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Sample quantiles of heavy tailed stochastic processes ☆

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Abstract

Distributions of sample quantiles of measurable stochastic processes are important for the purpose of rational pricing of "look-back" options. In this paper we compute the exact tail behavior of the sample quantile distribution for a large class of infinitely divisible stochastic processes with heavy tails.

Keywords: Sample quantiles; Look-back options; Regular variation; Infinitely divisible processes; Stable processes; Lévy measure; Tail behavior of the distribution

1. Introduction

Let $(\Omega,\mathscr{F},P)$ and $(T,\mathscr{T},m)$ be probability spaces. For a general measurable real valued stochastic process $\boldsymbol{Y} = (Y(\omega,t),\, t \in T)$ on $\Omega \times T$, we define for $0 \leqslant \rho < 1$ the random variable

$$Q_\rho(\boldsymbol{Y}) = \inf\left\{x \in \mathbb{R}\colon \int_T \mathbf{1}_{\{Y(t)\leqslant x\}} m(\mathrm{d}t) > \rho\right\}. \tag{1.1}$$

Here $\mathbf{1}_A$ denotes the indicator function of the event A. $Q_\rho(\boldsymbol{Y})$ is called the sample ρ-quantile of $\boldsymbol{Y}$ on T. By definition, $\boldsymbol{Y}$ spends at least $100\rho\%$ of its "time" at or below $Q_\rho(\boldsymbol{Y})$, and at least $100(1-\rho)\%$ of its "time" at or above $Q_\rho(\boldsymbol{Y})$. $Q_{1/2}(\boldsymbol{Y})$ is the median level of $\boldsymbol{Y}$ over T. These variables were introduced in the realm of mathematical finance by Miura (1992). In Akahori (1995) and Dassios (1995a), the law of $Q_\rho(\boldsymbol{Y})$ was determined for $\boldsymbol{Y}$ a Brownian motion with drift. Yor (1995) and Embrechts et al. (1995) give a more systematic treatment of the Brownian case. In

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particular, Dassios' result for $Y(t) = \mu t + \sigma B(t)$, $0 \leqslant t \leqslant 1$ with $\mu \in \mathbb{R}$, $\sigma > 0$ and $\boldsymbol{B}$ the standard Brownian motion on $[0,1]$ says that

$$Q_\rho(\boldsymbol{Y}) \stackrel{\mathrm{d}}{=} \sup_{s \leqslant \rho} Y(s) + \inf_{s \leqslant 1-\rho} Y'(s), \tag{1.2}$$

where $\boldsymbol{Y}'$ is an independent copy of $\boldsymbol{Y}$. From this result, an explicit expression for the law of $Q_\rho(\boldsymbol{Y})$ follows. Very recently Dassios (1995b) has extended (1.2) to general Lévy processes. Unfortunately, it is still not clear in general how to deduce the distribution of the sample quantile. To the best of our knowledge, no other (non-trivial) examples of processes are known for which such an explicit result has been established. In the present paper we shall find the asymptotic behavior of $P(Q_\rho(\boldsymbol{Y}) > \lambda)$ for a large class of infinitely divisible (i.d.) stochastic processes, some of which can be of interest as financial models. Even though the exact distribution of the sample quantiles is yet unknown for these processes, our results cover "the extreme oscillations" of the sample quantiles, and those are often relevant as measures of risk associated with certain path-dependent derivative financial instruments. Notice that even though setting $\rho = 1$ in (1.1) does not produce a meaningful object, all the discussion and the results of this paper hold for $Q_1(\boldsymbol{Y})$ interpreted as $\lim_{\rho \to 1} Q_\rho(\boldsymbol{Y}) = \operatorname{ess\,sup}_{t \in T} Y(t)$. The results corresponding to this case are known; see for instance Rosinski and Samorodnitsky (1993)

The processes under consideration are real valued infinitely divisible stochastic processes given by their integral representation

$$X(t) = \int_S f(t,s) M(\mathrm{d}s), \quad t \in T, \tag{1.3}$$

where $(S, \mathscr{A})$ is a measurable space and M is an i.d. random measure on $(S, \mathscr{A})$ with Lévy measure F. That is, F is a σ-finite measure on $(S \times \mathbb{R}, \mathscr{A} \times \mathscr{B})$, where $\mathscr{B}$ is the Borel σ-field on $\mathbb{R}$. The random measure M is a stochastic process of the type $(M(A), A \in \mathscr{A}_0)$, where

$$\mathscr{A}_0 = \left\{ A \in \mathscr{A}\colon\ \lambda(A) := \int_A \int_{\mathbb{R}} \min(1, x^2) F(\mathrm{d}s, \mathrm{d}x) < \infty \right\},$$

such that M is independently scattered (i.e. for any disjoint $\mathscr{A}_0$ sets $A_1, \ldots, A_n$, $M(A_1)$, $\ldots$, $M(A_n)$ are independent), σ-additive (i.e. for any disjoint $\mathscr{A}_0$ sets $A_1, A_2, \ldots$ such that $\cup_{i=1}^\infty A_i \in \mathscr{A}_0$ we have $M(\cup_{i=1}^\infty A_i) = \Sigma_{i=1}^\infty M(A_i)$ a.s.) and for every $A \in \mathscr{A}_0$, $M(A)$ is a real i.d. random variable with

$$E \exp(\mathrm{i}\theta M(A)) = \exp\left\{ \int_A \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta \tau(x)) F(\mathrm{d}s, \mathrm{d}x) \right\}, \tag{1.4}$$

where $\tau(x) = x/(1+x^2)$.

We refer the reader to Rajput and Rosinski (1989) for more details on i.d. random measures and on conditions on the kernel $f(t,s)$ in (1.3) ensuring that the stochastic integral is well defined. One should note at this point that the results of the paper remain valid if the random measure M and, consequently, the stochastic process $\{X(t),\ t \in T\}$ have a Gaussian component, for it introduces probability tails of a smaller order than the heavy tails we are considering.

Our main result is Theorem 2.1. As a special case of the latter theorem, we give the tail behavior of the sample quantiles in the case of α-stable processes. Two important examples are those of α-stable motions and α-stable Ornstein–Uhlenbeck processes. These processes are relevant within finance; the former are alternative models for asset (log-) returns, see for instance Mittnik and Rachev (1993), while the latter are the corresponding analogs of interest rate models (see, for instance, the Vasicek model in Lamberton and Lapeyre, 1991).

