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On hitting times for jump-diffusion processes with past dependent local characteristics

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It is well known how to apply a martingale argument to obtain the Laplace transform of the hitting time of zero (say) for certain processes starting at a positive level and being skip-free downwards. These processes have stationary and independent increments. In the present paper the method is extended to a more general class of processes the increments of which may depend both on time and past history. As a result a generalized Laplace transform is obtained which can be used to derive sharp bounds for the mean and the variance of the hitting time. The bounds also solve the control problem of how to minimize or maximize the expected time to reach zero.

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hitting times * Laplace transform * expectation and variance * martingales * diffusions * compound Poisson process * random walks * M/G/1 queue * perturbation * control

1. Introduction

Consider a process $X_t = S_t - ct + \sigma W_t$ where S is a compound Poisson process with Poisson parameter λ which is disturbed by an independent standard Wiener process W. This is the point of view of Dufresne and Gerber (1991). One can also look on X as a diffusion disturbed by jumps. This is the point of view of Ethier and Kurtz (1986, Section 4.10). Here σ and λ may be zero so that the jump term or the diffusion term may vanish. Thus $x+X$ can describe (i) the waiting time of an M/G/1 queue, (ii) the content of a dam, (iii) the level of a storage process, and (iv) $-(x+X)$ the surplus of an insurance company: at time t initiated by a level x (resp. $-x$ in case (iv)) at time $t=0$. Now let

$$\tau := \tau_x := \inf\{t \geq 0, x + X_t = 0\} \quad \text{where } x > 0.$$

Then τ may be the busy period and the wet period for an initial value x or the time when the surplus reaches the level $u+x$ for an initial surplus u. Let μ and φ denote

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the expectation and the Laplace transform of the jump size, respectively. Then it is known that if $c-\lambda\mu>0$,

$$E[\tau]=x/(c-\lambda\mu) \tag{1.1}$$

$$E[\exp\{-\psi(\alpha)\tau\}]=\exp\{-\alpha x\} \tag{1.2}$$

where $\psi(\alpha):=\lambda[\varphi(\alpha)-1]+c\alpha+\frac{1}{2}\sigma^2\alpha^2$ is invertible and hence $\exp\{-\psi^{-1}(\zeta)x\}$ is the Laplace transform of τ; cf. Williams (1979, p. 85), Prabhu (1980, Theorem 5, p. 79), Gerber (1990), Kella and Whitt (1991, Lemma 4.2). Now consider the case that the parameters c, λ, μ, σ vary in the course of time. For example, the arrival rate (exploration or claim rate) and the service rate (consumption or premium rate) may depend on time and queue length (storage or surplus level); cf. Soner (1985), Asmussen and Schock Petersen (1988). One can ask whether the formula (1.1) still holds approximately if the changes of parameters are only slight. This would be a sort of a perturbation result. In this paper a positive answer is given under a boundedness condition on the second moments of the increments. Indeed

$$\frac{x}{M}\leq E[\tau]\leq\frac{x}{m} \tag{1.3}$$

where m and M are the minimal and maximal values of $c-\lambda\mu$. These are inequalities in the sense of Dubins and Savage (1965) and show that in order to minimize the expectation of τ one cannot do better than choosing those parameters all the time which maximize the value of $c-\lambda\mu$. The inequalities (1.3) are proved by Heath, Orey, Pestien and Sudderth (1987) for the case of a controlled diffusion process (i.e., $\lambda=0$ and hence $S_t=0$). Here, a proof is given which provides the reduction to a martingale problem, unifies the proofs for the upper and lower bound and allows for jumps away from the goal zero. Moreover, a generalization of (1.2) is obtained which can be used to find bounds also for the variance of τ. In a final section we also consider a continuous-time random walk on the integer lattice which is skip-free downwards and the embedded Markov chain of the M/G/1 queue.

2. The model

The underlying process will be a jump-diffusion process X, i.e.

$$X_t=\int_0^t\sigma_s\,\mathrm{d}W_s-\int_0^t c_s\,\mathrm{d}s+\sum_{0<s\leq t}(X_s-X_{s-})=:Y_t+S_t \tag{2.1}$$

where W is a standard Wiener process, hence Y is an Itô process and S is a jump process with positive jumps $S_t-S_{t-}\geq 0$. There S is given by a jump rate $\lambda_t\geq 0$, i.e., $\lambda_t\,\mathrm{d}t+\mathrm{o}(\mathrm{d}t)$ is the probability of a jump in $(t,t+\mathrm{d}t]$ given the past, and by the jump distribution Q_t, i.e., Q_t is the distribution of X_t-X_{t-} given there is a jump at time t.

