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# Exit times for semimartingales under nonlinear expectation

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**Abstract**

Let $\hat{\mathbb{E}}$ be the upper expectation of a weakly compact but possibly non-dominated family $\mathcal{P}$ of probability measures. Assume that $Y$ is a $d$-dimensional $\mathcal{P}$-semimartingale under $\hat{\mathbb{E}}$. Given an open set $Q \subset \mathbb{R}^d$, the exit time of $Y$ from $Q$ is defined by

$$\tau_Q := \inf\{t \geq 0 : Y_t \in Q^c\}.$$

The main objective of this paper is to study the quasi-continuity properties of $\tau_Q$ under the nonlinear expectation $\hat{\mathbb{E}}$. Under some additional assumptions on the growth and regularity of $Y$, we prove that $\tau_Q \wedge t$ is quasi-continuous if $Q$ satisfies the exterior ball condition. We also give the characterization of quasi-continuous processes and related properties on stopped processes. In particular, we obtain the quasi-continuity of exit times for multi-dimensional $G$-martingales, which nontrivially generalizes the previous one-dimensional result of Song (2011).


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*Keywords:* Nonlinear expectation; $G$-expectation; Multi-dimensional nonlinear semimartingales; Exit times; Quasi-continuity

## 1. Introduction

On the space $\Omega$ of continuous paths, equipped with the topology of uniform convergence on compact sets and filtration generated by the canonical process, let $\mathcal{P}$ be a weakly compact but possibly non-dominated family of probability measures. We define the corresponding upper

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expectation and upper capacity by

$$\hat{\mathbb{E}}[\xi] := \sup_{P\in\mathcal{P}} E_P[\xi], \ c(A) := \sup_{P\in\mathcal{P}} P(A), \quad \text{for random variable } \xi \text{ and measurable set } A.$$

Assume that the process $Y$ is a $d$-dimensional (nonlinear) $\mathcal{P}$-semimartingale, i.e., $Y$ is a semimartingale under each $P \in \mathcal{P}$. A typical case of such kind of nonlinear expectation and nonlinear semimartingales is the notion of $G$-expectation and $G$-martingales proposed by Peng [15,18]. Given an open set $Q \subset \mathbb{R}^d$, we define the exit time of $Y$ from $Q$ by

$$\tau_Q(\omega) := \inf\{t \geq 0 : Y_t(\omega) \in Q^c\}, \quad \text{for } \omega \in \Omega.$$

The aim of this paper is to study the quasi-continuity problem of exit times $\tau_Q$ under the nonlinear expectation $\hat{\mathbb{E}}$.

We say that a random variable is quasi-continuous, if it is continuous outside an open set with any given small capacity, see Denis, Hu and Peng [1]. As is well-known, according to Lusin's theorem, all the Borel measurable random variables defined on a Polish space are quasi-continuous under the linear probability. This is the case that $\mathcal{P}$ is reduced to a single measure. But it is no longer obvious for the general case since the elements in the family $\mathcal{P}$ can be infinite, mutually singular and non-dominated. Roughly speaking, the quasi-continuous random variables are those that can be regarded as the limit of elements in $C_b(\Omega)$ in some proper sense, where $C_b(\Omega)$ is the set of bounded continuous functions on $\Omega$. Many important properties in the nonlinear expectation theory, for example, monotone convergence theorem for decreasing sequence (monotone convergence theorem for increasing sequence is trivial since $\hat{\mathbb{E}}$ is an upper expectation) and (forward and backward) stochastic differential equations driven by $G$-Brownian motion, only hold for random variables with such kind of regularity, see [1], Gao [4], Hu, Ji, Peng and Song [7] and Hu, Lin and Hima [8].

So one of the most important problems in the nonlinear expectation theory is to verify that whether a random variable is quasi-continuous, especially for stopping times in the form of $\tau_Q$ since such kind of problems keep occurring when we stop a process as we often do in the classical analysis. The first breakthrough on this direction was due to Song [22, 2011] (see also Song [24, 2014]) who solved the quasi-continuity problem of exit times when $Y$ is a one-dimensional $G$-martingale and $Q = (-\infty, a)$. But the method of Song relies on a very important observation that $Y_{\tau_Q\wedge t} \geq Y_{\tau_{\overline{Q}}\wedge t}$, which holds only when $d = 1$ and $Q = (-\infty, a)$, and hence cannot be applied to the more general situation. So it remains a fascinating and challenging open problem to establish the quasi-continuity of exit times for general dimension $d$ and domain $Q$.

The main purpose of this paper is to provide a general theory on the quasi-continuity properties of exit times $\tau_Q$, which allows us to maintain the regularity of random variables or processes when we employ the localization techniques. Under some additional assumptions on the growth and regularity for the process $Y$, we prove that $\tau_Q \wedge t$ is quasi-continuous if $Q$ satisfies the exterior ball condition (see Section 3 for the definition). Furthermore, we show that $\tau_Q$ itself is quasi-continuous if $Q$ is also bounded.

Our approach consists two key ingredients. One is to prove that $\tau_Q = \tau_{\overline{Q}}$ q.s. (we say that a property holds "quasi-surely" (q.s.) if it holds $P$-a.s. for each $P \in \mathcal{P}$) in Proposition 3.11, where

$$\tau_{\overline{Q}}(\omega) := \inf\{t \geq 0 : Y_t(\omega) \in \overline{Q}^c\}, \quad \text{for } \omega \in \Omega.$$

This is done by extending the auxiliary function argument in Lions and Menaldi [13] to the case that the quadratic variation of $Y$ has possibly unbounded rate of change and utilizing the tool of regular conditional probability distributions of Stroock and Varadhan [25]. The other key ingredient is to investigate the semi-continuities of $\tau_Q$ and $\tau_{\overline{Q}}$ when the process $Y$ is continuous in $(\omega, t)$ and apply a downward monotone convergence theorem for sets, see Lemma 3.13 and Proposition 3.14. First from the semi-continuities of exit times, take in to account the regularity assumption on $Y$, we deduce that $\tau_Q \wedge t$ is q.s. continuous on nearly all the domain $\Omega$. Then the semi-continuities of $\tau_Q$ and $\tau_{\overline{Q}}$ allow us to use the downward convergence theorem for upper capacity (see Song [22]) to obtain an open set, on the complement of which $\tau_Q \wedge t$ is continuous in $\omega$.

The rest of the paper is devoted to the study of the regularity for processes needed for the above quasi-continuity of exit times. We give a characterization theorem on the regularity of processes, thus generalized the one for random variables in [1]. We also investigate the quasi-continuity of stopped processes when the stopping rule is a quasi-continuous stopping time. Via the characterization theorem, we obtain some typical examples of multi-dimensional nonlinear semimartingales satisfying our assumptions, including $G$-martingales, solutions of stochastic differential equations driven by $G$-Brownian motion and the canonical processes under a family of so-called semimartingale measures. We present at the end of the paper several counterexamples to show that the exit times may not be quasi-continuous when our assumptions are violated.

The paper is organized as follows. In Section 2, we recall the probabilistic framework of nonlinear expectation and nonlinear semimartingales. The main results on quasi-continuity of exit times for nonlinear semimartingales are stated in Section 3. Section 4 is devoted to the research on the regularity of processes. Finally, in Section 5, we give some examples and counterexamples.

## 2. Nonlinear expectation on the path space

We present the basic setting and notations of the nonlinear expectation and nonlinear semimartingales. More relevant results can be found in [16–18,21].

Let $\Omega := C([0,\infty); \mathbb{R}^k)$ be the space of all $\mathbb{R}^k$-valued continuous paths $(\omega_t)_{t\geq 0}$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{N=1}^{\infty} 2^{-N} [(\sup_{t\in[0,N]} |\omega_t^1 - \omega_t^2|) \wedge 1],$$

Let $B_t(\omega) := \omega_t$ for $\omega \in \Omega$, $t \geq 0$ be the canonical process and $\mathcal{F}_t := \sigma\{B_s : s \leq t\}$ for $t \geq 0$ be the natural filtration of $B$. We denote $\mathcal{F} := (\mathcal{F}_t)_{t\geq 0}$. A mapping $\tau : \Omega \to [0, \infty]$ is called a stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$. Let $\mathcal{P}$ be a family of probability measures on $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(\Omega)$ is the $\sigma$-algebra of all Borel sets. We set

$$\mathcal{L}(\Omega) := \{X \in \mathcal{B}(\Omega) : E_P[X] \text{ exists for each } \ P \in \mathcal{P}\}.$$

We define the corresponding upper expectation by

$$\hat{\mathbb{E}}[X] := \sup_{P\in\mathcal{P}} E_P[X] \in [-\infty, \infty], \quad \text{for } X \in \mathcal{L}(\Omega). \tag{2.1}$$

Then it is easy to check that the triple $(\Omega, \mathcal{L}(\Omega), \hat{\mathbb{E}})$ forms a sublinear expectation space (see [18] for the definition).

For this $\mathcal{P}$, we define the corresponding upper capacity

$$c(A) := \sup_{P\in\mathcal{P}} P(A), \quad \text{for } A \in \mathcal{B}(\Omega).$$

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A set $A \in \mathcal{B}(\Omega)$ is said to be polar if $c(A) = 0$. We say a property holds q.s. (quasi-surely) if it holds outside a polar set. In the following, we do not distinguish two random variables if they coincide q.s.

We define the $L^p$-norm of random variables as $\|X\|_p := (\hat{\mathbb{E}}[|X|^p])^{\frac{1}{p}}$ for $p \geq 1$ and set

$$L^p(\Omega) := \{X \in \mathcal{B}(\Omega) : \|X\|_p < \infty\}.$$

Then $L^p(\Omega)$ is a Banach space under the norm $\|\cdot\|_p$. Let $C_b(\Omega)$ be the space of all bounded, continuous functions on $\Omega$. We denote the corresponding completion under the norm $\|\cdot\|_p$ by $L_C^p(\Omega)$.

**Definition 2.1.** A real function $X$ on $\Omega$ is said to be quasi-continuous if for each $\varepsilon > 0$, there exists an open set $O \subset \Omega$ with $c(O) < \varepsilon$ such that $X|_{O^c}$ is continuous.

**Definition 2.2.** We say that $X : \Omega \mapsto \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega \mapsto \mathbb{R}$ such that $X = Y$ q.s.

The following result characterizes the space $L_C^p(\Omega)$ in the measurable and integrable sense, which can be seen as a counterpart of Lusin's theorem in the nonlinear expectation theory.

**Theorem 2.3** ([1]). *For each $p \geq 1$, we have*

$$L_C^p(\Omega) = \{X \in \mathcal{B}(\Omega) \ : \ \lim_{N\to\infty} \hat{\mathbb{E}}[|X|^p I_{\{|X|\geq N\}}] = 0 \textit{ and } X \textit{ has a quasi-continuous version}\}.$$

Moreover, we have the following monotone convergence results. It worthy noting that different from the linear case, the downward convergence is quite restrictive.

**Proposition 2.4** ([1,22]). *Suppose $X_n$, $n \geq 1$ and $X$ are $\mathcal{B}(\Omega)$-measurable.*

(1) *Assume $X_n \uparrow X$ q.s. on $\Omega$ and $E_P[X_1^-] < \infty$ for all $P \in \mathcal{P}$. Then $\hat{\mathbb{E}}[X_n] \uparrow \hat{\mathbb{E}}[X]$.*
(2) *Assume $\mathcal{P}$ is weakly compact.*
   (a) *If $\{X_n\}_{n=1}^\infty$ in $L_C^1(\Omega)$ satisfies that $X_n \downarrow X$ q.s., then $\hat{\mathbb{E}}[X_n] \downarrow \hat{\mathbb{E}}[X]$.*
   (b) *For each closed set $F \in \mathcal{B}(\Omega)$, $c(F) = \inf\{c(O) : \ O$ open in $\mathcal{B}(\Omega), \ F \subset O\}$.*

**Definition 2.5.** An $\mathcal{F}$-adapted process $Y = (Y_t)_{t\geq 0}$ is called a $\mathcal{P}$-martingale ($\mathcal{P}$-supermartingale, $\mathcal{P}$-submartingale, $\mathcal{P}$- semimartingale resp.) if it is a martingale (supermartingale, submartingale, semimartingale resp.) under each $P \in \mathcal{P}$.

The following is the quasi-continuity concept for processes, which is slightly different from the one for random variables.

**Definition 2.6** ([22,23]). We say that a process $F = (F_t)_{t\geq 0}$ is quasi-continuous (on $\Omega\times[0,\infty)$) if for each $\varepsilon > 0$, there exists an open set $G \subset \Omega$ with $c(G) < \varepsilon$ such that $F_\cdot(\cdot)$ is continuous on $G^c \times [0, \infty)$.