As our results will mainly concentrate on heavy tailed stochastic processes, we have summarized below the key properties on the relevant classes of real functions. We call a Lebesgue measurable function L from $\mathbb{R}$ into $(0,\infty)$ *slowly varying* (denoted $L \in \mathscr{R}(0)$) whenever

$$\lim_{\lambda\to\infty} \frac{L(\lambda t)}{L(\lambda)} = 1 \quad \text{for every } t > 0.$$

The class of *regularly varying* functions $\mathscr{R}(p)$ with index $p \in \mathbb{R}$ is defined as

$$\mathscr{R}(p) = \left\{ h\colon \mathbb{R} \to (0,\infty),\ h(x) = x^p L(x), \text{ for some } L \in \mathscr{R}(0) \right\}.$$

The class of all regularly varying functions is denoted by $\mathscr{R}$. For a detailed discussion on these and related classes, see Bingham et al. (1987). We will often use the fact that $\mathscr{R} \subset \mathscr{L}$, where

$$\mathscr{L} = \left\{ h\colon \mathbb{R} \to (0,\infty), \text{ Lebesgue measurable, } \lim_{x\to\infty} \frac{h(x-y)}{h(x)} = 1 \text{ for all } y \in \mathbb{R} \right\}.$$

For a distribution function F so that $1 - F \in \mathscr{L}$ we shall say that F has the *long tail property*.

The paper is organized as follows. Section 2 contains our main result (Theorem 2.1) linking the tail behavior of the sample quantiles of $\boldsymbol{X}$ to that of the quantiles of the underlying Lévy measure of the process, as expressed through the quantiles of the kernel f in the integral representation (1.3). In Section 3 this result is applied to some specific examples. Section 4 should be viewed as an appendix, and contains some general results on the quantiles of measurable stochastic processes.

2. Tail behavior of sample quantiles

We begin with a proposition containing a special case of the main result. This proposition is a crucial ingredient in the proof of the latter, and the special case described in it is important enough to be displayed on its own. It treats the compound Poisson model, which is a one of the basic probabilistic models. In the insurance context for example, the process $\boldsymbol{Y}$ can be regarded as describing the claim settlement process associated with the jth claim. These so-called incurred but not yet fully settled claims are discussed in Klüppelberg and Mikosch (1993a, b).

We remind the reader that in the sequel $(T, \mathscr{T}, m)$ is a probability space.

Proposition 2.1. *Let $\boldsymbol{Y}_j = (Y_j(t), t \in T)$, $j = 1,2,\ldots$ be a sequence of i.i.d. measurable stochastic processes. Let N be a mean μ Poisson random variable independent of $\boldsymbol{Y}_j$, $j \geqslant 1$ and*

$$\boldsymbol{X} = \sum_{j=1}^{N} \boldsymbol{Y}_j. \tag{2.1}$$

Assume that

$$\operatorname{ess\,sup}_{t\in T}|Y_1(t)| < \infty \ a.s.. \tag{2.2}$$

Let $0 \leqslant \rho < 1$. Assume that

$$P(Q_\rho(\boldsymbol{Y}_1) > \lambda) \in \mathscr{R}(-p) \ as \ \lambda \to \infty \tag{2.3}$$

for some $p \geqslant 0$, while

$$\lim_{\lambda\to\infty} \frac{\left(P(\operatorname{ess\,sup}_{t\in T}|Y_1(t)| > \lambda)\right)^2}{P(Q_\rho(\boldsymbol{Y}_1) > \lambda)} = 0. \tag{2.4}$$

Then

$$\lim_{\lambda\to\infty} \frac{P(Q_\rho(\boldsymbol{X}) > \lambda)}{P(Q_\rho(\boldsymbol{Y}_1) > \lambda)} = \mu. \tag{2.5}$$

Proof. Starting with

$$P(Q_\rho(\boldsymbol{X}) > \lambda) = \sum_{n=1}^{\infty} \mathrm{e}^{-\mu}\frac{\mu^n}{n!} P\left(Q_\rho\left(\sum_{j=1}^{n} \boldsymbol{Y}_j\right) > \lambda\right) \tag{2.6}$$

for $\lambda > 0$, we use Lemma 4.1 to conclude that for every $n \geqslant 1$ and $M > 0$,

$$\begin{aligned} P\left(Q_\rho(\sum_{j=1}^{n} \boldsymbol{Y}_j) > \lambda\right) &\geqslant P\Big(\cup_{j=1}^{n}\{Q_\rho(\boldsymbol{Y}_j) > \lambda + (n-1)M, \operatorname{ess\,sup}_{t\in T}|Y_i(t)| \\ &\qquad \leqslant M \text{ for all } i \neq j\}\Big) \\ &= nP(Q_\rho(\boldsymbol{Y}_1) > \lambda + (n-1)M)\left(P(\operatorname{ess\,sup}_{t\in T}|Y_1(t)| \leqslant M)\right)^{n-1} \end{aligned}$$

for all $\lambda > M$. Therefore, by (2.6),

$$\begin{aligned} &P(Q_\rho(\boldsymbol{X}) > \lambda) \\ &\geqslant \sum_{n=1}^{\infty} \mathrm{e}^{-\mu}\frac{\mu^n}{n!} nP(Q_\rho(\boldsymbol{Y}_1) > \lambda + (n-1)M)P\left(\operatorname{ess\,sup}_{t\in T}|Y_1(t)| \leqslant M\right)^{n-1}. \end{aligned}$$

The regular variation assumption (2.3) implies, in particular, the long tail property of $Q_\rho(\boldsymbol{Y}_1)$ (see Section 1). Therefore, by Fatou's lemma,

$$\begin{aligned} \liminf_{\lambda\to\infty} \frac{P(Q_\rho(\boldsymbol{X}) > \lambda)}{P(Q_\rho(\boldsymbol{Y}_1) > \lambda)} &\geqslant \sum_{n=1}^{\infty} \mathrm{e}^{-\mu}\frac{\mu^n}{(n-1)!}\left(P(\operatorname{ess\,sup}_{t\in T}|Y_1(t)| \leqslant M)\right)^{n-1} \\ &= \mu \exp\left(-\mu(1 - P(\operatorname{ess\,sup}_{t\in T}|Y_1(t)| \leqslant M))\right). \end{aligned}$$

