



Intrinsic expansions for averaged diffusion processes

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Abstract

We show that the convergence rate of asymptotic expansions for solutions of SDEs is higher in the case of degenerate diffusion compared to the elliptic case, i.e. it is higher when the Brownian motion directly acts only along some directions. In the scalar case, this phenomenon was already observed in Gobet and Miri 2014 using Malliavin calculus techniques. Here, we provide a general and detailed analysis by employing the recent study of intrinsic functional spaces related to hypoelliptic Kolmogorov operators in Pagliarani et al. 2016. Applications to finance are discussed, in the study of path-dependent derivatives (e.g. Asian options) and in models incorporating dependence on past information.

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1. Introduction

We consider a continuous $\mathbb{R}^d$ -valued Feller process $X = (X_t)_{t \in [0, T]}$ defined, according to [16], on the space $(\Omega, \mathcal{F}, (\mathcal{F}_s^t)_{0 \leq t \leq s \leq T}, (P_{t,x})_{(t,x) \in [0, T] \times D})$ where D is a domain of $\mathbb{R}^d$ and $T > 0$ is fixed. We assume that the infinitesimal generator of X coincides on $[0, T] \times D$ with a

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second order differential operator of the form

$$\mathcal{A} = \frac{1}{2} \sum_{i,j=1}^{p_0} a_{ij}(t, x) \partial_{x_i} \partial_{x_j} + \sum_{i=1}^{p_0} a_i(t, x) \partial_{x_i} + \langle Bx, \nabla_x \rangle, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad (1.1)$$

where $p_0 \leq d$ and $\mathcal{A}$ verifies the following

Assumption 1.1. $A_0 := (a_{ij}(t, x))_{i,j=1,\dots,p_0}$ satisfies the non-degeneracy condition

$$\mu M |\xi|^2 < \sum_{i,j=1}^{p_0} a_{ij}(t, x) \xi_i \xi_j < M |\xi|^2, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad \xi \in \mathbb{R}^{p_0}, \quad (1.2)$$

for some positive constants M and μ ;

Assumption 1.2. B is a $(d \times d)$ -matrix with constant entries satisfying the following structural condition

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ B_1 & 0 & \cdots & 0 & 0 \\ 0 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_r & 0 \end{pmatrix} \quad (1.3)$$

where each B_j is a $(p_j \times p_{j-1})$ -matrix of rank p_j and

$$p_0 \geq p_1 \geq \cdots \geq p_r \geq 1, \quad \sum_{j=0}^r p_j = d.$$

Example 1.3. Let $X = (S, A)$ where

$$dS_t = \sigma S_t dW_t, \quad dA_t = S_t dt, \quad (1.4)$$

are the price and average processes in the Black & Scholes model for the pricing of arithmetic Asian options. In this case, we have $d = 2$, $p_0 = 1$ and

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The infinitesimal generator of X is

$$\frac{\sigma^2 x_1^2}{2} \partial_{x_1 x_1} + x_1 \partial_{x_2}, \quad (x_1, x_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0},$$

and clearly coincides, on any domain D compactly contained in $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$, with an operator of the form (1.1), that satisfies Assumptions 1.1 and 1.2. Clearly, more sophisticated multi-dimensional models, including stochastic and local volatility, as well as non-null interest rates, fall within the general framework above.

Assumption 1.2 implies that vector fields $\partial_{x_1}, \dots, \partial_{x_{p_0}}$ and

$$Y := \langle Bx, \nabla_x \rangle + \partial_t \quad (1.5)$$

satisfy the Hörmander condition (cf. [27]). Under suitable regularity conditions that will be specified later, the ultra-parabolic operator

$$\mathcal{K} := \mathcal{A} + \partial_t \quad (1.6)$$

admits a fundamental solution (see [38,8]). In the case $p_0 < d$, which is the focus of this work, this is a remarkable fact as the second order part of $\mathcal{A}$ is fully degenerate at any point. Operators $\mathcal{K}$ of this kind are often referred to as *Kolmogorov operators*.

In this paper, we study the asymptotic expansion of the conditional expectation

$$u(t, x) := \mathbb{E}_{t,x}[\varphi(X_T)], \quad (t, x) \in [0, T] \times D. \quad (1.7)$$

Our analysis takes advantage of the intrinsic geometry and the related regularity structures induced by the Kolmogorov operator $\mathcal{K}$. These features bring a number of benefits that are explained here below, and distinguish our approach from others in the literature. It is worth to emphasize further that our results are carried out under strictly *local assumptions* on the generator of the Feller diffusion X , which coincides with a Kolmogorov operator on a domain D , not necessarily equal to $\mathbb{R}^d$. The proof of our main result, [Theorem 3.5](#), will be split in two separate steps: first, in [Theorem 3.8](#), we consider the case $D = \mathbb{R}^d$ for which we employ some Gaussian upper bounds for the transition density of X ; second, we adapt a localization procedure originally introduced in [40] and lately extended in [4]. The latter localization procedure is coherent with what is known in the theory of diffusions as the *principle of not feeling the boundary* (cf. [23,18]). This allows to include in our analysis degenerate models such as the well-known CEV model (that is when σ in (1.4) is not a constant but a function of S of the form $\sigma(S) = S^\gamma$ for some $\gamma \in \mathbb{R}$) and the Heston stochastic volatility model as very particular cases.

1.1. Intrinsic Taylor-based asymptotic expansions

Intrinsic Hölder and Sobolev spaces for Kolmogorov operators were studied by several authors, among others [9,2,31,30,34,32]. In Section 2 we recall some results on the homogeneous Lie group and the related intrinsic Hölder spaces, hereafter denoted by $C_B^{n,\alpha}$, induced by the vector fields $\partial_{x_1}, \dots, \partial_{x_{p_0}}$ and Y in (1.5): in this framework, in [37] the authors proved a Taylor formula with reminder expressed in terms of the homogeneous norm of the group. Deferring precise definitions and statements until Section 2, the n th order intrinsic Taylor polynomial, centered at $\zeta = (s, \xi) \in \mathbb{R} \times \mathbb{R}^d$, of a function $f \in C_B^{n,\alpha}$ reads as

$$\begin{aligned} \mathbf{T}_n(f, \zeta)(z) &:= \sum_{2k+|\beta|_B \leq n} \frac{1}{k! \beta!} (Y^k \partial_\xi^\beta f(s, \xi))(t-s)^k (x - e^{(t-s)B} \xi)^\beta, \\ z = (t, x) &\in \mathbb{R} \times \mathbb{R}^d, \end{aligned} \quad (1.8)$$

where $|\beta|_B$, given in (2.4), is a suitable weight for the multi-index¹ $\beta \in \mathbb{N}_0^d$. Such Taylor expansion forms the cornerstone of the perturbation technique that we study in this paper. Here below we summarize the intuitive idea behind it and its primary features.

We recall that, under mild assumptions that will be specified in Section 3, the function u in (1.7) satisfies

$$\begin{cases} \mathcal{K}u = 0, & \text{on } [0, T] \times D, \\ u(T, \cdot) = \varphi, & \text{on } D. \end{cases} \quad (1.9)$$

¹ We denote by $\mathbb{N}$ the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Notice that (1.9) is not a standard Cauchy–Dirichlet problem since no lateral boundary conditions are imposed. In a series of papers, two of the authors propose a perturbative method to carry out a closed-form approximation of solutions to (1.9) under the assumption that $\mathcal{K}$ in (1.1)–(1.6) is locally parabolic, i.e. $p_0 = d$ and $B = 0$ in (1.3) (for a recent and thorough description the reader can refer to [29,35]). The basic idea is to approximate the generator by Taylor expanding its coefficients, and take advantage of some symmetry properties of Gaussian kernels. Sharp short-time/small-noise asymptotic estimates for the remainder of the expansion are then proved. In order to generalize the aforementioned technique to the case $p_0 < d$, we perform an expansion that is compatible with the sub-elliptic geometry induced by Kolmogorov operators. Assuming $a_{ij}, a_i \in C_B^{N,1}$, we expand the operator $\mathcal{K}$ through the sequence $(\mathcal{K}_n^{(\bar{z})})_{0 \leq n \leq N}$ defined as

$$\mathcal{K}_n^{(\bar{z})} = \frac{1}{2} \sum_{i,j=1}^{p_0} \mathbf{T}_n(a_{ij}, \bar{z})(z) \partial_{x_i x_j} + \sum_{i=1}^{p_0} \mathbf{T}_{n-1}(a_i, \bar{z})(z) \partial_{x_i} + Y, \\ z = (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad (1.10)$$

where $\mathbf{T}_n(a_{ij}, \bar{z})$ is the Taylor polynomial of a_{ij} , defined as in (1.8), centered at a fixed point $\bar{z} \in \mathbb{R} \times \mathbb{R}^d$, and $\mathbf{T}_{-1}(a_i, \bar{z}) \equiv 0$.

Remark 1.4. When $p_0 < d$, the intrinsic space $C_B^{n,\alpha}$ is strictly contained into the corresponding Euclidean Hölder space $C^{n,\alpha}$: for this reason, the regularity assumptions on the coefficients are weaker than in the parabolic case.

The leading term of the expansion, $\mathcal{K}_0^{(\bar{z})}$, is the Kolmogorov operator with constant coefficients

$$\mathcal{K}_0^{(\bar{z})} = \frac{1}{2} \sum_{i,j=1}^{p_0} a_{ij}(\bar{z}) \partial_{x_i x_j} + Y, \quad (1.11)$$

defined on $\mathbb{R} \times \mathbb{R}^d$. It is well-known that $\mathcal{K}_0^{(\bar{z})}$ admits a Gaussian fundamental solution that satisfies some remarkable symmetry properties written in terms of the increments appearing in the intrinsic Taylor polynomials in (1.8). The main result of this paper, Theorem 3.5, provides an explicit approximating expansion for $u(t, x)$ in (1.7), equipped with sharp short-time error bounds, and can be roughly summarized as:

$$u(t, x) = u_0(t, x) + \sum_{n=1}^N \mathcal{L}_n(t, T, x) u_0(t, x) + O\left((T-t)^{\frac{N+1+k}{2}}\right) \quad \text{as } t \rightarrow T^-, \quad (1.12)$$

uniformly with respect to $x \in D$, where:

- the leading term u_0 is the solution of the Cauchy problem for $\mathcal{K}_0^{(\bar{z})}$ with final datum φ ;
- $(\mathcal{L}_n)_{1 \leq n \leq N}$ is a family of differential operators, acting on x , that can be explicitly computed in terms of the intrinsic Taylor polynomials $\mathbf{T}_n(a_{ij}, \bar{z})$ and $\mathbf{T}_n(a_i, \bar{z})$ (see Theorem A.2);
- the positive exponent k , contributing to the asymptotic rate of convergence, is the intrinsic Hölder exponent of φ . Precisely, $\varphi \in C_B^k$ according to Definition 2.4.

Such approximation turns out to be optimal to several extents. In particular, the benefit in exploiting the intrinsic regularity is threefold: first, since the intrinsic Taylor polynomial is typically a projection of the Euclidean one, we avoid taking up terms in the expansion that do not improve

the quality of the approximation; secondly, the fact that the increments of the intrinsic Taylor polynomial appear in the symmetries of the fundamental solution of $\mathcal{K}_0^{(\bar{z})}$ allows to get compact approximation formulas; finally, the asymptotic rate of convergence of the expansion also depends on the *intrinsic regularity* of the datum φ , which is typically higher than the Euclidean regularity. This is particularly relevant in the financial applications (see [Remark 1.5](#)).

