|(\tau_f G)_{s,t}(0)| \leqslant \left| \int_s^t G_{\mathrm{d}u}(f_u) - G_{s,t}(f_s) \right| + |G_{s,t}(f_s)| \leqslant \|G\|_{\gamma,\eta}|t-s|^\gamma (\|f\|_\gamma^\eta + 1)$$

and by the embedding of Lemma 2.11, the result follows. □

The averaged translation is a suitable tool to control the difference of two non-linear Young integrals, as soon as we have enough regularity to estimate the integral. The following lemma states the estimation for generic functions.

Lemma 2.15. *Let $\gamma, \nu, \nu', \rho \in [0,1]$ such that $\gamma + \rho\nu > 1$ and $\gamma + \rho\nu' > 1$. Let $f^1, f^2 \in \mathscr{C}^\rho$, $G \in \mathscr{C}_b^{\gamma,\nu}$, and suppose that $\tau_{f^2} G \in \mathscr{C}_b^{\gamma,\nu'}$. Then*

$$\left\| \int_0^\cdot G_{\mathrm{d}u}(f_u^1) - G_{\mathrm{d}u}(f_u^2) \right\|_{\infty,[0,T]} \lesssim \|\tau_{f^2} G\|_{\gamma,\nu'} T^\gamma ([\![f^1 - f^2]\!]_\rho^{\nu'} + \|f^1 - f^2\|_\infty^{\nu'})$$

and

$$\left[\!\!\left[\int_0^\cdot G_{\mathrm{d}u}(f_u^1) - G_{\mathrm{d}u}(f_u^2) \right]\!\!\right]_{\gamma,[0,T]} \lesssim \|\tau_{f^2} G\|_{\gamma,\nu'} ([\![f^1 - f^2]\!]_\gamma^{\nu'} + \|f^1 - f^2\|_\infty^{\nu'}).$$

Furthermore when $1 \geqslant \eta > 0$ such that $\rho\eta + \gamma > 1$ and $G \in \mathscr{C}^{\gamma,\nu'+\eta}$

$$\left\| \int_0^\cdot G_{\mathrm{d}u}(f_u^1) - G_{\mathrm{d}u}(f_u^2) \right\|_{\infty,[0,T]}$$
$$\lesssim T^\gamma \|G\|_{\gamma,\nu'+\eta}(1 + \|f^2\|_\rho^\eta)([\![f^1 - f^2]\!]_\rho^{\nu'} + \|f^1 - f^2\|_\infty^{\nu'}).$$

Proof. It is a direct application of the definition of the averaged translation. Let $s, t \in [0,T]$, by definition we have $\int_s^t G_{\mathrm{d}u}(f_u^2) = \tau_{f^2} G_{s,t}(0)$. Hence

$$\left| \int_s^t G_{\mathrm{d}u}(f_u^1) - \int_s^t G_{\mathrm{d}u}(f_u^2) \right|$$
$$\leqslant \left| \int_s^t \tau_{f^2} G_{\mathrm{d}u}(f_u^1 - f_u^2) - \tau_{f^2} G_{s,t}(f_s^1 - f_s^2) \right| + |\tau_{f^2} G_{s,t}(f_s^1 - f_s^2) - \tau_{f^2} G_{s,t}(0)|$$
$$\lesssim [\![\tau_{f^2} G]\!]_{\gamma,\nu'} |t-s|^\gamma ([\![f^1 - f^2]\!]_\rho |t-s|^{\nu'\gamma} + |f_s^1 - f_s^2|^{\nu'}).$$

Q8 Hence,

$$\left\| \int_0^\cdot G_{\mathrm{d}u}(f_u^1) - G_{\mathrm{d}u}(f_u^2) \right\|_{\infty,[0,T]} \lesssim [\![\tau_{f^2} G]\!]_{\gamma,\nu'} T^\gamma [[\![f^1 - f^2]\!]_\rho^{\nu'} + \|f^1 - f^2\|_\infty^{\nu'}]$$

and

$$\left[\!\!\left[\int_0^\cdot G_{\mathrm{d}u}(f_u^1) - G_{\mathrm{d}u}(f_u^2) \right]\!\!\right]_{\gamma,[0,T]} \lesssim [\![\tau_{f^2} G]\!]_{\gamma,\nu'} ([\![f^1 - f^2]\!]_\rho^{\nu'} + \|f^1 - f^2\|_\infty^{\nu'}).$$

For the second part of the lemma, we use the bound of Lemma 2.14. □

We are now ready to prove a comparison principle between two solutions. In order to keep a high degree of generality, we do not use the estimation of Lemma 2.14, but prefer to state a general assumption for the regularity of the averaged translation of the first vector field.

Theorem 2.16. *Let $\gamma > \frac{1}{2}$, $\nu \in [0,1]$ such that $\gamma(1+\nu) > 1$. Let $G^1, G^2 \in \mathscr{C}_b^{\gamma,\nu}$, θ^1 (respectively θ^2) be a solution of the nonlinear Young differential equation driven by G^1 (respectively by G^2). Suppose that $\tau_{\theta^2} G^1 \in \mathscr{C}^{\gamma,1}$. Then*

$$\|\theta^1 - \theta^2\|_{\infty,[0,T]} \leqslant c_1 e^{c_2 \|\tau_{\theta^2} G^1\|_{\gamma,1}^{1/\gamma}} (\|\theta^2\|_\gamma^\nu + 1)(|\theta_0^1 - \theta_0^2| + \|G^1 - G^2\|_{\gamma,\nu}).$$

Proof. Let $t_0 \in [0,T]$, $S > 0$ and define $J = [t_0, (t_0 + S) \wedge 1]$. For $s \leqslant t \in J$ we have

$$\begin{aligned} \delta(\theta^1 - \theta^2)_{s,t} = &\int_s^t \tau_{\theta^2} G^1_{\mathrm{d}u}(\theta_u^1 - \theta_u^2) - \tau_{\theta^2} G^1_{s,t}(\theta_s^1 - \theta_s^2) + \tau_{\theta^2} G^1_{s,t}(\theta_s^1 - \theta_s^2) \\ &- \tau_{\theta^2} G^1_{s,t}(0) + \int_s^t (G^1 - G^2)_{\mathrm{d}u}(\theta_u^2) - (G^1 - G^2)_{s,t}(\theta_s^2) \\ &+ (G^1 - G^2)_{s,t}(\theta_s^2) - (G^1 - G^2)_{s,t}(0) + (G^1 - G^2)_{s,t}(0). \end{aligned}$$

Hence,

$$\begin{aligned} |\delta(\theta^1 - \theta^2)_{s,t}| \lesssim &\|\tau_{\theta^2} G^1\|_{\gamma,1} |t-s|^\gamma (S^\gamma [\![\theta^1 - \theta^2]\!]_\gamma + |\theta_s^1 - \theta_s^2|) \\ &+ \|G^1 - G^2\|_{\gamma,\nu} |t-s|^\gamma (S^{\gamma\nu} [\![\theta^2]\!]_\gamma^\nu + |\theta_s^2|^\nu + 1). \end{aligned}$$

When C_1 is the universal constant in the previous inequality and for S small enough such that $\frac{1}{4} \leqslant C_1 \|\tau_{\theta^2} G^1\|_{\gamma,1} S^\gamma \leqslant \frac{1}{2}$, there exists another constant C_2 such that

$$\begin{aligned} |\delta(\theta^1 - \theta^2)_{s,t}| \leqslant &\frac{1}{2} |t-s|^\gamma ([\![\theta^1 - \theta^2]\!]_\gamma + S^{-\gamma} |\theta_s^1 - \theta_s^2|) \\ &+ C_2 \|G^1 - G^2\|_{\gamma,\nu} |t-s|^\gamma (S^{\gamma\nu} \|\theta^2\|_\gamma^\nu + 1). \end{aligned}$$

Hence

$$[\![\theta^2 - \theta^2]\!]_\gamma \leqslant S^{-\gamma} \|\theta^1 - \theta^2\|_{\infty,J} + C_3 \|G^1 - G^2\|_{\gamma,\nu} (\|\theta^2\|_\gamma^\nu + 1)$$

and

$$\|\theta^1 - \theta^2\|_{\infty,J} \leqslant \frac{1}{2} (\|\theta^1 - \theta^2\|_{\infty,J} + |\theta_s^1 - \theta_s^2|) + C_4 \|G^1 - G^2\|_{\gamma,\nu} S^\gamma (\|\theta^2\|_\gamma^\nu + 1).$$

Finally

$$\|\theta^1 - \theta^2\|_{\infty,J} \leqslant 2|\theta_s^1 - \theta_s^2| + C_5 \|G^1 - G^2\|_{\gamma,\nu} S^\gamma (\|\theta^2\|_\gamma^\nu + 1).$$

By the same gluing argument as in Theorem 2.9, we have

$$\|\theta^1 - \theta^2\|_\infty \lesssim 2^{1/S}(|\theta_0^1 - \theta_0^2| + C_5\|G^1 - G^2\|_{\gamma,\nu} S^\gamma(\|\theta^2\|_\gamma^\nu + 1)).$$

Remind that $\frac{1}{4} \leqslant C_1\|\tau_{\theta^2}G^1\|_{\gamma,1}S^\gamma \leqslant \frac{1}{2}$, and there exist two universal constants (depending on γ, ν, T) c_1 and c_2 such that

$$\|\theta^1 - \theta^2\|_\infty \leqslant c_1 e^{c_2\|\tau_{\theta^2}G^1\|_{\gamma,1}^{1/\gamma}}(|\theta_0^1 - \theta_0^2| + \|G^1 - G^2\|_{\gamma,\nu}(\|\theta^2\|_\gamma^\nu + 1)),$$

which ends the proof. □

2.2.2. *Uniqueness of solutions*

We prove here the uniqueness of the solutions when G is regular enough using the comparison principle given in Theorem 2.16. In order to use the comparison principle, we ask that the vector field is regular enough (in space) and as stated in Lemma 2.14, we are able to estimate the averaged translation if we accept a additional loss of space regularity. Furthermore, as we will use uniqueness results in different contexts, especially in situation when we will have a priori regularity properties for the solutions, we will give a pretty general theorem of uniqueness, Theorem 2.17, and specialise it in the two following corollaries.

Theorem 2.17. *Let $\gamma > 1/2$, $\nu \in [0, 1]$ such that $\gamma(1 + \nu) > 1$ and $G \in \mathscr{C}_b^{\gamma,\nu}$. Suppose that there exists a sequence $(G^\varepsilon)_\varepsilon \in \mathscr{C}_b^{\gamma,\nu}$ such that*

i. *For all $\gamma' < \gamma$ and all, $\|G - G^\varepsilon\|_{\gamma',\nu} \to 0$.*
ii. *For all $\varepsilon > 0$ and $\|G^\varepsilon\|_{\gamma,\nu} \leqslant \|G\|_{\gamma,\nu}$.*
iii. *For all $\varepsilon > 0$ and all $x \in \mathbb{R}^d$ there exists a unique solution θ^ε for the equation*

$$\theta_t^\varepsilon(x) = x + \int_0^t G_{\mathrm{d}u}^\varepsilon(\theta_u^\varepsilon(x))\mathrm{d}u.$$

iv. *For all $\varepsilon > 0$, $\tau_{\theta^\varepsilon}G \in \mathscr{C}_b^{\gamma,1}$ and $\sup_{\varepsilon>0}\|\tau_{\theta^\varepsilon}G\|_{\gamma,1} < +\infty$.*

Then the solution of the nonlinear Young equation driven by G is unique.