Letting $M \to \infty$ we obtain the lower bound

$$\liminf_{\lambda\to\infty} \frac{P(Q_\rho(\boldsymbol{X}) > \lambda)}{P(Q_\rho(\boldsymbol{Y}_1) > \lambda)} \geqslant \mu. \tag{2.7}$$

For the upper bound note that by Lemma 4.1 for any $n \geqslant 1$ we have for all $0 < \varepsilon < 1$ and $\lambda > 0$

$$\begin{aligned} P\left(Q_\rho\left(\sum_{j=1}^n \boldsymbol{Y}_j\right) > \lambda\right) &\leqslant P\left(\cup_{j=1}^n \{Q_\rho(\boldsymbol{Y}_j) > \lambda(1-\varepsilon)\}\right) \\ &\quad + P\left(\cap_{j=1}^n \{\operatorname{ess\,sup}_{t\in T} \sum_{i\neq j} |Y_i(t)| \geqslant \lambda\varepsilon\}\right) \\ &\leqslant nP\left(Q_\rho(\boldsymbol{Y}_1) > \lambda(1-\varepsilon)\right) \\ &\quad + P\left(\operatorname{ess\,sup}_{t\in T}|Y_i(t)| \geqslant \lambda\varepsilon/n \text{ for at least two different } i\text{'s}\right) \\ &\leqslant nP\left(Q_\rho(\boldsymbol{Y}_1) > \lambda(1-\varepsilon)\right) \\ &\quad + \frac{n(n-1)}{2}\left(P\left(\operatorname{ess\,sup}_{t\in T}|Y_1(t)| \geqslant \lambda\varepsilon/n\right)\right)^2. \end{aligned}$$

We now use (2.6) and (2.4) to conclude that there is (for a fixed ε) a positive function c with $c(\lambda) \to 0$ as $\lambda \to \infty$ such that

$$\begin{aligned} P(Q_\rho(\boldsymbol{X}) > \lambda) \leqslant \sum_{n \leqslant \sqrt{\lambda}} \mathrm{e}^{-\mu}\frac{\mu^n}{n!}\Big(&nP\left(Q_\rho(\boldsymbol{Y}_1) > \lambda(1-\varepsilon)\right) \\ &+ c(\lambda)\frac{n(n-1)}{2}P\left(Q_\rho(\boldsymbol{Y}_1) \geqslant \lambda\varepsilon/n\right)\Big) + P(N > \sqrt{\lambda}). \end{aligned} \tag{2.8}$$

It follows from (2.3) that there exists a finite positive constant C such that for every $\lambda > 1$ (say) and $n \leqslant \sqrt{\lambda}$ we have

$$P\left(Q_\rho(\boldsymbol{Y}_1) \geqslant \lambda\varepsilon/n\right) \leqslant Cn^{p+1}\varepsilon^{-(1+p)}P\left(Q_\rho(\boldsymbol{Y}_1) > \lambda\right).$$

Therefore, by (2.8) we have

$$\begin{aligned} P(Q_\rho(\boldsymbol{X}) > \lambda) \leqslant \sum_{n=1}^{\infty} \mathrm{e}^{-\mu}\frac{\mu^n}{n!}\Big(&nP\left(Q_\rho(\boldsymbol{Y}_1) > \lambda(1-\varepsilon)\right) \\ &+ c(\lambda)\frac{n(n-1)}{2}Cn^{p+1}\varepsilon^{-(1+p)}P\left(Q_\rho(\boldsymbol{Y}_1) > \lambda\right)\Big) + P(N > \sqrt{\lambda}). \end{aligned}$$

Using once again (2.3) and recalling that the probability tail of a Poisson random variable decays faster than exponentially, we conclude that

$$\limsup_{\lambda\to\infty} \frac{P(Q_\rho(\boldsymbol{X}) > \lambda)}{P(Q_\rho(\boldsymbol{Y}_1) > \lambda)} \leqslant \mu(1-\varepsilon)^p. \tag{2.9}$$

Letting $\varepsilon \to 0$ in (2.9) and comparing it with (2.7) completes the proof of the proposition. □

Remark. (i) A straightforward modification of the proof shows that the assumption (2.3) of Proposition 2.1 may be replaced with a weaker assumption, that the right probability tail of $Q_\rho(Y_1)$ belongs only to the class of functions of *extended regular variation*. We have chosen the present formulation out of awareness that the latter class does not enjoy presently the wide recognition awarded to the class of regularly varying functions. See Bingham et al. (1987, p.65), for a discussion of extended regular variation.

(ii) The above-mentioned extension allows for a particular subset of the so-called class of *subexponential distributions*:

$$\mathscr{S} = \left\{ F \text{ d.f. on } [0,\infty)\text{: } \lim_{x\to\infty} \frac{1-F^{*2}(x)}{1-F(x)} = 2 \right\},$$

where F^{*2} denotes the second convolution of F. In the spirit of the techniques and results of Embrechts et al. (1979) and Rosinski and Samorodnitsky (1993) one could suspect that Proposition 2.1 holds when $\mathscr{R}(-p)$ is replaced by $\mathscr{S}$. We do not know whether this is in fact true, even though the lower bound (2.7) holds for subexponential tails as well, for such distribution functions have the long tail property.

(iii) A further generalization concerns the Poisson assumption on N. It is clear that the same result as in Proposition 2.1 holds whenever the generating function of N,

$$\hat{P}(s) = \sum_{n=0}^{\infty} P(N=n)s^n$$

is analytic in $s = 1$. The constant μ in the conclusion (2.5) has to be replaced by $\hat{P}'(1)$.

The following theorem is our main result.