**Remark 2.7.** From the definition, it is easy to see that, if the process $F = (F_t)_{t\geq 0}$ is quasi-continuous (in the process setting), then for each $t$, the random variable $F_t$ is quasi-continuous (in the random variable setting).

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**Definition 2.8.** Let $X, Y : \Omega \times [0, \infty) \to \mathbb{R}$ be two processes. We say $X$ is a modification of $Y$ if for each $t \in [0, \infty)$, $X_t = Y_t$ q.s. We say $X$ is indistinguishable from (or a version of) $Y$ if $X_t = Y_t$ for all $t \in [0, \infty)$, q.s.

## 3. Exit times for multi-dimensional nonlinear semimartingales

Let $Y$ be a $d$-dimensional continuous $\mathcal{P}$-semimartingale under a given weakly compact family $\mathcal{P}$ of probability measures. Assume that, under each $P \in \mathcal{P}$, we have the decomposition $Y_t = M_t^P + A_t^P$, where $M_t^P$ is a $d$-dimensional continuous local martingale and $A_t^P$ is a $d$-dimensional finite-variation process. We also denote by $\langle Y \rangle^P = \langle M^P \rangle^P$ the quadratic variation under $P$, and shall often omit the superscript $P$ on $\langle \cdot \rangle$ when there is no danger of ambiguity.

### 3.1. Quasi-continuity of exit times

For each set $D \subset \mathbb{R}^d$, we define the exit times of $Y$ from $D$ by

$$\tau_D(\omega) := \inf\{t \geq 0 : Y_t(\omega) \in D^c\}, \quad \text{for } \omega \in \Omega.$$

**Definition 3.1.** Let $E$ be a metric space. We say that a function $f : E \to [-\infty, \infty]$ is upper (lower resp.) semi-continuous if for each $x_0 \in E$,

$$\limsup_{x \to x_0} f(x) \leq f(x_0) \quad (\liminf_{x \to x_0} f(x) \geq f(x_0) \text{ resp.}).$$

**Definition 3.2.** An open set $O$ is said to satisfy the exterior ball condition at $x \in \partial O$ if there exists an open ball $U(z, r)$ with center $z$ and radius $r$ such that $U(z, r) \subset O^c$ and $x \in \partial U(z, r)$. If every boundary point $x \in \partial O$ satisfies the exterior ball condition, then $O$ is said to satisfy the exterior ball condition.

Given an open set $Q$ in $\mathbb{R}^d$, we denote

$$\Omega^\omega = \{\omega' \in \Omega : \omega'_t = \omega_t \text{ on } [0, \tau_Q(\omega)]\}, \quad \text{for each } \omega \in \Omega. \tag{3.1}$$

In this section, we shall mainly deal with nonlinear semimartingales $Y$ possessing a local growth condition at the boundary:

$(H)$ Given each $P \in \mathcal{P}$. For $P$-a.s. $\omega$ such that $\tau_Q(\omega) < \infty$, there exist some stopping time $\sigma^\omega$ and constants $\lambda^\omega, \varepsilon^\omega > 0$ (these three quantities may depend on $P, \omega$ but are supposed to be uniform for all $\omega' \in \Omega^\omega$) so that for $\omega' \in \Omega^\omega$,

- (i) $\sigma^\omega(\omega') > 0$;
- (ii) On the interval of $t \in [0, \sigma^\omega(\omega') \wedge (\tau_{\overline{Q}}(\omega') - \tau_Q(\omega'))]$, it holds that
  - (a) Non-degeneracy: $d\langle M^P \rangle_{\tau_Q(\omega)+t}(\omega') \geq \lambda^\omega \mathrm{tr}[d\langle M^P \rangle_{\tau_Q(\omega)+t}(\omega')] I_{d\times d}$ and $\mathrm{tr}[d\langle M^P \rangle_{\tau_Q(\omega)+t}(\omega')] > 0$,
  - (b) Controllability: $\mathrm{tr}[d\langle M^P \rangle_{\tau_Q(\omega)+t}(\omega')] \geq \varepsilon^\omega |dA^P_{\tau_Q(\omega)+t}(\omega')|$.

The following theorem is the main result of this section concerning the regularity of exit times.

**Theorem 3.3.** *Let $Q$ be an open set satisfying the exterior ball condition and let $\Omega^\omega$ be defined as in (3.1). Suppose that $Y$ is quasi-continuous and satisfies the local growth condition $(H)$. Then for any $\delta > 0$, there exists an open set $O \subset \Omega$ such that $c(O) \le \delta$ and on $O^c$, it holds that:*

(i) *$\tau_Q$ is lower semi-continuous and $\tau_{\overline{Q}}$ is upper semi-continuous;*
(ii) *$\tau_Q = \tau_{\overline{Q}}$.*

**Remark 3.4.** Let us explain the meaning of the three inequalities in (ii) of the condition $(H)$.

(a) We first give a general discussion. For two (signed) measures $\mu_1$ and $\mu_2$ on some subinterval $I$ of $\mathbb{R}$, by $d\mu_1 \ge d\mu_2$ we mean that $\mu_1(A) \ge \mu_2(A)$ for each $A \in \mathcal{B}(I)$. If $\mu_i, i = 1, 2$, are Lebesgue–Stieltjes measures corresponding to finite-variation functions $f_i$ respectively, $d\mu_1 \ge d\mu_2$ is equivalent to the assertion that $f_1 - f_2$ is non-decreasing. Such a discussion obviously also holds for the more general case that the two measures $\mu_1$ and $\mu_2$ take $\mathbb{S}(d)$-values, where $\mathbb{S}(d)$ is the set of $d \times d$ symmetric matrices endowed with the usual order, i.e., for $\gamma_1, \gamma_2 \in \mathbb{S}(d)$ we write $\gamma_1 \ge \gamma_2$ if $\gamma_1 - \gamma_2$ is nonnegative definite.

(b) On the interval of $t \in [0, \sigma^\omega(\omega') \wedge (\tau_{\overline{Q}}(\omega') - \tau_Q(\omega'))]$, the Lebesgue–Stieltjes measures $d\langle M^P\rangle_{\tau_Q(\omega)+t}(\omega')$, $\mathrm{tr}[d\langle M^P\rangle_{\tau_Q(\omega)+t}(\omega')]$ and $dA^P_{\tau_Q(\omega)+t}(\omega')$ are generated by finite-variation functions $t \to \langle M^P\rangle_{\tau_Q(\omega)+t}(\omega')$, $t \to \mathrm{tr}[\langle M^P\rangle_{\tau_Q(\omega)+t}(\omega')]$ and $t \to A^P_{\tau_Q(\omega)+t}(\omega')$ respectively.
The first inequality in (a) of $(H)$ means that the measure $d\langle M^P\rangle_{\tau_Q(\omega)+t}(\omega')$ is no smaller than $\lambda^\omega \mathrm{tr}[d\langle M^P\rangle_{\tau_Q(\omega)+t}(\omega')] I_{d\times d}$, and the one in (b) of $(H)$ has a similar meaning that the measure $\mathrm{tr}[d\langle M^P\rangle_{\tau_Q(\omega)+t}(\omega')]$ is no smaller than $\varepsilon^\omega |dA^P_{\tau_Q(\omega)+t}(\omega')|$, where $|dA^P_{\tau_Q(\omega)+t}(\omega')|$ is the total variation measure of $dA^P_{\tau_Q(\omega)+t}(\omega')$. Moreover, the second inequality in (a) of $(H)$ means that the function $t \to \mathrm{tr}[\langle M^P\rangle_{\tau_Q(\omega)+t}(\omega')]$ that generates the measure $\mathrm{tr}[d\langle M^P\rangle_{\tau_Q(\omega)+t}(\omega')]$ is strictly increasing.

**Remark 3.5.**

(i) A simple and sufficient condition of $(H)$ is the case that $\lambda, \varepsilon$ are independent of $\omega$ and the growth condition is global, i.e.,

$(H')$ for each $P$, there exist constants $\lambda, \varepsilon > 0$ (may depend on $P$) such that $d\langle M^P\rangle_t \ge \lambda \mathrm{tr}[d\langle M^P\rangle_t] I_{d\times d}$, $\mathrm{tr}[d\langle M^P\rangle_t] > 0$ and $\mathrm{tr}[d\langle M^P\rangle_t] \ge \varepsilon |dA^P_t|$ on $[0, \tau_{\overline{Q}}]$, $P$-a.s.

Indeed, we can take $\sigma^\omega \equiv t$ for any given $t > 0$ in this situation, and then $(H)$ holds.

(ii) If for each $P$, it holds that $\lambda I_{d\times d} \le \frac{d\langle M^P\rangle_t}{dt} \le \Lambda I_{d\times d}$ and $|\frac{dA^P_t}{dt}| \le C$ on $[0, \tau_{\overline{Q}}]$, for some constants $0 < \lambda \le \Lambda, C \ge 0$ (may depend on $P$), $P$-a.s., then $(H')$ is satisfied.

**Remark 3.6.** We discuss two special situations mainly based on the condition $(H')$. Similar results hold for $(H)$ by a straightforward modification. Since the symbol for the latter is more complicated and so is omitted.

(i) If $Y$ is a $\mathcal{P}$-martingale, i.e., $A^P \equiv 0$ for each $P \in \mathcal{P}$, then the inequality $\mathrm{tr}[d\langle M^P\rangle_t] \ge \varepsilon |dA^P_t|$ in $(H')$ holds trivially by taking $\varepsilon = 1$.

(ii) When $d = 1$, the inequality $d\langle M^P\rangle_t \geq \lambda \mathrm{tr}[d\langle M^P\rangle_t] I_{d\times d}$ in $(H')$ is just $d\langle M^P\rangle_t \geq \lambda d\langle M^P\rangle_t$, and thus holds for $\lambda = 1$. If moreover $A^P \equiv 0$, then $(H')$ reduces to $d\langle M^P\rangle_t > 0$ on $[0, \tau_{\overline{Q}}]$.

Before presenting the proof, we shall state a direct consequence of Theorem 3.3 concerning the quasi-continuity of exit times. Note that $\tau_Q$ and $\tau_{\overline{Q}}$ may take the value $+\infty$. The fact that $\tau_Q$ is lower semi-continuous, $\tau_{\overline{Q}}$ is upper semi-continuous and $\tau_Q = \tau_{\overline{Q}}$ does not imply that $\tau_Q$ and $\tau_{\overline{Q}}$ are continuous. In general, we can get the quasi-continuity by a truncation manipulation as follows.

**Corollary 3.7.** *Assume that the conclusion of Theorem 3.3 is true for $\tau_Q$ and $\tau_{\overline{Q}}$.*

(i) *If $X$ is a quasi-continuous random variable, then $\tau_Q \wedge X$ and $\tau_{\overline{Q}} \wedge X$ are both quasi-continuous.*
(ii) *If $X \in L^1_C(\Omega)$, then $\tau_Q \wedge X$ and $\tau_{\overline{Q}} \wedge X$ both belong to $L^1_C(\Omega)$.*

**Proof.** (i) By assumption, we can find an open set $O_1 \subset \Omega$ such that $c(O_1) \leq \varepsilon$ and $X$ is continuous on $(O_1)^c$. Moreover, from Theorem 3.3, we can choose an open set $O_2 \subset \Omega$ such that $c(O_2) \leq \varepsilon$ and on $(O_2)^c$, $\tau_Q$ and $\tau_{\overline{Q}}$ are lower and upper semi-continuous respectively, and $\tau_Q = \tau_{\overline{Q}}$. Denote $O = O_1 \cup O_2$. Then $c(O) \leq 2\varepsilon$ and on $O^c$, it holds that $\tau_Q \wedge X : \Omega \to \mathbb{R}$ is lower semi-continuous, $\tau_{\overline{Q}} \wedge X : \Omega \to \mathbb{R}$ is upper semi-continuous, and

$$\tau_Q \wedge X = \tau_{\overline{Q}} \wedge X.$$

From this, we deduce that $\tau_Q \wedge X$ and $\tau_{\overline{Q}} \wedge X$ are continuous on $O^c$.
(ii) From (i), $\tau_Q \wedge X$ is quasi-continuous. Noting that $|\tau_Q \wedge X| \leq |X|$, then

$$\hat{\mathbb{E}}[|\tau_Q \wedge X| I_{\{|\tau_Q \wedge X| > k\}}] \leq \hat{\mathbb{E}}[|X| I_{\{|X|>k\}}] \to 0, \quad \text{as } k \to \infty.$$

Now the desired result follows from Theorem 2.3. □

**Remark 3.8.** Typically, we shall often take $X \equiv t$, for any fixed $t$, in the above corollary.
Assume that $d = 1$ and $Y$ is a one-dimensional $\mathcal{P}$-martingale. Then from Corollary 3.7, we deduce that $\tau_Q \wedge t$ is quasi-continuous if $d\langle M^P\rangle_t > 0$ $P$-a.s., for each $P \in \mathcal{P}$, and $Q$ satisfies the exterior ball condition. In particular, if we take $Q = (-\infty, a)$ for $a \in \mathbb{R}$, then

$$\tau_Q(\omega) = \inf\{t \geq 0 : Y_t(\omega) > a\}$$

and we get the result in [22].