1.2. Applications to finance and comparison with the existing literature

The application of Kolmogorov operators in mathematical finance is particularly relevant in the pricing of Asian-style derivatives. These are financial claims whose payoff is a function not only of the terminal value of an underlying asset, but also of its average over a certain time-period. In most cases of interest, the problem of computing the conditional expectation (1.7), which defines the no-arbitrage price of such financial claims, is not known to have an explicit solution, and thus a considerably large amount of literature has been developed in the last decades in order to find accurate and quickly computable approximate solutions. Some of these approaches make use of asymptotic techniques that lead to semi-closed approximation formulas. In this section we aim at firming our results within the existing literature on analytical approximations of Asian-style derivatives. Before to proceed we recall that other financial applications, where averaged-diffusion processes are employed, include volatility models with path-dependent coefficients, e.g. the Hobson–Rogers model [22].

Let us resume [Example 1.3](#) and now assume that S follows the more general dynamics

$$dS_t = \sigma(t, S_t, A_t) dW_t.$$

In this case, $a_{11}(t, x_1, x_2) = \sigma^2(t, x_1, x_2)$ and its n th order intrinsic Taylor polynomial centered at $\zeta = (s, \xi_1, \xi_2)$ reads as

$$\begin{aligned} \mathbf{T}_n(a_{11}, \zeta)(t, x_1, x_2) &= \sum_{2k+\beta_0+3\beta_1 \leq n} \frac{(\partial_s + \xi_1 \partial_{\xi_2})^k \partial_{\xi_1}^{\beta_0} \partial_{\xi_2}^{\beta_1} a_{11}(s, \xi_1, \xi_2)}{k! \beta_0! \beta_1!} \\ &\quad \times (t-s)^k (x_1 - \xi_1)^{\beta_0} (x_2 - \xi_2 - (t-s)\xi_1)^{\beta_1}. \end{aligned}$$

More explicitly, up to order 3 we have

$$\begin{aligned} \mathbf{T}_0(a_{11}, \zeta)(t, x_1, x_2) &= a_{11}(\zeta), \\ \mathbf{T}_1(a_{11}, \zeta)(t, x_1, x_2) &= \mathbf{T}_0(a_{11}, \zeta)(t, x_1, x_2) + (x_1 - \xi_1) \partial_{\xi_1} a_{11}(\zeta), \\ \mathbf{T}_2(a_{11}, \zeta)(t, x_1, x_2) &= \mathbf{T}_1(a_{11}, \zeta)(t, x_1, x_2) \\ &\quad + \frac{(x_1 - \xi_1)^2}{2!} \partial_{\xi_1}^2 a_{11}(\zeta) + (t-s)(\partial_s + \xi_1 \partial_{\xi_2}) a_{11}(\zeta), \\ \mathbf{T}_3(a_{11}, \zeta)(t, x_1, x_2) &= \mathbf{T}_2(a_{11}, \zeta)(t, x_1, x_2) \\ &\quad + \frac{(x_1 - \xi_1)^3}{3!} \partial_{\xi_1}^3 a_{11}(\zeta) + (x_2 - \xi_2 - (t-s)\xi_1) \partial_{\xi_2} a_{11}(\zeta) \\ &\quad + (t-s)(x_1 - \xi_1)(\partial_s + \xi_1 \partial_{\xi_2}) \partial_{\xi_1} a_{11}(\zeta), \end{aligned}$$

which shows that the increment in the time variable appears only from the 2nd order on, whereas the increment along the average variable appears from the 3rd order on. As it was mentioned above, the operators $\mathcal{L}_n^{(\zeta)}$ appearing in the asymptotic expansion in (1.12) can be explicitly

computed by applying (A.6)–(A.7)–(A.4)–(3.2). In this case they read as

$$\begin{aligned}\mathcal{L}_n^{(\zeta)}(t, T, x) &= \frac{1}{2} \int_t^T (\mathbf{T}_n(a_{11}, \zeta) - \mathbf{T}_{n-1}(a_{11}, \zeta)) \\ &\quad \times (s, \mathcal{M}^{(\zeta)}(s - t, x_1, x_2)) (\partial_{x_1} - (s - t) \partial_{x_2})^2 ds, \\ \mathcal{M}^{(\zeta)}(t, x_1, x_2) &= \left(x_1 + a_{11}(\zeta) t \partial_{x_1} - a_{11}(\zeta) \frac{t^2}{2} \partial_{x_2}, \right. \\ &\quad \left. tx_1 + x_2 - a_{11}(\zeta) \frac{t^2}{2} \partial_{x_1} + a_{11}(\zeta) \frac{t^3}{6} \partial_{x_2} \right).\end{aligned}$$

In order to show an even more explicit sample, at order 1 we have:

$$\begin{aligned}\mathcal{L}_1^{(\zeta)}(t, T, x) &= \frac{\partial_{\xi_1} a_{11}(\zeta)}{2} \int_t^T \left((x_1 - \xi_1) + a_{11}(\zeta)(s - t) \partial_{x_1} - \frac{a_{11}(\zeta)}{2} (s - t)^2 \partial_{x_2} \right) \\ &\quad \times (\partial_{x_1} - (s - t) \partial_{x_2})^2 ds.\end{aligned}$$

Two typical arithmetic Asian options are the so-called *floating strike* and *fixed strike* Call options, whose payoffs are given respectively by

$$\varphi_{\text{float}}(x_1, x_2) = (x_1 - x_2/T)^+, \quad \varphi_{\text{fixed}}(x_1, x_2) = (x_2/T - K)^+,$$

where T is the maturity and K is the strike price.

Remark 1.5. The payoff φ_{fixed} is Lipschitz continuous in the standard Euclidean sense but has higher intrinsic regularity (namely, C_B^3 according to Definition 2.4, see also Example 2.5): this property reflects a higher rate of convergence of the asymptotic expansion (1.12) compared with other expansions based on standard Taylor polynomials.

Even in the simplest case of constant volatility, i.e. in the Black & Scholes model, both the marginal distribution of A_t and the joint distribution of (S_t, A_t) are difficult to characterize analytically. The distribution of A_t was given an integral representation in the pioneering work [44], though that result is of limited practical use in the valuation of Asian options. The approximation formulas that we propose in this paper were applied heuristically in [15], where intensive numerical tests were performed to confirm their accuracy. However, the general hypoelliptic framework that we consider here clearly allows for several generalization, including more general dynamics and more sophisticated Asian style-derivatives including stochastic local volatility models such as the CEV and the Heston models [21]. An interesting example is also given by a generalized type of Asian option, where the average is weighted w.r.t. the volume of traded assets: these options are written on the *Volume Weighted Average Price* (VWAP), a trading benchmark used especially in pension plans (see, for instance, [33]). The dynamics of the traded volume V are lead by an additional stochastic factor that has to be chosen as to reflect the corresponding volume statistics, and the average process A is then given by

$$A_t = \frac{\int_0^t S_\tau V_\tau d\tau}{\int_0^t V_\tau d\tau}.$$

As it was previously argued, our technique makes use of the intrinsic Taylor polynomials in (1.8) in order to be consistent with the subelliptic geometry induced by Kolmogorov operators. This differentiates our approach from others appearing in the literature that are based on classical

Euclidean expansions. In the relevant paper [20], Malliavin calculus techniques were employed to derive analytical approximations for the law of a general averaged diffusion. When applied to the pricing of arithmetic Asian options, the approach in [20] returns an expansion whose leading term is the price of a geometric Asian option. Correcting terms are computed by Taylor expanding the coefficients of the diffusion and error estimates depend on standard Euclidean regularity assumptions on the coefficients and on the payoff function. In [42,3], the authors followed a different approach and carried out a Taylor based-expansion of the joint distribution (S_t, A_t) to analytically approximate the price of an Asian option (possibly, forward-starting); this technique seems to be limited to the Black & Scholes dynamics. Other approximations, based on Taylor expansions and on Watanabe's theory, can be found in [26], though no rigorous error bounds are provided.

For sake of completeness, we also give a brief, and by no means exhaustive, overview of the existing literature concerning other approaches to the pricing of Asian options. Within the Black & Scholes framework, [19] derived an analytical expression for the Laplace transform of A_t . However, several authors pointed out some stability issues related to the numerical inversion of the Laplace transform, which lacks accuracy and efficiency in regimes of small volatility or short time-to-maturity. This is also a disadvantage of the Laguerre expansion proposed in [11]. [41] used a contour integral approach based on Mellin transforms to improve the accuracy of the results in the case of low volatilities, albeit at a higher computational cost. As opposed to numerical inversion, [28] derived an eigenfunction expansion of the transition density of A_t (see also [10]) by employing spectral theory of singular Sturm–Liouville operators. Although it returns in general very accurate results, Linetsky's series formula may converge slowly in the case of low volatility and become computationally expensive. Note that, by opposite, the analytical pricing formulas we propose here do not suffer any lack of accuracy or efficiency in these limiting cases. In actual fact, [Theorem 3.5](#) and [Remark 3.7](#) show that the accuracy improves as volatility and/or time to maturity get smaller. Again in the particular case of the Black & Scholes model, and for special homogeneous payoff functions, it is possible to reduce the pricing PDE in (1.9) to a one state variable PDE. PDE reduction techniques were initiated in [24] and applied to the problem of pricing Asian options by several authors, including [39,43,7]. Eventually, other approaches include the parametrix expansion in [5] and the moment-matching techniques in [12,6,17,14] among others.

2. Kolmogorov operators and intrinsic Hölder spaces

In this section we collect some known facts about the intrinsic geometry of Kolmogorov operators. We also recall the definition of intrinsic Hölder spaces and the Taylor formula recently proved in [37]. We consider the prototype Kolmogorov operator obtained by (1.1)–(1.6) with A_0 equal to a scalar $(p_0 \times p_0)$ -matrix and $a_i \equiv 0$, $i = 1, \dots, p_0$, i.e.

$$\mathcal{K}^A := \frac{\Lambda}{2} \sum_{i=1}^{p_0} \partial_{x_i}^2 + \langle Bx, \nabla_x \rangle + \partial_t, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad \Lambda > 0. \quad (2.1)$$

In this case we say that $\mathcal{K}^A$ is a *constant coefficients Kolmogorov operator*. By [Assumption 1.2](#), the vector fields $\partial_{x_1}, \dots, \partial_{x_{p_0}}$ and Y in (1.5) satisfy the Hörmander's condition and therefore $\mathcal{K}^A$ is hypoelliptic. As it was first observed in [27], $\mathcal{K}^A$ has remarkable invariance properties with respect to the homogeneous Lie group $\mathcal{G}_B = (\mathbb{R} \times \mathbb{R}^d, \circ, (D(\lambda))_{\lambda>0})$ where “ $\circ$ ” is the group law defined as

$$(t, x) \circ (s, \xi) = \left(t + s, e^{sB} x + \xi \right), \quad (t, x), (s, \xi) \in \mathbb{R} \times \mathbb{R}^d,$$

and $(D(\lambda))_{\lambda>0}$ are the dilations given by

$$D(\lambda) = \text{diag}(\lambda^2, \lambda I_{p_0}, \lambda^3 I_{p_1}, \dots, \lambda^{2r+1} I_{p_r}),$$

where I_{p_j} denote the $(p_j \times p_j)$ -identity matrices. Precisely, it was proved in [27] that $\mathcal{K}^A$ is invariant with respect to the left $\circ$ -translations and homogeneous of degree two with respect to the dilations $(D(\lambda))_{\lambda>0}$. Notice that $\mathcal{G}_B$ is completely determined by the matrix B ; moreover, the identity element in $\mathcal{G}_B$ is $\text{Id} = (0, 0)$ and the inverse is $(t, x)^{-1} = (-t, -e^{-tB}x)$. For convenience, we also denote by

$$D_0(\lambda) = \text{diag}(\lambda I_{p_0}, \lambda^3 I_{p_1}, \dots, \lambda^{2r+1} I_{p_r}),$$

the “spatial part” of $D(\lambda)$. A homogeneous norm on $\mathcal{G}_B$ is defined as follows:

$$\|(t, x)\|_B = |t|^{1/2} + [x]_B, \quad [x]_B := \sum_{j=1}^d |x_j|^{1/\sigma_j}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad (2.2)$$

where $(\sigma_j)_{1 \leq j \leq d}$ are the integers such that

$$D_0(\lambda) = \text{diag}(\lambda^{\sigma_1}, \dots, \lambda^{\sigma_d}). \quad (2.3)$$

In the general setting of homogeneous Lie groups, Hölder spaces and intrinsic Taylor polynomials can be defined as in [13,1]. For the particular case of homogeneous Lie groups induced by Kolmogorov operators, [37] provides a deeper analysis of the intrinsic Taylor polynomials under optimal regularity assumptions.