Theorem 2.1. *Let*

$$X(t) = \int_S f(t,s) M(\mathrm{d}s), \quad t \in T$$

be a measurable infinitely divisible stochastic process, where M *is an infinitely divisible random measure with a* σ*-finite Lévy measure* F*, and the kernel* $f: S\times T \to \mathbb{R}$ *is (jointly) measurable. Assume that*

$$\operatorname{ess\,sup}_{t\in T}|X(t)| < \infty \quad a.s.. \tag{2.10}$$

For $y > 0$ *denote*

$$H_\rho(y) = F\Big\{(s,x) \in S\times\mathbb{R}\text{: } Q_\rho\big(xf(\cdot,s)\big) > y\Big\}, \tag{2.11}$$

$0 \leqslant \rho < 1$. *Let further*

$$H_*(y) = F\Big\{(s,x) \in S\times\mathbb{R}\text{: } |x| \operatorname{ess\,sup}_{t\in T}\big(|f(t,s)|\big) > y\Big\}. \tag{2.12}$$

Assume that

$$H_\rho(\lambda) \in \mathscr{R}(-p) \text{ as } \lambda \to \infty \tag{2.13}$$

for some $p \geq 0$, *while*

$$\lim_{\lambda\to\infty} \frac{\left(H_*(\lambda)\right)^2}{H_\rho(\lambda)} = 0. \tag{2.14}$$

Then

$$\lim_{\lambda\to\infty} \frac{P(Q_\rho(\boldsymbol{X}) > \lambda)}{H_\rho(\lambda)} = 1. \tag{2.15}$$

Proof. The first step in the proof is to "discretize" the problem. Let $\tau_1, \tau_2, \ldots$ be a sequence of i.i.d. T-valued random variables with common law m, living on a probability space $(\Omega_1, \mathscr{F}_1, P_1)$ (while the original process $\boldsymbol{X}$ lives on a separate probability space $(\Omega, \mathscr{F}, P)$). Observe that for every $\omega \in \Omega$ by the strong law of large numbers

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbf{1}\left(X(\tau_i) \leqslant x\right)}{n} = \int_T \mathbf{1}_{\{X(t)\leqslant x\}} m(\mathrm{d}t)$$

for P_1-almost every $\omega_1 \in \Omega_1$, which implies that

$$Q_\rho(\boldsymbol{X}) = \inf\left\{x \in \mathbb{R}:\ \lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbf{1}\left(X(\tau_i) \leqslant x\right)}{n} > \rho\right\} \tag{2.16}$$

P_1-a.s., and so by Fubini's theorem there is an event $\Omega_{11} \subset \Omega_1$ with $P_1(\Omega_{11}) = 1$, such that for every $\omega_1 \in \Omega_{11}$ (2.16) holds P-a.s. We regard, therefore, (2.16) as an identity that holds on a set of $\omega \in \Omega$ of probability 1 (and this set may further depend on the particular sequence $\tau_1, \tau_2, \ldots$ we choose).

We denote the functional in the right hand side of (2.16) by $\tilde{Q}_\rho(\boldsymbol{X})$. Of course, $\tilde{Q}_\rho(\boldsymbol{X})$ depends on ω_1 through the sequence $\tau_1, \tau_2, \ldots$, but this dependence will be for now kept implicit. An argument identical to the one above shows that there is an event $\Omega_{12} \subset \Omega_1$ with $P_1(\Omega_{12}) = 1$, such that for every $\omega_1 \in \Omega_{12}$

$$H_\rho(y) = F\left\{(s,x) \in S \times \mathbb{R}:\ \tilde{Q}_\rho\left(xf(\cdot,s)\right) > y\right\}. \tag{2.17}$$

Indeed, we only have to replace $(\Omega, \mathscr{F}, P)$ with $(S \times \mathbb{R}, \mathscr{A} \times \mathscr{B}, F)$ and to recall that Fubini's theorem holds for σ-finite measures, to conclude that

$$Q_\rho(xf(\cdot,s)) = \tilde{Q}_\rho(xf(\cdot,s))$$

F-a.e., and so the two functions above have the same distribution under F, which is exactly (2.17). Similarly, since

$$\sup_{n\geqslant 1}\left(|f(\tau_n,s)|\right) = \operatorname{ess\,sup}_{t\in T}\left(|f(t,s)|\right)$$

P_1-a.s., we conclude in the same way that there is an event $\Omega_{13} \subset \Omega_1$ with $P_1(\Omega_{13}) = 1$, such that for every $\omega_1 \in \Omega_{13}$

$$H_*(y) = F\Big\{(s,x) \in S \times \mathbb{R}:\ |x| \sup_{n\geqslant 1}\big(|f(\tau_n,s)|\big) > y\Big\} \tag{2.18}$$

P-a.s. We fix once and for all an $\omega_1 \in \Omega_{11} \cap \Omega_{12} \cap \Omega_{13}$. Observe that (2.16)–(2.18) reduce effectively the problem to a stochastic process indexed by the countable set $\{\tau_1, \tau_2, \ldots\}$. For notational simplicity we will identify τ_n with n for all $n \geqslant 1$. That is, we consider a stochastic process

$$X(n) = \int_S f(n,s) M(\mathrm{d}s),\ \ n \geqslant 1,$$

and the functionals Q_ρ, H_ρ and H_* are now given by (2.16)–(2.18) correspondingly, with τ_n replaced by n for all $n \geqslant 1$.

In the second step of the proof we decompose the process $(X(n),\ n\geqslant 1)$ as follows. Let M_1 and M_2 be two independent infinitely divisible random measures with Lévy measures F_1 and F_2 given by

$$F_1(A) = F\left(A \cap \Big\{(s,x) \in S \times \mathbb{R}:\ |x| \sup_{n\geqslant 1} |f(n,s)| > 1\Big\}\right)$$

and

$$F_2(A) = F\left(A \cap \Big\{(s,x) \in S \times \mathbb{R}:\ |x| \sup_{n\geqslant 1} |f(n,s)| \leqslant 1\Big\}\right),$$

$A \in \mathscr{A} \times \mathscr{B}$. Observe that F_1 is a finite measure. See e.g. Araujo and Gine (1980) or Linde (1986). Define for $i = 1, 2$

$$X_i(n) = \int_S f(n,s) M_i(\mathrm{d}s), \quad n \geqslant 1.$$

The stochastic processes $\boldsymbol{X}_1 = (X_1(n),\ n \geqslant 1)$ and $\boldsymbol{X}_2 = (X_2(n),\ n \geqslant 1)$ are independent, and

$$\boldsymbol{X} \stackrel{\mathrm{d}}{=} \boldsymbol{X}_1 + \boldsymbol{X}_2. \tag{2.19}$$

Since the sample quantiles of measurable processes depend only on their finite dimensional distributions (see Lemma 4.2), we may regard (2.19) as an a.s. equality, and then it follows from Lemma 4.1 that

$$Q_\rho(\boldsymbol{X}_1) - \sup_{n\geqslant 1} |X_2(n)| \leqslant Q_\rho(\boldsymbol{X}) \leqslant Q_\rho(\boldsymbol{X}_1) + \sup_{n\geqslant 1} |X_2(n)|. \tag{2.20}$$