Now we proceed to the proof of Theorem 3.3. We first present a result which shows that $Y$ originating at the boundary point of $Q$ with exterior ball will exit $\overline{Q}$ immediately.

**Proposition 3.9.** *Let $Q$ be an open set satisfying the exterior ball condition at some $x \in \partial Q$. Assume $P$ is a probability measure such that $Y = M^P + A^P$ is a continuous semimartingale satisfying $Y_0 = x$ $P$-a.s and the following local growth assumption at $x$:*

(A) *There exist some stopping time $\sigma > 0$ $P$-a.s. and constants $\lambda, \varepsilon > 0$ such that $d\langle M^P\rangle_t \geq \lambda tr[d\langle M^P\rangle_t] I_{d\times d}$, $tr[d\langle M^P\rangle_t] > 0$ and $tr[d\langle M^P\rangle_t] \geq \varepsilon |dA^P_t|$ on $[0, \sigma \wedge \tau_{\overline{Q}}]$, $P$-a.s.*

*Then we have $\tau_{\overline{Q}} = 0$ $P$-a.s., i.e., $P$-a.s. for each $\delta > 0$, there exists a point $t \in (0, \delta]$ such that $Y_t \in \overline{Q}^c$.*

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**Proof.** Let $U(z,r)$ be the exterior ball of $Q$ at $x$. We set $h(y) := e^{-k|y-z|^2}$, where the constant $k$ will be determined in the sequel. Then

$$\begin{aligned}
&D_y h(y) = -2k(y-z)e^{-k|y-z|^2},\\
&D^2_{yy}h(y) = (4k^2(y_i-z_i)(y_j-z_j) - 2k\delta_{ij})e^{-k|y-z|^2}\\
&\quad = (4k^2(y-z)(y-z)^T - 2kI_{d\times d})e^{-k|y-z|^2}.
\end{aligned}$$

Let $\langle\cdot,\cdot\rangle$ be the Euclidian scalar product for vectors and matrices. Let any $R \geq 2r$ be given. By the assumption, for $P$-a.s. $\omega$, on $[0,\sigma]$, we have for all $y \in U(x,R)\cap\overline{Q}$,

$$\begin{aligned}
&\langle D^2_{yy}h(y), d\langle M^P\rangle_t\rangle + 2\langle D_y h(y), dA^P_t\rangle\\
&= (\langle 4k^2(y-z)(y-z)^T, d\langle M^P\rangle_t\rangle - \langle 2kI_{d\times d}, d\langle M^P\rangle_t\rangle - 4k\langle (y-z), dA^P_t\rangle)e^{-k|y-z|^2}\\
&\geq (\langle 4k^2(y-z)(y-z)^T, \lambda\mathrm{tr}[d\langle M^P\rangle_t]I_{d\times d}\rangle - \langle 2kI_{d\times d}, d\langle M^P\rangle_t\rangle - 4k\langle (y-z), dA^P_t\rangle)e^{-k|y-z|^2}\\
&\geq (4\lambda k^2|y-z|^2\mathrm{tr}[d\langle M^P\rangle_t] - 4k|y-z||dA^P_t| - 2k\mathrm{tr}[d\langle M^P\rangle_t])e^{-k|y-z|^2}\\
&\geq ((4\lambda k^2r^2 - 2k)\mathrm{tr}[d\langle M^P\rangle_t] - 4k(R+r)\frac{1}{\varepsilon}\mathrm{tr}[d\langle M^P\rangle_t])e^{-k|y-z|^2}\\
&= (((4\lambda k^2r^2 - 2k) - 4k(R+r)\frac{1}{\varepsilon})\mathrm{tr}[d\langle M^P\rangle_t])e^{-k|y-z|^2}.
\end{aligned}\tag{3.2}$$

Here we have used the well-known matrix inequality that $\langle A_1, B\rangle \geq \langle A_2, B\rangle$ if $A_1, A_2, B \in \mathbb{S}(d)$ such that $A_1 \geq A_2$ and $B \geq 0$ (recall that $\mathbb{S}(d)$ is the set of $d\times d$ symmetric matrices with the usual order).

Since $M^P$ is a local martingale, we can find a stopping times $\sigma_1 > 0$ such that $M^P_{\cdot\wedge\sigma_1}$ is a square-integrable martingale. For symbol simplicity, we still denote $\sigma\wedge\sigma_1$ by $\sigma$. For any given $t > 0$, applying Itô's formula, we obtain

$$\begin{aligned}
h(Y_{\tau_{\overline{Q}}\wedge\tau_{U(x,R)}\wedge\sigma\wedge t}) - h(x) &= \int_0^{\tau_{\overline{Q}}\wedge\tau_{U(x,R)}\wedge\sigma\wedge t}\langle D_y h(Y_s), dM^P_s\rangle\\
&\quad + \int_0^{\tau_{\overline{Q}}\wedge\tau_{U(x,R)}\wedge\sigma\wedge t}\langle D_y h(Y_s), dA^P_s\rangle\\
&\quad + \frac{1}{2}\int_0^{\tau_{\overline{Q}}\wedge\tau_{U(x,R)}\wedge\sigma\wedge t}\langle D^2_{yy}h(Y_s), d\langle M^P\rangle_s\rangle.
\end{aligned}$$

Taking expectation on both sides, we get

$$\begin{aligned}
&E_P[\int_0^{\tau_{\overline{Q}}\wedge\tau_{U(x,R)}\wedge\sigma\wedge t}(\frac{1}{2}\langle D^2_{yy}h(Y_s), d\langle M^P\rangle_s\rangle + \langle D_y h(Y_s), dA^P_s\rangle)]\\
&\quad = E_P[h(Y_{\tau_{\overline{Q}}\wedge\tau_{U(x,R)}\wedge\sigma\wedge t}) - h(x)] \leq 0,
\end{aligned}$$

since $h(y) - h(x) \leq 0$ for each $y \in (U(z,r))^c$. Combining this with inequality (3.2), we get

$$E_P[\int_0^{\tau_{\overline{Q}}\wedge\tau_{U(x,R)}\wedge\sigma\wedge t}(((2\lambda k^2r^2 - k) - 2k(R+r)\frac{1}{\varepsilon})\mathrm{tr}[d\langle M^P\rangle_s])e^{-k|Y_s-z|^2}] \leq 0.$$

This can be rewritten as

$$E_P[\int_0^{\tau_{\overline{Q}}\wedge\tau_{U(x,R)}\wedge\sigma\wedge t}(((2\lambda kr^2 - 1)\varepsilon - 2(R+r))\mathrm{tr}[d\langle M^P\rangle_s])e^{k((R+r)^2-|Y_s-z|^2)}] \leq 0.$$

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If $P(\tau_{\overline{Q}} > 0) > 0$, then $P(\tau_{\overline{Q}} \wedge \tau_{U(x,R)} \wedge \sigma \wedge t > 0) > 0$. In view of

$$((2\lambda k r^2 - 1)\varepsilon - 2(R+r))e^{k((R+r)^2 - |Y_s - z|^2)} \uparrow \infty, \quad \text{as } k_0 \le k \to \infty, \text{ for some } k_0 > 0,$$

we can apply the classical monotone convergence theorem to obtain

$$\lim_{k\to\infty} E_P[\int_0^{\tau_{\overline{Q}} \wedge \tau_{U(x,R)} \wedge \sigma \wedge t} (((2\lambda k r^2 - 1)\varepsilon - 2(R+r))\mathrm{tr}[d\langle M^P\rangle_s])e^{k((R+r)^2 - |Y_s - z|^2)}] = \infty,$$

which is a contradiction. So we must have $\tau_{\overline{Q}} = 0$. The proof is complete. $\square$

**Remark 3.10.**

(i) Surely the assumption $(A)$ is satisfied by the global growth condition that $(A)$ holds with $\sigma = \infty$.

(ii) The presence of $\sigma$ should be understood by the observation that the phenomenon of immediate exit from $\overline{Q}$ is a local behavior which is determined by the path property of $Y$ near time 0, i.e., the behavior of $Y$ on $[0, \sigma]$.

The immediate leaving property also holds for $Y$ with general initial points.

**Proposition 3.11.** *Let $Y$, $Q$ be assumed as in Theorem 3.3. Then*

$$\tau_Q = \tau_{\overline{Q}}, \quad q.s. \tag{3.3}$$

**Proof.** Given any $P \in \mathcal{P}$. Observe that if $Y_0 = x$ $P$-a.s. for some $x \in \partial Q$, from Proposition 3.9, we obviously have that $\tau_{\overline{Q}} = \tau_Q = 0$, $P$-a.s. If not, we will use the method of regular conditional expectations to restart $Y$ at the boundary as follows.

For $\mathcal{F}_{\tau_Q}$, from Theorem 1.3.4 in [25], there exists a regular conditional expectation $\{P^\omega\}_{\omega\in\Omega}$ such that

$$P^\omega(\Omega^\omega) = 1 \quad \text{and} \quad E_P[\cdot|\mathcal{F}_{\tau_Q}](\omega) = E_{P^\omega}[\cdot], \quad \text{for } P\text{-a.s. } \omega.$$

If $\tau_Q(\omega) = \infty$, it is obvious that $\tau_{\overline{Q}}(\omega) = \tau_Q(\omega)$.

For $P$-a.s. $\omega$, we have $\sigma^\omega > 0$ $P^\omega$-a.s. Moreover, for any given $\omega$, by Galmarino's test (see [20], Chap. I, Exercise 4.21 (3)), we have for $\omega' \in \Omega^\omega$, $Y_t(\omega) = Y_t(\omega')$, $t \le \tau_Q(\omega)$. This implies that $\tau_Q(\omega) = \tau_Q(\omega')$. Thus, for $\omega$ such that $P^\omega(\Omega^\omega) = 1$, under $P^\omega$, $\omega' \to \tau_{\overline{Q}}(\omega') - \tau_Q(\omega')$ is also the exit time of $\omega' \to (Y_{\tau_Q(\omega)+t}(\omega'))_{t\ge 0}$ from $\overline{Q}$.

Applying the following Lemma 3.12, we deduce that for $P$-a.s. $\omega$ such that $\tau_Q(\omega) < \infty$, under $P^\omega$, $\omega' \to (Y_{\tau_Q(\omega)+t}(\omega'))_{t\ge 0}$ is a semimartingale starting from $Y_{\tau_Q(\omega)} \in \partial Q$ and satisfying the assumption $(A)$ in Proposition 3.9. Therefore, by applying Proposition 3.9, we obtain

$$E_P[(\tau_{\overline{Q}} - \tau_Q)I_{\{\tau_Q<\infty\}}|\mathcal{F}_{\tau_Q}](\omega) = E_{P^\omega}[\tau_{\overline{Q}} - \tau_Q]I_{\{\tau_Q(\omega)<\infty\}} = 0, \quad \text{for } P\text{-a.s. } \omega.$$

Summarizing the above, we get

$$\tau_{\overline{Q}} = \tau_Q, \quad P\text{-a.s.},$$

which implies

$$\tau_{\overline{Q}} = \tau_Q, \quad \text{q.s.}$$

This completes the proof. $\square$

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**Lemma 3.12.** *Let $\tau : \Omega \to [0, \infty]$ be a stopping time. Given a local martingale $(M_t^P, \mathcal{F}_t)_{t\geq 0}$ under some probability measure $P$. Let $\{P^\omega\}_{\omega\in\Omega}$ be the corresponding regular conditional expectation of $P$ for $\mathcal{F}_\tau$. Then for $P$-a.s. $\omega$,*

(i) *Under $P^\omega$, $\omega' \to (M_t^P(\omega') - M^P_{\tau(\omega)\wedge t}(\omega'), \mathcal{F}_t)_{t\geq 0}$ is a local martingale, which can also be restated as that $\omega' \to (M^P_{\tau(\omega)+t}(\omega'), \mathcal{F}_{\tau(\omega)+t})_{t\geq 0}$ is a local martingale for $\tau(\omega) < \infty$.*

(ii) *If $\tau(\omega) < \infty$, then $\langle M^P_{\tau(\omega)+\cdot}\rangle_t^{P^\omega} = \langle M^P\rangle^P_{\tau(\omega)+t}$ for each $t \geq 0$, $P^\omega$-a.s. (recall that when necessary, we use the superscript $Q$ on the quadratic variation $\langle \cdot \rangle$ to indicate the dependence on a probability $Q$).*

**Proof.** (i) *Step 1.* If $M^P$ is a martingale under $P$, then by Theorem 1.2.10 in [25], for $P$-a.s. $\omega$, $\omega' \to (M_t^P(\omega') - M^P_{\tau(\omega)\wedge t}(\omega'))_{t\geq 0}$ is a $\mathcal{F}_t$-martingale under $P^\omega$.