Proof. This theorem is a direct consequence of the comparison principle of Theorem 2.16. Let $x \in \mathbb{R}^d$ be an initial condition, and let θ be a solution of the nonlinear Young differential equation with initial condition x. Furthermore let G^ε and θ^ε be as in the hypothesis of the theorem. Take $\frac{1}{2} < \gamma' < \gamma$ such that $\gamma(1 + \nu) > 1$. Remark that we can apply the comparison principle to θ and θ^ε with γ' instead of γ, and as $\|G^\varepsilon\|_{\gamma',\nu} \leqslant \|G\|_{\gamma',\nu} \lesssim \|G\|_{\gamma,\nu}$, we have

$$\begin{aligned}\|\theta^\varepsilon\|_{\gamma'} &\lesssim \|G^\varepsilon\|_{\gamma',\nu}^{1/\nu\gamma'}(1 + \|G^\varepsilon\|_{\gamma',\nu})^{\tilde{K}_2\|G^\varepsilon\|_{\gamma',\nu}^{1/\nu\gamma'}}(1 + |x|)\\ &\lesssim \|G\|_{\gamma,\nu}^{1/\nu\gamma'}(1 + \|G\|_{\gamma,\nu})^{\tilde{K}_2\|G\|_{\gamma,\nu}^{1/\nu\gamma'}}(1 + |x|),\end{aligned}$$

but also

$$\begin{aligned}\|\theta(x) - \theta^\varepsilon(x)\|_\infty &\lesssim e^{c_2\|\tau_{\theta^\varepsilon}G\|_{\gamma',1}^{1/\gamma}}\|G^1 - G^\varepsilon\|_{\gamma',\nu}(\|\theta^\varepsilon\|_{\gamma'}^\nu + 1)\\ &\lesssim \|G\|_{\gamma,\nu}^{1/\nu\gamma'}(1 + \|G\|_{\gamma,\nu})^{\tilde{K}_2\|G\|_{\gamma,\nu}^{1/\nu\gamma'}}(1 + |x|)e^{c_2\|\tau_{\theta^\varepsilon}G\|_{\gamma,1}^{1/\gamma}}\|G^1 - G^\varepsilon\|_{\gamma',\nu}.\end{aligned}$$

As $\sup_{\varepsilon>0}\|\tau_{\theta^\varepsilon}G\|_{\gamma,1} < +\infty$, we have $\|\theta(x) - \theta^\varepsilon(x)\|_\infty \to_{\varepsilon\to 0} 0$. As θ^ε is unique, and since this convergence holds true for every function $\theta(x)$ solution of the equation, the solution is unique. □

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We can now use the averaged translation operator to establish uniqueness in the case where we have a priori informations of the regularities of the solutions θ^ε.

Corollary 2.18. *Let $\gamma > 1/2$, $\delta > 0$ and $G \in \mathscr{C}_b^{\gamma,1+\eta}$. Suppose that there exists a sequence $(G^\varepsilon)_\varepsilon \in \mathscr{C}_b^{\gamma,1+\eta}$ such that*

i. *For all $\gamma' < \gamma$ and all, $\|G - G^\varepsilon\|_{\gamma',1} \to 0$.*
ii. *For all $\varepsilon > 0$ and $\|G^\varepsilon\|_{\gamma,1+\eta} \leqslant \|G\|_{\gamma,1+\eta}$.*
iii. *For all $\varepsilon > 0$ and all $x \in \mathbb{R}^d$ there exists a unique solution θ^ε for the equation*

$$\theta_t^\varepsilon(x) = x + \int_0^t G_{\mathrm{d}u}^\varepsilon(\theta_u^\varepsilon(x))\mathrm{d}u.$$

iv. *There exists $\rho > 0$ such that $\eta\rho + \gamma > 0$ and for which for all $\varepsilon > 0$, $\theta^\varepsilon(x) \in \mathscr{C}^\rho$ and $\sup_{\varepsilon>0} \|\theta^\varepsilon(x)\|_\rho < +\infty$.*

Then solution of the non-linear Young equation driven by G is unique.

Furthermore, when the function $x \to \sup_{\varepsilon>0} \|\theta^\varepsilon(x)\|_\rho$ is locally bounded in time, the flow $(t, x) \to \theta_t(x)$ of the equation is locally Lipschitz continuous in space, uniformly in time and

$$\|\theta(x) - \theta(y)\|_\infty \lesssim \exp(C(1 + \log(1 + \|G\|_{\gamma,1+\eta}) + (\sup_{\varepsilon>0} \|\theta^\varepsilon(y)\|_\rho)^\eta)\|G\|_{\gamma,1+\delta}^{1/\gamma})|x - y|(|y| + 1).$$

Proof. Condition i., ii. and iii. are the same of those of Theorem 2.17. We only have to prove that the point iv. of Theorem 2.17 is satisfied. But thanks to Lemma 2.14, we know that $\tau_{\theta^\varepsilon} G \in \mathscr{C}_b^{\gamma,1}$. Furthermore

$$\|\tau_{\theta^\varepsilon} G\|_{\gamma,1} \lesssim \|G\|_{\gamma,1+\eta}(\|\theta^\varepsilon\|_\rho^\eta + 1) \lesssim \|G\|_{\gamma,1+\eta}((\sup_{\varepsilon>0} \|\theta^\varepsilon\|_\rho)^\eta + 1)$$

and the uniqueness follows by Theorem 2.17. Furthermore, for $y \in \mathbb{R}^d$ since $\|G^\varepsilon\|_{\gamma,1+\eta} \leqslant \|G\|_{\gamma,1+\eta}$, we have

$$\|\tau_{\theta^\varepsilon(y)} G^\varepsilon\|_{\gamma,1} \lesssim \|G^\varepsilon\|_{\gamma,1+\eta}((\sup_{\varepsilon>0} \|\theta^\varepsilon\|_\rho)^\eta + 1) \lesssim \|G\|_{\gamma,1+\eta}((\sup_{\varepsilon>0} \|\theta^\varepsilon\|_\rho)^\eta + 1).$$

Since $\sup_{\varepsilon>0} \|G^\varepsilon\|_{\gamma,1} \lesssim \|G\|_{\gamma,1+\eta}$ we have, thanks to the a priori bounds for the solutions of Theorem 2.9, and Remark 2.12, that

$$\|\theta^\varepsilon(y)\|_\gamma \lesssim \|\theta\|_\infty \lesssim (1 + \|G\|_{\gamma,1+\eta})^{K(\|G\|_{\gamma,1+\eta})^{1/\gamma}}(|y| + 1),$$

which implies

$$\|\theta^\varepsilon(x) - \theta^\varepsilon(y)\|_\infty \lesssim \exp(C(1 + \log(1 + \|G\|_{\gamma,1+\eta}) + (\sup_{\varepsilon>0} \|\theta^\varepsilon(y)\|_\rho)^\eta)\|G\|_{\gamma,1+\delta}^{1/\gamma})|x - y|(|y| + 1),$$

the conclusion then easily follows when we let ε go to zero. □

Remark 2.19. Suppose furthermore that for all $t \in [0, T]$, $x \to \theta_t^\varepsilon(x)$ is differentiable in space. Then

$$\sup_{\varepsilon>0} |D\theta_t^\varepsilon(x)| \lesssim \exp(C(1 + \log(1 + \|G\|_{\gamma,1+\eta}) + (\sup_{\varepsilon>0} \|\theta^\varepsilon(x)\|_\rho)^\eta)\|G\|_{\gamma,1+\delta}^{1/\gamma})(|x| + 1).$$

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Finally, we state the more general results of uniqueness, where all the needed informations are the regularity of G.

Corollary 2.20. *Let $\gamma > \frac{1}{2}$, $\nu \in [0,1]$ such that $\gamma(1+\nu) > 0$ and suppose that $G \in \mathscr{C}_b^{\gamma,\nu+1}$, then there exists a unique solution $\theta(x)$ for the non-linear Young equation with initial condition x. Furthermore θ is locally Lipschitz continuous in space uniformly in time.*

Proof. We only have to check the conditions of Corollary 2.18 with $\nu = \eta$ and $\gamma = \rho$. Let $G^\varepsilon \in \mathscr{C}^{\gamma,1+\nu}$ such that the time derivative $(G^\varepsilon)'$ exists and lies in $\mathscr{C}_b^{\gamma,1+\nu}$, and such that $\|G^\varepsilon\|_{\gamma,1+\nu} \leqslant \|G\|_{\gamma,1+\nu}$ and for all $\gamma' < \gamma$, $\|G - G^\varepsilon\|_{\gamma',1+\nu} \to 0$. In that case, $\theta^\varepsilon(x)$ is the solution of

$$\theta^\varepsilon(x) = x + \int_0^t (G_r^\varepsilon)'(\theta_r^\varepsilon(x))\mathrm{d}r = x + \int_0^t G_{\mathrm{d}r}^\varepsilon(\theta_r^\varepsilon(x)).$$

As $G^\varepsilon \in \mathscr{C}_b^{\gamma,1+\nu}$, θ^ε is unique and furthermore θ^ε is differentiable in space, and the differential is the solution of the following equation

$$D\theta^\varepsilon(x) = \mathrm{id} + \int_0^t D(G_r^\varepsilon)'(\theta_r^\varepsilon(x))\mathrm{d}r.$$

Thanks to Remark 2.12 we have

$$\|\theta^\varepsilon(x)\|_\infty + \|\theta^\varepsilon(x)\|_\gamma \lesssim (1 + \|G\|_{\gamma,1}^b)^{K(\|G\|_{\gamma,1}^b)^{1/\gamma}}(|x|+1).$$

Hence $x \to \sup_\varepsilon \|\theta^\varepsilon(x)\|_\gamma$ is locally bounded in space. All the conditions of Corollary 2.18 are fulfilled, and the result follows. □

2.3. Localisation of unbounded vector fields

In order to give a complete survey of the question, we need to go back to the weighted spaces $\mathscr{C}^{\gamma,\nu,\psi}$ and to state the existence and uniqueness theorems in that case.

Let $\gamma > 1/2$, $\nu < 1$ and $\gamma(1+\nu) > 1$ and $\psi > 0$ a ν-weight. Let $r > 0$ and $r = K_1(1 + \|G\|_{\gamma,1}^b)^{K_2(\|G\|_{\gamma,1}^b)^{1/\gamma}}(r+1)$, where K_1 and K_2 are define as in Theorem 2.9 and depend on ψ. As we intend to use the averaged translation operator, and since any solution lies in balls of radius R, we need to localise G on balls B of centre 0 and of radius $2R$. We then let $G_{|R} \in \mathscr{C}_b^{\gamma,\nu}([0,T], B)$ the restriction of G on $[0,T] \times B$. We have of course $\|G_{|R}\|_{\gamma,\nu}^b \lesssim \psi(2R)\|\tilde{G}\|_{\gamma,\nu,\psi}$. Furthermore, as all the arguments hold locally, as we have done all the estimations for $x, y \in B(0,r)$ in the previous section.