For the approach of the present paper however, it is more convenient to start with the assumption that X is the solution of a martingale problem rather than starting with (2.1) directly. Since we are interested in Laplace transforms we can restrict attention to the small class of test functions

$$f^{\alpha}(x):=e^{-\alpha x}, \quad x\in\mathbb{R},\ \alpha\geqslant 0.$$

To the Itô process there belongs the generator

$$D(c,\sigma)f(x):=-cf'(x)+\tfrac{1}{2}\sigma^2 f''(x) \quad \text{for } c\in\mathbb{R},\ \sigma\geqslant 0,$$

where f', f'' denote the derivatives of f, to the jump process there belongs the generator

$$D(\lambda,Q)f(x):=\lambda\left[\int f(x+y)Q(\mathrm{d}y)-f(x)\right].$$

Hence we obtain for our test functions

$$D(c,\sigma)f^{\alpha}(x)=[c\alpha+\tfrac{1}{2}\sigma^2\alpha^2]f^{\alpha}(x)\ ,$$

$$D(\lambda,Q)f^{\alpha}(x)=\lambda[\varphi_Q(\alpha)-1]f^{\alpha}(x)\ ,$$

where φ_Q is the Laplace transform of Q, and finally for the generator of the jump-diffusion process

$$L_t f^{\alpha}(x):=\{D(\lambda_t,Q_t)+D(c_t,\sigma_t)\}f^{\alpha}(x)=e^{-\alpha x}\psi_t(\alpha)$$

where

$$\psi_t(\alpha):=\lambda_t[\varphi_t(\alpha)-1]+c_t\alpha+\tfrac{1}{2}\sigma_t^2\alpha^2 \quad \text{with } \varphi_t:=\varphi_{Q_t}.$$

Further set

$$\Psi_t(\alpha):=\int_0^t \psi_s(\alpha)\,\mathrm{d}s.$$

Assumption. Let there be given a probability space $(\Omega,\mathscr{F},P)$ endowed with a right-continuous filtration $\mathbb{F}=\{\mathscr{F}_t,\ t\geqslant 0\}$, a progressively measurable real process $\{c_t\}$, two progressively measurable non-negative real processes $\{\sigma_t\}$ and $\{\lambda_t\}$ such that

$$P\left[\int_0^t (\lambda_s+|c_s|+\sigma_s^2)\,\mathrm{d}s<\infty\right]=1 \quad \text{for } 0\leqslant t<\infty$$

as well as a progressively measurable process $\{Q_t\}$ with values in the space of probability measures on $[0,\infty)$. We assume that $X=\{X_t\}$ is an adapted process with right-continuous trajectories which are skip-free downwards and with initial condition $X_0=0$ such that for all $\alpha\geqslant 0$,

$$f^{\alpha}(X_t)-\int_0^t L_s f^{\alpha}(X_s)\,\mathrm{d}s=\exp\{-\alpha X_t\}-\int_0^t \exp\{-\alpha X_s\}\psi_s(\alpha)\,\mathrm{d}s, \quad t\geqslant 0,$$

is a local martingale with respect to $\mathbb{F}$. Here $\{(c_t,\sigma_t,\lambda_t,Q_t),)\ t\geqslant 0\}$ are called the local characteristics of X.

3. Some martingales

Lemma 3.1. *For all* $\alpha \geqslant 0$, $M_t^\alpha := \exp\{-\alpha X_t - \Psi_t(\alpha)\}$ *is a local martingale.*

The lemma is known at least for special cases. For an Itô process cf. Karlin and Taylor (1881, p. 166), Rogers and Williams (1987, p. 77).

Proof. Using our Assumption, this is a direct consequence of Etheir and Kurtz (1986, Corollary 3.3, p. 66). There the assumption $\inf_{s \leqslant t} \exp\{-\alpha X_s\} > 0$ is made. It is satisfied if X has only finitely many jumps in compact intervals, a harmless condition with respect to applications. However, in our special case we even don't need that assumption because dividing $\exp\{-\alpha X_s\}\psi_s(\alpha)$ by $\exp\{-\alpha X_s\}$ causes no troubles. □

Now fix some $x > 0$ which will be our initial point and define

$$\tau := \tau_x := \inf\{t \geqslant 0;\ x + X_t < 0\}$$

which is a stopping time with respect to $\mathbb{F}$ because of the right-continuity of $\mathbb{F}$ (cf. Ethier and Kurtz, 1986, p. 54). Since X is skip-free downwards we know that

$$x + X_\tau = 0 \quad \text{on } \{\tau < \infty\}. \tag{3.1}$$

Lemma 3.2. *Let Q be a probability measure on $[0, \infty)$ with Laplace transform φ, first and second moments μ and γ^2 and*

$$R(\alpha) := \int_0^\infty \mathrm{d}y\, \mathrm{e}^{-\alpha y} \int_y^\infty \mathrm{d}s\, Q[(s, \infty)].$$