*Step 2.* Now suppose that $M^P$ is a local martingale under $P$. Let $T_n$ be localization sequence of stopping times for $M^P$ such that $T_n \uparrow \infty$ $P$-a.s. and $(M^P_{t\wedge T_n})_{t\geq 0}$ is a martingale under $P$. For any given $n$, since $M^P_{t\wedge T_n}$ is a martingale under $P$, applying Step 1 yields that $\omega' \to M^P_{t\wedge T_n(\omega')}(\omega') - M^P_{\tau(\omega)\wedge t\wedge T_n(\omega')}(\omega')$ is a martingale under $P^\omega$, for $P$-a.s. $\omega$. Since $T_n \uparrow \infty$ $P^\omega$-a.s., for $P$-a.s. $\omega$, we can find a set $N \subset \Omega$ such that $P(N) = 0$ and for $\omega \in N^c$, $\omega' \to M^P_{t\wedge T_n(\omega')}(\omega') - M^P_{\tau(\omega)\wedge t\wedge T_n(\omega')}(\omega')$ is a martingale under $P^\omega$ for each $n$ and $T_n \uparrow \infty$ $P^\omega$-a.s. Let any $\omega \in N^c$ be given such that $\tau(\omega) < \infty$ and $\omega' \to M^P_{(\tau(\omega)+t)\wedge T_n(\omega')}(\omega')$ is a $\mathcal{F}_{\tau(\omega)+t}$-martingale under $P^\omega$ for each $n$. We define

$$\sigma_m(\omega') := \inf\{t \geq 0 : |M_{\tau(\omega)+t}(\omega')| \geq m\}, \quad m \geq 1,$$

which is a $\mathcal{F}_{\tau(\omega)+t}$-stopping time. We claim that $\sigma_m$ is a localization sequence for $M^P_{\tau(\omega)+t}$. Indeed, note that

$$\sup_{t\geq 0} |M^P_{\tau(\omega)+t\wedge\sigma_m}| \leq m,$$

then we can apply the dominated convergence theorem to derive that, for $s \leq t$,

$$\begin{aligned} E_{P^\omega}[M^P_{\tau(\omega)+t\wedge\sigma_m}|\mathcal{F}_{\tau(\omega)+s}] &= E_{P^\omega}[\lim_{n\to\infty} M^P_{(\tau(\omega)+t\wedge\sigma_m)\wedge T_n}|\mathcal{F}_{\tau(\omega)+s}] \\ &= \lim_{n\to\infty} E_{P^\omega}[M^P_{(\tau(\omega)+t\wedge\sigma_m)\wedge T_n}|\mathcal{F}_{\tau(\omega)+s}] \\ &= \lim_{n\to\infty} M^P_{(\tau(\omega)+s\wedge\sigma_m)\wedge T_n} \\ &= M^P_{\tau(\omega)+s\wedge\sigma_m}, \end{aligned}$$

where the third equality is due to the fact that $M^P_{(\tau(\omega)+t\wedge\sigma_m)\wedge T_n}$ is a $\mathcal{F}_{\tau(\omega)+t}$-martingale by the classical optional sampling theorem. Therefore, $(M^P_{\tau(\omega)+t}, \mathcal{F}_{\tau(\omega)+t})_{t\geq 0}$ is a local martingale under $P^\omega$.

(ii) Note that $(M_t^P)^2 - \langle M^P\rangle_t^P$ is a local martingale under $P$. Then from Step 2 in (i), we obtain that for $P$-a.s. $\omega$, if $\tau(\omega) < \infty$, then $(M^P_{\tau(\omega)+t})^2 - \langle M^P\rangle^P_{\tau(\omega)+t}$ is also a local martingale under $P^\omega$. This implies that

$$\langle M^P_{\tau(\omega)+\cdot}\rangle_t^{P^\omega} = \langle M^P\rangle^P_{\tau(\omega)+t} \quad \text{for each } t \geq 0,\ P^\omega\text{-a.s.},$$

as desired. □

The following lemma concerns the semi-continuities of exit times when the process is bi-continuous.

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**Lemma 3.13.** *Let $E$ be a metric space and $(\omega, t) \to F_t(\omega)$ be a continuous mapping from $E \times [0, \infty)$ to $\mathbb{R}^d$. Define, for each set $D \subset \mathbb{R}^d$, the exit times of $F$ from $D$ by*

$$\sigma_D(\omega) := \inf\{t \geq 0 : F_t(\omega) \in D^c\}, \quad \text{for } \omega \in \Omega.$$

*Assume $Q$ is an open set. Then $\sigma_Q$ is lower semi-continuous and $\sigma_{\overline{Q}}$ is upper semi-continuous.*

**Proof.** We first show that $\sigma_{\overline{Q}}$ is upper semi-continuous. For any given $\omega \in E$, set $t_0 := \sigma_{\overline{Q}}(\omega)$. Noting that the case $t_0 = \infty$ is trivial, we may assume that $t_0 < \infty$. Then we can find an arbitrarily small $\varepsilon > 0$ such that $F_{t_0+\varepsilon}(\omega) \in \overline{Q}^c$. Since $\overline{Q}^c$ is open, there exists an open ball $U(F_{t_0+\varepsilon}(\omega), r)$ with center $F_{t_0+\varepsilon}(\omega)$ and radius $r$ such that $U(F_{t_0+\varepsilon}(\omega), r) \subset \overline{Q}^c$. For each $\omega'$ whose distance with $\omega$ is sufficiently small, we will have

$$F_{t_0+\varepsilon}(\omega') \in U(F_{t_0+\varepsilon}(\omega), r) \subset \overline{Q}^c$$

by the continuity of $F$. That is,

$$\sigma_{\overline{Q}}(\omega') \leq t_0 + \varepsilon,$$

as desired.

Now we consider the second part. Given any $\omega \in E$, we first prove the assertion that for any given $t \in [0, \infty)$, if $\sigma_Q(\omega) \geq t$, then

$$\liminf_{\omega' \to \omega} \sigma_Q(\omega') \geq t. \tag{3.4}$$

If not, we can find a sequence $\omega^n \in E$ and $t_n \in [0, t - \varepsilon]$ for some $\varepsilon > 0$ such that

$$\omega^n \to \omega \quad \text{and} \quad F_{t_n}(\omega^n) \in Q^c.$$

We can extract a subsequence of $\{t_n\}$, which is still denoted by $\{t_n\}$, such that $t_n \to t'$ for some $t' \in [0, t - \varepsilon]$. Then by the continuity assumption on $F$,

$$F_{t'}(\omega) = \lim_{n \to \infty} F_{t_n}(\omega^n) \in Q^c,$$

which is a contradiction. Thus we have proved the assertion. Now set $t_0 := \sigma_Q(\omega)$. If $t_0 < \infty$, the conclusion follows from taking $t = t_0$ in (3.4). If $t_0 = \infty$, we can apply (3.4) to each $t < \infty$ to show that

$$\liminf_{\omega' \to \omega} \sigma_Q(\omega') \geq t, \quad \text{for every } t > 0,$$

which implies

$$\liminf_{\omega' \to \omega} \sigma_Q(\omega') = \infty.$$

The proof is now complete. □

Now we can complete the proof of Theorem 3.3. For this purpose, it suffice to prove the following proposition which is stated in a slightly more general form, without the specific assumptions on $Q$ and $Y$ as in Theorem 3.3. It can be useful in the future work.

**Proposition 3.14.** *Let $Q$ be an open set. Suppose that $Y$ is quasi-continuous with exit times satisfying $\tau_Q = \tau_{\overline{Q}}$ q.s. Then the conclusion in Theorem 3.3 holds: for any $\delta > 0$, there exists an open set $O \subset \Omega$ such that $c(O) \leq \delta$ and on $O^c$, $\tau_Q$ and $\tau_{\overline{Q}}$ are lower and upper semi-continuous respectively, and $\tau_Q = \tau_{\overline{Q}}$.*

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**Proof.** Set $\Gamma = \{\tau_Q = \tau_{\overline{Q}}\}$. Then $c(\Gamma^c) = 0$ by the assumption. Since the process $Y$ is quasi-continuous, for any $\delta > 0$, we can find an open set $G \subset \Omega$ such that $c(G) \leq \frac{\delta}{2}$ and $Y$ is continuous on $G^c \times [0, \infty)$. From Lemma 3.13, $\tau_Q$ and $\tau_{\overline{Q}}$ are lower and upper semi-continuous respectively on $G^c$. Moreover, we can write the polar set

$$\Gamma^c \cap G^c = \{\tau_Q < \tau_{\overline{Q}}\} \cap G^c = \bigcup_{s<r;s,r\in\mathbb{Q}} (\{\tau_Q \leq s\} \cap \{\tau_{\overline{Q}} \geq r\}) \cap G^c.$$

For every $s, r$, from the semi-continuities of $\tau_Q$ and $\tau_{\overline{Q}}$ on $G^c$, we deduce that $(\{\tau_Q \leq s\} \cap \{\tau_{\overline{Q}} \geq r\}) \cap G^c$ is closed. Then according to Proposition 2.4(2) (b), there exists an open set with any given small capacity such that

$$O_{sr} \supset (\{\tau_Q \leq s\} \cap \{\tau_{\overline{Q}} \geq r\}) \cap G^c.$$

From this, we can find an open set $O_1 \supset \Gamma^c \cap G^c$ such that $c(O_1) \leq \frac{\delta}{2}$. Denote the open set $O = O_1 \cup G$. Then on $O^c$, $\tau_Q$ is lower semi-continuous and $\tau_{\overline{Q}}$ is upper semi-continuous, and $\tau_Q = \tau_{\overline{Q}}$. □

**Remark 3.15.** In Proposition 3.9, the condition in $(A)$ that there exist some constant $\varepsilon > 0$ such that

$$\mathrm{tr}[d\langle M^P\rangle_t] \geq \varepsilon |dA_t^P|, \tag{3.5}$$

can be relaxed in two one-dimensional cases. Note that we use inequality (3.5) to guarantee that, in (3.2) in the proof of Proposition 3.9,

$$\mathrm{tr}[d\langle M^P\rangle_t] \geq \varepsilon \langle \frac{y-z}{|y-z|}, dA_t^P\rangle, \quad \text{for each } y \in \overline{Q}. \tag{3.6}$$

Assume that $d = 1$ and $Q = (-\infty, a)$ for some $a \in \mathbb{R}$. We take the exterior ball $U(a+1, 1) = (a, a+2)$. Then the inequality (3.6) reduces to

$$d\langle M^P\rangle_t \geq \varepsilon \langle \frac{y-a-1}{|y-a-1|}, dA_t^P\rangle, \quad \text{for each } y \leq a,$$

which is just

$$d\langle M^P\rangle_t \geq -\varepsilon dA_t^P. \tag{3.7}$$

Similar analysis shows that when $d = 1$ and $Q = (a, +\infty)$ for some $a \in \mathbb{R}$, the inequality (3.6) reduces to

$$d\langle M^P\rangle_t \geq \varepsilon dA_t^P. \tag{3.8}$$

In these two situations respectively, we can use (3.7) and (3.8) to replace (3.5) and get the conclusion of Proposition 3.9. We can also similarly modify the assumption $(H)$ in Theorem 3.3 and repeat the proofs as before, to recover all the corresponding results in this subsection.

### 3.2. Integrability of exit times

When a certain integrability condition imposed, $\tau_Q$ and $\tau_{\overline{Q}}$ themselves can be quasi-continuous.

**Theorem 3.16.** *Assume that the conclusion of Theorem 3.3 is true for $\tau_Q$ and $\tau_{\overline{Q}}$, i.e., for any $\delta > 0$, we can find an open set $O \subset \Omega$ with $c(O) \leq \delta$ such that on $O^c$, $\tau_Q$ is lower semi-continuous, $\tau_{\overline{Q}}$ is upper semi-continuous and $\tau_Q = \tau_{\overline{Q}}$.*

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(i) *If*

$$c(\{\tau_{\overline{Q}} > k\}) \to 0, \quad \text{as } k \to \infty, \tag{3.9}$$

*then* $\tau_Q$ *and* $\tau_{\overline{Q}}$ *are quasi-continuous.*

(ii) *Assume that*

$$\hat{\mathbb{E}}[\tau_{\overline{Q}} I_{\{\tau_{\overline{Q}} > k\}}] \to 0, \quad \text{as } k \to \infty. \tag{3.10}$$

*Then* $\tau_Q$ *and* $\tau_{\overline{Q}}$ *both belong to* $L^1_C(\Omega)$.