When $\nu = 1$, it is necessary to have the existence and a bound for the solution in order to localise. As this holds only for $\nu < 1$, the good hypothesis is that there exists $\nu < 1$, $\tilde{\psi}$ a ν-weight such that $G \in \mathscr{C}^{\gamma,\nu,\tilde{\psi}} \cap \mathscr{C}^{\gamma,1,\psi}$. In that case, we are again able to localise and to use the result of the previous section. The following theorem holds:

Theorem 2.21. *Let $\gamma > \frac{1}{2}$, $1 \geqslant \nu > 0$ with $\gamma(1+\nu) > 1$ and ψ a weight. Let $1 < \nu' \leqslant \nu$ with $\nu' < 1$ such that $\gamma(1+\nu') > 1$, and ψ' a ν'-weight. Let $G \in \mathscr{C}^{\gamma,\nu',\psi'} \cap \mathscr{C}^{\gamma,\nu,\psi}$. For all $x \in \mathbb{R}^d$ there exists a solution $\theta(x) \in \mathscr{C}^\gamma([0,T])$ to the equation*

$$\theta_t(x) = x + \int_0^t G_{\mathrm{d}u}(\theta_u).$$

Furthermore, there exist K_1 and K_2 two constants depending on γ, ν' and ψ' such that

$$\|\theta\|_{\gamma,[0,T]} \leqslant K_1 \|G\|^{1/\nu'\gamma} (1 + \|G\|_{\gamma,\nu',\psi'})^{K_2 \|G\|^{1/\nu'\gamma}_{\gamma,\nu',\psi'}} (1 + |x|).$$

Let $r > 0$, $R = K_1 \|G\|^{1/\nu'\gamma} (1 + \|G\|_{\gamma,\nu',\psi'})^{K_2 \|G\|^{1/\nu'\gamma}_{\gamma,\nu',\psi'}} (1 + |r|)$ and $B = B(0, 2R)$. Let us take $\tilde{G} \in \mathscr{C}^{\gamma,\nu',\psi'} \cap \mathscr{C}^{\gamma,\nu,\psi}$ such that $\|\tilde{G}\|_{\gamma,\nu,\psi'} \leqslant \|G\|_{\gamma,\nu,\psi'}$, suppose furthermore that for $y \in B(0, r)$, $\tau_{\tilde{\theta}(y)} G \in \mathscr{C}^{\gamma,1,\psi}$ where $\tilde{\theta}$ is the solution of the following equation

$$\tilde{\theta}_t(y) = y + \int_0^t \tilde{G}_{\mathrm{d}u}(\tilde{\theta}_u).$$

Then

$$\|\theta(x) - \tilde{\theta}(y)\|_\infty \lesssim \varphi(R) e^{c_2 \varphi(R) \|\tau_{\tilde{\theta}(y)} G\|^{1/\gamma}_{\gamma,1,\psi}} (|x - y| + \|G - \tilde{G}\|_{\gamma,\nu,\psi}),$$

where $\varphi(R) = (R + 1)\psi(R)$.

3. Averaging of paths

We turn now to the study of the averaging operator T^w proper. One of our main results is a proof that fBm paths are ρ-irregular for any $\rho < 1/2H$ and as a consequence that the averaging operator T^w is bounded from the Fourier–Lebesgue space $\mathscr{F}L^\alpha$ to $\mathscr{C}^\gamma \mathscr{F}L^{\alpha+\rho}$ for any $\alpha \in \mathbb{R}$ and for almost every fBm path w. This result was one of our main reasons to look at the scale of Fourier–Lebesgue spaces.

For the scale of Besov spaces $(\mathscr{C}^\alpha)_\alpha$ we were unable to prove similar results and we limited ourselves to study the averaged vector-fields $T^w f$ for fixed $f \in \mathscr{C}^\alpha$.

In this section we will first study the almost-sure irregularity of fBm paths. This study proceeds in two steps: first we use well known chaining arguments (essentially going back to Kolmogorov lemma in the form given to it by Garsia, Rodemich and Rumsey) to go from supremum norm to “integral” norms more suitable to probabilistic estimates and then use Hoeffding inequality to prove these estimates.

The use of Hoeffding inequality replaces what in Davie’s paper [7] are explicit and painful computations on Brownian motions (relying on the Markov property) and what in other works (e.g. in [9]) is achieved via stochastic calculus (and thus martingale properties). In the fBm context neither technique is applicable and explicit computations using Gaussian tools, while possible are quite cumbersome and moreover we were unable to use them to obtain the exponential square integrability we show here to be valid. So we think that our observation that discrete martingale techniques like Hoeffding inequality are useful in the fBm context is one of the interesting points of our research.

3.1. Chaining lemmas

To see the average properties of the fractional Brownian path, we will need some chaining lemmas, to infer global estimates from pointwise ones.

Lemma 3.1. *Let X from I^2 to $\mathbb{R}^d$ such that for all $s \leqslant u \leqslant t$*

$$|X_{s,t}| \leqslant |X_{s,u}| + |X_{u,t}| \quad \text{and} \quad X_{s,s} = 0.$$

And let us define for $\mu > 0$,

$$R_\mu(X) = \sum_{n\in\mathbb{N}} \sum_{k=0}^{2^n-1} 2^{-2n} \exp(\mu 2^n |X_{k2^{-n},(k+1)2^{-n}}|^2).$$

Then there exists a constant $K > 0$ such that for all $s \leqslant t$,

$$\exp(\mu|X_{s,t}|^2/|t-s|) \lesssim |t-s|^{-K} R_{\mu K}(X).$$

Proof. Let $0 \leqslant s < t \leqslant 1$, $n \in \mathbb{N}$ be the largest $n' \in \mathbb{N}$ such that $2^{-(n'+1)} \leqslant t-s \leqslant 2^{-n'}$. By definition of n there exists l such that $l/2^n \leqslant s < t \leqslant (l+1)/2^n$. We can find some sequences $(s_k)_{k\geq 1}$ and $(t_k)_{k\geq 1}$ such that (s_k) decreases, (t_k) increases, $s_1 = t_1 = (2l+1)/2^{n+1}$, $\lim_{k\to\infty} s_k = s$, $\lim_{k\to\infty} t_k = t$, $s_{k+1} - s_k \leqslant 2^{n+k+1}$, $t_{k+1} - t_k \leqslant 2^{n+k+1}$ and $2^{n+k} t_k \in \mathbb{Z}$ and $2^{n+k} s_k \in \mathbb{Z}$. Hence $[s,t) = \cup_{k\geq 1}[s_{k+1}, s_k) \cup \cup_{k\geq 1}[t_k, t_{k+1})$ and thanks to the definition of the sequences, the following inequalities hold for s_k, but also for t_k.

First, if $s_{k+1} = s_k$, $\sqrt{\mu}|X_{s_{k+1},s_k}| = 0$. Now, if $s_{k+1} < s_k$ then there exists $l_k \in \{0,\dots,n+k\}$ such that $s_{k+1} = (2l_k - 1)/2^{n+k+1}$ and $s_k = l_k/2^{n+k}$. Hence

$$\begin{aligned}\sqrt{\mu}|X_{s_{k+1},s_k}| &= 2^{-(n+k+1)/2} \log(2^{(n+k+1)} 2^{-(n+k+1)} \exp(\mu 2^{k+n+1} |X_{s_{k+1},s_k}|^2))^{1/2} \\ &\lesssim 2^{-(n+k)/2}\{(n+k) + \log(R_\mu(X))\}^{1/2}.\end{aligned}$$

But $2^{-(n+1)} \leqslant |t-s| \leqslant 2^{-n}$, hence

$$\sqrt{\mu}|X_{s_{k+1},s_k}| \lesssim |t-s|^{1/2} 2^{-k/2}\{k + \log(1/|t-s|) + \log(R_\mu(X))\}^{1/2}.$$

Thanks to the definition of $(s_k)_k$ and $(t_k)_k$, we have

$$\begin{aligned}\sqrt{\mu}|X_{s,t}| &\leqslant \sum_{k\geqslant 1} \sqrt{\mu}|X_{s_{k+1},s_k}| + \sqrt{\mu}|X_{t_k,t_{k+1}}| \\ &\lesssim |t-s|^{1/2}\{1 + \log\big(1/|t-s|\big) + \log(R_\mu(X))\}^{1/2}.\end{aligned}$$

Hence,

$$\begin{aligned}\exp(\mu|X_{s,t}|^2/|t-s|) &\lesssim \exp(K\log(1/|t-s|) + K\log R_\mu(X)) \\ &\lesssim |t-s|^{-K} R_{\mu K}(X)\end{aligned}$$

and by Jensen inequality $R_\mu(X)^K \leqslant R_{\mu K}(X)$. □

In the following, to approach a point of $\mathbb{R}^d$ we will use a similar argument. Namely we will use the graph $(2^{-m}\mathbb{Z})^d$ as a good approximation of $\mathbb{R}^d$. Hence we need to have an approximation of the biggest error we can make using such an approximation. It is well known that for all d and all $m \in \mathbb{N}$, $\sup_{x\in\mathbb{R}^d} \inf_{y\in(2^{-m}\mathbb{Z})^d} |x-y| = \sqrt{d}/2^{m+1}$.

Lemma 3.2. *Let X be a function from $\mathbb{R}^d$ to $\mathbb{R}^d$ and g such that $g \geqslant 1$, $\sup_{|\zeta-\zeta'|\leqslant\sqrt{d}/2} g(\zeta')/g(\zeta) < \infty$ and with $\|g^{-1}\|_{L^1(\mathbb{R}^d)} < +\infty$. Suppose furthermore that the following quantity is finite*

$$C_X := \sup_{\substack{m\in\mathbb{N}\\ \zeta: 2\geqslant g(\zeta)/2^m\geqslant 1/2 \\ |\zeta-\zeta'|\leqslant\sqrt{d}/2}} |X(\zeta) - X(\zeta')|/(2^m|\zeta-\zeta'|) < +\infty.$$

Let

$$S_\mu(X) = \sum_{n\in\mathbb{N}} \sum_{\zeta\in(2^{-n}\mathbb{Z})^d} 2^{-(d+1)n} g(\zeta')^{-1} \exp(\mu|X(\zeta')|^2).$$

Then, there exists a constant $C \geqslant 1$ be such that

$$\exp(\mu|X(\zeta)|^2) \lesssim g(\zeta)^{-C} \exp(\mu K C_X^2) S_{\mu C}(X).$$

Proof. Let $\zeta \in \mathbb{R}^d$ and m such that $g(\zeta) \sim 2^m$. Let $\zeta' \in (2^{-m}\mathbb{Z})^d$ such that $|\zeta - \zeta'| \leqslant 2^{-m}\sqrt{d}/2$ then $|X(\zeta) - X(\zeta')| \leqslant C_X 2^m |\zeta - \zeta'| \lesssim C_X$. Furthermore, the hypothesis on g gives us that $\log(g(\zeta')) \lesssim 1 + \log(g(\zeta))$. Hence

$$\begin{aligned} \sqrt{\mu}|X(\zeta)| &\leqslant \sqrt{\mu}|X(\zeta) - X(\zeta')| + \sqrt{\mu}|X(\zeta')| \\ &\leqslant \sqrt{\mu}C_X + \{\log(2^m 2^{-m} g(\zeta')g(\zeta')^{-1} \exp(\mu|X(\zeta')|^2))\}^{1/2} \\ &\lesssim \sqrt{\mu}C_X + \{m + \log(g(\zeta')) + \log(S_\mu(X))\}^{1/2} \\ &\lesssim \sqrt{\mu}C_X + \{1 + \log(g(\zeta)) + \log(S_\mu(X))\}^{1/2}. \end{aligned}$$

Finally we have

$$\begin{aligned} \exp(\mu|X(\zeta)|^2) &\leqslant \exp(K\{\mu C_X^2 + 1 + \log(g(\zeta)) + \log S_\mu(X)\}) \\ &\lesssim g(\zeta)^K \exp(\mu C C_X^2) S_{\mu C}(X). \quad \square \end{aligned}$$

We can think of g as $g(\zeta) = (1 + |\zeta|)^{d+1}$.