Now $\boldsymbol{X}_1$ is an infinitely divisible sequence with Lévy measure (regarded as a cylindrical measure on $\mathbb{R}^\infty$) given by

$$\nu_1 = F_1 \circ V^{-1},$$

where

$$V:\ S \times \mathbb{R} \to \mathbb{R}^\infty$$

is given by

$$V(s,x) = xf(\cdot,s),$$

see Rajput and Rosinski (1989). Observe that ν_1 is a finite measure. We denote the total mass of ν_1 by μ_1. Let $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots$ be i.i.d. $\mathbb{R}^\infty$-valued random variables with common law $(1/\mu_1)\nu_1$, and let N be an independent of it Poisson random variable with mean μ_1. Then $\boldsymbol{X}_1^*$ defined by

$$X_1^*(n) = \sum_{j=1}^{N} Y_j(n), \quad n \geqslant 1 \tag{2.21}$$

is equal in distribution to $\boldsymbol{X}_1$, and so

$$Q_\rho(\boldsymbol{X}_1) \stackrel{\mathrm{d}}{=} Q_\rho(\boldsymbol{X}_1^*). \tag{2.22}$$

We now verify that the process $\boldsymbol{X}_1^*$ defined by (2.21) satisfies the assumptions of Proposition 2.1. First of all, (2.10) implies that $\sup_{n \geqslant 1} |X_n| < \infty$ a.s., and so

$$\sup_{n \geqslant 1} |f(n,s)| < \infty$$

for F_1-almost all $s \in S$ (one should say F_1-almost all $(s,x) \in S \times \mathbb{R}$ to be precise, but we use the shorter statement). See Rosinski (1986). Therefore, condition (2.2) of Proposition 2.1 holds. Furthermore, by construction, for every $\lambda > 1$,

$$\begin{aligned} P\Big(Q_\rho(\boldsymbol{Y}_1) > \lambda\Big) &= \frac{1}{\mu_1} F_1\Big((s,x) \in S \times \mathbb{R}\colon\ Q_\rho(xf(\cdot,s)) > \lambda\Big) \\ &= \frac{1}{\mu_1} H_\rho(\lambda). \end{aligned}$$

Therefore, condition (2.3) of Proposition 2.1 follows from (2.13). Similarly, for every $\lambda > 1$,

$$P\Big(\sup_{n \geqslant 1}(|Y_1(n)|) > \lambda\Big) = \frac{1}{\mu_1} H_*(\lambda),$$

and so condition (2.4) of Proposition 2.1 follows from (2.14).

Therefore, we are in a position to apply (the "density version" of) Proposition 2.1, which gives us

$$\lim_{\lambda \to \infty} \frac{P(Q_\rho(\boldsymbol{X}_1) > \lambda)}{H_\rho(\lambda)} = 1. \tag{2.23}$$

Now we recall that the Lévy measure $\nu_2 = F_2 \circ V^{-1}$ of $\boldsymbol{X}_2$ is supported by a set in $\mathbb{R}^\infty$ which is bounded in the L^∞ norm. Therefore, for every $a > 0$ there is a $c > 0$ such that

$$P\Big(\sup_{n \geqslant 1} |X_2(n)| > \lambda\Big) \leqslant c\mathrm{e}^{-a\lambda} \tag{2.24}$$

for all $\lambda > 0$ (deAcosta, 1980; Braverman and Samorodnitsky, 1995). Now the conclusion of the theorem is an immediate consequence of (2.20), (2.23), (2.24) and the well known properties of distributions with subexponential (and, in particular, regularly varying) tails. See e.g. Embrechts et al. (1979). □

Note that the first two remarks made at the end of the proof of Proposition 2.1 apply fully here as well. In particular, the lower bound for the tail probabilites holds in the subexponential case as well.

An important particular case of Theorem 2.1 is described in the following corollary. It applies to the processes with stationary independent increments – Lévy processes. In this case the time space is taken to be $T = [0,1]$, and the measure m is the Lebesgue measure on it. We remind the reader that the integral representation (1.3) takes now a particularly simple form. Here $S = [0,1]$ as well,

$$f(t,s) = \mathbf{1}(t > s), \quad t,s \in [0,1],$$

and

$$F(\mathrm{d}s, \mathrm{d}x) = \gamma(\mathrm{d}s)\mu(\mathrm{d}x),$$

where γ is the Lebesgue measure on $[0,1]$, and μ is a one-dimensional Lévy measure.

Corollary 2.1. *Let $\{X(t), t \in [0,1]\}$ be a Lévy process with the Lévy measure μ, such that*

$$\mu((\lambda,\infty)) \in \mathscr{R}(-p) \tag{2.25}$$

for some $p \geqslant 0$. Then for every $0 \leqslant \rho < 1$

$$\lim_{\lambda\to\infty} \frac{P(Q_\rho(X) > \lambda)}{\mu((\lambda,\infty))} = \rho. \tag{2.26}$$

Proof. In the present case, for every $0 \leqslant \rho < 1$,

$$Q_\rho(xf(\cdot,s)) = x\mathbf{1}(\rho \geqslant s)$$

for $x > 0$ and

$$Q_\rho(xf(\cdot,s)) = 0$$

for $x < 0$, $0 < s < 1$. Therefore, for any $y > 0$

$$H_\rho(y) = \rho\mu((y,\infty)),$$

$0 \leqslant \rho < 1$, and it is easy to see that

$$H_*(y) = \mu((y,\infty)).$$

This verifies immediately the assumptions (2.13) and (2.4) of Theorem 2.1, and (2.26) follows immediately from (2.15). □

Remark. It is well known that under assumptions weaker than those of Corollary 2.1 one has

$$\lim_{\lambda\to\infty} \frac{P(\operatorname{ess\,sup}_{t\in[0,1]} X(t) > \lambda)}{\mu((\lambda,\infty))} = \lim_{\lambda\to\infty} \frac{P(\sup_{t\in[0,1]} X(t) > \lambda)}{\mu((\lambda,\infty))} = 1,$$

see e.g. Marcus (1987) and Rosinski and Samorodnitsky (1993). We conclude that for any $0 \leqslant \rho \leqslant 1$

$$\lim_{\lambda \to \infty} \frac{P(Q_\rho(X) > \lambda)}{P(\operatorname{ess\,sup}_{t \in [0,1]} X(t) > \lambda)} = \rho. \tag{2.27}$$

Similarly,

$$\lim_{\lambda \to \infty} \frac{P(Q_\rho(X) < \lambda)}{P(\operatorname{ess\,inf}_{t \in [0,1]} X(t) < \lambda)} = 1 - \rho. \tag{2.28}$$

Compare this to Dassios (1995b).