**Proof.** Since $\tau_Q = \tau_{\overline{Q}}$ q.s., we may mainly prove the conclusions for $\tau_Q$.

(i) By the assumption, we can choose an open set $O_1$ such that $c(O_1) \le \varepsilon$ and on $(O_1)^c$, $\tau_Q$ is lower semi-continuous, $\tau_{\overline{Q}}$ is upper semi-continuous and $\tau_Q = \tau_{\overline{Q}}$. From (3.9), we can also take $k$ sufficiently large such that $c(\{\tau_Q > k\}) \le \varepsilon$. Utilizing the semi-continuity of $\tau_Q$ on $(O_1)^c$, we deduce that $(O_1)^c \cap \{\tau_Q \le k\}$ is a closed set, and thus, $O := O_1 \cup \{\tau_Q > k\}$ is an open set. It is easy to see that $c(O) \le 2\varepsilon$, and $\tau_Q$ and $\tau_{\overline{Q}}$ are continuous on $O^c$.

(ii) Note that

$$c(\tau_Q > k) = \hat{\mathbb{E}}[1 \cdot I_{\{\tau_Q > k\}}] \le \hat{\mathbb{E}}[\tau_Q I_{\{\tau_Q > k\}}] \to 0, \quad \text{as } 1 \le k \to \infty.$$

Then $\tau_Q$ is quasi-continuous and the conclusion now follows directly from the characterization theorem of $L^1_C(\Omega)$ (Theorem 2.3). □

Obviously, (3.10) implies (3.9). Now we provide a sufficient condition for (3.10).

**Proposition 3.17.** *Let* $Q$ *be a bounded open set and* $Y$ *be a* $\mathcal{P}$*-semimartingale. Assume that, for some* $1 \le l \le d$*, there exist some constants* $\varepsilon > 0$ *and* $\lambda \neq 0$ *such that*

$$\lambda dA^{P,l}_t + d\langle M^{P,l}\rangle_t \ge \varepsilon dt \quad \text{on } [0, \tau_{\overline{Q}}], \quad P\text{-a.s., for each } P \in \mathcal{P},$$

*where* $M^{P,l}$ *and* $A^{P,l}$ *are the* $l$*th components of* $M^P$ *and* $A^P$*, respectively. Then there exists a constant* $C > 0$ *depending only on* $\lambda, \varepsilon$ *and the diameter of* $Q$ *such that,*

$$\hat{\mathbb{E}}[(\tau_{\overline{Q}})^2] \le C. \tag{3.11}$$

**Proof.** We mainly use an auxiliary function from [3, p. 145]. Without loss of generality, we can assume $0 \in Q$ and $l = 1$.

*Step 1.* Let $P \in \mathcal{P}$ be given. Let $h(y) := \beta e^{\frac{2y_1}{\lambda}}$, and take $\beta > 0$ large enough such that $P$-a.s. for each $y \in \overline{Q}$,

$$\frac{2}{\lambda} h(y)(dA^{P,1}_t + \frac{1}{\lambda} d\langle M^{P,1}\rangle_t) = \frac{2}{\lambda^2} h(y)(\lambda dA^{P,1}_t + d\langle M^{P,1}\rangle_t) \ge dt \quad \text{on } [0, \tau_{\overline{Q}}].$$

By Itô's formula, we have

$$\begin{aligned} h(Y_{\tau_{\overline{Q}} \wedge t}) - h(Y_0) &= \int_0^{\tau_{\overline{Q}} \wedge t} \frac{2}{\lambda} h(Y_s) dM^{P,1}_s + \int_0^{\tau_{\overline{Q}} \wedge t} \frac{2}{\lambda} h(Y_s) dA^{P,1}_s + \frac{1}{2} \int_0^{\tau_{\overline{Q}} \wedge t} \frac{4}{\lambda^2} h(Y_s) d\langle M^{P,1}\rangle_s \\ &= \int_0^{\tau_{\overline{Q}} \wedge t} \frac{2}{\lambda} h(Y_s) dM^{P,1}_s + \int_0^{\tau_{\overline{Q}} \wedge t} \frac{2}{\lambda^2} h(Y_s)(\lambda dA^{P,1}_s + d\langle M^{P,1}\rangle_s). \end{aligned}$$

Taking expectation on both sides and using a standard localization argument when necessary,

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we get

$$2C_h \geq E_P[\tau_{\overline{Q}} \wedge t].$$

where $C_h$ is the bound of $h$ on $\overline{Q}$, which is independent of $P \in \mathcal{P}$ and $t$.

*Step 2.* Consider $th(y)$, where $h$ with $\beta$ is given as in Step 1. Applying Itô's formula, we have

$$\begin{aligned}(\tau_{\overline{Q}} \wedge t)h(Y_{\tau_{\overline{Q}}\wedge t}) &= \int_0^{\tau_{\overline{Q}}\wedge t} h(Y_s)ds + \int_0^{\tau_{\overline{Q}}\wedge t} \frac{2s}{\lambda} h(Y_s)dM_s^{P,1} + \int_0^{\tau_{\overline{Q}}\wedge t} \frac{2s}{\lambda} h(Y_s)dA_s^{P,1} \\ &\quad + \frac{1}{2}\int_0^{\tau_{\overline{Q}}\wedge t} \frac{4s}{\lambda^2} h(Y_s)d\langle M^{P,1}\rangle_s \\ &\geq \int_0^{\tau_{\overline{Q}}\wedge t} \frac{2s}{\lambda} h(Y_s)dM_s^{P,1} + \int_0^{\tau_{\overline{Q}}\wedge t} \frac{2s}{\lambda^2} h(Y_s)(\lambda dA_s^{P,1} + d\langle M^{P,1}\rangle_s).\end{aligned}$$

Taking expectation on both sides, we get

$$C_h E_P[\tau_{\overline{Q}} \wedge t] \geq E_P[(\tau_{\overline{Q}} \wedge t)h(Y_{\tau_{\overline{Q}}})] \geq E_P[\int_0^{\tau_{\overline{Q}}\wedge t} s ds] = \frac{1}{2}E_P[(\tau_{\overline{Q}} \wedge t)^2],$$

which together with Step 1 implies

$$E_P[(\tau_{\overline{Q}} \wedge t)^2] \leq 4(C_h)^2.$$

Taking supremum over $P \in \mathcal{P}$ and then letting $t \to \infty$, we obtain

$$\hat{\mathbb{E}}[(\tau_{\overline{Q}})^2] \leq 4(C_h)^2,$$

as desired. $\square$

**Remark 3.18.** If $\hat{\mathbb{E}}[(\tau_{\overline{Q}})^2] < \infty$, then by the Markov inequality, we obtain that (ii) in Theorem 3.16 holds: $\hat{\mathbb{E}}[\tau_{\overline{Q}} I_{\{\tau_{\overline{Q}} > k\}}] \leq \frac{\hat{\mathbb{E}}[(\tau_{\overline{Q}})^2]}{k} \to 0$, as $k \to \infty$.

## 4. Quasi-continuous processes

In the previous section, the regularity theorem for exit times (Theorem 3.3) was established under the assumption that the $\mathcal{P}$-semimartingale $Y$ has some kind of regularity which is called quasi-continuity in the process sense. In the present section, we shall give a characterization theorem on the quasi-continuity of processes as well as some related properties of stopped processes.

### *4.1. Characterization of quasi-continuous processes*

Assume that $\mathcal{P}$ is a family of probability measures on $\Omega$, $c$ and $\hat{\mathbb{E}}$ are the corresponding upper capacity and expectation, respectively.

Now we give a general criterion (characterization) on the quasi-continuity of processes. It is convenient to first introduce the notion of quasi-continuity on the finite interval. We say that a process $F = (F_t)_{t\in[0,\infty)}$ is quasi-continuous on $\Omega \times [0, T]$ if for each $\varepsilon > 0$, there exists an open set $G \subset \Omega$ with $c(G) < \varepsilon$ such that $F_\cdot(\cdot)$ is continuous on $G^c \times [0, T]$. Obviously, if $F$ is quasi-continuous on $\Omega \times [0, \infty)$, then $F$ is quasi-continuous on $\Omega \times [0, T]$, for each $T > 0$.

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**Theorem 4.1.** *Let $X : \Omega \times [0, \infty) \to \mathbb{R}$ be a stochastic process, i.e., $X_t$ is $\mathcal{B}(\Omega)$-measurable for each $t \geq 0$.*

(i) *$X$ has a quasi-continuous version on $\Omega \times [0, T]$ if and only if we can find a sequence $X^n \in C(\Omega \times [0, T])$ such that, for each $\varepsilon > 0$,*

$$c(\{\sup_{0\leq t\leq T} |X_t^n - X_t| > \varepsilon\}) \to 0, \quad \text{as } n \to \infty. \tag{4.1}$$

*Moreover, $X$ and this version are q.s. continuous in $t \in [0, T]$, i.e., continuous in $t \in [0, T]$ for q.s. $\omega \in \Omega$.*

(ii) *$X$ has a quasi-continuous version on $\Omega \times [0, \infty)$ if and only if for each $T > 0$, there exists a sequence $X^n \in C(\Omega \times [0, T])$ such that (4.1) holds. Also, $X$ and this version are q.s. continuous in $t \in [0, \infty)$.*

**Proof.** (i) Note that

$$c(\{\sup_{0\leq t\leq T} |X_t^n - X_t^m| > \varepsilon\}) \to 0, \quad \text{as } n, m \to \infty.$$

We can find a sequence $(X_t^{n_k})_{k\geq 1}$ such that

$$c(\{\sup_{0\leq t\leq T} |X_t^{n_{k+1}} - X_t^{n_k}| > \frac{1}{2^k}\}) \leq \frac{1}{2^k}, \quad \forall k \geq 1.$$

Denote

$$A_k = \{\sup_{0\leq t\leq T} |X_t^{n_{k+1}} - X_t^{n_k}| > \frac{1}{2^k}\}.$$

Then

$$\sum_{k=1}^{\infty} c(A_k) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

As a consequence, by Borel–Cantelli Lemma, $D_T := \limsup_{k\to\infty} A_k$ is polar. Since $X_t^{n_k}$ is continuous on $\Omega$, for each $t$ and $k \geq 1$, then $\sup_{0\leq t\leq T} |X_t^{n_{k+1}} - X_t^{n_k}|$ is lower semi-continuous on $\Omega$, and thus, the set $A_k$ is open on $\Omega$. Therefore, $\cup_{k\geq k_0} A_k \supset D_T$ is an open set and can have any sufficient small capacity when $k_0$ large enough. We define the limit of $X^{n_k}$ on $[0, T]$ by

$$I_t^T(\omega) = \limsup_{k\to\infty} X_t^{n_k}(\omega).$$

As each $X^{n_k}$ is continuous in $(\omega, t)$, for all $k \geq 1$, and $X^{n_k}$ converges uniformly on $(\cup_{k\geq k_0} A_k)^c \times [0, T]$. Therefore, $I_\cdot^T(\cdot)$ is continuous on $(\cup_{k\geq k_0} A_k)^c \times [0, T]$. Thus, the process $I^T$ is quasi-continuous on $\Omega \times [0, T]$.

Moreover, note that for each $\omega \in (D_T)^c$, $t \to X_t^{n_k}(\omega)$ converges to $t \to I_t^T(\omega)$ uniformly, thus q.s. $t \to I_t^T(\omega)$ is continuous on $[0, T]$. So is $X$ since they are versions of each other on $[0, T]$.