For any Lipschitz vector field Z on $\mathbb{R} \times \mathbb{R}^d$, we denote by $\delta \mapsto e^{\delta Z}(z)$ the integral curve of Z starting from z : in particular, we have

$$e^{\delta \partial_{x_i}}(t, x) = (t, x + \delta \mathbf{e}_i), \quad i = 1, \dots, p_0, \quad e^{\delta Y}(t, x) = (t + \delta, e^{\delta B}x),$$

where $\mathbf{e}_i$ denotes the i th element of the natural Euclidean basis of $\mathbb{R}^d$. We say that a function u is Z -differentiable at z if $\delta \mapsto u(e^{\delta Z}(z))$ is differentiable at 0 and in that case $\frac{d}{d\delta} u(e^{\delta Z}(z)) \big|_{\delta=0}$ is referred to as the *Lie derivative of u at z along Z* . Since the vector fields $\partial_{x_1}, \dots, \partial_{x_{p_0}}$ and Y are $D(\lambda)$ -homogeneous of degree one and two respectively, we associate to $\partial_{x_1}, \dots, \partial_{x_{p_0}}$ and Y the *formal degrees* one and two respectively. In general, if a Lipschitz vector field Z has formal degree $m_Z > 0$ and u is a function on $\mathbb{R} \times \mathbb{R}^d$, then for any $\alpha \in]0, m_Z]$ we say that $u \in C_Z^\alpha \equiv C_Z^\alpha(\mathbb{R} \times \mathbb{R}^d)$ if the norm

$$\|u\|_{C_Z^\alpha} := \sup_{\mathbb{R} \times \mathbb{R}^d} |u| + \sup_{\substack{\delta \in \mathbb{R} \setminus \{0\} \\ z \in \mathbb{R} \times \mathbb{R}^d}} \frac{|u(e^{\delta Z}(z)) - u(z)|}{|\delta|^{\frac{\alpha}{m_Z}}}$$

is finite. Now we define the intrinsic Hölder spaces on the homogeneous group $\mathcal{G}_B$.

Definition 2.1. Let $\alpha \in]0, 1]$ and $n \in \mathbb{N}$ with $n \geq 2$, then:

- (i) $u \in C_B^{0,\alpha}$ if $u \in C_Y^\alpha$ and $u \in C_{\partial_{x_i}}^\alpha$ for any $i = 1, \dots, p_0$;
- (ii) $u \in C_B^{1,\alpha}$ if $u \in C_Y^{1+\alpha}$ and $\partial_{x_i} u \in C_B^{0,\alpha}$ for any $i = 1, \dots, p_0$;
- (iii) $u \in C_B^{n,\alpha}$ if $Yu \in C_B^{n-2,\alpha}$ and $\partial_{x_i} u \in C_B^{n-1,\alpha}$ for any $i = 1, \dots, p_0$.

We also introduce the norms:

$$\begin{aligned}\|u\|_{C_B^{0,\alpha}} &:= \|u\|_{C_Y^\alpha} + \sum_{i=1}^{p_0} \|u\|_{C_{\partial x_i}^\alpha}, \\ \|u\|_{C_B^{1,\alpha}} &:= \|u\|_{C_Y^{1+\alpha}} + \sum_{i=1}^{p_0} \|\partial_{x_i} u\|_{C_B^{0,\alpha}}, \\ \|u\|_{C_B^{n,\alpha}} &:= \|Yu\|_{C_B^{n-2,\alpha}} + \sum_{i=1}^{p_0} \|\partial_{x_i} u\|_{C_B^{n-1,\alpha}}.\end{aligned}$$

Remark 2.2. Notice that $C_B^{n+1,\alpha} \subseteq C_B^{n,\alpha}$ for any $n \in \mathbb{N}_0$.

For any multi-index $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}_0^d$, we define the B -length of β as

$$|\beta|_B := \sum_{j=1}^d \sigma_j \beta_j, \quad (2.4)$$

with σ_j as in (2.3). We are now in position to state the intrinsic Taylor theorem that was proved in [37].

Theorem 2.3. Let $\alpha \in]0, 1]$ and $n \in \mathbb{N}_0$. If $u \in C_B^{n,\alpha}$ then the derivatives

$$Y^k \partial_x^\beta u \in C_B^{n-2k-|\beta|_B, \alpha} \quad \text{for } 0 \leq 2k + |\beta|_B \leq n,$$

exist and therefore, for any point $\zeta = (s, \xi)$, the n th order B -Taylor polynomial $\mathbf{T}_n(u, \zeta)(\cdot)$ in (1.8) is well defined. Moreover, we have

$$|u(z) - \mathbf{T}_n(u, \zeta)(z)| \leq c_B \|u\|_{C_B^{n,\alpha}} \|\zeta^{-1} \circ z\|_B^{n+\alpha}, \quad z, \zeta \in \mathbb{R} \times \mathbb{R}^d,$$

where c_B is a positive constant that only depends on B .

Definition 2.1 and **Theorem 2.3** will be used in the next section, respectively, to specify suitable regularity conditions on the coefficients of $\mathcal{K}$ in (1.1)–(1.6), and to expand them as in (1.10). However, as anticipated in Section 1.1, the intrinsic regularity of the terminal datum φ plays as well a key role in the error analysis of the expansion (1.12). This motivates the following

Definition 2.4. Let $k \in]0, 2r + 1]$. We denote by $C_B^k(\mathbb{R}^d)$ the space of functions φ on $\mathbb{R}^d$ such that

$$|\varphi(x) - \varphi(y)| \leq C[x - y]_B^k, \quad x, y \in \mathbb{R}^d,$$

for some positive constant C , where $[\cdot]_B$ is the norm on $\mathbb{R}^d$ defined in (2.2). We also set

$$\|\varphi\|_{C_B^k(\mathbb{R}^d)} = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{[x - y]_B^k}.$$

Moreover, by convention, $C_B^0(\mathbb{R}^d)$ is the set of bounded and continuous functions on $\mathbb{R}^d$ and $\|\varphi\|_{C_B^0(\mathbb{R}^d)} = \|\varphi\|_{L^\infty(\mathbb{R}^d)}$.

Example 2.5. Consider the case of arithmetic Asian options with fixed strike discussed in Section 1.2, i.e.

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \varphi_{\text{fixed}}(x_1, x_2) = (x_2/T - K)^+.$$

According to Definition 2.4, $\varphi_{\text{fixed}} \in C_B^3(\mathbb{R}^2)$ even if it is only Lipschitz continuous in the Euclidean sense.

3. Approximate solutions and error bounds

Let X be a Feller process as defined in the introduction: in particular, we assume that the infinitesimal generator of X coincides with operator $\mathcal{A}$ in (1.1) on a fixed domain D of $\mathbb{R}^d$. Moreover, $\mathcal{A}$ satisfies Assumptions 1.1 and 1.2. Throughout this section $N \in \mathbb{N}_0$ and $T > 0$ are fixed and we also require the following assumptions to be in force:

Assumption 3.1. The coefficients a_{ij}, a_i of $\mathcal{A}$ belong to $C_B^{N,1}$ and

$$\|a_{ij}\|_{C_B^{N,1}}, \|a_i\|_{C_B^{N,1}} \leq M,$$

with M as in (1.2).

Assumption 3.2. The final datum φ is a continuous function with sub-exponential growth such that $u = u(t, x)$ in (1.7) is well defined and belongs to $L^\infty([0, T] \times D)$. Moreover, there exists $\psi \in C_B^k(\mathbb{R}^d)$, with $k \in [0, 2r + 1]$, such that $\varphi = \psi$ on D .

The following preliminary result can be proved as in [25] or [36], using the Schauder estimates and the results on Green functions proved in [9].

Proposition 3.3. Let Assumptions 1.1, 1.2, 3.1 and 3.2 be in force. Then, $u \in C([0, T] \times D) \cap C_{B,\text{loc}}^{N+2,1}$ and satisfies (1.9).

As was mentioned in the introduction, the idea behind our approximation of $u = u(t, x)$ in (1.7) is to expand the generator of X by approximating the coefficients a_{ij} and a_j in (1.1) by means of their *intrinsic* Taylor polynomials in (1.8). Thus we fix $\bar{z} = (\bar{t}, \bar{x}) \in \mathbb{R} \times \mathbb{R}^d$ and consider the sequence $(\mathcal{K}_n^{(\bar{z})})_{0 \leq n \leq N}$ in (1.10). We recall that, by Assumptions 1.1 and 1.2, $\mathcal{K}_0^{(\bar{z})}$ in (1.11) has a fundamental solution $\Gamma_0^{(\bar{z})}$ that is the d -dimensional Gaussian density

$$\begin{aligned} \Gamma_0^{(\bar{z})}(t, x; T, y) \\ = \frac{1}{\sqrt{(2\pi)^d |\mathbf{C}_{\bar{z}}(T-t)|}} \exp \left(-\frac{1}{2} \langle \mathbf{C}_{\bar{z}}^{-1}(T-t)(y - e^{(T-t)B}x), (y - e^{(T-t)B}x) \rangle \right) \end{aligned} \quad (3.1)$$

with covariance matrix $\mathbf{C}_{\bar{z}}(t)$ given by

$$\mathbf{C}_{\bar{z}}(t) = \int_0^t e^{sB} A(\bar{z}) e^{sB^*} ds, \quad A(\bar{z}) := \begin{pmatrix} A_0(\bar{z}) & 0_{p_0 \times (d-p_0)} \\ 0_{(d-p_0) \times p_0} & 0_{(d-p_0) \times (d-p_0)} \end{pmatrix}. \quad (3.2)$$

Next we formally expand the expected value u in (1.7) as

$$u \approx U_N^{(\bar{z})} := \sum_{n=0}^N u_n^{(\bar{z})}. \quad (3.3)$$

Inserting (1.10), (3.3) into (1.9) and formally collecting terms of the same order, we find that the functions $u_n^{(\bar{z})}$ satisfy the following sequence of nested Cauchy problems

$$\begin{cases} \mathcal{K}_0^{(\bar{z})} u_0^{(\bar{z})} = 0, & \text{on } [0, T] \times \mathbb{R}^d, \\ u_0^{(\bar{z})}(T, \cdot) = \varphi, & \text{on } \mathbb{R}^d, \end{cases} \quad (3.4)$$

and

$$\begin{cases} \mathcal{K}_0^{(\bar{z})} u_n^{(\bar{z})} = - \sum_{h=1}^n (\mathcal{K}_h^{(\bar{z})} - \mathcal{K}_{h-1}^{(\bar{z})}) u_{n-h}^{(\bar{z})}, & \text{on } [0, T] \times \mathbb{R}^d, \\ u_n^{(\bar{z})}(T, \cdot) = 0, & \text{on } \mathbb{R}^d. \end{cases} \quad (3.5)$$

The explicit representation of the terms $u_n^{(\bar{z})}$ of the expansion is given in Theorem A.2.