Then

$$\varphi(\alpha) = 1 - \mu\alpha + \alpha^2 R(\alpha)$$

where for $s > 0$,

$$0 \leqslant [\mathrm{e}^{-\alpha s} - 1 + s\alpha] Q[(s, \infty)] \leqslant \alpha^2 R(\alpha) \leqslant \alpha^2 R(0) = \tfrac{1}{2}\gamma^2\alpha^2. \qquad \square$$

The lemma is well known and elementary. Now define μ_t as the first moment of Q_t.

Proposition 3.3. *For all* $\alpha \geqslant 0$:

(a) $M^{\alpha,\tau} := \{M_{t\wedge\tau}^\alpha,\ t \geqslant 0\}$ *is a supermartingale;*

(b) *if $c_t - \lambda_t\mu_t \geqslant 0$ for all $t < \tau$, then $M^{\alpha,\tau}$ is a martingale.*

Proof. By Lemma 3.1 and the optional sampling theorem, $M^{\alpha,\tau}$ is again a local martingale (cf. Ethier and Kurtz, 1986, p. 64). Since $M^{\alpha,\tau}$ is non-negative, part (a) follows from Lemma 3.1 and Fatou's lemma (cf. Rogers and Williams, 1987, 14.3, p. 22). If $c_t - \lambda_t\mu_t \geqslant 0$, then, by Lemma 3.2, $\psi_t \geqslant 0$. Now, part (b) follows from

$$0 \leqslant M_t^\alpha \wedge \tau \leqslant \mathrm{e}^{\alpha x}. \qquad \square \tag{3.2}$$

4. The main result

In this section a generalization of (1.2) will be proved upon considering the generalized Laplace transform

$$E\left[\exp\left\{\int_0^\tau \psi_s(\alpha)\,\mathrm{d}s\right\}\right] = E[\exp\{-\Psi_\tau(\alpha)\}].$$

Theorem 4.1. *For all* $\alpha > 0$:

(a) $E[\exp\{-\Psi_\tau(\alpha)\};\ \tau < \infty] \leq \mathrm{e}^{-\alpha x}$;

(b) *if either*

(i) $\inf_{t<\tau}\{c_t - \lambda_t\mu_t\} > 0$, *or*

(ii) $c_t - \lambda_t\mu_t \geq 0$ *for all* $t < \tau$ *and* $\inf_{t<\tau}\{\lambda_t Q_t[(s,\infty)] + \frac{1}{2}\sigma_t^2\} > 0$ *for some* $s > 0$,

then $E[\exp\{-\Psi_\tau(\alpha)\}] = \mathrm{e}^{-\alpha x}$.

Proof. From Proposition 3.3(a) and Fatou's lemma one obtains part (a) according to

$$\begin{aligned} &E[\exp\{-\Psi_\tau(\alpha)\};\ \tau<\infty] \\ &\quad\leq E[\liminf_{t\to\infty} \exp\{-\alpha(x + X_{t\wedge\tau}) - \Psi_{t\wedge\tau}(\alpha)\}] \\ &\quad\leq \liminf \mathrm{e}^{-\alpha x} E[M_t^{\alpha,\tau}] \leq \mathrm{e}^{-\alpha x} E[M_0^{\alpha,\tau}] = \mathrm{e}^{-\alpha x}. \end{aligned}$$

From Proposition 3.3(b) and the dominated convergence theorem combined with (3.2) one obtains $1 = \lim_{t\to\infty} E[M_t^{\alpha,\tau}] = E[\lim_t M_{t\wedge\tau}^\alpha]$. Under the conditions (i) or (ii) one has $\Psi_\tau(\alpha) = \infty$ on $\{\tau = \infty\}$ and thus by (3.1), $\lim_t M_{t\wedge\tau}^\alpha = \exp\{\alpha x - \Psi_\tau(\alpha)\}$. Now part (b) follows. □

Define γ_t^2 as the second moment of Q_t and

$$m = \inf\{(c_t - \lambda_t\mu_t)(\omega),\ 0 \leq t < \tau(\omega),\ \omega\in\Omega\}, \qquad M = \sup\{(c_t - \lambda_t\mu_t)(\omega), \ldots\},$$

$$k = \inf\{(\lambda_t\gamma_t^2 + \sigma_t^2)(\omega),\ 0 \leq t < \tau(\omega),\ \omega \in \Omega\}, \qquad K = \sup\{(\lambda_t\gamma_t^2 + \sigma_t^2)(\omega), \ldots\}.$$

Corollary 4.2. (a) *If* $M < \infty$ *and* $K < \infty$, *then* $E[\tau] \geq x/M^+$.