3. Applications to stable processes

An important class of heavy tailed stochastic processes is that of α-stable processes, $0 < \alpha < 2$. In that case the random measure M in (1.3) is α-stable. That is, its Lévy measure F is given in the form

$$F(A \times B) = \int_A \left[\frac{1+\beta(s)}{2} \int_{(-\infty,0) \cap B} |x|^{-(1+\alpha)}\, \mathrm{d}x + \frac{1-\beta(s)}{2} \int_{(0,\infty) \cap B} x^{-(1+\alpha)}\, \mathrm{d}x \right] \gamma(\mathrm{d}s), \tag{3.1}$$

where γ is a σ-finite measure on $(S, \mathscr{A})$, called the *control measure*, and $\beta : S \to [-1, 1]$ is a measurable function, called the *skewness intensity* of M. See Samorodnitsky and Taqqu (1994) and Janicki and Weron (1994) for comprehensive reference on stable processes and measures. A straightforward computation shows that in this case

$$H_\rho(y) = K_\rho y^{-\alpha}, \quad y > 0,$$

where

$$K_\rho = \alpha^{-1} \int_S \left[\frac{1+\beta(s)}{2} \Big(Q_\rho(f(\cdot,s))_+ \Big)^\alpha + \frac{1-\beta(s)}{2} \Big(Q_\rho(-f(\cdot,s))_+ \Big)^\alpha \right] \gamma(\mathrm{d}s). \tag{3.2}$$

Similarly,

$$H_*(y) = K_* y^{-\alpha}, \quad y > 0,$$

where

$$K_* = \alpha^{-1} \int_S \operatorname{ess\,sup}_{t \in T} |f(t,s)|^\alpha \gamma(\mathrm{d}s). \tag{3.3}$$

Therefore, conditions (2.13) and (2.14) of Theorem 2.1 hold automatically for bounded α-stable processes. We immediately obtain the following corollary.

Corollary 3.1. *Let X be a measurable α-stable process given in the form*

$$X(t) = \int_S f(t,s) M(\mathrm{d}s), \quad t \in T,$$

where M is an α-stable random measure with control measure γ and skewness intensity β, and $f : S \times T \to \mathbb{R}$ is measurable. If

$$\operatorname{ess\,sup}_{t\in T}|X(t)| < \infty \quad a.s.$$

then for every $0 \leqslant \rho < 1$

$$\lim_{\lambda\to\infty} \lambda^\alpha P(Q_\rho(X) > \lambda) = K_\rho, \tag{3.4}$$

where K_ρ *is given by* (3.2).

Remark. In the context of α-stable random measures it is somewhat more common to use the normalized control measure, $\tilde{\gamma}$, such that for every $A \in \mathscr{A}$

$$|E e^{iM(A)}| = e^{-\tilde{\gamma}(A)},$$

see e.g. Samorodnitsky and Taqqu (1994, Chapter 3). The two measures, γ and $\tilde{\gamma}$, differ by a constant factor, and K_ρ in (3.4) can be expressed in terms of $\tilde{\gamma}$ as

$$K_\rho = C_\alpha \int_S \left[\frac{1+\beta(s)}{2}\Big(Q_\rho(f(\cdot,s))_+\Big)^\alpha + \frac{1-\beta(s)}{2}\Big(Q_\rho(-f(\cdot,s))_+\Big)^\alpha\right]\tilde{\gamma}(\mathrm{d}s),$$

where

$$C_\alpha = \left(\int_0^\infty x^{-\alpha}\sin x\,\mathrm{d}x\right)^{-1}.$$

We provide two examples.

Example 3.1 (*α-stable motion*). Let $(X(t), 0\leqslant t\leqslant 1)$ be an α-stable motion. Here, as in the context of Corollary 2.1, $S = [0,1]$ equipped with the Borel σ-field, the skewness intensity $\beta(\cdot)$ is identically equal to $\beta \in [-1,1]$, and the control measure γ is just the Lebesgue measure on $[0,1]$. The parameter space is once again $T = [0,1]$ with the Borel σ-field, and m is the Lebesgue measure on it. As before we have

$$f(t,s) = \mathbf{1}(t > s), \quad t,s \in [0,1],$$

and so

$$Q_\rho(f(\cdot,s)) = \mathbf{1}(\rho \geqslant s),$$

and

$$Q_\rho(-f(\cdot,s)) = 0,$$

$0 < s < 1$. It follows that

$$K_\rho = \frac{\rho(1+\beta)}{2\alpha},$$

so that

$$\lim_{\lambda\to\infty} \lambda^\alpha P(Q_\rho(X) > \lambda) = \frac{\rho(1+\beta)}{2\alpha}. \tag{3.5}$$

Example 3.2 (α-*stable Ornstein–Uhlenbeck process*). This process is commonly described in the form

$$X(t) = \int_{-\infty}^{t} e^{-\mu(t-s)} M(ds), \quad 0 \leqslant t \leqslant 1,$$

where M is an α-stable random measure on $S = (-\infty, 1]$, with the Lebesgue control measure γ and a constant skewness intensity β. μ is a positive parameter.