Lemma 3.3. *For all $\beta \in \mathbb{R}$ and all $R > 0$ there exists a constant $C(\beta, R)$ such that for all $\zeta' \in B(\zeta, R)$*

$$|(1 + |\zeta|)^\beta - (1 + |\zeta'|)^\beta| \leqslant C(\beta, R)(1 + |\zeta|)^{\beta-1}|\zeta - \zeta'|.$$

Proof. Let us suppose first that $|\zeta'| \geqslant |\zeta|$ by the choice of ζ' we have $0 \leqslant \frac{|\zeta'|-|\zeta|}{1+|\zeta|} \leqslant R$. Then

$$\begin{aligned} |(1 + |\zeta|)^\beta - (1 + |\zeta'|)^\beta| &= (1 + |\zeta|)^\beta \left| \left(1 + \frac{|\zeta'| - |\zeta|}{1 + |\zeta|}\right)^\beta - 1 \right| \\ &\leqslant (1 + |\zeta|)^\beta \sup_{x\in[1,R]} |f_\beta'(x)| \frac{||\zeta'| - |\zeta||}{1 + |\zeta|} \\ &\leqslant \sup_{x\in[0,R]} |f_\beta'(x)|(1 + |\zeta|)^{\beta-1}|\zeta - \zeta'|, \end{aligned}$$

where the function f_β is define from $[0, R]$ to $\mathbb{R}$ by $f_\beta(x) = (1 + x)^\beta$. We have

$$|f_\beta'(x)| = |\beta(1 + x)^{\beta-1}| \leqslant |\beta|((1 + R)^{\beta-1} \vee 1).$$

If $|\zeta| > |\zeta'|$, the same computation gives

$$|(1 + |\zeta|)^\beta - (1 + |\zeta'|)^\beta| \lesssim_{R,\beta} (1 + |\zeta'|)^{\beta-1}|\zeta - \zeta'|.$$

When $\beta - 1 \geqslant 0$, the result follows. Suppose now that $\beta - 1 < 0$, we have to prove that $(1 + |\zeta|) \lesssim (1 + |\zeta'|)$. When $|\zeta| \leqslant 2R$, then we have

$$(1 + |\zeta'|)/(1 + |\zeta|) \geqslant 1/(1 + 2R).$$

When $|\zeta| > 2R$,

$$(1+|\zeta'|)/(1+|\zeta|) \geqslant (1+|\zeta|-|\zeta'-\zeta|)/(1+|\zeta|) \geqslant 1-|\zeta'-\zeta|/(1+|\zeta|) \geqslant 1/2$$

and the result follows. □

3.2. *Application of the chaining lemmas, control of the averaging along curves*

The last lemmas allows us to control the average of a function (or a distribution) along the curve w. Indeed, to estimate on the quantity $\int_s^t f_u(x+w_u)\mathrm{d}u$ it will be enough to have a control on simpler quantities. We will apply those lemmas in two similar situations, namely when $f \in \mathscr{C}^\alpha$ and when $f \in \mathscr{F}L^\alpha$. In this latter case, we will see that it is enough to control Φ^w.

3.2.1. *Averaging property of the occupation measure*

Recall that we have already defined $\Phi^w_t(\xi) = \int_0^t e^{i\langle \xi, w_r\rangle}\mathrm{d}r$ and

$$\|\Phi^w\|_{\mathcal{W}^{\rho,\gamma}_T} = \sup_{\xi\in\mathbb{R}^d}\sup_{0\leqslant s<t\leqslant T}(1+|\xi|)^\rho \frac{|\Phi^w_t(\xi)-\Phi^w_s(\xi)|}{|s-t|^\gamma}.$$

Lemma 3.4. *For all $-\beta < \alpha$ there exist a constant $a > 0$ and $\gamma > 0$ such that for all $\lambda > 0$,*

$$|\Phi^w_t(\xi)-\Phi^w_s(\xi)| \lesssim |t-s|^\gamma (1+|\xi|)^{-\alpha'}(1+\log^{1/2}(e^{a\mu\|w\|_\infty}K^w_\alpha(\lambda))),$$

where

$$K^w_\alpha(\lambda) = \sum_{\substack{n,m\in\mathbb{N}\\ 0\leqslant k\leqslant 2^n-1\\ \xi'\in(2^{-m}\mathbb{Z})^d}} 2^{-2n+(d+1)m}(1+|\xi'|)^{-(d+1)}\exp(\lambda 2^n(1+|\xi'|)^{2\beta}|\Phi^w_{k2^{-n},(k+1)2^{-n}}(\xi')|^2).$$

Proof. We apply Lemmas 3.1 and 3.2 to

$$X_{s,t}(\xi) = (1+|\xi|)^\beta|\Phi^w_{s,t}(\xi)|/|t-s|^{1/2}.$$

Thanks to Lemma 3.3 and the definition of $\Phi^w_{s,t}$, for all $\xi \in \mathbb{R}^d$, and all $\xi' \in B\left(\xi, \sqrt{d}/2\right)$, we have

$$\begin{aligned}|X_{s,t}(\xi)-X_{s,t}(\xi')| &\leqslant |(1+|\xi|)^\beta-(1+|\xi'|)^\beta||\Phi^w_{s,t}(\xi')|/|t-s|^{1/2}\\ &\quad+(1+|\xi|)^\beta|\Phi^w_{s,t}(\xi)-\Phi^w_{s,t}(\xi')|/|t-s|^{1/2}\\ &\lesssim (1+|\xi|)^\beta|\xi-\xi'|(1+\|w\|_\infty)|t-s|^{1/2}.\end{aligned}$$

Here we take $\zeta = \xi$, $g(\zeta) = (1+|\zeta|)^{\beta+Cd}$ such that $\beta + Cd \geqslant d+1$. With those choices, $X_{s,t}$ and g verify the hypothesis of Lemma 3.2, furthermore $C_X \lesssim (1+\|w\|_\infty)$, hence

$$\begin{aligned}\exp(\mu|X_{s,t}(\xi)|^2) &= \exp(\mu(1+|\xi|)^{2\beta}|\Phi^w_{s,t}(\xi)|^2/|t-s|)\\ &\lesssim (1+|\xi|)^{C(d+1)}S_{C\mu}(X_{s,t})\exp(\mu C\|w\|^2_\infty).\end{aligned}$$

Now, let us apply Lemma 3.1 to $(1+|\xi|)^\beta\Phi^w_{s,t}(\xi)$, then

$$\exp(\mu|X_{s,t}(\xi)|^2) \lesssim |t-s|^{-K}R_{K\mu}((1+|\xi|)^\beta\Phi^w_\cdot(\xi)).$$

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But

$$\begin{aligned}
&R_{K\mu}((1+|\xi|)^{\beta}\Phi_{\cdot}(\xi)) \\
&= \sum_{n\in\mathbb{N}}\sum_{k=0}^{2^n-1} 2^{-2n}\exp(\mu K 2^n|\Phi^w_{k2^{-n},(k+1)2^{-n}}(\xi)|^2(1+|\xi|)^{2\beta}) \\
&\lesssim (1+|\xi|)^{C(d+1)}\sum_{n\in\mathbb{N}}\sum_{k=0}^{2^n-1} 2^{-2n} S_{\mu C K}(X_{k2^{-n},(k+1)2^{-n}})\exp(\mu C K\|w\|_\infty^2) \\
&\lesssim (1+|\xi|)^{C(d+1)}\exp(\lambda\|w\|^2_\infty)K^w_\beta(\lambda).
\end{aligned}$$

When we take the logarithm, we have

$$\begin{aligned}
|\Phi^w_{s,t}(\xi)| \lesssim \mu^{-1/2}&|t-s|^{1/2}(1+|\xi|)^{-\beta}(1+\log(1/|t-s|) \\
&+\log(1+|\xi|)+\log(\exp(\lambda\|w\|^2_\infty)K^w_\beta(a\mu))).
\end{aligned}$$

Hence, for all $\varepsilon_1,\varepsilon_2>0$, we have

$$|\Phi^w_{s,t}(\xi)| \lesssim_{\varepsilon_1,\varepsilon_2} \mu^{-1/2}|t-s|^{1/2-\varepsilon_1}(1+|\xi|)^{-\beta+\varepsilon_2}(1+\log(\exp(\lambda\|w\|^2_\infty)K^w_\beta(a\mu))).$$

Furthermore, by interpolating with the trivial estimate $|\Phi_{s,t}(\xi)|\leqslant|t-s|$, for all $-\beta<\alpha$, there exists $\gamma>1/2$, and a constant $a>0$ such that

$$|\Phi^w_{s,t}(\xi)| \lesssim |t-s|^{\gamma}(1+|\xi|)^{\alpha}(1+\log^{1/2}(\exp(a\mu\|w\|^2_\infty)K^w_\beta(a\mu))). \quad \square$$

3.2.2. Averaging of Besov functions along paths

In this section we analyse the averaging effect of paths on functions belonging to the scale of Besov spaces $(\mathscr{C}^\alpha)_\alpha$. Note the following. If we write $\tilde{\Delta}_i=\sum_{j:|i-j|\le 1}\Delta_j$, we have $\tilde{\Delta}_i\Delta_i=\Delta_i$ for all $i\ge -1$ and then $T^w_{s,t}(\Delta_i f)(x)=T^w_{s,t}(\tilde{\Delta}_i\Delta_i f)(x)=(T^w_{s,t}(\tilde{K}_i)*\Delta_i f)(x)$ where $\tilde{K}_i$ is the integral kernel corresponding to the operator $\tilde{\Delta}_i$. In this case

$$\|T^w_{s,t}(\Delta_i f)\|_{L^\infty} \lesssim \|\Delta_i f\|_{L^\infty}\|T^w_{s,t}(\tilde{K}_i)\|_{L^1}.$$

So any control of quantities like $\sum_{i\ge -1}\Psi(2^{\alpha i}\|T^w_{s,t}(\tilde{K}_i)\|_{L^1})$ for increasing functions Ψ will imply boundedness properties of T^w in Hölder–Besov spaces. However in the case of fractional Brownian sample path (or even just in the case of Brownian motion) we were unable to devise useful estimates for this kind of quantities. Due to this difficulty which prevents us from having (useful) estimates which are uniform in $\mathscr{C}^\alpha$, the chaining argument now depends on the chosen function f and the computations follows closely those in the previous section.

Lemma 3.5. *For all $-\beta<\alpha$ there exists $\gamma>1/2$ such that for all $f\in\mathcal{S}'(\mathbb{R}^d)$, all $\lambda>0$ and all i,*

$$|T^w_{s,t}(\Delta_i f)(x)| \lesssim_\lambda 2^{\alpha i}\|\Delta_i f\|_\infty|t-s|^{\gamma}(1+\log^{1/2}(1+|x|)+\log^{1/2}(K^w_{f,\beta}(\lambda))),$$

where

$$K^w_{f,\beta}(\lambda) = \sum_{\substack{n,m\in\mathbb{N}\\ i\geqslant -1\\ 0\leqslant k\leqslant 2^n-1\\ x'\in(2^{-m}\mathbb{Z})^d}} \frac{2^{-c_\beta(m+n+i)}}{(1+|x'|)^{(d+1)}}\exp(\lambda 2^{n+2i\beta}|T^w_{k/2^n,(k+1)/2^n}(\Delta_i f)(x')|^2/\|\Delta_i f\|^2_\infty)$$

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and c_β is a constant depending only of β and d such that the sum without the exponential is finite.