(b) *If* $m > 0$, *then* $E[\tau] \leq x/m$.

(c) *If* $m > 0$, $M < \infty$, $K < \infty$, *then* $E[\int_0^\tau (c_s - \lambda_s\mu_s)\,\mathrm{d}s] = x$.

Proof. In view of Lemma 3.2 we have

$$\psi_t(\alpha)(\omega) \leq M\alpha + \tfrac{1}{2}K\alpha^2 \quad \text{for } \alpha > 0,\ 0 \leq t < \tau(\omega),\ \omega \in \Omega. \tag{4.1}$$

For a proof of (a) choose ε such that $M + \varepsilon > 0$. From Theorem 4.1(a) we conclude that

$$\begin{aligned} 1 - \mathrm{e}^{-\alpha x} &\leq E[1 - \exp\{-((M+\varepsilon)\alpha + \tfrac{1}{2}K\alpha^2)\tau\}] \\ &\leq ((M+\varepsilon)\alpha + \tfrac{1}{2}K\alpha^2)\cdot E[\tau]. \end{aligned}$$

Upon dividing by α and taking the limit as $\alpha \to 0$ we obtain the assertion of (a).

Under the assumption of (b), now Theorem 4.1(b) applies and we obtain

$$x = \lim_{\alpha \to 0} \frac{1}{\alpha} E[1 - \exp\{-\Psi_\tau(\alpha)\}]$$
$$\geq \liminf \frac{1}{\alpha} E[1 - \mathrm{e}^{-m\tau\alpha}] \geq E\left[\liminf \frac{1}{\alpha}\{1 - \mathrm{e}^{-m\tau\alpha}\}\right],$$

hence $x \geq mE[\tau]$ which proves (b).

In order to prove (c) we want to apply the dominated convergence theorem rather than Fatou's lemma in the preceding argument. This can be done since now, by (4.1),

$$(1 - \exp\{-\Psi_\tau(\alpha)\})/\alpha \leq \Psi_\tau(\alpha)/\alpha \leq \tau(M + \tfrac{1}{2}K),$$

$0 \leq \alpha \leq 1$ and τ is integrable by part (b). Further, since again by (4.1) $\psi_s(\alpha)/\alpha \leq M + \frac{1}{2}K$, we conclude that

$$\lim_{\alpha \to 0} \Psi_\tau(\alpha)/\alpha = \Psi'_\tau(0) = \int_0^\tau \psi'_s(0)\,\mathrm{d}s = \int_0^\tau (c_s - \lambda_s\mu_s)\,\mathrm{d}s. \tag{4.2}$$

Now $x = E[\exp\{-\Psi_\tau(0)\}\Psi'_\tau(0)] = E[\Psi'_\tau(0)]$ which proves part (c). □

Remark 4.3. In Corollary 4.2(a) one cannot dispense with the assumption $K < \infty$. Heath, Orey, Prestien and Sudderth (1987) constructed a diffusion process for any $\varepsilon > 0$ where even $M = 0$ but $E[\tau] < \varepsilon$ and of course $K = \infty$.

Corollary 4.4. *Suppose that* $m > 0$, $M < \infty$, $K < \infty$. *Then*:

(a) $$\mathrm{Var}\left[\int_0^\tau (c_s - \lambda_s\mu_s)\,\mathrm{d}s\right] = E\left[\int_0^\tau (\lambda_s\gamma_s^2 + \sigma_s^2)\,\mathrm{d}s\right];$$

(b) $$kM^{-3}x - (m^{-2} - M^{-2})x^2 \leq \mathrm{Var}[\tau] \leq Km^{-3}x + (m^{-2} - M^{-2})x^2.$$

Proof. We will use the relation $\mathrm{e}^{-z} - 1 + z = \frac{1}{2}z^2\eta(z)$ where $0 \leq \eta(z) \leq 1$ for $z \geq 0$ and $\eta(z) \to 1$ as $z \to 0$. Then, by use of Theorem 4.1(b) and Corollary 4.2(c),

$$\tfrac{1}{2} E[\{\Psi_\tau(\alpha)/\alpha\}^2\eta(\Psi_\tau(\alpha))]$$
$$= \alpha^{-2}E[\exp\{-\Psi_\tau(\alpha)\} - 1 + \Psi_\tau(\alpha)]$$
$$= \alpha^{-2}[\mathrm{e}^{-\alpha x} - 1 + \alpha x] + E\left[\int_0^\tau \alpha^{-2}\{\psi_s(\alpha) - \alpha(c_s - \lambda_s\mu_s)\}\,\mathrm{d}s\right].$$