Remark 3.4. In the above construction, the approximation in (3.3) is defined in terms of a sequence of Cauchy problems that admit a unique non-rapidly increasing solution. Conversely, Eqs. (1.9) do not have a unique solution unless additional lateral boundary conditions are posed. Nevertheless, Theorem 3.5 states that the above expansion is asymptotically convergent in the limit of short-time, uniformly on compact subsets of D . This is in line with the so-called *principle of not feeling the boundary* (cf. [23,18]). Basically, the same asymptotic result would hold for any bounded solution of equations (1.9), with error bounds depending on the L^∞ -norm of the solution. Of course, knowing the boundary conditions would allow to construct an approximate sequence that is also accurate near the boundary; this is the case of barrier options in the financial applications.

The choice of the basis point $\bar{z}$ is somewhat arbitrary, but only some particular choices allow for performing a rigorous error analysis. For instance, here below we consider the case $\bar{z} = z = (t, x)$. However, although we omit to write separate proofs, the same results hold by setting $\bar{z} = (T, x)$. In the following statement, we put

$$U_N(z) := U_N^{(z)}(z), \quad z \in [0, T] \times D, \quad (3.6)$$

with $U_N^{(z)}$ defined by (3.3)–(3.5).

Theorem 3.5. Let Assumptions 1.1, 1.2, 3.1 and 3.2 be in force. Then for any compact subset K of D , we have

$$|u(t, x) - U_N(t, x)| \leq C(T - t)^{\frac{N+k+1}{2}}, \quad (t, x) \in [0, T] \times K, \quad (3.7)$$

where C is a positive constant that depends only on $M, \mu, B, T, N, K, \|\psi\|_{C_B^k(\mathbb{R}^d)}$ and $\|u\|_{L^\infty([0, T] \times D)}$.

Theorem 3.5 will be proved in Section 3.2.

Remark 3.6. As shown in Example 2.5, for a fixed-strike Asian option we have $\varphi \in C_B^3(\mathbb{R}^2)$ and therefore we get $(T - t)^{\frac{N+4}{2}}$ in the error estimate (3.7). This is coherent with the previous results proved in [20] in the scalar case for $N \leq 2$, and sheds some light on why the order of convergence of Asian call options is improved w.r.t. their European counterparts, for which the error is of order $(T - t)^{\frac{N+2}{2}}$. When placed within our framework, this improvement of convergence can be seen as part of a wider phenomenon related to the intrinsic geometry of Kolmogorov operators.

Remark 3.7. If the coefficients a_{ij} , a_i only depend on the first p_0 variables, then it is possible to prove the error bounds in (3.7) to be also asymptotic in the limit of small M . Precisely,

$$|u(t, x) - U_N(t, x)| \leq C(M(T-t))^{\frac{N+k+1}{2}}, \quad (t, x) \in [0, T] \times K,$$

with C independent of M as $M \rightarrow 0^+$. This is the case, for instance, of classical volatility models for Asian options where the volatility coefficient depends at most on the underlying asset S_t (local volatility) and on some exogenous factors (stochastic volatility), but not on the average process A_t .

In the global case, when $D = \mathbb{R}^d$, we have some stronger results. Aside from the error bounds in (3.7) becoming global in space, we are also able to obtain analogous asymptotic error bounds for the transition density of X . We start by observing that when $D = \mathbb{R}^d$ our assumptions imply that X has a transition density Γ that coincides with the fundamental solution of $\mathcal{K}$ as in (1.1)–(1.6) (see, for instance, [38]). We denote by Γ_N the N th order approximation of Γ defined as

$$\Gamma_N(t, x; T, y) = \sum_{n=0}^N u_n(t, x; T, y) \quad 0 \leq t < T, \quad x, y \in \mathbb{R}^d,$$

where $u_0(t, x; T, y) = \Gamma_0^{(t,x)}(t, x; T, y)$ in (3.1), and the correcting terms $u_n(t, x; T, y)$ are defined recursively by (3.5) with $\bar{z} = (t, x)$. We have the following

Theorem 3.8. Let Assumptions 1.1, 1.2, 3.1 and 3.2 be in force with $D = \mathbb{R}^d$. Then, we have

$$|u(t, x) - U_N(t, x)| \leq C(T-t)^{\frac{N+k+1}{2}}, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \quad (3.8)$$

where C depends only on M, μ, B, T, N and $\|\varphi\|_{C_B^k(\mathbb{R}^d)}$. Moreover, for any $c > 1$, we have

$$|\Gamma(t, x; T, y) - \Gamma_N(t, x; T, y)| \leq C(T-t)^{\frac{N+1}{2}} \Gamma^{cM}(t, x; T, y), \quad (t, x) \in [0, T] \times \mathbb{R}^d, \quad (3.9)$$

where, for any $\Lambda > 0$, Γ^Λ denotes the fundamental solution of the constant-coefficient Kolmogorov operator $\mathcal{K}^\Lambda$ as defined in (2.1), and C is a positive constant that depends only on M, μ, B, T, N and c .

3.1. Proof of Theorem 3.8

The proof of Theorem 3.8 is based on the following two propositions. The first one provides some Gaussian estimates for the fundamental solution $\Gamma = \Gamma(t, x; T, y)$ of the operator $\mathcal{K}$ in (1.6)–(1.1): for the proof see [38,8]. Throughout this section we suppose the assumptions of Theorem 3.8 to be in force.

Proposition 3.9. For any $k \in \mathbb{R}_{\geq 0}$, $c > 1$ and $\beta \in \mathbb{N}_0^d$, with $|\beta|_B \leq N+2$, we have

$$\begin{aligned} & \left| y - e^{(T-t)B} x \right|_B^k \left| D_x^\beta \Gamma(t, x; T, y) \right| \leq C(T-t)^{\frac{k-|\beta|_B}{2}} \Gamma^{cM}(t, x; T, y), \\ & 0 \leq t < T, \quad x, y \in \mathbb{R}^d, \end{aligned}$$

where Γ^{cM} is the fundamental solution of the operator in (2.1) and C is a positive constant, only dependent on M, μ, B, T, N, k and c .

The following result is proved in Appendix B.

Proposition 3.10. Let $\varphi \in C_B^k(\mathbb{R}^d)$ with $k \in [0, 2r + 1]$ and $n \in \mathbb{N}$ with $n \leq N$. Then we have

$$|D_x^\beta u_n^{(\bar{z})}(t, x)| \leq C(T-t)^{\frac{k-|\beta|_B}{2}} \left((T-t)^{\frac{n}{2}} + [x - e^{(t-\bar{t})B} \bar{x}]_B^n \right), \quad 0 \leq t < T, \quad x \in \mathbb{R}^d,$$

where C is a constant that depends only on $M, \mu, B, T, N, |\beta|_B$ and $\|\varphi\|_{C_B^k(\mathbb{R}^d)}$.

Proof of Theorem 3.8. To keep formulas at a reasonable size we suppose that the functions a_i , $i = 1, \dots, p_0$, in (1.1) are identically zero. We first remark that a straightforward computation (see Lemma 6.3 in [29]) shows that

$$u(t, x) - U_N(t, x) = \sum_{n=0}^N E_n^{(\bar{z})}(t, x) \Big|_{\bar{z}=(t,x)} \quad (3.10)$$

where

$$\begin{aligned} E_n^{(\bar{z})}(t, x) &:= \int_t^T \int_{\mathbb{R}^d} \Gamma(t, x; s, \xi) \left(\mathcal{K} - \mathcal{K}_n^{(\bar{z})} \right) u_{N-n}^{(\bar{z})}(s, \xi) d\xi ds \\ &= \frac{1}{2} \sum_{i,j=1}^{p_0} \int_t^T \int_{\mathbb{R}^d} \Gamma(t, x; s, \xi) (a_{ij}(s, \xi) - \mathbf{T}_n(a_{ij}, \bar{z})(s, \xi)) \\ &\quad \times \partial_{\xi_i \xi_j} u_{N-n}^{(\bar{z})}(s, \xi) d\xi ds. \end{aligned} \quad (3.11)$$

Now, if $k > 0$, by Theorem 2.3 and Proposition 3.10 we have

$$\begin{aligned} |E_n^{(t,x)}(t, x)| &\leq C \int_t^T \int_{\mathbb{R}^d} \Gamma(t, x; s, \xi) \left\| (t, x)^{-1} \circ (s, \xi) \right\|_B^{n+1} \\ &\quad \times (T-s)^{\frac{k-2}{2}} \left((T-s)^{\frac{N-n}{2}} + [\xi - e^{(s-t)B} x]_B^{N-n} \right) d\xi ds \end{aligned}$$

(by Proposition 3.9)

$$\begin{aligned} &\leq C \int_t^T (s-t)^{\frac{n+1}{2}} (T-s)^{\frac{k-2}{2}} \left((T-s)^{\frac{N-n}{2}} + (s-t)^{\frac{N-n}{2}} \right) ds \\ &\leq C(T-t)^{\frac{N+k+1}{2}} \end{aligned}$$

where we have used the identity

$$\int_t^T (T-s)^n (s-t)^k ds = \frac{\Gamma_E(k+1) \Gamma_E(n+1)}{\Gamma_E(k+n+2)} (T-t)^{k+n+1}, \quad n, k > -1,$$

with Γ_E denoting the Euler Gamma function. The case $k = 0$ can be handled similarly performing first an integration by parts in (3.11).

Finally, estimate (3.9) can be proved by a straightforward modification of the proof of (3.8), using also the Chapman–Kolmogorov equation. We omit the details for brevity. $\square$

Remark 3.11. Under the assumptions of Theorem 3.8, we have also error bounds for the approximation of the derivatives of u ; precisely, we have

$$|D_x^\alpha u(t, x) - D_x^\alpha U_N^{(\bar{z})}(t, x)|_{\bar{z}=(t,x)} \leq C(T-t)^{\frac{N+k+1-|\alpha|_B}{2}}, \quad |\alpha|_B \leq N. \quad (3.12)$$

The proof of this formula is analogous to the proof of Theorem 3.8, once D_x^α is applied to the representation formulas (3.10) and (3.11). When $u(t, x)$ represents the price of an arithmetic

Asian option, formula (3.12) provides error bounds on the approximate sensitivities or, as they are usually called in finance, the Greeks. For instance, in the case of a fixed-strike Asian option (see Example 2.5), we have $k = 3$ and thus

$$\begin{aligned} |\text{Delta} - \partial_{x_1} U_N^{\bar{z}}|_{\bar{z}=(t,x_1,x_2)} &\leq C(T-t)^{\frac{N+3}{2}}, \\ |\text{Gamma} - \partial_{x_1,x_1} U_N^{\bar{z}}|_{\bar{z}=(t,x_1,x_2)} &\leq C(T-t)^{\frac{N+2}{2}}, \end{aligned}$$

Delta := $\partial_{x_1} u$ and Gamma := $\partial_{x_1,x_1} u$.