To prove the reverse direction, we can assume that $X$ itself is quasi-continuous on $\Omega\times[0, T]$, since if $X'$ is the quasi-continuous version of $X$ on $\Omega \times [0, T]$, then $\sup_{0\leq t\leq T} |X_t - X_t'| = 0$ q.s. For any $\varepsilon > 0$, we can find an open set $G \subset \Omega$ with $c(G) < \varepsilon$ such that $X_\cdot(\cdot)$ is continuous on $G^c \times [0, T]$. By the Tietze's extension theorem, there exists a $Y$ which is continuous on $\Omega \times [0, T]$ such that $X = Y$ on $G^c \times [0, T]$. Then

$$c(\{\sup_{0\leq t\leq T} |Y_t - X_t| > \varepsilon\}) \leq c(\{\sup_{0\leq t\leq T} |Y_t - X_t| > \varepsilon\} \cap G) \leq c(G) \leq \varepsilon.$$

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(ii) For each $k \geq 1$, from (i), we get a quasi-continuous version $I^k$ of $X$ on $[0, k]$. Then $I_t^k = I_t^{k'}, 0 \leq t \leq k \wedge k'$ q.s. for each $k, k' \geq 1$. Denote the polar sets

$$F^{k,k'} := \{\omega \in \Omega : I_t^k(\omega) = I_t^{k'}(\omega), 0 \leq t \leq k \wedge k' \text{ does not hold}\} \quad \text{and} \quad F := \cup_{k,k' \geq 1} F^{k,k'}.$$

Then we can define

$$I_t(\omega) = \begin{cases} I_t^k(\omega), \ t \leq k; & \omega \in F^c, \\ 0; & \omega \in F. \end{cases}$$

For any given $\varepsilon > 0$ and for each $k \geq 1$, from (i), we can find an open set $O_k$ such that $c(O_k) \leq \frac{\varepsilon}{2^k}$ and $I^k$ is continuous on $(O_k)^c \times [0, k]$. Denoting the open set $O' = \cup_{k \geq 1} O_k$, then $c(O') \leq \varepsilon$. We also denote $O = O' \cup F$ and obviously that $c(O) \leq \varepsilon$. It is easy to see that $I$ is continuous on $O^c \times [0, \infty)$. Now it remains to prove that $O$ is open. To that end, it suffices to show that $O^c$ is closed. Given any $k, k' \geq 1$. For every $t \in [0, k \wedge k']$, since $I_t^k, I_t^{k'}$ is continuous on $(O')^c$, then $\{\omega \in \Omega : I_t^k(\omega) = I_t^{k'}(\omega)\} \cap (O')^c = \{\omega \in \Omega : I_t^k(\omega) - I_t^{k'}(\omega) = 0\} \cap (O')^c$ is a closed set. Thus,

$$\begin{aligned} (F^{k,k'})^c \cap (O')^c &= (\cap_{t \in [0, k \wedge k']} \{\omega \in \Omega : I_t^k(\omega) = I_t^{k'}(\omega)\}) \cap (O')^c \\ &= \cap_{t \in [0, k \wedge k']} (\{\omega \in \Omega : I_t^k(\omega) = I_t^{k'}(\omega)\} \cap (O')^c) \end{aligned}$$

is closed. This implies

$$O^c = (O')^c \cap F^c = (O')^c \cap (\cap_{k,k' \geq 1} (F^{k,k'})^c) = \cap_{k,k' \geq 1} ((F^{k,k'})^c \cap (O')^c)$$

is closed, as desired.

The q.s. continuity of $I$ in $t$ on $[0, \infty)$ follows from the above definition of $I$ and the q.s. continuity of $I^k$ in $t$ on $[0, k]$ for each $k \geq 1$. Moreover, $X$ is also q.s. continuous since it is a version of $I$ on $[0, \infty)$.

Now we prove the reverse direction. If $X$ is quasi-continuous on $\Omega \times [0, \infty)$, then $X$ is quasi-continuous on $\Omega \times [0, T]$, for each $T > 0$, and the conclusion follows from (i). $\square$

**Remark 4.2.** From the proof of the above theorem, we know that the direction of obtaining the approximation property in the form of (4.1) from the quasi-continuity is always trivial. In fact, we can get a better approximation property in (ii): If $X$ is quasi-continuous on $\Omega \times [0, \infty)$, then there exists a sequence $X^n \in C(\Omega \times [0, \infty))$ such that, for each $\varepsilon > 0$,

$$c(\{\sup_{0 \leq t < \infty} |X_t^n - X_t| > \varepsilon\}) \to 0, \quad \text{as } n \to \infty. \tag{4.2}$$

Indeed, for any $\varepsilon > 0$, by a similar analysis as in the proof of (i), we can find an open set $G \subset \Omega$ with $c(G) < \varepsilon$ and a function $Y$ which is continuous on $\Omega \times [0, \infty)$ such that $X = Y$ on $G^c \times [0, \infty)$. Then

$$c(\{\sup_{0 \leq t < \infty} |Y_t - X_t| > \varepsilon\}) \leq c(\{\sup_{0 \leq t < \infty} |Y_t - X_t| > \varepsilon\} \cap G) \leq c(G) \leq \varepsilon.$$

So we can see that the weak form of approximation condition in (ii) of Theorem 4.1 that for each $T > 0$, (4.1) holds for some sequence $X^n$, is equivalent to the stronger form (4.2), which does not seem obvious. For Theorem 4.1, we mainly use its nontrivial direction of obtaining the quasi-continuity from the approximation property in the concrete problems. So in (ii), we prefer the weak form condition since, by the observation that it is verified on finite time interval with approximation sequence $X^n$ that can depend on $T$, it is more convenient to apply.

In particular, taking $T = 0$ in Theorem 4.1(i), we get the corresponding quasi-continuity characterization theorem for random variables, which also generalizes Theorem 2.3 a bit.

**Corollary 4.3.** *Let $X : \Omega \to \mathbb{R}$ be a random variable. Then $X$ is quasi-continuous if and only if there exists a sequence $X^n \in C(\Omega)$ such that, for each $\varepsilon > 0$,*

$$c(\{|X^n - X| > \varepsilon\}) \to 0, \quad \text{as } n \to \infty. \tag{4.3}$$

The following two results concern the quasi-continuity of stopped processes.

**Proposition 4.4.** *Let $X = (X_t)_{t\in[0,\infty)}$ be a process. The random variable $X_\tau$ is quasi-continuous if $X$ is a quasi-continuous process on $\Omega\times[0, T]$ ($\Omega\times[0, \infty)$ resp.) and $\tau : \Omega \to [0, T]$ ($[0, \infty)$ resp.) is a quasi-continuous stopping time.*

**Proof.** We just prove the conclusion on $[0, T]$, and the proof for the other part is similar. For any $\varepsilon > 0$, we can find an open set $G \subset \Omega$ such that $c(G^c) \leq \varepsilon$ and on $G^c \times [0, T]$, $X$ is continuous. Moreover, we can also find an open set $O \subset \Omega$ such that $\tau$ is continuous on $O^c$. Then on $G^c \cap O^c = (G \cup O)^c$, it is easy to see that $X_\tau$ is continuous. □

**Proposition 4.5.** *Let $X = (X_t)_{t\in[0,\infty)}$ be a process. Then the process $(X_{\tau\wedge t})_{t\in[0,T]}$ ($(X_{\tau\wedge t})_{t\in[0,\infty)}$ resp.) is quasi-continuous on $\Omega\times[0, T]$ ($\Omega\times[0, \infty)$ resp.) if $X$ is and $\tau$ is a quasi-continuous stopping time.*

**Proof.** The proof is similar to that of Proposition 4.4, so we omit it. □

**Remark 4.6.** We remark that Proposition 4.4 is a special case of Proposition 4.5 from Remark 2.7. But it should be beneficial to give Proposition 4.4 explicitly as above due to its potential broader use.

### *4.2. Application to G-expectation space*

For any given family of probability measures, the canonical process $B$ is continuous in $(\omega, t)$, and thus is trivially quasi-continuous. Now we shall use Theorem 4.1 to obtain some non-trivial quasi-continuous processes in the case that $\mathcal{P}$ is the $G$-expectation family, i.e., the upper expectation of $\mathcal{P}$ is a $G$-expectation. Let us first briefly review the construction of $G$-expectation, and more details can be found in [1,18].

Let $\Gamma$ be a bounded and closed subset of $\mathbb{S}_+(k)$, where $\mathbb{S}_+(k)$ is the collection of nonnegative $k \times k$ symmetric matrices. The $G$-expectation $\hat{\mathbb{E}}$ is the upper expectation of the probability family

$$\mathcal{P} = \left\{P : P \text{ is a probability measure on } \Omega \text{ such that } B \text{ is a martingale and } \frac{d\langle B\rangle_t^P}{dt} \in \Gamma\right\},$$

under which the canonical process $B$ is called $G$-Brownian motion. In the $G$-expectation case, $L_C^1(\Omega)$ is usually denoted by $L_G^1(\Omega)$ and the conditional $G$-expectation $\hat{\mathbb{E}}_t[\cdot]$ is well-defined on $L_G^1(\Omega)$.

An adapted process $(M_t)_{t\geq 0}$ is called a $G$-martingale if for each $s \leq t$, $M_t \in L_G^1(\Omega_t)$ and $\hat{\mathbb{E}}_s[M_t] = M_s$, where $\Omega_t = \{\omega_{\cdot\wedge t} : \omega \in \Omega\}$ and $L_G^1(\Omega_t)$ is defined similar to $L_G^1(\Omega)$ with $\Omega$ replaced by $\Omega_t$. Furthermore, a $G$-martingale $M$ is called symmetric if $-M$ is also a

$G$-martingale. We remark that, if $M$ is a symmetric $G$-martingale, then it is a $\mathcal{P}$-martingale, i.e., it is a martingale under each $P \in \mathcal{P}$. In general, a $G$-martingale is a $\mathcal{P}$-supermartingale, see [12,19] for more discussions.

Let $M_G^0(0,T)$ be the collection of processes in the form: for a given partition $\{t_0, \cdots, t_N\}$ of $[0,T]$,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})}(t),$$

where $\xi_j \in C_b(\Omega_{t_j})$, $j = 0, 1, 2, \cdots, N-1$. For $p \geq 1$ and $\eta \in M_G^0(0,T)$, let $\|\eta\|_{M_G^p} = \{\hat{\mathbb{E}}[\int_0^T |\eta_s|^p ds]\}^{1/p}$, $\|\eta\|_{H_G^p} = \{\hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}$, and denote by $M_G^p(0,T)$ and $H_G^p(0,T)$ the completions of $M_G^0(0,T)$ under the norms $\|\cdot\|_{M_G^p}$ and $\|\cdot\|_{H_G^p}$, respectively. For each $1 \leq i, j \leq k$, we denote by $\langle B^i, B^j \rangle$ the cross-variation process of $B$. Then for two processes $\eta \in H_G^p(0,T)$ and $\xi \in M_G^p(0,T)$, the $G$-Itô integrals $\int_0^t \eta_s dB_s^i$ and $\int_0^t \xi_s d\langle B^i, B^j \rangle_s$, $\int_0^t \xi_s ds$ are well-defined, and $\int_0^t \eta_s dB_s^i$ is a symmetric $G$-martingale.

In the following of this subsection, we always assume that $\mathcal{P}$ is a family of probability measures corresponding to $G$-expectation as above.

Theorem 4.1 contains the following three typical processes in the $G$-expectation space.

**Proposition 4.7.** *We have:*

(i) *$G$-martingale $M$ has a quasi-continuous modification on $\Omega \times [0, \infty)$.*
(ii) *If $\eta \in M_G^1(0,T)$ ($\cap_{T>0} M_G^1(0,T)$ resp.), then the process $A_t := \int_0^t \eta_s ds$ has a quasi-continuous modification on $\Omega \times [0,T]$ ($\Omega \times [0,\infty)$ resp.).*
(iii) *If $\eta \in M_G^1(0,T)$ ($\cap_{T>0} M_G^1(0,T)$ resp.), then the process $A_t := \int_0^t \eta_s d\langle B^i, B^j \rangle_s$ has a quasi-continuous modification on $\Omega \times [0,T]$ ($\Omega \times [0,\infty)$ resp.).*

**Proof.** (i). For each $T$, since $M_T \in L_G^1(\Omega_T)$, according to [1], we can find $\xi^n \in L_{ip}(\Omega_T)$ such that $\xi^n \to M_T$ under the norm $\hat{\mathbb{E}}[|\cdot|]$, where

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, B_{t_2} - B_{t_1} \cdots, B_{t_n} - B_{t_{n-1}}) : n \in \mathbb{N}, 0 \leq t_1 < t_2 \cdots < t_n \leq T, \varphi \in C_{b.Lip}(\mathbb{R}^{k\times n})\}.$$

From the definition of conditional $G$-expectation (c.f. Chapter III of [18]), we can see that the process $\hat{\mathbb{E}}_t[\xi^n]$ is continuous on $\Omega \times [0,T]$. By the following Lemma 4.9, we can take the continuous modifications of $G$-martingales $M_t = \hat{\mathbb{E}}_t[M_T]$ and $\hat{\mathbb{E}}_t[|\xi^n - M_T|]$. Since for any given $P \in \mathcal{P}$, $\hat{\mathbb{E}}_t[|\xi^n - M_T|]$ is a supermartingale, we can apply the Doob's martingale inequality (see, e.g., Theorem 2.42 of [5]) to obtain that for each $\varepsilon > 0$,

$$\begin{aligned} P(\{\sup_{0\leq t\leq T} |\hat{\mathbb{E}}_t[\xi^n] - M_t| > \varepsilon\}) &= P(\{\sup_{0\leq t\leq T} |\hat{\mathbb{E}}_t[\xi^n] - \hat{\mathbb{E}}_t[M_T]| > \varepsilon\}) \\ &\leq P(\{\sup_{0\leq t\leq T} \hat{\mathbb{E}}_t[|\xi^n - M_T|] > \varepsilon\}) \\ &\leq \frac{1}{\varepsilon}\hat{\mathbb{E}}[|\xi^n - M_T|]. \end{aligned}$$

Taking supremum over $P \in \mathcal{P}$, we obtain

$$c(\{\sup_{0\leq t\leq T} |\hat{\mathbb{E}}_t[\xi^n] - M_t| > \varepsilon\}) \leq \frac{1}{\varepsilon}\hat{\mathbb{E}}[|\xi^n - M_T|] \to 0, \quad \text{as } n \to \infty.$$

Now applying Theorem 4.1, we deduce that $M$ is quasi-continuous.