Now, by Lemma 3.2, $\alpha^{-2}\{\psi_s(\alpha) - \alpha(c_s - \lambda_s\mu_s)\}\uparrow\frac{1}{2}(\lambda_s\gamma_s^2 + \alpha_s^2)$ and we conclude from the monotone convergence theorem that

$$E\left[\int_0^\tau \alpha^{-2}\{\psi_s(\alpha) - \alpha(c_s - \lambda_s\mu_s)\}\,\mathrm{d}s\right]\uparrow\frac{1}{2}E\left[\int_0^\tau (\lambda_s\gamma_s^2 + \sigma_s^2)\,\mathrm{d}s\right] =: \frac{1}{2}L.$$

Relation (4.2) and Fatou's lemma yield

$$\begin{aligned} m^2 E[\tau^2] &\leq E[\{\Psi'_\tau(0)\}^2] \\ &= E[\lim\{\Psi_\tau(\alpha)/\alpha\}^2\eta(\Psi_\tau(\alpha))] \\ &\leq \liminf E[\{\Psi_\tau(\alpha)/\alpha\}^2\eta(\Psi_\tau(\alpha))] = x^2 + L. \end{aligned}$$

Hence $\tau^2(M+\frac{1}{2}K)^2$ is integrable and dominates $\{\Psi_\tau(\alpha)/\alpha\}^2\eta(\Psi_\tau(\alpha))$ for $0 \leq \alpha \leq 1$ by (4.1). Therefore we can replace Fatou's lemma by the dominated convergence theorem and obtain

$$\begin{aligned} \mathrm{Var}[\Psi'_\tau(0)] &= E[\{\Psi'_\tau(0)\}^2] - E[\{\Psi'_\tau(0)\}]^2 \\ &= x^2 + L - x^2 = L \geq m^2 E[\tau^2] - x^2 \\ &= m^2(\mathrm{Var}[\tau] + E[\tau]^2) - x^2 \geq m^2\,\mathrm{Var}[\tau] + m^2 x^2 M^{-2} - x^2 \end{aligned}$$

by use of Corollary 4.2(a). On the other side, by Corollary 4.2(b), $L \leq KE[\tau] \leq Kx/m$ which proves part (a) and the upper bound of $\mathrm{Var}[\tau]$. The lower bound is proved in the same way. □

5. Examples

5.1. Itô's processes ($\lambda_t \equiv 0$)

Let $b(t, \xi)$ and $a(t, \xi)$ be progressively measurable functionals from $[0, \infty) \times C[0, \infty)$ into $\mathbb{R}$ or $[0, \infty)$, respectively. The measurability assumption implies that $b(t, \xi)$, e.g., depends on the continuous function ξ on $[0, \infty)$ only through its past at time t, i.e., only on the restriction of ξ to $[0, t]$. In the most important case, $b(t, \xi)$ will depend only on $\xi(t)$, then $b(t, x)$ is just a function on $[0, \infty) \times \mathbb{R}$. Suppose that (X, W), $(\Omega, \mathscr{F}, P)$, $\mathbb{F}$ is a weak solution to the functional stochastic differential equation

$$\mathrm{d}X_t = b(t, X)\,\mathrm{d}t + a(t, X)\,\mathrm{d}W_t, \quad 0 \leq t < \infty.$$

Then W is a standard Wiener process and X and W have continuous paths. As a consequence we know (Karatzas and Shreve, 1988, Problem 4.3, p. 313) that for all $f \in C^2(\mathbb{R})$,

$$f(X_t) - f(X_0) - \int_0^t \left[\frac{1}{2} a^2(s, X) f''(X_s) + b(s, X) f'(X_s)\right] \mathrm{d}s \quad \text{is a local martingale.}$$