3.2. Proof of Theorem 3.5

Throughout this section we suppose the assumptions of Theorem 3.5 to be in force. The proof of Theorem 3.5 is based on some estimates on short cylinders initially introduced in [40] for uniformly parabolic operators and later generalized to Kolmogorov operators in [4].

First, we introduce the “cylinder” of radius R and height h centered in $(s, \xi) \in \mathbb{R} \times \mathbb{R}^d$ and its lateral and parabolic boundaries, respectively:

$$\begin{aligned} H_{h,R}(s, \xi) &:= \{(t, x) \in \mathbb{R} \times \mathbb{R}^d | s - h < t < s, [x - e^{(t-s)B} \xi]_B < R\}, \\ \Sigma_{h,R}(s, \xi) &:= \{(t, x) \in \mathbb{R} \times \mathbb{R}^d | s - h < t < s, [x - e^{(t-s)B} \xi]_B = R\}, \\ \partial_P H_{h,R}(s, \xi) &:= \Sigma_{h,R}(s, \xi) \cup \{(s, x) \in \mathbb{R} \times \mathbb{R}^d | [x - \xi]_B < R\}. \end{aligned}$$

We explicitly observe that the cylinders are invariant with respect to the left translations in $\mathcal{G}_B$, meaning that $z \circ H_{h,R}(\zeta) = H_{h,R}(z \circ \zeta)$ for any $z, \zeta \in \mathbb{R} \times \mathbb{R}^d$. We also recall the following inequality (see Proposition 2.1 in [31]):

$$\|z \circ \zeta\|_B \leq c_B (\|z\|_B + \|\zeta\|_B), \quad z, \zeta \in \mathbb{R} \times \mathbb{R}^d, \quad (3.13)$$

where $c_B \geq 1$ is a constant that depends only on the matrix B . In particular, taking $z = (0, x)$ and $\zeta = (t, 0)$, (3.13) implies that

$$[e^{tB} x]_B \leq \|(t, e^{tB} x)\|_B = \|z \circ \zeta\|_B \leq c_B \left(|t|^{\frac{1}{2}} + [x]_B \right), \quad t \in \mathbb{R}, x \in \mathbb{R}^d. \quad (3.14)$$

Lemma 3.12. *There exist $C > 0$, $\varepsilon \in]0, 1[$, only dependent on M, μ, B , and a nonnegative function $v \in C([0, T] \times \mathbb{R}^d) \cap C_{B,loc}^{2,1}$ such that, for every $R > 0$ we have*

$$\mathcal{K}v(t, x) = 0 \quad (t, x) \in H_{\varepsilon R^2, R}(T, 0), \quad (3.15)$$

$$v(t, x) \geq 1, \quad (t, x) \in \Sigma_{\varepsilon R^2, R}(T, 0), \quad (3.16)$$

$$v(t, x) \leq C \exp\left(-\frac{R^2}{C(T-t)}\right) \quad (t, x) \in H_{\varepsilon R^2, \frac{R}{8c_B^2}}(T, 0), \quad (3.17)$$

where c_B is the constant in (3.13).

Proof. Let Γ denote the fundamental solution of $\mathcal{K}$. The proof is based on the Gaussian upper bound for Γ in Proposition 3.9, which we recall here for greater convenience: there exists a positive constant c^+ , only depending on M, μ and B , such that

$$\Gamma(t, x; s, \xi) \leq c^+ \Gamma^\Lambda(t, x; s, \xi), \quad 0 \leq t < s \leq T, x, \xi \in \mathbb{R}^d, \quad (3.18)$$

where Γ^Λ is the fundamental solution of the constant coefficients Kolmogorov operator in (2.1) and Λ is strictly greater than M , say $\Lambda = 2M$.

Next, we set

$$v(t, x) = 2 \int_{\mathbb{R}^d} \Gamma(t, x; T, y) \chi_R(y) dy, \quad t < T, \quad x \in \mathbb{R}^d,$$

where $\chi_R \in C^\infty(\mathbb{R}^d, [0, 1])$ is a cut-off function such that $\chi_R(y) = 0$ if $[y]_B < \frac{R}{2}$ and $\chi_R(y) = 1$ if $[y]_B > \frac{3}{4}R$. By definition, it is clear that v satisfies (3.15). Moreover, we have

$$\lim_{t \rightarrow T^-} v(t, x) = 2\chi_R(x) = 2, \quad (3.19)$$

uniformly w.r.t. $x \in \mathbb{R}^d$ such that $[x]_B = R$: this follows by noting that

$$\begin{aligned} |v(t, x) - 2\chi_R(x)| &\leq 2 \int_{\mathbb{R}^d} \Gamma(t, x; T, y) |\chi_R(y) - \chi_R(x)| dy \\ &\leq 2c^+ \int_{\mathbb{R}^d} \Gamma^A(t, x; T, y) |\chi_R(y) - \chi_R(x)| dy. \end{aligned} \quad (\text{by (3.18)})$$

Now, by (3.19) there exists $\varepsilon > 0$, which we can safely assume to be less than $\frac{1}{16c_B^4}$ and $\frac{1}{64c_B^2}$, such that (3.16) holds.

The proof of (3.17) depends on the reverse triangle inequality for the norm $[\cdot]_B$:

$$[y - e^{tB}x]_B \geq \frac{1}{c_B} [y]_B - c_B \left(|t|^{\frac{1}{2}} + [x]_B \right), \quad t \in \mathbb{R}, \quad x, y \in \mathbb{R}^d,$$

whose proof is an easy consequence of (3.14). In particular, if $[y]_B \geq \frac{R}{2}$ and $(t, x) \in H_{\varepsilon R^2, \frac{R}{8c_B^2}}(T, 0)$, then in light of the first bound for ε we get

$$[y - e^{(T-t)B}x]_B \geq \frac{R}{8c_B}. \quad (3.20)$$

Hence, for such (t, x) we get

$$\begin{aligned} v(t, x) &\leq 2c^+ \int_{\mathbb{R}^d} \Gamma^A(t, x; T, y) \chi_R(y) dy \leq 2c^+ \int_{[y]_B \geq \frac{R}{2}} \Gamma^A(t, x; T, y) dy \\ &= \frac{2c^+(2\pi)^{-\frac{d}{2}}}{\sqrt{|\mathbf{C}(T-t)|}} \\ &\quad \times \int_{[y]_B \geq \frac{R}{2}} \exp \left(-\frac{1}{2} \langle \mathbf{C}^{-1}(T-t)(y - e^{(T-t)B}x), (y - e^{(T-t)B}x) \rangle \right) dy \end{aligned}$$

(by (3.20) and denoting by $\mathbf{C}$ the matrix in (3.2) with $A_0 = \Lambda I_{p_0}$ and I_{p_0} being the $(p_0 \times p_0)$ identity matrix)

$$\begin{aligned} &\leq \frac{2c^+(2\pi)^{-\frac{d}{2}}}{\sqrt{|\mathbf{C}(T-t)|}} \\ &\quad \times \int_{[y - e^{(T-t)B}x]_B \geq \frac{R}{8c_B}} \exp \left(-\frac{1}{2} \langle \mathbf{C}^{-1}(T-t)(y - e^{(T-t)B}x), (y - e^{(T-t)B}x) \rangle \right) dy \end{aligned}$$

(by the change of variables $\eta = D_0(\frac{1}{\sqrt{T-t}})(y - e^{(T-t)B}x)$ and the homogeneity relation (B.1))

$$= \frac{2c^+(2\pi)^{-\frac{d}{2}}}{\sqrt{|\mathbf{C}(1)|}} \int_{[\eta]_B \geq \frac{R}{8c_B\sqrt{T-t}}} \exp \left(-\frac{1}{2} \langle \mathbf{C}^{-1}(1)\eta, \eta \rangle \right) d\eta. \quad (3.21)$$

Since we are assuming $T - t \leq \varepsilon R^2$, thanks to the second bound on ε we have $[\eta]_B \geq \frac{R}{8c_B\sqrt{T-t}} \geq 1$ and thus, there exists $C_0 > 0$ only dependent on μ, M, B , such that

$$\begin{aligned} \langle \mathbf{C}^{-1}(1)\eta, \eta \rangle &\geq C_0 |\eta|^2 = C_0 \sum_{j=1}^d \frac{|\eta_j|^2}{[\eta]_B^{2\sigma_j}} [\eta]_B^{2\sigma_j} = C_0 \sum_{j=1}^d \left(\frac{|\eta_j|^{1/\sigma_j}}{[\eta]_B} \right)^{2\sigma_j} [\eta]_B^{2\sigma_j} \\ &\geq C_0 [\eta]_B^2 \sum_{j=1}^d \left(\frac{|\eta_j|^{1/\sigma_j}}{[\eta]_B} \right)^{2(2r+1)} \\ &\geq \frac{C_0}{d^{4r+1}} [\eta]_B^2 \left(\sum_{j=1}^d \frac{|\eta_j|^{1/\sigma_j}}{[\eta]_B} \right)^{2(2r+1)} \\ &= \frac{C_0}{d^{4r+1}} [\eta]_B^2. \end{aligned}$$

Setting $C_1 := \frac{C_0}{d^{4r+1}}$ we get

$$\begin{aligned} &\int_{[\eta]_B \geq \frac{R}{8c_B\sqrt{T-t}}} \exp\left(-\frac{1}{2} \langle \mathbf{C}^{-1}(1)\eta, \eta \rangle\right) d\eta \\ &\leq \int_{[\eta]_B \geq \frac{R}{8c_B\sqrt{T-t}}} \exp\left(-\frac{1}{2} C_1 [\eta]_B^2\right) d\eta \\ &\leq \max_{[y]_B \geq \frac{R}{8c_B\sqrt{T-t}}} \exp\left(-\frac{1}{4} C_1 [y]_B^2\right) \int_{[\eta]_B \geq \frac{R}{8c_B\sqrt{T-t}}} \exp\left(-\frac{1}{4} C_1 [\eta]_B^2\right) d\eta \\ &\leq \exp\left(-\frac{C_1 R^2}{2^8 c_B^2 (T-t)}\right) \int_{\mathbb{R}^d} \exp\left(-\frac{1}{4} C_1 [\eta]_B^2\right) d\eta, \end{aligned}$$

which, combined with (3.21), proves (3.17). $\square$

Proof of Theorem 3.5. Since the statement is a short-time estimate on a compact subset, it is enough to prove (3.7) for $(t, x) \in H_{\varepsilon R^2, R}(T, \xi) \subseteq]0, T[\times D$ for suitably small $\varepsilon, R > 0$. Secondly, we can suppose $\xi = 0$. In fact, if u is a solution to $\mathcal{K}u = 0$ in $H_{\varepsilon R^2, R}(T, \xi)$ then $w(t, x) = u(t, x - e^{-TB}\xi)$ solves on $H_{\varepsilon R^2, R}(T, 0)$ the operator obtained through $\mathcal{K}$ by translating its coefficients.