Proof. The proof is very similar to the proof of Lemma 3.4. We will apply Lemmas 3.1 and 3.2 to

$$X^i_{s,t}(x) = 2^{i\alpha}|T^w_{s,t}(\Delta_i f)(x)|/(\|\Delta_i f\|_\infty |t-s|^{1/2}),$$

with the convention that $X^i = 0$ when $\Delta_i f = 0$. We have, thank to the definition of T^w,

$$|T^w_{s,t}(\Delta_i f)(x)| \leqslant \|\Delta_i f\|_\infty |t-s|. \tag{12}$$

Furthermore, as the Fourier transform of $\Delta_i f$ is compactly supported in an annulus, we have the obvious estimate

$$\begin{aligned}|X^i_{s,t}(x) - X^i_{s,t}(x')| &\lesssim 2^{i\beta}\|\nabla \Delta_i f\|_\infty |t-s|^{1/2}|x-x'|/\|\Delta_i f\|_\infty \\ &\lesssim 2^{i(\beta+1)}|x-x'|.\end{aligned}$$

Let us take $g_i(x) = 2^{(a+\beta)i}(1+|x|)^{d+1}$ with $a \geqslant 1$ and $a+\beta \geqslant c_\beta$. Hence, $C_{X^i_{s,t}} \lesssim 1$. By Lemma 3.2, there exists $b, c > 1$ such that

$$\exp(\mu |X^i_{s,t}(x)|^2) \lesssim 2^{b(a+\beta)i}(1+|x|)^{b(d+1)} S_{\mu b}(X^i_{s,t}).$$

Now, thanks to Lemma 3.1, there exist a', b', c', d' such that

$$\exp(\mu |X^i_{s,t}(x)|^2) \leqslant 2^{a'i}(1+|x|)^{b'}|t-s|^{-c'} K^w_{f,\beta}(d'\mu).$$

Hence by taking the logarithm, and by losing a small power of time and on i, we have

$$\begin{aligned}|T^w_{s,t}\Delta_i f(x)| \lesssim 2^{(-\beta+\varepsilon_1)i}\|\Delta_i f\|_\infty |t-s|^{1/2-\varepsilon_2}(1 + \log^{1/2}(1+|x|) \\ + \log^{1/2}(K^w_{f,\beta}(d'\mu))).\end{aligned} \tag{13}$$

Now we interpolate (13) and (12) and for all $\alpha > -\beta$, there exist $\rho > 0$ and $\gamma > 1/2$ such that

$$|T^w_{s,t}(\Delta_i f)(x)| \lesssim 2^{\alpha i}\|\Delta_i f\|_\infty |t-s|^\gamma (1 + \log^{1/2}(1+|x|) + \log(K^w_{f,\beta}(d'\mu)))^{1/2}. \quad \square$$

3.2.3. The operator T^w

We are now able to define the function $T^w f$ for all $f \in \mathscr{C}^\alpha$ (respectively $\mathscr{F}L^\alpha$) for all $\alpha > -\beta$, as soon as there exists $\lambda > 0$ small enough such that $K^w_{f,\beta}(\lambda)$ (respectively $K^w_\beta(\lambda)$) is finite. As already mentioned, it remains an open problem to study the boundedness of T^w as an operator with range in Besov spaces so we restrict ourselves to study the image of $T^w f$ for fixed f and with w in the support of the fBm law without any attempt to obtain estimates which are uniform in f. On the contrary, for $\mathscr{F}L^\alpha$, the estimate on Φ^w are good enough to define T^w as an operator on the whole space.

Definition 3.6. Let $\beta \in \mathbb{R}$, $\alpha > -\beta$ and let $f \in \mathscr{C}^\alpha$. We define

$$T^w_{s,t} f(x) = \sum_{i \geqslant -1} T^w_{s,t}(\Delta_i f)(x) = \lim_{N\to\infty} T^w_{s,t}(\pi_{\leqslant N} f)(x)$$

and

$$T^w_{s,t} g(x) = \lim_{\substack{h \in \mathscr{F}L^{0\vee\alpha} \\ h \xrightarrow{\mathscr{F}L^\alpha} g}} T^w_{s,t} h(x).$$

As these objects are defined by some limiting procedures, it is not straightforward that they exist. Furthermore, for the consistency of the definition, we must show that when $f \in \mathscr{F}L^\alpha$ these two limiting procedures give the same object, and that the limit does not depend of the choice of sequence (Δ_i). This is the purpose of the following theorem.

Theorem 3.7. *Let $\beta \in \mathbb{R}$ and let $\alpha > -\beta$.*

i. *Suppose that there exists λ_0 such that $K^w_\beta(\lambda_0) < +\infty$. Then for all $g \in \mathscr{F}L^\alpha$, $T^w g$ exists and does not depend on the choice of the sequence. Furthermore, for all $\lambda \leqslant \lambda_0$ we have*

$$|T^w_{s,t} g(x)| \lesssim |t-s|^\gamma N_\alpha(g)(1 + \log^{1/2} K^w_\beta(\lambda)).$$

Hence, $(T^w_{s,t})_{0 \leqslant s \leqslant t \leqslant 1}$ is well defined as a family of operators on $\mathscr{F}L^\alpha$.

ii. *For $f \in \mathscr{C}^\alpha$ suppose that there exists λ_0 such that $K^w_{f,\beta}(\lambda_0) < +\infty$. Then $T^w f$ exists and the following bound holds*

$$|T^w f_{s,t}(x)| \lesssim_\lambda |t-s|^\gamma \|f\|_\alpha (1 + \log^{1/2}(1+|x|) + \log^{1/2} K^w_{f,\beta}(\lambda)).$$

Furthermore let us suppose that for $g \in \mathscr{C}^\alpha$, $K^w_{g,\beta}(\lambda_0)$ is also finite, then for all $\lambda \leqslant \lambda_0/2$

$$|T^w_{s,t}(f-g)(x)| \lesssim |t-s|^\gamma \|f-g\|_\alpha (1 + \log^{1/2}(1+|x|) + (K^w_{f,\beta}(\lambda) + K^w_{g,\beta}(\lambda))).$$

iii. *These two limiting procedures are compatible when $f \in \mathscr{F}L^\alpha$.*

Proof. The proof is quite straightforward when $g \in \mathscr{F}L^\alpha$. Indeed, for h^1 and h^2 in $\mathscr{F}L^0 \cap \mathscr{F}L^\alpha$, we have

$$T^w_{s,t}(h_1 - h_2)(x) = \int_{\mathbb{R}^d} \mathrm{d}\xi (\hat{h}_1 - \hat{h}_2)(\xi) \exp(i\xi \cdot x) \Phi^w_{s,t}(\xi),$$

hence

$$|T^w_{s,t}(h_1 - h_2)(x)| \lesssim N_\alpha(h_1 - h_2)|t-s|^\gamma (1 + \log^{1/2}(K^w_\beta(\lambda_0)))$$

and the result of (i) follows.

Let us prove (ii). For $f \in \mathscr{C}^\alpha$ let us show as the quantity $T^w_{s,t}(\pi_{\leqslant N} f)(x)$ converges when $N \to +\infty$. Indeed, thanks to Lemma 3.5, for $\varepsilon > 0$ such that $-\beta < \alpha - \varepsilon < \alpha$, we have

$$\begin{aligned} |T^w_{s,t}(\pi_{\leqslant N} f)(x) - T^w_{s,t}(\pi_{\leqslant N+M} f)(x)| &\lesssim \sum_{i=N+1}^{N+M} |T^w_{s,t}(\Delta_i f)(x)| \\ &\lesssim C_{s,t,x} (\log^{1/2} K^w_{f,\beta}(\lambda_0)) \|f\|_\alpha 2^{-\varepsilon N}. \end{aligned}$$

Hence, $\left(T^w_{s,t}(\pi_{\leqslant N} f)(x)\right)_{N \geqslant 0}$ is Cauchy, and then the limit $T^w f$ exists. Furthermore we have the straightforward bound for all $\lambda < \lambda_0$

$$|T^w_{s,t}(\pi_{\leqslant N} f)(x)| \lesssim |t-s|^\gamma \|f\|_\alpha (1 + \log^{1/2}(1+|x|) + \log^{1/2}(K^w_{f,\beta}(\lambda)))$$

and the same bound holds as $N \to +\infty$. For f and g we have

$$|T^w_{s,t}(f-g)(x)| \lesssim |t-s|^\gamma \|f-g\|_\alpha (1 + \log^{1/2}(1+|x|) + \log^{1/2}(K^w_{f-g,\beta}(\lambda))),$$

but thanks to the definition of the constants, $K^w_{f-g,\beta}(\lambda) \lesssim K^w_{f,\beta}(2\lambda) + K^w_{g,\beta}(2\lambda)$, and the result follows. For (iii) let us consider $f \in \mathscr{F}L^\alpha$, we have

$$\begin{aligned} |T^w_{s,t}(\pi_{\leqslant N} f)(x) - T^w_{s,t}(\pi_{\leqslant N+M} f)(x)| &= |T^w_{s,t}(\pi_{\leqslant N} f - \pi_{\leqslant N+M} f)(x)| \\ &\lesssim C_{s,t,x} N_\alpha(\pi_{\leqslant N} f - \pi_{\leqslant N+M} f) \log^{1/2} K^w_\beta(\lambda_0) \end{aligned}$$

and as the sequence $(\pi_{\leqslant N} f)_N$ converges in $\mathscr{F}L^\alpha$ the limit also exists, and of course it is the same. Furthermore, for two functions in $\mathscr{F}L^{\alpha\vee 0}$,

$$|T^w_{s,t}(\pi_{\leqslant N} f)(x) - T^w_{s,t}(\pi_{\leqslant N+M} f)(x)| \lesssim_{s,t,x} N_\alpha(f-g)$$

and the limiting procedure in the $\mathscr{F}L^\alpha$ case is correct. $\square$

Remark 3.8. The definition of $T^w f$ given above seems to depend on the choice of the Littlewood–Paley decomposition $(\Delta_i)_i$. It is indeed the fact. When we will consider w being a stochastic process, this will lead us to a choice of a version of this averaging process defined almost surely. In fact, if $(\widetilde{\Delta}_i)_i$ is another sequence of Littlewood–Paley operators, and $\tilde{K}_i$ is the associated integral kernels, we have

$$\begin{aligned}
|K_j| * \log(1+|\cdot|)(x) &\leqslant \log(1+|x|) \\
&\qquad + \int_{\mathbb{R}^d} \mathrm{d}y |\tilde{K}_j(y)| \, |\log(1+|x-y|) - \log(1+|x|)| \\
&\lesssim \log(1+|x|) + 2^{-j} \int_{\mathbb{R}^d} |2^j y| 2^{jd} \tilde{K}(2^j y) \mathrm{d}y \\
&\lesssim 1 + \log(1+|x|).
\end{aligned}$$

Hence, there exist two constants $c < C$ such that for all $\alpha > -\beta$ and for all $\varepsilon > 0$ small enough, we have

$$\begin{aligned}
|(T^w_{s,t} \tilde{\Delta}_j f)(x)| &\leqslant \sum_{c+j \leqslant i \leqslant C+j} |(\tilde{\Delta}_j T^w \Delta_i f)(x)| \\
&\lesssim 2^{-j\varepsilon} \|f\|_\alpha |t-s|^\gamma \sum_{c+j \leqslant i \leqslant C+j} \int_{\mathbb{R}^d} \tilde{K}_j * (1 + \log(1+|\cdot|) \\
&\quad + K^w_{f,\beta}(\lambda_0))(x) \\
&\lesssim 2^{-j\varepsilon} \|f\|_\alpha (1 + \log(1+|x|) + K^w_{f,\beta}(\lambda_0)),
\end{aligned}$$

which gives the convergence of $T^w \tilde{\pi}_{\leqslant N} f$ to a limit we called $T^{w,\tilde{\Delta}} f$, and with the same stochastic constant $K^w_{f,\beta}(\lambda_0)$.

In order to apply the results of the section related to the Young integral, it is necessary to have a better understanding of the space regularity of an average function. Thanks to the property of the operator T^w, as soon as we ask f to be regular, this regularity will hold. Furthermore the definition of T^w allows us to differentiate it whenever f is regular enough, and the constant is finite. Namely we have the following propositions.