Hence our assumption of Section 2 is satisfied upon choosing $\lambda_t = 0$, Q_t arbitrarily and

$$-c_t(\omega) = b(t, X(\omega)), \qquad \sigma_t(\omega) = a(t, X(\omega)).$$

5.2. Piecewise-deterministic processes ($\sigma_t \equiv 0$)

Let $\lambda(t, x)$ be a measurable function from $[0, \infty) \times \mathbb{R}$ into $[0, \infty)$; let q be a transition probability from $[0, \infty) \times \mathbb{R}$ into $\mathbb{R}$ such that $q(\cdot \mid t, x)$ is concentrated on $[x, \infty)$ and let $b(t, x)$ be a continuous function on $[0, \infty) \times \mathbb{R}$ satisfying the global Lipschitz condition of the theorem of Picard–Lindelöf. Suppose that X is a piecewise-deterministic Markov process living on $\mathbb{R}$ with local characteristics (b, λ, q) in the sense of Davis (1984), i.e., X has jumps determined by (λ, q) and in between the jumps the trajectories behave according to the differential equation $(\partial/\partial t)\xi(t) = b(t, \xi(t))$. It is allowed that the local characteristics contain explicit time variation, since time may be incorporated in the state. As a consequence we know (Davis, 1984, Theorem 5.5)

$$f(X_t) - f(X_0) - \int_0^t \left\{ b(s, X_s) f'(X_s) + \lambda(s, X_s) \int [f(y) - f(X_s)] q(\mathrm{d}y \mid s, X_s) \right\} \mathrm{d}s$$

is a local martingale for all f in the domain $\mathscr{D}$ of the extended generator. It is easy to see that the functions f^α defined in Section 2 are members of $\mathscr{D}$. (In order to prove the local integrability condition one can take $\{\tau_n \wedge \sigma_n\}$ as localizing sequence where τ_n is defined as in Sections 1 and 3 and σ_n is the nth jump time.) Hence our assumption of Section 2 is satisfied if we take $\sigma_t = 0$ and

$$-c_t = b(t, X_t), \qquad \lambda_t = \lambda(t, X_t), \qquad Q_t[\mathrm{d}y] = q(x + \mathrm{d}y \mid t, X_t) \quad \text{where } x = X_t.$$

Non-Markovian piecewise-deterministic processes are apparently not considered as yet in the literature. For the case $b \equiv 0$ one has just a pure jump process.

5.3. Superposition of processes

The most important case where jump-diffusion processes arise stems from a superposition of a jump process or piecewise-deterministic process with an independent diffusion process. Such a superposition can be carried through for two independent general jump-diffusion processes, however. It is based on the following product rule for independent martingales which seems to be well known (cf. Ethier and Kurtz, 1986, Section 4.10, p. 253):

Lemma 5.1. *Suppose that $Z_{it} - \int_0^t Y_{is}\,\mathrm{d}s$, $t \geq 0$, are right continuous (local) martingales with respect to the filtrations $\mathbb{F}_i$ for $i = 1, 2$ such that $\mathscr{F}_{it}$, $i = 1, 2$, are independent for all $t \geq 0$. Then, with respect to $\mathbb{F}_1 \vee \mathbb{F}_2 := \{\mathscr{F}_{1t} \vee \mathscr{F}_{2t}, t \geq 0\}$,*

$$Z_{1t} Z_{2t} - \int_0^t [Y_{1s} Z_{2s} + Z_{1s} Y_{2s}]\,\mathrm{d}s, \quad t \geq 0, \quad \textit{is a (local) martingale.} \qquad \square$$

Now let there be given two independent jump-diffusion processes X_1 and X_2 satisfying the assumption of Section 2. The quantities referring to X_i are marked by an index i. Choose

$$Z_{it} := \exp\{-\alpha X_{it}\} \quad \text{and} \quad Y_{is} := \exp\{-\alpha X_{is}\} \psi_{is}(\alpha)$$

then one obtains

$$Z_{1t}Z_{2t} = \exp\{-\alpha(X_{1t} + X_{2t})\}$$

and

$$Y_{1s}Z_{2s} + Z_{1s}Y_{2s} = \exp\{-\alpha(X_{1s} + X_{2s})\}[\psi_{1s}(\alpha) + \psi_{2s}(\alpha)]$$

where

$$\psi_{1s}(\alpha) + \psi_{2s}(\alpha) = (\lambda_{1s} + \lambda_{2s})\{(\lambda_{1s} + \lambda_{2s})^{-1}[\lambda_{1s}\varphi_{1s}(\alpha) + \lambda_{2s}\varphi_{2s}(\alpha)] - 1\}$$
$$+ (c_{1s} + c_{2s})\alpha + \tfrac{1}{2}(\sigma_{1s}^2 + \sigma_{2s}^2)\alpha^2.$$

Now set $Q_s := (\lambda_{1s} + \lambda_{2s})^{-1}[\lambda_{1s}Q_{1s} + \lambda_{2s}Q_{2s}]$ on $\{(\lambda_{1s} + \lambda_{2s}) > 0\}$ and $Q_s := \delta_0$ elsewhere where δ_0 is the dirac measure on 0, then we obtain from Lemma 5.1:

Proposition 5.2. *If X_1 and X_2 are two independent jump-diffusion processes satisfying the assumption of Section 2 then $X_1 + X_2$ is again such a process with local characteristics*

$$c_t := c_{1t} + c_{2t}, \qquad \sigma_t^2 := \sigma_{1t}^2 + \sigma_{2t}^2, \qquad \lambda_t := \lambda_{1t} + \lambda_{2t} \quad \textit{and} \quad Q_t \textit{ as above.} \qquad \square$$

In the special case where $c_{2t} \equiv 0$, $\sigma_{2t} \equiv 0$, $\lambda_{1t} \equiv 0$, i.e., X_1 is an Itô process and X_2 is a jump process, it follows for $X_1 + X_2$ that c_t, σ_t depend only on the past $\mathscr{F}_{1t}$ of X_1 and λ_{2t}, Q_{2t} depend only on the past $\mathscr{F}_{2t}$ of X_2. The case where all local characteristics of $X_1 + X_2$ depend on the whole past $\mathscr{F}_{1t} \vee \mathscr{F}_{2t}$ of $X_1 + X_2$ is also interesting. Such Markov processes were constructed by Ethier and Kurtz (1986, proof of 4.10.2) and Stroock (1975, Theorem 2.1).

5.4. Continuous-time random walks on the integer lattice

The techniques of Sections 3, 4 also apply to processes with values in the set $\mathbb{Z}$ of integers; though the results don't apply directly. Such processes are naturally pure jump processes and are skip-free downwards if $X_t - X_{t-} + 1$ has values in the set $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ of non-negative integers. In that case it is usual to work with generating functions rather than with Laplace transforms. Now consider a Markovian jump process given by (λ, q) as in Section 5.2, but now $q(\cdot \mid t, x)$ is concentrated on $\{x-1, x, x+1, \ldots\}$. Or more generally, consider a (not necessarily Markovian) jump process X with state space $\mathbb{Z}$ and right-continuous trajectories on $(\Omega, \mathscr{F}, P)$ endowed with a right-continuous filtration $\mathbb{F}$ with local characteristics $\{\lambda_t, Q_t, t \geq 0\}$ where $\{\lambda_t\}$ is an adapted non-negative process and $\{Q_t\}$ is an adapted process with values in the space of probability measures on $\mathbb{N}_0$. Let $\varphi_t(\beta)$, $0 \leq \beta \leq 1$, be the generating function of Q_t, μ_s its mean and γ_s^2 the corresponding second momentum around one, i.e., $\gamma_t^2 = \int (y-1)^2 Q_t(\mathrm{d}y)$. Then $\beta^{-1}\varphi_t(\beta)$ is the conditional expectation of $\beta^{X_t - X_{t-}}$ given the past $\mathscr{F}_t$ and that there is a jump at time t. Assume now that for all $0 \leq \beta \leq 1$,

$$\beta^{X_t} - \int_0^t \beta^{X_{s-}} \cdot \lambda_s[\beta^{-1}\varphi_s(\beta) - 1]\,\mathrm{d}s, \quad t \geq 0,$$

is a local martingale. We want to give now the results without proof and will only consider equalities and omit the inequalities for the hitting time

$$\tau := \tau_x := \inf\{t \geq 0, x + X_t = 0\} \quad \text{for some fixed } x \in \mathbb{N}.$$

Theorem 5.3. *If either*

(i) $\inf_{t<\tau} \lambda_t(1-\mu_t) > 0$, *or*

(ii) $\lambda_t(1-\mu_t) \geq 0$ *for all* $t < \tau$ *and* $\inf_{t<\tau} \lambda_t Q_t[\{1,2,\ldots\}] > 0$;

then $E[\exp\{-\int_0^\tau \lambda_s[\beta^{-1}\varphi_s(\beta) - 1]\,\mathrm{d}s\}] = \beta^x$. □

Corollary 5.4. *If for some* $\varepsilon > 0$, $\lambda_t(1-\mu_t) \geq \varepsilon$ *and* $\lambda_t\gamma_t^2 \leq \varepsilon^{-1}$, $0 \leq t < \tau$, *then*

$$E\left[\int_0^\tau \lambda_s(1-\mu_s)\,\mathrm{d}s\right] = x \quad \textit{and} \quad \mathrm{Var}\left[\int_0^\tau \lambda_s(1-\mu_s)\,\mathrm{d}s\right] = E\left[\int_0^\tau \lambda_s\gamma_s^2\,\mathrm{d}s\right] \qquad \square$$