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(ii). We can find a sequence $\eta^n \in M_G^0(0,T)$ such that $\eta^n \to \eta$ in $M_G^1(0,T)$. Then the conclusion follows from the observation that the process $(\int_0^t \eta_s^n ds)_{t\geq 0}$ is continuous on $\Omega \times [0,T]$ and

$$\hat{\mathbb{E}}[\sup_{0\leq t\leq T} |\int_0^t \eta_s^n ds - \int_0^t \eta_s ds|] \leq \hat{\mathbb{E}}[|\int_0^T |\eta_s^n - \eta_s|ds] \to 0, \quad \text{as } n \to \infty.$$

(iii). Note that $M_t := \int_0^t \eta_s d\langle B^i, B^j\rangle_s - \int_0^t 2G(\tilde{\eta}_s)ds$ is a $G$-martingale (see Chapter IV of [18]), where $\tilde{\eta} = (\tilde{\eta}^{ml})_{m,l=1}^k$ is defined by

$$\tilde{\eta}_t^{ml} = \begin{cases} \eta_t; & m = i \text{ and } l = j, \\ 0; & \text{otherwise.} \end{cases}$$

Then we deduce the result from (i) and (ii). □

**Remark 4.8.** We remark that the result (i) on finite interval $[0,T]$ has already been obtained in [23]. Compared with this, our proof is simple and different, and moreover, does not rely on the non-degeneracy assumption on $\Gamma$.

In the above proof, the following continuity modification theorem for $G$-martingales $M$ is needed. It corresponds to the classical fact in the linear stochastic analysis that every martingale for Brownian motion filtration has a continuous modification, and partial result under the additional assumption that $M_T \in L_G^p(\Omega)$ for $T > 0$, for some $p > 1$, on this direction has already been proved as a byproduct in the $G$-martingale representation theorem (see [21,23]).

**Lemma 4.9.** *Any $G$-martingale $M$ has a continuous modification.*

**Proof.** We employ the notation in the proof of Proposition 4.7(i). For any given $T > 0$ and $M_T \in L_G^1(\Omega_T)$, there exists some $\xi^n \in L_{ip}(\Omega_T)$ such that $\xi^n \to M_T$ under the norm $\hat{\mathbb{E}}[|\cdot|]$. Then by the definition of conditional $G$-expectation, $t \to \hat{\mathbb{E}}_t[\xi^n]$ and $t \to \hat{\mathbb{E}}_t[|\xi^n - \xi^m|]$ are continuous on $[0,T]$, for each $\omega \in \Omega$, and by a similar calculation as in the proof of Proposition 4.7(i), we have

$$c(\{\sup_{0\leq t\leq T} |\hat{\mathbb{E}}_t[\xi^n] - \hat{\mathbb{E}}_t[\xi^m]| > \varepsilon\}) \to 0, \quad \text{as } n, m \to \infty.$$

Now from a similar analysis as in the proof of Theorem 4.1(i), we can extract a q.s uniformly convergent subsequence $\hat{\mathbb{E}}_t[\xi^{n_k}]$ such that $t \to \limsup_{k\to\infty} \hat{\mathbb{E}}_t[\xi^{n_k}]$ is continuous and it is a modification of $M$. □

A $G$-martingale stopped at a quasi-continuous stopping time is still a $G$-martingale.

**Corollary 4.10.** *Let $\tau$ be a quasi-continuous stopping time. If $(M_t)_{t\geq 0}$ is a $G$-martingale (symmetric $G$-martingale resp.), then $(M_{t\wedge\tau})_{t\geq 0}$ is still a $G$-martingale (symmetric $G$-martingale resp.).*

**Proof.** We just prove the $G$-martingale case, from which the symmetric case follows by applying the conclusion to $M$ and $-M$.

For any $t$ and stopping time $\sigma \leq t$, let $\hat{\mathbb{E}}_\sigma$ be the conditional $G$-expectation at $\sigma$ as defined in [6,14]. By the optional sampling theorem for $G$-martingales (see [14]), we have

$$\hat{\mathbb{E}}_\sigma[M_t] = M_\sigma. \tag{4.4}$$

From Proposition 4.4, the random variable $M_{t\wedge\tau}$ is quasi-continuous. Moreover, note that from (4.4) and the properties of conditional $G$-expectation,

$$\begin{aligned} c(\{|M_{t\wedge\tau}|>N\}) &\leq \frac{\hat{\mathbb{E}}[|M_{t\wedge\tau}|]}{N}=\frac{\hat{\mathbb{E}}[|\hat{\mathbb{E}}_{t\wedge\tau}[M_t]|]}{N} \\ &\leq \frac{\hat{\mathbb{E}}[\hat{\mathbb{E}}_{t\wedge\tau}[|M_t|]]}{N}=\frac{\hat{\mathbb{E}}[|M_t|]}{N}\to 0, \quad \text{as } N\to\infty. \end{aligned}$$

Then applying Proposition 19 in [1] yields that

$$\begin{aligned} \hat{\mathbb{E}}[|M_{t\wedge\tau}|I_{\{|M_{t\wedge\tau}|>N\}}] &= \hat{\mathbb{E}}[|\hat{\mathbb{E}}_{t\wedge\tau}[M_t]|I_{\{|M_{t\wedge\tau}|>N\}}] \\ &\leq \hat{\mathbb{E}}[\hat{\mathbb{E}}_{t\wedge\tau}[|M_t|]I_{\{|M_{t\wedge\tau}|>N\}}] \\ &= \hat{\mathbb{E}}[|M_t|I_{\{|M_{t\wedge\tau}|>N\}}] \\ &\to 0, \quad \text{as } N\to\infty. \end{aligned}$$

Therefore, by the characterization Theorem 2.3, we deduce that $M_{t\wedge\tau}\in L^1_G(\Omega_t)$.

Now it remains to show the martingale property. Indeed, from (4.4) and the properties of conditional $G$-expectation, for each $s\geq t$, we have

$$\begin{aligned} \hat{\mathbb{E}}_t[M_{s\wedge\tau}] &= \hat{\mathbb{E}}_t[M_{s\wedge\tau}I_{\{\tau\geq t\}}]+\hat{\mathbb{E}}_t[M_{s\wedge\tau}I_{\{\tau<t\}}] \\ &= \hat{\mathbb{E}}_t[M_{(s\wedge\tau)\vee t}]I_{\{\tau\geq t\}}+\hat{\mathbb{E}}_t[M_{\tau\wedge t}]I_{\{\tau<t\}} \\ &= \hat{\mathbb{E}}_t[\hat{\mathbb{E}}_{(s\wedge\tau)\vee t}[M_s]]I_{\{s\wedge\tau\geq t\}}+M_{\tau\wedge t}I_{\{\tau<t\}} \\ &= \hat{\mathbb{E}}_t[M_s]I_{\{s\wedge\tau\geq t\}}+M_{\tau\wedge t}I_{\{\tau<t\}} \\ &= M_tI_{\{s\wedge\tau\geq t\}}+M_\tau I_{\{s\wedge\tau<t\}} \\ &= M_{\tau\wedge t}. \end{aligned}$$

This completes the proof. □

We close this section with a regularity theorem for the stopping of stochastic integrals.

**Proposition 4.11.** *Let $\tau\leq T$ be a quasi-continuous stopping time. Then for each $p\geq 1$, we have*

$$I_{[0,\tau]}\in M^p_G(0,T). \tag{4.5}$$

**Proof.** Without loss of generality, we assume that $\tau\leq T$. For each $k\in\mathbb{N}$, by the partition of unit theorem, we can find a sequence of continuous functions $\{\phi_i^k\}_{i=1}^{n_k}$ with $n_k=2^k+1$ such that:

(i) the diameter of support $\lambda(\mathrm{supp}(\phi_i^k))\leq\frac{2}{2^k}$ and $0\leq\phi_i^k\leq 1$;
(ii) $\sum_{i=1}^{n_k}\phi_i^k(t)=1$, for each $t\in[0,1]$;
(iii) $\phi_i^k(t)>0$ for some $t\in[\frac{i-1}{2^k},\frac{i}{2^k})$ but $\phi_i^k(t)\equiv 0$ for $t\geq\frac{i}{2^k}$, for $1\leq i\leq n_k+1$.

It is easy to check that

$$\sum_{i=1}^{n_k}I_{[0,\frac{i}{2^k}]}\phi_i^k(\tau)\to I_{[0,\tau]} \text{ in } M^p_G(0,T), \quad \text{as } k\to\infty.$$

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Then it remains to show that $\sum_{i=1}^{n_k} I_{[0,\frac{i}{2^k}]}\phi_i^k(\tau) \in M_G^p(0,T)$. A rewriting gives

$$\begin{aligned}\sum_{i=1}^{n_k} I_{[0,\frac{i}{2^k}]}\phi_i^k(\tau) &= \sum_{i=1}^{n_k}(\sum_{j=1}^{i} I_{(\frac{j-1}{2^k},\frac{j}{2^k}]} + I_{\{0\}})\phi_i^k(\tau)\\ &= \sum_{j=1}^{n_k}\sum_{i=j}^{n_k} I_{(\frac{j-1}{2^k},\frac{j}{2^k}]}\phi_i^k(\tau) + \sum_{i=1}^{n_k} I_{\{0\}}\phi_i^k(\tau)\\ &= \sum_{j=1}^{n_k} I_{(\frac{j-1}{2^k},\frac{j}{2^k}]}\sum_{i=j}^{n_k}\phi_i^k(\tau) + I_{\{0\}}.\end{aligned}$$

Noting that $1 \geq \sum_{i=j}^{n_k}\phi_i^k \geq I_{[\frac{j-1}{2^k},1]}$, then

$$\sum_{i=j}^{n_k}\phi_i^k(\tau) = \sum_{i=j}^{n_k}\phi_i^k(\tau \wedge \frac{j-1}{2^n})I_{[\tau\leq\frac{j-1}{2^k}]} + I_{[\tau>\frac{j-1}{2^k}]} \in \mathcal{F}_{\frac{j-1}{2^k}}.$$

Since $\phi_i^k$ is continuous, thus $\sum_{i=j}^{n_k}\phi_i^k(\tau)$ is quasi-continuous. Then by applying Theorem 2.3, we deduce that $\sum_{i=j}^{n_k}\phi_i^k(\tau) \in L_G^p(\Omega_{\frac{j-1}{2^k}})$. This completes the proof. □

**Remark 4.12.**

(i) Similar argument shows that $I_{[0,\tau]} \in H_G^p(0,T)$ under the same assumptions.

(ii) One of the referees provides an alternative short and novel proof to the above proposition. Indeed, first from Proposition 4.4, the random variable $B_\tau$ is quasi-continuous. Then by a similar analysis as in the proof of Corollary 4.10, we have

$$\hat{\mathbb{E}}[|B_\tau|^{p+2}] = \hat{\mathbb{E}}[|\hat{\mathbb{E}}_\tau[B_T]|^{p+2}] \leq \hat{\mathbb{E}}[|B_T|^{p+2}] < \infty.$$

Thus $B_\tau \in L_G^{p+1}(\Omega_T)$. Now we can apply the $G$-martingale representation theorem (see [21,23]) to obtain a process $h \in H_G^p(0,T)$ such that

$$B_\tau = \int_0^T h_s dB_s.$$

This implies that $\|I_{[0,\tau]} - h\|_{H_G^p(0,T)} = 0$, and thus $I_{[0,\tau]} \in H_G^p(0,T)$. In particular, $I_{[0,\tau]} \in H_G^2(0,T) = M_G^2(0,T)$. Combining this with the characterization theorem of $M_G^p(0,T)$ (see [9]), we obtain that $I_{[0,\tau]} \in M_G^p(0,T)$.

But we can still keep our original proof because it is more direct and constructive, and since it does not rely on the $G$-martingale representation theorem which is from the structure of the probability family for $G$-expectation, it is applicable to the more general case that the $G$-expectation probability family is replaced by an arbitrary given family of probability measures on $\Omega$.

**Remark 4.13.** Let $\tau \leq T$ be a stopping time and $\eta \in H_G^p(0,T)$. From [11], we have

$$\int_0^\tau \eta_s dB_s^i = \int_0^T \eta_s I_{[0,\tau]}(s) dB_s^i.$$

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If $\tau$ is quasi-continuous, then by the above Proposition 4.11 (see also Remark 4.12(i)), we derive that $\eta I_{[0,\tau]} \in H_G^p(0,T)$. Such kind of conclusions may be useful in the localization argument for the stochastic integrals.