Let us denote by u^ψ the unique solution (with polynomial growth) to the Cauchy problem

$$\begin{cases} \mathcal{K}f = 0, & \text{on } [0, T[\times \mathbb{R}^d, \\ f(T, \cdot) = \psi, & \text{on } \mathbb{R}^d, \end{cases}$$

with ψ as in Assumption 3.2, and by U_N^ψ its N th order approximation as defined in Section 3. By triangular inequality we have

$$|u - U_N| \leq |u - u^\psi| + |u^\psi - U_N^\psi| + |U_N^\psi - U_N|. \quad (3.22)$$

We now aim at estimating each of the terms in the sum above.

We start with $|u - u^\psi|$. Let v be the function appearing in Lemma 3.12. By Proposition 3.3 and (3.15), $u - u^\psi$ and v solve $\mathcal{K}w = 0$ in $H_{\varepsilon R^2, R}(T, 0)$ and are continuous on $\overline{H_{\varepsilon R^2, R}(T, 0)}$.

Moreover, $(u - u^\psi)(T, x) = 0$ if $[x]_B < R$. Therefore, by setting

$$C_1 := \max_{\Sigma_{\varepsilon R^2, R}(T, 0)} |u - u^\psi|,$$

we get $|u - u^\psi| \leq C_1 v$ on $\partial_P H_{\varepsilon R^2, R}(T, 0)$ and, by the maximum principle combined with estimate (3.17) of Lemma 3.12, we have

$$|(u - u^\psi)(t, x)| \leq C_1 C_2 \exp\left(-\frac{R^2}{C_2(T-t)}\right), \quad (t, x) \in H_{\varepsilon R^2, \frac{R}{8c_B^2}}(T, 0), \quad (3.23)$$

with $C_2 > 0$ depending only on M, μ, B .

We continue by estimating $|u^\psi - U_N^\psi|$. By Theorem 3.8 there exists $C_3 > 0$, only dependent on M, μ, B, T, N and $\|\psi\|_{C_B^k(\mathbb{R}^d)}$, such that

$$|u^\psi(t, x) - U_N^\psi(t, x)| \leq C_3(T-t)^{\frac{N+k+1}{2}}, \quad (t, x) \in [0, T] \times \mathbb{R}^d. \quad (3.24)$$

We conclude by estimating $|U_N^\psi - U_N|$. First observe that, by (A.1), for any multi-index $\alpha \in \mathbb{N}_0^d$ we have

$$\begin{aligned} D_x^\alpha (u_0^{(\bar{z})} - u_0^{(\bar{z}), \psi})(t, x) &= D_x^\alpha \int_{\mathbb{R}^d} \Gamma_0^{(\bar{z})}(t, x; T, y) (\varphi(y) - \psi(y)) dy \\ &= \int_{\mathbb{R}^d} D_x^\alpha \Gamma_0^{(\bar{z})}(t, x; T, y) (\varphi(y) - \psi(y)) dy, \end{aligned}$$

with $\Gamma_0^{(\bar{z})}$ as in (3.1). Now, $\Gamma_0^{(\bar{z})}$ is the fundamental solution of the constant-coefficients Kolmogorov operator $\mathcal{K}_0^{(\bar{z})}$ in (1.11), for which Assumptions 1.1, 1.2 and 3.1 are trivially satisfied. Therefore, the bounds in Proposition 3.9 also apply to $\Gamma_0^{(\bar{z})}$ and yield

$$|D_x^\alpha (u_0^{(\bar{z}), \psi} - u_0^{(\bar{z})})(t, x)| \leq C_4(T-t)^{-\frac{|\alpha|_B}{2}} w(t, x), \quad \bar{z} \in \mathbb{R} \times \mathbb{R}^d, \quad (t, x) \in [0, T[\times \mathbb{R}^d, \quad (3.25)$$

with

$$w(t, x) := \int_{\mathbb{R}^d} \Gamma^{2M}(t, x; T, y) |(\varphi(y) - \psi(y))| dy,$$

where Γ^{2M} is the fundamental solution of the Kolmogorov operator $\mathcal{K}^{2M}$ as in (2.1), and $C_4 > 0$ only depends on $M, \mu, B, T, |\alpha|_B$. Now note that, by (3.6) and (A.5), we have

$$(U_N^\psi - U_N)(t, x) = (u_0^{(\bar{z}), \psi} - u_0^{(\bar{z})})(t, x) + \sum_{n=1}^N \mathcal{L}_n^{(\bar{z})}(u_0^{(\bar{z}), \psi} - u_0^{(\bar{z})})(t, x) \Big|_{\bar{z}=(t, x)}.$$

Thus by Lemma B.7 with (3.25) we get

$$|(U_N^\psi - U_N)(t, x)| \leq C_5 |w(t, x)|, \quad (t, x) \in [0, T[\times \mathbb{R}^d,$$

where $C_5 > 0$ only depends on M, μ, B, T and N . By repeating step by step the same proof of (3.23) it is straightforward to obtain an estimate for $|w(t, x)|$ analogous to (3.23), which finally yields

$$|(U_N^\psi - U_N)(t, x)| \leq C_5 C_6 C_7 \exp\left(-\frac{R^2}{C_7(T-t)}\right), \quad (t, x) \in H_{\varepsilon R^2, \frac{R}{8c_B^2}}(T, 0), \quad (3.26)$$

with $C_7 > 0$ depending only on M, μ, B, T, N , and

$$C_6 := \max_{\Sigma_{\varepsilon R^2, R}(T, 0)} |w|.$$

Plugging (3.23)–(3.24)–(3.26) into (3.22) yields (3.7) for $(t, x) \in H_{\varepsilon R^2, \frac{R}{8c_B^2}}(T, 0)$ and concludes the proof. $\square$

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Appendix A. Analytical approximation formulas

We show that the functions $u_n^{(\bar{z})}$ in (3.4)–(3.5) can be explicitly computed at any order. It is clear that the leading term $u_0^{(\bar{z})}$ is given by

$$u_0^{(\bar{z})}(t, x) = \int_{\mathbb{R}^d} \Gamma_0^{(\bar{z})}(t, x; T, y) \varphi(y) dy, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \quad (\text{A.1})$$

where $\Gamma_0^{(\bar{z})}$ is the Gaussian density in (3.1). For $n \in \mathbb{N}$ with $n \leq N$, the explicit representation for the correcting terms $u_n^{(\bar{z})}$ can be derived using the following notable symmetry properties of $\Gamma_0^{(\bar{z})}$.

Lemma A.1. *For any $x, y \in \mathbb{R}^d$, $t < s$ and $\bar{z} = (\bar{t}, \bar{x}) \in \mathbb{R} \times \mathbb{R}^d$, we have*

$$\nabla_x \Gamma_0^{(\bar{z})}(t, x; s, y) = -e^{(s-t)B^*} \nabla_y \Gamma_0^{(\bar{z})}(t, x; s, y), \quad (\text{A.2})$$

$$y \Gamma_0^{(\bar{z})}(t, x; s, y) = \mathcal{M}^{(\bar{z})}(s - t, x) \Gamma_0^{(\bar{z})}(t, x; s, y), \quad (\text{A.3})$$

where $\mathcal{M}^{(\bar{z})}(t, x)$ is the operator defined as

$$\mathcal{M}^{(\bar{z})}(t, x) = e^{tB} (x + \mathbf{M}_{\bar{z}}(t) \nabla_x), \quad \mathbf{M}_{\bar{z}}(t) = e^{-tB} \mathbf{C}_{\bar{z}}(t) e^{-tB^*}. \quad (\text{A.4})$$

Proof. Using the explicit expression of $\Gamma_0^{(\bar{z})}$, the proof is a direct computation. $\square$

The following result provides an explicit representation of $u_n^{(\bar{z})}$ in (3.3): remarkably, it can be written as a finite sum of spatial derivatives acting on $u_0^{(\bar{z})}$.

Theorem A.2. *Let Assumptions 1.1, 1.2 and 3.1 be in force. Then, for any $n \in \mathbb{N}$ with $n \leq N$, and for any $\bar{z} \in \mathbb{R} \times \mathbb{R}^d$, we have*

$$u_n^{(\bar{z})}(t, x) = \mathcal{L}_n^{(\bar{z})}(t, T, x) u_0^{(\bar{z})}(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d. \quad (\text{A.5})$$

In (A.5), $\mathcal{L}_n^{(\bar{z})}(t, T, x)$ denotes the differential operator

$$\mathcal{L}_n^{(\bar{z})}(t, T, x) = \sum_{h=1}^n \int_t^T ds_1 \int_{s_1}^T ds_2 \cdots \int_{s_{h-1}}^T ds_h \sum_{i \in I_{n,h}} \mathcal{G}_{i_1}^{(\bar{z})}(t, s_1, x) \cdots \mathcal{G}_{i_h}^{(\bar{z})}(t, s_h, x), \quad (\text{A.6})$$

where

$$I_{n,h} = \{i = (i_1, \dots, i_h) \in \mathbb{N}^h \mid i_1 + \dots + i_h = n\}, \quad 1 \leq h \leq n,$$

and

$$\begin{aligned} \mathcal{G}_n^{(\bar{z})}(t, s, x) &= \frac{1}{2} \sum_{i,j=1}^{p_0} (\mathbf{T}_n(a_{ij}, \bar{z}) - \mathbf{T}_{n-1}(a_{ij}, \bar{z}))(s, \mathcal{M}^{(\bar{z})}(s-t, x)) \\ &\quad \times (e^{-(s-t)B^*} \nabla_x)_i (e^{-(s-t)B^*} \nabla_x)_j \\ &\quad + \sum_{i=1}^{p_0} (\mathbf{T}_{n-1}(a_i, \bar{z}) - \mathbf{T}_{n-2}(a_i, \bar{z}))(s, \mathcal{M}^{(\bar{z})}(s-t, x)) (e^{-(s-t)B^*} \nabla_x)_i, \end{aligned} \quad (\text{A.7})$$

with $\mathcal{M}^{(\bar{z})}(t, x)$ as in (A.4) and, by convention, $\mathbf{T}_{-1}f \equiv 0$.

Next, we sketch the proof of [Theorem A.2](#) that is based on the symmetry properties of the Gaussian density Γ_0 in (3.1), combined with an extensive use of other very general relations such as the Duhamel's principle and the Chapman–Kolmogorov equation. Since the choice of $\bar{z}$ is unimportant throughout this section, we drop the explicit dependence on $\bar{z}$ in the following formulas. First, we generalize formula (A.3) to polynomial functions p with time-dependent coefficients, that is $p = p(t, \cdot)$ is a polynomial for every fixed $t \in \mathbb{R}$: this will be used to deal with the operators $\mathcal{K}_n$ in (1.10) that have coefficients of this form.