5.5. Discrete-time random walks on the integer lattice: the M/G/1 *queue*

Let us consider the embedded queue length process X_n, $n \in \mathbb{N}_0$, where X_n denotes the queue size for an M/G/1 queue immediately after the nth departure of a customer. At time 0, the queue length is $x \in \mathbb{N}$ and the service of a customer begins. Let τ denote the number of customers served during a busy period initiated by x customers. Then τ may be viewed as the total number of customers served during x independent (complete) busy periods. Up to the discrete time τ, the queue length process coincides with a discrete-time random walk on $\mathbb{Z}$ or $\mathbb{N}_0$,

$$X_n = [X_{n-1} - 1]^+ + N_n, \quad n \in \mathbb{N},$$

where N_n is the number of arrivals during the nth service time. Let Q denote the distribution of N_n. Now we want to allow that Q depends on time and past. Such a situation may arise when the queue is controlled by a past dependent control via the arrival rate or the service rate. Given transition probabilities Q_n from $\mathbb{N}_0^{n+1}$ into $\mathbb{N}_0$, we construct a probability measure P_x by Ionescu Tulcea's theorem such that

$$P_x[X_0 = x] = 1,$$

$$P_x[X_{n+1} = [X_n - 1]^+ + y \mid X_n, \ldots, X_0] = Q_n[\{y\} \mid X_n, \ldots, X_0].$$

Let $\varphi_n(\beta, \omega)$, $0 \leq \beta \leq 1$, $\omega \in \Omega$, denote the generating function of $Q_n[\cdot \mid \cdots](\omega)$, μ_n, γ_n^2 its mean and variance. Note the slightly different definition of γ_n^2 in Sections 4 and 5.4. Now set

$$M_n^\beta := \beta^{X_n} \cdot \prod_{j=0}^{n-1} \frac{\beta}{\varphi_j(\beta)}, \quad n \in \mathbb{N}_0.$$

Then it is easily shown that $E_x[M_{n+1}^\beta \mid X_n, \ldots, X_0] = M_n^\beta$ on $\{X_n > 0\}$. Hence $M_n^{\beta,\tau} := M_{n\wedge\tau}^\beta$ defines a martingale where

$$\tau := \inf\{n \geq 0, X_n = 0\}.$$

Based on M^{β}, Kennedy (1976) succeeded in constructing a martingale beyond time τ. As Section 5.4 we will give the results without proof and only for the equalities.

Theorem 5.5. *If either*
(i) $\inf_{n<\tau} 1-\mu_n > 0$, *or*
(ii) $1-\mu_n \geq 0$ *for all* $n<\tau$ *and* $\inf_{n<\tau} Q_n[\{1,2,\ldots\}]>0$;
then $E_x[\Pi_{j=0}^{\tau-1} \beta/\varphi_j(\beta)] = \beta^x, 0\leq\beta\leq 1$. □

Corollary 5.6. *If for some* $\varepsilon>0$: $1-\mu_n\geq\varepsilon$ *and* $\gamma_n^2\leq\varepsilon^{-1}$ *for all* $n<\tau$, *then*

$$E\left[\sum_{n=0}^{\tau-1}(1-\mu_n)\right]=x \quad \text{and} \quad \operatorname{Var}\left[\sum_{n=0}^{\tau-1}(1-\mu_n)\right]=E\left[\sum_{n=0}^{\tau-1}\gamma_n^2\right]. \qquad \square$$

Let us only give the following relation for the proof of the variance formula, which follows from Taylor's theorem. For $0\leq\varepsilon_n:=\varphi_n(\beta)/\beta-1, 0\leq n<\tau\in\mathbb{N}_0$, $\varepsilon=(\varepsilon_0,\ldots,\varepsilon_{\tau-1})$ one has

$$\prod_{0\leq n<\tau}(1+\varepsilon_n)^{-1}-1+\sum_{0\leq n<\tau}\varepsilon_n=\tfrac{1}{2}\sum_{0\leq m,n<\tau}\varepsilon_m\varepsilon_n\eta_{m,n}(\varepsilon)+\tfrac{1}{2}\sum_{0\leq n<\tau}\varepsilon_n^2\eta_{n,n}(\varepsilon),$$

where

$$0\leq\eta_{m,n}(\varepsilon)\leq 1=\lim_{\varepsilon\to 0}\eta_{m,n}(\varepsilon).$$

For the classical M/G/1 queue, Prahu (1980, Theorem 12, p. 58) and in a different form Cohen (1969, p. 251-252) give the generating function of τ and its mean.

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