## 5. Examples and counterexamples

We first present some examples of nonlinear semimartingales $Y$ satisfying the assumptions in Theorem 3.3.

**Example 5.1.**

(i) Let $\mathcal{P}$ be the weakly compact family of probability measures corresponding to $G$-expectation, under which canonical process $B$ is a $G$-Brownian motion. Assume that $B$ satisfies $\frac{d\langle B\rangle_t}{dt} \geq \underline{\sigma}^2 I_{k\times k}$ for some $\underline{\sigma}^2 > 0$. Then $B$ is quasi-continuous and satisfies the assumption $(H')$.
More generally, let $Q$ be the open set we concern. We take $Y$ as the solution of a $d$-dimensional SDEs driven by $G$-Brownian motion $B$:

$$X_t^x = x + \int_0^t b(s, X_s^x)ds + \sum_{i,j=1}^k \int_0^t h_{ij}(s, X_s^x)d\langle B^i, B^j\rangle_s + \sum_{j=1}^k \int_0^t \sigma_j(s, X_s^x)dB_s^j, \quad t \geq 0,$$

where $x \in \mathbb{R}^d$, $b(t,x), h_{ij}(t,x), \sigma_j(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are deterministic functions continuous in $t$ and Lipschitz in $x$, with coefficient $L$. We also assume that $\sigma := (\sigma_1 \cdots, \sigma_k)$ is non-degenerate, i.e., there exists a constant $\lambda > 0$ such that

$$\lambda I_{d\times d} \leq \sigma(t,y)\sigma(t,y)^T, \quad \text{for all } y \in \overline{Q}.$$

First from Proposition 4.7, the process $X^x$ is quasi-continuous. Then we show that the assumption $(H)$ holds. Indeed, we can fix any $R > 0$ and define, for $\omega$ satisfying $\tau_Q(\omega) < \infty$, the stopping times

$$\sigma^\omega(\omega') = \inf\{t \geq \tau_Q(\omega) : X_t^x(\omega') \in (U(X^x_{\tau_Q(\omega)}(\omega), R))^c\},$$

where $(U(X^x_{\tau_Q(\omega)}(\omega), R))$ is the open ball with center $X^x_{\tau_Q(\omega)}(\omega)$ and radius $R$. Fix any $\delta > 0$, then for $\omega' \in \Omega^\omega$, $\phi(\tau_Q(\omega)+t, X^x_{\tau_Q(\omega)+t}(\omega'))$, for $\phi = b, h_{ij}, \sigma_j$, are bounded for $t \in [0, \sigma^\omega(\omega') \wedge \delta]$:

$$\begin{aligned}
&|\phi(\tau_Q(\omega)+t, X^x_{\tau_Q(\omega)+t}(\omega'))| \\
&\leq |\phi(\tau_Q(\omega)+t, X^x_{\tau_Q(\omega)+t}(\omega')) - \phi(\tau_Q(\omega)+t, 0)| + |\phi(\tau_Q(\omega)+t, 0)| \\
&\leq L|X^x_{\tau_Q(\omega)+t}(\omega')| + |\phi(\tau_Q(\omega)+t, 0)|.
\end{aligned}$$

Now by the non-degeneracy assumption on $B$ and $\sigma$, it is easy to check that the assumption $(H)$ is satisfied.
It worthy pointing out that whereas $(H')$ in Remark 3.5 may not hold. This case is one of the main motivation for our general condition $(H)$. For the above $G$-SDE, if moreover $b, h_{ij}, \sigma_j$ are bounded (globally on $Q$), then the stronger assumption $(H')$ is also satisfied.

(ii) In the $G$-expectation space, we take a $d$-dimensional process $Y = M + A$, where $M$ is a symmetric $G$-martingale and $A$ is a quasi-continuous finite variation process, such that $(H)$ or $(H')$ is satisfied (in the one-dimensional case, this assumption can be weakened, see Remark 3.15).

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(iii) Let $Y = B$ and $\mathcal{P}$ be a weakly compact family of probability measures such that under each $P \in \mathcal{P}$, $Y = M^P + A^P$ is a semimartingale satisfying

$$\lambda I_{k\times k} \le \frac{d\langle M^P\rangle_t}{dt} \le \Lambda I_{k\times k},\ |\frac{dA_t^P}{dt}| \le C \quad on\ \overline{Q},\ P\text{-a.s., for some constants } 0 < \lambda \le \Lambda, C \ge 0,$$

as a case considered in [2]. Then $(H')$ is satisfied and obviously $Y$ is quasi-continuous.

We then consider several counterexamples which showing that the exit times may not possess the quasi-continuity if the condition $(H')$ is violated. Here we mainly confine the discussions to the condition $(H')$ for the sake of symbol simplicity, although the condition $(H)$ can also be checked.

The first example concerns on the case that the assumption $\mathrm{tr}[d\langle Y\rangle_t] > 0$, for each $P$, in $(H')$ does not hold.

**Example 5.2.**

(i) Let $k = 1$ and denote $\omega^x$ the path with constant value $x \in \mathbb{R}$, i.e., $\omega_t^x \equiv x$ for each $t \ge 0$. We consider the family $\mathcal{P} = \{P_x : x \in [-1, 1]\}$ of probability measures such that

$$P_x(\{\omega^x\}) = 1, \quad \text{for each } x \in [-1, 1].$$

Take $Q = (-\infty, 0)$ and $Y = B$. It is easy to see that $\mathcal{P}$ is weakly compact and $\langle B\rangle_t^P \equiv 0$ for each $P \in \mathcal{P}$. Note that

$$(\tau_Q \wedge 1)(\omega^x) = 0 \text{ for } x \in [0, 1], \quad \text{and} \quad (\tau_Q \wedge 1)(\omega^x) = 1 \text{ for } x \in [-1, 0).$$

So $\omega^0$ is a discontinuity point of $\tau_Q \wedge 1$. Assume on the contrary that we can find a set $E$ such that $c(E) \le \frac{1}{2}$ and $\tau_Q \wedge 1$ is continuous on $E^c$. For each $x \in [-1, 1]$, since $c(\{\omega^x\}) = 1$, so it must hold that $\omega^x \in E^c$. But this contradicts to the assumption that $\tau_Q \wedge 1$ is continuous on $E^c$. Therefore, $\tau_Q \wedge 1$ is not quasi-continuous

(ii) Let $k = 1$ and $\mathcal{P}$ be a weakly compact family of probability measures, under which $B$ is a one-dimensional $G$-Brownian motion with $\Gamma = [0, \overline{\sigma}^2]$ for some $\overline{\sigma}^2 > 0$. Assume that under $P_\sigma \in \mathcal{P}$, $B$ is a linear Brownian motion such that $\langle B\rangle_t^{P_\sigma} = \sigma^2 t$, for each $\sigma \in [0, \overline{\sigma}]$. Take $Q = (-\infty, 0)$ and $Y = B$. In this $G$-Brownian motion case, we need to consider another kind of neighborhood for $\omega^0$, where $\omega^0$ is defined as in (i). Let us denote

$$A := \{\omega \in \Omega : \omega_0 = 0,\ (\omega_t)_{t\ge 0} \text{ changes sign infinitely many times in } [0, \varepsilon],\ \text{for each } \varepsilon > 0\}.$$

Then

$$(\tau_{\overline{Q}} \wedge 1)(\omega) = 0 \text{ for } \omega \in A, \quad \text{and} \quad (\tau_{\overline{Q}} \wedge 1)(\omega^0) = 1.$$

This means that $\tau_{\overline{Q}} \wedge 1$ is not continuous at $\omega^0$. Now we show that $\tau_{\overline{Q}} \wedge 1$ is not quasi-continuous. Indeed, for any given $T > 0$ and $\varepsilon > 0$, since $\frac{B_t}{\sigma}$ is a standard Brownian motion, then

$$P_\sigma(\{\sup_{0\le t\le T} |B_t| \le \varepsilon\}) = P_\sigma(\{\sup_{0\le t\le T} |\frac{B_t}{\sigma}| \le \frac{\varepsilon}{\sigma}\}) \to 1, \quad \text{as } 0 < \sigma \downarrow 0.$$

Thus,

$$P_\sigma(\{\omega \in \Omega : \rho(\omega, \omega^0) \le \varepsilon\}) \to 1, \quad \text{as } 0 < \sigma \downarrow 0.$$

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Therefore, by the path property of linear Brownian motion (see Problem 2.7.18 of [10]),

$$P_\sigma(\{\omega \in A : \rho(\omega, \omega^0) \leq \varepsilon\}) \to 1, \quad \text{as} \quad 0 < \sigma \downarrow 0.$$

This implies

$$c(A_\varepsilon) = 1, \text{ for each } \varepsilon > 0, \quad \text{where } A_\varepsilon := \{\omega \in A : \rho(\omega, \omega^0) \leq \varepsilon\}. \tag{5.1}$$

Assume on the contrary that we can find a set $E$ such that $c(E) \leq \frac{1}{2}$ and $\tau_{\overline{Q}} \wedge 1$ is continuous on $E^c$. Since $P_0(\{\omega^0\}) = 1$, so $c(\{\omega^0\}) = 1$, and thus $\omega^0 \in E^c$. Note that $\omega^0 \in E^c$ is a limit point of $A \cap E^c$, since if not, there exists some $\varepsilon > 0$ such that $A_\varepsilon \subset E$, which is impossible by equality (5.1). Thus we have reached a contradiction. So $\tau_{\overline{Q}} \wedge 1$ is not quasi-continuous.

Now we give an example in which $d\langle Y\rangle_t \geq \varepsilon \operatorname{tr}[d\langle Y\rangle_t] I_{d\times d}$ for some $\varepsilon > 0$, for each $P$, in $(H')$ is not met.

**Example 5.3.** Let $k = 2$ and $\mathcal{P}$ be the weakly compact family of probability measures such that $B$ is a two-dimensional $G$-Brownian motion with

$$\Gamma = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{bmatrix} : 0 \leq \alpha \leq 1 \right\}.$$

Then $\operatorname{tr}[\langle B\rangle_t^P] = t$, for each $P \in \mathcal{P}$. Assume that under $P_\alpha \in \mathcal{P}$, $B$ is a linear Brownian motion with $\langle B\rangle_t^{P_\alpha} = t \begin{bmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{bmatrix}$, for each $0 \leq \alpha \leq 1$. Let us take $Q = (-\infty, \infty) \times (0, 1)$ and $Y = B$. We identify $\omega = (\omega^1, \omega^2)$, where $\omega^j$, $j = 1, 2$ are the corresponding scalar components. In this example, we need to consider the following set of points of discontinuity:

$$\Omega_0 := \{\omega = (\omega^1, \omega^2) \in \Omega : \omega_t^2 \equiv 0, \ t \geq 0\}.$$

We define

$$A := \{\omega \in \Omega : \omega_0^2 = 0, \ (\omega_t^2)_{t\geq 0} \text{ changes sign infinitely many times in } [0, \varepsilon], \text{ for each } \varepsilon > 0\}.$$

Since $(\omega_t^2)_{t\geq 0}$ is a linear Brownian motion under $P_\alpha$, then $P_\alpha(A) = 1$ for $\alpha < 1$. It is easy to see that

$$(\tau_{\overline{Q}} \wedge 1)(\omega) = 0 \text{ for } \omega \in A, \quad \text{and} \quad (\tau_{\overline{Q}} \wedge 1)(\omega) = 1 \text{ for } \omega \in \Omega_0,$$

which means that each $\omega \in \Omega_0$ is a discontinuity point of $\tau_{\overline{Q}} \wedge 1$. Assume that we can find a set $E$ such that $c(E) \leq \frac{1}{2}$ and $\tau_{\overline{Q}} \wedge 1$ is continuous on $E^c$. If $\omega \in \Omega_0$ is a limit point $A \cap E^c$, since $E^c$ is closed, we will have $\omega \in E^c$, which leads to the discontinuity of $\tau_{\overline{Q}} \wedge 1$ on $E^c$. So any $\omega \in \Omega_0$ should not be a limit point of $A \cap E^c$, and thus there exists an open set $O \subset \Omega$ such that $O \supset \Omega_0$ and $O \cap (A \cap E^c) = \emptyset$. Now we claim that $c(O \cap A) = 1$. Indeed, since $P_\alpha$ converges to $P_1$ weakly, as $\alpha \to 1$, then

$$\liminf_{1>\alpha\to 1} P_\alpha(O \cap A) = \liminf_{1>\alpha\to 1} P_\alpha(O) \geq P_1(O) = P_1(\Omega_0) = 1.$$

This implies

$$c(O \cap A) = 1,$$

which is a contradiction since $O \cap A \subset E$. Therefore, $\tau_Q \wedge 1$ is not quasi-continuous.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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