Since

$$f(t,s) = e^{-\mu(t-s)} \mathbf{1}(t > s), \quad 0 \leqslant t \leqslant 1$$

for $s \in (-\infty, 1]$, it is straightforward to compute that for every $0 \leqslant \rho < 1$,

$$Q_\rho\Big(f(\cdot,s)\Big) = \begin{cases} e^{-\mu(1-\rho-s)} & \text{if } s \leqslant 0, \\ e^{-\mu(1-\rho)} & \text{if } 0 < s \leqslant \rho, \\ 0 & \text{if } \rho < s \leqslant 1, \end{cases} \tag{3.6}$$

while

$$Q_\rho\Big(-f(\cdot,s)\Big) \equiv 0$$

for all $s \in (-\infty, 1]$. Therefore, by Corollary 3.1,

$$\lim_{\lambda \to \infty} \lambda^\alpha P(Q_\rho(X) > \lambda) = \left(\frac{(\rho + 1/(\alpha\mu))(1+\beta)}{2\alpha}\right) e^{-\alpha\mu(1-\rho)}. \tag{3.7}$$

4. Appendix: Quantiles of measurable processes

In this section we collect the results on measurable functions, processes and their quantiles that underly the discussion in the previous sections. We will state some of the results in two versions: one, concerned with measurable functions and stochastic processes with parameter belonging to an arbitrary probability space $(T, \mathscr{T}, m)$ (the "measure version"), and the other dealing with deterministic and random sequences of the type $(X_n, n \geqslant 1)$, in which case the quantiles are defined by

$$Q_\rho(X) = \inf\left\{x \in \mathbb{R}:\ \limsup_{n \to \infty} \frac{\sum_{i=1}^n \mathbf{1}(X_i \leqslant x)}{n} > \rho\right\}$$

(the "density version"). In the latter case ess sup and ess inf should be understood as the usual sup and inf correspondingly. The first lemma is elementary.

Lemma 4.1. *Let $f, g: T \to \mathbb{R}$ be two measurable functions. Then for every $0 \leqslant \rho < 1$,*

$$Q_\rho(f) + \operatorname{ess\,inf}_{t \in T} g(t) \leqslant Q_\rho(f+g) \leqslant Q_\rho(f) + \operatorname{ess\,sup}_{t \in T} g(t) \tag{4.1}$$

("measure version"). Moreover, if $T = \mathbb{N}$, then the corresponding "density version" of (4.1) *holds.*

The next lemma shows that sample quantiles of measurable stochastic processes are well defined random variables, whose distributions are determined by the finite dimensional distributions of the process.

Lemma 4.2. *Let $\boldsymbol{X} = (X(t),\, t \in T)$ be a measurable stochastic process that lives on a probability space $(\Omega, \mathscr{F}, P)$. Then $Q_\rho(\boldsymbol{X})$ is $\mathscr{F}$-measurable for every $0 \leqslant \rho < 1$. Moreover, if $\boldsymbol{Y} = (Y(t),\, t \in T)$ is another measurable process such that $\boldsymbol{X} \stackrel{\mathrm{d}}{=} \boldsymbol{Y}$, then $Q_\rho(\boldsymbol{X}) \stackrel{\mathrm{d}}{=} Q_\rho(\boldsymbol{Y})$.*

Proof. The first statement of the lemma is obvious. For the second one, we may assume, without loss of generality, that $\boldsymbol{X}$ and $\boldsymbol{Y}$ live on the same probability space. The argument leading to (2.16) shows that there is a a sequence $(\tau_1, \tau_2, \ldots)$ of points in T such that (2.16) holds a.s. and also

$$Q_\rho(\boldsymbol{Y}) = \inf\left\{x \in \mathbb{R}:\ \lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbf{1}\Big(Y(\tau_i) \leqslant x\Big)}{n} > \rho\right\}.$$

Since

$$\Big(X(\tau_1), X(\tau_2), \ldots\Big) \stackrel{\mathrm{d}}{=} \Big(Y(\tau_1), Y(\tau_2), \ldots\Big),$$

the statement of the lemma follows. □

In this paper we treat sample quantiles of particular infinitely divisible processes. The following proposition gives necessary and sufficient conditions on the integral representation of the process, for the latter to have a measurable version. This proposition makes the description of the problem in Theorem 2.1 meaningful. It also generalizes the corresponding results for α-stable processes known since Rosinski and Woyczynski (1986) and Samorodnitsky and Taqqu (1994). For this proposition we assume that T is a separable metric space, and $\mathscr{T}$ is the Borel σ-field on it. Note that this assumption is used only in the proof of the necessity part.

Proposition 4.1. *Let T be a separable metric space, and $\mathscr{T}$ the Borel σ-field on it. Let*

$$X(t) = \int_S f(t,s) M(\mathrm{d}s), \quad t \in T,$$

where M is an i.d. random measure on $(S, \mathscr{A})$ with σ-finite Lévy measure F. Then $(X(t),\, t \in T)$ has a measurable version if and only if there exists a measurable function $g: T \times S \to \mathbb{R}$ such that for every $t \in T$,

$$F\Big\{(s,x) \in S \times \mathbb{R}:\ x \neq 0,\ f(t,s) \neq g(t,s)\Big\} = 0. \tag{4.2}$$

Proof. *Sufficiency.* We start with the case of finite Lévy measure F. Observe that finiteness of F implies that the integral

$$W(t) = \int_S \int_{\mathbb{R}} f(t,s) \frac{x}{1+x^2} F(\mathrm{d}s, \mathrm{d}s) = \int_S \int_{\mathbb{R}} g(t,s) \frac{x}{1+x^2} F(\mathrm{d}s, \mathrm{d}s)$$

is well defined for all $t \in T$ (see Rajput and Rosinski, (1989)), and then by Fubini's theorem the function $W : T \to \mathbb{R}$ is measurable. Let μ be the total mass of F, and $(U_j, Z_j), j \geqslant 1$ be a sequence of i.i.d. random vectors in $S \times \mathbb{R}$ with common law $\mu^{-1}F$. Finally, let N be a Poisson random variable with mean μ independent of the i.i.d. sequence $(U_j, Z_j), j \geqslant 1$. Then $(X(t), t \in T) \stackrel{d}{=} (Y(t), t \in T)$, where

$$Y(t) = \sum_{j=1}^{N} U_j g(t, Z_j) - W(t), \ t \in T. \tag{4.3}$$

Now, the right-hand side of (4.3) is, obviously, a measurable stochastic process, and as such supplies a measurable version of $(X(t), t \in T)$.