Proposition 3.9. *Let $\nu \in [0,1]$, $\alpha > -\beta$, and $f \in \mathscr{F}L^{\alpha+\nu}$ (respectively in $\mathscr{C}^{\alpha+\nu}$). Furthermore we suppose that there exists $\lambda_0 > 0$ such that $K^w_\beta(\lambda_0) < \infty$ (respectively $K^w_{\nabla f,\beta}(\lambda_0) < +\infty$). Then $T^w f \in \mathscr{C}^{\gamma,\nu}_b$ (respectively $T^w f \in \mathscr{C}^{\gamma,\nu,\psi}$ where $\psi(r) = 1 + \log^{1/2}(1+r)$) and the following bounds hold*

$$|T^w_{s,t} f(x) - T^w_{s,t} f(y)| \lesssim N_{\alpha+\nu}(b) |t-s|^\gamma |x-y|^\nu (1 + \log^{1/2} K^w_\beta(\lambda_0))$$

Q9 *(respectively*

$$\begin{aligned}
|T^w_{s,t} f(x) - T^w_{s,t} f(y)| &\lesssim |t-s|^\gamma |x-y|^\nu \|f\|_{\alpha+\nu} (\psi(|x|+|y|) \\
&\qquad + \log^{1/2} K^w_{\nabla f,\beta}(\lambda_0) + \log^{1/2} K^w_{f,\beta}(\lambda_0))).
\end{aligned}$$

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Proof. For $f \in \mathscr{C}^{\alpha+\nu}$, and for all $\beta < \alpha' < \alpha$

$$
\begin{aligned}
|T^w \Delta_i f(x) - T^w \Delta_i f(y)| &= \left| \int_0^1 \nabla T^w \Delta_i f(r(x-y)+y) \cdot (x-y) \mathrm{d}r \right| \\
&\leqslant |x-y| \sup_{r\in[0,1]} |T^w \Delta_i \nabla f(r(x-y)+y)| \\
&\lesssim 2^{i\alpha} \|\Delta_i \nabla f\|_\infty |x-y||t-s|^\gamma (1 + \sup_{r\in[0,1]} \log^{1/2}(1+|r(x-y)+y|) + K^w_{\nabla f,\beta}(\lambda_0)) \\
&\lesssim 2^{i(\alpha+1)} \|\Delta_i f\|_\infty |x-y|\,|t-s|^\gamma (1 + \log^{1/2}(1+|x|+|y|) + K^w_{\nabla f,\beta}(\lambda_0)).
\end{aligned}
$$

Furthermore, we also have

$$
\begin{aligned}
|T^w \Delta_i f(x) - T^w \Delta_i f(y)| \lesssim 2^{i\alpha} \|\Delta_i f\|_\infty |t-s|^\gamma (1 + \log^{1/2}(1+|x|) \\
+ \log^{1/2}(1+|y|) + K^w_{f,\beta}(\lambda_0)),
\end{aligned}
$$

and by interpolation we have the bound for $T^w \Delta_i f$. The argument of Theorem 3.7 gives us the result. A similar argument holds when $f \in \mathscr{F}L^\alpha$. □

The next proposition shows that the definition of the averaging operator T is compatible with the space differential in the Hölder spaces.

Proposition 3.10. *Let $r \in \mathbb{N}^d$, $|r| = r_1 + \cdots .r_d$, and $\alpha > -\beta$. Suppose furthermore that for $f \in \mathscr{C}^{\alpha+|r|}$ there exists λ_0 such that $K^w_{D^{|r|} f,\lambda_0}(\lambda_0) < +\infty$ respectively there exists $\lambda_0 > 0$ such that $K^w_\beta(\lambda) < +\infty$. Then the derivative $\partial^r T^w f$ is well defined and we have $\partial^r T^w f = T^w \partial^r f$.* Q10

Proof. First, let us take $f \in \mathscr{F}L^{\alpha+r}$. For $N \geqslant 0$, the projection $\pi_{\leqslant N}$ is a convolution operator, hence

$$
\partial^r T^w(\pi_{\leqslant N} f)(x) = T^w \pi_{\leqslant N}(\partial^r f)(x).
$$

But for $f \in \mathscr{F}L^{\alpha+|r|}$, $\pi_{\leqslant N}(\partial^r f) \to^{\mathscr{F}L^\alpha} \partial^r f$, which gives the result for $f \in \mathscr{F}L^\alpha$. Now take $-\beta < \alpha' < \alpha$. We know that for all $f \in \mathscr{C}^{\alpha+r}$, $\pi_{\leqslant N}\partial^r f \to^{\mathscr{C}^{\alpha'}} \partial^r f$, and the result follows for $f \in \mathscr{C}^\alpha$. □

4. Averages along fractional Brownian paths

4.1. Fractional Brownian motion case

The results of Lemmas 3.4 and 3.5 show that in order to control the irregularity constant of fBm paths it is enough to prove that there exist $\lambda > 0$ and $\alpha \in \mathbb{R}$ such that the random variable $e^{\lambda \|B^H\|^2_\infty} K^{B^H}_\alpha(\lambda)$ is almost surely finite when B^H is a continuous random path with the law of the fBm. Then we only have to consider the following two quantities:

$$
\exp(\lambda \|B^H\|^2_\infty), \qquad \exp(\lambda(1+|\xi|)^\alpha |\Phi^{B^H}_{s,t}(\xi)|^2/|t-s|).
$$

If the expectation of those quantities are bounded independently of s, t, x, ω, ξ then the expectations of $K^{B^H}_f$ and $e^{\lambda\|B^H\|_\infty} K^{B^H}_\alpha(\lambda)$ are finite and the variable are finite almost surely. For $\exp(\lambda\|B^H\|^2_\infty)$ it is an application of a well known theorem due to Fernique.

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Theorem 4.1. *Let X be a Gaussian random variable which takes values in a Banach space $(\mathcal{B}, \|.\|)$. Then there exists a constant $\mu > 0$ such that*

$$\mathbb{E}[\exp(\mu\|X\|^2)] < +\infty.$$

Remark 4.2. This holds for the fractional Brownian motion of Hurst parameter $H \in (0, 1)$ and for the Banach spaces $\left(\mathscr{C}^{H-\varepsilon}([0, 1], \mathbb{R}^d), \|.\|_{0,H-\varepsilon}\right)$ and for $(C([0, 1], \mathbb{R}^d), \|.\|_{\infty,[0,1]})$.

To control the square exponential integrability of $\Phi^{B^H}_{s,t}(\xi)$ we devised a novel technique based on an elementary application of Hoeffding inequality for discrete martingale increments. This bypasses the explicit Gaussian computations or the computations based on Malliavin calculus usual in the studies involving the fBm. The following theorem then gives general estimates for additive functionals of the fBm of the form

$$\int_s^t f_u(B^H_u)\mathrm{d}u,$$

where $f : [0, 1] \times \mathbb{R}^d \to \mathbb{R}$ is a measurable and bounded function. Note that the following theorem suggest in general that such functionals have the same Gaussian deviation behaviour of Brownian martingales.

Theorem 4.3. *Let $B^H = (B^{H,(1)}, \ldots, B^{H,(d)})$ be a d-dimensional fractional Brownian motion of Hurst parameter $H \in (0, 1)$ and let f be a function bounded by 1 and such that*

$$C_f := \sup_u \int_0^\infty |P_{t^{2H}} f_u|_\infty \mathrm{d}t < +\infty,$$

where P is the heat kernel on $\mathbb{R}^d$. Then for $\mu > 0$ small enough independent of f we have

$$\sup_{t\neq s} \mathbb{E}\left[\exp\left(\mu\left|\int_s^t f_u(B^H_u)\mathrm{d}u\right|^2 \Big/ (|t-s|C_f)\right)\right] < +\infty.$$

Proof. The fBm B^H can be represented as a stochastic integral over a d-dimensional standard Brownian motion $W = (W^{(1)}, \ldots, W^{(d)})$ defined on the whole $\mathbb{R}$ (with $W_0 = 0$):

$$B^{H,(i)}_u = \int_{-\infty}^u (K(u, r) - K(0, r))\mathrm{d}W^{(i)}_r,$$

where $K(u, r) = (u - r)_+^{H-1/2}/\Gamma(H + 1/2)$. Let $(\mathscr{F}_t)_{t\in\mathbb{R}}$ be the natural filtration of $(W_t)_{t\in\mathbb{R}}$. For $v \leqslant u$ we have the decomposition

$$\begin{aligned} B^{H,(i)}_u &= \int_{-\infty}^u (K(u, r) - K(0, r))\mathrm{d}W^{(i)}_r \\ &= \int_v^u K(u, r)\mathrm{d}W^{(i)}_r + \int_{-\infty}^v (K(u, r) - K(0, r))\mathrm{d}W^{(i)}_r \\ &= W^{1,(i)}_{u,v} + W^{2,(i)}_{u,v}, \end{aligned}$$

where the random variable $W^{1,(i)}_{u,v}$ is independent of $\mathcal{F}_v$ and $W^{2,(i)}_{u,v}$ is $\mathcal{F}_v$ measurable. We define $W^j = (W^{j,(1)}, \ldots, W^{j,(d)})$ for $j \in \{1, 2\}$ and in the following we will note abusively

$\mathrm{Var}(W^j) = \mathrm{Var}(W^{j,(1)})$. Now there is two cases we have to consider. Suppose first that $t - s/C_f \leqslant 1$. Then

$$\left|\int_s^t f_u(B_u^H)\mathrm{d}u\right| \leqslant |t-s|^2 \leqslant |t-s|C_f$$

and the result follows in that case. Suppose now that $|t-s|C_f^{-1} \geqslant 1$. Let $N \in \mathbb{N}$ to be specify later. For $n \in \{0,\dots,N\}$, let us define $t_n = s + (t-s)n/N$ and

$$Z_n = \mathbb{E}\left[\int_s^t f_u(B_u^H)\mathrm{d}u|\mathcal{F}_{t_{n+1}}\right] - \mathbb{E}\left[\int_s^t f_u(B_u^H)\mathrm{d}u|\mathcal{F}_{t_n}\right].$$

Thanks to the previous decomposition of the fractional Brownian motion, we are able to bound Z_n and to apply Hoeffding lemma to the sum of the martingale increments $(Z_n)_{1\leqslant n\leqslant N}$. Let $S_N = \sum_{n=0}^{N-1} Z_n$, then

$$\int_s^t f_u(B_u^H)\mathrm{d}u = S_N + \mathbb{E}\left[\int_s^t f_u(B_u^H)\mathrm{d}u\middle|\mathcal{F}_s\right]. \tag{14}$$

Let us first estimate the conditional expectation in Eq. (14): for all $u \geqslant 0$ we have

$$\begin{aligned}\left|\mathbb{E}\left[\int_s^t f_u(B_u^H)\mathrm{d}u\middle|\mathcal{F}_s\right]\right| &= \left|\int_s^t P_{Var(W_{u,s}^1)} f_u(W_{u,s}^2)\mathrm{d}u\right| \\ &\leqslant \int_s^t |P_{\mathrm{Var}(W_{u,s}^1)} f_u|_\infty \mathrm{d}u \leqslant C_f < +\infty\end{aligned}$$

since $\mathrm{Var}(W_{u,s}^1) = C(u-s)^{2H}$. But we also have the trivial bound

$$\left|\mathbb{E}\left[\int_s^t f_u(B_u^H)\mathrm{d}u\middle|\mathcal{F}_s\right]\right| \leqslant |t-s|.$$

Hence

$$\left|\mathbb{E}\left[\int_s^t f_u(B_u^H)\mathrm{d}u\middle|\mathcal{F}_s\right]\right| \leqslant |t-s|^{1/2} C_f^{1/2}. \tag{15}$$

Next we bound Z_n by decomposing it into three pieces which are easier to estimate. We have

$$\begin{aligned}U_n &= \int_{t_n}^t \mathbb{E}[f_u(B_u^H)|\mathcal{F}_{t_n}]\mathrm{d}u = \int_{t_n}^t \mathbb{E}[f_u(W_{u,t_n}^1 + W_{u,t_n}^2)|\mathcal{F}_{t_n}]\mathrm{d}u \\ &= \int_{t_n}^t P_{\mathrm{Var}(W_{u,t_n}^1)} f_u(W_{u,t_n}^2)\mathrm{d}u \\ &= \int_{t_n}^{t_{n+1}} P_{\mathrm{Var}(W_{u,t_n}^1)} f_u(W_{u,t_n}^2)\mathrm{d}u + \int_{t_{n+1}}^t P_{\mathrm{Var}(W_{u,t_n}^1)} f_u(W_{u,t_n}^2)\mathrm{d}u.\end{aligned}$$

Hence

$$Z_n = \int_{t_n}^{t_{n+1}} f_u(B_u^H)\mathrm{d}u + U_{n+1} - U_n,$$

moreover

$$|U_n| \leqslant \int_{t_n}^t |P_{\mathrm{Var}(W_{u,t_n}^1)} f_u(W_{u,t_n}^2)|\mathrm{d}u \leqslant \int_{t_n}^t |P_{\mathrm{Var}(W_{u,t_n}^1)} f_u|_\infty \mathrm{d}u \leqslant C_f < +\infty$$

and of course $|\int_{t_n}^{t_{n+1}} f_u(B^H_u)\mathrm{d}u| \leqslant (t-s)/N$, which implies that $|Z_n| \lesssim (t-s)/N + C_f$. By the standard Hoeffding inequality we obtain

$$\mathbb{P}(|S_N| > \lambda) \lesssim \exp(-2\lambda^2/((t-s)N^{-1/2} + N^{1/2}C_f)^2).$$

Hence for $0 \leqslant \nu < 1$, we have

$$\mathbb{E}[\exp(2\nu|S_N|^2/((t-s)N^{-1/2} + N^{1/2}C_f)^2)] \lesssim \nu/(1-\nu) + 1$$

Now, we can choose $N = [1 + |t-s|/C_f]$, hence

$$((t-s)N^{-1/2} + N^{1/2}C_f) \lesssim |t-s|C_f,$$

and thanks to (15) we have

$$\mathbb{E}\left[\exp\left(\mu\left|\int_s^t f_u(B^H_u)\mathrm{d}u\right|^2 \Big/ (|t-s|C_f)\right)\right]$$
$$\lesssim \mathbb{E}[\exp(C\mu|S_N|^2/(|t-s|C_f))] \lesssim_\mu 1. \quad \square$$

As an immediate corollary we have the wanted result for the ρ-irregularity constant for the fractional Brownian motion.