Proposition A.3. *For any $t, s, s_1 \in [0, T]$, with $t < s$, $x, y \in \mathbb{R}^d$, we have*

$$p(s_1, y) \Gamma_0(t, x; s, y) = p(s_1, \mathcal{M}(s-t, x)) \Gamma_0(t, x; s, y). \quad (\text{A.8})$$

Proof. Let us recall that operator $\mathcal{M}(t, x)$ acts only on the variable x . First, we prove that the components $\mathcal{M}_j(t, x)$, $i = 1, \dots, d$, commute when applied to $\Gamma_0 = \Gamma_0(t, x; s, y)$ and to its derivatives (notice however that this is not true in general when they are applied to a generic function). Notice also that formula (A.2) expresses an x -derivative as a linear combination of y -derivatives with coefficients that depend only on t and s . This is obviously true also for higher orders and we express it through the differential operator $S_y^\beta(s-t)$, acting on y , defined by

$$D_x^\beta \Gamma_0(t, x; s, y) = S_y^\beta(s-t) \Gamma_0(t, x; s, y).$$

Now we have

$$\begin{aligned} &\mathcal{M}_i(s-t, x) \mathcal{M}_j(s-t, x) D_x^\beta \Gamma_0 \\ &= \mathcal{M}_i(s-t, x) \mathcal{M}_j(s-t, x) S_y^\beta(s-t) \Gamma_0 && \text{(by the definition above)} \\ &= S_y^\beta(s-t) (\mathcal{M}_i(s-t, x) \mathcal{M}_j(s-t, x) \Gamma_0) && (S_y^\beta \text{ and } \mathcal{M}_j \text{ commute)} \\ &= S_y^\beta(s-t) (\mathcal{M}_i(s-t, x) y_j \Gamma_0) && \text{(by (A.3))} \\ &= S_y^\beta(s-t) (y_j \mathcal{M}_i(s-t, x) \Gamma_0) \\ &= S_y^\beta(s-t) (y_j y_i \Gamma_0) && \text{(again, by (A.3))} \\ &= \mathcal{M}_j(s-t, x) \mathcal{M}_i(s-t, x) D_x^\beta \Gamma_0. && \text{(by reversing the steps above)} \end{aligned}$$

Since $p(s_1, \cdot)$ is a polynomial by definition, we therefore have that the operators $p(s_1, \mathcal{M}(s - t, x))$ are defined unambiguously when applied to $\Gamma_0(t, x; s, y)$ and to its derivatives. Moreover, clearly (A.8) is now a straightforward consequence of (A.3). $\square$

Remark A.4. By Proposition A.3, the operators $\mathcal{G}_n(t, s, x)$ are defined unambiguously when applied to $\Gamma_0 = \Gamma_0(t, x; s, y)$, to its derivatives and, more generally, by the representation formula (A.1), to solutions of the Cauchy problem (3.4).

The next proposition, essentially based on the symmetries of Lemma A.1, is the key of the proof of Theorem A.2.

Proposition A.5. For any $x, y \in \mathbb{R}^d$, $t < s$ and $n \in \mathbb{N}$ with $n \leq N$, we have

$$\int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) ((\mathcal{K}_n - \mathcal{K}_{n-1}) f)(s, \xi) d\xi = \mathcal{G}_n(t, s, x) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) f(\xi) d\xi, \quad (\text{A.9})$$

for any $f \in C_0^2(\mathbb{R}^d)$.

Proof. To keep formulas at a reasonable size we suppose that the functions a_i , $i = 1, \dots, p_0$, in (1.1) are identically zero. By the definition (1.10) we have

$$\begin{aligned} & \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) ((\mathcal{K}_n - \mathcal{K}_{n-1}) f)(s, \xi) d\xi \\ &= \frac{1}{2} \sum_{i,j=1}^{p_0} \int_{\mathbb{R}^d} (\mathbf{T}_n(a_{ij}, \bar{z}) - \mathbf{T}_{n-1}(a_{ij}, \bar{z})) \\ & \quad \times (s, \xi) \Gamma_0(t, x; s, \xi) \partial_{\xi_i \xi_j} f(\xi) d\xi \\ &= \frac{1}{2} \sum_{i,j=1}^{p_0} (\mathbf{T}_n(a_{ij}, \bar{z}) - \mathbf{T}_{n-1}(a_{ij}, \bar{z})) (s, \mathcal{M}(s - t, x)) \\ & \quad \times \int_{\mathbb{R}^d} \Gamma_0(t, x; y; s, \xi, \omega) \partial_{\xi_i \xi_j} f(\xi) d\xi \quad (\text{by (A.8)}) \\ &= \frac{1}{2} \sum_{i,j=1}^{p_0} (\mathbf{T}_n(a_{ij}, \bar{z}) - \mathbf{T}_{n-1}(a_{ij}, \bar{z})) (s, \mathcal{M}(s - t, x)) \\ & \quad \times \int_{\mathbb{R}^d} \partial_{\xi_i \xi_j} \Gamma_0(t, x; s, \xi) f(\xi) d\xi \quad (\text{by parts}) \\ &= \mathcal{G}_n(t, s, x) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) f(\xi) d\xi. \quad (\text{by (A.2) and (A.7)}). \quad \square \end{aligned}$$

The proof of Theorem A.2 consists of mostly formal and tedious computations that are totally analogous to those given for the parabolic case in Section 5 in [29]. This may not be surprising since our framework contains the parabolic one as a special case. Therefore, we only give a proof for $n = 1$, which still sheds light on the origin of the operators $\mathcal{L}_n$.

By definition, u_1 is the solution of the Cauchy problem (3.5) with $n = 1$. By Duhamel's principle we have

$$\begin{aligned} u_1(t, x) &= \int_t^T \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \\ & \quad \times ((\mathcal{K}_1 - \mathcal{K}_0) u_0)(s, \xi) d\xi ds \end{aligned}$$

$$\begin{aligned}
 &= \int_t^T \mathcal{G}_1(t, s, x) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) u_0(s, \xi) d\xi ds \quad (\text{by (A.9) with } n = 1) \\
 &= \int_t^T \mathcal{G}_1(t, s, x) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \\
 &\quad \times \int_{\mathbb{R}^d} \Gamma_0(s, \xi; T, y) \varphi(y) dy d\xi ds \quad (\text{by (A.1)}) \\
 &= \int_t^T \mathcal{G}_1(t, s, x) \int_{\mathbb{R}^d} \varphi(y) \\
 &\quad \times \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \Gamma_0(s, \xi; T, y) d\xi dy ds \quad (\text{Fubini's theorem}) \\
 &= \int_t^T \mathcal{G}_1(t, s, x) ds u_0(t, x) \quad (\text{Chapman–Kolmogorov and (A.1)}) \\
 &= \mathcal{L}_1(t, T, x) u_0(t, x). \quad (\text{by (A.6)}) \quad \square
 \end{aligned}$$

Appendix B. Proof of Proposition 3.10

In this section we prove some preliminary estimates on the spatial derivatives of solutions of constant coefficient-Kolmogorov operators: in particular, we prove estimates for the derivatives of $u_n^{(\bar{z})}$ defined by (3.4)–(3.5). Throughout this section $\bar{z} \in \mathbb{R} \times \mathbb{R}^d$ is fixed.

Proposition B.1. *Let $k \in [0, 2r + 1]$, $\beta \in \mathbb{N}_0^d$ with $|\beta|_B > 0$. If $\psi \in C_B^k(\mathbb{R}^d)$ then the solution $u_0^{(\bar{z})}$ of the Cauchy problem (3.4) satisfies*

$$|D_x^\beta u_0^{(\bar{z})}(t, x)| \leq C(T - t)^{\frac{k - |\beta|_B}{2}}, \quad 0 \leq t < T, \quad x \in \mathbb{R}^d,$$

where C is a positive constant that depends only on M, μ, B, T, β and $\|\psi\|_{C_B^k(\mathbb{R}^d)}$.

Proof. We prove the case $k \in]0, 2r + 1]$ since the case $k = 0$ is straightforward. We first note that, since $\Gamma_0^{(\bar{z})}$ is a density and $|\beta|_B > 0$, we have

$$D_x^\beta \int_{\mathbb{R}^d} \Gamma_0^{(\bar{z})}(t, x; T, y) dy = 0.$$

and therefore

$$\begin{aligned}
 D_x^\beta u_0^{(\bar{z})}(t, x) &= \int_{\mathbb{R}^d} \psi(y) D_x^\beta \Gamma_0^{(\bar{z})}(t, x; T, y) dy \\
 &= \int_{\mathbb{R}^d} \left(\psi(y) - \psi(e^{(T-t)B}x) \right) D_x^\beta \Gamma_0^{(\bar{z})}(t, x; T, y) dy.
 \end{aligned}$$

Since $\psi \in C_B^k(\mathbb{R}^d)$, we obtain

$$\begin{aligned}
 |D_x^\beta u_0^{(\bar{z})}(t, x)| &\leq \|\psi\|_{C_B^k(\mathbb{R}^d)} \int_{\mathbb{R}^d} [y - e^{(T-t)B}x]_B^k |D_x^\beta \Gamma_0^{(\bar{z})}(t, x; T, y)| dy \\
 &\leq C \|\psi\|_{C_B^k(\mathbb{R}^d)} (T - t)^{\frac{k - |\beta|_B}{2}} \int_{\mathbb{R}^d} \Gamma^{2M}(t, x; T, y) dy,
 \end{aligned}$$

where the second inequality follows from a direct estimate on the derivatives of $\Gamma_0^{(\bar{z})}$ (see, for example, Section 2 in [38]) and Γ^{2M} is the fundamental solution of the Kolmogorov operator $\mathcal{K}^{2M}$ as defined in (2.1). $\square$

In the next lemmas we will use the following result proved in [27].

Lemma B.2. *The following homogeneity relations hold*

$$C_{\bar{z}}(t) = D_0(\sqrt{t}) C_{\bar{z}}(1) D_0(\sqrt{t}), \quad (\text{B.1})$$

$$M_{\bar{z}}(t) = D_0(\sqrt{t}) M_{\bar{z}}(1) D_0(\sqrt{t}), \quad (\text{B.2})$$

$$e^{tB} = D_0(\sqrt{t}) e^B D_0\left(\frac{1}{\sqrt{t}}\right), \quad (\text{B.3})$$

for any $t > 0$.

Notation B.3. From now to the end of this section, we use the Greek letters $\alpha, \beta, \gamma, \delta, \nu$ to denote multi-indexes in $\mathbb{N}_0^d$, and $|\alpha| = \sum_{i=1}^d \alpha_i$ is the standard Euclidean height of α . To simplify notations, if I is any family of indexes, we use the unconventional notation

$$\sum_{\ell \in I} \pi_{\ell} = \sum_{\ell \in I} c_{\ell} \pi_{\ell}$$

for a sum where the constants c_{ℓ} depend only on $\bar{z}, B, N, T, a_{ij}, a_i$ and are uniformly bounded by a constant that depends only on M, μ, B, T, N and B .