In the general case of a σ-finite F, let $S \times \mathbb{R} = \cup_{j=1}^{\infty} A_j$, where A_j's are pairwise disjoint, and $F(A_j) < \infty$ for all $j \geqslant 1$. Let $F_j = F\mathbf{1}_{A_j}, j \geqslant 1$, and let $(M_j, j \geqslant 1)$ be independent infinitely divisible random measures , with M_j having finite Lévy measure F_j. The stochastic processes

$$X_j(t) = \int_S f(t,s) M_j(\mathrm{d}s), \quad t \in T,$$

$j \geqslant 1$, are well defined by Theorem 2.7 of Rajput and Rosinski (1989), independent, and moreover,

$$\Big(X(t), t \in T\Big) \stackrel{d}{=} \left(\sum_{j=1}^{\infty} X_j(t), t \in T\right),$$

with the sum converging a.s. for every $t \in T$. We have already proved that each $(X_j(t), t \in T)$ has a measurable version. Since the pointwise limit of measurable functions is measurable, we have constructed our measurable version of $(X(t), t \in T)$. This proves the sufficiency part of the proposition.

Necessity. Suppose that $\boldsymbol{X} = (X(t), t \in T)$ has a measurable version. Letting $\boldsymbol{X}_1$ be an independent copy of $\boldsymbol{X}$, we conclude that the process $\boldsymbol{Y} = \boldsymbol{X} - \boldsymbol{X}_1$ has a measurable version as well. The latter process has the same integral representation as $\boldsymbol{X}$ does, except that its Lévy measure $\tilde{F}$ is now symmetric in the sense that

$$\tilde{F}(A \times (-B)) = \tilde{F}(A \times B)$$

for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$. Indeed, for all such A and B

$$\tilde{F}(A \times B) = F(A \times (-B)) + F(A \times B).$$

In particular, F satisfies (4.2) if and only if $\tilde{F}$ does. We may assume, therefore, that the Lévy measure F of M is symmetric to start with.

We can represent the symmetric Lévy measure F in the form

$$F(A \times B) = 2 \int_A \left[\int_0^{\infty} \mathbf{1}_B(x) \eta(s, \mathrm{d}x) \right] \gamma(\mathrm{d}s), \tag{4.4}$$

$A \in \mathscr{A}$ and $B \in \mathscr{B}$ (see Rajput and Rosinski, 1989). Here γ is a control measure of M, and it can be chosen to be a probability measure. The family of measures

$(\eta(s,\cdot),\, s \in S)$ is a measurable family of one dimensional Lévy measures concentrated on $(0,\infty)$. Following Rosinski (1990) we define for $u > 0$ and $s \in S$

$$R(u,s) = \inf\{x > 0:\ \eta(s,(x,\infty)) \leqslant u\}.$$

Then a version of $\boldsymbol{X}$ is $\boldsymbol{Z}$, where

$$Z(t) = \sum_{n=1}^{\infty} \varepsilon_n R(\Gamma_n, \tau_n) f(t, \tau_n), \quad t \in T, \tag{4.5}$$

where $(\varepsilon_i,\, i \geqslant 1)$ is a sequence of i.i.d. Rademacher random variables: $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$, $(\Gamma_i,\, i \geqslant 1)$ is a sequence of arrival times of a unit rate Poisson process on $(0,\infty)$, and $(\tau_i,\, i \geqslant 1)$ is a sequence of i.i.d. S-valued random variables with common law γ. All three sequences are independent.

In particular, $\boldsymbol{Z}$ has a measurable version. Therefore, so does $\tilde{\boldsymbol{Z}}$, where

$$\tilde{Z}(t) = -\varepsilon_1 R(\Gamma_1, \tau_1) f(t, \tau_1) + \sum_{n=2}^{\infty} \varepsilon_n R(\Gamma_n, \tau_n) f(t, \tau_n), \quad t \in T.$$

Now, since $(T, \mathscr{T})$ is a separable metric space equipped with its Borel σ-field, it follows from Hoffmann-Jørgensen (1973) that measurable versions $\boldsymbol{Z}_1$ and $\tilde{\boldsymbol{Z}}_1$ of $\boldsymbol{Z}$ and $\tilde{\boldsymbol{Z}}$ can be chosen in such a way that $(\boldsymbol{Z}, \tilde{\boldsymbol{Z}}) \stackrel{\mathrm{d}}{=} (\boldsymbol{Z}_1, \tilde{\boldsymbol{Z}}_1)$. Therefore, the process $(\frac{1}{2}(Z(t) - \tilde{Z}(t)),\, t \in T)$ has a measurable version as well. Obviously,

$$\tfrac{1}{2}(Z(t) - \tilde{Z}(t)) = \varepsilon_1 R(\Gamma_1, \tau_1) f(t, \tau_1),\ t \in T.$$

Assume, for convenience, that the sequence $(\varepsilon_i,\, i \geqslant 1)$ lives on a probability space $(\Omega_1, \mathscr{F}_1, P_1)$, the sequence $(\Gamma_i,\, i \geqslant 1)$ lives on a probability space $(\Omega_2, \mathscr{F}_2, P_2)$, and the sequence $(\tau_i,\, i \geqslant 1)$ lives on the probability space $(S, \mathscr{A}, \gamma)$. It follows from Hoffmann-Jørgensen (1973) that there is a jointly measurable process $(U(t; \omega_1, \omega_2, s),\, t \in T, \omega_i \in \Omega_i,\, i = 1, 2,\, s \in S)$, such that for every $t \in T$,

$$\varepsilon_1(\omega_1) R(\Gamma_1(\omega_2), \tau_1(s)) f(t, \tau_1(s)) = U(t, \omega_1, \omega_2, s) \tag{4.6}$$

$P_1 \times P_2 \times \gamma$-a.s. In particular, for any $t \in T$ and γ-almost every $s \in S$, (4.6) holds $P_1 \times P_2$-a.s. Define

$$g(t,s) = \Big(P(R(\Gamma_1, s) > 0)\Big)^{-1} \times \int_{\Omega_1} \int_{\Omega_2} \frac{U(t; \omega_1, \omega_2, s)}{\varepsilon_1(\omega_1) R(\Gamma_1(\omega_2), \tau_1(s))} \mathbf{1}(R(\Gamma_1(\omega_2), \tau_1(s)) > 0) P_1(\mathrm{d}\omega_1) P_2(\mathrm{d}\omega_2),$$

$t \in T, s \in S$. Then by Fubini's theorem, g is a jointly measurable function of its variables, and for every $t \in T$,

$$\gamma\Big\{s \in S:\ f(t,s) \neq g(t,s)\Big\} = 0. \tag{4.7}$$

Now (4.2) follows from (4.7) and (4.4). This completes the proof. □

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