Corollary 4.4. *For λ small enough,*

$$\mathbb{E}\exp(\lambda(1+|\xi|)^{1/H}|\Phi^{B^H}_{s,t}(\xi)|^2/|t-s|) \leqslant C < +\infty$$

uniformly in ξ, t, s.

Proof. When $\xi \leqslant 1$, we have

$$\exp(\lambda(1+|\xi|)^{1/H}|\Phi^{B^H}_{s,t}(\xi)|^2/|t-s|) \leqslant \exp(\lambda 2^{1/H}).$$

For $|\xi| \geqslant 1$, we have $(1+|\xi|)^{1/H} \lesssim |\xi|^{1/H}$. But we also have

$$|P_{t^{2H}} f(x)| = |\mathbb{E}[\exp(i\xi(x + B^H_t))]| = \exp(-|\xi|^2 t^{2H}/2),$$

therefore $|\xi|^{-1/H} \lesssim C_{f_\xi} \lesssim |\xi|^{-1/H}$. Finally there exists a constant $C > 0$ such that

$$\mathbb{E}\exp(\lambda(1+|\xi|)^{1/H}|\Phi^{B^H}_{s,t}(\xi)|^2/|t-s|) \leqslant \mathbb{E}\exp(C\lambda|\Phi^{B^H}_{s,t}(\xi)|^2/(|t-s|C_{f_\xi}))$$

and for λ small enough, thanks to Theorem 4.3 the right hand side is bounded by a constant independent of ξ, s, t. $\square$

We are now in condition to prove Theorem 1.4 (ρ-irregularity of the fBm paths for all $\rho < 1/2H$).

Proof of Theorem 1.4. By Lemma 3.4 we have

$$\|\Phi^{B^H}\|_{\mathcal{W}^{\rho,\gamma}_1} = \sup_{\xi\in\mathbb{R}^d}\sup_{0\leqslant s<t\leqslant 1}(1+|\xi|)^\rho\frac{|\Phi^{B^H}_t(\xi) - \Phi^{B^H}_s(\xi)|}{|s-t|^\gamma}$$
$$\lesssim 1 + \log^{1/2}(e^{\lambda\|B^H\|_\infty}K^{B^H}_{1/2H}(\lambda)).$$

Moreover by Theorem 4.1 the quantity $e^{\lambda \|B^H\|_\infty^2}$ is almost surely finite and by Corollary 4.4 we readily have that

$$\mathbb{E}[K_{1/2H}^{B^H}(\lambda)] < +\infty$$

as soon as λ is small enough. $\square$

Remark 4.5. A byproduct of the proof of Theorem 1.4 is that the irregularity constant $\|\Phi^{B^H}\|_{\mathcal{W}_1^{\rho,\gamma}}$ is exponentially square integrable, as easily shown: for small $\lambda > 0$ and all $\gamma > 1/2$ and $\rho < 1/2H$ we have

$$\mathbb{E}\left[e^{\lambda \|\Phi^W\|^2_{\mathcal{W}_1^{\rho,\gamma}}}\right] < +\infty.$$

When we consider the spaces $\mathscr{C}^\alpha$ instead of $\mathscr{F}L^\alpha$,the ρ-irregularity of the fractional Brownian path is not enough. Nevertheless Theorem 4.3 allows us to give the correct bound for $T^{B^H}\Delta_i f$ in order to define $T^{B^H} f$ for any $f \in \mathscr{C}^\alpha$.

Corollary 4.6. *Let B^H be a d-dimensional fractional Brownian motion of Hurst parameter $H \in (0,1)$. There exists $\lambda > 0$ such that for $f \in C([0,T]; \mathcal{S}'(\mathbb{R}^d))$, all $i \geqslant -1$*

$$\sup_{x\in\mathbb{R}^d, i\geqslant -1, s\neq t} \mathbb{E}[\exp(\lambda 2^{i/H} |T^w(\Delta_i f)(x)|^2 / (|t-s| \, \|\Delta_i f\|_\infty^2))] < +\infty.$$

Proof. The function $g_u = \Delta_i f_u / \|\Delta_i f\|_\infty$ is bounded by 1. Furthermore, for $i \geqslant 0$, $\operatorname{supp}\hat{g}_u \subset 2^i \mathcal{A}$ where $\mathcal{A}$ is an annulus. Hence by the Lemma 2.4 page 54 of [3], there exists a constant $c > 0$ independent of g such that

$$\|P_{t^{2H}} g_u\|_\infty \lesssim \exp(-ct^{2H} 2^{2i}),$$

hence $C_g \lesssim 2^{-i/H}$ and the result follows immediately by applying Theorem 4.3 to g.

When $i = -1$, we have

$$\left|\int_s^t \Delta_{-1} f(B_u^H) \mathrm{d}u\right|^2 \leqslant 2^{1/H} |t-s| \, \|\Delta_{-1} f\|_\infty^2$$

and the result follows. $\square$

The result is exactly the needed hypothesis of Theorem 3.7, but also Propositions 3.9 and 3.10, depending on the regularity of f. Hence, the averaging operator T^{B^H}, or its finite dimensional marginals depending of the space, is well defined and has the right range for applying results of Section 2. This operator is defined almost surely, hence, almost surely (depending of b when $b \in \mathscr{C}^\alpha$) the following equation has a solution

$$\theta_u = \theta_0 + \int_0^t T_{\mathrm{d}u}^{B^H} b(\theta_u),$$

when $b \in \mathscr{C}^\alpha$. The existence of such a solution is guaranteed by Theorem 2.9. Furthermore, when $b \in \mathscr{C}^{\alpha+2}$, $\alpha > -1/2H$ (or $\mathscr{C}^{\alpha+\gamma+1}$) there is uniqueness and the flow is Lipschitz-continuous, thanks to Corollary 2.18. As the set where $T^w b$ is not defined does not depend on b when $b \in \mathscr{F}L^\alpha$, those results does not depend on b. The uniqueness for $b \in \mathscr{C}^{\alpha+1}$ or $b \in \mathscr{F}L^{\alpha+1}$ is a more probabilistic argument and is the subject of the following section.

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Remark 4.7. If the regularity of b is not enough to guarantee uniqueness by the above arguments the solution constructed via Theorem 2.9 lacks, a priori, measurability with respect to B^H. If a measurable solution is needed the fix-point argument of Theorem 2.9 has to be repeated in a space of random processes, for example in $L^p(\Omega, \mathscr{C}^\gamma([0,1];\mathbb{R}^d))$.

4.2. *Averaging for absolutely continuous perturbations of the fBm*

In this section, we analyse the properties of the averaging operator along a path of the form $B^H + \theta^n$ where θ^n is a solution to the approximate equation $d\theta^n = T^{B^H}_{du} b^n(\theta^n)$. In order to do so we will use a version of the Girsanov theorem for fractional Brownian motion. The results holds of course for both types of functions spaces $\mathscr{C}^{\alpha+1}$ and $\mathscr{F}L^{\alpha+1}$. We will only give the proof for $b \in \mathscr{C}^{\alpha+1}$, and give some comments for the case $\mathscr{F}L^{\alpha+1}$. Let $(b^n)_{n\geq 1}$ be a sequence of smooth vector fields such that $\|b^n\|_\alpha \leqslant C$ uniformly in $n \geq 1$. By a standard fixed point argument, it is well known that the following equation

$$X^n_t = x_0 + \int_0^t b^n_s(X^n_s)ds + B^H_t \tag{16}$$

has an adapted solution X^n (to the standard filtration of the fractional Brownian motion).

Here we analyse the averaging constant of X_n and we prove that it satisfy the requirements of Theorem 2.17 implying uniqueness of the limit ODE for $b \in \mathscr{C}^{\alpha+1}$ and convergence of X^n to this unique solution. Furthermore, if we consider, as in Section 2.2.1, the averaged translation by $\theta^n = X^n - B^H$, we only have to check the hypothesis of Theorem 2.17. Indeed, if θ is the solution of

$$\theta_t = \theta_0 + \int_0^t T^{B^H}_{du} b(\theta_u)$$

and θ^n is the solution of

$$\theta^n_t = \theta_0 + \int_0^t T^{B^H}_{du} b^n(\theta^n_u),$$

then $\theta^n + B^H_u = X^n$ and by the averaged translation by θ^n, as $\tau_{\theta^n} T^{B^H} b = T^{X_n} b$. Then $\theta - \theta^n$ is the solution of the following Young equation

$$(\theta - \theta^n)_t = \int_0^t T^{X^n}_{du} b(\theta_u - \theta^n_u) + \int_0^t T^{B^H}_{du} b^n(\theta^n_u).$$

These considerations are the motivation to introduce the comparison principle based on averaged translations in the proof of uniqueness in Section 2.2.

Below we will take advantage of the absolute continuity of the law of X^n w.r.t. the law of the fractional Brownian motion B^H to transfer the averaging properties of the fractional Brownian motion to the stochastic process X^n. This approach is an extension of an observation of Davie [7] to the fractional Brownian motion's context.

A drawback of this approach is that the exceptional set will necessarily depend on the initial point x_0 and on the vector field b. This prevents us from easily applying the uniqueness result to the case of random b and to the analysis of the flow of the ODE.

The computation of the Radon–Nikodym derivative between the law of X^n and the law of B^H will result in a Girsanov transform. For technical reasons we will do this transformation only on

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a subinterval $[0, T_{Gir}] \subset [0, 1]$. For b^n regular enough, and as X^n is regular enough, according to Nualart and Ouknine [18], there exist a Brownian motion W adapted to the filtration associated with B^H and a probability $\mathbb{P}_n$ such that the process $(X^n_t)_{t\in[0,T_{Gir}]}$ is a fractional Brownian motion of Hurst parameter H, where

$$\frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}} = \exp\left(-\int_0^{T_{Gir}} H^n_t \cdot \mathrm{d}W_t - \frac{1}{2}\int_0^{T_{Gir}} |H^n_t|^2 \mathrm{d}t\right),$$

where for $H \geq \frac{1}{2}$

$$H^n_t = \frac{t^{H-\frac{1}{2}}}{\Gamma\left(\frac{3}{2}-H\right)}\left(t^{1-2H} b^n(t, X^n_t) + \left(H-\frac{1}{2}\right)\int_0^t \frac{t^{\frac{1}{2}-H} b^n_t(X^n_t) - s^{\frac{1}{2}-H} b^n_s(X^n_s)}{(t-s)^{H+\frac{1}{2}}}\mathrm{d}s\right)$$

and for $H < \frac{1}{2}$

$$H^n_t = \frac{t^{H-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-H\right)}\int_0^t (s(t-s))^{\frac{1}{2}-H} b^n_s(X^n_s)\mathrm{d}s.$$

Thanks to that Girsanov transform, the almost sure bound for $T^{B^H}_b$ can be used to estimate $T^{X_n}_b$ since $\mathbb{P}_n$ and $\mathbb{P}$ are equivalent.