Lemma B.4. *Let*

$$\mathcal{W}(t) = e^{-tB^*} \nabla_x, \quad t \in \mathbb{R},$$

denote the differential operators appearing in (A.7) and by $\mathcal{W}^{\alpha}(t)$ the composition²

$$\mathcal{W}^{\alpha}(t) = \mathcal{W}_1^{\alpha_1}(t) \cdots \mathcal{W}_d^{\alpha_d}(t). \quad (\text{B.4})$$

The following representation holds true:

$$\mathcal{W}^{\beta}(t) = \sum_{\substack{|\alpha| = |\beta| \\ |\alpha|_B \geq |\beta|_B}} t^{\frac{|\alpha|_B - |\beta|_B}{2}} D_x^{\alpha}.$$

Proof. It suffices to prove the statement for a single $\mathcal{W}_i(t)$. Using the relations in Lemma B.2, we have

$$\begin{aligned} \mathcal{W}_i(t) &= \sum_{j=1}^d D_0\left(\frac{1}{\sqrt{t}}\right)_{ii} e_{ij}^{-B^*} D_0(\sqrt{t})_{jj} \partial_{x_j} \\ &= t^{-\frac{\alpha_i}{2}} \sum_{j=1}^d e_{ij}^{-B^*} t^{\frac{\sigma_j}{2}} \partial_{x_j}, \end{aligned}$$

² Operator $\mathcal{W}^{\alpha}(t)$ in (B.4) is well defined since the components of $\mathcal{W}(t)$ commute.

with σ_i as in (2.3). The result follows noting that the intrinsic order of ∂_{x_j} is exactly σ_j . Moreover, as the matrix e^{-B^*} is upper triangular the sum actually ranges over $j = i, \dots, d$ and thus $\sigma_j - \sigma_i$ is always a nonnegative integer. $\square$

Next step is the study of the operator $\mathcal{M}^{(\bar{z})}(t, x)$: we recall that, by Proposition A.3, the components of $\mathcal{M}^{(\bar{z})}(t, x)$ commute when applied to $I_0^{(\bar{z})}$ and more generally to $u_n^{(\bar{z})}$ and its derivatives.

Lemma B.5. *For any $\beta \in \mathbb{N}_0^d$, we have*

$$\left(\mathcal{M}^{(\bar{z})}(s-t, x) - e^{(s-\bar{t})B} \bar{x}\right)^\beta = \sum_{\substack{|\delta|+|\alpha|\leq|\beta| \\ |\delta|_B-|\alpha|_B\leq|\beta|_B}} (s-t)^{\frac{|\beta|_B+|\alpha|_B-|\delta|_B}{2}} \left(x - e^{(t-\bar{t})B} \bar{x}\right)^\delta D_x^\alpha. \quad (\text{B.5})$$

Proof. First of all, let us note that

$$\mathcal{M}^{(\bar{z})}(s-t, x) - e^{(s-\bar{t})B} \bar{x} = e^{(s-t)B} \left(x - e^{(t-\bar{t})B} \bar{x} + \mathbf{M}_{\bar{z}}(s-t) \nabla_x\right),$$

and it is not restrictive to take $\bar{x} = 0$ and $t = 0$. We proceed now by induction on $|\beta|$. If $|\beta| = 1$ then $\beta = \mathbf{e}_i$ where $\mathbf{e}_i$ is the i th element of the canonical basis of $\mathbb{R}^d$. A direct computation shows

$$\begin{aligned} (\mathcal{M}^{(\bar{z})}(s, x))^{\mathbf{e}_i} &= \sum_{\substack{|\delta|=1 \\ |\delta|_B \leq |\mathbf{e}_i|_B}} s^{\frac{|\mathbf{e}_i|_B-|\delta|_B}{2}} (x^\delta + (\mathbf{M}_{\bar{z}}(s) \nabla_x)^\delta) \quad (\text{by (B.3)}) \\ &= \sum_{\substack{|\delta|=1 \\ |\delta|_B \leq |\mathbf{e}_i|_B}} s^{\frac{|\mathbf{e}_i|_B-|\delta|_B}{2}} \left(x^\delta + s^{\frac{|\delta|_B}{2}} \sum_{|\nu|=1} s^{\frac{|\nu|_B}{2}} D_x^\nu\right), \quad (\text{by (B.2)}) \end{aligned}$$

which proves (B.5) with $\beta = \mathbf{e}_i$. We now assume the statement to hold for $|\beta| \leq n$, and prove it true for $\beta + \mathbf{e}_i$. By inductive hypothesis applied to both β and $\mathbf{e}_i$ we get

$$\begin{aligned} (\mathcal{M}^{(\bar{z})}(s, x))^{\beta+\mathbf{e}_i} &= \sum_{\substack{|\delta^1|+|\alpha^1|\leq 1 \\ |\delta^1|_B-|\alpha^1|_B\leq|\mathbf{e}_i|_B}} \sum_{\substack{|\delta^2|+|\alpha^2|\leq|\beta| \\ |\delta^2|_B-|\alpha^2|_B\leq|\beta|_B}} s^{\frac{|\mathbf{e}_i|_B+|\alpha^1|_B-|\delta^1|_B}{2}} s^{\frac{|\beta|_B+|\alpha^2|_B-|\delta^2|_B}{2}} x^{\delta^1} D_x^{\alpha^1} \left(x^{\delta^2} D_x^{\alpha^2}\right) \\ &= \sum_{\substack{|\delta|+|\alpha|\leq|\beta+\mathbf{e}_i| \\ |\delta|_B-|\alpha|_B\leq|\beta+\mathbf{e}_i|_B}} s^{\frac{|\beta+\mathbf{e}_i|_B+|\alpha|_B-|\delta|_B}{2}} x^\delta D_x^\alpha, \quad (\text{setting } \delta = \delta^1 + \delta^2 \text{ and } \alpha = \alpha^1 + \alpha^2). \quad \square \end{aligned}$$

Lemma B.6. *For any $n \in \mathbb{N}$, with $n \leq N$, we have the following representation*

$$\mathcal{G}_n^{(\bar{z})}(t, s, x) = \sum_{(\alpha, \delta) \in I_n} (s-t)^{\frac{|\alpha|_B-|\delta|_B+n-2}{2}} (x - e^{(t-\bar{t})B} \bar{x})^\delta D_x^\alpha, \quad (\text{B.6})$$

where

$$I_n = \{(\alpha, \delta) \in \mathbb{N}_0^d \times \mathbb{N}_0^d \mid 1 \leq |\alpha| \leq n+2, |\delta|_B \leq n, |\alpha|_B - |\delta|_B + n - 2 \geq 0\}.$$

Proof. Using the definition of $\mathcal{G}_n^{(\bar{z})}(t, s, x)$ in (A.7), the proof is a straightforward application of Lemmas B.4 and B.5. $\square$

Lemma B.7. For any $n \in \mathbb{N}$, with $n \leq N$, we have the following representation

$$\mathcal{L}_n^{(\bar{z})}(t, T, x) = \sum_{(\alpha, \delta) \in J_n} (T - t)^{\frac{|\alpha|_B - |\delta|_B + n}{2}} (x - e^{(t-\bar{t})B} \bar{x})^\delta D_x^\alpha, \quad (\text{B.7})$$

where

$$J_n = \{(\alpha, \delta) \in \mathbb{N}_0^d \times \mathbb{N}_0^d \mid 1 \leq |\alpha| \leq 3n, |\delta|_B \leq n, |\alpha|_B - |\delta|_B + n \geq 0\}. \quad (\text{B.8})$$

Proof. For greater convenience we recall the expression of $\mathcal{L}_n^{(\bar{z})}(t, T, x)$ as given in (A.6):

$$\mathcal{L}_n^{(\bar{z})}(t, T, x) = \sum_{h=1}^n \sum_{i \in I_{n,h}} L_{h,i}(t, T, x),$$

where

$$L_{h,i}(t, T, x) := \int_t^T ds_1 \int_{s_1}^T ds_2 \cdots \int_{s_{h-1}}^T ds_h \mathcal{G}_{i_1}^{(\bar{z})}(t, s_1, x) \cdots \mathcal{G}_{i_h}^{(\bar{z})}(t, s_h, x),$$

and $I_{n,h} = \{i = (i_1, \dots, i_h) \in \mathbb{N}^h \mid i_1 + \dots + i_h = n\}$, for $1 \leq h \leq n$. We prove that, for fixed $h \in \{1, \dots, n\}$ and $i \in I_{n,h}$ it holds

$$L_{h,i}(t, T, x) = \sum_{(\alpha, \delta) \in J_n} (T - t)^{\frac{|\alpha|_B - |\delta|_B + n}{2}} (x - e^{(t-\bar{t})B} \bar{x})^\delta D_x^\alpha,$$

the result will then readily follow. We only consider the case $\bar{x} = 0$. Plugging equation (B.6) into the definition of $L_{h,i}$ we obtain

$$\begin{aligned} L_{h,i}(t, T, x) &= \sum_{(\alpha^1, \delta^1) \in I_{i_1}} \cdots \sum_{(\alpha^h, \delta^h) \in I_{i_h}} x^{\delta^1} D_x^{\alpha^1} \left(x^{\delta^2} D_x^{\alpha^2} \left(\cdots \left(x^{\delta^h} D_x^{\alpha^h} \right) \right) \right) \\ &\quad \times \int_t^T \cdots \int_{s_{h-1}}^T \prod_{j=1}^h (s_j - t)^{\frac{|\alpha^j|_B - |\delta^j|_B + i_j - 2}{2}} ds_1 \cdots ds_h. \end{aligned}$$

Now, setting $\alpha = \alpha^1 + \dots + \alpha^h$, $\delta = \delta^1 + \dots + \delta^h$ and recalling that $i_1 + \dots + i_h = n$, the integral above can be easily computed to be equal to

$$(T - t)^{\frac{|\alpha|_B - |\delta|_B + n}{2}},$$

times a constant. The statement follows applying Leibniz rule and noticing that $(\alpha, \delta) \in J_n$ if $(\alpha^j, \delta^j) \in I_{i_j}$ for $j = 1, \dots, h$. $\square$

Proof of Proposition 3.10. By (A.5)–(B.7), we get

$$D_x^\beta u_n^{(\bar{z})}(t, x) = D_x^\beta \sum_{(\alpha, \delta) \in J_n} (T - t)^{\frac{|\alpha|_B - |\delta|_B + n}{2}} (x - e^{(t-\bar{t})B} \bar{x})^\delta D_x^\alpha u_0^{(\bar{z})}(t, x)$$

(by applying Leibniz rule and reordering the indexes of J_n in (B.8))

$$= \sum_{\substack{(\alpha, \delta) \in J_n \\ \nu \leq \min\{\beta, \delta\}}} (T-t)^{\frac{|\alpha|_B - |\delta|_B + n}{2}} (x - e^{(t-\bar{t})B} \bar{x})^{\delta-\nu} D_x^{\alpha+\beta-\nu} u_0^{(\bar{z})}(t, x),$$

where $\nu \leq \min\{\beta, \delta\}$ means that $\nu_i \leq \min\{\beta_i, \delta_i\}$ for any $i = 1, \dots, d$. Now, by applying Proposition B.1 and the property

$$|y^\delta| = \prod_{i=1}^d |y_i|^{\delta_i} \leq \prod_{i=1}^d [y]_B^{\sigma_i \delta_i} = [y]_B^{|\delta|_B}, \quad y \in \mathbb{R}^d,$$

we obtain

$$\begin{aligned} |D_x^\beta u_n^{(\bar{z})}(t, x)| &\leq \sum_{\substack{(\alpha, \delta) \in J_n \\ \nu \leq \min\{\beta, \delta\}}} (T-t)^{\frac{-|\delta|_B + n + k - |\beta|_B + |\nu|_B}{2}} [x - e^{(t-\bar{t})B} \bar{x}]_B^{|\delta|_B - |\nu|_B} \\ &= \sum_{0 \leq m \leq n} (T-t)^{\frac{-m + n + k - |\beta|_B}{2}} [x - e^{(t-\bar{t})B} \bar{x}]_B^m, \end{aligned}$$

and the statement follows by the elementary inequality

$$a^m b^{n-m} \leq a^n + b^n, \quad a, b \in \mathbb{R}_{>0}, \quad 0 \leq m \leq n. \quad \square$$

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