Lemma 4.8. *Let $0 \geqslant \alpha > -1/2H$. There exist a constant $\lambda > 0$ and a constant C_λ independent of n such that for all $b \in \mathscr{C}^{H-\varepsilon}\left([0, 1]; \mathscr{C}^{\alpha+1}\right) \cap \mathscr{C}^{\alpha+1}\left(\mathbb{R}^d, \mathscr{C}^{H-1/2+\varepsilon}([0, T])\right)$,*

$$\mathbb{E}[K^{X_n}_{b,1/2H}(\lambda)] \leqslant C_\lambda.$$

Until the end of the section, we will only consider $K^{X_n}_{b,1/2H}$. For simplicity we only write it as $K^{X_n}_b$.

Proof. Let $K > 0$. By using the notation above, we have

$$\begin{aligned}
\mathbb{E}[K^{X_n}_b(\lambda)]^2 &= \mathbb{E}_{\mathbb{P}_n}\left[K^{X_n}_b(\lambda)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_n}\right]^2 \\
&\leqslant \mathbb{E}_{\mathbb{P}_n}[K^{X_n}_b(\lambda)^2]\mathbb{E}_{\mathbb{P}_n}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_n}\right)^2\right] \\
&\lesssim \mathbb{E}[K^{B^H}_b(2\lambda)]\mathbb{E}_{\mathbb{P}_n}\left[\exp\left(2\int_0^T H^n_t \mathrm{d}W_t + \int_0^T H^n_t \mathrm{d}t\right)\right]
\end{aligned}$$

where we have used that under $\mathbb{P}_n$, X^n is a fractional Brownian motion of same Hurst parameter H. If ρ is small enough the first term is finite by the above results. To prove the lemma, it is sufficient to prove that

$$\mathbb{E}_{\mathbb{P}_n}\left[\exp\left(2\int_0^T H^n_t \mathrm{d}W_t + \int_0^T |H^n_t|^2 \mathrm{d}t\right)\right]$$

is bounded by a constant independent of n. As W is a Brownian motion, it is enough to bound

$$\mathbb{E}_{\mathbb{P}_n}\left[\exp\left(\int_0^T |H^n_t|^2 \mathrm{d}t\right)\right].$$

The arguments are quite different depending whether $H > 1/2$ or $H < 1/2$. First suppose that $H < \frac{1}{2}$.

$$\begin{aligned}
|H_t^n|^2 &= \left| K_H^{-1}\left(\int_0^{\cdot} b_s^n(X_s^n)\mathrm{d}s\right)(t)\right|^2 \\
&= \left| \frac{t^{H-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-H\right)} \int_0^t (s(t-s))^{\frac{1}{2}-H} b_s^n(X_s^n)\mathrm{d}s \right|^2 \\
&= \left| \frac{-\left(\frac{1}{2}-H\right)t^{H-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-H\right)} \int_0^t (s(t-s))^{-\left(\frac{1}{2}+H\right)}(t-2s) \underbrace{\int_0^s b_u^n(X_u^n)\mathrm{d}u}_{(T_s^{X^n}b^n)(0)}\,\mathrm{d}s \right|^2 \\
&\lesssim t^{2H}\int_0^t (s(t-s))^{-(1+2H)}|t-2s|\,|(T_s^{X^n}b^n)(0)|^2\mathrm{d}s \\
&\lesssim \|b^n\|_{\mathscr{C}^\alpha}^2(1+\log^{1/2}(K_b^{X_n}(\lambda)))^2 t^{2H}\int_0^t (s(t-s))^{-(1+2H)}|t-2s|s^{2\gamma}\mathrm{d}s \\
&\lesssim \|b^n\|_{\mathscr{C}^\alpha}^2(1+\log(K_b^{X_n}(\lambda)))t^{2(\gamma-H)}\int_0^1 (u(1-u))^{-(1+2H)}|1-2u|u^{2\gamma}\mathrm{d}s \\
&\leqslant C(b,H,\gamma,\lambda)(1+\log(K_b^{X_n}(\lambda))).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_n}\left[\exp\left(\int_0^T |H_t^n|^2\mathrm{d}t\right)\right] &\lesssim \mathbb{E}_{\mathbb{P}_n}[K_b^{X_n}(C\lambda)] \\
&\lesssim \mathbb{E}[K_b^{B^H}(C\lambda)].
\end{aligned}$$

For λ small enough, this quantity is bounded, and the lemma is proved in this case.

For $H \geqslant \frac{1}{2}$, b^n is $(1+\alpha)$-Hölder continuous and $\|b^n\|_{\mathscr{C}^{\alpha+1}} \lesssim \|b\|_{\mathscr{C}^{\alpha+1}}$. Furthermore, $|(b^n(0,0))_n|$ is uniformly bounded, then

$$\begin{aligned}
|H_t^n| &\lesssim \left| t^{H-1/2}\left(t^{1-2H}b^n(t,X_t^n) + \int_0^t \frac{t^{1/2-H}b_t^n(X_t^n) - s^{1/2-H}b_s^n(X_s^n)}{(t-s)^{H+1/2}}\mathrm{d}s\right)\right| \\
&\lesssim t^{1/2-H}(\|b^n\|_{\alpha+1} + \|b_\cdot^n(0)\|_\infty)(|X_t^n - X_0^n|^{1+\alpha}+1) \\
&\quad + t^{H-1/2}\int_0^t (t-s)^{-(H+1/2)}|(t^{1/2-H}-s^{1/2-H})(b_t^n(X_t^n)+b_s^n(X_s^n))|\mathrm{d}s \\
&\quad + t^{H-1/2}\int_0^t |(t-s)^{-(H+1/2)}(t^{1/2-H}+s^{1/2-H})(b_t^n(X_t^n)-b_s^n(X_s^n))|\mathrm{d}s.
\end{aligned}$$

The first term is bounded by $(\|b^n\|_\alpha + \|b^n_\cdot(0)\|_\infty)(\|X^n\|^{1+\alpha}_{H-\varepsilon} + 1)t^{1/2-H}$ and is integrable. The second is bounded by

$$
\begin{aligned}
& t^{H-1/2} \int_0^t \mathrm{d}s (t-s)^{-(H+1/2)} |t^{1/2-H} - s^{1/2-H}| \\
& \quad \times (|b^n_t(X^n_t) - b^n(t,0)| + |b^n_s(X^n_s) - b^n_t(0)| + 2\|b_\cdot(0)\|_\infty) \\
& \lesssim t^{H-1/2+1-H-1/2+1/2-H} \int_0^1 (1-u)^{-H-1/2}(1-u^{1/2-H}) \\
& \quad \times (\|b^n\|_\alpha + \|b^n_\cdot(0)\|_\infty)\|X^n\|^{\alpha+1}_\infty \\
& \lesssim_{x_0} t^{1/2-H}((\|b^n\|_\alpha + \|b^n_\cdot(0)\|_\infty)(\|X^n\|^{\alpha+1}_{H-\varepsilon} + 1) \\
& \quad + \|b_\cdot(X^n_s)\|_{H-1/2+\varepsilon}|t-s|^{H-1/2+\varepsilon}).
\end{aligned}
$$

The third term is bounded by

$$
\begin{aligned}
& t^{H-\frac{1}{2}} \int_0^t \left| (t-s)^{-\left(H+\frac{1}{2}\right)} s^{\frac{1}{2}-H} (|b^n_t(X^n_t) - b^n_t(X^n_s)| + |b^n_t(X^n_s) - b^n_s(X^n_s)|) \right| \mathrm{d}s \\
& \lesssim (\|b^n\|_\alpha + \|b(.,0)\|_{H-1/2+\varepsilon})(\|X^n\|^{1+\alpha}_{H-\varepsilon} + 1)t^{H-\frac{1}{2}} \\
& \quad \times \int_0^t (t-s)^{-\left(H+\frac{1}{2}\right)} s^{\frac{1}{2}-H} \left(|t-s|^{H-\frac{1}{2}+\varepsilon} + |t-s|^{(\alpha+1)(H-\varepsilon)} \right) \mathrm{d}s \\
& \lesssim (\|b^n\|_\alpha + \|b^n_\cdot(0)\|_{H-1/2+\varepsilon} + \|b^n_\cdot(0)\|_\infty) \\
& \quad \times (\|X^n\|^{1+\alpha}_{H-\varepsilon} + 1)t^\varepsilon \int_0^1 (1-u)^{-1+\varepsilon} u^{-H+1/2} \mathrm{d}u.
\end{aligned}
$$

Finally, we choose b^n such that $(\|b^n\|_\alpha + \|b^n_\cdot(0)\|_{H-1/2+\varepsilon} + \|b^n_\cdot(0)\|_\infty) \lesssim_b 1$, hence

$$
|H^n_t| \lesssim C_{b,x_0}(\|X^n\|^{1+\alpha}_{H-\varepsilon} + 1)t^\varepsilon.
$$

Under $\mathbb{P}_n$, X^n is a fractional Brownian motion of Hurst parameter H. Thanks to Fernique theorem, $\|X^n\|^{1+\alpha}_{H-\varepsilon}$ is exponentially integrable in $\mathbb{P}_n$ and the result follows. □

This bound in L^1 is not enough to use Theorem 2.17, as we need an almost surely, uniformly in n, bound for $\|T^{X^n} b\|_{\mathscr{C}^{\gamma,1,\psi}}$. Nevertheless, by using the results of Section 3.2.3, we already know that

$$
\exp\left(C\|T^{X^n} b\|_{\mathscr{C}^{\gamma,1,\psi}} \right) \lesssim 1 + K^{X^n}_b(\lambda).
$$

We have all the tools to prove the following theorem

Theorem 4.9. *Assume that $b \in \mathscr{C}^{\alpha+1}$. There then there exists $\lambda > 0$ and sequence of smooth vector fields $(b^n)_n$ such that $b^n \to b$ in $\mathscr{C}^{\alpha'}$ for all $\alpha' < \alpha$ and almost surely*

$$
K^{X_n}_b(\lambda)\|b - b^n\|_{\alpha'+1} \to 0,
$$

which implies uniqueness of the Young equation for b by Theorem 2.17.

Proof. By the previous result we have that the L^1 norm of $K^{X_n}_b(\lambda)$ is uniformly bounded in n. Moreover consider b^n such that $b^n = \rho_n * b$ where $\rho_n(x) = \frac{1}{2}n^d \exp(-n|x|)$. Then b^n is smooth, $b^n \to b$ in $\mathscr{C}^{1+\alpha'}$ for all $\alpha' < \alpha$ by the dominated convergence theorem and there exists a

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subsequence which will still denote with b^n such that $\|b - b^n\|_{\alpha'+1} \lesssim n^{-2}$. On this subsequence (which depends on b) consider the random variable

$$D = \sum_{n \geq 1} K_b^{X_n}(\lambda) \|b - b^n\|_{\alpha'+1}.$$

Then

$$\mathbb{E}D = \sum_{n \geq 1} \mathbb{E}[K_b^{X_n}(\lambda)] \|b - b^n\|_{\alpha'+1} \lesssim \sum_{n \geq 1} n^{-2} \lesssim 1,$$

so that almost surely $D < \infty$ which implies that $K_b^{X_n}(\lambda) \|b - b^n\|_{\alpha'+1} \to 0$. $\square$

Note that this argument give an exceptional set of zero measure which a priori depends on b (and on the sequence $(b^n)_n$) and of x_0. As remarked previously, this fact prevents straightforward extension of the uniqueness results in $\mathscr{C}^\alpha$ to random b. Furthermore, it also prevent to consider the regularity of the flow of the equation by pathwise methods.

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